Summer school on Moduli of curves and Gromov–Witten theory

Institut Fourier, Grenoble, June 20th - July 1st, 2011

Abstracts of lectures

Carel FABER and Dimitri ZVONKINE

Title: Introduction to moduli spaces and their tautological cohomology and Chow ring

Abstract: This introduction to the intersection theory on moduli spaces of curves is meant to be as elementary as possible, but still reasonably short.

The intersection theory of an algebraic variety M looks for answers to the following questions: What are the interesting cycles (algebraic subvarieties) of M and what cohomology classes do they represent? What are the interesting vector bundles over M and what are their characteristic classes? Can we describe the full cohomology ring of M and identify the above classes in this ring? Can we compute their intersection numbers? In the case of moduli space, the full cohomology ring is still unknown. We are going to study its subring called the "tautological ring" that contains the classes of most interesting cycles and the characteristic classes of most interesting vector bundles.

To give a sense of purpose to the audience, we assume the following goal: after having followed the course, one should be able to write a computer program evaluating all intersection numbers between the tautological classes on the moduli space of stable curves. And to understand the foundation of every step of these computations.

Gavril FARKAS

Title: Birational geometry of moduli spaces of curves with level structure

Abstract: I will discuss the problem of describing moduli spaces of curves with various level structure, concentrating on the case of (higher order) Prym and spin moduli spaces. Topics to be treated include (i) compactifications of level structures, (ii) intersection theory on the moduli stack, (iii) singularities of coarse moduli spaces and (iv) explicit geometry of these moduli spaces in small genus.

Bibliography:

1. G. Farkas and K. Ludwig, The Kodaira dimension of the moduli space of Prym curves, JEMS 12(2010), 755-795.
2. G. Farkas, The birational geometry of the moduli space of even spin curves, Advances in Math 223(2010), 433-443.
3. G. Farkas and A. Verra, Moduli of theta-characteristics via Nikulin surfaces, arXiv:1104.0273
4. G. Farkas, Prym varieties and their moduli, arXiv:1104.2886.

Motohico MULASE

Title: Hurwitz numbers and new recursion formulae in GW theory

Abstract: Hurwitz numbers are simple objects, yet they lead us to many important ideas in Gromov-Witten theory. In this series of lectures, we start with defining these numbers, and then study the key equations that Hurwitz numbers satisfy. These equations include the Kadomtsev-Petviashvili equations and the combinatorial "cut-and-join" equations. Our goal is to illustrate the mathematical idea behind the solution to the Hurwitz number version of the celebrated Remodeling Conjecture on Gromov-Witten invariants of toric Calabi-Yau threefolds.

Rahul PANDHARIPANDE

Title: Stable quotients and relations in the tautological ring

Abstract: The topic concerns relations among the kappa classes in the tautological ring of the moduli space of genus g curves. After a discussion of classical constructions in Wick form, we derive an explicit set of relations obtained from the virtual geometry of the moduli space of stable quotients. In a series of steps, the stable quotient relations are transformed to simpler and simpler forms. Our final result establishes a previously conjectural set of tautological relations proposed a decade ago by Faber–Zagier.

Yuan-Pin LEE

Title: Introduction to Gromov-Witten theory and the crepant transformation conjecture

Abstract: In these lectures, Gromov–Witten theory will be introduced, assuming only basic moduli theory covered in the first week of the School. Then the Crepant Transformation Conjecture will be explained. Some examples, with emphasis on the projective/global cases, will be given.

Note: The construction of virtual fundamental class, which forms the foundation of the GW theory, will be given in Jun Li's concurrent lectures and will not be explained here.

Alessandro CHIODO and Yongbin RUAN

Title: Towards global mirror symmetry

Abstract: Mirror symmetry is a phenomenon which inspired fundamental progress in a wide range of disciplines in mathematics and physics in the last twenty years; we will review here a number of results going from the enumerative geometry of curves to homological algebra. These advances justify the introduction of new techniques, which are interesting in their own right. Among them, Gromov–Witten theory and its variants allow us to provide a refined statement of mirror symmetry. Of course this leads to further open questions (despite much effort and progress, Gromov–Witten theory remains unknown in high genus for the quintic threefold). In this course, we will illustrate the natural problem of moving beyond the local mirror symmetry statement and completing a framework of global mirror symmetry which is gradually taking shape. We will show how the missing piece in this picture comes unexpectedly from a classical subject in algebraic geometry: the theory of curves with level structures.

Plan of the course. The notes cover the material of the course of the summer school and more; they also contain a detailed discussion of several crucial points and many examples. Lecture 1 will present the problem of stating mirror symmetry beyond the local setup. Level structures are introduced as the geometric object completing the picture; they will be preliminarily approached via examples. Lecture 2 will present some of the material covered in Chapter I of the notes: the compactification of moduli of curves with level structures, the enumerative geometry, the Grothendieck–Riemann–Roch formula (this is interesting in its own right and is related to the first week course by Gavril Farkas). Lecture 3 will fit the theory of level curves into the mirror symmetry framework; this is the so-called Landau–Ginzburg model set up in Chapter II. Finally, Lecture 4 will provide a more general treatment of this global mirror symmetry framework, beyond the case of Calabi–Yau hypersurfaces, moving from a construction due to Berglund, Hübsch and Krawitz (Chapters III and IV in the notes).

Alessio CORTI

Title: Extremal Laurent Polynomial

Abstract: The course is an elementary introduction to my experimental work in progress with T. Coates, S. Galkin, V. Golyshev and A. Kasprzyk (http://coates.ma.ic.ac.uk/fanosearch/)

Topics:

1. Given a Laurent polynomial f, I explain how to construct the Picard-Fuchs differential operator L_f and its natural solution, the principal period. By definition, f is extremal if L_f has minimal ramification. I explain the general theory and give some examples. In particular I describe an interesting class of (conjecturally) extremal Laurent polynomials, called Minkowski polynomials.
2. I briefly summarize quantum cohomology of a Fano manifold X and quick-and-dirty methods of calculation. Much of the structure is encoded in a differential operator Q_X^{reg} and power series solution I_X^{reg}. I motivate with examples the conjecture that Q_X^{reg} is of small (often minimal) ramification.
3. A Fano manifold X is mirror-dual to a Laurent polynomial f if $Q_X^{\text{reg}} = L_f$. I demonstrate how to derive the classification of Fano 3-folds (Iskovskikh, Mori-Mukai) from the classification of 3-variable Minkowski polynomials. I outline a program to use these ideas in 4 dimensions.

Jun LI

Title: Cosection localized virtual cycle

Abstract: I will review the recent construction of localized virtual class via a cosection of the obstruction sheaf. This is built up on a reduction result on virtual normal cone and on the algebraic construction of localized Gysin maps. One of its application is that in some ideal case one can construct properly supported virtual class of a non-proper DM stack. This construction potentially will have a wide range of applications. For instance, the Witten's perturbed equation can be understood in this framework.

Davesh MAULIK

Title: Introduction to Donaldson-Thomas theory

Abstract: We will give an introduction to Donaldson-Thomas theory and some basic tools and computations. In the last lecture, we hope to explain some aspects of the proof of the GW/DT correspondence for toric threefolds.

Johan MARTENS

Title: Toroidal orbifold compactifications of reductive groups and moduli of framed bundles

Abstract: We will introduce a class of moduli problems for any reductive group G, whose moduli stacks provide us with (toroidal) equivariant compactifications of G. All toric varieties and orbifolds are special cases of these, as are the "wonderful compactifications" of semi-simple groups of adjoint type constructed by De Concini - Procesi. Our construction further provides a canonical orbifold compactification for any semi-simple group. We shall discuss how these moduli spaces are related to the concept of non-abelian symplectic cutting in symplectic geometry, and indicate connections with moduli spaces arising in gauge theory. This is joint work with Michael Thaddeus (Columbia).