Lectures on quasi-isometric rigidity of 3-manifold groups

Misha Kapovich

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1 Plan

1. What do 3-dimensional manifolds (3-manifolds) look like?
2. Metrics of non-positive curvature on 3-manifolds.
3. Asymptotic cones of the fundamental groups of 3-manifolds.
4. Topological rigidity of the asymptotic cones.
5. Quasi-isometric rigidity of the geometric decompositions of 3-manifold groups.
6. Quasi-isometric rigidity of 3-manifold groups.

2 Introduction

One of the central goals of the geometric group theory is to see how much algebraic information about a group could be recovered from the geometric information about it Cayley graph.

The prototypical examples go back to Stallings's work on the ends of groups in late 1960-s:

Theorem 2.1. *(J.Stallings) A (finitely generated) group G splits nontrivially as an amalgam over finite groups if and only if G has infinitely many ends. In particular, if groups G, G' are quasi-isometric then G splits if and only if G' does.*

This theorem has a further refinement: Quasi-isometries of G preserve its decomposition as an amalgam (over finite groups) in the following sense:

Suppose that a finitely generated group G is isomorphic to $\pi_1(\Gamma, G_v, G_e)$, the fundamental group of a graph of groups where each edge group is finite and each vertex group has at most 2 ends.

Theorem 2.2. *Suppose that $f : G \to G$ is a quasi-isometry. Then f sends each 1-ended vertex group G_v of G to a subset $f(G_v) \subset G$ so that there exists a 1-ended vertex subgroup G_w and its conjugate gG_wg^{-1} in G so that the Hausdorff distance between $f(G_v)$ and gG_wg^{-1} is finite.*

This result was very recently generalized by Panos Papasoglou [7] to the case of decompositions of 1-ended groups over 2-ended subgroups.

The goal of these lectures is to explain another generalization of this theorem, namely in the context of 3-manifolds. Since freely decomposable 3-manifold groups are already covered by Stallings' theorem (and the groups with ≤ 2 ends are easy to understand), we will concentrate on the freely indecomposable ones. They are fundamental groups of *aspherical* 3-manifolds, i.e. 3-manifolds with contractible universal covers. According to Thurston's Geometrization Conjecture (GC), such manifolds are *geometrizable*, i.e. admit a *geometric decomposition* which will be defined in the next section.

Theorem 2.3. *(M.Kapovich, B.Leeb, [5]) Let M be a closed aspherical 3-manifold satisfying GC. Then the geometric decomposition of the universal cover of M is preserved by quasi-isometries.*

Corollary 2.4. *(M.Kapovich, B.Leeb, [5]) Suppose that G is the fundamental group of a manifold M which appears in the above theorem and which is not a Sol-manifold. Let G' be a group which is quasi-isometric to G. Then there is a short exact sequence*

$$1 \to K \to G' \to \bar{G} \to 1,$$

with K a finite group and $\bar{G}$ the fundamental group of a geometrizable 3-dimensional orbifold. E.g., $\bar{G}$ contains a finite index subgroup which is the fundamental group of a closed geometrizable aspherical 3-manifold.

There are many words in this section which are probably unfamiliar to many of you, see the next section for the explanations.

3 Geometric decomposition of 3-manifolds

Throughout the rest of these notes we will consider only aspherical 3-manifolds.

Definition 3.1. *A manifold is said to be* closed *if it is compact and has empty boundary.*

Most of what I say will hold for compact 3-manifolds whose boundary consists of tori and Klein bottles, but I will stick to closed manifolds to simplify the language.

In dimension 3 TOP=PL=DIFF (Moise), i.e. each topological 3-manifold admits a unique PL/smooth structure. Hence throughout I will be working in the category of differentiable manifolds, assuming for simplicity that all 2- and 3-manifolds are orientable.

Loosely speaking, the goal of the Geometrization Conjecture (GC) is to generalize the classification of surfaces by their genus.

Definition 3.2. *A* **geometry** *is a simply-connected homogeneous unimodular Riemannian manifold X. Unimodularity means that X admits a discrete group of isometries with compact quotient.*

A lá Felix Klein we will be identifying geometry with its group of isometries.

Definition 3.3. *A compact manifold M is called* **geometric** *if $int(M) = X/\Gamma$ has finite volume, where X is a geometry and Γ is a discrete group of isometries of X acting freely.*

3-dimensional geometries (the first 5 are symmetric spaces):

- $S^3, \mathbb{E}^3, \mathbb{H}^3$, are the constant (sectional) curvature geometries.
- $S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$ are the product geometries.
- $Nil, Sol, \widetilde{SL_2}(\mathbb{R})$ are the twisted product geometries.

Note that only the spherical geometry is compact. The hyperbolic geometry is the most interesting one. See [10] for a detailed discussion of these geometries.

Decomposition of 3-manifolds:

Assume that M is closed (compact, no boundary).

Step 1: Connected sum decomposition of M into *prime* pieces (closed manifolds which cannot be decomposed further).

Step 2. If M is prime, consider a *toral* decomposition of M along *incompressible* [1] tori into *simple* pieces (the ones which cannot be decomposed further). Note that simple pieces typically have nonempty toral boundary.

Both decomposition processes terminate (Kneser, Haken: theory of normal surfaces).

Uniqueness of the decompositions: (1) Components of the connected sum decomposition are uniquely determined by M (Milnor). (2) The toral decomposition is unique up to isotopy if we consolidate simple pieces into maximal geometric pieces (Jaco, Shalen; Johannson).

Similar decompositions exist for compact manifolds with boundary.

Thurston's Geometrization Conjecture (GC): *Each prime closed 3-manifold M is either geometric or its simple pieces are geometric.*

A similar conjecture can be stated (and is proven by Thurston!) if M has nonempty boundary.

A restatement of the GC: Each closed prime 3-manifold is either geometric or it splits along disjoint incompressible tori as $M_{thick} \cup M_{thin}$, where M_{thick} is a disjoint

[1] I.e. π_1-injective.

union of hyperbolic manifolds, and M_{thin} is a *graph-manifold*, i.e. a manifold obtained by gluing along boundary tori of geometric 3-manifolds which are **not** modeled on $\mathbb{H}^3$.

Graph-manifolds are interesting and well-understood objects, they appear for instance in theory of complex surface singularities. Example of a graph-manifold: let Σ be a surface of genus ≥ 1 with one boundary circle, M_1, M_2 are copies of $\Sigma \times S^1$. Now glue M_1, M_2 along their boundary tori.

Omnibus Theorem (Thurston et al.):

(1) GC is equivalent to the conjunction of PC (Poincare conjecture), SSFC (spherical space form conjecture) and HC (Hyperbolization conjecture).

(2) (Thurston) GC holds if M is prime but not simple.

(3) (Thurston) GC holds for *Haken manifolds*[2]. (For proofs of this theorem, which includes (2) as a special case, see [6], [3].)

(4) If M is (prime) aspherical then GC holds for $M \iff$ GC holds for all manifolds finitely covered by $M \iff$ GC holds for all (prime) manifolds which are homotopy-equivalent to M.

(5) GC holds if $\pi_1(M)$ contains $\mathbb{Z} \times \mathbb{Z}$ or has infinite center.

Explanation:

PC: If M is homotopy-equivalent to the sphere then it is diffeomorphic to the sphere. Equivalently, if M is (closed) simply-connected, then $M = S^3$.

SSFC: If the universal cover of M is the 3-sphere then M admits a metric of (positive) constant curvature, i.e. it is geometric, modeled on S^3.

HC: If M is prime, aspherical (i.e. its universal cover is contractible) and $\pi_1(M)$ does not contain $\mathbb{Z} \times \mathbb{Z}$ then M is hyperbolic.

HC is the most interesting (although, not the most famous) of the 3 parts of the geometrization conjecture.

A confidence-building exercise: GC implies PC. Indeed, suppose that M is closed and simply-connected. Consider connected sum decomposition of M into prime components $M_1, ..., M_k$. Then each M_i is also closed and simply-connected. Since $\pi_1(M_i)$ is trivial, M_i contain no incompressible tori, hence, by GC, M_i is geometric. Since the only compact 3-dimensional geometry is spherical, we conclude that $M_i = S^3$ for each i. Hence $M = S^3$ as well.

The above described the status of the GC until November of 2002, when Grisha Perelman announced a proof of the GC.

Definition 3.4. *A compact (aspherical) 3-manifold is called* Seifert *if any of the following equivalent properties hold:*

1. M admits a foliation by circles.

[2] I.e. M is prime and contains an incompressible surface: a π_1-injective surface which is not S^2.

2. M admits a finite cover which is a circle bundle over a surface.

3. M is irreducible and there is a short exact sequence

$$1 \to \mathbb{Z} \to \pi_1(M) \to F \to 1,$$

where F is a discrete isometry group of $\mathbb{H}^2$.

4. M admits a geometric structure modelled on $\mathbb{R}^3, Nil, \mathbb{H}^2 \times R$ or $\widetilde{SL}_2(\mathbb{R})$.

Example 3.5. *Let S be a surface and $\phi : S \to S$ be a finite order diffeomorphism. Then the mapping torus $M = S \times [0,1]/\phi$ is a Seifert manifold. To see the foliation by circles consider the projections to M of the segments $\{x\} \times [0,1]$: Since ϕ has finite order, finite union of such projections is a circle in M.*

The geometric decomposition of M lifts to a *geometric decomposition* of the universal cover $X = \tilde{M}$ of the manifold M: The tori and Klein bottles lift to 2-planes in X which split X into *geometric* components of either hyperbolic or Seifert type. We will refer to the above planes as *splitting planes.*

Definition 3.6. *Suppose that X, X' are universal covers of geometrizable 3-manifolds. We say that an (L, A)–quasi-isometry $f : X \to X'$* preserves the geometric decomposition *if there exists a number $C = C(L, A)$ such that:*

For each geometric component Y of X (resp. a splitting plane) there exists a unique geometric component Y' of X' (resp. a splitting plane) such that Hausdorff distance between $f(Y)$ and Y' is $\leq C$.

We observe that a quasi-isometry f which preserves the geometric decomposition, induces an isomorphism $f_* : T \to T'$ of the trees T, T' dual to the geometric decompositions of X, X'.

Definition 3.7. *A* graph-manifold *is a closed geometrizable 3-manifold M without hyperbolic components, i.e. all its geometric components are Seifert or Sol-manifolds. M is called a* proper *graph manifold if it is not geometric (i.e if it neither Seifert nor Sol-manifold).*

Conjecture 3.8. *Fundamental groups of all proper graph-manifolds are quasi-isometric to each other.*

A closed geometrizable manifold M is called a *flip-manifold* if the following holds:

Each Seifert component M_j of M is a product of a compact orientable surface and a circle: $M_j = S_j \times S^1$. This decomposition defines a two circle foliations on the boundary of M into horizontal circles (contained in $\partial S_j \times \{t\}$, $t \in S^1$) and vertical circles (of the form the $\{x\} \times S^1$, $x \in \partial S_j$). *We now require that each gluing map between boundary tori of Seifert components of M interchanges (flips) vertical and horizontal foliations.*

The first step towards proving Theorem 2.3 is the following

Theorem 3.9. *([4]) Let M be a closed geometrizable manifold which is not a Sol or Nil manifold. Then there exists a flip-manifold M' and a quasi-isometry $f : \tilde{M} \to \tilde{M}'$ which preserves the geometric decomposition.*

The most basic special case of this theorem is the following, which was independently observed by Epstein, Gersten and Mess in late 1980-s:

Proposition 3.10. *$\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL}_2(\mathbb{R})$ are quasi-isometric.*

Proof: Note that $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL}_2(\mathbb{R})$ are cyclic isometric covers of $\mathbb{H}^2 \times S^1$ and of the unit tangent bundle $U\mathbb{H}^2$, respectively. I will describe a bilipschitz homeomorphism $h : U\mathbb{H}^2 \to \mathbb{H}^2 \times S^1$ which then will lift to a quasi-isometry between the universal covers.

Given a pair of points $x, y \in \mathbb{H}^2$ let $\Pi_{xy} : U_x\mathbb{H}^2 \to U_y\mathbb{H}^2$ denote the parallel transport of the unit tangent spaces along the unique geodesic from x to y.

Pick a base-point $o \in \mathbb{H}^2$ and define the map h by sending

$$(x, v) \mapsto (x, \Pi_{xo}(v)),$$

where $v \in U_x\mathbb{H}^2$. Let's check that this map is Lipschitz. Let $(x, v), (y, w)$ be nearby points in $U\mathbb{H}^2$:

$$d((x, v), (y, w)) = d(x, y) + \angle(v, \Pi_{yx}(w)),$$

where $\alpha = \angle(v, \Pi_{yx}(w))$ stands for the angle in the unit circle $U_x\mathbb{H}^2$. (This ad hoc metric is invariant under isometries and thus is as good as any.) Then the angle between the vectors

$$\Pi_{xo}(v), \Pi_{yo}(w) \in U_o\mathbb{H}^2$$

is at most $\alpha + \delta$, where δ is the angle deficit of the geodesic triangle $\Delta = \Delta(oxy)$, i.e. the difference between π and the sum of the angles of this triangle. However, by the Gauss-Bonnet formula,

$$\delta = Area(\Delta).$$

Moreover, there exists a constant C such that if one side of Δ is $\leq d$ then

$$Area(\Delta) \leq Cd.$$

Therefore the mapping h is Lipschitz.

4 Metrics of non-positive curvature on geometrizable 3-manifolds

Not all geometrizable 3-manifolds admit metrics of non-positive curvature. First of all, by Cartan-Hadamard theorem the universal cover of such a manifold has to be contractible, which is yet another reason to stick to aspherical manifolds. There are however more interesting obstructions. Suppose that $t \in \pi_1(M) = G$ is a nontrivial element whose centralizer in G is a subgroup $H \subset G$. Consider the isometric action of G on $X = \tilde{M}$. Since M is compact, g has an *axis*, i.e. an invariant geodesic l. Recall that the union of axes of t is a closed convex subset which splits as a product $Y \times l$. This set has to be preserved by the group H, since for each $h \in H$, $h(l)$ is an axis of $hth^{-1} = t$. For the same reason, the action of H on $Y \times l$ preserves the splitting as well as the orientation on l. Thus each element of the commutator subgroup of H acts trivially on the l-factor. Note also that t acts trivially on Y (since t is a translation along each of its axes). To be more specific, assume now that $Q = H/\langle g\rangle$ is the fundamental group of a closed oriented surface, and hence H is a finite index subgroup in $\pi_1(M)$ has the presentation

$$\langle a_1, b_1, ..., a_n, b_n, t | [a_i, t] = 1, [b_i, t] = 1, \prod_{i=1}^{n} [a_i, b_i] = t^m \rangle.$$

However the action of $[a_i, b_i]$ is trivial on l, the action of t^m is trivial on Y. Therefore $t^m = 1$. Since G is torsion-free, it follows that $m = 0$ and hence H splits as a direct product of $\mathbb{Z}$ and the surface group.

We therefore conclude that Nil and $\widetilde{SL}_2(\mathbb{R})$ manifolds do not admit metrics of non-positive curvature. A minor variation on the above argument shows that Sol manifolds do not admit metrics of non-positive curvature either.

Once you understand the above example you should have no difficulty generalizing it to

Example 4.1. *Suppose that S_1, S_2 are closed oriented surfaces (of genus ≥ 1) each having one boundary component; let $M_i := S_i \times S^1$. Consider graph-manifolds M obtained by gluing M_1, M_2 along the boundary tori. Then M admits a metric of non-positive curvature if and only if it is either product of a surface and a circle or it is a flip-manifold. I.e., if and only if the gluing map either matches horizontal and vertical foliations of the boundary tori (possibly reversing their orientation) or it flips them.*

Hint: 1. Let $t_i \in \pi_1(M)$ be the generator of the center of $\pi_1(M_i)$ and let $h_i \in \pi_1(M)$ be the element corresponding to the boundary circle of S_i. Suppose that M does admit a metric of non-positive curvature and l_i, m_i are axes of t_i, h_i in $\tilde{M}$ which happen to belong to a common 2-flat. Then l_i is orthogonal to m_i.

2. Let T^2 be flat torus with the "rectangular" metric of product of two circles. Then each isometry of T^2 preserves the product structure of the torus or "flips" it interchanging the factors.

More generally one has

Theorem 4.2. *Suppose that M is a flip-manifold. Then M admits a metric of non-positive curvature.*

It turns out that one can completely classify geometrizable 3-manifolds M which admit metrics of non-positive curvature:

Theorem 4.3. *(B. Leeb, [8]) Suppose that M contains at least one hyperbolic component. Then M admits a metric of non-positive curvature.*

The situation in the case of graph-manifolds is more subtle, I refer you to the paper of Buyalo and Kobelsky [1].

In any case, Theorems 3.9 and 4.2 reduce Theorem 2.3 to the case when M admits a metric of non-positive curvature.

5 Asymptotic cones

We first have to figure out how asymptotic cones of geometric components look like.

Case 1. Let M be an $\mathbb{H}^2 \times \mathbb{R}$–Seifert manifold, possibly with nonempty boundary. Then each asymptotic cone X_ω of $X = \tilde{M}$ is isometric to the product of a tree T with $\mathbb{R}$. The tree T branches at every point. By abusing language we will refer to the cone X_ω as "Seifert".

Proposition 5.1. *Each topological 2-flat F in X_ω is a flat.*

Proof: Pick a point $x \in F$, then $x \in \{y\} \times \mathbb{R}$, where $y \in T$. Then, since $L = \{y\} \times \mathbb{R}$ separates $T \times \mathbb{R}$, it has to separate F as well. On the other hand, if L is not entirely contained in $Y \times \mathbb{R}$, the intersection $L \cap F$ is a proper subset of the straight line. Therefore

$$H^1_c(L \cap F) = 0.$$

Now, Alexander-Poincare duality implies that $L \cap F$ cannot separate F. Therefore, if P is the projection of F to T, then $F = P \times \mathbb{R}$. It is now clear that P is a line and thus F is a flat. □

Case 2. Suppose that M is a compact manifold with nonempty boundary, whose interior admits a complete hyperbolic metric. Then M admits a metric of non-positive curvature with flat totally-geodesic boundary. Then each asymptotic cone X_ω of $X = \tilde{M}$ is a "tree of flats":

1. If p_α, p_β are the nearest-point projections to F_α, F_β then $p_\alpha(F_\beta) = \{x_\alpha\}, p_\beta(F_\alpha) = \{x_\beta\}$.

2. Every continuous path in X_ω connecting F_α to F_β passes through the points $x_\alpha \in F_\alpha$, $x_\beta \in F_\beta$.

3. Each flat $F_\alpha \in \mathbb{F}$ is represented by a sequence of peripheral planes of X.

By abusing language we will refer to the cone X_ω as "hyperbolic".

Proposition 5.2. *Each topological 2-flat F in X_ω is one of the flats F_α.*

Proof: Otherwise there exists a point $x \in X$ such that F intersects two distinct connected components of $X_\omega \setminus \{x\}$. However a point cannot separate a plane. □

Corollary 5.3. *There are no topological embeddings to Seifert asymptotic cones into hyperbolic asymptotic cones.*

Global picture: *Geometric components* of X_ω are ultralimits of sequences of geometric components of X. *Splitting flats* in X_ω are ultralimits of sequences of splitting flats in X.

The ultralimits F_ω of sequences of splitting flats in X come equipped with *coorientation*: $X_\omega \setminus F_\omega$ splits into *positive and negative sides.*

A subset $A \subset X_\omega$ is *essentially split* by F_ω if it contains points which lie in the opposite sides. Similarly one defines essential splitting by geometric components of X_ω.

Lemma 5.4. *Suppose that $A \subset X_\omega$ is not essentially split by any splitting flat. Then A is contained in a single geometric component of X_ω.*

Lemma 5.5. *Each 2-flat in X_ω is contained in a single geometric component and appears as ultralimit of a sequence of flats in X.*

Similar assertion fails for bilipschitz 2-flats. However we have

Lemma 5.6. *Let $B \subset X_\omega$ is a bilipschitz 2-flat. Then the following are equivalent:*

1. *The intersection of B with each bilipschitz 2-flat B' contains at most 1 point.*
2. *B is a splitting flat which is not contained in any Seifert component.*

Lemma 5.7. *Suppose that $B \subset X_\omega$ is a bilipschitz flat. Then*

1. *B is contained in a finite union of flats $F_i, i = 1, ..., m$, in X_ω.*
2. *Each $F_i \subset X_{i\omega}$ where $X_{i\omega}$ are consecutive Seifert components.*

Corollary 5.8. *A bilipschitz flat in X_ω is not contained in the sublevel set of any Busemann function on X_ω.*

We have a better control on behavior of bilipschitz embeddings of $T \times \mathbb{R}$:

Lemma 5.9. *Suppose that T is a geodesically complete tree which branches at each point. Then for each bilipschitz embeddings $f : T \times \mathbb{R} \to X_\omega$ the image of f is contained in a single Seifert component and the map f preserves fibration of $T \times \mathbb{R}$ by lines.*

We now can prove rigidity of bilipschitz homeomorphisms of X_ω:

Theorem 5.10. *Let X, X' be universal covers of geometrizable nonpositively curved 3-manifolds. Suppose that $f : X_\omega \to X'_\omega$ is a bilipschitz homeomorphism. Then:*

1. *Each splitting flat which is not contained in a Seifert component is mapped to a splitting flat of the same kind.*
2. *The image of each Seifert component of X_ω is again a Seifert component.*
3. *f maps flats to flats.*

Proof: 1. Suppose that $F \subset X_\omega$ is a splitting flat which is not contained in any Seifert component. Then it follows from Lemma 5.7 that F intersects each bilipschitz flat in at most 1 point. This property is clearly preserved by f. Hence, according to Lemma 5.6, the image $f(F)$ is a splitting flat which is not contained in any Seifert component.

2. By Lemma 5.9, the image of each Seifert component of X_ω is contained in a Seifert component in X'_ω. By applying the same argument to f^{-1} we see that the image is the entire Seifert component of X'_ω.

3. If F is a splitting flat which is not contained in any Seifert component of X_ω then the $f(F)$ is a splitting flat by (1).

Otherwise F is contained in a Seifert component $Y_\omega \subset X_\omega$. By (2), the mapping f sends $Y\omega$ onto a Seifert component Y'_ω. However each bilipschitz flat in a geometric component of X'_ω is again a flat. □

Observe that at this stage we do not know if the image of a hyperbolic component of X_ω is a hyperbolic component of X'_ω. The reason for this is quite simple: Our technique was mostly topological. However topologically we cannot tell apart a hyperbolic component of X_ω from a union of two such adjacent components.

6 Down to earth

Commercial of the day:

Never underestimate the power of asymptotic cone. Buy infinitely many for the price of one!

It's time now to get back from asymptotic cones to the geometry of universal covers X of 3-manifolds.

The main problem in deriving information about the properties of X from the properties of the asymptotic cones X_ω is the nature of the transition from X to X_ω: It is obtained by rescaling. If something is constant in the ultralimit, it may be far from being constant in X. Consider for instance the graph of the function $\sqrt{t}$ in $\mathbb{R}^2$: It is not contained within finite distance from any geodesic ray in $\mathbb{R}^2$. However, if we pass the asymptotic cone of $\mathbb{R}^2$, the *sublinear* function $\sqrt{t}$ becomes constant, so its graph degenerates into a geodesic ray. At the first glance it spells real trouble: How could we hope to control quasiflats in X using its asymptotic cones if we cannot do it for $\mathbb{R}^2$? However, by picking an appropriate rescaling and using the fact that asymptotic cones of X contain no 3-flats one gets:

Lemma 6.1. *(Divergence lemma) Suppose that an (L, A)- quasi-flat Q in X diverges sublinearly from a flat F. Then the Hausdorff distance between Q and F is at most $C(L, A)$.*

Combining this lemma with Theorem 5.10 we obtain:

Theorem 6.2. *Suppose that $f : X \to X'$ is an (L, A)-quasi-isometry. Then f maps each flat in X within distance $\leq C(L, A)$ from a flat $f(F)^*$ in X.*

To conclude from this that f preserves the geometric decomposition of X we use

Lemma 6.3. *Suppose that F_1, F_2, F_3 are splitting flats in X which do not separate each other. Then the flats $f(F_i)^* \subset X'$ do not separate each other either.*

This implies Theorem 2.3.

One can also classify quasi-flats in X using classification of bilipschitz flats in X_ω, we will discuss this if the time permits...

7 Quasi-isometric rigidity of 3-manifold groups

Suppose that M is a non-positively curved geometrizable 3-manifold with nontrivial geometric decomposition, $G = \pi_1(M)$, and suppose that G' is a group quasi-isometric to G, in particular it is quasi-isometric to $X = \tilde{M}$ via a quasi-isometry $f : X' \to X$, where X' is a Cayley graph of G'. Let $\bar{f} : X \to X'$ denote a quasi-inverse to f. Using this one defines a *quasi-action* of G' on X, a map $\phi : G' \to QI(X)$ to the set of quasi-isometries of X given by

$$\phi(g) = f \circ g \circ \bar{f}.$$

The map ϕ satisfies

$$d(\phi(g_1 g_2), \phi(g_1) \circ \phi(g_2)) \leq Const, \forall g_1, g_2 \in G'.$$

Moreover, since the action of G' on X' is isometric, the elements $\phi(g), g \in G'$ are (L, A)-quasi-isometries of X for uniform constants L, A.

Although the quasi-action of G' is not an action, using the main theorem from previous section, we get an actual action of G' on the simplicial tree T dual to the geometric decomposition of X. This determines a structure of a fundamental group of a graph of groups on G':

$$G' = \pi_1(\Gamma, G'_v, G'_e)$$

where the edge groups are "quasi-stabilizers" of the splitting flats in X and the vertex groups are quasi-stabilizers of the geometric components of X.

Therefore each edge group is commensurable to $\mathbb{Z}^2$ and each vertex group is quasi-isometric to the fundamental group of the corresponding geometric component of M. The key now is the following

Theorem 7.1. *(R. Schwartz, [9]) Suppose that M is a complete noncompact hyperbolic n-manifold of finite volume and G is a group quasi-isometric to $\pi_1(M)$. Then there is a short exact sequence*

$$1 \to K \to G \to \bar{G} \to 1$$

where K is finite and $\bar{G}$ is a nonuniform lattice in $Isom(\mathbb{H}^n)$ which is commensurable to $\pi_1(M)$.

A similar (and easier) result holds when M is a Seifert 3-manifold with nonempty boundary, it is essentially due to E. Rieffel (who proved such statement in the case of manifolds without boundary).

By applying these results to the group G' as above we obtain

Corollary 7.2. *There is a short exact sequence*

$$1 \to K \to G' \to \bar{G} \to 1,$$

with K a finite group and $\bar{G}$ the fundamental group of a geometrizable 3-dimensional orbifold.

Problem 7.3. *Suppose that G is a group quasi-isometric to Sol. Then G is commensurable to a lattice in Sol.*

Note that virtual solvability is not a quasi-isometry invariant, see [2].

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