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Contents

INTRODUCTION (en français).

0.1. Le rôle des variétés Grassmanniennes et des variétés de Fano.

L étude d'une variété peut se faire moyennant ses sousvariétés. Par exemple, une hypersurface de degré 2 dans un espace projectif peut être etudiée par la famille des droites qu'elle contient. On peut procéder à une utilisation systématique des espaces linéaires de dimension s dans une variété X d'un espace projectif donné $\mathbf{P}^n$ en introduisant la Grassmannienne $\mathrm{Gr}(s, \mathbf{P}^n)$ des s-plans dans $\mathbf{P}^n$ et la *variété de Fano* associée:

$$F(s, X) := \{\Gamma \in \mathrm{Gr}(s, \mathbf{P}^n) | \Gamma \subset X\}$$

On sait par exemple que pour une hypersurface générique X de degré d (avec $d \geq 3$) on a:

$$\varphi(s, n, d) := \dim F(s, X) = (s+1)(n-s) - \begin{pmatrix} s+d \\ s \end{pmatrix}$$

Cela veut dire que si X est une hypersurface générique de degré d telle que $\varphi(s, n, d) < 0$, alors elle ne contient pas de s-plans; et que si par contre $\varphi(s, n, d) \geq 0$, la variété de Fano n'est pas vide et a la dimension $\varphi(s, n, d)$.

Donc, *grosso modo*, plus d est petit, et plus il y aura de variétés linéaires dans une hypersurface de degré d.

Une question plus fine se pose alors: à l'équivalence rationnelle près, peut-on dire que tout s-cycle est une combinaison linéaire de s-plans? Cela implique une étude des groupes de Chow $\mathrm{CH}_s(X)$ des s-cycles modulo l'équivalence rationnelle (plus de détails en Ch. 1.2). En effet, Roitman a étudié les 0-cycles sur des variétés $X \subset \mathbf{P}^n$ données par des équations de petit degré. La variété X n'a pas besoin d' être lisse ni d' être une intersection complète. Le résultat est le suivant:

Théorème 0.1.1.[RO]*Soit X une variété projective irréductible de dimension $n-r$, définie par r équations de degré $d_1, \ldots, d_r$, tels que $\sum_i d_i \leq n$. Alors:*

$$\mathrm{CH}_0(X) = \mathbf{Z}$$

□

On verra plus bas une idée de la démonstration pour $\mathrm{CH}_0(X) \otimes \mathbf{Q}$ et pour une hypersurface X de degré d.

Inspiré par cette démonstration géométrique, [ELV] étend ce résultat aux groupes de Chow de degrés supérieurs. Les variétés de Fano jouent ici un rôle essentiel. Voici le résultat:

Théorème 0.1.2. [ELV] *Soit X une variété projective irréductible de dimension $(n-r)$, définie par r équations de degré $d_1, \dots, d_r$ tels que $d_1 \geq d_2 \geq \dots \geq d_r \geq 2$. Soit:*

$$l = \max_{k \in \mathbf{N}} \left\{ k \middle| \sum_{i=1}^{r} \begin{pmatrix} k+d_i \\ k+1 \end{pmatrix} \leq n \right\}$$

Soit $d_1 \geq 3$ ou $r \geq l+1$. Alors:

$$\mathrm{CH}_s(X) \otimes \mathbf{Q} = \mathbf{Q} \qquad \forall s \leq l$$

□

Donc $\mathrm{CH}_s(X) \otimes \mathbf{Q}$ est engendré par n'importe quel s-plan contenu dans X, mais en fait la démonstration utilise des proprietés de la variété de Fano $F(s, X)$.

On trouvera plus bas l'idée de la démonstration dans le cas d'une hypersurface et pour $l = 1$, ce qui est le premier cas dans le prolongement du travail de Roitman.

0.2. Généralisation des résultats aux espaces projectifs tordus.

Il est naturel de se demander s'il est possible de généraliser les résultats de [ELV] au cas des espaces projectifs tordus et des variétés définies par des polynômes Q-homogènes de bas degrés. C'est ce que nous faisons dans le Chapitre 2 pour des hypersurfaces de degré d dans $\mathbf{P}(q_0, \dots, q_n)$. Voici le résultat principal:

Théorème 0.2.1. *Pour une hypersurface ponderée $X = X'/\mu \subset \mathbf{P}(Q)$ de Q-degré d, $d \geq 3$, quotient d'une hypersurface lisse X' sous l'action du groupe produit des groupes des racines des poids q_j, on a:*

$$\mathrm{CH}_s(X) \otimes \mathbf{Q} = \mathbf{Q} \qquad \forall s \text{ tel que } \begin{pmatrix} s+d \\ s+1 \end{pmatrix} \leq \sum_{i=0}^{n} q_i - 1$$

□

Le théorème est réduit au théorème de [ELV] pour une hypersurface de degré d dans $\mathbf{P}^N$, où $N := \sum_{i=0}^{n} q_i - 1$, ce qui explique la borne.

L'idée est de dominer l'hypersurface par une hypersurface de $\mathbf{P}^N$, bien choisie, et du même degré. En fait, on a une application:

$$\sigma_Q : \tilde{X} \to X$$

qui, sous des conditions assez générales, ici satisfaites, induit une application surjective:

$$\sigma_{Q*} : \mathrm{CH}_*(\tilde{X}) \otimes \mathbf{Q} \to \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

permettant de conclure.

Cette démonstration ne dit pourtant rien sur les espaces "linéaires" qui seraient contenus dans X. En fait, on devrait d'abord bien définir ce que l'on entend par "espace linéaire" dans $\mathbf{P}(Q)$. Pour cela, on peut utiliser l'application quotiente:

$$\mathbf{P}^n \xrightarrow{p_Q} \mathbf{P}(Q) = \mathbf{P}^n/\mu$$

où $\mu = \mu_{q_0} \times \ldots \times \mu_{q_n}$ agit sur la coordonnée x_j par produit avec une q_j-racine de l'unité. On définit alors un s-plan de $\mathbf{P}(Q)$ comme le quotient d'un s-plan de $\mathbf{P}^n$. Cela conduit à une "petite" Grassmannienne qui pourtant est aisément comprise, ainsi qu'à une généralisation des résultats de [ELV], mais l'amélioration de la borne est perdue.

Dans le dernier paragraphe du Chapitre 2, nous avons fait une comparaison entre ces deux résultats et avons montré que le remplacement de n par N est une réelle amélioration: les degrés admis sont plus grands si $N - n$ est assez grand.

On obtient une définition plus naturelle du plan dans $\mathbf{P}(Q)$ en considérant l'action de $\mathbf{C}^*$ sur $\mathbf{C}^{n+1} - \{0\}$. Si $p_0, \ldots, p_s$ sont $s+1$ points indépendants dans $\mathbf{C}^{n+1}$, on définit un s-plan par:

$$x_j = \sum_{k=0}^{s} p_j \cdot \lambda_j$$

Ici, la notation "$\cdot$"signifie que l'on utilise l'action de $\mathbf{C}^*$.

Malheuresement, il est difficile de savoir quel genre de varieté nous obtenons ainsi. A l'exception de quelques cas particuliers, nous ne connaissons même pas sa dimension.

0.3. Les étapes essentielles dans les preuves des résultats de [ELV].

L'argument de Roitman:

Lemme 0.3.1.[RO] *Soit X une hypersurface de degré d dans $\mathbf{P}^n$ et soit $x \in X$. Si $d \leq n$, il existe une droite $l \subset \mathbf{P}^n$ telle que $x \in l$ et:*

$$(*) \qquad \text{soit } X \cap l = \{x\}, \text{ soit } l \subset X.$$

Démonstration:

On choisit des coordonnées $x_0, \ldots, x_n$ telles que $x = [1 : 0 : \ldots : 0]$. Une droite passant par l'origine est paramétrisée par $[\lambda_1 : \ldots : \lambda_n] \in \mathbf{P}^{n-1}$ puisqu' on a:

$$l = \{x_j = \lambda_j t | t \in \mathbf{C}\}$$

Dans les coordonnées affines X sera définie par:

$$f_d(x_1, \ldots, x_n) + f_{d-1}(x_1, \ldots, x_n) + \ldots + f_1(x_1, \ldots, x_n) = 0$$

où $f_a(x_1, \ldots, x_n)$ est homogène de degré a. Dans le $\mathbf{P}^{n-1}$ des droites qui passent par x, la condition $(*)$ équivaut à $f_a(\lambda_1, \ldots, \lambda_n) = 0 \ \forall a = 1, \ldots, d-$ 1. En fait, $X \cap l$ est définie par:

$$0 = \sum_{a=1}^{d} f_a(\lambda_1 t, \ldots, \lambda_n t) = \sum_{a=1}^{d} t^a f_a(\lambda_1, \ldots, \lambda_n)$$

Certes, $l \subset X$ signifie $f_a(\lambda_1, \ldots, \lambda_n) = 0 \ \forall a = 1, \ldots, d$; par contre, $X \cap l = \{0\}$ signifie que la seule solution doit être $t = 0$, et celle-là doit avoir multiplicité d. Donc $f_a(\lambda_1, \ldots, \lambda_n) = 0 \ \forall a = 1, \ldots, d-1$.

On a donc d equations, donc si $d \leq n$ on a des solutions non banales.

QED

Corollaire 0.3.1.[RO] *Si $d \leq n$,* $\mathrm{CH}_0(X) \otimes \mathbf{Q} = \mathbf{Q}$.

Démonstration:

Dans le lemme précédent, si $l \not\subset X$, on a $l \cdot X = dx$ dans $\mathrm{CH}_0(l \cap X)$ (on utilise ici la théorie de l'intersection refinie, voir [FU]). Si, au contraire, $l \subset X$, puisque $\mathrm{CH}_0(l \cap X) \otimes \mathbf{Q} = \mathrm{CH}_0(l) \otimes \mathbf{Q}$ est engendré par x (en effet $\mathrm{CH}_0(\mathbf{P}^1) \otimes \mathbf{Q} = \mathbf{Q}$), on a $l \cdot X = ax$ dans $\mathrm{CH}_0(l)$ avec $a \in \mathbf{Q}$, et comme $\deg(l \cdot X) = d$, on en déduit que $d = a$. Pour toute autre droite l', comme $\mathrm{CH}_1(\mathbf{P}^n) \otimes \mathbf{Q} = \mathbf{Q}$, on a $l' = al$ et donc $adx = al \cdot X = l' \cdot X = dx'$ dans $\mathrm{CH}_0(X) \otimes \mathbf{Q}$, pour tout autre $x' \in X$. Donc $\mathrm{CH}_0(X) \otimes \mathbf{Q} = \mathbf{Q}$.

QED

L'argument de [ELV]**:**

Passons à la généralisation de [ELV]. D'abord, on dit qu'une variété Y est engendrée par s-plans s'il existe une variété $Z \subset \mathrm{Gr}(s, \mathbf{P}^n)$ telle que $\dim Z = \dim Y - s$ et que la projection:

$$\Lambda_Z(s) := \{(\Gamma, x) \in Z \times \mathbf{P}^n | x \in \Gamma\} \to \mathbf{P}^n$$

est surjective sur Y. $\Lambda_Z(s)$ joue le rôle des droites l dans l'argument de Roitman, où l'on considérait seulement les droites l qui passaient par x. (Ici, c'est Y qui remplace x). Le lemme devient:

Lemme 0.3.2. *Soit:*

$$\mathbf{H}(s, X) := \{(\Gamma, \Gamma') \in Z \times \mathrm{Gr}(s+1, \mathbf{P}^n) | \Gamma \subset \Gamma' \subseteq X \text{ ou } \Gamma = \Gamma' \cap X\}$$

et soit θ la projection vers Z. Alors si:

$$\binom{s+d}{s+1} \leq n - s$$

it θ est surjective.

□

La démonstration est une généralisation immédiate de celle de Roitman. Bien sûr, $\mathbf{H}(0, X)$ est l'ensemble des droites l telles que $x \in l$ et ou bien $l \subset X$, ou bien $l \cap X = \{x\}$. On peut aussi généraliser le corollaire de Roitman:

Corollaire 0.3.2. *Sous les hypothèses precédentes, si:*

$$\binom{l+d}{l+1} \leq n - l$$

on a $\mathrm{CH}_l(X) \otimes \mathbf{Q} = \mathbf{Q}$.

□

Nous donnons la stratégie de la démonstration pour $l = 1$.

Soit $Y \subset X$ une courbe irréductible, supposons d'abord qu'il ne s'agit pas d'une droite, donc Y n'est pas engendrée par 1-plans (droites) tandis que, évidemment, elle est engendrée par 0-plans, (toute sous-varieté est engendrée par ses points). Il y a $Z \subset \mathrm{Gr}(0, \mathbf{P}^n) = \mathbf{P}^n$, $\dim Z = 1$, telle que:

$$\Lambda_Z(0) = \{(y, x) \in Z \times \mathbf{P}^n | x = y\} \to Y$$

est surjective; donc $Z = Y$. Puisque:

$$\binom{d+1}{2} \le n-1 \Rightarrow d \le n$$

on déduit du lemme précédent que l'application:

$$\mathbf{H}(0,X) := \{(x,\Gamma) \in Y \times \mathrm{Gr}(1,\mathbf{P}^n) | x \in \Gamma \subseteq X \text{ ou } x = \Gamma \cap X\} \to Y$$

est surjective. En fait, il s'agit tout simplement du résultat de Roitman: $\forall y \in Y \exists \Gamma \in \mathrm{Gr}(1,\mathbf{P}^n)$ telle que $y \in \Gamma$ et soit $\Gamma \subseteq X$ soit $x = \Gamma \cap X$.

Soit Σ une courbe de $\mathbf{H}(0,X)$ qui se projette sur Y de façon génériquement finie et soit $\sigma : \tilde{\Sigma} \to \Sigma$ une desingularisation. Posons:

$$\Lambda'_{\tilde{\Sigma}} := \{(x,\Gamma,z) \in \tilde{\Sigma} \times \mathbf{P}^n | z \in \Gamma\} \xrightarrow{Pr} \mathbf{P}^n$$

$Pr(\Lambda'_{\tilde{\Sigma}})$ est la surface rulée de $\mathbf{P}^n$ obtenue en prenant, $\forall x \in Y$, “la” droite L dans $\mathbf{P}^n$ telle que $(x,L) \in \tilde{\Sigma}$ et $x \in L$. (Certes, $\tilde{\Sigma} \subset \mathbf{H}(0,X)$ implique soit $L \subset X$, soit $x = L \cap X$). Par construction on en deduit $Y = X \cap Pr(\Lambda'_{\tilde{\Sigma}})$ et donc, comme classes rationnelles:

$$aY = X \cdot Pr(\Lambda'_{\tilde{\Sigma}}) \text{ pour quelque } a \in \mathbf{Q}$$

Remarquons que $Pr(\Lambda'_{\tilde{\Sigma}}) \neq 0$ parce que $Pr|_{\Lambda'_{\tilde{\Sigma}}}$ est génériquement finie sur l'image.

Or, $Pr(\Lambda'_{\tilde{\Sigma}}) \in \mathrm{CH}_2(\mathbf{P}^n) \otimes \mathbf{Q} \simeq \mathbf{Q}$, donc $aY = bX$ pour quelque $b \in \mathbf{Q}$. Ça veut dire que toute curbe irréductible est rationnellement équivalente au cycle X et donc elles sont toutes rationnellement équivalentes entre elles, sauf peut-être les droites.

On considère alors le cas d'une droite. Soit $Y = L$ une droite; certes elle est engendrée par 1-plans: disons $Z \subset \mathrm{Gr}(1,\mathbf{P}^n)$, $\dim Z = 0$, une sous-varieté pour qui l'application:

$$\Lambda_Z(1) := \{(\Gamma,x) \in Z \times \mathbf{P}^n | x \in \Gamma\} \to L = Y$$

est surjective. Z est un ensemble fini de droites et la condition précédente nous permet de choisir $Z = \{L\}$. Comme:

$$\binom{d+1}{2} \le n-1$$

le lemme précédent donne la surjectivité de l'application:

$$\mathbf{H}(1,X) := \{(\Gamma,\Gamma') \in Z \times \mathrm{Gr}(2,\mathbf{P}^n) | \Gamma \subset \Gamma' \subset X \text{ or } \Gamma = \Gamma' \cap X\} \to Z = \{L\}$$

Remarquons que $(\Gamma, \Gamma') \in \mathbf{H}(1, X) \Rightarrow \Gamma = L$. Soit $\Sigma \subset \mathbf{H}(1, X)$ un couple (L, L') qui se projette sur Z, on peut choisir $L' \cap X = L$. Posons:

$$\Lambda'_\Sigma := \{(L, L', x) \in \Sigma \times \mathbf{P}^n | x \in L'\} \equiv \{(L, L', L')\} \xrightarrow{Pr} L' \subset \mathbf{P}^n$$

Puisque $L' \cap X = L$, il doit y avoir un rationnel a pour qui $L' \cdot X = aL$ comme cycles rationnels; on utilise alors le même argument pour en déduire que toute droite est rationnellement équivalente au cycle X et donc à tout cycle provénant d'une courbe irréductible: donc $\mathrm{CH}_1(X) \otimes \mathbf{Q} = \mathbf{Q}$.

En ce qui concerne la démonstration que $\mathrm{CH}_0(X) \otimes \mathbf{Q} = \mathbf{Q}$, il suffit de voir que $d \leq n$ et appliquer le résultat de Roitman.

□

Pour que la borne soit ameliorée à n, on utilise ce résultat, toujours dû à [ELV]:

Lemme 0.3.3. *Soit l, d, n des naturels tels que:*

$$\binom{l+d}{l+1} \leq n$$

On suppose que $d \geq 3$. Alors, pour tout $s < l$:

$$\binom{s+d}{s+1} < n - s$$

□

Dans le théorème 4.6 de [ELV], on démontre que pour l comme avant on a encore $\mathrm{CH}_l(X) \otimes \mathbf{Q} = \mathbf{Q}$.

0.4. La structure de la thèse.

Dans le **premier chapitre**, nous rappellons des notions générales sur les espaces tordus (la référence principale est alors [DO]) et sur les groupes de Chow (référence principale: [FU]).

Dans le **deuxième chapitre**, nous prouvons le résultat principal. Bien que implicite en [KO], l'application σ_Q apparaît pour la première fois et, en effet, tout le chapitre est nouveau. Le dernier paragraphe est consacré à l'étude de l'amélioration de la borne.

Dans le **chapitre 3**, nous voulons étudier la géométrie des "plans" dans l'espace projectif tordu. La définition la plus intuitive semble satisfaisante pour l'aspect géométrique, et on prouve (avec des toutes petites modifications) que les résultats de [ELV] sont encore valables; mais la borne n'est pas ameliorée. Pour concilier le point de vue géométrique et la nouvelle borne, nous avons essayé de définir la Grassmannienne dans $\mathbf{P}(Q)$ avec l'application σ_Q. Le cas où $Q = (1^{(r)}, q_r, \ldots, q_n)$ et $s \leq r-1$ est traité avec soin (le cas $Q = (1, \ldots, 1, q)$ est un sous-cas). Dans ce cas, on peut prouver la nouvelle borne en donnant une démonstration qui suit celle de [ELV]. Mais nous ne savons pas si cette "Grassmannienne" est une variété projective ou non. Nous pouvons seulement démontrer qu'il existe un sous-ensemble dense, qui est obtenu comme quotient d'un ouvert de Zariski de la grande Grassmannienne $\mathrm{Gr}(s, \mathbf{P}^N)$ par un groupe fini.

Dans le **chapitre 4** sont regroupés des résultats de moindre importance de même que des problèmes ouverts. La tentative de définir une Grassmannienne dans l'espace projectif tordu par l'application σ_Q aboutit à un problème de taille: dans le cas général, on ne connait pas la dimension de la fibre sur un "s-plan" générique. Sous un point de vue strictement géométrique, cette définition n'est pas satisfaisante: par exemple, quand il est possible de plonger l'espace projectif tordue dans un espace projectif plus grand par une application de Veronese, les "plans" ainsi définis n'apparaissent pas comme l'intersection d'un plan usuel avec l'espace plongé. En outre, $s+1$ points indépendents ne définissent pas un seul "s-plan". Pour ces raisons nous avons introduit une nouvelle définition du plan qui prend en considération l'action de $\mathbf{C}$ (ces plans, nous les appellons K_p). Hélas, cela ne résoud pas le problème: les propriétés géométriques précédentes ne sont toujours pas atteintes et, ce qui est pire, nous ne connaissons pas la dimension de cet espace, essentiellement parce que l'on n'a pas la linéarité. Néanmoins, cette definition ne donne pas la même que l'autre, le seul cas où il y a une relation entre les deux étant le cas des droites: les droites K_p sont des droites particulières $\sigma_Q(L)$, pour $L \in \mathrm{Gr}(1, \mathbf{P}^N)$ si $Q = (1, \ldots, 1, q)$.

INTRODUCTION.

0.1. The role of Grassmannian and Fano varieties.

One way to understand a variety is by means of the special subvarieties lying on it. For instance, a quadric hypersurface in projective space can be studied by means of the family of lines on it. A systematic use of the linear spaces of dimension s on a subvariety X of a given projective space $\mathbf{P}^n$ can be done using the Grassmannian $\mathrm{Gr}(s, \mathbf{P}^n)$ of s-planes in $\mathbf{P}^n$ and the associated *Fano variety*:

$$F(s, X) := \{\Gamma \in \mathrm{Gr}(s, \mathbf{P}^n) | \Gamma \subset X\}$$

One knows classically, for instance, that for a degree d (with $d \geq 3$) generic hypersurface X, one has:

$$\varphi(s, n, d) := \dim F(s, X) = (s+1)(n-s) - \left(\begin{array}{c} s+d \\ d \end{array} \right)$$

This has to be interpreted as follows: a generic X of degree d such that $\varphi(s, n, d) < 0$ contains no s-planes; if $\varphi(s, n, d) \geq 0$, the Fano variety is actually non-empty and has dimension $\varphi(s, n, d)$.

Therefore, loosely speaking, the smaller d, the more linear subvarieties a degree d hypersurface contains.

A coarser question is wether all s-dimensional cycles on X are linear combinations of s-planes, at least up to rational equivalence. This leads to the study of the Chow groups $\mathrm{CH}_s(X)$ of s-cycles modulo rational equivalence (see Ch. 1.2 for more details). In fact Roitman studied 0-cycles on varieties $X \subset \mathbf{P}^n$ given by equations of small degree. The variety X needs not to be smooth, nor a complete intersection. The upshot is:

Theorem 0.1.1.[RO] *Let X be an irreducible projective variety of dimension $n-r$, defined by r equations of degree $d_1, \ldots, d_r$ such that $\sum_i d_i \leq n$. Then:*

$$\mathrm{CH}_0(X) = \mathbf{Z}$$

□

Below we sketch a proof of this for $\mathrm{CH}_0(X) \otimes \mathbf{Q}$ and for a hypersurface X of degree d. This captures the essential idea of the proof.

Based on this geometric proof, [ELV] extends this to higher Chow groups. Here Fano varieties form an essential argument. The result is:

Theorem 0.1.2.[ELV] *Let X be an irreducible $(n-r)$-dimensional projective variety defined by r equations of degree $d_1, \dots, d_r$ with $d_1 \geq d_2 \geq \dots d_r \geq 2$. Let:*

$$l := \max_{k \in \mathbf{N}} \left\{ k | \sum_{i=1}^{r} \binom{k+d_i}{k+1} \leq n \right\}$$

If either $d_1 \geq 3$ or $r \geq l+1$, then:

$$\mathrm{CH}_s(X) \otimes \mathbf{Q} = \mathbf{Q} \qquad \forall s \leq l$$

□

It follows that $\mathrm{CH}_s(X) \otimes \mathbf{Q}$ is generated by any s-plane lying on X, but in fact the proof heavily uses properties of the Fano varieties $F(s, X)$.

Below we give a sketch of the proof of this theorem in the case of a hypersurface and $l = 1$, the first case beyond Roitman. This also captures the essential idea of the proof.

0.2. Generalizing the results to the weighted projective spaces.

It is natural to ask for the generalization of the results of [ELV] to the case of weighted projective spaces and varieties given by weighted homogeneous polynomials of small degree. We do this in chapter 2 for hypersurfaces of degree d in $\mathbf{P}(q_0, \dots, q_n)$. The main result here is:

Theorem 0.2.1. *For a weighted hypersurface $X := X'/\mu \subset \mathbf{P}(Q)$ of Q-degree d, $d \geq 3$, given as the quotient of a smooth hypersurface X' under the action of the product of the groups of unity roots of the weights, we have:*

$$\mathrm{CH}_s(X) \otimes \mathbf{Q} = \mathbf{Q} \qquad \forall s \text{ such that } \binom{s+d}{s+1} \leq \sum_{i=0}^{n} q_i - 1$$

□

This theorem is reduced to the theorem of [ELV] for a hypersurface of degree d in $\mathbf{P}^N$, $N := \sum_{i=0}^{n} q_i - 1$, explaining the bound. The idea is of dominating the hypersurface by a carefully chosen hypersurface in $\mathbf{P}^N$ of the same degree. Say we have a map:

$$\sigma_Q : \tilde{X} \to X$$

Under fairly general conditions which are satisfied here, the induced map:

$$\sigma_{Q*} : \mathrm{CH}_*(\tilde{X}) \otimes \mathbf{Q} \to \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

is surjective and the result follows.

This proof, however, says nothing about the "linear" spaces contained in X. In fact, one should first properly define what one means by this in a weighted projective space. One way of defining these is to look at the quotient map:

$$\mathbf{P}^n \xrightarrow{p_Q} \mathbf{P}(Q) = \mathbf{P}^n/\mu$$

Here $\mu = \mu_{q_0} \times \ldots \times \mu_{q_n}$ acts coordinate-wise by multiplication of q_j-rooth of unity on $\mathbf{P}^n$. One then simply defines an s-plane on $\mathbf{P}(Q)$ as the quotient of an s-plane in $\mathbf{P}^n$. This leads to a "small" Grassmannian which however is easily understood. It leads also to a second, more geometric proof of the main result but with N replaced by n.

In the last section of chapter 2 we have made a comparison between these two results, showing that replacing n by N indeed gives many more degrees when $N - n$ gets bigger and bigger.

A more natural way to define an s-plane is by using the weighted $\mathbf{C}^*$-action on $\mathbf{C}^{n+1} - \{0\}$. Any $s+1$ independent points in $\mathbf{C}^{n+1}$, say $p_0, \ldots, p_s$, define an s-plane by:

$$x_j = \sum_{k=0}^{s} p_j \cdot \lambda_j$$

where the dot means that we're using the weighted $\mathbf{C}^*$-action.

It is however hard to see in general what sort of variety this leads to. We are not able to determine its dimension, except in special cases.

0.3. Sketch of the proof of the results of [ELV]

Roitman argument:

Lemma 0.3.1.[RO] *Let X be a degree d hypersurface of $\mathbf{P}^n$. Let $x \in X$. If $d \leq n$, there exists a line $l \subset \mathbf{P}^n$ such that $x \in l$ and:*

$$(*) \qquad \text{either } X \cap l = \{x\} \text{ or } l \subset X.$$

Proof:

Choose coordinates $x_0, \ldots, x_n$ centered at x. A line l through $x = [1 : 0 : \ldots 0]$ is parametrized by $[\lambda_1 : \ldots : \lambda_n] \in \mathbf{P}^{n-1}$ such that:

$$l = \{x_j = \lambda_j t | t \in \mathbf{C}\}$$

Write the equation for X in affine coordinates as:

$$f_d(x_1,\dots,x_n)+f_{d-1}(x_1,\dots,x_n)+\dots+f_1(x_1,\dots,x_n)=0$$

where $f_a(x_1,\dots,x_n)$ is homogeneous of degree a. Inside the $\mathbf{P}^{n-1}$ of the lines through x, the condition $(*)$ is equivalent to $f_a(\lambda_1,\dots,\lambda_n)=0\ \forall a=1,\dots,d-1$. Indeed, $X\cap l$ is defined by:

$$0=\sum_{a=1}^{d} f_a(\lambda_1 t,\dots,\lambda_n t)=\sum_{a=1}^{d} t^a f_a(\lambda_1,\dots,\lambda_n)$$

Certainly, $l\subset X$ means $f_a(\lambda_1,\dots,\lambda_n)=0\ \forall a=1,\dots,d$; while $X\cap l=\{0\}$ implies that the only solution must be $t=0$, with multeplicity d. So $f_a(\lambda_1,\dots,\lambda_n)=0\ \forall a=1,\dots,d-1$.

One has d equations, so if $d\le n$, we have non-trivial solutions.

QED

Corollary 0.3.1.[RO] *If $d\le n$, $\mathrm{CH}_0(X)\otimes\mathbf{Q}=\mathbf{Q}$.*

Proof:

In the previous lemma, if $l\not\subset X$, clearly $l\cdot X=dx$ inside $\mathrm{CH}_0(X\cap l)$ (this is the refined intersection theory of [FU]). If $l\subset X$, since $\mathrm{CH}_0(l\cap X)\otimes\mathbf{Q}=\mathrm{CH}_0(l)\otimes\mathbf{Q}$ is generated by x (recall that $\mathrm{CH}_0(\mathbf{P}^1)\otimes\mathbf{Q}=\mathbf{Q}$), we also have $l\cdot X=ax$ in $\mathrm{CH}_0(l)$ for some a, and since $\deg l\cdot X=d$, we deduce $d=a$. But $al\cdot X=l'\cdot X$ in $\mathrm{CH}_0(X)$ for any other line section l', because $l\sim l'$ inside $\mathbf{P}^n$. So $d\cdot x=adx'$ for any $x'\in X$. Therefore $\mathrm{CH}_0(X)\otimes\mathbf{Q}=\mathbf{Q}$.

QED

[ELV] **argument:**

As for the generalization by [ELV], one first defines Y as *spanned by s-planes* if there is a variety $Z\subset\mathrm{Gr}(s,\mathbf{P}^n)$ such that $\dim Z=\dim Y-s$ and the projection:

$$\Lambda_Z(s):=\{(\Gamma,x)\in Z\times\mathbf{P}^n|x\in\Gamma\}\to\mathbf{P}^n$$

is onto Y. This $\Lambda_Z(s)$ plays the role of the set of all l in Roitman's case, because there we only considered lines such that $x\in l$. (Y here replaces x). Then the lemma becomes:

Lemma 0.3.2. *Let:*

$$\mathbf{H}(s,X):=\{(\Gamma,\Gamma')\in Z\times\mathrm{Gr}(s+1,\mathbf{P}^n)|\Gamma\subset\Gamma'\subseteq X\text{ or }\Gamma=\Gamma'\cap X\}$$

The projection θ into Z is onto whenever

$$\binom{s+d}{s+1} \le n-s$$

The proof of this lemma is an immediate generalization of Roitman's proof for the case $s=0$. Obviously $\mathbf{H}(0,X)$ is the set of lines such that $x \in l$ and either $l \subset X$ or $l \cap X = \{x\}$. Roitman's corollary can also be generalized:

Corollary 0.3.2. *In the previous hypothesis, let:*

$$\binom{l+d}{l+1} \le n-l$$

Then $\mathrm{CH}_l(X) \otimes \mathbf{Q} = \mathbf{Q}$.
Here is the outline of the proof for $l=1$.

If $Y \subset X$ is an irreducible curve, we may first suppose that it is not a line, so it is not spanned by 1-planes (lines) while of course it is spanned by 0-planes, (trivially, any subvariety is spanned by its points). In other words, there is $Z \subset \mathrm{Gr}(0,\mathbf{P}^n) = \mathbf{P}^n$, $\dim Z = 1$, such that:

$$\Lambda_Z(0) = \{(y,x) \in Z \times \mathbf{P}^n | x = y\} \to Y$$

is onto; so $Z = Y$ itself. Since:

$$\binom{d+1}{2} \le n-1 \Rightarrow d \le n$$

from the previous lemma we deduce that the map:

$$\mathbf{H}(0,X) := \{(x,\Gamma) \in Y \times \mathrm{Gr}(1,\mathbf{P}^n) | x \in \Gamma \subseteq X \text{ or } x = \Gamma \cap X\} \to Y$$

is onto. Indeed, this is just Roitman result: $\forall y \in Y \exists \Gamma \in \mathrm{Gr}(1,\mathbf{P}^n)$ with $y \in \Gamma$ and either $\Gamma \subseteq X$ or $x = \Gamma \cap X$.

Say Σ a curve in $\mathbf{H}(0,X)$ projecting onto Y in a generically finite way and let $\sigma : \tilde{\Sigma} \to \Sigma$ be a desingularization. Set:

$$\Lambda'_{\tilde{\Sigma}} := \{(x,\Gamma,z) \in \tilde{\Sigma} \times \mathbf{P}^n | z \in \Gamma\} \xrightarrow{Pr} \mathbf{P}^n$$

Now $Pr(\Lambda'_{\tilde{\Sigma}})$ is the ruled surface of $\mathbf{P}^n$ obtained by taking, $\forall x \in Y$, "the" line L in $\mathbf{P}^n$ such that $(x,L) \in \tilde{\Sigma}$ and $x \in L$. (Of course, the fact that $\tilde{\Sigma} \subset \mathbf{H}(0,X)$ means that $L \subset X$ or $x = L \cap X$). By construction itself we deduce $Y = X \cap Pr(\Lambda'_{\tilde{\Sigma}})$ and therefore, as rational classes:

$$aY \equiv X \cdot Pr(\Lambda'_{\tilde{\Sigma}}) \text{ for some } a \in \mathbf{Q}$$

Remark that $Pr(\Lambda'_{\tilde{\Sigma}}) \neq 0$ because $Pr|_{\Lambda'_{\tilde{\Sigma}}}$ is generically finite on it.

Now $Pr(\Lambda'_{\tilde{\Sigma}}) \in \mathrm{CH}_2(\mathbf{P}^n) \otimes \mathbf{Q} \simeq \mathbf{Q}$, so $aY = bX$ for some $b \in \mathbf{Q}$. Therefore, all the irreducible curves are rationally equivalent cycles, with the only exception (for the moment) of lines, because they're all rationally equivalent to the cycle X.

It suffices now to consider the case of a line. Let $Y = L$ a line, which is therefore trivially spanned by 1-planes: say $Z \subset \mathrm{Gr}(1, \mathbf{P}^n)$, $\dim Z = 0$, for which:

$$\Lambda_Z(1) := \{(\Gamma, x) \in Z \times \mathbf{P}^n | x \in \Gamma\} \to L = Y$$

is onto. Z is a finite set of lines and the previous condition means that we can choose $Z = \{L\}$. Since

$$\binom{d+1}{2} \leq n - 1$$

the previous lemma asserts that the map:

$$\mathbf{H}(1, X) := \{(\Gamma, \Gamma') \in Z \times \mathrm{Gr}(2, \mathbf{P}^n) | \Gamma \subset \Gamma' \subset X \text{ or } \Gamma = \Gamma' \cap X\} \to Z = \{L\}$$

is onto. Remark that $(\Gamma, \Gamma') \in \mathbf{H}(1, X) \Rightarrow \Gamma = L$. Let $\Sigma \subset \mathbf{H}(1, X)$ be a couple (L, L') projecting onto Z, we may assume $L' \cap X = L$. Set:

$$\Lambda'_{\Sigma} := \{(L, L', x) \in \Sigma \times \mathbf{P}^n | x \in L'\} \equiv \{(L, L', L')\} \xrightarrow{Pr} L' \subset \mathbf{P}^n$$

Since $L' \cap X = L$, as rational cycles there must be a rational a for which $L' \cdot X = aL$; then the same argument as before works; all the lines are rationally equivalent to X and hence to all the cycles issuing from irreducible curves: therefore $\mathrm{CH}_1(X) \otimes \mathbf{Q} = \mathbf{Q}$.

As for the fact that $\mathrm{CH}_0(X) \otimes \mathbf{Q} = \mathbf{Q}$, this is true because of Roitman since $d \leq n$.

$\square$

In order to improve the bound to n, [ELV] proves this lemma:

Lemma 0.3.3. *Let l, d, n be natural numbers such that:*

$$\binom{l+d}{l+1} \leq n$$

Suppose $d \geq 3$. Then, for every $s < l$:

$$\binom{s+d}{s+1} < n - s$$

□

In the theorem 4.6 of [ELV], one shows that for l as before, one still has $\mathrm{CH}_l(X) \otimes \mathbf{Q} = \mathbf{Q}$.

0.4. The structure of the thesis.

The thesis is structured as follows.

In the first chapter we recall some generalities about weighted projective spaces (the basic reference being [DO])and Chow groups (the basic reference being [FU]).

In the second chapter we prove the main result. Even if implicit in [KO], the map σ_Q appears for the first time and indeed all the chapter is new. We conclude the chapter with a section dedicated to the study of the improvement of the bound.

In chapter 3 we want to study the geometry of "planes" in the weighted projective space. The most naive definition appears to be satisfactory for such a geometrical insight, and it's easy to prove (with minor modifications) the results of [ELV] for such a Grassmannian; but the bound is not improved. To couple the geometrical point of view and the new bound, we have tried to define the Grassmannian in $\mathbf{P}(Q)$ by means of the map σ_Q. In this chapter, the easier case $Q = (1^{(r)}, q_r, \ldots, q_n)$ is treated in detail for s-planes such that $s \leq r - 1$ (the case $Q = (1, \ldots, 1, q)$ being a subcase). One can prove the new bound by giving a proof along the lines of [ELV], getting even rid of the smoothness assumptions. But as for the structure of such a Grassmannian, we don't know if it is a projective variety or not. We are only able to prove that a dense subset of it is obtained as the quotient by a finite group of a Zariski-open subset of the big Grassmannian $\mathrm{Gr}(s, \mathbf{P}^N)$.

In chapter 4, various minor results and open questions are described. The tentative of defining a Grassmannian in the weighted projective space by means of the map σ_Q reveals, in the general case, a major problem: this time we don't even know the dimension of the fibre over a generic "weighted" plane. As a matter of fact, such a definition of Grassmannian is not very satisfactory under the geometrical point of view: for example, when it is possible to embed the weighted projective space in a bigger projective space by a Veronese type map, the planes don't come as intersection of the imbedded space with some usual plane. The usual property of $s + 1$ linearly independent points of uniquely defining a plane is also lost. For these reasons we introduced a new, parametric definition of plane based on the weighted actions (the planes that we call K_p.) But things don't go better: the two previous geometric reasonable requests are still not attained and, what's more, we don't know the dimension of the set of all such planes, essentially because of the lack of linearity. Nonetheless, one can see that this new definition doesn't agree with the σ_Q-one, the two being in general quite different 'the only case in which there is a relation is for lines: then lines K_p are particular lines $\sigma_Q(L)$ for $L \in \mathrm{Gr}(1, \mathbf{P}^N)$.

Chapter 1

Preliminaries

Dans ce chapitre, nous rappelons les notions fondamentales que nous allons utiliser par la suite. Le premier paragraphe est donc consacré à la définition et aux propriétés de l'espace projectif tordu. Nous donnons aussi des exemples où cet espace peut se réaliser comme un cône dans un espace projectif classique par une application de type Veronese. Dans le deuxième paragraphe, nous rappelons les résultats fondamentaux concernant les groupes de Chow, en particulier la formule de projection, qui sera nécessaire dans la démonstration du résultat principal de cette thèse, au chapitre suivant.

1.1 Weighted projective spaces

We start with an $(n+1)$-tuple of positive integers $Q := (q_0, \ldots, q_n) \in \mathbf{N}^{n+1}$, the so called **set of weights**.

Let K be a field and consider the ring of polynomials in $n+1$ variables in K:

$$S(Q) := K[t_0, \ldots, t_n]$$

graded by the conditions $\deg(t_j) := q_j \; \forall j = 0, \ldots, n$.

Definition 1.1.1

A polynomial $f \in S(Q)$ is **weighted homogeneous of degree** d iff $f(lt) = l^d f(t) \; \forall l \in K$.

$\square$

Explicitly, a homogeneous polynomial of degree d is of the form:

$$f(t) = \sum_{\underline{d} \in \mathbf{N}_d^{n+1}(Q)} a_{d_0 \ldots d_n} \prod_{j=0}^{n} t_j^{d_j}$$

where for each $d \in \mathbf{N}$:

$$\mathbf{N}_d^{n+1}(Q) := \{\underline{d} = (d_0, \ldots, d_n) | \sum_{j=0}^{n} d_j q_j = d\}$$

Hence a weighted homogeneous polynomial of degree d is the sum of monomials which in turn are weighted homogeneous of degree d. The subring of weighted d-homogeneous polynomials will be denoted by $S_d(Q)$. Such polynomials are parametrized by an affine space and the corresponding projective space parametrizes the corresponding hypersurfaces.

Definition 1.1.2

The **weighted projective space of type** Q is the scheme:

$$\mathbf{P}(Q) := \mathrm{Proj}(S(Q))$$

□

One should immediately remark that $\mathbf{P}(1, \ldots, 1)$ is the usual projective space $\mathbf{P}^n$, and also that:

Lemma 1.1.1 *Let $Q := (q_0, \ldots, q_n)$ and $Q' := (q'_0, \ldots, q'_n)$ be two sets of weights, and suppose that $q'_k = aq_k \ \forall k = 0, \ldots, n$. Then $\mathbf{P}(Q) \simeq \mathbf{P}(Q')$.*

Proof:
One has a canonical isomorphism (see [GD], 2.4.7.):

$$\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(a)})$$

where A is a ring and $A^{(a)}$ is the **shift of degree** a of A. Now observe that $S_m(Q') = S_{am}(Q)$ and deduce from the previous isomorphism that:

$$\mathbf{P}(Q) = \mathrm{Proj}(S(Q)) \simeq \mathrm{Proj}(S(Q)^{(a)}) = \mathbf{P}(Q')$$

QED

It is interesting to remark that even if we assume that the weights are relatively prime, that is if $(q_0, \ldots, q_n) = 1$, the spaces $\mathbf{P}(Q)$ and $\mathbf{P}(Q')$ can still be isomorphic:

Lemma 1.1.2 *Let $Q := (q_0, \ldots, q_n)$ with $(q_0, \ldots, q_n) = 1$, and set:*

$$\begin{aligned} d_i &:= (q_0, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) \\ a_i &:= l.c.m.(d_0, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n) \\ a &:= l.c.m.(d_0, \ldots, d_n) \end{aligned}$$

Let $Q' := (\frac{q_0}{a_0}, \ldots, \frac{q_n}{a_n})$. Then there is a natural isomorphism:

$$\mathbf{P}(Q) \ \simeq \ \mathbf{P}(Q').$$

Proof:

First, remark that $a_i|q_i$: because d_j is the maximum common divisor of all the weights except the one labelled by j, there are naturals m_j such that $q_i = m_j d_j \ \forall j \neq i$. So the usual property of the least common divisor implies that a_i also divides q_i, since this last one is divided by each $d_j : j \neq i$. Moreover, $(a_i, d_i) = 1$: suppose $s|d_i$, then by the definition of d_i this means that $s|q_j \forall j \neq i$; and if $s|a_i$ also, by definition there is a $j \neq i$ such that $s|d_j$. This means that $s|q_i$ also, so that $(q_0, \ldots, q_n) = s = 1$ by hypothesis. Finally, this obviously implies that $a_i d_i = a$. For the proof of the lemma, define $S' := \oplus_{k=0}^{\infty} S_{ak}(Q)$ so that S' is a subring of $S(Q)$ isomorphic to $K[X_0, \ldots, X_n]$ where the variable $X_i := t_i^{d_i}$ has degree $d_i q_i = \frac{a q_i}{a_i}$. Then $S(Q') = S'^{(a)}$ so that:

$$\mathrm{Proj}(S(Q')) = \mathrm{Proj}(S') \simeq \mathrm{Proj}(S(Q)^{(a)}) = \mathrm{Proj}(S(Q)).$$

QED

Corollary 1.1.1 *Let* $d'_i := (q'_0, \ldots q'_{i-1}, q'_{i+1}, \ldots, q'_n) = 1,\ \forall i = 0, \ldots, n,$ *with* $q'_j = \frac{q_j}{a_j}$. *Then* $\mathbf{P}(Q) \simeq \mathbf{P}(Q')$.

Proof:

One has $a'_i = a' \ \forall i$, so $q_k = \frac{q'_k}{a'_k} = \frac{q'_k}{a'}$

QED

Corollary 1.1.2 *Assume* $q_i = a_i \ \forall i = 0, \ldots, n$. *Then* $\mathbf{P}(Q) \simeq \mathbf{P}^n$.

Proof:

$q'_i = \frac{q_i}{a_i} = 1.$

QED

Corollary 1.1.3 $\mathbf{P}(q_0, q_1) \simeq \mathbf{P}^1$.

Proof:

Certainly $q_i = a_i$ so it suffices to apply the previous corollary.

QED

There are two possible concrete constructions of the weighted projective space.

First Construction:

Consider the action

$$K^* \times (K^{n+1} - 0) \to (K^{n+1} - 0)$$

defined by:

$$(l, t_0, \ldots, t_n) \mapsto (l^{q_0}t_0, \ldots, l^{q_n}t_n).$$

Then one sets:

$$\mathbf{P}(Q) = (K^{n+1} - 0)/K^*$$

the space of the orbits. This construction is certainly consistent with the previous general definition of weighted projective space, since it is clear that the variable t_j has degree q_j.

We shall denote by $[x]_Q$ the class of $x \in K^{n+1^*}$ in $\mathbf{P}(Q)$. Also,

$$\pi_Q : K^{n+1^*} \to \mathbf{P}(Q)$$

will be the natural projection.

Second Construction:
For every $q \in \mathbf{N}$, let $\mu_q := \{\epsilon \in K : \epsilon^q = 1\}$ be the finite group of the q-roots of unity. Then put:

$$\mu := \mu_{q_0} \times \ldots \times \mu_{q_n}$$

and consider the action:

$$\mu \times \mathbf{P}^n_K \to \mathbf{P}^n_K$$

defined by:

$$(\epsilon_0, \ldots, \epsilon_n; t_0, \ldots, t_n) \to (\epsilon_0 t_0, \ldots, \epsilon_n t_n)$$

One has the space of the orbits and the projection map:

$$p_Q : \mathbf{P}^n_K \to \mathbf{P}^n_K/\mu$$

The class of $[t] \in \mathbf{P}^n_K$ will be denoted by $[t]_\mu$.

Consider the morphism:

$$\varphi_Q : [t_0 : \ldots : t_n]_\mu \mapsto [t_0^{q_0}, \ldots, t_n^{q_n}]_Q$$

One easily verifies that φ_Q establishes an isomorphism

$$\mathbf{P}^n_K/\mu \simeq \mathbf{P}(Q)$$

and in the sequel we use this isomorphism to switch between the two descriptions.

Remark 1.1.1 *In the following, we shall always assume* $(q_0, \ldots, q_n) = 1$ *and* $q_j \le q_k \forall j < k$.

Definition 1.1.3

A *V-variety* is a variety which is locally the quotient of a smooth variety by a finite group acting on it.

Lemma 1.1.3 $\mathbf{P}(Q)$ *is a V-variety.*

Proof:

One has $\mathbf{P}(Q) = \cup_{k=0}^{n} U_k$ where $U_k := \{x \in \mathbf{P}(Q) : x_k \neq 0\}$. Since $\forall [a_0, \ldots, a_n]_Q \in U_k$ we have:

$$[a_0, \ldots, a_n]_Q = [\frac{a_0}{b_k^{q_0}}, \ldots, \frac{a_{k-1}}{b_k^{q_{k-1}}}, 1, \frac{a_{k+1}}{b_k^{q_{k+1}}}, \ldots, \frac{a_n}{b_k^{q_n}}]$$

where $b_k^{q_k} = a_k$ is defined up to an element $\epsilon_k \in \mu_{q_k}$, setting:

$$V_k := \{x \in K^{n+1} - \{0\} : x_k = 1\}$$

we get: $U_k = V_k / G_k$ with $G_k \subset \mathrm{Aut} K^{n+1}$ such that:

$$G_k := \left\{ \begin{pmatrix} \epsilon_k^{q_0} & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & \epsilon_k^{q_n} \end{pmatrix} : \epsilon_k \in \mu_{q_k} \right\}$$

QED

Example 1.1.1 ([HA], pag. 128).

One can see $\mathbf{P}(1^{(n)}, q) := \mathbf{P}(1, \ldots, 1, q)$ (1 appears n times) as a cone over the Veronese embedded $\mathbf{P}^{n-1}$ of degree q in $\mathbf{P}^{n(q)}$, where:

$$n(q) := \begin{pmatrix} n+q-1 \\ n-1 \end{pmatrix}$$

is the number of monomials of degree q in the variables $t_0, \ldots, t_{n-1}$.

Indeed, for a multindex $I = (i_0, \ldots, i_{n-1})$ having $|I| := \sum_{j=0}^{n-1} i_j = q$ one writes:

$$t^I := \prod_{j=0}^{n} t_j^{i_j}$$

Then the classical Veronese map is:

$$\phi_{n-1,q} : K^n \to K^{n(q)}$$

$$\phi_{n-1,q}(t_0,\ldots,t_{n-1}) := (\ldots,t^I,\ldots)$$

Clearly:

$$\phi_{n-1,d}(t) = \mu\phi_{n-1,d}(r) \Leftrightarrow t = \lambda r \text{ with } \lambda^d = \mu$$

Therefore one has an immersion:

$$\phi_{n-1,q} : \mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^{n(q)-1}$$

Now let:

$$\Psi_{n,q} : \mathbf{P}^n \to \mathbf{P}^{n(q)}$$

defined by:

$$\Psi_{n,q}([t_0 : \ldots : t_n]) := [\phi_{n-1,q}(t) : t_n^q]$$

This map is well defined and becomes an immersion after replacing $\mathbf{P}^n$ with the weighted projective space $\mathbf{P}(1,\ldots,1,q)$, because the last coordinate is only characterized up to a q-root of unity. Therefore it descends to a morphism:

$$\chi_{n,q} : \mathbf{P}(1,\ldots,1,q) \to \mathbf{P}^{n(q)}$$

which identifies $\mathbf{P}(1,\ldots,1,q)$ to a cone over $\phi_{n-1,q}(\mathbf{P}^n)$ and where the vertex is $[0 : \ldots : 0 : 1]$.

□

Example 1.1.2

In the same spirit, consider $Q = (1,1,q^{(n-1)}) := (1,1,q,\ldots,q)$, where there are exactly $n-1$ weights equal to q. We can consider again a Veronese type map:

$$\Psi_{n,q} : \mathbf{P}(Q) \to \mathbf{P}^{n+q-1}$$

defined by:

$$\begin{aligned}\Psi_{n,q}([t_0 : \ldots : t_n]_\mu) := [t_0^q : t_0^{q-1}t_1 : \ldots : t_0t_1^{q-1} : t_1^q : t_2^q : \ldots : t_n^q] = \\ = [\phi_{1,q}(t_0,t_1) : t_2^q : \ldots : t_n^q]\end{aligned}$$

Then the same diagram as before works. So $\mathbf{P}(Q)$ is now a subvariety of degree q in $\mathbf{P}^{n+q-1}$.

□

Example 1.1.3

Here is an example in which two nonequal weights are different from 1. Let $n = 3$ and $Q = (1, 1, q, aq)$. Then we have a Veronese type map:

$$\Psi_{3,a,q} : \mathbf{P}(Q) \to \mathbf{P}^{\eta_{aq}}$$

where $\eta_{aq} := (a+1)(\frac{aq}{2}+1) \in \mathbf{N}$, defined by:

$\Psi_{3,a,q}([t_0 : \ldots : t_3]_\mu) =$
$= [\phi_{1,aq}(t_0,t_1) : \phi_{1,(a-1)q}(t_0,t_1)t_2^q : \ldots : \phi_{1,q}(t_0,t_1)t_2^{(a-1)q} : t_2^{aq} : t_3^{aq}]$

Remark that for $a = 1$ one has $Q = (1, 1, q, q)$ and so one has to embed $\mathbf{P}(Q)$ into $\mathbf{P}^{q+2}$, since the number of monomials in t_0 and t_1 of degree q is $q + 1$ and one also has t_2^q and t_3^q, as in **Example 1.1.2.** And indeed, from the previous formula: $\eta_{aq}|_{a=1} = q + 2$.

Remark also that it is necessary to work with at least two weights equal to 1 in order to get a new example; indeed, by **Lemma 1.1.2** one has $\mathbf{P}(1, q, aq) \simeq \mathbf{P}(1, 1, a)$.

□

The map $\varphi_Q p_Q : \mathbf{P}^n \to \mathbf{P}(Q) = K^{n+1} - \{0\}/K^*$ establishes a one-to-one correspondence between hypersurfaces $X = \{f = 0\}$ of $\mathbf{P}(Q)$, where $f \in S(Q)_d$ and those of $\mathbf{P}^n$ that are defined by homogeneous polynomials of the form:

$$\hat{f}(t_0 : \ldots : t_n) := f([t_0^{q_0}, \ldots, t_n^{q_n}]_Q)$$

so that $X' := (\varphi_Q p_Q)^{-1}(X) = \{\hat{f} = 0\}$.

In general this defines singular hypersurfaces, but if $\hat{f}$ is smooth, the singularities are modest: the cone $\mathcal{C}_X := \pi_Q^{-1}(X)$ defined by $f = 0$ inside the affine space K^{n+1} has its only singularity at 0. The corresponding hypersurface in $\mathbf{P}(Q)$ is then said *quasi-smooth.* Indeed, observe that:

$$\frac{\partial \hat{f}}{\partial t_j}(t) = \frac{\partial f}{\partial x_j}(\varphi_Q p_Q(t)) q_j t_j^{q_j - 1}$$

so that $\nabla \hat{f}(t) \neq 0 \Rightarrow \nabla f(\varphi_Q p_Q(t)) \neq 0$, and since $\mathcal{C}_X$ is defined by $f = 0$ in K^{n+1}, then the only singularity of $\mathcal{C}_X$ can be the origin.

If X is a quasismooth subvariety of $\mathbf{P}(Q)$, the *quasicone* $\mathcal{C}_X \subset K^{n+1} - \{0\}$ is smooth, but one cannot conclude that X' is smooth:

$$\frac{\partial \hat{f}}{\partial t_j}(t) = \frac{\partial f}{\partial x_j}(\varphi_Q p_Q(t)) q_j t_j^{q_j - 1}$$

so that $\varphi_Q p_Q(\mathrm{Sing}(X'))$ is the set of $x \in \mathbf{P}(Q)$ such that:

$$\frac{\partial f}{\partial x_j}(x) = 0 \quad \forall j \text{ such that } q_j = 1$$

and:

$$\frac{\partial f}{\partial x_j}(x) x_j = 0 \quad \forall j \text{ such that } q_j > 1$$

which may be nonempty. For example, let $Q = (1, \ldots, 1, q)$, then for a quasismooth $X = \{f = 0\} \subset \mathbf{P}(Q)$ we have:

$$\varphi_Q p_Q(\mathrm{Sing}(X')) = \{x \in \mathbf{P}(Q) | \frac{\partial f}{\partial x_j}(x) = 0 \quad \forall j = 1, \ldots, n-1;\ x_n = 0\} \cap X$$

Differentiating the identity $t^d f(x_0, \ldots, x_n) = f(t^{q_0} x_0, \ldots, t^{q_n} x_n)$ with respect to t and then putting $t = 1$ yelds the **Euler's formula:**

$$d\ f(x) = \sum_{j=0}^{n} q_j x_j \frac{\partial f}{\partial x_j}(x)$$

So in our situation a point $x \in X$ such that $\frac{\partial f}{\partial x_j}(x) = 0\ \forall j = 0, \ldots, n-1$, even if $\frac{\partial f}{\partial x_n}(x) \neq 0$ and so is nonsingular, automatically satisfies $x_n = 0$ and hence comes from a singular point of X'.

Lemma 1.1.4 [LA] *A quasismooth hypersurface of* $\mathbf{P}(Q)$ *is a* V*-variety.*

Proof:

Let $X = \{f = 0\}$ with $f \in S(Q)_d$ and let $U_i := \{x_i \neq 0\}$ the standard cover of $\mathbf{P}(Q)$ and $V_i := \{x \in K^{n+1} : x_i = 1\}$. Set $W_i := \mathcal{C}_X \cap V_i$. [LA] proves that W_i is a smooth hypersurface of $V_i \simeq K^n$. Indeed, its jacobian is:

$$\mathrm{Jac}(W_i) = \left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_{i-1}}, \frac{\partial f}{\partial x_{i+1}}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

A point x such that this vector is zero would give $\frac{\partial f}{\partial x_i} = 0$ also by using $f = 0$ in the Euler's formula. But then this point would let the jacobian of X be zero, which is against the hypothesis of X quasismooth.

Now let $\pi_{Qi} : W_i \to \mathbf{P}(Q)$ be the restriction of π_Q to W_i; its image is $U_i \cap X$. So, as in the proof of the fact that $\mathbf{P}(Q)$ is a V-variety, $X \cap U_i$ is the quotient of a smooth variety W_i by a finite group.

QED

The following lemma is useful to give conditions about quasismoothness of weighted hypersurfaces.

Lemma 1.1.5 [LA] *Let $X \subset \mathbf{P}(Q)$ be a quasismooth weighted hypersurface of Q-degree $d > \max q_i = q_n$. Then, $\forall q_i \in Q$, either $d = r_i q_i$ for some $r_i \in \mathbf{N}$ or there exists $q_{k_i} \in Q - \{q_i\}$ such that $d = r_i q_i + q_{k_i}$. Moreover, q_{k_i} is uniquely determined.*

Conversely, if d satisfies the previous condition, then the generic weighted hypersurface of $\mathbf{P}(Q)$ is quasismooth.

Proof:

Suppose X is quasismooth but there is i such that q_i doesn't satisfy the condition. Then f must have the form:

$$f(x) = x_a x_b f_1(x_0, \ldots, x_n) + f_2(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

where $a, b \neq i$. Then $(1, 0, \ldots, 0)$ is a singular point different from the vertex. To prove the unicity, suppose q_0 appears both as q_{k_1} and q_{k_2}, that is $d = q_1 r_1 + q_0 = r_2 q_2 + q_0$. Then:

$$f(x) = x_0(a_1 x_1^{r_1} + a_2 x_2^{r_2}) + x_a x_b f_1(x) + f_2(x_0, x_3, \ldots, x_n)$$

where $a, b > 2$. Let (s_1, s_2) be a nonzero solution of $(a_1 x_1^{r_1} + a_2 x_2^{r_2}) = 0$. Then f and all its partial derivatives vanish at $(0, s_1, s_2, 0, \ldots, 0)$.

The converse statement follows from the same considerations.

QED

1.2 Chow groups

Definition of Chow groups.

A *k-cycle* on an algebraic scheme X is a finite formal sum:

$$\sum_i n_i V_i$$

of subvarieties $V_i \subset X$ with integral coefficients $n_i \in \mathbf{N}$. The free abelian group generated by k cycles is denoted by $Z_k(X)$.

Let W be a $(k+1)$ dimensional subvariety of X and let $f \in K(W)^*$ be a nonzero rational function on W. One defines the cycle:

$$(f) := \sum_V \mathrm{ord}_V(f) V$$

where the sum is taken over subvarieties V_i of codimension 1 in W. As for the order, recall that for $K = \mathbf{C}$, one starts by defining the order of a holomorphic function $f \in \mathcal{O}(W)$; in this case one observes that, near a point $x \in W$, $V = \{g_V = 0\}$; then $f = g_V^a h$ for some $a \in \mathbf{N}$, and $\mathrm{ord}_{Vx}(f) := a$; this doesn't depend on x. When $f \in \mathcal{M}(W)$ is meromorphic, $f = p/q$ and $\mathrm{ord}_V(f) := \mathrm{ord}_V(p) - \mathrm{ord}_V(q)$.

For example, if $f : X \to \mathbf{P}^1$ is a dominant map, that is $f \in K(X)$, and if $k := \dim X$, $Y_0 := f^{-1}(0)$ and $Y_\infty := f^{-1}(\infty)$ are subschemes of dimension $k-1$ and one has $(f) = Y_0 - Y_\infty$.

One says that $a \in Z_k(X)$ is *rationally equivalent to* 0 and writes $a \sim 0$, if there is a finite number of subvarieties $W_i \subset X$ of dimension $k+1$ and $f_i \in K(W_i)^*$ such that:

$$a = \sum_i (f_i)$$

The subset $RZ_k(X) := \{a \in Z_k(X) : a \sim 0\}$ is a subgroup of $Z_k(X)$.

Definition 1.2.1

The *Chow group of order* k in X is defined to be:

$$\mathrm{CH}_k(X) := Z_k(X)/RZ_k(X)$$

One also sets:

$$\mathrm{CH}(X) := \oplus_{k=0}^{\dim X} \mathrm{CH}_k(X)$$

□

Remark 1.2.1

- If $X = \cup_{i=1}^t X_i$ with $X_i \cap X_j = \emptyset\ \forall i \neq j$, one has

$$\mathrm{CH}_k(X) = \oplus_{i=1}^t \mathrm{CH}_k(X_i)$$

- If $X_1, X_2 \subset X$ are closed subschemes, the following sequence is exact:

$$\mathrm{CH}_k(X_1 \cap X_2) \xrightarrow{l} \mathrm{CH}_k(X_1) \oplus \mathrm{CH}_k(X_2) \xrightarrow{m} \mathrm{CH}_k(X_1 \cup X_2)$$

  with $l(\alpha) = (\alpha, \alpha)$ and $m(\alpha, \beta) = \alpha - \beta$.

□

The push-forward and the pull-back.

Let $f : X \to Y$ be a proper morphism and let $V \subset X$ be a subvariety of dimension k. Then $f(V)$ is a closed subvariety of Y and one sets:

$$\deg(V, f(V)) := \begin{cases} [K(V) : K(f(V))] & \text{if } \dim(f(V)) = \dim V \\ 0 & \text{if } \dim(f(V)) < \dim V \end{cases}$$

Define:

$$f_*[V] := \deg(V, f(V))[f(V)]$$

for $[V] \in \mathrm{CH}_k(X)$. This gives a morphism:

$$f_* : \mathrm{CH}_k(X) \to \mathrm{CH}_k(Y)$$

which is called *push-forward by f*.

When $K = \mathbf{C}$, for example, the degree is just the degree of the covering f, that is its number of sheets.

If f is flat, and if $W \subset Y$ is a dimension k subvariety, one defines:

$$f^*([W]) := [f^{-1}(W)]$$

Supposing $\dim X = a + \dim Y$, one has $\dim f^{-1}(W) = \dim W + a = k + a$ and this gives:

$$f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+a}(X)$$

The map f^* is called *pull-back by f*.

Lemma 1.2.1 *Consider a fibre square:*

$$\begin{array}{ccc} X \times_Y Y' & \stackrel{g'}{\to} & X \\ f' \downarrow & & \downarrow f \\ Y' & \stackrel{g}{\to} & Y \end{array}$$

in which f is proper and g is flat. Then f' is proper, g' is flat and:

$$f'_* g'^* = g^* f_* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y')$$

$\square$

One can also define a pull-back when

$$f : X \to Y$$

is a morphism between smooth projective varieties (see [FU], chapter 8).

The action of a finite group.

Let G be a finite group acting on a variety X and define an action of G on $\mathrm{CH}_*(X)$ by:

$$g \cdot [V] := [g \cdot V]$$

for every subvariety $V \subset X$. Here of course $g \cdot V := \{g \cdot x : x \in V\}$. Let $f \in K(X)$ be a rational function on X. Then $f^g(x) := f(g \cdot x)\ \forall x \in V$ and $\forall g \in G$. One easily sees that $g \cdot (f) = (f^g)$ and hence:

$$g : \mathrm{CH}_*(X) \to \mathrm{CH}_*(X)$$

Let:

$$\mathrm{CH}_*(X)^G := \{\alpha \in \mathrm{CH}_*(X) : g \cdot \alpha = \alpha\, \forall g \in G\}$$

Lemma 1.2.2

$$\mathrm{CH}_*(X/G) \otimes \mathbf{Q} = \mathrm{CH}_*(X)^G \otimes \mathbf{Q}$$

Proof:

Let $\pi : X \to X/G$ be the projection. For every $V \subset X$, set:

$$I_V := \{g \in G : g \cdot x = x\, \forall x \in V\}$$

$$\deg_i(V, \pi(V)) := \text{degree of inseparability of } K(V) \text{ over } K(\pi(V))$$

Then one has a morphism:

$$\pi^* : Z^*(X/G) \otimes \mathbf{Q} \to Z^*(X)^G \otimes \mathbf{Q}$$

by defining, $\forall W \subset X/G$:

$$\pi^*([W]) := \sum_{V \,\text{irr.comp.} \subset \pi^{-1}(W)} \frac{|I_V|}{\deg_i(V, \pi V)} V$$

It is clear that $g \cdot \pi^*(W) = \pi^*(W)$ so the map is well defined. Indeed, π^* is an isomorphism. If $V \in \mathrm{CH}_*(X)^G \otimes \mathbf{Q}$, one has $\pi^*(\pi(V)) = |G|V$, because $I_V = G$. This isomorphism descends to the quotient by rational equivalence, so that:

$$\mathrm{CH}_*(X)^G \otimes \mathbf{Q} \overset{\pi^*}{\simeq} \mathrm{CH}_*(X/G) \otimes \mathbf{Q}$$

QED

A ring structure and the projection formula.

One can give $\mathrm{CH}_l(Y)$ a ring structure, where Y is a smooth projective variety. One denotes the ring operation with a dot. This construction can be found in [FU], Ch. 6 and 8.

Lemma 1.2.3 (Projection formula.) *Let $f : X \to Y$ be a morphism between smooth projective varieties. Let $\alpha \in \mathrm{CH}_*(X)$ and $\beta \in \mathrm{CH}_*(Y)$. Then:*

$$f_*(\alpha \cdot f^*\beta) = \beta \cdot f_*\alpha$$

The proof of this *projection formula* can be found in [FU], prop. 8.3(c).

Let now Z be smooth projective and Z/G be its quotient by a finite group. One can define a ring structure in $\mathrm{CH}_*(Z/G) \otimes \mathbf{Q}$ by setting:

$$a \cdot b := \pi_* \left(\frac{\pi^* a \cdot \pi^* b}{|G|} \right)$$

The ring structure a priori depends on the choice of the presentation of $Y = Z/G$ as a quotient variety. That this is NOT the case is shown in [FU], ex. 16.1.13. Here it is in fact shown that for such varieties a pull-back ring homomorphism is defined such that the projection formula holds *after tensoring with* $\mathbf{Q}$:

Lemma 1.2.4 *Let X and Y be varieties which are quotients of smooth projective varieties by finite groups. If:*

$$f : X \to Y$$

is a morphism, there is a ring homomorphism:

$$f^* : \mathrm{CH}_*(Y) \otimes \mathbf{Q} \to \mathrm{CH}_*(X) \otimes \mathbf{Q}$$

such that the projection formula holds.

□

Mumford has shown that there is a ring structure on $\mathrm{CH}_*(X) \otimes \mathbf{Q}$ for any orbifold X. See [MU]. In particular, this lemma is then valid for quasi-smooth projective hypersurfaces.

Chapter 2

Chow groups of weighted hypersurfaces of small degree.

Dans le premier paragraphe, nous définissons une application rationnelle $\sigma_Q : \mathbf{P}^N \to \mathbf{P}(Q)$ que nous allons utiliser pour démontrer le théorème principal dans le seconde. Dans le troisième paragraphe nous étudions l'amélioration, par rapport au cas $\mathbf{P}^n$, de la borne qu'on obtient dans le résultat principal.

2.1 The map σ_Q.

Let $Q = (q_0, \ldots, q_n) \in \mathbf{N}^{n+1}$. Let:

$$N := (\sum_{j=0}^{n} q_j) - 1$$

$$N_r := \begin{cases} 0 & r = -1 \\ \sum_{j=0}^{r} q_j & \forall r = 0, \ldots, n \end{cases}$$

Remark in particular that $N_n = N + 1$, and that $N_r - N_{r-1} = q_r \; \forall r = 0, \ldots, n$. Define a rational map:

$$\sigma_Q : \mathbf{P}^N \to \mathbf{P}(Q)$$

by:

$$(\sigma_Q([t_0 : \ldots : t_N]))_r := \prod_{j=N_{r-1}}^{N_r - 1} t_j \; \forall r = 0, \ldots, n.$$

Set:

$$\mathcal{J}_Q := \{(j_0,\dots,j_n)\in \mathbf{N}^{n+1} : N_{r-1}\le j_r\le N_r-1\,\forall r=0,\dots,n\}$$

and consider $\forall J\in\mathcal{J}_Q$, the subvarieties:

$$Z_J := \left\{t\in\mathbf{P}^N : t_{j_0}=\dots=t_{j_n}=0\right\}$$

$$Z_Q := \cup_{J\in\mathcal{J}_Q} Z_J$$

It is clear that σ_Q is only defined on $\mathbf{P}^N - Z_Q$., because $\forall z\in Z_Q$ every coordinate of $\sigma_Q(z)$ is zero, so such a point is not defined in $\mathbf{P}(Q)$. This map is well-defined on $\mathbf{P}^N - Z_Q$: indeed, if one considers lt_j instead of t_j for a nonzero l, one has:

$$\prod_{j=N_{r-1}}^{N_r-1} lt_j = l^{q_r}\prod_{j=N_{r-1}}^{N_r-1} t_j$$

so that modulo the weighted action of $\mathbf{C}^*$ these two quantities coincide.

Also, σ_Q is onto, since if $x\in\mathbf{P}(Q)$ and $(x_0,\dots,x_n)$ is a representative in $\mathbf{C}^{n+1^*}$, one may choose, $\forall r=0,\dots,n$, some q_r-1 variables freely and the last one such that $x_r=\prod_{j=N_{r-1}}^{N_r-1} t_j$. So:

$$\forall x\in\mathbf{P}(Q), \dim(\sigma_Q^{-1}(x)) = \sum_{r=0}^{n}(q_r-1) = N-n$$

2.2 The Main result.

Let $X\subset\mathbf{P}(Q)$ be a weighted homogeneous hypersurface of Q-degree $d\ge 3$. If X is defined by the weighted homogeneous polynomial $f=f(x_0,\dots,x_n)$, we define $\tilde{X}$ in $\mathbf{P}^N$ by the polynomial $\tilde{f}=\tilde{f}(t_0:\dots:t_N)$, of the same degree, obtained by replacing x_k by $\prod_{j=N_{k-1}}^{N_k-1} t_j$. The map σ_Q induces a rational map:

$$\sigma_Q : \tilde{X}\to X.$$

Remark 2.2.1

Recall that a weighted homogeneous hypersurface $X\subset\mathbf{P}(Q)$ defined by a weighted polynomial $f=f(x_0,\dots,x_n)$, lifts back to homogeneous $X'\subset\mathbf{P}^n$ defined by $\hat{f}=f([z_0^{q_0}:\dots:z_n^{q_n}]_Q)$. Since:

$$\frac{\partial\hat{f}}{\partial z_j}(z) = \frac{\partial f}{\partial x_j}(x)q_j z_j^{q_j-1}$$

this implies that for any point $x \in X$ coming from a $z \in \mathrm{Sing}(X')$ we have:

$$\frac{\partial f}{\partial x_j}(x) = 0 \text{ if } q_j = 1; \qquad \frac{\partial f}{\partial x_j}(x)x_j = 0 \text{ if } q_j \geq 2$$

But we have, if $a \in [N_{r-1}, N_r - 1]$:

$$\frac{\partial \tilde{f}}{\partial t_a} = \frac{\partial f}{\partial x_r} t_{N_{r-1}} \ldots \hat{t}_a \ldots t_{N_r - 1}$$

In particular, $\frac{\partial \tilde{f}}{\partial t_j} = \frac{\partial f}{\partial x_j}$ if $q_j = 1$. To see this, remember that $\tilde{f} = f \circ \sigma_Q$, so that $\nabla \tilde{f} = \nabla f \nabla \sigma_Q$. Now:

$$\nabla \sigma_Q = \begin{pmatrix} * & \ldots & * & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & \ldots & 0 & * & \ldots & * & \ldots & 0 & \ldots & 0 \\ & \vdots & & & \vdots & & \vdots & & \vdots & \\ 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & * & \ldots & * \end{pmatrix} \in M_{n+1,N+1}$$

and the assertion follows. Since $x_r = \prod_{j=N_{r-1}}^{N_r - 1} t_j$, it follows that if $t \in \tilde{X} \cap \mathrm{Sing}\tilde{X}$ and $t \notin Z_Q$ (where the map σ_Q is not defined), then:

$$0 = \frac{\partial \tilde{f}}{\partial t_a}(t) = \begin{cases} \frac{\partial f}{\partial x_r} & q_r = 1 \\ \frac{\partial f}{\partial x_r} x_r & q_r \geq 2 \end{cases}$$

Then $x = \sigma_Q(t)$ comes from a $z \in \mathrm{Sing}(X')$. Therefore:

$$\sigma_Q(\mathrm{Sing}(\tilde{X} - \tilde{X} \cap Z_Q)) \subseteq p_Q(\mathrm{Sing}(X'))$$

or, equivalently:

$$X' \text{ smooth } \Rightarrow \mathrm{Sing}(\tilde{X}) \subseteq Z_Q.$$

□

Theorem 2.2.1 *For a smooth irreducible weighted hypersurface X' of degree $d \geq 3$ in $\mathbf{P}^n$ and $\forall l \in \mathbf{N}$ such that:*

$$\binom{d+l}{1+l} \leq N := \sum_{j=0}^{n} q_j - 1$$

one has:

$$\mathrm{CH}_l(X) \otimes \mathbf{Q} = \mathbf{Q}$$

where $X = X'/\mu$.

Proof:
Let R be the plane in $\mathbf{P}^N$ defined by the equations:

$$t_{N_{r-1}} = \ldots = t_{N_r - 1} \quad \forall r = 0, \ldots, n$$

The number of equations which define it is:

$$\sum_{r=0}^{n} (N_r - 1 - N_{r-1}) = \sum_{r=0}^{n} q_r - (n+1) = N - n$$

Let $S := R \cap \tilde{X}$. Then this linear space has dimension n and has the fundamental property that $S \cap Z_Q = \emptyset$, by construction itself:

$$\forall t \in Z_Q, \forall r | 0 \leq r \leq n, \exists i \text{ such that } N_{r-1} \leq i \leq N_r - 1 \text{ for which } t_i = 0$$

But then in S, $t_{N_{r-1}} = 0$ also and all the other t_j with j in the rth string are also zero. This for every r.

Let

$$u : \mathrm{Bl}_{Z_Q}(\tilde{X}) \to \tilde{X}$$

be the blow-up along Z_Q turning σ_Q into a morphism:

$$\begin{array}{ccc} \mathrm{Bl}_{Z_Q}(\tilde{X}) & \stackrel{\hat{\sigma}_Q}{\to} & X \\ \downarrow u & & || \\ \tilde{X} & \stackrel{\sigma_Q}{\to} & X \end{array}$$

Here, in contrast with σ_Q, $\hat{\sigma}_Q$ is a morphism. Let:

$$l_0 := \max_{l \in \mathbf{N}} \left\{ \begin{pmatrix} l+d \\ l+1 \end{pmatrix} \leq N \right\} \cap \{ l \in \mathbf{N} | l \leq n \}$$

We know from [ELV], Lemma 1.1, that if $s < l_0$, then:

$$\begin{pmatrix} s+d \\ s+1 \end{pmatrix} \leq N - s$$

and so, defining:

$$\mathbf{H}(s, \tilde{X}) := \{ (\Gamma, \Gamma') \in \mathrm{Gr}(s, \tilde{X}) \times \mathrm{Gr}(s+1, \mathbf{P}^N) | \Gamma \subset \Gamma' \subset \tilde{X} \text{ or } \Gamma = \Gamma' \cap \tilde{X} \}$$

the map:

$$\mathbf{H}(s, \tilde{X}) \stackrel{\theta}{\to} \mathrm{Gr}(s, \tilde{X})$$

is onto. So let's take $\gamma \in \mathrm{CH}_s(X) \otimes \mathbf{Q}$ where $s < l_0$. Set $\tilde{\gamma} := \tau^{-1}(\gamma)$, being $\tau := \sigma_Q|_S$

Certainly $\tilde{\gamma}$ is an s-cycle on $\tilde{X}$ which is supported on S. By [ELV], Prop 2.2, since:

$$\left(\begin{array}{c} s+d \\ s+1 \end{array} \right) \leq N-s$$

one has $\mathrm{CH}_s(\tilde{X}) \otimes \mathbf{Q} = \mathbf{Q}$. Therefore there is some $a \in \mathbf{Q}$ and a $\Gamma \in \mathrm{Gr}(s, \tilde{X})$ such that:

$$\tilde{\gamma} \sim_{\tilde{X}} a\Gamma$$

By the surjectivity of θ, there is $\Gamma' \in \mathrm{Gr}(s+1, \mathbf{P}^N)$ such that:

$$\Gamma = \Gamma' \cap \tilde{X}$$

and therefore we have:

$$\tilde{\gamma} \sim_{\tilde{X}} b\Gamma' \cdot \tilde{X}$$

for some $b \in \mathbf{Q}$. Since $\mathrm{CH}_s(\mathbf{P}^N) \otimes \mathbf{Q} = \mathbf{Q}$, one can eventually replace Γ' by another $(s+1)$-plane which is transversal to Z_Q. Therefore we may assume that the proper transform of Γ' under the blow-up along Z_Q, which I note by $\hat{\Gamma}'$, is isomorphic to Γ' itself. Certainly $\hat{\tilde{\gamma}} \simeq \tilde{\gamma}$ because $\tilde{\gamma} \cap Z_Q = \emptyset$. Therefore we deduce:

$$\hat{\tilde{\gamma}} \sim_{\mathrm{Bl}_{Z_Q}(\tilde{X})} b\hat{\Gamma}' \cdot \mathrm{Bl}_{Z_Q}(\tilde{X})$$

Since X' is smooth, and since μ is a finite group, by [FU], Ex 11.4.7., we have a "moving lemma" on $X = X'/\mu$. Therefore we can move γ inside X in such a way that that it is not in the ramification locus of $\hat{\sigma}_Q$. Hence $\hat{\sigma}_Q$ is finite of a certain nonzero degree, say e. [1]

So we deduce:

$$\hat{\sigma}_{Q*}\hat{\tilde{\gamma}} = e\gamma$$

while:

$$\hat{\sigma}_{Q*}(\hat{\Gamma}' \cdot \mathrm{Bl}_{Z_Q}(\tilde{X})) = eH_{s+1} \cdot eX$$

being H_{s+1} the generator of $\mathrm{CH}_{s+1}(\mathbf{P}(Q)) \otimes \mathbf{Q} = \mathbf{Q}$.

Therefore $\gamma \sim_X be^2 H_{s+1} \cdot X = tH_s$, with H_s generator of $\mathbf{P}(Q) \otimes \mathbf{Q} = \mathbf{Q}$. This shows $\mathrm{CH}_s(X) \otimes \mathbf{Q} = \mathbf{Q}$ $\forall s < l_0$. We are left with the case $s = l_0$.

In this case, one knows by [ELV], Prop. 4.4, that *all l_0-planes are equivalent in $\tilde{X}$*. Take therefore $\Gamma' \in \mathrm{Gr}(l_0+1, \mathbf{P}^N)$ such that Γ' is transversal to Z_Q. The intersection $\Gamma' \cap \tilde{X}$ is a l_0-cycle contained in $\tilde{X}$ and therefore

[1] Indeed:

$$\tau^{-1}(x_0, \ldots, x_n) = \{[t_0 : \ldots : t_0 : \ldots : t_n : \ldots : t_n] \in \mathbf{P}^N | t_j^{q_j} = x_j \; \forall j\}$$

so τ is finite.

is here equivalent to $\tilde{\gamma}$, because by the Main Theorem of[ELV] one has $\mathrm{CH}_{l_0}(\tilde{X}) \otimes \mathbf{Q} = \mathbf{Q}$.

Now one can repeat *ad litteram* the previous reasoning for Γ' and conclude that:

$$\gamma \sim_X \hat{\hat{\sigma}}_{Q*}\tilde{\gamma} \sim_X \hat{\sigma}_{Q*}\hat{\Gamma}' \cdot \hat{\sigma}_{Q*}(\mathrm{Bl}_{Z_Q}(\tilde{X})) = e^2 H_{s+1} \cdot X$$

QED

2.3 Comparison between the bounds for $\mathbf{P}^n$ and for $\mathbf{P}(Q)$

Suppose we want to compare the degrees d of a hypersurface in $\mathbf{P}^n$ with that in $\mathbf{P}(Q)$ for which the Chow groups up to a given rank s are small. For every $(s,n) \in \mathbf{N}^2 : s \leq n-1$, and $\forall q \in \mathbf{N}$ we introduce:

$$\delta(s, n+q-1) = \delta(s,n,q) := \sup_{d \in \mathbf{N}} = \left\{ \left(\begin{array}{c} s+d \\ s+1 \end{array} \right) \leq n+q-1 \right\}$$

Remark that $N = \sum_{i=0}^n q_i - 1 = n + \sum_{i=0}^n (q_i - 1)$, so the definition of $\delta(s,n,q)$ is valid in any weighted projective space $\mathbf{P}(Q)$ such that $\sum_{i=0}^n (q_i - 1) = q - 1$. We denote $\delta(s,n) := \delta(s,n,1)$.

Lemma 2.3.1 *The following properties are valid:*

- *1) Fix $s \in \mathbf{N}$. Let $\delta \in \mathbf{N} : \delta \geq 2$. Then there is an $n \in \mathbf{N}$ such that $\delta = \delta(s,n)$ and $s \leq n-1$.*
- *2) Let $q, \delta \in \mathbf{N}$. Then there exists a couple of integers (s_0, n_0) such that $s_0 \leq n_0 - 1$ and $\delta = \delta(s_0, n_0, q)$.*
- *3) $\delta(s,n) \leq \delta(s,n,q)$ $\forall s,n,q$.*
- *4) $\delta(0,n) = n$ and $\delta(0,n,q) = n+q-1$*
- *5) Suppose $\delta(s,n) < \delta(s,n+1)$. Then $\delta(s,n+1) = \delta(s,n)+1$ and:*

$$\left(\begin{array}{c} s+\delta(s,n)+1 \\ s+1 \end{array} \right) = n+1$$

- *6) Suppose $\delta(s,n,q) < \delta(s,n+1,q)$. Then $\delta(s,n+1,q) = \delta(s,n,q)+1$ and:*

$$\left(\begin{array}{c} s+1+\delta(s,n,q) \\ s+1 \end{array} \right) = n+q$$

- *7) Let* $\delta \in \mathbf{N}^*$ *such that, for some* $(s,n) \in \mathbf{N}^2$ *satisfying* $s \leq n-1$*:*

$$\binom{s+\delta+1}{s+1} = n+1$$

Then $\delta = \delta(s,n)$ *and* $\delta(s,n+1) = \delta+1$.

- *8) Let* $\delta \in \mathbf{N}^*$ *such that, for some* $(s,n) \in \mathbf{N}^2$ *with* $s \leq n-1$*:*

$$\binom{s+\delta+1}{s+1} = n+q$$

Then $\delta = \delta(s,n,q)$ *and* $\delta(s,n+1,q) = \delta+1$.

Proof:
1) Let:

$$n_0 := \binom{s+\delta}{s+1}$$

Then of course $\delta \leq \delta(s,n_0)$ and:

$$\binom{s+\delta+1}{s+1} > n_0$$

So $\delta = \delta(s,n)$ by definition. Recalling that:

$$\binom{a+b+1}{a+1} = \binom{a+b}{a+1} + \binom{a+b}{a}$$

and since $\delta \geq 2$, one gets:

$$\binom{s+\delta}{s+1} \geq \binom{s+2}{s+1} = s+2$$

so that $s \leq n_0 - 2 < n_0 - 1$.

2) Define:

$$s_0 := \inf_{s \in \mathbf{N}} \left\{ \binom{s+\delta}{s+1} - q + 1 > 0 \right\}$$

So, if $\delta \geq q$, then $s_0 = 0$. In this case take

$$n_0 := \delta - q + 1$$

Then $\delta = \delta(0, \delta+q-1, q)$.

If $\delta \leq q-1$, then s_0 satisfies:

$$\binom{s_0+\delta}{s_0+1} = q+l \qquad \text{for some } l \geq 0$$

Take now $n_0 := s_0 + 1 + l$. Certainly $s_0 < n_0$ and:

$$\binom{s_0+\delta}{s_0+1} = q + l \leq s_0 + q + l = n_0 + q - 1$$

So $\delta \leq \delta(s_0, n_0, q)$ and it suffices to see that:

$$\binom{s_0+\delta+1}{s_0+1} \geq n_0 + q = s_0 + q + 1 + l$$

Indeed, if not:

$$\frac{(s_0+\delta+1)(q+l)}{\delta} = \left(\frac{s_0+\delta+1}{\delta}\right)\binom{s_0+\delta}{s_0+1} = \binom{s_0+\delta+1}{s_0+1} \leq s_0+q+l$$

which happens if and only if:

$$s_0(q + l - \delta) + q + l \leq 0$$

which cannot be the case because $\delta \leq q - 1$. So $\delta = \delta(s_0, s_0 + 1 + l, q)$.

3) Suppose $\delta^Q := \delta(s, n, q) = \delta(s, n) + k =: \delta + k$. Then:

$$n \geq \binom{s+\delta}{s+1} = \binom{s+\delta^Q+k}{s+1} > \binom{s+\delta^Q}{s+1} > n + 1 + q - 1 = n + q$$

which is absurd.

4)

$$\binom{s+\delta}{s+1}_{s=0} = d$$

5) Let $\delta := \delta(s, n)$ and $\delta(s, n+1) = \delta + k$. By the definition of δ:

$$\binom{s+\delta}{s+1} \leq n\,, \binom{s+\delta+1}{s+1} > n.$$

So:

$$n < \binom{s+\delta+1}{s+1} \leq \binom{s+\delta+k}{s+1} \leq n+1$$

because:

$$\binom{s+\delta+k}{s+1} = \binom{s+\delta(s,n+1)}{s+1} \leq n+1$$

Therefore[2] $k = 1$ and:

$$\binom{s+\delta+1}{s+1} = n+1$$

[2]Here one uses that $\binom{a}{b} \in \mathbf{N} \forall a, b \in \mathbf{N}$ and that, $\forall b > 0$, $\binom{a}{b} < \binom{a+1}{b}$

6) As for the previous one.
7) Since:

$$\binom{s+\delta+1}{s+1} = n+1$$

the second assertion is clear: if one takes $\delta+2$ instead than $\delta+1$, the binomial coefficient will be grater than $n+1$, so $\delta+2$ will be larger than $\delta(s, n+1)$. Moreover, since $n+1 > n$, $\delta(s, n) < \delta+1$. But of course:

$$\binom{s+\delta}{s+1} \leq \binom{s+\delta+1}{s+1} - 1 \leq n+1-1 = n$$

8) As for the previous point.

QED

Corollary 2.3.1 *Let* $s, n, q \in \mathbf{N}$ *such that* $s \leq n-1$ *and* $q \geq 2$. *Suppose* $\delta = \delta(s, n)$. *Then:*

$$\delta = \delta(s, n+1) = \delta(s, n, q) = \delta(s, n+1, q) - 1 \Leftrightarrow \binom{s+\delta+1}{s+1} = n+q$$

Proof:
$\Rightarrow$: Apply the point 6) in the previous lemma.
$\Leftarrow$: Certainly $\delta+1 = \delta(s, n+1, q)$. Moreover, $\delta+1 > \delta(s, n, q)$, otherwise:

$$\binom{s+\delta+1}{s+1} \leq n+q-1$$

So by the point 6) we have $\delta = \delta(s, n+1, q) - 1 = \delta(s, n, q)$. Since $q \geq 2$, $\delta+1 \geq \delta(s, n+1)$, but $\delta = \delta(s, n)$ by hypothesis, so $\delta(s, n+1) = \delta$ also.

QED

Remark 2.3.1

The usefulness of this lemma is in the fact that if we know, for some fixed s, n, q, that $\delta(s, n) = \delta(s, n, q)$, then passing to $n+1$ there is an amelioration of δ in the weighted projective space iff the binomial coefficient satisfies a precise equation. Remark that it cannot happen that δ progresses at the same time for both the nonweighted and the weighted case, for otherwise the binomial coefficient should be equal to both $n+1$ and $n+q$, which cannot happen if $q \geq 2$.

$\square$

Remark 2.3.2 *Let $\delta = \delta(s,n)$ and $\delta^Q = \delta(s,n,q)$ for fixed s,n,q.*

- *If $\delta = \delta^Q$, then:*
$$\binom{s+\delta}{s} \geq q$$
- *If $q = 2$, then either $\delta = \delta^Q$ or $\delta^Q = \delta + 1$.*

To see this, remark that:

$$\binom{s+\delta+1}{s+1} \geq n+1, \ \binom{s+\delta}{s+1} \leq n$$

while:
$$\binom{s+\delta^Q+1}{s+1} \geq n+q, \ \binom{s+\delta^Q}{s+1} \leq n+q-1$$

So, setting $\delta^Q = \delta + \rho$:

$$0 \leq \binom{s+\delta+\rho}{s+1} - \binom{s+\delta+1}{s+1} \leq n+q-1-(n-1) = q-2$$

So if $q = 2$ we have $\rho = 1$. As for the second point:

$$\binom{s+\delta}{s} = \binom{s+\delta+1}{s+1} - \binom{s+\delta}{s+1} \leq n+q-n = q$$

□

Here's the table for the nonweighted case; the first row represents s, while the first column is n. In the crossing, the value of $\delta(s,n)$.

$\downarrow n, s \rightarrow$	0	1	2	3	4	5	6	7
5	5	2	2	2	1	1	1	1
6	6	3	2	2	2	1	1	1
7	7	3	2	2	2	2	1	1
8	8	3	2	2	2	2	2	1
9	9	3	2	2	2	2	2	2
10	10	4	3	2	2	2	2	2
12	12	4	3	2	2	2	2	2
15	15	4	3	3	2	2	2	2
18	18	4	3	3	2	2	2	2
20	20	4	4	3	2	2	2	2
25	25	5	4	3	3	2	2	2
30	30	5	4	3	3	3	2	2
50	50	9	5	4	3	3	3	3
100	100	13	7	5	4	4	3	3

The following table is for $q = 2$:

$\downarrow n, s \rightarrow$	0	1	2	3	4	5	6	7
5	6	3	2	2	2	1	1	1
6	7	3	2	2	2	2	1	1
7	8	3	2	2	2	2	2	1
8	9	3	2	2	2	2	2	2
9	10	4	3	2	2	2	2	2
10	11	4	3	2	2	2	2	2
12	13	4	3	2	2	2	2	2
15	16	5	3	3	3	2	2	2
18	19	5	3	3	3	2	2	2
20	21	6	4	3	3	2	2	2
25	26	6	4	3	3	2	2	2
30	31	7	4	3	3	3	2	2
50	51	9	5	4	3	3	3	3
100	101	13	7	5	4	4	3	3

This other one for $q = 10$:

$\downarrow n, s \rightarrow$	0	1	2	3	4	5	6	7
5	14	4	3	2	2	2	2	2
6	15	5	3	3	2	2	2	2
7	16	5	3	3	2	2	2	2
8	17	5	3	3	2	2	2	2
9	18	5	3	3	2	2	2	2
10	19	5	3	3	2	2	2	2
12	21	6	4	3	3	2	2	2
15	24	6	4	3	3	2	2	2
18	27	6	4	3	3	2	2	2
20	29	7	4	3	3	3	2	2
25	34	7	4	3	3	3	2	2
30	39	8	5	4	3	3	3	2
50	59	10	6	4	4	3	3	3
100	109	14	7	5	4	4	3	3

For $q = 21$ one gets:

$\downarrow n, s \rightarrow$	0	1	2	3	4	5	6	7
5	25	6	4	3	3	2	2	2
6	26	6	4	3	3	2	2	2
7	27	6	4	3	3	2	2	2
8	28	7	4	3	3	3	2	2
9	29	7	4	3	3	3	2	2
10	30	7	4	3	3	3	2	2
12	32	7	4	3	3	3	2	2
15	35	7	5	4	3	3	2	2
18	38	8	5	4	3	3	3	2
20	40	8	5	4	3	3	3	2
25	45	9	5	4	3	3	3	3
30	50	9	5	4	3	3	3	3
50	70	11	6	5	4	3	3	3
100	120	15	8	5	4	4	4	3

We denote the difference from the weighted and the non-weighted case as follows:

$$\epsilon_\delta(s,n,q) := \delta(s,n,q) - \delta(s,n)$$

Then we get the following three tables:

$\epsilon_\delta(s,n,2)$:

$\downarrow n, s \rightarrow$	0	1	2	3	4	5	6	7
5	1	1	0	0	1	0	0	0
6	1	0	0	0	0	1	0	0
7	1	0	0	0	0	0	1	0
8	1	0	0	0	0	0	0	1
9	1	1	1	0	0	0	0	0
10	1	0	0	0	0	0	0	0
12	1	0	0	0	0	0	0	0
15	1	1	0	0	1	0	0	0
18	1	1	0	0	1	0	0	0
20	1	2	0	0	1	0	0	0
25	1	1	0	0	0	0	0	0
30	1	2	0	0	0	0	0	0
50	1	0	0	0	0	0	0	0
100	1	0	0	0	0	0	0	0

$\epsilon_\delta(s, n, 10)$:

$\downarrow n, s \rightarrow$	0	1	2	3	4	5	6	7
5	9	2	1	0	1	1	1	1
6	9	2	1	1	0	1	1	1
7	9	2	1	1	0	0	1	1
8	9	2	1	1	0	0	0	1
9	9	2	1	1	0	0	0	0
10	9	1	0	1	0	0	0	0
12	9	2	1	1	1	0	0	0
15	9	2	1	1	1	0	0	0
18	9	2	1	1	1	0	0	0
20	9	3	0	1	1	1	0	0
25	9	2	0	0	1	1	0	0
30	9	3	1	1	0	0	1	0
50	9	1	1	0	1	0	0	0
100	9	1	1	0	0	0	0	0

$\epsilon_\delta(s, n, 21)$:

$\downarrow n, s \rightarrow$	0	1	2	3	4	5	6	7
5	20	4	2	1	2	1	1	1
6	20	3	2	1	1	1	1	1
7	20	3	2	1	1	0	1	1
8	20	4	2	1	1	1	0	1
9	20	4	2	1	1	1	0	0
10	20	3	1	1	1	1	0	0
12	20	3	1	1	1	1	0	0
15	20	3	2	1	1	1	0	0
18	20	4	2	1	1	1	1	0
20	20	4	1	1	1	1	1	0
25	20	4	1	1	0	1	1	1
30	20	4	1	1	0	0	1	1
50	20	2	1	1	1	0	0	0
100	20	2	1	0	0	0	0	0

Another comparison problem arises when we fix the degree and the dimension and look at the range of small Chow groups. So now $d \geq 2$, $q, n \in \mathbf{N}^*$ are fixed. We define:

$$\sigma(d, n+q-1) = \sigma(d, n, q) := \sup_{s \leq n-1} \left\{ \begin{pmatrix} s+d \\ s+1 \end{pmatrix} \leq n+q-1 \right\}$$

The following hypothesis will be necessary:

$$d \leq n+q-1$$

If not, $\forall s \geq 0$:

$$n + q \leq d = \binom{d}{1} \leq \binom{d+s}{1+s}$$

so that $\sigma(d,n,q) \notin \mathbf{N}$.

The following properties are obvious (for the first one, recall that it must be $\sigma \leq n-1$):

Lemma 2.3.2 *One has:*

- $\sigma(2,n,q) = \begin{cases} n-1 & \forall q \geq 2 \\ n-2 & q = 1 \end{cases}$
- $\sigma(d,n,q) \leq \sigma(d,n,q')$ $\forall q < q'$.
- $\sigma(d,n,q) \leq \sigma(d,n',q)$ $\forall n < n'$.

I write $\sigma(d,n) := \sigma(d,n,1)$.

Lemma 2.3.3 *Let $\sigma \in \mathbf{N}^*$. Then there exists $(d,n) \in \mathbf{N}^2$ with $2 \leq d \leq n$ such that $\sigma = \sigma(d,n)$.*

Proof:

Fix $d \geq 2$. Define:

$$n_0 := \binom{\sigma+d}{\sigma+1} > d$$

Since:

$$\binom{\sigma+1+d}{\sigma+2} > n_0$$

$\sigma = \sigma(d,n_0)$, unless $\sigma = n_0 + k$ for some $k \geq 0$. But then, since $d \geq 2$:

$$\sigma = \binom{\sigma+d}{\sigma+1} + k > \sigma$$

QED

Lemma 2.3.4 *Let $\sigma := \sigma(d,n,q) < \sigma(d,n+1,q)$, where $q \geq 1$ is fixed. Then $\sigma(d,n+1,q) = \sigma(d,n,q)+1$ and:*

$$\binom{d+\sigma+1}{\sigma+2} = n+q$$

Viceversa, suppose the previous equation holds for some $\sigma \in \mathbf{N}$ and some $n,q \in \mathbf{N}$ such that $2 \leq d \leq n+q-1$ and $\sigma \leq n-1$. Then

$$\sigma(d,n,q) = \sigma = \sigma(d,n+1,q) - 1$$

Proof:

Let $\sigma(d, n+1, q) := \sigma + \rho$. Then, since σ is the largest integer satisfying the given bound:

$$\begin{pmatrix} d+\sigma+1 \\ \sigma+2 \end{pmatrix} \geq n+q$$

But $\forall \rho \geq 1$:

$$\begin{pmatrix} d+\sigma+1 \\ \sigma+2 \end{pmatrix} \leq \begin{pmatrix} d+\sigma+\rho \\ \sigma+\rho+1 \end{pmatrix} \leq n+q$$

so we deduce $\rho = 1$ and the given equation for the binomial coefficient.

On the contrary, if such an equation holds, then necessarily $\sigma(d, n, q) \leq \sigma + 1$. Since:

$$\begin{pmatrix} d+\sigma \\ 1+\sigma \end{pmatrix} \leq \begin{pmatrix} d+\sigma+1 \\ \sigma+2 \end{pmatrix} - 1 \leq n+q-1$$

and since by hypothesis $\sigma \leq n-1$, we deduce $\sigma = \sigma(d, n, q)$. That $\sigma(d, n+1, q) = \sigma + 1$ is obvious.

QED

Corollary 2.3.2 *Let* $\sigma := \sigma(d, n)$ *and* $d \geq 2$.

- *If* $\sigma(d, n, q) = \sigma$ *while* $\sigma(d, n+1) < \sigma(d, n+1, q)$ *then* $\sigma(d, n+1) = \sigma = \sigma(d, n+1, q) - 1$ *and:*

$$\begin{pmatrix} d+\sigma+1 \\ 2+\sigma \end{pmatrix} = n+q$$

- *If:*

$$\begin{pmatrix} d+\sigma+1 \\ 2+\sigma \end{pmatrix} = n+q$$

then:

$$\sigma = \sigma(d, n, q) = \sigma(d, n+1) = \sigma(d, n+1, q) - 1$$

Proof:

The first part immediately follows from the previous lemma. For the second, it is clear that:

$$\begin{pmatrix} d+\sigma+1 \\ 2+\sigma \end{pmatrix} = n+q \geq n+1$$

so that $\sigma \leq \sigma(d, n+1) < \sigma + 1$. The other two assertions also follow immediately from the lemma. The fact that $d \geq 2$ assures us that $\sigma \leq n-2$, because:

$$\sigma + 2 = \begin{pmatrix} \sigma+2 \\ \sigma+1 \end{pmatrix} \leq \begin{pmatrix} \sigma+d \\ \sigma+1 \end{pmatrix} \leq n$$

QED

Here are the tables for q respectevely equal to 1, 10 and 21.

$q = 1$

$\downarrow n, d \rightarrow$	3	4	5	6
6	2	0	0	–
7	2	0	0	0
8	2	0	0	0
9	2	0	0	0
10	2	1	0	0
15	3	1	1	0
20	3	2	1	0
50	7	3	2	1
100	11	5	3	2

$q = 10$

$\downarrow n, d \rightarrow$	3	4	5	6
6	3	1	1	0
7	3	1	1	0
8	3	1	1	0
9	3	1	1	0
10	3	1	1	0
15	4	2	1	1
20	5	2	1	1
50	8	3	2	2
100	12	5	3	2

$q = 21$

$\downarrow n, d \rightarrow$	3	4	5	6
6	4	2	1	1
7	4	2	1	1
8	5	2	1	1
9	5	2	2	1
10	5	2	2	1
15	6	3	2	1
20	6	3	2	1
50	9	4	3	2
100	13	6	3	2

We introduce the difference:

$$\epsilon_\sigma(d, n, q) := \sigma(d, n, q) - \sigma(d, n)$$

Then we get the following tables:

$\epsilon_\sigma(d, n, 10)$

$\downarrow n, d \rightarrow$	3	4	5	6
6	1	1	1	–
7	1	1	1	0
8	1	1	1	0
9	1	1	1	0
10	1	0	1	0
15	1	1	0	1
20	2	0	0	1
50	1	0	0	1
100	1	0	0	0

$\epsilon_\sigma(d, n, 21)$

$\downarrow n, d \rightarrow$	3	4	5	6
6	2	2	1	–
7	2	2	1	1
8	3	2	1	1
9	3	2	2	1
10	3	1	2	1
15	3	2	1	1
20	3	1	1	1
50	2	1	1	1
100	2	1	0	0

Suppose $\sigma = \sigma(d, n, q)$ for some $q \geq 1$, $2 \leq d \leq n + q - 1$. The question is: for which $q' > q$ do we have $\sigma(d, n, q') > \sigma$? Here is a way to find it.

Corollary 2.3.3 *Let* $\sigma = \sigma(d, n, q)$. *For* q' *to be such that* $\sigma(d, n, q') = \sigma(d, n, q) + 1$ *while* $\forall q''$ *satisfying* $q < q'' < q'$, $\sigma(d, n, q'') = \sigma$ *it is sufficient and necessary that:*

$$\binom{\sigma + d + 1}{\sigma + 2} = n + q' - 1$$

Proof:

If $q' > q + 1$, then $\sigma(d, n, q + 1) = \sigma$ so we can apply lemma 2.3.4. with $\sigma(d, n+1, q) = \sigma(d, n, q+1)$ and putting $q+1$ in the place of q. One goes on this way until the equation is solved for some q' which will be the requested one.

QED

Chapter 3

Geometric approach to weighted linear spaces and Fano varieties.

Après avoir rappelé la notion de Grassmannienne dans l'espace $\mathbf{P}^n$ (paragraphe 1), nous définissons une Grassmannienne "naive" (paragraphe 2) et prouvons dans les paragraphes 3,4 et 5 qu' à peu près tous les résultats de [ELV] restent valables; mais la borne n'est pas améliorée et pour cette raison nous cherchons à définir la Grassmannienne en utilisant l'application σ_Q, introduite au chapitre précédent et nécéssaire pour la démonstration du résultat principal. Nous démontrons que dans le cas $Q = (1^{(r)}, q_r, \ldots, q_n)$ avec $s < r$, il y a un ouvert de Zariski $U_*^Q(s, \mathbf{P}^N) \subset \mathrm{Gr}(s, \mathbf{P}^N)$ tel que son quotient par le group fini $\mathcal{S} := \mathcal{S}_{q_r} \times \ldots \times \mathcal{S}_{q_n}$ est en bijection avec son image. On peut définir $T(s, Q)$ comme $\mathrm{Gr}(s, \mathbf{P}^N)/\sim$ oú $L \sim L' \Leftrightarrow \sigma_Q(L) = \sigma_Q(L')$: on obtient un éspace topologique. Dans les deux derniers paragraphes, nous prouvons le résultat principal avec la méthode de [ELV] dans ce cas particulier.

3.1 Non-weighted case.

Let V be a K-vector space of dimension $n+1$. Recall that $\mathrm{Gr}(s+1, V)$ is the set of $(s+1)$-planes in V and it coincides with the set $\mathrm{Gr}(s, \mathbf{P}(V))$ of s-planes in $\mathbf{P}(V)$ by the usual K^*-action on $V - \{0\}$.

Any $(s+1)$-plane Γ can be represented by $(s+1)$ linearly independent vectors $p_0, \dots, p_s \in V$ and so by a $(n+1) \times (s+1)$ matrix:

$$p = \begin{pmatrix} \vdots & & \vdots \\ p_0 & \dots & p_s \\ \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} p_{00} & \dots & p_{0s} \\ \vdots & \dots & \vdots \\ p_{n0} & \dots & p_{ns} \end{pmatrix}$$

and p, p' represent the same $(s+1)$-plane if and only if there is a matrix $G \in \mathrm{GL}(s+1, K)$ such that $p = p'G$. Hence I have:

$$\mathrm{Gr}(s+1, n+1) = \mathcal{M}_{n+1,s+1}/\mathrm{Aut}(K^{s+1})$$

where $\mathcal{M}_{n+1,s+1}$ represents the Zariski open subset of $M_{n+1,s+1}(K)$ consisting of matrices of maximal rank . The dimension of the Grassmannian is therefore $(s+1)(n+1) - (s+1)^2 = (n-s)(s+1)$.

Recall that an s-plane in $\mathbf{P}^n$ is the image of an $(s+1)$-plane in K^{n+1} by the projection map. In other words, letting:

$$\pi : K^{n+1} - \{0\} \to \mathbf{P}^n := \frac{K^{n+1} - \{0\}}{K^*}$$

then any s-plane H in $\mathbf{P}^n$ is given by:

$$H := \pi(\Gamma - \{0\})$$

where Γ is an $(s+1)$-plane in K^{n+1}.

One can see the Grassmannian as an union of charts:

$$\mathrm{Gr}(s, \mathbf{P}^n) = \cup_I U_I$$

where $I = (k_0, \dots, k_s)$ is a multiindex of length $s+1$ and by definition:

$$U_I := \left\{ [u] \in \mathcal{M}_{n+1,s+1}(K)/\mathrm{Aut}(K^{s+1}) : \det(u_{jk})_{\left\{ \substack{j \in I \\ k = 0, \dots, s} \right\}} \neq 0 \right\}$$

After acting with $\mathrm{Aut}(K^{s+1})$, one may assume that the minor of maximal rank is just the unit matrix $\mathbf{1}_{s+1}$. This representation explicitly gives the dimension of $u \in U_I$: the entries in the left $n+1-(s+1)$ columns are free; each column having $s+1$ elements, the dimension of U_I must be $(n-s)(s+1)$. See [HA], 193-211 for more informations on Grassmannians.

Remark also that the complement of a chart is a Zariski closed subset of the Grassmannian: namely, the complement to the chart U_I is defined by the equation:

$$\det(u_{jk})_{\left\{ \substack{j \in I \\ k = 0, \dots, s} \right\}} = 0.$$

3.2 The “naive” definition and some properties.

Here is the most naive definition of an s-plane in a weighted projective space: consider the representation

$$\mathbf{P}(Q) = \mathbf{P}^n/\mu$$

where $\mu := \mu_{q_0} \times \ldots \times \mu_{q_n}$, with $\mu_{q_j} := \{z \in K : z^{q_j} = 1\}$ the group of q_j-roots of unity, a group isomorphic to $\mathbf{Z}_{q_j}$. Then one may set:

Definition 3.2.1 *An s-plane in $\mathbf{P}(Q)$ is the image by the projection map*[1] *of an s-plane in $\mathbf{P}^n$.*

$$\mathrm{Gr}^Q(s,n) := \{H = p_Q(\Gamma) : \Gamma \in \mathrm{Gr}(s, \mathbf{P}^n)\} = \mathrm{Gr}(s, \mathbf{P}^n)/\mu$$

□

Here the action of μ on $\mathrm{Gr}(s, \mathbf{P}^n)$ is defined as follows: let Γ be defined by the equations $\sum_{k=0}^n a_{jk} t_k = 0 \; \forall j = s+1, \ldots, n$ and let $\epsilon = (\epsilon_0, \ldots, \epsilon_n)$. Then $\epsilon \cdot \Gamma$ is defined by $\sum_{k=0}^n a_{jk} \epsilon_k^{-1} \cdot t_k = 0 \; \forall j = s+1, \ldots, n$.

In particular we have:

$$\dim \mathrm{Gr}^Q(s,n) = \dim \mathrm{Gr}(s, \mathbf{P}^n) = \dim \mathrm{Gr}(s+1, n+1) = (s+1)(n-s).$$

Example 3.2.1

Let $Q = (1^{(n)}, q)$. Then we have a diagram:

$$\begin{array}{ccc} \mathbf{P}^n \xrightarrow{\pi_Q} \mathbf{P}^n/\mu \xrightarrow{\varphi_Q} & \mathbf{P}(1^{(n)}, q) \\ & \downarrow \chi_{n,q} \\ & \mathbf{P}^{n(q)} \end{array}$$

where $n(q) := \begin{pmatrix} n+q-1 \\ n-1 \end{pmatrix}$.

[1] Using the other representation of $\mathbf{P}(Q)$, this definition would even give the wrong dimension: if $\Gamma \in \mathrm{Gr}(s+1, K^{n+1})$ is not stable under the action of K^*, then $H = \Gamma/K^*$ has the wrong dimension $s+1$. As an example, take $Q = (1,1,2,2)$ and $\Gamma := \{x_0 = x_2\}$; then on $x_0 \neq 0$:

$$\Gamma/K^* = \left\{ [x_0, x_1, x_0, x_3]_Q = \left[1, \frac{x_1}{x_0}, \frac{1}{x_0}, \frac{x_3}{x_0^2}\right]_Q \right\}_{(x_0,x_1,x_3) \in K^3 - \{0\}}$$

Let:

$$L := \left\{ \sum_{k=0}^{n} a_k t_k = 0 \right\} \in \mathrm{Gr}(n-1, \mathbf{P}^n)$$

and denote by H its class (of course this exists topologically, but the equations coming from upstairs are not stable by the action, so they don't describe H). φ_Q carries H onto:

$$\varphi_Q(H) = \{[t_0, \ldots, t_{n-1}, t_n^q] | t \in L\} = \left\{ (-a_n)^q x_n = (\sum_{k=0}^{n-1} a_k x_k)^q \right\} \subset \mathbf{P}(Q)$$

Now:

$$(\sum_{k=0}^{n-1} a_k x_k)^q = \sum_{\sum_{k=0}^{n-1} \rho_k = q} a_{\rho_0 \ldots \rho_{n-1}} \prod_{k=0}^{n-1} x_k^{\rho_k}$$

and since:

$$\chi_{n,q} : [x_0, \ldots, x_n]_Q \mapsto [\ldots : \prod_{k=0}^{n-1} x_k^{\rho_k} : \ldots : x_n]$$

one has:

$$\chi_{n,q}(\varphi_Q(H)) = \left\{ [z_0 : \ldots : z_{n(q)}] | z_{n(q)} = \sum_{\sum_{k=0}^{n-1} \rho_k = q} a_{\rho_0 \ldots \rho_{n-1}} z_{u(\rho_0 \ldots \rho_{n-1})} \right\}$$

where $u(\rho_0 \ldots \rho_{n-1})$ is the index such that:

$$z_u = \prod_{k=0}^{n-1} x_k^{\rho_k}$$

In particular, when $n = 2$, one has:

$$(\sum_{k=0}^{1} a_k x_k)^q = \sum_{\rho=0}^{q} \left(\begin{array}{c} q \\ \rho \end{array} \right) a_0^{q-\rho} a_1^{\rho} x_0^{q-\rho} x_1^{\rho}$$

The Veronese map, in this case, carries $[x_0, x_1, x_2]_Q$ to $[z_0 : \ldots : z_{q+1}] \in \mathbf{P}^{q+1}$ where $z_k = x_0^{q-k} x_1^k \ \forall k \leq q$ and $z_{q+1} = x_2$. Therefore:

$$\chi_{n,q}(\varphi_Q(H)) = \left\{ z_{q+1} = \sum_{k=0}^{q} \left(\begin{array}{c} q \\ k \end{array} \right) a_0^{q-k} a_1^k z_k \right\} \in \mathrm{Gr}(q, \mathbf{P}^{q+1})$$

.

3.3 Fano varieties

Let Y be a subvariety of $\mathbf{P}^n$ invariant by μ. Define:

$$\mathrm{Gr}^Q(s,Y) := \{H \in \mathrm{Gr}^Q(s,n) \text{ such that } H \subset Y/\mu\}$$

One has:

Lemma 3.3.1 *$H \in \mathrm{Gr}^Q(s,Y) \Leftrightarrow \forall \Gamma \in \mathrm{Gr}(s,\mathbf{P}^n)$ such that $p_Q(\Gamma) = H$, $\cup_{\epsilon\in\mu}\, \epsilon \cdot \Gamma \subset Y$.*

Proof:
It is clear that if Γ is a representative of H, then any other representative is $\epsilon \cdot \Gamma$, so it suffices to see the lemma for just one representative. Indeed, $H \subset Y/\mu$ implies that there is an $\epsilon_1 \in \mu$ such that $\epsilon_1 \cdot \Gamma \subset Y$, and then $\forall \epsilon \in \mu$, saying $\epsilon_2 \in \mu$ the element for which $\epsilon = \epsilon_1\epsilon_2$, we get $\epsilon \cdot \Gamma = \epsilon_2 \cdot (\epsilon_1 \cdot \Gamma) \subset \epsilon_2 Y = Y$. In particular, of course, $\Gamma \subset Y$. Viceversa, let $\Gamma \subset Y$, so that $\epsilon \cdot \Gamma \subset Y\ \forall \epsilon \in \mu$, and hence the union is also contained in Y. Then $H = p_Q(\Gamma) \subset Y/\mu$.

QED

Remark in particular that if Y is a hypersurface or a complete intersection of $\mathbf{P}^n$:

$$\mathrm{Gr}^Q(s,Y) = \mathrm{Gr}(s,Y)/\mu$$

Let's now consider $H \in \mathrm{Gr}^Q(s,n)$ and $H' \in \mathrm{Gr}^Q(s+1,n)$. When is $H \subset H'$? The answer is easily given:

Lemma 3.3.2 *Say Γ and Γ' two representatives of H and of H' respectively. Then $H \subset H' \Leftrightarrow \exists \epsilon \in \mu$ such that $\Gamma \subset \epsilon \cdot \Gamma'$.*

Proof:
$\Leftarrow)$: $H = p_Q(\Gamma) \subset p_Q(\epsilon \cdot \Gamma') = p_Q(\Gamma') = H'$.
$\Rightarrow)$: obvious.

QED

Remark 3.3.1

1. Note that there may very well be $\epsilon_1 \in \mu$ such that $\Gamma \not\subset \epsilon_1 \cdot \Gamma'$. For example, consider $Q = (1,1,1,2)$ and let $\Gamma = \{t_0 = 0; t_2 = t_3\}$, $\Gamma' = \{t_2 = -t_3\}$. Certainly $\Gamma \not\subset \Gamma'$, for example the point $[0:0:1:1]$ is in Γ but not in Γ', nontheless taking $\epsilon = (1,1,1,-1) \in \mu = \{1\} \times \{1\} \times \{1\} \times \{\pm 1\}$ one has $\epsilon \cdot \Gamma' = \{t_2 = t_3\}$ and it contains Γ.
2. If $\Gamma_1' = \epsilon_1 \cdot \Gamma'$ and $\Gamma_2 = \epsilon_2 \cdot \Gamma$ are two other representatives, then $\Gamma_2 = \epsilon_2 \cdot \Gamma \subset \epsilon_2\epsilon \cdot (\epsilon_1^{-1} \cdot \Gamma_1') = (\epsilon_2\epsilon\epsilon_1^{-1}) \cdot \Gamma_1'$.

□

Definition 3.3.1

Let Y be a subvariety of $\mathbf{P}^n$ invariant under the action of μ and set:

$$\begin{aligned} F^Q(s,Y) &:= \big\{(H,H') \in \mathrm{Gr}^Q(s,Y) \times \mathrm{Gr}^Q(s+1,n) \text{ such that } H \subset H'\big\} \\ \Lambda^Q(s) &:= \big\{(H,[t]_\mu) \in \mathrm{Gr}^Q(s,n) \times \mathbf{P}^n/\mu \text{ such that } [t]_\mu \in H\big\} \\ \Lambda^Q(s,Y) &:= \big\{(H,[t]_\mu) \in \Lambda^Q \text{ such that } H \subset Y/\mu\big\} \\ \mathbf{K}^Q(s,Y) &:= \big\{(H,H') \in \mathrm{Gr}^Q(s,Y) \times \mathrm{Gr}^Q(s+1,n) \text{ such that } H \subset H' \subset Y\big\} \\ \mathbf{H}^Q(s,Y) &:= \big\{(H,H') \in \mathrm{Gr}^Q(s,Y) \times \mathrm{Gr}^Q(s+1,n) | H = H' \cap Y\big\} \cup \mathbf{K}^Q \end{aligned}$$

I will call $\tilde{\theta}_Q$ and θ_Q the projections of $\mathbf{K}^Q$ resp. of $\mathbf{H}^Q$ to the first factor $\mathrm{Gr}^Q(s,Y)$, defined by $(H,H') \mapsto H$.

□

Let us determine the fibre $\tilde{\theta}_Q^{-1}(H)$. Let Γ be a representative of H. Then $\tilde{\theta}_Q^{-1}(H)$ is the set of all the elements $p_Q(\Gamma')$ such that:

- 1. $\Gamma' \in \mathrm{Gr}(s+1,n+1)$
- 2. $\exists \epsilon \in \mu$ such that $\Gamma \subset \epsilon \cdot \Gamma'$
- 3. $\epsilon \cdot \Gamma' \subset Y \; \forall \epsilon \in \mu$

Since Y is invariant under the action of μ, the previous conditions may immediately be reduced to the following one:

$$\exists \epsilon \in \mu : \epsilon \cdot \Gamma \subset \Gamma' \subset Y$$

where $p_Q(\Gamma) = H$ and $p_Q(\Gamma') = H'$. Hence:

$$\tilde{\theta}_Q^{-1}(H) = p_Q\left(\cup_{\epsilon\in\mu}(\tilde{\theta}^{-1}(\epsilon \cdot \Gamma))\right)$$

where I have to recall that, setting

$$\mathbf{K} := \{(\Gamma,\Gamma') \in \mathrm{Gr}(s,Y) \times \mathrm{Gr}(s+1,\mathbf{P}^n) | H \subset H' \subset Y\}$$

I called $\tilde{\theta} : \mathbf{K} \to \mathrm{Gr}(s,\mathbf{P}^n)$ the projection on the first factor. It is clear that:

$$p_Q\left(\cup_{\epsilon\in\mu}(\tilde{\theta}^{-1}(\epsilon \cdot \Gamma))\right) = p_Q(\tilde{\theta}^{-1}(\epsilon \cdot \Gamma))$$

for any $\epsilon \in \mu$. Indeed, the inclusion of the righthand side in the lefthand one is obvious; while if $H' \in p_Q\left(\cup_{\tau\in\mu}(\tilde{\theta}^{-1}(\tau\cdot\Gamma))\right)$ then $H' = p_Q(\Gamma')$ where $\tau\cdot\Gamma \subset \Gamma' \subset Y$ for some $\tau \in \mu$. Then $H' = p_Q(\epsilon\tau^{-1}\cdot\Gamma')$ also, with $\epsilon\tau^{-1}\cdot\Gamma' \in \tilde{\theta}^{-1}(\epsilon\cdot\Gamma)$ because $\epsilon\cdot\Gamma = \epsilon\tau^{-1}\tau\cdot\Gamma \subset \epsilon\tau^{-1}\cdot\Gamma' \subset \epsilon\tau^{-1}Y = Y$.

Now the result from [ELV] is that $\tilde{\theta}^{-1}(\epsilon\cdot\Gamma)$ is a subvariety of $\mathbf{P}^{n-s-1}$ defined by $\left(\begin{array}{c} s+d \\ s+1 \end{array}\right) - 1$ equations, hence we have:

Lemma 3.3.3

$$\dim(\tilde{\theta}_Q^{-1}(H)) \geq n-s-1-\left[\left(\begin{array}{c} s+d \\ s+1 \end{array}\right)-1\right] = n-s-\left(\begin{array}{c} s+d \\ s+1 \end{array}\right).$$

In particular: If $n-s \geq \left(\begin{array}{c} s+d \\ s+1 \end{array}\right)$, *then* $\tilde{\theta}_Q$ *is onto.*

Lemma 3.3.4 *Let* $H \in \mathrm{Gr}^Q(s,Y)$ *and* $H' \in \mathrm{Gr}^Q(s+1,n)$. *Let* Γ' *and* Γ *be two representatives of* H' *and* H *resp. Then:*

$$H'\cap Y = H \Leftrightarrow \cup_{\epsilon\in\mu}(\epsilon\cdot\Gamma'\cap Y) = \cup_{\epsilon\in\mu}\epsilon\cdot\Gamma.$$

Proof:

$\Leftarrow)$: applying p_Q to

$$\cup_{\epsilon\in\mu}(\epsilon\cdot\Gamma'\cap Y) = \cup_{\epsilon\in\mu}\epsilon\cdot\Gamma$$

gives certainly $H'\cap Y = H$.

$\Rightarrow)$

$$H'\cap Y = H \Rightarrow p_Q\left(\cup_{\epsilon\in\mu}(\epsilon\cdot\Gamma'\cap Y)\right) = p_Q(\cup_{\epsilon\in\mu}\epsilon\cdot\Gamma)$$

So let $t \in \cup_{\epsilon\in\mu}\epsilon\cdot\Gamma$ be such that $t \notin \cup_{\epsilon\in\mu}(\epsilon\cdot\Gamma'\cap Y)$, then either $t \notin Y$ or $t \notin \cup_{\epsilon\in\mu}\epsilon\cdot\Gamma'$; in the first case, $[t]_\mu \notin Y/\mu$, while the hypothesis $H'\cap Y = H$ assures that this is the case, since $[t]_\mu \in p_Q(\cup_{\epsilon\in\mu}\epsilon\cdot\Gamma) = H$; in the second case, $[t]_\mu \notin H' = p_Q(\cup_{\epsilon\in\mu}\epsilon\cdot\Gamma')$, and the same contradiction appears.

QED

Corollary 3.3.1 *Under the same hypothesis of the previous lemma:*

$$H = H'\cap Y \Leftrightarrow \exists\epsilon\in\mu \text{ such that } \epsilon\cdot\Gamma = \Gamma'\cap Y$$

Proof:

$\Rightarrow)$: $p_Q(\Gamma'\cap Y) = H'\cap Y = H = p_Q(\Gamma)$, so there is $\epsilon \in \mu$ such that $\Gamma'\cap Y = \epsilon\cdot\Gamma$.

$\Leftarrow)$: $\forall\tau\in\mu$ one has $\tau\epsilon\cdot\Gamma = \tau\cdot\Gamma'\cap Y$, so:

$$\cup_{\tau\in\mu}(\tau\cdot\Gamma'\cap Y) = \cup_{\tau\in\mu}\tau\epsilon\cdot\Gamma = \cup_{\sigma\in\mu}\sigma\cdot\Gamma = \cup_{\tau\in\mu}\tau\cdot\Gamma$$

QED

Let $H \in \mathrm{Gr}^Q(s,Y)$, then I want to see what $\tilde{\theta}_Q^{-1}(H)$ is. Let as usual $p_Q(\Gamma) = H$ and $p_Q(\Gamma') = H'$, then $\Gamma \in \tilde{\theta}_Q^{-1}(H)$ if and only if, either:

1. $\exists \epsilon \in \mu$ such that $\epsilon \cdot \Gamma \subset \Gamma' \subset Y$, which would give $H \subset H' \subset Y$.

or:

2. $\exists \epsilon \in \mu$ such that $\epsilon \cdot \Gamma = \Gamma' \cap Y$, which would give $H = H' \cap Y$.

We therefore conclude that:

Lemma 3.3.5

$$\tilde{\theta}_Q^{-1}(H) = p_Q(\tilde{\theta}^{-1}(\epsilon \cdot \Gamma))$$

for some $\epsilon \in \mu$ and some Γ representative of H.

Corollary 3.3.2

$$\dim(\tilde{\theta}_Q^{-1}(H)) = \dim(p_Q(\tilde{\theta}^{-1}(\epsilon \cdot \Gamma))) = \dim(\tilde{\theta}^{-1}(\epsilon \cdot \Gamma)) \geq n - s - \binom{s+d}{s+1}$$

In particular, $\tilde{\theta}_Q$ is onto if $n - s \geq \binom{s+d}{s+1}$.

3.4 Weighted hypersurfaces of small degree

Definition 3.4.1 *Let Y be a projective subvariety of $\mathbf{P}^n$ invariant under μ and contained in a weighted hypersurface X and consider the quotient $Y/\mu \subset \mathbf{P}^n/\mu$. Such a Y/μ is called* **spanned by s-planes** *if and only if there exists a subvariety $\tilde{Z} \subset \mathrm{Gr}^Q(s,X)$ such that the three following properties are satisfied:*

1. $Y/\mu \subset X/\mu \subset \mathbf{P}^n/\mu$.

2. $\dim(\tilde{Z}) = \dim Y - s$.

3. The map:

$$\tilde{\sigma} : \Lambda^Q_{\tilde{Z}}(s) := \left\{(H,[t]_\mu) \in \tilde{Z} \times (\mathbf{P}^n/\mu) \text{ such that } [t]_\mu \in H\right\} \to \mathbf{P}^n/\mu$$

defined by $(H,[t]_\mu) \mapsto [t]_\mu$, has $\mathrm{Im}(\tilde{\sigma}) = Y/\mu$.

Lemma 3.4.1 *$Y \subset X$ is spanned by s-planes if and only if $Y/\mu \subset X/\mu$ is spanned by s-planes.*

Proof:

$\Rightarrow)$: There exists $Z \subset \mathrm{Gr}(s,X)$ with $\dim Z = \dim Y - s$ and such that:

$$\sigma : \Lambda^Q_Z(s) := \{(\Gamma,t) \in Z \times \mathbf{P}^n \text{ such that } t \in \Gamma\} \to \mathbf{P}^n$$

defined by $(\Gamma, t) \mapsto t$ has $\mathrm{Im}(\sigma) = Y$.

Certainly $Y/\mu \subset X/\mu$, while defining $\tilde{Z} := Z/\mu$ (where μ acts on the planes in the previously defined way) gives $\dim \tilde{Z} = \dim Z = \dim Y - s = \dim(Y/\mu) - s$, and we now prove that:

$$\tilde{\sigma} : \Lambda^Q_{\tilde{Z}} \to \mathbf{P}^n/\mu$$

has image Y/μ. Indeed, let $[t]_\mu \in Y/\mu$, then $t \in Y$ and hence there is some $\Gamma \in Z$ such that $t \in \Gamma$. Then $[t]_\mu \in H$ with $H = p_Q(\Gamma) \in \tilde{Z}$, so that $(H, [t]_\mu) \in \Lambda^Q_{\tilde{Z}}(s)$. So $[t]_\mu = \tilde{\sigma}(H, [t]_\mu) \in \mathrm{Im}\tilde{\sigma}$. This proves $Y/\mu \subset \mathrm{Im}(\tilde{\sigma})$.

Conversely, let $[t]_\mu \in \mathrm{Im}(\tilde{\sigma})$. Then there is $H \in \tilde{Z}$ such that $[t]_\mu \in H$. Hence there is $\epsilon \in \mu$ for which $\epsilon t \in \Gamma$ where Γ is a representative of H such that $\Gamma \in Z$. Then $(\Gamma, \epsilon t) \in \Lambda_Z(s)$ and so $\epsilon t \in Y$, which implies $t \in Y$ (You should remember that Y weighted subvariety is equivalent to Y stable by the action of μ.)

$\Leftarrow)$: Call $\tilde{Z}$ the subvariety of $\mathrm{Gr}^Q(s, X)$ satisfying the conditions of the definition. Say Z a representative of $\tilde{Z}$ which is a subvariety of $\mathrm{Gr}(s, X)$, because of the invariance of X by μ. Defining $\Lambda_Z(s)$ and σ as before, I say that certainly $\mathrm{Im}(\sigma) \subset Y$. Indeed, if not, there would be $(\Gamma, t) \in \Lambda_Z(s)$ such that $t \notin Y$. But $t \in \Gamma$ and hence $[t]_\mu \in H = p_Q(\Gamma)$; moreover $(H, [t]_\mu) \in \tilde{Z} \subset \mathrm{Gr}^Q(s, X)$ and hence $[t]_\mu \in \mathrm{Im}(\tilde{\sigma}) = Y/\mu$, so there is $\epsilon \in \mu$ such that $\epsilon \cdot t \in Y$. But then $t \in Y$ also.

The same reasoning shows of course that $\sigma(\Lambda_{\epsilon \cdot Z}(s)) \subset Y \ \forall \epsilon \in \mu$.

Conversely, if $y \in Y$, then $[y]_\mu \in Y/\mu = \mathrm{Im}(\tilde{\sigma})$ so there is $H \in \tilde{Z}$ such that $[y]_\mu \in H$. Say $\Gamma \in Z$ a representative of H, then $y \in \epsilon \cdot \Gamma$ for some $\epsilon \in \mu$. This doesn't mean of course that $y \in \Gamma$, but certainly gives:

$$Y \subset \sigma\left(\cup_{\epsilon \in \mu} \{(\epsilon \cdot \Gamma, y) | y \in \epsilon \cdot \Gamma \,;\, \Gamma \in Z\}\right) = \sigma\left(\cup_{\epsilon \in \mu} \Lambda_{\epsilon \cdot Z}\right)$$

We deduce that the variety $\cup_{\epsilon \in \mu} \epsilon \cdot Z$ has the same dimension as Z and as $\tilde{Z}$ because finite union of subvarieties of that same dimension, it is contained in $\mathrm{Gr}(s, X)$ and has a map

$$\sigma : \Lambda_{\cup_{\epsilon \in \mu} \epsilon \cdot Z}(s) \to \mathbf{P}^n$$

whose image is Y.

QED

I shall indicate by $\mathrm{CH}^s_l(X)$ the Chow group of l-cycles of X spanned by s-planes and by $\mathrm{CH}^s_l(X/\mu)$ the Chow group of l-cycles in X/μ spanned by s-planes.

I now come to the important:

Theorem 3.4.1 *Let X be an irreducible weighted homogeneous hypersurface of $\mathbf{P}^n$ of degree d, and let Y be a weighted subvariety $Y \subset X$, such that Y/μ is spanned by s-planes but not by $(s+1)$-planes. Suppose:*

$$\binom{l+d}{l+1} \leq n-l$$

Then there exists a weighted homogeneous subvariety T of dimension $l+1$ and an $\alpha \in \mathbf{Q}$ such that:

$$[T/\mu] \cdot [X/\mu] \equiv \alpha[Y/\mu] \pmod{\mathrm{CH}_l^{(s+1)}(X/\mu)}$$

Proof:

Because of the previous lemma, Y is spanned by s-planes but not by $(s+1)$-planes, and if $\tilde{Z} \subset \mathrm{Gr}^Q(s,X)$ is the subvariety satisfying the conditions of the definitions, we can take $\cup_{\epsilon\in\mu}\epsilon \cdot Z$ as the corresponding subvariety of $\mathrm{Gr}(s,X)$.

From [ELV] we know that there is an $a \in \mathbf{Q}$ and a subvariety T of $\mathbf{P}^n$ such that:

$$[T] \cdot [X] \equiv a[Y] \pmod{\mathrm{CH}_l^{(s+1)}(X)}$$

By [FU 1.7.6] we know that:

$$\mathrm{CH}_t(R/G) \otimes \mathbf{Q} = \mathrm{CH}_t(R)^G \otimes \mathbf{Q}$$

$\forall t = 0, \ldots, n$, with R any subvariety of $\mathbf{P}^n$ on which a finite group G acts. This is our case, so we deduce that *up to a rational multiplicative constant*:

$$[X] \equiv [X/\mu]$$

$$[Y] \equiv [Y/\mu]$$

So the only thing I have to show is that T is an invariant cycle, because if this was the case, we would have:

$$[T/\mu] \cdot [X/\mu] = \rho[T] \cdot [X]$$

for some $\rho \in \mathbf{Q}$, and so the thesis, using the previous lemma to conclude that subvarieties spanned by $(s+1)$-planes upstairs are still so downstairs. Now recall from the proof given in [ELV] that we had $T = Pr_*(\Lambda'_{\tilde{\Sigma}})$, where $W \subset \mathrm{Gr}(s,X)$ is the subvariety coming from the definition of subvariety spanned by s-planes, and where:

$$\tilde{\Sigma} := \{(\Gamma, \Gamma_1) \in \mathbf{H} \text{ such that } \Gamma \in W\}$$

$$Pr : \Lambda'_W := \{(\Gamma, \Gamma_1, t) \in \tilde{\Sigma} \times \mathbf{P}^n \text{ such that } t \in \Gamma_1\} \to \mathbf{P}^n$$

$$(\Gamma, \Gamma_1, t) \mapsto t.$$

In our case, $W = \cup_{\epsilon \in \mu} \epsilon \cdot Z$. Hence:

$$Pr(\Lambda'_{\tilde{\Sigma}}) = \{t \in \mathbf{P}^n \text{ such that } \exists (\Gamma, \Gamma_1) \in \tilde{\Sigma}, t \in \Gamma_1\} = \cup_{(\Gamma,\Gamma_1) \in \tilde{\Sigma}} \Gamma_1$$

Now let $\epsilon \in \mu$, then:

$$\epsilon \cdot Pr(\Lambda'_{\tilde{\Sigma}}) = \cup_{(\Gamma,\Gamma_1) \in \tilde{\Sigma}} \epsilon \cdot \Gamma_1 = \cup \epsilon \cdot \Gamma_1$$

where the last union is over all couples $(\Gamma, \Gamma_1) \in \mathbf{H}$ such that $\Gamma \in W = \cup_{\tau \in \mu} \tau \cdot Z$.

Now $(\Gamma, \Gamma_1) \in \mathbf{H} \Leftrightarrow \Gamma \subset \Gamma_1 \subset X$ or $\Gamma = \Gamma_1 \cap X$. This is equivalent to $\epsilon \cdot \Gamma \subset \epsilon \cdot \Gamma_1 \subset X$ or $\epsilon \cdot \Gamma = \epsilon \cdot \Gamma_1 \cap X$, that is $(\epsilon \cdot \Gamma, \epsilon \cdot \Gamma_1) \in \mathbf{H}$. So varying Γ in $\cup_{\tau \in \mu} \tau \cdot Z$ is the same that varying $\epsilon \cdot \Gamma$ in $\cup_{\tau \cdot \mu} \tau \cdot Z$.

Finally:

$$\bigcup_{\epsilon \cdot \Gamma \in \cup_{\tau \in \mu} \tau \cdot \ Z \,;\, (\epsilon \cdot \Gamma, \epsilon \cdot \Gamma_1) \in \mathbf{H}} \epsilon \cdot \Gamma_1 = \bigcup_{\omega \in \cup_{\tau \in \mu} \tau \cdot Z \,;\, (\omega, \omega_1) \in \mathbf{H}} \omega_1 = \bigcup_{(\omega,\omega_1) \in \tilde{\Sigma}} \omega_1 = Pr(\Lambda'_{\tilde{\Sigma}})$$

QED

Lemma 3.4.2 *Let $X \subset \mathbf{P}^n/\mu$ be an irreducible weighted hypersurface of degree d and suppose:*

$$\binom{l+d}{l+1} \leq n - l$$

Then:

$$\mathrm{CH}_l(X/\mu) \otimes \mathbf{Q} \simeq \mathbf{Q}$$

Proof:

Since X is invariant by μ by hypothesis, the result from Fulton gives:

$$\mathbf{Q} = \mathrm{CH}_l(X)^\mu \otimes \mathbf{Q} = \mathrm{CH}_l(X/\mu) \otimes \mathbf{Q}$$

This lemma can also be proved in the same way as it was proven the analogous one on $\mathbf{P}^n$, that is as a corollary of the previous theorem. In this case, the technique is to show first that $\mathrm{CH}_l^{(l')}(X/\mu) \otimes \mathbf{Q} \simeq \mathbf{Q}$ for some well-chosen l' and then that, $\forall s \leq l'$:

$$\mathrm{CH}_l^{(s)}(X/\mu) \otimes \mathbf{Q} \simeq \mathrm{CH}_l^{(l')}(X/\mu) \otimes \mathbf{Q}$$

For the first step one chooses l' as the maximal integer such that there is a $Y' \subset \mathbf{P}^n$, stable under μ, with $\dim Y' = l$ and spanned by l'-planes. Then $l' \leq l$ or $\dim Z = l - l' < 0$, being Z the subvariety of $\mathrm{Gr}(s, \mathbf{P}^n)$

satisfying the definition for which Y' is spanned by l'-planes for Y'. Using the hypothesis:

$$\binom{l'+d}{l'+1} \leq \binom{l+d}{l+1} \leq n-l \leq n-l'$$

and because of the previous result, there must be $T' \subset \mathbf{P}^n$ which is stable under the action of μ and satisfies:

$$T' \cdot X = \rho' Y' \pmod{\mathrm{CH}_l^{(l'+1)}(X/\mu)}$$

for some $\rho' \in \mathbf{Q}$. By the definition of l' one gets $\mathrm{CH}_l^{(l'+1)}(X/\mu) = 0$. Moreover, $\mathrm{CH}_{l+1}(\mathbf{P}^n) \otimes \mathbf{Q} \simeq \mathbf{Q}$ and hence in such a space one has $T' \sim \eta' H$ for some $\eta' \in \mathbf{Q}$ and H linear generator of $\mathrm{CH}_{l+1}(\mathbf{P}^n) \otimes \mathbf{Q}$; therefore:

$$\rho' Y' = \eta' X \cdot H$$

This proves:

$$\mathrm{CH}_l^{(l')}(X/\mu) \otimes \mathbf{Q} \simeq \mathbf{Q}$$

Now let $Y \in \mathrm{CH}_l^{(s)}(X/\mu)$ for some $s \leq l$. Certainly $s \leq l'$, so:

$$\binom{s+d}{s+1} \leq \binom{l'+d}{l'+1} \leq n-l' \leq n-s$$

so there is $T_Y \subset \mathbf{P}^n$, stable under the action of μ, such that:

$$T_Y \cdot X = \rho_Y Y \pmod{\mathrm{CH}_l^{(s+1)}(X/\mu)}$$

for some $\rho_Y \in \mathbf{Q}$. Still, $T_Y = \eta_Y H$ and so:

$$\eta_Y X \cdot H = \rho_Y Y \pmod{\mathrm{CH}_l^{(s+1)}(X/\mu)}$$

Then:

$$H \cdot X = \frac{\rho_Y}{\eta_Y} Y = \frac{\rho'}{\eta'} Y' \pmod{\mathrm{CH}_l^{(s+1)}(X/\mu)}$$

and so:

$$\mathrm{CH}_l^{(s)}(X/\mu) \otimes \mathbf{Q} = \mathrm{CH}_l^{(s+1)}(X/\mu) \otimes \mathbf{Q} = \ldots = \mathrm{CH}_l^{(l')}(X/\mu) \otimes \mathbf{Q} \simeq \mathbf{Q}$$

QED

3.5 Complete intersections of small degree

As in the nonweighted case, one may extend the results to weighted complete intersections of small weighted degree:

Theorem 3.5.1 *Let $X_1, \ldots, X_r$ be projective hypersurfaces in $\mathbf{P}^n$ invariant under the action of μ and of Q-weighted degrees $d_1, \ldots, d_r$ respectively, with $d_j \geq d_{j+1}$ $\forall j = 1, \ldots, r$ and $d_r \geq 2$. Let X be a union of irreducible components of $X_1 \cap \ldots \cap X_r$ equidimensional of dimension $n - r$. Let Y be an l-dimensional subvariety of X stable under μ and spanned by s-planes. Then if*

$$\sum_{i=1}^{r} \left(\begin{array}{c} s + d_i \\ s + 1 \end{array} \right) \leq n - s$$

there exists an effective cycle $T/\mu \in \mathrm{CH}_l^{(s+1)}(\mathbf{P}^n/\mu)$ and a rational number γ such that:

$$T/\mu \cdot X/\mu \equiv \gamma Y/\mu \pmod{\mathrm{CH}_{l,Q}^{(s+1)}(X/\mu)}$$

Moreover, if :

$$\sum_{i=1}^{r} \left(\begin{array}{c} s + d_i \\ s + 1 \end{array} \right) \leq n - l$$

then one may choose T as an $(l + r)$-dimensional subvariety of $\mathbf{P}^n$ stable under μ. So, $\mathrm{CH}_l(X/\mu) \otimes \mathbf{Q} \simeq \mathbf{Q}$.

Proof: Starting from the results 3.1 and 3.2 in [ELV], one has the existence of such a T and only has to verify that this T is stable under the action of μ, which can be done as in the previous theorem. Finally, $\mathrm{CH}_l(X/\mu) \otimes \mathbf{Q} \simeq \mathrm{CH}_l(X)^{\mu} \otimes \mathbf{Q}$.

□

As in the nonweighted case, one has the:

Theorem 3.5.2 *Let $(d_1, \ldots, d_r) \in \mathbf{N}^r$ be naturals such that $d_1 \geq \ldots \geq d_r$. Suppose $d_1 \geq 3$ or $r \geq l + 1$. Let:*

$$\sum_{i=1}^{r} \left(\begin{array}{c} d_i + l \\ 1 + l \end{array} \right) \leq n.$$

Then:

- *1) Let* $(f_1,\ldots,f_r)\in\prod_{i=1}^r S_{d_i}(Q)$ *with* $\deg f_i = d_i$ *and* $d_r\geq 2$ *and let* $X_{f_1\ldots f_r} := \{f_j = 0\,\forall j = 1,\ldots,r\}$*. Then* $\forall [t]_\mu\in X_{f_1\ldots f_r}/\mu\subset \mathbf{P}(Q)$ *there is* $H\in \mathrm{Gr}^Q(l, X_{f_1\ldots f_r})$ *such that:*
$$[t]_\mu\in H.$$
In particular:
$$\mathrm{Gr}^Q(l, X_{f_1\ldots f_r})\neq\emptyset$$

- *2)If* $\mathrm{char} k = 0$*, there is a Zariski open subset*
$$U\subset\prod_{i=1}^r S_{d_i}(Q)$$
such that $\mathrm{Gr}^Q(l, X_{f_1\ldots f_r})$ *is irreducible of codimension*
$$\sum_{i=1}^r \binom{d_i + l}{l}$$
in $\mathrm{Gr}^Q(\mathbf{P}(Q))$*,* $\forall (f_1,\ldots,f_r)\in U$*.*

Proof:

The first assertion is a corollary of the corresponding result in [ELV], where it is proved under the same hypothesis that $\mathrm{Gr}(l, X_{f_1\ldots f_r})\neq\emptyset$.

For the second one we have from [ELV] the same results for $\mathrm{Gr}(l, X_{f_1\ldots f_r})$ so, apart from smoothness, they are true also for the quotent $\mathrm{Gr}^Q(l, X_{f_1\ldots f_r}) = \mathrm{Gr}(l, X_{f_1\ldots f_r})/\mu$. As for the smoothness, it is in general false that $\mathrm{Gr}^Q(s)$ is smooth, because the action of μ may have fixed points.[2] (See Prop. 5.3 of [WE]).

QED

We set:
$$\mathrm{A}_0(Y) := \{\alpha\in \mathrm{CH}_0(Y) : \deg\alpha = 0\}$$

[2]Indeed, let:
$$\Gamma := \{\sum_{k=0}^n a_{jk}t_k = 0\,\forall j = s+1,\ldots,n\}$$
be an s-plane; if $\epsilon = (\epsilon_0,\ldots,\epsilon_n)\in\mu$ one has:
$$\epsilon^{-1}\cdot\Gamma = \{\sum_{k=0}^n a_{jk}\epsilon_k t_k\,\forall j = s+1,\ldots,n\}$$
But this doesn't mean that $\epsilon_k = \epsilon_0\ \forall k$, because it may for example happen that for some k $a_{jk} = 0\ \forall j$.

Lemma 3.5.1 *Let:*

$$X := \{f_j = 0\, \forall j = 1, \ldots, r\}$$

be a subvariety of $\mathbf{P}^n$ invariant under the action of μ. Let:

$$\sum_{i=1}^{r} \left(\begin{array}{c} d_i + l \\ 1 + l \end{array} \right) \leq n.$$

Suppose that either $d_1 \geq 3$ or that $r \geq l + 1$ Then:

$$\mathrm{A}_0(\mathrm{Gr}(l, X)/\mu) = 0.$$

Proof:
In [ELV], under the same hypothesis, one has:

$$\mathrm{A}_0(\mathrm{Gr}(l, X)) = 0.$$

QED

The following result descends from the equivalent one in [ELV] simply by dividing by μ.

Theorem 3.5.3 *Let $d_1 \geq \ldots \geq d_r \geq 2$ be integers and let $l \in \mathbf{N}$. Suppose either $d_1 \geq 3$ or $r \geq l + 1$. Suppose also that:*

$$\sum_{i=1}^{r} \left(\begin{array}{c} d_i + l \\ 1 + l \end{array} \right) \leq n$$

Then $\forall (f_1, \ldots, f_r) \in \prod_{i=1}^{r} S_{d_i}(Q)$ there is $H \in \mathrm{Gr}^Q(l, X_{f_1 \cdots f_r}/\mu)$.

Moreover:

$$\mathrm{CH}_s(X_{f_1, \ldots, f_r}/\mu) \otimes \mathbf{Q} \simeq \mathbf{Q}\ \forall s \leq l.$$

If $d_1 = 2$, $1 \leq r \leq l$ and:

$$r(l+1) = \sum_{i=1}^{r} \left(\begin{array}{c} d_i + l \\ 1 + l \end{array} \right) - r \leq n - l - 1$$

the same conclusions hold.

3.6 Weighted case using σ_Q.

In order to have a geometric approach with the better bound for the Chow groups, I want to apply σ_Q to s-planes of $\mathbf{P}^N$. For this purpose, the very first thing to remark is that *changes of coordinates in $\mathbf{P}^N$ are not allowed* because they may even change the dimension of the image. Here's an easy example:

Example 3.6.1

Let $n = 3$ and $Q = (1, 1, 2, 2)$ so that $N = 5$. Take $s = 2$ and let:

$$L := \{t_0 = 0;\ t_1 = 0;\ t_3 + t_4 = 0\} \in \mathrm{Gr}(2, \mathbf{P}^5).$$

Since:

$$\sigma_Q([t_0 : \ldots : t_5]) = [t_0, t_1, t_2 t_3, t_4 t_5]$$

we have:

$$H := \sigma_Q(L) = \{[0, 0, t_2 t_3, -t_3 t_5]\} = \{x_0 = x_1 = 0\}$$

Now apply the following change of coordinates:

$$t'_k := \begin{cases} t_k & \forall k \neq 4 \\ t_3 + t_4 & k = 4 \end{cases}$$

Then the image of L will be defined by the conditions $t'_0 = t'_1 = t'_4 = 0$ and its image in $\mathbf{P}(Q)$ will be $[0, 0, 1, 0]$.

□

Remark 3.6.1

I show now that the action of $\mathrm{Aut}(\mathbf{C}^{s+1})$ on $\mathcal{M}_{N+1,s+1}$ is consistent with the map σ_Q, so that if u has a nonzero minor corresponding to the multiindex I, I can assume that such a minor is the identity; in fact, this is not a change of coordinates, but simply a good choice of the parameters describing the plane. Indeed, let:

$$x_r = \prod_{j=N_{r-1}}^{N_r - 1} \left(\sum_{k=0}^{s} u_{jk} \lambda_k \right)$$

and suppose $u = vA$ with $A \in \mathrm{Aut}(\mathbf{C}^{s+1})$. Then:

$$x_r = \prod_{j=N_{r-1}}^{N_r - 1} \left[\sum_{k=0}^{s} \left(\sum_{i=0}^{s} v_{ji} A_{ik} \right) \lambda_k \right]$$

that is:

$$x_r = \prod_{j=N_{r-1}}^{N_r - 1} \left[\sum_{i=0}^{s} v_{ji} \left(\sum_{k=0}^{s} A_{ik} \lambda_k \right) \right]$$

and so:

$$x_r = \prod_{j=N_{r-1}}^{N_r - 1} \left(\sum_{i=0}^{s} v_{ji} \mu_i \right)$$

where $\mu = A\lambda$ also varies on all of $\mathbf{C}^{s+1} - \{0\}$.

□

Convention: Since the interest is in the image of the planes, and since the target space has dimension n, from now on *I shall assume* $s \leq n-1$.

□

Remark 3.6.2

To avoid confusion, I recall that any $L \in \mathrm{Gr}(s, \mathbf{P}^N)$ is associated to a matrix $u \in \mathcal{M}_{N+1,s+1}$ and viceversa any such u defines a plane L_u, and u, v define the same plane if and only if $v = ug$ for some $g \in \mathrm{Aut}(\mathbf{C}^{s+1})$. If:

$$u = (u_{jk})_{\substack{j=0,\ldots,N \\ k=0,\ldots,s}}$$

then the plane L_u is given, in parametric equations, by:

$$L_u := \left\{ t_j = \sum_{k=0}^{s} u_{jk}\lambda_k \; \forall j = 0, \ldots, N \right\}_{\lambda \in \mathbf{C}^{s+1}}$$

and so:

$$H_u := \sigma_Q(L_u) = \left\{ x_r = \prod_{j=N_{r-1}}^{N_r-1} (\sum_{k=0}^{s} u_{jk}\lambda_k) \; \forall r = 0, \ldots, n \right\}_{\lambda \in \mathbf{C}^{s+1}}$$

Hence the q_r rows of u going from the one labelled by N_{r-1} to the one labelled $N_r - 1$ are the ones occurring in defining parametrically the variable x_r in H_u. I often call *string* the set of rows occurring to define the same variable. For example, when $Q = (1, \ldots, 1, q)$, I have n strings of length 1, namely the rows from 0 to $n-1$, and a string of length q, the one composed by the rows from n to $N = n + q - 1$. In this case I shall write

$$u = \begin{pmatrix} u(1) \\ u(2) \end{pmatrix}$$

with $u(1) \in M_{n,s+1}$ and $u(2) \in M_{q,s+1}$.

□

Set:

$$\mathrm{Gr}(s, Z_Q) := \{L \in \mathrm{Gr}(s, \mathbf{P}^N) \text{ such that } L \subset Z_Q\}$$

This is the subset of s-planes in $\mathbf{P}^N$ which lie in the locus Z_Q where σ_Q is not defined. Since Z_Q is a union of linear spaces, the s-planes completely contained in it sweep out a proper closed subset of $\mathrm{Gr}(s, \mathbf{P}^N)$. The next lemma shows this:

Lemma 3.6.1 *Let* $Q = (q_0, \dots, q_n)$. *Then:*

$$\dim \mathrm{Gr}(s, Z_Q) = (N - s - n - 1)(s + 1).$$

The complement is a (Zariski) open subset of $\mathrm{Gr}(s, \mathbf{P}^N)$.

Remark that if $s > N - n - 1$, one has negative dimension; and indeed, let $L = L_u \in \mathrm{Gr}(s, Z_Q)$; $n+1$ of the $N+1$ rows must be zero, namely one for each string. But then only $N - n$ rows are left, while we have to introduce an identity matrix of order $s + 1$, which is impossible if $s > N - n - 1$. If $N = s + n + 1$, $\mathrm{Gr}(s, Z_Q)$ has dimension zero; this means that it has a finite number of points. For example, when $Q = (1, \dots, 1, q)$, one has to put all the rows in $u(1)$ equal to zero, and one of the last q rows equal to zero also (to kill the last coordinate x_n in the image). Since $s = N - n - 1 = q - 2$ one has precisely q possibilities for $u(2)$:

$$\begin{pmatrix} 0 & \dots & 0 \\ & \mathbf{1}_{s+1} & \end{pmatrix}, \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & \vdots & \dots & \vdots & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} & \mathbf{1}_{s+1} & \\ 0 & \dots & 0 \end{pmatrix}$$

Proof:

For every string there must be a zero row, so that the left rows are $N - n$. Now insert the identity matrix of order $s + 1$ to have the nonzero maximal minor, this makes the number of free rows decrease to $N - n - s - 1$. Each row has $s + 1$ entries, so we get the dimension as stated.

QED

Remark 3.6.3

It may very well happen that $L \cap Z_Q \neq \emptyset$ even if $L \not\subset Z_Q$. In this case, one adopts the very standard process in Algebraic Geometry, namely one replaces $\sigma_Q(L - L \cap Z_Q)$ by its closure inside $\mathbf{P}(Q)$.

$\square$

3.7 The case $Q = (1, \dots, 1, q)$

Lemma 3.7.1 *Let* $Q = (1, \dots, 1, q)$ *and let* $L_u \in \mathrm{Gr}(s, \mathbf{P}^N)$.
Then $\dim \sigma_Q(L_u) < s$ *if and only if either* $\mathrm{rk}(u(1)) < s$ *or* $\mathrm{rk}(u(1)) = s$ *and* $\exists j_0 \in [n, \dots, N] \cap \mathbf{N}$ *such that* $u_{j_0 k} = 0$ $\forall k = 0, \dots, s$.

Corollary 3.7.1 *Let:*

$$V^Q(s, \mathbf{P}^N) := \{L_u \in \mathrm{Gr}(s, \mathbf{P}^N) \text{ such that } \dim \sigma_Q(L_u) < s\}$$

Then:

$$U^Q(s, \mathbf{P}^N) := \mathrm{Gr}(s, \mathbf{P}^N) - V^Q(s)$$

is a Zariski open subset of $\mathrm{Gr}(s, \mathbf{P}^N)$.

Proof:

$\Leftarrow$): Suppose $\mathrm{rk}(u(1)) < s$. For example, let it be equal to $s-1$ and let $t_0, \ldots, t_{s-2}$ be the independent variables. Then:

$$t_j = \sum_{k=0}^{s-2} \rho_{jk} t_k \ \forall j = s-1, \ldots, N.$$

And:

$$\sigma_Q(L_u) = \left\{ [t_0, \ldots, t_{s-2}, \sum_{k=0}^{s-2} \rho_{s-1,k} t_k, \ldots, \sum_{k=0}^{s-2} \rho_{n-1,k} t_k, x_n]_Q \right\}$$

that is:

$$\sigma_Q(L_u) = \left\{ x_j = \sum_{k=0}^{s-2} \rho_{jk} x_k \ \forall j = s-1, \ldots, n-1 \right\}$$

Therefore, $\dim \sigma_Q(L_u) = n - (n - 1 - (s-2)) = s - 1$.

If $\mathrm{rk}(u(1)) = s$, and there is a zero column between the last q ones, certainly $x_n = 0$ so that:

$$\sigma_Q(L_u) = \left\{ x_n = 0;\ x_j = \sum_{k=0}^{s-1} \rho_{jk} x_k \forall j = s, \ldots, n-1 \right\}$$

which is still defined by $n - s + 1$ independent equations.

$\Rightarrow$): Suppose that $\dim \sigma_Q(L_u) < s$ and $\mathrm{rk}(u(1)) = s + 1$. Say $t_0, \ldots, t_s$ the independent variables, then so are $x_0, \ldots, x_s$, so that $\dim \sigma_Q(L_u) \geq s$ and so is s. This shows that $\dim \sigma_Q(L_u) < s$ may happen only if $\mathrm{rk}(u(1)) \leq s$. If such a rank is exactly s, then we have to see that one of the last q rows is zero. Otherwise, we may assume:

$$u(1) = \begin{pmatrix} & & & 0 \\ & \mathbf{1}_s & & \vdots \\ & & & 0 \\ * & \ldots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \ldots & * & 0 \end{pmatrix}$$

while $u(2)$ has at least one row having a nonzero entry in the last column (the one labelled by s), because anyway $\mathrm{rk} u = s+1$. Hence x_n, if it is not zero, has the parameter λ_s which never appeared before and is therefore independent. Therefore the dimension would be s.

QED

Let now $u, v \in U^Q(s, \mathbf{P}^N)$. The next step is to see when $\sigma_Q(L_u) = \sigma_Q(L_v)$.

Suppose first $\mathrm{rk}(u(1)) = s+1$. Then we may suppose:

$$u(1) = \begin{pmatrix} \mathbf{1}_{s+1} \\ *_1 \end{pmatrix}$$

while:

$$u(2) = (*_2).$$

So if $\sigma_Q(L_u) = \sigma_Q(L_v)$, for all values of the parameters λ_j's one has:

$$\sum_{k=0}^{s} u_{jk}\lambda_k = x_j = \sum_{k=0}^{s} v_{jk}\lambda_k$$

$\forall j = s+1, \ldots, n-1$, and this implies that $*_1$ is the same for the matrices u and v; and:

$$\prod_{j=n}^{N} \left(\sum_{k=0}^{s} u_{jk}\lambda_k \right) = x_n = \prod_{j=n}^{N} \left(\sum_{k=0}^{s} v_{jk}\lambda_k \right)$$

and since two polynomials on $\mathbf{C}$ differ by a multiplicative constant if and only if their roots are the same, the part $*_2$ in v is just a permutation of the rows of the same part $*_2$ in u. In other words:

$$\exists \tau \in \mathcal{S}_q \text{ such that } u_{jk} = v_{\tau(j)k} \ \forall j = n, \ldots, N.$$

Of course this is not necessary if u has a zero row among the last q ones, because in that case it only suffices that v has a zero row in its last string too: in both cases $x_n = 0$.

Now suppose $\mathrm{rk}(u(1)) = s$ and there is no zero row among the last q ones. Then it is clear that $x_n \neq 0$ is the last independent variable in $\sigma_q(L_u)$, no matter how $*_2$ is done. Hence, in this case, the condition reduces to $*_1(u) = *_1(v)$. To resume:

Lemma 3.7.2 *Let $Q = (1, \ldots, 1, q)$ and let $u \in U^Q(s)$. Then:*

- *If* $\mathrm{rk}(u(1)) = s + 1$ *and there is no zero row among the last q ones, $\sigma_Q(L_u) = \sigma_Q(L_v)$ if and only if $u(1) = v(1)$ and $v(2) = \tau u(2)$ for some $\tau \in \mathcal{S}_q$. So for any such u there is only a finite number of other matrices defining the same image (namely, $\mid \mathcal{S}_q \mid -1 = q! - 1$ ones).*

- *If* $\mathrm{rk}(u(1)) = s+1$ *and there is a zero row among the last q ones, then $\sigma_Q(L_u) = \sigma_Q(L_v)$ if and only if $u(1) = v(1)$ and there is a zero row in $v(2)$. So in this case, there is a subvariety of dimension $(q-1)(s+1)$ of $U^Q(s)$, such that any of its elements v has the same image as u.*

- *If* $\mathrm{rk}(u(1)) = s$*, then any other v with $v(1) = u(1)$ will have the same image. In this case, the dimension of this subvariety is therefore $(q-1)(s+1)$.*

Indeed, in the last case, one of the rows in $u(2)$ completes the unitary minor.

This discussion leads to the following:

Definition 3.7.1

Let $Q = (1, \dots, 1, q)$. We denote by $U_*^Q(s)$ the set of all $u \in \mathrm{Gr}(s, \mathbf{P}^N)$ such that $\mathrm{rk}(u(1)) = s + 1$ and $\forall j \geq n, \exists k \in [0, s]$ such that $u_{jk} \neq 0$.

□

It is clear that $U_*^Q(s)$ is a (Zariski) open subset of $\mathrm{Gr}(s, \mathbf{P}^N)$ and hence it has the same dimension. Moreover, defining an action (permutation of rows in the same string):

$$\mathcal{S}_q \times U_*^Q(s) \to U_*^Q(s)$$

by:

$$(\tau, u) \mapsto v$$

where:

$$v_{jk} := \begin{cases} u_{jk} & \forall j = 0, \dots, n-1;\ \forall k = 0, \dots, s \\ u_{\tau(j)k} & \forall j = n, \dots, N; \qquad \forall k = 0, \dots, s \end{cases}$$

we obtain a bijection:

$$\sigma_Q : U_*^Q(s)/\mathcal{S}_q \xrightarrow{\simeq} \sigma_Q(U_*^Q(s))$$

Therefore, letting:

$$T(s, Q) := \{\sigma_Q(L) \text{ such that } L \in U^Q(s, \mathbf{P}^N)\} = U^Q(s, \mathbf{P}^N)/\sim$$

where $L \sim L' \Leftrightarrow \sigma_Q(L) = \sigma_Q(L')$, we have:

Lemma 3.7.3 *There is a Zariski open subset $U_*^Q(s) \subset \mathrm{Gr}(s, \mathbf{P}^N)$ and a dense subset $W_*^Q(s) \subset T(s,Q)$ such that:*

- $\forall H \in W_*^Q(s)$, $\dim H = s$.
- *The map:*
$$\sigma_Q : U^Q(s, \mathbf{P}^N) \to T(s,Q)$$
gives a bijection:
$$U_*^Q(s)/\mathcal{S}_q \simeq W_*^Q(s)$$

□

Remark 3.7.1

Observe that $U_*^Q(s) = \cup_I U_I^*$ where the charts U_I are as in paragraph 1, chapter 2, the multiindices I have the last index $k_s < n$ and the upper $*$ means that we're taking the elements of the chart having no zero line between the last q lines.

□

Remark 3.7.2

If $L \in U_*^Q(s)$, certainly $\dim \sigma_Q^{-1}(\sigma_Q(L_u)) \geq N - n - s$. This is a general result in Algebraic Geometry. We have proved that each element L, in a Zariski open subset of $\mathrm{Gr}(s, \mathbf{P}^N)$, is such that $\sigma_Q^{-1}(\sigma_Q(L))$ contains only finitely many other planes.

□

There is a case where a generalization of the previous reasoning is possible.

Let $Q = (1, \ldots, 1, q_r, \ldots, q_n)$ and suppose $s \leq r$. Consider the set:

$$\mathcal{J}_{s,r}(Q) := \left\{(k_0, \ldots, k_s) \in \mathbf{N}^{s+1} \text{ such that } 0 \leq k_0 < \ldots < k_s \leq r\right\}$$

and let:

$$U(s, \mathbf{P}^N) := \cup_{I \in \mathcal{J}_{sr}(Q)} U_I$$

This is a union of charts of $\mathrm{Gr}(s, \mathbf{P}^N)$ and therefore Zariski open in it. Suppose for example that $I = (0, \ldots, s)$. Then any $u \in U_I$ is of the form:

$$u = \begin{array}{cc} \left(\begin{array}{c} \\ \mathbf{1}_{s+1} \\ \\ \\ * \\ \vdots \\ * \\ \vdots \\ \vdots \\ \vdots \end{array}\right) & \begin{array}{c} 0 \\ \vdots \\ \vdots \\ s \\ s+1 \\ \vdots \\ r-1 \\ q_r\text{terms} \\ \\ q_n\text{terms} \end{array} \end{array}$$

So that $x_k = \lambda_k \ \forall k = 0, \ldots, s$ and so H_u has dimension s. It is also clear that $\forall \nu = s+1, \ldots, n$:

$$x_\nu = \prod_{j=N_{\nu-1}}^{N_\nu - 1} \left(\sum_{k=0}^{s} u_{jk} x_k \right)$$

so that if x_ν is nonzero, all the other possible v's giving the same x_ν are obtained by a permutation of the rows of u in the ν-th string. If the string is made of just one row, that is if $s+1 \leq j \leq r-1$, then it can also be zero: the only v's having $x_j = 0$ are such that the j-th row is also zero. This of course is no longer true if $r \leq j \leq N$. Therefore, setting:

$$U_*^Q(s) := \{u \in U(s, \mathbf{P}^N) \text{ such that } \forall j, r \leq j \leq N \, \exists k \text{ for which } u_{jk} \neq 0\}$$

and defining an action:

$$\mathcal{S}_{q_r} \times \ldots \times \mathcal{S}_{q_n} \times U_*^Q(s) \to U_*^Q(s)$$

by $(\tau_r, \ldots, \tau_n; u) \mapsto v$ with:

$$v_{jk} := \begin{cases} v_{jk} & \forall j \leq r-1 \\ v_{\tau_b(j)k} & \forall j \in [N_{b-1}, N_b - 1] \cap \mathbf{N} \end{cases}$$

we get a bijection:

$$\sigma_Q : U_*^Q(s) / \left(\mathcal{S}_{q_r} \times \ldots \times \mathcal{S}_{q_n}\right) \to \sigma_Q(U_*^Q(s))$$

As before, $U_*^Q(s)$ is Zariski open in $\mathrm{Gr}(s, \mathbf{P}^N)$, and therefore:

Lemma 3.7.4 *Let $Q = (1, \ldots, 1, q_r, \ldots, q_n)$ and let $s < r$. Then there is a Zariski open subset $U_*^Q(s, \mathbf{P}^N) \subset \mathrm{Gr}(s, \mathbf{P}^N)$ and a dense subset $W_*^Q(s) \subset T(s, Q)$ such that:*

- $\forall H \in W_*^Q(s), \dim H = s.$
- *the map σ_Q gives a bijection:*

$$U_*^Q(s)/\left(\mathcal{S}_{q_r} \times \ldots \times \mathcal{S}_{q_n}\right) \simeq W_*^Q(s)$$

□

3.8 Fano varieties in $\mathbf{P}(1, \ldots, 1, q)$

Lemma 3.8.1 *Let $Q = (1, \ldots, 1, q)$. Let X be a weighted homogeneous hypersurface of Q-degree d and let $u \in U_*^Q(s)$ be such that $H_u \subset X$. Then the set:*

$$\{H_v \in W_*^Q(s+1) | H_u \subset H_v \subset X\}$$

is nonempty if:

$$\binom{s+d}{s+1} \leq n + q - s - 2$$

It suffices that:

$$\binom{s+d}{s+1} \leq n + q - s - 1$$

for the set:

$$\{H_v \in W_*^Q(s+1) | H_u = H_v \cap X \text{ or } H_u \subset H_v \subset X\}$$

to be nonempty.

Proof:

Since changes of coordinates not affecting the last q coordinates are allowed in $\mathbf{P}^N$ (the weighted action only acts on the q last coordinates), we can assume , after setting without loss of generality $u \in U_{(0,\ldots,s)}$, that the row labelled by $s+1$ is zero in u. Then certainly:

$$H_u = \left\{ \begin{array}{lll} x_{s+1} = & 0 & \\ x_n = & \prod_{j=n}^{N} (\sum_{k=0}^{s} x_k u_{jk}) & \\ x_j = & \sum_{k=0}^{s} x_k u_{jk} & \forall j = s+2, \ldots, n-1 \end{array} \right\}$$

Let $v \in U_{(0,\ldots,s+1)}$, then we may assume that v is the unit matrix in the first $s+2$ rows, the one labelled from 0 to $s+1$, and so:

$$H_v = \left\{ \begin{array}{lcll} x_n & = & \prod_{j=n}^{N}(\sum_{k=0}^{s} x_k v_{jk}) & \\ x_j & = & \sum_{k=0}^{s} x_k v_{jk} & \forall j = s+2,\ldots,n-1 \end{array} \right\}$$

If we require $H_u \subset H_v$, then of course we must have:

$$H_u \subseteq H_v \cap \{x_{s+1} = 0\}$$

so that, being closed, connected and with the same dimension, these two sets must be equal:

$$H_u = H_v \cap \{x_{s+1} = 0\}.$$

Therefore, $\forall k = 0,\ldots,s$ and $\forall j = s+2,\ldots,n-1$, I have:

$$\sum_{k=0}^{s} x_k u_{jk} = \sum_{k=0}^{s} x_k v_{jk}$$

and applying the identity principle I deduce $u_{jk} = v_{jk}$. Also:

$$\prod_{j=n}^{N}(\sum_{k=0}^{s} x_k u_{jk}) = \prod_{j=n}^{N}(\sum_{k=0}^{s} x_k v_{jk})$$

and so there must be a permutation $\tau \in \mathcal{S}_q$ such that $v_{jk} = u_{\tau(j)k}$ $\forall k = 0,\ldots,s$ and $j = n\ldots,N$. This shows that the only unknowns are the entries v_{js+1} such that $j = s+2,\ldots,N$.

Let $X = \{f = 0\}$. Since $H_u \subset H_v \cap X$, one has:

$$H_v \cap X = \{x_{s+1} g = 0\}$$

where g has now Q-degree $d-1$. So:

$$g(x) = \sum_{\mathbf{p}\in\mathbf{N}^{n+1}_{d-1}(Q)} a_{p_0\ldots p_n} x_n^{p_n q} \prod_{j=0}^{n-1} x_j^{p_j} = \sum_{\sum_j r_j = d-1} b_{r_0\ldots r_{s+1}} \prod_{j=0}^{s+1} t_j^{r_j} =: \tilde{g}(t)$$

One must have $\tilde{g}(t) = 0$ $\forall t \in H_v$, if $H_u \subset H_v \subset X$, and $\tilde{g}(t) = a t_{s+1}^{d-1}$ for some $a \in \mathbf{C}$ if $H_u = H_v \cap X$. The first case implies that all the coefficients $b_{r_0\ldots r_{s+1}}$ are zero; the second, that the same is true except for $b_{0\ldots 0 d-1}$. These coefficients are certainly polynomials in the variables v_{jk}. The number of these coefficients is:

$$\binom{s+d}{s+1}$$

so that we have this number of equations in the first case and hence

$$\binom{s+d}{s+1} - 1$$

equations in the second one. The number of variables is the number of the free v_{jk} and we have seen that it is $N-(s+1)$. Hence the previous equations are defined in $\mathbf{C}^{n+q-1-(s+1)} = \mathbf{C}^{n+q-s-2}$.

QED

Remark 3.8.1

When $q=1$, the nonweighted case, immediately one gets back the results of [ELV].

$\square$

Remark 3.8.2

Janos Kollár asserts in [KO] that if X is a weighted homogeneous subvariety of $\mathbf{P}(Q)$ of Q-degrees $d_1,\ldots,d_r$, if:

$$\sum_j d_j \leq \sum_i q_i - 2$$

then $\forall x \in X$ there is a rational curve C such that $x \in C$ and $\mathcal{O}_X(1)\cdot C \leq 1$. In particular, if X is a hypersurface and $d_1 = d$, and in the case $Q = (1,\ldots,1,q)$, the estimate reads $d \leq n+q-2$, which coincides with the lemma taken for $s=0$ (the case of a point).

$\square$

Definition 3.8.1

Let Y be a weighted subvariety of $\mathbf{P}(Q)$ and set:
$\mathrm{Gr}^Q(s,Y) := \{H \in T(s,Q) \text{ such that } H \subseteq Y\}$
$F^Q(s,s+1,Y) := \{(H,H') \in T(s,Q,Y) \times T(s+1,Q) | H \subset H'\}$
$\Lambda^Q(s) := \{(H,x) \in T(s,Q) \times \mathbf{P}(Q) \text{ such that } x \in H\}$
$\Lambda^Q(s,Y) := \left\{(H,x) \in \Lambda^Q(s) \text{ such that } H \subset Y\right\}$
$\mathbf{K}^Q := \{(H,H') \in T(s,Q,Y) \times T(s+1,Q) \text{ such that } H \subset H' \subset Y\}$
$\mathbf{H}^Q := \{(H,H') \in T(s,Q,Y) \times T(s+1,Q) \text{ such that } H = H' \cap Y\}$
$\cup \mathbf{K}^Q$

I will call $\tilde{\theta}_Q$ and θ_Q the projections of $\mathbf{K}^Q$ resp. of $\mathbf{H}^Q$ to the first factor $\mathrm{Gr}^Q(s,Y)$, defined by $(H,H') \mapsto H$.

$\square$

The previous lemma then says that if $\binom{s+d}{s+1} \leq n+q-s-2$ one has $\tilde{\theta}_Q^{-1}(H) \neq \emptyset$, and that if $\binom{s+d}{s+1} \leq n+q-s-2$, then $\theta_Q^{-1}(H) \neq \emptyset$.

Remark 3.8.3

- $\mathcal{S}_q$ acts on the matrices by permuting the last q rows. This corresponds to an action of $\mathcal{S}_q$ on all of $\mathbf{P}^N$ by permuting the last q coordinates. Of course such an action has no influence on the image by σ_Q: the last q coordinates go to the same last coordinate x_n. In particular, letting Y be a Q-weighted projective subvariety defined by:

$$Y = \left\{ f_h = \sum_{\sum_{j=0}^{n-1} d_j + d_n q = d} a^h_{d_0\ldots d_n} \prod_{j=0}^{n} x_j^{d_j} = 0 \,\forall h \right\}$$

and:

$$\tilde{Y} := \sigma_Q^{-1}(Y) = \left\{ \sum_{\sum_{j=0}^{n-1} d_j + d_n q = d} a^h_{d_0\ldots d_n} \prod_{j=0}^{n-1} t_j^{d_j} \prod_{j=n}^{N} t_j^{d_n} = 0 \,\forall h \right\}$$

one has:

$$\prod_{j=n}^{N} t_j^{d_n} = \prod_{j=n}^{N} t_{\tau(j)}^{d_n}$$

$\forall \tau \in \mathcal{S}_q$, so that $\tau \cdot \tilde{Y} = \tilde{Y}$.

- $H \equiv H_u \in T := (s, Q, Y) := \{H \in T(s,Q) \text{ such that } H \subseteq Y\} \Leftrightarrow \forall \tau \in \mathcal{S}_q,\ L_{\tau\cdot u} \equiv \tau \cdot L_u \subset \tilde{Y}$.

  Indeed, let $H_u = \sigma_Q([L_u]_{\mathcal{S}_q})$, where $[L_u]_{\mathcal{S}_q} = \{L_{\tau\cdot u} \forall \tau \in \mathcal{S}_q\}$. Then for every permutation τ, $\sigma_Q(L_{\tau\cdot u}) = H_u$, so that there must be a permutation ρ so that $L_{\tau\cdot u} \subset \rho\tilde{Y} = \tilde{Y}$.

  The converse is immediate by definition.

- By the previous point, we deduce that there is a Zariski open subset of $\mathrm{Gr}^Q(s, Y)$ which is the isomorphic image of $U_*^Q(s, \tilde{Y})/\mathcal{S}_q$.

- Let $H \in T(s,Q)$ and $H' \in T(s+1,Q)$. Then $H \subset H' \Leftrightarrow \exists \tau \in \mathcal{S}_q$ such that, for every representatives L and L' respectively of H and H', $L \subset \tau \cdot L'$.

  To see this, one side is obvious by definition: $L \subset \tau \cdot L'$ clearly implies $H \subset H'$ by applying σ_Q. On the contrary, it is clear that the hypothesis $H \subset H'$, implies $L \subset \cup_\tau \tau \cdot L'$, and so there must be one τ for which $L \subset \tau \cdot L'$. This is equivalent, of course, to $\tau \cdot L \subset L'$.

These remarks allow us to say that, given an s-plane H and an $(s+1)$-plane H' in $\mathbf{P}(Q)$, $H' \in \tilde{\theta}_Q^{-1}(H) = \{H \subset H' \subset Y\}$ if and only if for every representatives L and L' of H and H' resp., there must be a $\tau \in \mathcal{S}_q$ such that:

$$\tau \cdot L \subset L' \subset \tilde{Y}.$$

So:

$$\tilde{\theta}_Q^{-1}(H) = \sigma_Q \left(U_*^Q(s+1, \mathbf{P}^N) \cap \bigcup_{\tau \in \mathcal{S}_q} \tilde{\theta}^{-1}(\tau \cdot L) \right) =$$

$$= \sigma_Q \left(U_*^Q(s+1, \mathbf{P}^N) \cap \tilde{\theta}^{-1}(\tau \cdot L) \right)$$

for some and hence all $\tau \in \mathcal{S}_q$.

Since:

$$\dim \tilde{\theta}^{-1}(\tau \cdot L) \geq N - s - 1 - \left(\begin{array}{c} s+d \\ s+1 \end{array} \right)$$

and since σ_Q is injective up to $\mathcal{S}_q$ on $U_*^Q(s+1, \mathbf{P}^N)$, I deduce that, *generically*:

$$\dim \tilde{\theta}^{-1}(H) \geq N - s - 1 - \left(\begin{array}{c} s+d \\ s+1 \end{array} \right)$$

I say generically because it may happen, for some L, that:

$$\tilde{\theta}^{-1}(\tau \cdot L) \subset \mathrm{Gr}(s+1, \mathbf{P}^N) - U_*^Q(s+1, \mathbf{P}^N)$$

$\forall \tau \in \mathcal{S}_q$.

It is clear that the same proof works for the case $Q = (1^{(r)}, q_r, \ldots, q_n)$ and $s \leq n$. We have indeed:

Lemma 3.8.2 *Let $Q = (1^{(r)}, q_r, \ldots, q_n)$, X be a weighted homogeneous hypersurface of Q-degree d and $u \in U^Q(s, \mathbf{P}^N)$ be such that $H_u \subset X$. Then, whenever $s \leq r-1$, the set:*

$$\{H_v \in W_*^Q(s+1) | H_u \subset H_v \subset X\}$$

is nonempty if:

$$\binom{s+d}{s+1} \leq r - 1 + (\sum_{j=r}^{n} q_j) - s - 2$$

It suffices that:

$$\binom{s+d}{s+1} \leq r - 1 + (\sum_{j=r}^{n} q_j) - s - 1$$

for the set:

$$\{H_v \in W_*^Q(s+1) | H_u = H_v \cap X \ \text{or} \ H_u \subset H_v \subset X\}$$

to be nonempty.

The proof of this lemma is omitted because it is the same as for the case $Q = (1, \ldots, 1, q)$ and because the lemma will be proved in a more general case when we shall have introduced $T(s, Q)$ for the general Q (See Ch. 4.1).

3.9 Another proof of the main result for $Q = (1^{(r)}, q_r, \ldots, q_n)$.

The main result of this thesis can be reproved by using the same arguments as [ELV] as well as the previous lemma when $Q = (1^{(r)}, q_r, \ldots, q_n)$ and $s \leq r - 1$. Indeed, in this case we may set:

Definition 3.9.1 *Let X be a weighted hypersurface in $\mathbf{P}(1^{(r)}, q_r, \ldots, q_n)$ of Q-degree d, and let Y be an l-dimensional subvariety of X stable under the weighted action. We say that Y is* **spanned by s-planes** *if there is a variety $Z \subset W_*^Q(s)$ of dimension $l - s$, such that the projection:*

$$\Lambda_Z(s) := \{(H, x) \in Z \times \mathbf{P}(Q) \ \text{such that} \ x \in H\} \to \mathbf{P}(Q)$$

has image Y.

The subgroup of $\mathrm{CH}_l(X)$ generated by subvarieties spanned by s-planes is denoted by $\mathrm{CH}_l^{(s)}(X)$.

Remark that in the special cases we're considering,
$W_*^Q(s) = U_*^Q(s, \mathbf{P}^n)/(\mathcal{S}_{q_r} \times \ldots \times \mathcal{S}_{q_n})$ is the quotient of a Zariski open subset of a smooth variety by a finite group, hence it is a variety (we lack this structure in the general case). Observe that W_*^Q may be nonconnected.

We have:

Theorem 3.9.1 *Let $Q = (1^{(r)}, q_r, \ldots, q_n)$ and let X be a weighted hypersurface of Q-degree d. Then:*

$$\mathrm{CH}_s(X) \otimes \mathbf{Q} = \mathbf{Q} \qquad \forall s < r \text{ such that } \binom{s+d}{s+1} \leq N = \sum_{j=r}^{n} q_j - 1$$

To prove the theorem one proves this lemma first:

Lemma 3.9.1 *In the hypothesis of the theorem, suppose Y is a subvariety of dimension l of X which is stable under the weighted action, is spanned by s-planes and is not spanned by $(s+1)$-planes. If:*

$$\binom{s+d}{s+1} \leq N - s$$

there exists a cycle $T \in \mathrm{CH}_{l+1}(\mathbf{P}(Q))$ and a nonzero rational $a \in \mathbf{Q}$ such that $T \cdot X = aY \pmod{\mathrm{CH}_l^{(s+1)}(X)}$

Proof:

First of all, one uses the lemma of the previous section to see that there is a subvariety $\Sigma \subset \mathbf{H} = \mathbf{H}(s, X)$ such that:

$$\begin{array}{ccccc} \tilde{\Sigma} \stackrel{\sigma}{\to} & \Sigma & \stackrel{\pi_1}{\to} & Z \\ & \downarrow & & \downarrow \\ & \mathbf{H} & \stackrel{\pi_1}{\to} & W_*^Q(s, X) \end{array}$$

where π_1 is proper and generically finite over Z, while σ is a blow-up making Σ smooth. (This passage to the blow-up is used later to apply a projection formula).

Define:

$$\begin{array}{ccc} \Lambda_{\tilde{\Sigma}} := \{(H, H', x) \in \tilde{\Sigma} \times \mathbf{P}(Q) | x \in H\} & \stackrel{Pr}{\to} & Y \\ \downarrow & & \downarrow \\ \Lambda'_{\tilde{\Sigma}} := \{(H, H', x) \in \tilde{\Sigma} \times \mathbf{P}(Q) | x \in H'\} & \stackrel{Pr}{\to} & Pr(\Lambda'_{\tilde{\Sigma}}) \quad \subset \mathbf{P}(Q) \end{array}$$

The cycle T that we're looking for will be $Pr(\Lambda'_{\tilde{\Sigma}})$.

On the first line, the image of Pr is Y because Y is generated by s-planes : if there was $y \in Y$ such that $y \notin \mathrm{Im}(Pr)$, then $\forall H \in Z$ we would have $y \notin H$, forbidden by the role of Z in the definition.

It is clear that $\Lambda_{\tilde{\Sigma}} \subset \Lambda'_{\tilde{\Sigma}}$ by definition of $\mathbf{H}$.

To have $T \neq 0$, by the definition of push-forward of cycles, it suffices to see that $\Lambda'_{\tilde{\Sigma}}$ is generically finite over T. This is immediate because otherwise, calling R the open subset of T such that any of its points has an infinite number of preimages, then $R \cap Y$ would be a Zariski open of Y which is spanned by $(s+1)$-planes.

Let:

$$\psi : \Lambda'_{\tilde{\Sigma}} \to \tilde{\Sigma}$$

$$\varphi : \Lambda_{\tilde{\Sigma}} \to \tilde{\Sigma}$$

defined by the projections:

$$(H, H', x) \mapsto (H, H')$$

Then either

$$\Lambda'_{\tilde{\Sigma}} \not\subset \tilde{\Sigma} \times X$$

or not. In the first case:

$$(*) \quad \forall (H, H') \in \tilde{\Sigma}, \psi^{-1}(H, H') \cdot X = d\varphi^{-1}(H, H')$$

This is intersection theory; recall that:

$$\psi^{-1}(H, H') = \{x \in \mathbf{P}(Q) | x \in H'\}$$

$$\varphi^{-1}(H, H') = \{x \in \mathbf{P}(Q) | x \in H\}$$

while $x \in \psi^{-1}(H, H') \cap X$ if and only if $f(x) = 0$, being $\deg f = d$.

Therefore there are principal divisors D_j in $\tilde{\Sigma}$ for which the following equation is valid:

$$(**)\, \Lambda'_{\tilde{\Sigma}} \cdot (\tilde{\Sigma} \times X) = d\Lambda_{\tilde{\Sigma}} + \sum_j a_j \psi^{-1}(D_j) \in \mathrm{CH}_l(\Lambda'_{\tilde{\Sigma}} \cap (\tilde{\Sigma} \times X))$$

If instead:

$$\Lambda'_{\tilde{\Sigma}} \subset \tilde{\Sigma} \times X$$

one can decompose $\mathrm{CH}_l(\Lambda'_{\tilde{\Sigma}})$ into the union of l-cycles contained in $\Lambda'_{\tilde{\Sigma}}$ and in $\Lambda_{\tilde{\Sigma}}$ respectevely. The dimension of $\Lambda_{\tilde{\Sigma}}$ is l and so $\Lambda_{\tilde{\Sigma}}$ defines an l-cycle in $\mathrm{CH}_l(\Lambda'_{\tilde{\Sigma}})$. The other l-cycles must be pull-back of divisors of $\tilde{\Sigma}$. So:

$$\mathrm{CH}_l(\Lambda'_{\tilde{\Sigma}}) =< \{\Lambda_{\tilde{\Sigma}}, \psi^{-1}(\mathrm{PDiv}(\tilde{\Sigma}))\} >$$

Finally, since the fibre of $\Lambda'_{\tilde{\Sigma}} \to \tilde{\Sigma}$ must intersect X in a cycle of degree d, the coefficient of $\Lambda_{\tilde{\Sigma}}$ in $\Lambda'_{\tilde{\Sigma}} \cdot (\tilde{\Sigma} \times X)$ will be d.

Apply the projection formula to $(**)$ and get[3] , in $\mathrm{CH}_l(X \cap T)$:

[3] We are indeed applying the projection formula to the projection:

$$\tilde{\Sigma} \times \mathbf{P}(Q) \to \mathbf{P}(Q)$$

Recalling that $\mathbf{P}(Q) = \mathbf{P}^n/\mu$ where μ is a finite group, we are allowed to use it, see [FU], Ex. 16.1.13.

$$T \cdot X = Pr_*(\Lambda'_{\tilde{\Sigma}} \cdot (\tilde{\Sigma} \times X)) = dPr_*(\Lambda_{\tilde{\Sigma}}) + \sum_j a_j Pr_* \psi^{-1}(D_j)$$

Since Y is not spanned by $(s+1)$- planes, $\Lambda_{\tilde{\Sigma}}$ is generically finite over Y, and so there is some $a \in \mathbf{Q}$ for which:

$$Pr_*(\Lambda_{\tilde{\Sigma}}) = aY$$

On the contrary, since each $\psi^{-1}(D_j)$ is spanned by $(s+1)$-planes, we get the result.

QED

Corollary 3.9.1 *When $d \geq 3$, the lemma is valid if we replace $N - s$ with N.*

Proof:
One uses lemma 4.5 of [ELV] which shows that if $d \geq 3$, then

$$\binom{l+d}{l+1} \leq N \Rightarrow \binom{s+d}{s+1} < N - s \quad \forall s < l$$

while for $s = l$ one reasons as in theorem 4.6 of [ELV].

QED

Now we have:

Corollary 3.9.2 *In the same hypothesis of the theorem,* $\mathrm{CH}_s(X) \otimes \mathbf{Q} = \mathbf{Q}$

The proof of this corollary is the same as in [ELV]. It goes on like for the corresponding corollary in the case of the "naive" definition, so we don't repeat it.

Chapter 4

Miscellaneous results and open questions.

Nous voyons d'abord pourquoi le cas général ne peut pas être traité de la même façon que le cas particulier du chapitre précédent (paragraphe 1), bien que le lemme "base" de la démonstration de [ELV] puisse être généralisé (paragraphe 2). Ensuite, nous procédons à la critique de notre définition de Grassmannienne $T(s,Q)$ (paragraphe 3) et en proposons une autre (paragraphe 4), qui malheuresement ne permet pas de meilleurs résultats. Nous comparons les deux définitions dans le paragraphe 5.

4.1 The general case

In order to consider the general case, $Q = (q_0, \ldots, q_n)$, let me recall the following very well-known result:

Lemma 4.1.1 *Let:*

$$\gamma : \mathbf{C}^{s+1} \to \mathbf{C}^{n+1}$$

be defined by:

$$\gamma(t_0, \ldots, t_s) = [\gamma_0(t) : \ldots : \gamma_n(t)]$$

where the γ_j's are polynomials. Then:

$$\dim(\mathrm{Im}\gamma) = s+1 \Leftrightarrow \left\{ \begin{array}{c} \textit{there is a Zariski open subset } U \subset \mathbf{C}^{n+1} \\ \textit{such that } \forall x \in U, \mathrm{rk}(\mathrm{Jac}_x \gamma) = s+1 \end{array} \right\}$$

This is standard and I omit the proof.

Consider $u \in U_I$ for some multiindex I of length $s+1$. Define:

$$\omega_{QuI} : \mathbf{C}^{s+1} \to \mathbf{C}^{n+1}$$

by:

$$(\omega_{QuI}(t_{j_0},\ldots,t_{j_s}))_r := \prod_{j=N_{r-1}}^{N_r-1} t_j \ \forall r=0,\ldots,n$$

where:

$$t_j := \begin{cases} t_{j_\alpha} & \forall j = j_\alpha \in I \\ \sum_{\alpha=0}^{s} u_{j\alpha} t_{j_\alpha} & \forall j \notin I \end{cases}$$

Since one has:

$$L_u = \left\{ \begin{array}{ccll} t_{j_\alpha} & = & \lambda_\alpha & \forall \alpha = 0,\ldots,s \\ t_j & = & \sum_{\alpha=0}^{s} u_{j\alpha}\lambda_\alpha & \forall j \notin I \end{array} \right\} = \left\{ t_j = \sum_{\alpha=0}^{s} u_{j\alpha} t_{j_\alpha} \ \forall j \notin I \right\}$$

it is clear that:

$$\omega_{QuI}(\mathbf{C}^{s+1}) - \{0\}/\mathbf{C}^* \ = \sigma_Q(L_u) = H_u$$

where $\mathbf{C}^*$ is the equivalence relation defined by the weighted action.

The previous lemma asserts therefore that:
$\dim H_u = s \Leftrightarrow \mathrm{rk}(\mathrm{Jac}(\omega_{QuI})) = s+1$ *on a Zariski open subset of* $\mathbf{C}^{s+1}$.

Remark that the subset V of $\mathbf{C}^{s+1}$ where the rank of the Jacobian matrix is not maximal corresponds to a Zariski closed subset of $L_u \simeq \mathbf{C}^{s+1}$. Indeed, it is $V \cap L_u$.

Definition 4.1.1

We denote by $U^Q(s,\mathbf{P}^N)$ the set of all $u \in \mathrm{Gr}(s,\mathbf{P}^N)$ for which there exists I such that $u \in U_I$ and a Zariski closed subset V such that, $\forall x \in L_u - V$, $\mathrm{rk}(\mathrm{Jac}_x(\omega_{QuI})) = s+1$.

□

Remark that this subset is *nonempty*; consider for example the matrix u having in the j-th string all rows with zero entries on all the columns except the j-th one and 1 on the row j, $\forall j \leq s$; and all the other entries are 0:

$$u_{jk} = \left\{ \begin{array}{cl} 0 & \forall j \geq N_s \text{ and } \forall j \in [N_{r-1}, N_r - 1] \cap \mathbf{N},\ k \neq r \\ 1 & \forall (j,k) \text{ such that } j \in [N_{r-1}, N_r - 1] \cap \mathbf{N}, k = r \end{array} \right\}$$

Certainly $u \in U_{(0,q_0,\ldots,N_s)}$ and one has:

$$\omega_{QuI}(t_0,\ldots,t_s) = [t_0^{q_0},\ldots,t_s^{q_s},0,\ldots,0]_Q$$

So that the jacobian is:

$$\mathrm{Jac}(\omega_{QuI}) = \begin{pmatrix} q_0 t_0^{q_0-1} & 0 & \dots & 0 \\ 0 & q_1 t_1^{q_1-1} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & q_s t_s^{q_s-1} \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The upper minor obviously has maximal rank on the Zariski open subset $\{t_0 t_{q_0} \dots t_{N_s} \neq 0\}$. Hence $u \in U^Q(s, \mathbf{P}^N)$.

We show that $U^Q(s, \mathbf{P}^N)$ is *open*. Let $V^Q(s, \mathbf{P}^N)$ be its complement in $\mathrm{Gr}(s, \mathbf{P}^N)$ and suppose $u \in V^Q(s, \mathbf{P}^N)$. Certainly $u \in U_I$ for some I. For any such I, there is a Zariski open subset $A_I \subset \mathbf{C}^{s+1}$ such that $\mathrm{rk}(\mathrm{Jac}_t(\omega_{QuI})) \leq s \ \forall t \in A_I$. Hence, if we call f^j_{QuI} the determinants of the maximal minors of $\mathrm{Jac}(\omega_{QuI})$, then these polynomials are zero for almost every choice of t, so that they must be identically equal to zero, and therefore their coefficients are zero (principle of identity). But the coefficients are now polynomials in the entries of u: hence $u \in V^Q(s, \mathbf{P}^N)$ if and only if the entries of u satisfy some polynomial relations, as was to show. More clearly, the determinants of the maximal minors of the Jacobian matrix of ω_{QuI} are polynomials in both the entries of u and the variables t, say $l_a(u,t)$, and since $l_a(u,t) = 0 \ \forall t \in A_I$, I get $0 \equiv l_a(u,t) = \sum_a l'_a(u) l''_a(t)$, from which $l'_a(u) = 0$. This defines the complement of $U^Q(s, \mathbf{P}^N)$.

Consider:

$$T(s, Q) := \left\{\sigma_Q(L_u) | u \in U^Q(s, \mathbf{P}^N)\right\} = U^Q(s, \mathbf{P}^N)/\sim$$

where $L \sim L' \Leftrightarrow \sigma_Q(L) = \sigma_Q(L')$.

One can be sure that every element in this set has the right dimension s. Unfortunately, one cannot easily distinguish the fiber; namely, if $u, v \in U^Q(s, \mathbf{P}^N)$, when is $H_u := \sigma_Q(L_u) = \sigma_Q(L_v) =: H_v$?

The following examples will show that in general the previous reasoning for the injectivity up to a permutation group cannot be repeated.

Example 4.1.1

Let $Q=(1,1,1,2,2,2)$, $n=5$ and $s=3$. So $N=8$. Let:

$$u := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{Then: } L_u = \left\{ \begin{array}{ccc} t_4 & = & t_0+t_3 \\ t_5 & = & t_0 \\ t_6 & = & t_1 \\ t_7 & = & t_1+t_2 \\ t_8 & = & t_0 \end{array} \right\}$$

and, in parametric equations:

$$H_u = \left\{ \begin{array}{ccc} x_0 & = & t_0 \\ x_1 & = & t_1 \\ x_2 & = & t_2 \\ x_3 & = & t_3(t_0+t_3) \\ x_4 & = & t_0t_1 \\ x_5 & = & t_0(t_1+t_2) \end{array} \right\}$$

which can be written in cartesian, Q-weighted equations as:

$$H_u = \left\{ \begin{array}{ccl} x_4 & = & x_0x_1 \\ x_5 & = & x_0(x_1+x_2) \end{array} \right\}$$

It is then clear that:

$$u := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ * & * & * & * \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

goes to the same H_u with the only caution that the "variable" row is nonzero. Remark that the variable x_3, relative to a "nontrivial" string, that is to a string with at least two elements, doesn't appear in the defining equations.

□

Example 4.1.2

Let $Q = (1, 1, 2, 2)$, $s = 2$. Let:

$$u := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \\ a & b & c \\ 0 & 0 & 1 \end{pmatrix}$$

Then in parametric form:

$$H_u = \left\{ \begin{array}{ll} x_j & = \lambda_j \, \forall j = 0, 1 \\ x_2 & = \lambda_2(a\lambda_0 + b\lambda_1 + c\lambda_2) \\ x_3 & = \lambda_2(a\lambda_0 + b\lambda_1 + c\lambda_2) \end{array} \right\}$$

where $(a, b, c) \neq 0$. So:

$$H_u = \{x_2 = x_3\}$$

Here all the variables appear; nontheless, changing (a, b, c) will change L_u without changing H_u. Of course one may always choose different (a, b, c) such that u is changed into some v which is not a permutation of the columns of u in the same string.

□

This happens because it's not so important that the variables, expressed parametrically, are the same (which of course happens if the defining matrices differ by a permutation of the rows in the same string): the very fundamental fact are the *relations* between the variables. A moment's thought and it will be clear that in the case $Q = (1, \ldots, 1, q)$ the relations are strictly determined: there is just one string. So in that case this problem doesn't appear. The same can be said for the more general case $Q = (1^{(r)}, q_r, \ldots, q_n)$ and $s \leq r - 1$, since there the $n - s$ defining equations are of type $x_j = x_j(x_0, \ldots, x_s)$ $\forall j = s + 1, \ldots, n$ (assuming for ease of notation that $u \in U_{(0,\ldots,s)}$).

4.2 Fano varieties in the general case.

The following lemma can be proved:

Lemma 4.2.1 *Let X be a weighted homogeneous hypersurface of Q-degree d and let $u \in U^Q(s, \mathbf{P}^N)$ be such that $H_u \subset X$. Then the set:*

$$\{H_v \in W_*^Q(s+1) | H_u \subset H_v \subset X\}$$

is nonempty if:

$$\binom{s+d}{s+1} \leq (\sum_{j=0}^{n} q_j) - s - 2$$

It suffices that:

$$\binom{s+d}{s+1} \leq (\sum_{j=1}^{n} q_j) - s - 1$$

for the set:

$$\{H_v \in W_*^Q(s+1) | H_u = H_v \cap X \ \textit{or}\ H_u \subset H_v \subset X\}$$

to be nonempty.

Proof:
In parametric equations:

$$H_u := \left\{ x_r = \prod_{j=N_{r-1}}^{N_r - 1} (\sum_{k=0}^{s} u_{jk} \lambda_k) \right\}$$

$$H_v := \left\{ x_r = \prod_{j=N_{r-1}}^{N_r - 1} (\sum_{k=0}^{s+1} v_{jk} \lambda_k) \right\}$$

So for $H_u \subset H_v$ is sufficient (but not necessary!) that

$$u_{jk} = v_{jk}\, \forall j = 0, \ldots, N; \forall k = 0, \ldots, s.$$

This corresponds to $H_u = H_v \cap \{\lambda_{s+1} = 0\}$. The remaining variables for v are therefore $N - s$: namely, v_{js+1} $\forall j = 0, \ldots, N$ minus the $s+1$ which are used to parametrize a minor of order $s+1$. We have to verify that $v \in U^Q(s+1, \mathbf{P^N})$. For this, we only need to throw away those values of the remaining variables v_{js+1} which set equal to zero all the $(s+1)$-minor of the Jacobian of $\omega_{QvI'}$, where I' is a multindex of length $s+2$. Now the proof proceeds as for the case $Q = (1, \ldots, 1, q)$, and the result is proven.

QED

Remark 4.2.1

For $s = 0$ one has the estimate:

$$d \leq \sum_{j=0}^{n} q_j - 2$$

which was present in Kollár's book (see [KO]).

4.3 $T(s,Q)$ can hardly be considered a Grassmannian.

The first default of $T(s,Q)$ is that we don't even know whether it is a variety or not. For the moment it is just a topological space: a (smooth) variety divided by an equivalence relation. In the particular case $Q = (1^{(r)}, q_r, \ldots, q_n)$ we know that a dense subset of it is homeomorphic to the quotient variety $U_*^Q(s, \mathbf{P}^N)/(\mathcal{S}_{q_r} \times \ldots \times \mathcal{S}_{q_n})$, where $U_*^Q(s, \mathbf{P}^N)$ is a Zariski-open subset of $\mathrm{Gr}(s, \mathbf{P}^N)$.

An s-plane is determined by $s+1$ linearly independent points. Here, on the contrary, $s+1$ points of $\mathbf{P}(Q)$ which are images of $s+1$ independent points of $\mathbf{P}^N$ under σ_Q may very well belong to an infinity of different elements of $T(s,Q)$.

Let $Q = (1,1,2,2)$ and let:

$$u := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \\ A & B & C \\ \alpha/A & \beta/B & \gamma/C \end{pmatrix} = (u_0, u_1, u_2)$$

where we have chosen of course $ABC \neq 0$ (this is possible on a Zariski open subset). Then the images $s_j := \sigma_Q(u_j)$ are three points in $\mathbf{P}(1,1,2,2)$ and we have:

$$s_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \alpha \end{pmatrix} \quad s_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \beta \end{pmatrix} \quad s_2 = \begin{pmatrix} 0 \\ 0 \\ c \\ \gamma \end{pmatrix}$$

The parameters are now only 4, namely c, α, β, γ. So all the different u's having these four parameters equal, no matter how the other ones are, should give the same image H_u. But in fact H_u may vary nonetheless. Indeed, this only proves that if the 4 parameters are the same, then the images H_u all have to contain the same three points s_0, s_1 and s_2. But the planes H_u are nonetheless different, since:

$$H_u = \left\{ \begin{array}{l} x_0 = \lambda_0 \\ x_1 = \lambda_1 \\ x_2 = \lambda_2(a\lambda_0 + b\lambda_1 + c\lambda_2) \\ x_3 = (A\lambda_0 + B\lambda_1 + C\lambda_2)(\frac{\alpha}{A}\lambda_0 + \frac{\beta}{B}\lambda_1 + \frac{\gamma}{C}\lambda_2) \end{array} \right\}$$

which clearly depends on a and b.

Recall that we have a Veronese type imbedding:

$$v : \mathbf{P}(1,1,2,2) \hookrightarrow \mathbf{P}^4$$

defined by $[x_0, \ldots, x_3]_Q \mapsto [x_0^2 : x_0x_1 : x_1^2 : x_2 : x_3]$. We see that $v(H_u)$ may very well be different from the intersection of a linear space in $\mathbf{P}^4$ with the imbedded $\mathbf{P}(Q)$. (The candidate should be a 3-plane of $\mathbf{P}^4$). Indeed $v(H_u)$ is defined by the parametric equations:

$$\begin{aligned} z_0 &= \lambda_0^2 \\ z_1 &= \lambda_0\lambda_1 \\ z_2 &= \lambda_1^2 \\ z_3 &= \lambda_2(a\lambda_0 + b\lambda_1 + c\lambda_2) \\ z_4 &= (A\lambda_0 + B\lambda_1 + C\lambda_2)(\frac{\alpha}{A}\lambda_0 + \frac{\beta}{B}\lambda_1 + \frac{\gamma}{C}\lambda_2) \end{aligned}$$

One can always make λ_2 explicit by subtracting γz_3 from cz_4. Then putting this expression of λ_2 in the expression of z_3 or of z_4 would give the cartesian equation defining H_u. However, let' s make an example where minor calculations are required, namely take:

$$u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ giving } H_u = \left\{ \begin{array}{c} x_j = \lambda_j \forall j = 0,1 \\ x_2 = \lambda_2(\lambda_1 + \lambda_2) \\ x_3 = \lambda_1(\lambda_0 + \lambda_1 + \lambda_2) \end{array} \right\}$$

Getting $\lambda_2 = \frac{x_3}{x_1} - (x_0 + x_1)$, we immediately arrive to:

$$H_u = \left\{ x_2x_1^2 = [x_3 - x_1(x_0 + x_1)][x_3 - x_0x_1] \right\}$$

Recall that $v(\mathbf{P}(Q)) = \{z_0z_2 = z_1^2\}$. Then:

$$v(H_u) = v(\mathbf{P}(Q)) \cap \mathcal{C}$$

where $\mathcal{C} := \{z_3z_2 = (z_4 - z_1 - z_2)(z_4 - z_1)\}$ is a quadric in $\mathbf{P}^4$.

□

What about the intersection of two elements H_u and H_v of $T(s, Q)$? Here are two examples:

Example 4.3.1

Let $Q=(1,1,1,2)$, so $n=3$, $N=4$ and let $s=2$. Let:

$$u := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad v := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Then:

$$H_u = \{x_3 = x_1(x_0+x_2)\}$$
$$H_v = \{x_3 = x_0(x_1+x_2)\}$$

The intersection is therefore:

$$\begin{aligned} &H_u \cap H_v = \{x_3 = x_1(x_0+x_2); \quad x_2(x_0-x_1)=0\} = \\ &= \{x_2 = 0; \quad x_3 = x_0x_1\} \cup \{x_1 = x_0; \quad x_3 = x_0(x_0+x_2)\} = \\ &= H_{u'} \cup H_{v'} \end{aligned}$$

where:

$$u' := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad v' := \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

So the intersection of two “planes” in $\mathbf{P}(1,1,1,2)$ may be two (disjoint) “lines”.

□

Example 4.3.2

Take the same case as before but with:

$$u := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad v := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then:

$$H_u = \{x_3 = x_0^2\} \qquad H_v = \{x_3 = x_1x_2\}$$

The intersection is therefore:

$$H_u \cap H_v = \{x_3 = x_0^2; \quad x_0^2 = x_1x_2\}$$

This is not in $T(1,Q)$! Indeed, there cannot be any normalized minor. In other words, if $w \in U_*^Q(1,\mathbf{P}^4)$ such that $H_u \cap H_v = \sigma_Q(L_w)$, it cannot happen that $w \in U_{(01)}$ because of the second equation, and for the same reason, $w \notin U_{(02)}$, while if $w \in U_{(12)}$ the variable $x_0 = \sqrt{x_1 x_2}$. What is impossible is indeed that there is a degree 2 equation concerning only the variables of Q-degree 1.

Indeed, let:

$$A := \begin{pmatrix} a & b \\ c & d \\ e & f \\ a & b \\ a & b \end{pmatrix}$$

be a matrix giving L_A with $H_u \cap H_v = \sigma_Q(L_A)$. The choice of the last two lines is dictated by the request $x_3 = x_0^2$. We now have to give conditions to satisfy $x_0^2 = x_1 x_2$. This turns into:

$$(a\lambda_0 + b\lambda_1)^2 = (c\lambda_0 + d\lambda 1)(e\lambda_0 + f\lambda_1)$$

for every choice of λ_0 and λ_1, which gives:

$$\begin{cases} df = b^2 \\ ce = a^2 \\ 2ab = de + cf \end{cases}$$

One can see that, because of these relations, any two minor of A is zero, contrary to the assuption. Indeed:

$$(ad - bc)^2 = a^2d^2 + b^2c^2 - 2abcd = ced^2 + dfc^2 - cd(de + cf) = 0$$

$$(af - be)^2 = a^2f^2 + b^2e^2 - 2abef = cef^2 + dfe^2 - ef(de + cf) = 0$$

$$(cf - de)^2 = (cf + de)^2 - 4cedf = (2ab)^2 - 4a^2b^2 = 0$$

□

4.4 The planes K_p.

One has:

$$\pi_Q : K^{n+1} - \{0\} \xrightarrow{\pi_Q} \mathbf{P}(Q)$$

and has to consider the K^*-invariant subsets of K^{n+1} to have a sensible definition of s-plane. Given $p \in \mathcal{M}_{n+1,s+1}(K)$, we shall consider:

$$K_p := \left\{ \sum_{j=0}^{s} \lambda_j \cdot p_j | \lambda_j \in K \right\}$$

where the dot means "weighted action". This is the equivalent of the parametric representation in the unweighted case, except that the action is not the same. Equivalently, K_p is the set defined by the parametric equations:

$$x_j = \sum_{k=0}^{s} p_{jk} \lambda_k^{q_j}$$

for every $j = 0, \dots, n$ and any value of $\lambda_0, \dots, \lambda_s$.

A basic question is: when is $K_p = K_r$? One may reason as in the linear case: two planes are equal iff there is an isomorphism tying the parameters. Since one shouldn't destroy the degree of the coefficients, such an isomorphism must be linear; so we may conclude that the planes are the same if and only if there is an automorphism g of K^{s+1} such that, $\forall j = 0, \dots, n$ and $\forall \lambda \in K^{s+1} - 0$:

$$\sum_{k=0}^{s} \lambda_k^{q_j} p_{jk} = \sum_{i=0}^{s} \mu_i^{q_j} r_{ji} = \sum_{i=0}^{s} r_{ji} (\sum_{k=0}^{s} g_{ik} \lambda_k)^{q_j} \tag{4.1}$$

One may define a "Grassmannian" of the weighted projective space as:

$$\mathrm{Gr}(s, Q) := \mathcal{M}_{n+1,s+1}/\mathrm{GL}(s+1)$$

where the group $\mathrm{GL}(s+1)$ acts on the matrices as in (4.1).

Example 4.4.1

Let $p \in U_I$ where I is a multindex of length $s+1$. One can suppose for semplicity $I = (0, \dots, s)$; the matrix p can then be normalized:

$$p_{jk} = \begin{cases} 0 & \forall j \neq k : j \leq s \\ 1 & \forall j = k : j \leq s \end{cases}$$

Then:

$$K_p = \left\{ \begin{array}{ll} x_j = \lambda_j^{q_j} & \forall j = 0, \dots, s \\ x_j = \sum_{k=0}^{s} p_{jk} \lambda_k^{q_j} & \forall j = s+1, \dots, n \end{array} \quad \forall \lambda \in K^{n+1} - 0 \right\}$$

We denote the space $\mathbf{P}(1, \dots, 1, q)$ in which the unit weights are n by $\mathbf{P}(1^{(n)}, q)$. Then this can be seen as a cone over the Veronese embedded $\mathbf{P}^{n-1}$ in $\mathbf{P}^{n(q)}$ where:

$$n(q) := \binom{n+q-1}{n-1}$$

In this case, *and under the hypothesis that the multindex I is such that its last index k_s is not n*, one gets from the previous calculation:

$$K_p = \left\{ \begin{array}{ll} x_j = \sum_{k=0}^{s} p_{jk} x_k & \forall j = s+1, \ldots, n-1 \\ x_n = \sum_{k=0}^{s} p_{nk} x_k^q & \end{array} \right\}$$

Notice that K_p is the zero set of some $n-s$ sections $\sigma_{s+1}, \ldots, \sigma_n \in$ $\in \mathbf{P}(H^0(\mathbf{P}(Q), \mathcal{O}(q)))$.

Remark also that $K_p = K_r$ if and only if there is a nonzero constant ρ acting on the entries of the matrices with the weighted action on the columns, such that $r = \rho \cdot p$.

So indeed the set K_p is the intersection of an s-plane in $\mathbf{P}^{n(q)}$ with the embedded $\mathbf{P}(Q)$: the previous equations can in fact be expressed in terms of the variables $[z_0 : \ldots : z_{n(q)}]$ of $\mathbf{P}^{n(q)}$.

When $p \in U_I$ only for I such that $k_s = n$ (a Zariski closed condition), then one has "degenerate" s-planes passing through the vertex; for example, letting $n = 2$ and $s = 1$ gives:

$$\det \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = 0$$

and so there is an $a \in \mathbf{C}$ such that $p_{1k} = a p_{0k}$ $\forall k = 0, 1$. Then:

$$K_p = \{x_1 = a x_0\}$$

□

Example 4.4.2

Let $n = 3$, $Q = (1, 1, q, aq)$ as in **Example 1.1.3.** and let $s = 2$. Let $p \in U_{(012)} \subset \mathcal{M}_{4,3}(K)$. If one has $p_{jk} = \delta_{jk}$ $\forall j, k$ such that $j \neq 3$, that is if the minor is "normalized", then:

$$K_p = \left\{ \begin{array}{c} x_0 = \lambda_0 \\ x_1 = \lambda_1 \\ x_2 = \lambda_2^q \\ x_3 = \sum_{k=0}^{2} p_{3k} \lambda_k^{aq} \end{array} \right\} = \{x_3 = p_{30} x_0^{aq} + p_{31} x_1^{aq} + p_{32} x_2^{a}\}$$

and so

$$\Psi_{3aq}(K_p) = \Psi_{3aq}(\mathbf{P}(Q)) \cap \{z_{\eta_{aq}} = p_{30} z_0 + p_{31} z_{aq} + p_{32} z_{a(q+1)}\}$$

Unfortunately, *even if $p \in U_{(0,1,2)}$, p cannot be "normalized" as in the linear case, that is one cannot replace p by an r such that $r_{jk} = \delta_{jk}$ $\forall j, k \leq 2$.*

Indeed, this operation is possible if and only if one can find $g \in \operatorname{Aut}(\mathbf{C}^{s+1})$ such that (4.1) is valid; with r of the desired form, and with the given weights, this means that g must satisfy, $\forall \lambda \in \mathbf{C}^3 - \{0\}$:

$$\sum_{k=0}^{2} p_{jk}\lambda_k = \sum_{k=0}^{2} g_{jk}\lambda_k \ \forall j = 0, 1$$

$$\sum_{k=0}^{2} p_{2k}\lambda_k^q = (\sum_{k=0}^{2} g_{2k}\lambda_k)^q = \sum_{k=0}^{2}(g_{2k}^q\lambda_k^q) + \text{mixed terms}$$

and a fourth noninteresting equation for the last coordinate (noninteresting because the entries r_{3k} are free). The first two equations give, by the principle of identity of polynomials:

$$g_{jk} = p_{jk} \ \forall j = 0, 1; \forall k = 0, 1, 2$$

In the last one, the coefficient of any of the mixed terms has to be zero, while

$$g_{2k}^q = p_{2k} \,\forall k = 0, 1, 2$$

Let $k \neq j$. The coefficient of $\lambda_k^{q-1}\lambda_j$ is $qg_{2k}^{q-1}g_{2j}$. Hence there is at most one k such that $g_{2k} \neq 0$. More precisely there is one such k, because $\det g \neq 0$. But then for the other two indices $j \neq k$ one has $p_{2j} = g_{2j}^q = 0$, which in general is not the case.

Let for example:

$$p := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Certainly $p \in U_{(012)}$ because

$$\det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = -1$$

Nonetheless, the last condition becomes:

$$\lambda_0^q + \lambda_1^q = (g_{20}\lambda_0 + g_{21}\lambda_1 + g_{22}\lambda_2)^q$$

and already for $q = 2$ this becomes:

$$\lambda_0^2 + \lambda_1^2 = g_{20}^2\lambda_0^2 + g_{21}^2\lambda_1^2 + g_{22}^2\lambda_2^2 + 2\lambda_0\lambda_1 g_{20}g_{21} + 2\lambda_0\lambda_2 g_{20}g_{22} + 2\lambda_2\lambda_1 g_{22}g_{21}$$

Hence

$$g_{20}^2 = g_{21}^2 = 1$$

and

$$g_{22} = g_{20}g_{21} = 0$$

which is clearly absurd.

□

Remark 4.4.1

One may ask whether the fundamental property of the planes in linear spaces, namely the fact that $s+1$ independent points uniquely determine an s-plane, is preserved for the planes K_p. The answer is no, as one should expect from the lack of linearity in the equations. Here is an example. Let $Q = (1,2,2,2)$ and consider the hyperplanes ($s = 2$). Let

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \text{ so that } K_p = \left\{ \begin{array}{l} x_0 = \lambda_0 \\ x_1 = \lambda_0^2 + \lambda_1^2 \\ x_2 = \lambda_0^2 + 2\lambda_1^2 + \lambda_2^2 \\ x_3 = 2\lambda_0^2 + \lambda_1^2 \end{array} \right\}$$

Take:

$$r_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}, r_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, r_2 = \begin{pmatrix} 1 \\ 2 \\ 7 \\ 3 \end{pmatrix}$$

Certainly r_0, r_1, r_2 are linearly independent and they belong to K_p because:

$$r_0 = K_p \cap \{\lambda_0 = \lambda_1, \lambda_2 = 0\} \quad ; \quad r_1 = K_p \cap \{\lambda_0 = \lambda_2, \lambda_1 = 0\}$$
$$r_2 = K_p \cap \{\lambda_0 = \lambda_1, \lambda_2 = 2\lambda_0\}$$

Nonetheless we have $K_r \neq K_p$: the point $[5, 21, 68, 32] \in K_r$, obtained from the parametric equations of K_r by putting $\lambda_2 = 3\lambda_0 = 3\lambda_1$, is not in K_p, because otherwise there would be $(\lambda_0, \lambda_1, \lambda_2) \in \mathbf{C}^3$ such that $5 = \lambda_0$, $21 = \lambda_0^2 + \lambda_1^2$, from which $\lambda_1 = \pm 2i$, and also $32 = 2\lambda_0^2 + \lambda_1^2 = 50 - 4$.

□

4.5 Relation with σ_Q in the case $Q = (1, \ldots, 1, q)$.

In this section, I would like to see whether there is some relation between the set $T(s, Q)$ of the last chapter and these sets K_p. Let $u \in U_*^Q(s, \mathbf{P}^N)$, and say $u \in U_I$ where $I = (j_0, \ldots, j_s)$ is a multiindex of length $s+1$ such that $j_s \leq n-1$. We may assume without loss of generality that $I = (0, \ldots, s)$. Then:

$$L_u = \left\{ t_j = \sum_{k=0}^{s} u_{jk} t_k \, \forall j = s+1, \ldots, N \right\}$$

and so:

$$H_u = \left\{ \begin{array}{ll} x_j = \sum_{k=0}^{s} u_{jk} x_k & \forall j = s+1, \ldots, n-1 \\ x_n = \prod_{j=n}^{N} (\sum_{k=0}^{s} u_{jk} x_k) & \end{array} \right\}$$

Let $p \in I$. Then:

$$K_p = \left\{ \begin{array}{ll} x_j = \sum_{k=0}^{s} p_{jk} x_k & \forall j = s+1, \ldots, n-1 \\ x_n = \sum_{k=0}^{s} p_{nk} x_k^q) & \end{array} \right\}$$

Hence $K_p = H_u$ if and only if, $\forall (x_0, \ldots, x_s) \in \mathbf{C}^{(s+1)} - 0$:

$$\left\{ \begin{array}{ll} \sum_{k=0}^{s} p_{jk} x_k = \sum_{k=0}^{s} u_{jk} x_k & \forall j = s+1, \ldots, n-1 \\ \sum_{k=0}^{s} p_{nk} x_k^q = \prod_{j=n}^{N} (\sum_{k=0}^{s} u_{jk} x_k) & \end{array} \right.$$

By the principle of identity of polynomials, the first equations give:

$$p_{jk} = u_{jk} \, \forall j = s+1, \ldots, n-1; \forall k = 0, \ldots, s$$

using the fact that:

$$\prod_{j=n}^{N} (\sum_{k=0}^{s} u_{jk} x_k) = \sum_{k=0}^{s} (\prod_{j=n}^{N} u_{jk} x_k^q) + \text{mixed terms}$$

one gets the condition:

$$p_{nk} = \prod_{j=n}^{N} u_{jk} \, \forall k = 0, \ldots, s$$

while the coefficients of the mixed terms have to be zero. These coefficients are of course as many as the monomials in $x_0, \ldots, x_s$ of degree q, minus the $s+1$ "pure" ones: $x_0^q \ldots, x_s^q$. So they are:

$$\binom{s+q}{s} - (s+1)$$

So we have this number of homogeneous equations of degree q in the variables u_{jk} with $j = n, \dots, N$ and $k = 0, \dots, s$. Since the previous condition

$$p_{nk} = \prod_{j=n}^{N} u_{jk} \; \forall k = 0, \dots, s$$

eats $s+1$ variables, the number of variable is therefore $q(s+1) - (s+1) = (q-1)(s+1)$. Therefore *a solution certainly exists if*:

$$\begin{pmatrix} s+q \\ s \end{pmatrix} - (s+1) < (q-1)(s+1) \Leftrightarrow \begin{pmatrix} s+q \\ s \end{pmatrix} < q(s+1)$$

Remark that if there were a zero row j_0 among the last q ones in u, we would have $p_{nk} = 0 \; \forall k = 0, \dots, s$, which cannot happen; hence the condition also implies that $u \in U_*^Q(s, \mathbf{P}^N)$.

When $q = 1$, that is in the unweighted case, one has

$$\begin{pmatrix} s+1 \\ s \end{pmatrix} = s+1$$

and we deduce that the set of all K_p coincides with that of all H_u, as of course was easy to deduce considering that there is no mixed term and that σ_q this time is the identity (in fact, they coincide with the standard planes).

Claim 4.5.1 *The previous estimate only holds when $s = 0, 1$. In other words, to each point or line of type K_p there corresponds some point or line of type H_u. But this is no longer true for higher dimensional subspaces.*

Proof:
One has:

$$\begin{pmatrix} s+q \\ s \end{pmatrix} - q(s+1) = \frac{(s+q)\dots(s+1)}{q!} - q(s+1) =$$
$$= \left(\frac{s+1}{q!}\right)[(s+q)\dots(s+2) - q!q]$$

The estimate holds when this quantity is negative, that is when:

$$f(s,q) := (\prod_{j=2}^{q}(s+j)) - q!q < 0$$

Now

$$f(0,q) = q! - q!q = q!(1-q) < 0$$

$$f(1,q) = (q+1)\dots 3 - q!q = \frac{q!(q+1)}{2} - q!q = q!\left[\frac{q+1}{2} - q\right] = q!\frac{1-q}{2} < 0$$

So the estimate always holds for points and lines.

When $q = 2$, one has:

$$\binom{s+2}{s} - 2(s+1) = \frac{(s+2)(s+1)}{2} - 2(s+1) \le 0 \Leftrightarrow s+2 \le 4 \Leftrightarrow s \le 2$$

Remark however that for $s = 2$ we have an equality: this doesn't suffice because the system is not linear.

Let $Q(1,1,1,2)$ and $s = 2$. Then:

$$K_p = \{x_3 = \sum_{k=0}^{2} p_{3k} x_k^2\}$$

$\forall p \in U_{(0,1,2)}$. If $u \in U_{(0,1,2)}$,

$$H_u = \{x_3 = (\sum_{k=0}^{2} u_{3k} x_k)(\sum_{k=0}^{2} u_{4k} x_k)\}$$

and in the hypothesis $p_{3k} \neq 0\ \forall k = 0,1,2$, since $u_{3k}u_{4k} = p_{3k}$, one has all the unknowns $u_{jk} \neq 0$. The remaining equations to satisfy are:

$$u_{3k}u_{4l} + u_{3l}u_{4k} = 0\,\forall k < l$$

So:

$$\begin{cases} u_{30}u_{41} + u_{31}u_{40} = 0 \\ u_{30}u_{42} + u_{32}u_{40} = 0 \\ u_{31}u_{42} + u_{32}u_{41} = 0 \end{cases}$$

from which:

$$u_{30} = -\frac{u_{31}u_{40}}{u_{41}} = -\frac{u_{32}u_{40}}{u_{42}}$$

which gives:

$$\frac{u_{31}}{u_{41}} = \frac{u_{32}}{u_{42}}$$

so that the last equation cannot be satisfied. This proves that for $q = 2$ already the case $s = 2$ may have a negative answer.

I now prove that $\forall q \ge 3$ and $\forall s \ge 2$, the estimate is not attained. Of course $f(s,q) < f(s+1,q)$ and $f(s,q) < f(s,q+1)$[1], so it suffices to see that $f(2,3) > 0$. Indeed:

$$\binom{2+3}{2} - 3(2+1) = 10 - 9 > 0$$

[1] $f(s,q+1) = \prod_{j=2}^{q+1}(s+j) - (q+1)!(q+1) =$

□

Remark that if u is fixed, we cannot hope to find a p such that $H_u = K_p$ in general. It sufficies that one of the mixed terms be nonzero.

$= (s+q+1)\prod_{j=2}^{q}(s+j) - q!(q+1)q - (q+1)q! =$
$= \prod_{j=2}^{q}(s+j) + (s+q)\prod_{j=2}^{q}(s+j) - (q+1)q!q - qq! - q! =$
$= f(s,q) + (s+q)\prod_{j=2}^{q}(s+j) - (q+1)q!q - q! =$
$= f(s,q) + (q+1)(\prod_{j=2}^{q}(s+j) - qq!) - q! + (s-1)\prod_{j=2}^{q}(s+j) =$
$= f(s,q) + (q+1)f(s,q) + (s-2)\prod_{j=2}^{q}(s+j) + \prod_{j=2}^{q}(s+j) - q! <$
$< (q+2)f(s,q) + (s-2)\prod_{j=2}^{q}(s+j) + f(s,q) =$
$= (q+3)f(s,q) + (s-2)\prod_{j=2}^{q}(s+j)$

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