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Introduction

Cette thèse traite de deux problèmes liés aux groupes de Chow d'une variété algébrique.

Rappellons que les *groupes de Chow* A_*X d'une variété algébrique X sont définis comme

$$A_iX := z_iX/b_iX \ ,$$

où z_iX désigne le groupe abélien libre engendré par les sous–variétés fermées de dimension i dans X, et $b_iX \subset z_iX$ est le sous–groupe engendré par les diviseurs de fonctions rationnelles sur les sous–variétés fermées de dimension $i+1$ dans X [Fu 3, Chapter 1]. Si X est lisse de dimension n, on peut définir un produit d'intersection qui munit les groupes de Chow d'une structure d'anneau

$$A_iX \otimes A_jX \longrightarrow A_{i+j-n}X$$

[Fu 3, Chapter 8].

La *première question* liée aux groupes de Chow est de trouver un "bon" anneau d'intersection pour les variétés arbitraires (i.e. éventuellement singulières). On appelle *anneau d'intersection* un anneau provenant d'un foncteur contravariant

$$R^*\colon \{\text{variétés}\} \to \{\text{anneaux gradués}\} \ ,$$

tel que pour X lisse de dimension n, il existe un isomorphisme d'anneaux $R^*X \xrightarrow{\sim} A_{n-*}X$. Il est naturel de souhaiter, de plus, qu'un anneau d'intersection satisfasse aux propriétés suivantes:

(1) *Fonctorialité.* Il existe des homomorphismes de Gysin fonctoriels $f_*\colon R^*X \to R^{*+d}Y$ pour un morphisme propre et localement d'intersection complète $f\colon X \to Y$ de dimension relative d.

(2) *Naturalité.* (a). Pour chaque bonne théorie de cohomologie H^*, il existe une transformation naturelle $R^* \to H^{2*}$ qui coincide avec l'application "classe d'un cycle" dans le cas lisse. (b). Pour toute variété X, il y a un isomorphisme $\operatorname{Pic} X \xrightarrow{\sim} R^1X$.

(Pour une définition de morphisme localement d'intersection complète, on renvoit à [Fu 3, Appendix B.7]; quant à H^*, le lecteur est invité à penser à la cohomologie singulière dans le cas complexe; plus généralement, on peut penser à une théorie cohomologique au sens de (1.3) ci–dessous.) Si on est moins exigeant, on peut souhaiter ces propriétés seulement avec des coefficients rationnels:

(1′) *Fonctorialité faible.* Il existe des homomorphismes de Gysin fonctoriels $f_*\colon R^*X\otimes\mathbb{Q} \to R^{*+d}Y \otimes \mathbb{Q}$.

(2′) *Naturalité faible.* (a). Il existe une transformation naturelle $R^* \otimes \mathbb{Q} \to H^{2*} \otimes \mathbb{Q}$. (b). Il y a un isomorphisme $\operatorname{Pic} X \otimes \mathbb{Q} \xrightarrow{\sim} R^1X \otimes \mathbb{Q}$.

Pour le moment, on ne connait pas de théorie d'intersection R^* qui possède à la fois les propriétés (1) et (2).

Parmi les divers anneaux d'intersection qui ont été proposés, il y a notamment la *cohomologie de Chow* CH^* de Fulton [Fu 1] et la *cohomologie de Chow opérationnelle* A^* de Fulton–MacPherson [F–M], [Fu 3].

La théorie CH^*, pour une variété quasi–projective X, est définie comme

$$CH^*X := \lim_{\rightarrow} CH^*Y \ ,$$

la limite portant sur tous les morphismes de X vers une variété lisse Y, où on pose $CH^*Y := A_{\dim Y - *}Y$ pour toute Y lisse. Cette théorie vérifie (2), mais ne vérifie pas (1) (par contre, elle vérifie (1′) [Fu 1, §3.3 Remark]). Un inconvénient de la théorie CH^* est de plus qu'elle se prête difficilement aux calculs explicites (pour donner un exemple très simple: si un groupe fini G agit sur une variété lisse X de dimension n, on ne sait pas si on a $CH^*(X/G) \otimes \mathbb{Q} \cong A_{n-*}(X/G) \otimes \mathbb{Q}$).

La théorie A^*, pour toute variété X, est définie de la façon suivante: un élément de A^iX est une collection "bien–formée" d'homomorphismes

$$A_kX' \longrightarrow A_{k-i}X' \ ,$$

pour tout morphisme $X' \to X$, et tout k (cf. [Fu 3, Chapter 17] pour plus de détails). Grâce à cette définition "opérationnelle", A^* possède des propriétés formelles extraordinaires, en particulier la propriété (1) [Fu 3, p. 328]. Bien que sa définition puisse paraître particulièrement compliquée à première vue, la théorie A^* se prête bien aux calculs explicites (par exemple, pour le quotient X/G comme ci–dessus, on a $A^*(X/G) \otimes \mathbb{Q} \cong A_{n-*}(X/G) \otimes \mathbb{Q}$ [Fu 3, Example 17.4.10]). Par contre, un inconvénient majeur est que A^* ne satisfait pas la propriété (2), ni même (2′) (pour (2′)(a), cf. [To 2, Theorem 7]; pour (2′)(b), cf. [Fu 3, 17.4.9]).

On peut dire que CH^* est la théorie d'intersection la plus fine, tandis que A^* est la théorie la moins fine [Fu 2, Chapter 10].

Après avoir vu ces exemples, on peut expliciter la question de trouver un "bon" anneau d'intersection: est–ce qu'on peut construire un anneau qui se situe entre CH^* et A^* ? Cet anneau devrait raffiner A^* assez pour satisfaire (2), tout en restant assez proche de A^* afin de conserver ses bonnes propriétés fonctorielles.

La *seconde question* liée aux groupes de Chow est motivée par les groupes de Chow supérieurs de Bloch [Bl 1], [Bl 2], [Bl 3]. Ces groupes forment une théorie bigraduée $A_*(X,*)$ comprenant les groupes de Chow: $A_i(X,0) = A_iX$. Bloch les a introduits afin de donner une description de la K–théorie supérieure en termes de cycles: pour X quasi-projective, on a un isomorphisme

$$(*) \qquad \bigoplus_i \mathrm{Gr}_i^\gamma K_jX_{\mathbb{Q}} \xrightarrow{\sim} K_jX_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_i A_i(X,j)_{\mathbb{Q}} \ ,$$

où K_jX désigne la K–théorie associée à la catégorie des faisceaux cohérents sur X [Qu 2], et γ dénote la γ–filtration de [So 2], [Kr], [Gra]. Grâce à cet isomorphisme, on appelle les $A_i(X,j)$ parfois homologie "motivique" (ou "absolue") [B–M–S], [Be], [De 3], [Su].

La construction de groupes de Chow supérieurs soulève la question: est–ce qu'il existe une théorie *co*homologique duale de $A_*(-,*)$? Au vu du résultat $(*)$, on s'attend à ce que cette nouvelle théorie coïncide sur $\mathbb{Q}$ avec une certaine K–théorie contravariante (i.e. "cohomologique"). De façon approximative, cette seconde question se traduirait donc comme la recherche d'une *cohomologie* motivique pour les variétés éventuellement singulières.

Pour tenter de répondre à ces deux questions, j'introduis la *cohomologie de Chow bigraduée* $A^*(-,*)$. La définition est une extrapolation des deux constructions A^* et $A_*(-,*)$: le groupe $A^i(X,j)$ est défini en termes de certaines collections d'opérations

$$A_*(X',*)\longrightarrow A_{*-i}(X',*+j)$$

pour $X' \to X$. La définition se sert aussi de la théorie de descente, ce qui nous oblige à supposer qu'on dispose d'une résolution des singularités.

On verra que $A^*(X,*)$ est un anneau bigradué, que $A_*(X,*)$ est un $A^*(X,*)$–module bigradué, et que pour X lisse de dimension n il y a un isomorphisme "dualité de Poincaré"

$$A^i(X,j)\xrightarrow{\sim}A_{n-i}(X,j)\ .$$

Il existe une relation avec la K–théorie, duale de la relation homologique $(*)$ démontrée par Bloch: il y a des isomorphismes

$$ch\colon\ K_f^j X_{\mathbb{Q}}\xrightarrow{\sim}\bigoplus_i A^i(X,j)_{\mathbb{Q}}\ ,$$

où K_f^* est une nouvelle K–théorie contravariante, inspirée par (et pour X complète, égale à) une construction de Gillet et Soulé [G–S]. Cette théorie répond donc à la seconde question posée ci–dessus.

Quant à la première question, le sous–anneau $A^*(-,0) \subset A^*(-,*)$ est un anneau d'intersection, qui (grâce à l'aspect opérationnel de la définition) possède de bonnes propriétés fonctorielles, en particulier une version faible de la propriété (1), cf. (4.4) et (4.5). D'autre part, l'anneau $A^*(-,0)$ satisfait la propriété (2)(a), donc (au vu du résultat de Totaro [To 2] mentionné ci–dessus) c'est une théorie strictement plus fine que A^*.

On remarque que dans la définition de $A^*(-,0)$, la K–théorie supérieure fait son apparition, mais déguisée en cycles par l'isomorphisme $(*)$.

Evidemment, $A^*(-,0)$ est loin d'être l'anneau d'intersection idéal dont on rêve (par exemple, on déplore l'absence d'une interprétation géométrique). Toutefois, il semble intéressant de voir qu'il est possible de raffiner la théorie A^* sans tout à fait quitter le cadre opérationnel de Fulton–MacPherson [F–M].

Résumé

Ce travail comprend six chapitres, chacun possédant une courte introduction séparée. Nous résumons ces différents chapitres.

Le *premier chapitre* regroupe notations, définitions et rappels sur des résultats de la littérature existante dont on aura besoin.

Le §1.2 formalise le concept d'une "théorie de dualité de Poincaré", de sorte que les couples homologie plus cohomologie qu'on connait forment des théories de dualité de Poincaré. Les axiomes d'une "théorie de dualité de Poincaré" sont proches des axiomes de Bloch et Ogus [B–O], mais diffèrent dans le sens qu'ils s'appliquent non seulement aux groupes d'homologie et cohomologie, mais plutôt aux complexes de faisceaux sous–jacents. Ce point de vue, qu'on retrouve par exemple dans [Gi 1] et [G–N], est essentiel pour ce qui suivra. Ensuite, §1.2 traite de la descente cubique de Navarro Aznar et alii [G–N–P–P], variante de la descente simpliciale de Deligne et Saint–Donat [De 2].

Le §1.3 rappelle quelques résultats de base d'algèbre homotopique, y compris les limites et colimites homotopiques d'ensembles simpliciaux et de spectres [B–K], [Th 1, §5]; ces (co)limites homotopiques nous fourniront une version "non–abélienne" de descente, adaptée notamment à la K–théorie (cf. par exemple la preuve de (2.24), ou la définition (4.8) de la K–théorie formelle K_f^*).

Dans le *deuxième chapitre*, on rencontre l'*homologie de Chow bigraduée*, dont la définition est une (légère) modification de celle des groupes de Chow supérieurs de Bloch [Bl 1], [Bl 2], [Bl 3]. Pour une variété quasi–projective X, le groupe $A_i(X, j)$ est défini comme homologie H_j d'un complexe $z_i(X, *)$, défini de la façon suivante: On désigne par Δ^j l'espace affine $\mathbb{A}^j$ muni d'une structure simpliciale (2.3). Alors le groupe $z_i(X, j)$ est engendré par les sous–variétés ψ dans X de dimension $i + j$, pour lesquelles ψ et $\delta_j(\psi)$ rencontrent les faces de codimension au plus 2 proprement (2.9). Ici le bord δ_j provient d'un complexe plus grand $Z_i(X, *) \supset z_i(X, *)$ (2.8), et les faces sont définies par la structure simpliciale de Δ^j. Pour la relation et la différence entre cette définition et la définition originale de Bloch, cf. (2.15). Il est facile à voir que $A_i(X, 0) = A_iX$, le i–ème groupe de Chow de X (2.10).

Avec cette définition, on sait démontrer l'importante propriété de localisation (2.14); c'est une suite exacte longue

$$\cdots \longrightarrow A_i(Y, j) \longrightarrow A_i(X, j) \longrightarrow A_i(U, j) \longrightarrow A_i(Y, j-1) \longrightarrow \cdots$$

pour $Y \subset X$ fermée avec complément $U = X \setminus Y$ (cette suite exacte étend la suite exacte standard [Fu 3, Chapter 1] pour les groupes de Chow).

Malheureusement, la preuve de cette suite exacte de localisation a besoin de l'hypothèse de quasi–projectivité. Pour cette raison, on se sert du lemme de Chow et de la descente cubique pour étendre la définition de $A_*(-, *)$ au cas non quasi–projectif (2.20), de telle

sorte qu'on a encore l'exactitude de la suite de localisation sans l'hypothèse de quasi–projectivité (2.21). Comme corollaire, on démontre la relation avec la K–théorie $(*)$, sans hypothèse de quasi–projectivité (2.24).

Le §2.3 traitant des fonctorialités de $A_*(-,*)$ est guidé par le traitement des groupes de Chow par Fulton [Fu 3, Chapters 1–6]; à l'aide de la déformation vers le cône normal [B–F–M], [Ve 4], [Fu 3, Chapter 5] on construit notamment des *homomorphismes de Gysin raffinés* pour $A_*(-,*)$ (2.32).

Avant de clôre le deuxième chapitre, on jette (dans §2.4) un bref coup d'oeil aux conjectures de Beilinson et Lichtenbaum et l'homologie motivique conjecturelle; ces idées, et ces éventuelles applications arithmétiques, ont été une motivation importante pour l'introduction des groupes de Chow supérieurs.

Le *troisième chapitre* propose une théorie cohomologique $A^*_{\mathrm{op}}(-,*)$ duale à $A_*(-,*)$; notons que cette théorie n'est pas encore la "cohomologie de Chow bigraduée" discutée dans l'introduction. La définition de $A^*_{\mathrm{op}}(-,*)$ s'inspire de la cohomologie de Chow opérationnelle A^* de Fulton–MacPherson: un élément dans $A^i(X,j)$ est par définition une collection bien–formée d'opérations

$$A_k(X',l) \longrightarrow A_{k-i}(X',l+j)$$

pour toute $X' \to X$ et tout $(k,l) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ (voir (3.13) pour une explication de ce choix de numérotation). Tout comme A^*, la théorie $A^*_{\mathrm{op}}(-,*)$ a de merveilleuses propriétés formelles.

On aura besoin de définir aussi un complexe $z^i_{\mathrm{op}}(X,*)$ dont la j–ème homologie est le groupe $A^i_{\mathrm{op}}(X,j)$; la définition de ce complexe est (malheureusement) plus compliquée (3.2).

Cette théorie $A^*_{\mathrm{op}}(-,*)$ a un inconvénient; c'est qu'on ne réussit pas à démontrer une certaine propriété de descente (cf. remarque (3.15.3)), et par conséquent il n'y a pas de relation avec la K–théorie K^*_f. Pour cette raison, on introduit (dans le chapitre 4) une autre théorie cohomologique duale à $A_*(-,*)$; la définition se sert de $A^*_{\mathrm{op}}(-,*)$.

Le *chapitre 4* donne une définition de type descente de $A^*(-,*)$, la *cohomologie de Chow bigraduée.* On suppose qu'on dispose d'une résolution des singularités (1.2), et on introduit pout toute variété X un complexe (homologique) $z^i_{\mathrm{op}}(X,j)$ par

$$z^i(X,-*) := s\bigl(z^i_{\mathrm{op}}(X_*,-*)\bigr) \ ,$$

où $X_* \to X$ est une hyperrésolution cubique (1.7), et s désigne le complexe simple d'un complexe cubique (1.9). Ensuite la cohomologie de Chow bigraduée est définie comme

$$A^i(X,j) := H_j\bigl(z^i(X,*)\bigr) \ ,$$

et à l'aide d'un résultat de Gillet et Soulé on démontre (4.3) que cette définition ne dépend pas du choix de l'hyperrésolution.

Les propriétés formelles des complexes $z^i_{\mathrm{op}}(-,*)$ étudiés dans le chapitre 3 servent maintenant à démontrer des propriétés de $A^*(-,*)$ (4.4).

La K–théorie formelle K^*_f est également définie par voie de descente (4.8); ainsi on trouve un isomorphisme “à la Riemann–Roch”

$$ch^j \,:\, K^j_f X_{\mathbb{Q}} \cong \bigoplus_i A^i(X,j)_{\mathbb{Q}}$$

(4.9). On introduit également des versions “à supports compacts” K^*_c et $A^*_c(-,*)$ (la première est d'ailleurs la théorie définie par Gillet et Soulé [G–S, §5]); de nouveau il y a un isomorphisme “à la Riemann–Roch” (4.9).

Dans un esprit bien plus concret que les autres chapitres, le *chapitre 5* contient des résultats explicites sur les diverses théories (co)homologiques pour les *variétés linéaires*, inspirés par Totaro [To 2]. La classe des variétés linéaires (introduite par Jannsen [Ja 2]) est une classe de variétés éventuellement singulières, mais faciles à étudier grâce à leur définition récurrente (5.1).

Bienque ce cinquième chapitre dépende essentiellement des constructions des chapitres 1–4, il peut dans une large mesure être lu indépendamment et comme une motivation de ces constructions.

Finalement, le *chapitre 6* traite de la partie en codimension 1, $A^1(-,0)$, de l'anneau d'intersection $A^*(-,0)$. On trouve que l'isomorphisme

$$A^1(X,0) \cong \mathrm{Pic}\, X$$

(qui était une des propriétés qu'on désirait d'un anneau d'intersection, cf. (2)(b) ci–dessus) n'existe pas pour toute variété X (même pas à coefficients rationnels). Par contre, $A^1(-,0)$ satisfait plus souvent cette propriété que ne le fait le groupe de Fulton–MacPherson A^1 (6.3).

Introduction

This thesis addresses two questions, motivated by two constructions involving Chow groups of varieties.

Recall that the *Chow groups* A_*X of a variety X are defined as

$$A_iX := z_iX/b_iX \ ,$$

where z_iX denotes the free abelian group on i–dimensional closed subvarieties of X, and $b_iX \subset z_iX$ is the subgroup generated by divisors of rational functions on $(i+1)$–dimensional closed subvarieties [Fu 3, Chapter 1]. For a smooth variety X of dimension n, intersecting subvarieties defines a ring structure on the Chow groups

$$A_iX \otimes A_jX \longrightarrow A_{i+j-n}X$$

[Fu 3, Chapter 8].

The *first question* raised by Chow groups is that of finding a good intersection ring on possibly singular varieties. Let's agree that an *intersection ring* should come from a contravariant functor

$$R^*\colon \{\text{varieties}\} \to \{\text{graded rings}\} \ ,$$

such that for X smooth of dimension n, there is an isomorphism of rings $R^*X \xrightarrow{\sim} A_{n-*}X$. It seems reasonable, moreover, to wish that an intersection ring R^* satisfies the following properties:

(1) *Functoriality.* There are (functorial) Gysin maps $f_*\colon R^*X \to R^{*+d}$ for a proper l.c.i. morphism $f\colon X \to Y$ of relative dimension d.

(2) *Naturality.* (a). For any adequate cohomology theory H^*, there exists a natural transformation $R^* \to H^{2*}$, coinciding with the cycle class maps in the smooth case. (b). There is an isomorphism $\mathrm{Pic}X \xrightarrow{\sim} R^1X$.

(For a definition of l.c.i. morphisms, cf. [Fu 3, Appendix B.7]; for H^*, the reader may think about singular cohomology in the complex case; more generally, take any cohomology theory in the sense of (1.3) below.) Or one could weaken these requirements to

(1′) *Weak functoriality.* There are Gysin maps $R^*X \otimes \mathbb{Q} \to R^{*+d}Y \otimes \mathbb{Q}$ for proper l.c.i. morphisms.

(2′) *Weak naturality.* (a). There is a natural transformation $R^* \otimes \mathbb{Q} \to H^{2*} \otimes \mathbb{Q}$. (b). One has an isomorphism $\mathrm{Pic}X \otimes \mathbb{Q} \xrightarrow{\sim} R^1X \otimes \mathbb{Q}$.

For the moment, no intersection theory R^* is known which satisfies properties (1) and (2).

Two of the most important intersection rings that have been proposed are Fulton's Chow cohomology CH^* [Fu 1] and Fulton–MacPherson's operational Chow cohomology A^* [F–M], [Fu 3].

The theory CH^*, for a quasi–projective variety X, is defined as

$$CH^*X := \lim_{\rightarrow} CH^*Y \ ,$$

where the limit is over all morphisms from X to smooth varieties Y, and $CH^*Y := A_{m-*}Y$ for Y smooth of dimension m. This theory satisfies (2) but not (1) (it does, however, satisfy (1′) [Fu 1, §3.3 Remark]). A serious drawback of CH^* is moreover that this theory is difficult to handle in practice (For instance, if a finite group G acts on a smooth n–dimensional X, I do not know a proof that $CH^*(X/G) \otimes \mathbb{Q} \cong A_{n-*}(X/G) \otimes \mathbb{Q}$).

The theory A^*, for any variety X, is defined as follows: an element in A^iX is a well–formed collection of homomorphisms

$$A_kX' \longrightarrow A_{k-i}X' \ ,$$

for all $X' \to X$, and all k. Thanks to this "operational" definition, A^* enjoys extraordinary functorial properties, in particular (1) holds [Fu 3, p. 328]. Furthermore, although the definition might seem impossibly complicated at first sight, it is actually possible to *work* with A^* (For example, for the quotient X/G as above, one does have $A^*(X/G) \otimes \mathbb{Q} \cong A_{n-*}(X/G) \otimes \mathbb{Q}$ [Fu 3, Example 17.4.10]). A drawback is that A^* does not satisfy property (2), nor even (2′) (for (2′)(a), cf. [To 2, Theorem 7]; for (2′)(b), cf. [Fu 3, 17.4.9]).

In a certain precise sense, CH^* is the *finest* possible intersection ring, A^* the *coarsest* [Fu 2, Chapter 10].

Now we can formulate the question of finding a good intersection ring as follows: Can one construct an intersection ring that lies in between CH^* and A^* ? This ring should refine A^* enough for (2) to hold, while at the same time conserving the extraordinary functorial properties of A^*.

The *second question* raised by Chow groups comes from Bloch's construction of higher Chow groups [Bl 1], [Bl 2], [Bl 3]. These are groups $A_i(X, j)$ extending the Chow groups: $A_i(X, 0) \cong A_iX$. Bloch has introduced them to give a cycle–theoretic description of higher K–theory: for X quasi–projective, one has an isomorphism

$$(*) \qquad \bigoplus_i \mathrm{Gr}_i^{\gamma} K_jX_{\mathbb{Q}} \xrightarrow{\sim} K_jX_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_i A_i(X, j)_{\mathbb{Q}} \ ,$$

where K_jX is K–theory associated to the category of coherent sheaves on X [Qu 2], and γ is the filtration coming from the Adams operations [So 2], [Kr], [Gra]. In view of this result, $A_i(X, j)$ is sometimes called "motivic" (or "absolute") homology [B–M–S], [Be], [De 3], [Su].

This construction naturally raises the question: can one construct a cohomology theory to pair with $A_*(-, *)$? Because of the result $(*)$, this theory should coincide over $\mathbb{Q}$ with some contravariant (i.e. "cohomological") K–theory. Very loosely, this question might thus be described as the search for a motivic cohomology on (possibly singular) varieties.

To address the above two questions, I introduce *bigraded Chow cohomology* $A^*(-,*)$. The definition is an extrapolation of the two constructions A^* and $A_*(-,*)$: the group $A^i(X,j)$ involves certain collections of operations

$$A_*(X',*) \longrightarrow A_{*-i}(X',*+j)$$

for $X' \to X$. However, the definition also involves descent theory, so we need to assume resolution of singularities throughout.

Then $A^*(X,*)$ is a bigraded ring, $A_*(X,*)$ is a bigraded $A^*(X,*)$–module, and for X smooth of dimension n there is a "Poincaré duality" isomorphism

$$A^i(X,j) \xrightarrow{\sim} A_{n-i}(X,j) \ .$$

There is also a relation with K–theory, dual to the homology type relation proven by Bloch: one has isomorphisms

$$\mathrm{ch}\colon\ K^j_f X_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_i A^i(X,j)_{\mathbb{Q}} \ ,$$

where K^*_f is a new contravariant K–theory inspired by (and in case X complete coinciding with) work of Gillet and Soulé [G–S].

As for the first question, the subring $A^*(-,0)$ is an intersection ring, which (thanks to the operational definition) has the excellent functorial properties of A^*; in particular, (a weak version of) property (1) holds, cf. (4.4) and (4.5). On the other hand, I prove that $A^*(-,0)$ satisfies property (2)(a), so (in view of Totaro's result [To 2] mentioned above) it is a strictly *finer* theory than A^*.

Note that higher K–theory enters in the definition of $A^*(-,0)$, but disguised as cycles via the isomorphism $(*)$.

To be sure, $A^*(-,0)$ is not the ideal intersection ring (for one thing, a geometric interpretation is lacking). Still, it seems interesting to see that it is possible to refine A^* without leaving the convenient operational framework.

The work is organised in six chapters. In the rest of the introduction, I will describe the contents of each chapter in more detail.

The first chapter contains some preliminary results from the existing literature. Some highlights of descent theory are presented, in particular the so called *cubical descent* of Navarro Aznar et alii [G–N–P–P], that will play a key role later on. Basic definitions and constructions from homotopical algebra are recalled, up to the homotopy (co)limit of simplicial sets and spectra (we shall see how this provides a non–abelian version of descent).

The second chapter introduces *bigraded Chow homology* $A_*(-,*)$, slightly modifying Bloch's construction of higher Chow groups. For a quasi–projective variety X, the bigraded Chow homology group $A_i(X,j)$ is defined as the j–th homology of a complex $z_i(X,*)$, defined as follows: We write Δ^j to denote affine space $\mathbb{A}^j$ endowed with a simplicial structure (2.3). Then the group $z_i(X,j)$ is generated by $(i+j)$–dimensional closed subvarieties ψ of $X \times \Delta^j$, for which ψ and $\delta_j(\psi)$ meet faces of codimension at most 2 properly (2.9); here the boundary δ_j comes from a larger complex $Z_i(X,*)$ (2.8) and the faces come from the simplicial structure of Δ^j. For the relation and the difference between this definition of $A_*(-,*)$ and Bloch's original definition, cf. (2.15).

With this definition, one can prove (following Bloch [Bl 1]) the important localization sequence (2.14), which is a long exact sequence in $A_*(-,*)$ extending the standard short exact sequence of Chow groups. However, the proof of this localization sequence does not work in the non quasi–projective case, so we use Chow's lemma and a descent trick (2.20) to extend $A_*(-,*)$ from quasi–projective to general varieties, and prove that the localization sequence continues to hold (2.21). As a consequence, the relation with K–theory mentioned above $(*)$ can now be proven without quasi–projectivity hypothesis (2.24).

The numbers (2.29)–(2.35) treating functoriality of $A_*(-,*)$ are guided by Fulton's treatment of Chow groups [Fu 3, Chapters 1–6]; in particular *refined Gysin homomorphisms* are constructed in bigraded Chow homology. This will be useful in chapter 3, when an operational theory is introduced.

Before closing chapter 2, we cast a brief glance at the Beilinson–Lichtenbaum conjectures and the conjectural motivic homology (2.39)–(2.42), which constituted some of the main motivation for the introduction of higher Chow groups.

The third chapter proposes a first cohomology theory $A^*_{\mathrm{op}}(-,*)$ to pair with $A_*(-,*)$; however this is not yet the bigraded Chow cohomology discussed above. The definition of $A^*_{\mathrm{op}}(-,*)$ is similar to the definition of operational Chow cohomology A^*: An element in $A^i_{\mathrm{op}}(X,j)$ is a well–formed collection of operations

$$A_k(X',l) \longrightarrow A_{k-i}(X',l+j)$$

for all X' mapping to X, and all $(k,l) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ (cf. (3.13) for a motivation of the choice of indices). Just as A^*, the theory $A^*_{\mathrm{op}}(-,*)$ has very good formal properties.

For later use, it is also necessary to define a complex $z^i_{\mathrm{op}}(X,*)$ of which the j–th homology is $A^i_{\mathrm{op}}(X,j)$; the definition is more complicated (3.2).

A disadvantage of this theory $A^*_{\mathrm{op}}(-,*)$ is that we are not able to prove a certain descent property (remark (3.15.3)), and hence we cannot prove a relation between $A^*_{\mathrm{op}}(-,*)$ and K^*_f. This is the reason for introducing (in chapter 4) a different cohomology theory to pair with $A_*(-,*)$.

Chapter 4 gives a descent style definition of $A^*(-,*)$, the *bigraded Chow cohomology*. That is, we suppose we have resolution of singularities (1.2), and we define a (homological) complex $z^i(X,*)$, for any variety X, by

$$z^i(X,-*) := s\big(z^i_{\mathrm{op}}(X_*,-*)\big) \ ,$$

where $X_* \to X$ is a cubical hyperresolution (1.7), and s denotes the single complex associated to a cubical complex (1.9). Bigraded Chow cohomology is then defined as

$$A^i(X,j) := H_j\big(z^i(X,*)\big) \ ,$$

and using a result of Gillet and Soulé we prove (4.3) that these groups $A^i(X, j)$ are independent of choice of hyperresolution.

The good formal properties of the complexes $z^i_{\mathrm{op}}(-,*)$ that were established in chapter 3 can now be used to prove properties of $A^*(X,*)$ (4.4).

"Formal K–theory" K^*_f is also defined by descent (4.8) (however, mimicking descent in a non–abelian context is more complicated; here the homotopy (co)limit of spectra (1.20) is used); this observation immediately gives a "Riemann–Roch–like" isomorphism

$$ch^j : K^j_f X_{\mathbb{Q}} \cong \bigoplus_i A^i(X, j)_{\mathbb{Q}}$$

(4.9). We also introduce compactly supported versions K^*_c and $A^*_c(-,*)$ that have the expected properties (K^*_c is actually the theory defined by Gillet and Soulé [G–S, §5]); again there is a Riemann–Roch–like isomorphism (4.9).

In an effort to counterbalance the abstractness of the first four chapters, chapter 5 proves some down to earth results on the various (co)homology theories for *linear varieties*, inspired by work of Totaro [To 2]. The class of linear varieties (introduced by Jannsen [Ja 2]) is a class of varieties that may be singular, but which are easy to study because of their inductive definition (5.1). Though this fifth chapter essentially relies on what has gone on before, it can in a large measure be read independently as a motivation for many of the constructions of chapters 1—4.

Finally, in chapter 6 the codimension 1 part $A^1(-,0)$ of the intersection ring $A^*(-,0)$ is discussed. We find that the isomorphism

$$A^1(X, 0) \cong \mathrm{Pic}\, X$$

(which was one of the desired properties of an intersection ring, cf. (2)(b) above) does not always hold (not even after tensoring with $\mathbb{Q}$), but holds more often than it does for the Fulton–MacPherson Chow cohomology group A^1X (6.3).

Chapter 1: Preparations

This first chapter collects, for the reader's convenience, some notations and results from the existing literature. Neither claims of originality, nor pretentions of completeness are being made.

The reader is invited to start with chapter 2, and only refer back to this first chapter when the need arises.

(1.0) *Conventions.* Let $\mathcal{SCH}_S$ denote the category of schemes over a fixed base S, and let $\mathcal{VAR}_k$ denote the category of *varieties*, i.e. reduced separated equidimensional schemes of finite type over a field k. If mention of the base field is not essential, we will sometimes write $\mathcal{VAR}$ to indicate $\mathcal{VAR}_k$ for a certain field k.

(Note that contrary to Hartshorne [Ha 2], we do not require varieties to be irreducible. The equidimensionality hypothesis is made merely for notational ease; in most (co)homological results that we will give, this hypothesis can be removed using Mayer–Vietoris for a union of closed subschemes.)

§1.1. Descent

(1.1) *M–V.* Let $\mathcal{V} \subset \mathcal{SCH}_S$ be a full subcategory. An *M–V diagram* (short for "Mayer–Vietoris diagram") in $\mathcal{V}$ will be a cartesian diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\widetilde{\tau}} & \widetilde{X} \\ \Big\downarrow{\scriptstyle \pi_Y} & & \Big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\tau} & X \end{array}$$

in $\mathcal{V}$ with τ and $\widetilde{\tau}$ closed immersions, and π a proper morphism inducing an isomorphism

$$\widetilde{X} \setminus \widetilde{Y} \xrightarrow{\sim} X \setminus Y \ .$$

(1.2) *ROS.* Let $\mathcal{V} \subset \mathcal{SCH}_S$ be a full subcategory. The assumption "$\mathcal{V}$ has ROS" (short for "resolution of singularities") means: for any $X \in \mathcal{V}$ there exists an M–V diagram in $\mathcal{V}$ with π a sequence of blow–ups with non–singular center, Y equal to the singular locus of X, and $\widetilde{X}$ regular.

Hironaka's famous result [Hi] asserts that the category of varieties over a field of characteristic 0 has ROS.

Descent only uses some formal properties of (co)homology. For this reason, we axiomatize these properties in a "Poincaré duality theory". The axioms are close to those in

[B–O] and [Ja 2], but different in the sense that they apply not merely to the homology and cohomology groups, but rather to the underlying complexes of sheaves—in this, we closely follow the axiomatizations of [Gi 1] and [G–N].

(1.3) *Poincaré duality theory.*
1. *Cohomology.* A cohomology theory on a category $\mathcal{V}$ of schemes over a fixed base S consists of objects $\Gamma(j)$, $j \in \mathbb{Z}$ in the derived category of sheaves of abelian groups on the big T site of $\mathcal{V}$ (T: some topology, at least as fine as the Zariski topology), with a pairing in the derived category

$$\Gamma(j) \otimes^{\mathbb{L}} \Gamma(k) \longrightarrow \Gamma(j+k) ,$$

which is associative with unit and commutative.

Given a cohomology theory, for a pair of schemes (Y, X) in $\mathcal{V}$ with $Y \subset X$ closed one defines cohomology groups with supports as

$$H^i_Y\big(X, \Gamma(j)\big) := \mathbb{H}^i_Y\big(X, \Gamma(j)\big) .$$

Note that the groups $H^i_Y\big(X, \Gamma(j)\big)$ are contravariant in (X, Y), and that $H^*\big(X, \Gamma(*)\big)$ has a commutative ring structure compatible with pull–backs.

2. *Homology.* A homology theory on $\mathcal{V}$ associates to each $X \in \mathcal{V}$ objects $\Gamma^X_h(j)$, $j \in \mathbb{Z}$, in the derived category of sheaves of abelian groups on the small T site of X, such that

$$\begin{aligned} \mathcal{V}_* &\longrightarrow D_a(Ab) , \\ X &\mapsto R\Gamma\big(X, \Gamma^X_h(j)\big) \end{aligned}$$

($\mathcal{V}_* \subset \mathcal{V}$ is the subcategory whose arrows are proper morphisms; $D_a(Ab)$ is the derived category of ascending chain complexes of abelian groups; $R\Gamma$ means: sections of an injective resolution) is a covariant functor, and the $\Gamma^X_h(j)$ satisfy moreover:

(a) For each open immersion $\phi: U \to X$ in $\mathcal{V}$, there are quasi–isomorphisms

$$\Gamma^U_h(j) \cong R\phi^*\Gamma^X_h(j)$$

for all $j \in \mathbb{Z}$; for each pair X, $Y \in \mathcal{V}$ there are quasi–isomorphisms

$$\Gamma^{X \amalg Y}_h(j) \cong R(i_X)_*\Gamma^X_h(j) \oplus R(i_Y)_*\Gamma^Y_h(j)$$

(where $i_X: X \to X \amalg Y$ resp. $i_Y: Y \to X \amalg Y$ are the obvious maps).

(b) For each cartesian diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array}$$

with vertical arrows proper morphisms and horizontal arrows open immersions, the diagram

$$\begin{array}{ccc} R\Gamma\big(X, \Gamma^X_h(j)\big) & \longrightarrow & R\Gamma\big(U, \Gamma^U_h(j)\big) \\ \downarrow & & \downarrow \\ R\Gamma\big(Y, \Gamma^Y_h(j)\big) & \longrightarrow & R\Gamma\big(V, \Gamma^V_h(j)\big) \end{array}$$

is homotopy commutative.

(c) For each closed immersion $Y \subset X$ in $\mathcal{V}$ with complement U, there is a triangle in $D_a(Ab)$

$$R\Gamma(Y, \Gamma_h^Y(j)) \longrightarrow R\Gamma(X, \Gamma_h^X(j)) \longrightarrow R\Gamma(U, \Gamma_h^U(j)) \xrightarrow{[1]} ,$$

natural in (Y, X).

(d) For each $X \in \mathcal{V}$ purely of relative dimension n over S, $H_{dn}(X, \Gamma(n))$ considered as Zariski presheaf on X has a global section ν_X (the *fundamental class* of X); here the integer d equals 1 or 2 and is fixed for the homology theory.

(e) For all X, $Y \in \mathcal{V}$, there are external product maps

$$R(p_X)^*\Gamma_h^X(j) \otimes^{\mathbb{L}} R(p_Y)^*\Gamma_h^Y(k) \xrightarrow{\times} \Gamma_h^{X\times Y}(j+k) ,$$

compatible with proper push–forwards.

Given a homology theory, for each $X \in \mathcal{V}$ the homology groups are defined as

$$H_i(X, \Gamma(j)) := \mathbb{H}^{-i}(X, \Gamma_h^X(j)) .$$

3. *Poincaré duality theory.* A Poincaré duality theory is a cohomology theory plus a homology theory satisfying the following compatibilities:

(a) For each closed immersion $\tau: Y \to X$ in $\mathcal{V}$ there is a cap product pairing in the derived category of sheaves on X

$$\Gamma_X(j) \otimes^{\mathbb{L}} R\tau_*\Gamma_h^X(k) \xrightarrow{\cap} R\tau_*\Gamma_h^Y(k-j)$$

(here and elsewhere $\Gamma_X(j)$ denotes the restriction of $\Gamma(j)$ to X), such that for each cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tau} & X \\ \downarrow{\scriptstyle f_Y} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\tau'} & X' \end{array}$$

with horizontal arrows closed immersions and vertical arows proper morphisms, the diagram

$$\begin{array}{cccc} & \Gamma_X(j) \otimes^{\mathbb{L}} \Gamma_h^X(k) & \xrightarrow{\cap} & R\tau_*\Gamma_h^Y(j-k) \\ {\scriptstyle f^*\otimes 1} \nearrow & & & \\ \Gamma_{X'}(j) \otimes^{\mathbb{L}} \Gamma_h^X(k) & & & \downarrow{\scriptstyle (f_Y)_*} \\ {\scriptstyle 1\otimes f_*} \searrow & & & \\ & \Gamma_{X'}(j) \otimes^{\mathbb{L}} \Gamma_h^{X'}(k) & \xrightarrow{\cap} & R(\tau')_*\Gamma_h^{Y'}(j-k) \end{array}$$

in the derived category is commutative (this gives the usual projection formula after taking $\mathbb{H}^*$).

(b) Suppose $X \in \mathcal{V}$ is smooth of relative dimension n over S, and let $\tau: Y \to X$ be a closed immersion. Then there is a quasi–isomorphism

$$R\tau^!\Gamma_X(j) \xrightarrow{\sim} \Gamma_h^Y(n-j)[-2n]$$

($\tau^!$ is the functor "sections with support in Y"), inducing the "Poincaré duality map"

$$\cap\nu_X : H^i_Y(X,\Gamma(j)) \xrightarrow{\sim} H_{dn-i}(Y,\Gamma(n-j)) .$$

The class

$$\nu_X \in H_{dn}(X\Gamma(n)) \cong H^0(X,\Gamma(0))$$

corresponds to the unit in the ring structure of $H^*(X,\Gamma(*))$.

(c) For any $X \in \mathcal{V}$ there are isomorphisms

$$p^* : H^i(X,\Gamma(j)) \xrightarrow{\sim} H^i(\mathbb{A}^1_X,\Gamma(j)) .$$

(d) For any $m \geq 1$ and any $X \in \mathcal{V}$, there are isomorphisms

$$\sum_{p=0}^{m} \psi^p \cap \pi^*(\ ) : \bigoplus_{p=0}^{m} H_{i-dp}(X,\Gamma(pj)) \xrightarrow{\sim} H_i(\mathbb{P}^m_X,\Gamma(j)) .$$

(e) There is a natural transformation of contravariant functors on $\mathcal{V}$

$$\mathrm{Pic}(\ ) \longrightarrow H^d(\ ,\Gamma(1)) ,$$

compatible with the cycle class map for smooth effective Cartier divisors on smooth schemes induced by the cap–product.

The following are examples of Poincaré duality theories:

(1.4) *Examples.*
1. *Singular (co)homology:* Let $\mathcal{V}$ be the category of varieties of finite type over $\mathbb{C}$, and let T be the classical (i.e. Euclidean) topology. Then one defines

$$\Gamma(j) := \mathbb{Z}$$

(the constant sheaf $\mathbb{Z}$) for all $j \in \mathbb{Z}$. As for homology, for each $X \in \mathcal{V}$ one defines

$$\Gamma^X_h(j) := R\pi^!\mathbb{Z}_X$$

for all $j \in \mathbb{Z}$, where $\pi: X \to \mathrm{Spec}\,\mathbb{C}$ is the structure morphism (this gives the so–called "Borel–Moore homology" [B–M], constructed somewhat differently in [F–M, 3.1] [Fu 3, Chapter 19]).

The axioms are proven in [Ve 1], [Ve 3].

2. *De Rham (co)homology:* Let $\mathcal{V}$ be the category of quasi–projective varieties over a field k of characteristic 0, and let T be the Zariski topology. For $X \in \mathcal{V}$ and $j \in \mathbb{Z}$, one defines

$$\Gamma_X(j) := \Omega^*_{X/k} .$$

For a definition of homology and proofs of the axioms, cf. [Ha 1].

3. *Etale (co)homology:* Let $\mathcal{V}$ be the category of schemes over $S := \operatorname{Spec}(\mathbb{Z}[1/\ell])$ for a fixed $\ell \in \mathbb{N}$, and let T be the étale topology. One defines

$$\begin{aligned} \Gamma(j) &:= \mu_\ell^{\otimes j} \; ; \\ \Gamma_h^X(j) &:= R\pi^! \mu_\ell^{\otimes -j} \; , \end{aligned}$$

where $\mu_\ell^{\otimes j}$ is the jth Tate twist of the étale sheaf of ℓth roots of unity, and for $X \in \mathcal{V}$, $\pi: X \to S$ is the structure morphism.

For proofs of the axioms, cf. [B–O, Example 2.1], [De 2], [La].

4. *Deligne (co)homology:* $\mathcal{V}$ is the category of varieties over $\mathbb{C}$, T is the classical topology. For smooth and complete X, the complex $\Gamma_X(j)$ is defined as

$$\Gamma_X(j) := \left(\mathbb{Z} \xrightarrow{\exp} \mathcal{O}_X \longrightarrow \Omega_X^1 \longrightarrow \cdots \longrightarrow \Omega_X^{j-1} \right)$$

for $j \geq 0$. Homology is defined using currents. For details, and the extension to singular and open varieties, cf. [Gi 2], [E–V], [Ja 1]. The Deligne (co)homology can also be defined for varieties over $\mathbb{R}$ by taking invariants under the complex conjugation.

5. *Chow theory.* Let $\mathcal{V}$ be a category of varieties for which we suppose ROS. We shall see that the bigraded Chow homology and cohomology

$$A_*(-,*) \; , \qquad A^*(-,*)$$

(to be introduced in chapter 2 resp. chapter 4) form a Poincaré duality theory on $\mathcal{V}$.

(Note: For the first 4 examples, the integer d equals 2; for the last example $d = 1$.)

(1.5) *Remarks:*
1. At certain places, we will need to assume that the Poincaré duality theory satisfies Gillet's axioms [Gi 1, §1], in particular for the construction of a Chern character from algebraic K–theory into cohomology. These axioms are as above, with the addition that cohomology can be computed from Zariski sheaves (i.e. we have $T = \text{Zar}$ in (1.3).1). All the examples (1.4) satisfy this extra hypothesis (for instance, for the singular cohomology one can consider

$$\Gamma(j) := Ru_* \mathbb{Z}$$

for all j, where

$$u : (\mathcal{VAR}_{\mathbb{C}})_{\text{class}} \longrightarrow (\mathcal{VAR}_{\mathbb{C}})_{\text{Zar}}$$

is the canonical morphism of sites, and similarly for the étale theory; cf. [Gi 1, 1.4 (iii) and (iv)]; for Deligne cohomology the required statement is proven in [Be, 1.6.5]).

2. Note that it follows from (1.3).2(c) that *homology has descent*, by which we mean that for any M–V diagram in $\mathcal{V}$

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

there is a long exact sequence

$$H_i(\widetilde{Y},\Gamma(j)) \longrightarrow H_i(\widetilde{X} \amalg Y,\Gamma(j)) \longrightarrow H_i(X,\Gamma(j)) \longrightarrow H_{i-1}(\widetilde{Y},\Gamma(j)) \longrightarrow \cdots \ .$$

This descent property is going to play an essential role in what follows. Sometimes we will also need the assumption that *cohomology has descent*, by which we mean that for any M–V diagram as above, there is a triangle

$$R\Gamma(X,\Gamma(j)) \longrightarrow R\Gamma(\widetilde{X} \amalg Y,\Gamma(j)) \longrightarrow R\Gamma(\widetilde{Y},\Gamma(j)) \xrightarrow{[1]}$$

in the derived category of abelian groups. (It follows from this assumption that there also exists a compactly supported version $H^*_c(X,\Gamma(*))$, with the usual long exact sequence for closed immersions.)

In all of the above examples, cohomology has descent. In fact, we will see that in examples 4 and 5, cohomology is actually *defined* by this property.

3. The homotopy axioms 3(c) and 3(d) in (1.3) are very restrictive: for instance, axiom 3(c) does not hold generally for the K–theory of vector bundles [C–S], nor does it hold for étale cohomology with μ_ℓ–coefficients if ℓ is not prime to the characteristic.

The reader is referred to [C–H–K, Part 2], where in giving a modern and generalized treatment of [B–O], a list of axioms is written down that applies more generally than those of [B–O]; in particular, the above two examples are cohomology theories in the sense of [C–H–K].

4. The ultimate hope is that the objects $\Gamma(j)$, $\Gamma_h(j)$ in the various examples are various manifestations of objects in a conjectural triangulated category D, the triangulated category of mixed motives. Analogs of the above axioms (1.3) (but perhaps excluding the axioms 3(c) and 3(d), cf. remark 3 above) should hold for these objects in D, and the "realization" maps to the various cohomology theories (1.4) should come from various forgetful functors; cf. [De 3, §3] and the references given there.

(1.6) *Cubes.* (for all of this and more, cf. [G–N–P–P, Exposé 1].) For any $n \in \mathbb{Z}_{\geq -1}$, let $\square_n^+$ denote the product category of $n+1$ copies of the category $\underline{2}$; objects of $\square_n^+$ can be identified with ordered sequences $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \{0,1\}$. For such an α, we write

$$|\alpha| := \alpha_0 + \cdots + \alpha_n \ .$$

Let $\square_n$ denote the full subcategory of $\square_n^+$ of objects different from the initial object $(0,0,\ldots,0)$.

A *cubical object or equivalently* $\square_n$*–object* (resp. an *augmented cubical object or equivalently* $\square_n^+$*–object*) of a category $\mathcal{C}$ is a contravariant functor from $\square_n$ (resp. from $\square_n^+$) to $\mathcal{C}$, for some n. For $m < n$, there is a natural face functor from $\square_m$ (or $\square_m^+$) to $\square_n$ (or $\square_n^+$), allowing one to consider a $\square_m$–object also as a $\square_n$–object (resp. a $\square_m^+$–object as a $\square_n^+$–object). Then cubical objects (resp. augmented cubical objects) form a category by defining the morphisms as natural transformations; this category will be denoted $\mathcal{C}^{\square^{\mathrm{opp}}}$ (resp. $\mathcal{C}^{\square^{+\,\mathrm{opp}}}$).

(1.7) We shall be interested in the category of (augmented) cubical schemes. A *cubical hyperresolution* of a scheme X is an augmented cubical scheme X_*, such that X_α is X for $\alpha = (0,\ldots,0)$, X_α is a regular scheme for all $\alpha \neq (0,\ldots,0)$, and all arrows are proper morphisms. We also write this as "$X_* \to X$", where it is now understood that X_* is a cubical scheme.

The prime example of a cubical hyperresolution is an M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

in which Y, $\widetilde{Y}$ and $\widetilde{X}$ are smooth (this is a $\square_1^+$–scheme). In fact, Navarro Aznar et alii have developed an inductive method of constructing cubical hyperresolutions of any variety X over a field of characteristic 0, which is based on ROS and M–V diagrams: for a given X, there exists an M–V diagram in which $\widetilde{X}$, but not necessarily Y and $\widetilde{Y}$ are smooth. If Y and $\widetilde{Y}$ are smooth, we are done; otherwise we consider M–V diagrams for Y and $\widetilde{Y}$ in such a way that these $\square_1^+$–varieties can be "pasted together" to a $\square_2^+$–variety (see [G–N–P–P, Exposé 1] for details, or [G–P] for a more visual explanation). Then we continue with certain M–V diagrams for the varieties that are still singular; we get a $\square_3^+$–variety, etcetera. The final result is a $\square_n^+$–variety which is a cubical hyperresolution.

In what follows, we will always suppose that cubical hyperresolutions come from M–V diagrams in this sense. Since clearly at each step the dimension strictly decreases, a cubical hyperresolution $X_* \to X$ satisfies

$$\dim X_\alpha \leq \dim X - |\alpha| + 1$$

[G–N–P–P, Exposé 1 Th. 2.15].

(1.8) Cubical descent is a variant of the better known simplicial descent (used for instance by Deligne in [De 1]). Although the first is more restrictive than the latter (every proper surjective map from a smooth variety gives a simplicial hyperresolution, whereas to get a *cubical* hyperresolution, the map has to be birational. In particular, de Jong's work [dJ] gives simplicial hyperresolutions, but unfortunately not cubical hyperresolutions), the cubical descent has some explicit advantages that mainly stem from the above finiteness result (for applications, for instance to Zeeman's filtration, cf. [G–P]).

(1.9) *Cubical descent.* Let $\Gamma_{X_*^+}$ be a cubical complex of abelian sheaves on an augmented cubical scheme X_*^+. The cohomology of X_*^+ with values in $\Gamma_{X_*^+}$ is by definition

$$H^i(X_*^+, \Gamma_*) := \mathbb{H}^i(X, sR\pi_*\Gamma_{X_*^+}) \ ;$$

here s denotes the single complex associated to the double complex K^{**} that is obtained by "contracting the cubical index":

$$K^{*p} := \bigoplus_{|\alpha|=p} R\pi_*\Gamma_{X_\alpha^+} \ .$$

Cubical descent can be used in two different ways:

1. Suppose $\Gamma = \Gamma(j)$ is a cohomology theory on $\mathcal{V}$ that has descent (in the sense of (1.5.2)). Let $X_* \to X$ be a cubical hyperresolution in $\mathcal{V}$, and let X_*^+ denote the augmented cubical scheme consisting of X_* and X. The descent condition on Γ implies that $sR\pi_*\Gamma_{X_*^+}$ is acyclic, or equivalently that

$$\Gamma_X \longrightarrow sR\pi_*\Gamma_{X_*}$$

is a quasi–isomorphism. Hence the cohomology of X can be expressed in the cohomology of the X_α, $|\alpha| \neq 0$ through a spectral sequence

$$E_1^{pq} = \bigoplus_{|\alpha|=p+1} H^q(X_\alpha, \Gamma) \Rightarrow H^{p+q}(X, \Gamma)\ , \quad p \geq 0\ .$$

For X a complete variety, the induced filtration on $H^*(X, \Gamma)$ (one can show this filtration does not depend on choice of X_*; cf. [G–N], [G–S]) is called the *weight filtration*, because for singular cohomology with $\mathbb{Q}$–coefficients on a complex variety it coincides with Deligne's weight filtration [De 1].

Passing to a compactification, and using a "cubical hyperresolution of an arrow $X \hookrightarrow \bar{X}$", one can also treat non–complete varieties; there are spectral sequences inducing a weight filtration on homology and on cohomology (with or without compact supports) [G–N–P–P], [G–N], [G–S].

2. Another use of descent theory is to extend a given cohomology theory $\Gamma = \Gamma(j)$ from smooth complete to arbitrary varieties, supposing Γ has descent on the category of smooth complete varieties; this is done in [G–N]. The idea is as follows: suppose (for simplicity) X is complete, and let $X_* \to X$ be a cubical hyperresolution. Then the cohomology of X is defined as

$$H^i(X, \Gamma) := \mathbb{H}^i(X, sR\pi_*\Gamma_{X_*})\ .$$

It follows from the descent property of Γ for smooth varieties that this is independent of X_*. Also, one easily extends the functorial properties of Γ to the singular case, and one can show the extended theory associates long exact sequences

$$\cdots \longrightarrow H^i(X, \Gamma) \longrightarrow H^i(\widetilde{X} \amalg Y, \Gamma) \longrightarrow H^i(\widetilde{Y}, \Gamma) \longrightarrow H^{i+1}(X, \Gamma) \longrightarrow \cdots$$

to M–V diagrams of possibly singular varieties.

(1.10) *Variant.* Chow's lemma [Ha 2, Chapter II Exercice 4.10] says that for any variety X, there exists an M–V diagram with $\widetilde{X}$ quasi–projective. Following the inductive procedure mentioned in (1.8), but with Chow's lemma instead of ROS as input, we find that for any variety X there exists a quasi–projective augmented cubical variety $X_* \to X$ (i.e. X_α is quasi–projective for each $\alpha \neq (0, \ldots, 0)$), satisfying

$$\dim \mathrm{X}_\alpha \leq \dim \mathrm{X} - |\alpha| + 1 \ \ \forall \alpha\ .$$

We will call such $X_* \to X$ a *cubical Chow envelope of* X.

§1.2. Homological algebra

(1.11) *Notation.* $C(Ab)$ (resp. $C^+(Ab)$) will denote the category of homological complexes of abelian groups (resp. the subcategory of complexes that are 0 in negative degree). The associated derived categories (cf. [Ve 2], [Iv]) will be denoted $D(Ab)$ (resp. $D^+(Ab)$).

(1.12) *Notation.* For any complex $A_* \in C^+(Ab)$ of *free* abelian groups, the category $\{A_*\}$ is defined as follows: Objects are complexes $B_* \in C^+(Ab)$ of *free* abelian groups such that the images of A_* and B_* in $D^+(Ab)$ are isomorphic; morphisms are morphisms in $C^+(Ab)$ that are quasi–isomorphisms.

The category $\{A_*\}_h \supset \{A_*\}$ is obtained by adding the objects $H(B_*) \in C^+(Ab)$ (the homology of B_*) for all $B_* \in \{A_*\}$, all isomorphisms $H(B^1_*) \cong H(B^2_*)$ and all homomorphisms $B_* \to H(B_*)$ that are quasi–isomorphisms (For any $B_* \in \{A_*\}$, such a homomorphism exists (non–canonically), since there is a splitting $B_j \cong \mathrm{Ker}(\delta_j) \oplus \mathrm{Compl}_j$ [Iv, Chapter I, Corollary 10.2]).

(1.13) **Lemma:** With notation as above, for any B^1_*, $B^2_* \in \{A_*\}$ there exist arrows

$$B^1_* \longrightarrow B^2_* \quad \text{and} \quad B^2_* \longrightarrow B^1_*$$

in the category $\{A_*\}$.

Proof: It is well–known that any two quasi–isomorphic complexes B^1, B^2 (of not necessarily free groups) can be joined in a diagram

$$\begin{array}{ccc} B^1 & & B^2 \\ & \searrow \quad \swarrow & \\ & B^3 & \end{array}$$

with quasi–isomorphisms as arrows [Iv, Chapter XI]. If B^1, B^2 are complexes of *free* abelian groups, we can even suppose $B^3 \in \{A_*\}$ by replacing B^3 by the complex generated by the images of the two arrows.

Then the arrow $B^2 \longrightarrow B^3$ admits a homotopy inverse [Iv, Chapter I, Corollary 10.4], and composing with $B^1 \longrightarrow B^3$ gives a quasi–isomorphism. □

(1.14) Given $\{A^1_*\}$, $\{A^2_*\}$, and a homomorphism $f: B^1_* \to B^2_*$ for some $B^i_* \in \{A^i_*\}$, $i = 1, 2$, the set

$$\{f\} \subset \amalg_{C^i \in \{A^i_*\}} \mathrm{Mor}_{C^+(Ab)}\big(C^1, C^2\big) \qquad (i = 1, 2)$$

is constructed as follows: For any $C \to B^2$ in $\{A^2_*\}$, the induced homomorphism

$$\mathrm{Cone}\big(B^1 \oplus C \longrightarrow B^2\big)[-1] \longrightarrow C$$

is in $\{f\}$.

By construction, there is a $g: C^1 \to C^2$ in $\{f\}$ for any $C^2 \in \{A^2\}$, and all $g \in \{f\}$ induce (up to isomorphism) the same homomorphism on homology. Also it is easy to see that if f is surjective, then all $g \in \{f\}$ are surjective.

The set

$$\{f\}_h \subset \amalg_{C^i \in \{A^i_*\}_h} \mathrm{Mor}_{C^+(Ab)}(C^1, C^2)$$

is obtained by adding to $\{f\}$ induced homomorphisms on homology.

§1.3. Homotopical algebra

(1.15) **Definitions:** Let Δ denote the category of finite, non–empty, totally ordered sets of non–negative integers, with as morphisms the *face* and *degeneracy* maps

$$d_i \colon [n] = \{0, 1, \ldots, n\} \longrightarrow [n+1] \ , \quad 0 \le i \le n+1;$$
$$s_i \colon [n+1] \longrightarrow [n] \ , \quad 0 \le i \le n,$$

characterized by the fact that they are increasing and $i \notin d_i[n]$, $s_i(i) = s_i(i+1)$.

For $n \ge 1$, the category $\Delta_n \subset \Delta$ is defined as the full subcategory of sets of $\le n$ elements.

For a category $\mathcal{C}$, a simplicial object in $\mathcal{C}$ is by definition a contravariant functor $\Delta \to \mathcal{C}$; a cosimplicial object is a covariant functor. Simplicial objects (resp. cosimplicial objects) form a category $\mathcal{C}^{\Delta^{\mathrm{opp}}}$ (resp. $\mathcal{C}^{\Delta}$), in which the morphisms are defined as natural transformations.

(1.16) *Simplicial sets.* We will be interested in the category $\mathcal{SET}^{\Delta^{\mathrm{opp}}}$ of *simplicial sets*, which is denoted $\mathcal{S}$. The category $\mathcal{S}_*$ of *pointed* simplicial sets is obtained by choosing a base point (an arbitrary fixed vertex), and asking that maps preserve this base point.

There is a realization functor B from $\mathcal{S}$ (resp. $\mathcal{S}_*$) to $\mathcal{T}$, the category of topological spaces (resp. to $\mathcal{T}_*$, the category of pointed topological spaces), which has an adjoint Sin, the singular functor.

One can define homotopy groups of objects in $\mathcal{S}$ and $\mathcal{S}_*$, in such a way that

$$\pi_i(S) \cong \pi_i(BS)$$

for any $S \in \mathcal{S}$ or $\mathcal{S}*$ [May], [B–K, Chapter VIII 3.1]. A map $f \colon X \to Y$ in $\mathcal{S}$ or $\mathcal{T}$ (resp. in $\mathcal{S}_*$ or $\mathcal{T}_*$) is called a *weak equivalence* if f induces isomorphisms

$$\pi_i(X, *) \xrightarrow{\sim} \pi_i(Y, *)$$

for all $i \ge 0$ and every choice of base point (resp. for all $i \ge 0$ and the distinguished base points).

One can define fibrations and cofibrations in $\mathcal{S}$, $\mathcal{T}$, $\mathcal{S}_*$, $\mathcal{T}_*$ [B–K, Ch. VIII 3.3 and 3.4]. A *fibrant object* in one of these 4 categories is an object such that the map to a point is a fibration. With these notions of weak equivalences and (co)fibrations, these 4 categories are *closed (simplicial) model categories* in the sense of Quillen [Qu 1] [B–K, Ch. VIII 3.5]. This fact implies that one can localize with respect to the weak equivalences (i.e. formally make the weak equivalences invertible [Qu 1]); the resulting categories are called the *(stable) homotopy categories* and denoted

$$\mathrm{Ho}(\mathcal{S})\ ,\quad \mathrm{Ho}(\mathcal{T})\ ,\quad \mathrm{Ho}(\mathcal{S}_*)\quad \text{and}\quad \mathrm{Ho}(\mathcal{T}_*)\ .$$

A fundamental fact is that at the localized level the realization and singular functors induce equivalences of categories

$$\begin{array}{ccc} \mathrm{Ho}(\mathcal{S}) & \rightleftarrows & \mathrm{Ho}(\mathcal{T})\ ; \\ \mathrm{Ho}(\mathcal{S}_*) & \rightleftarrows & \mathrm{Ho}(\mathcal{T}_*) \end{array}$$

[B–K, VIII 3.6, 4.6]. In fact, all homotopical notions in $\mathcal{S}$ (resp. $\mathcal{S}_*$) correspond to homotopical notions in $\mathcal{T}$ (resp. $\mathcal{T}_*$), and vice versa.

A *fibre sequence* in $\mathrm{Ho}(\mathcal{S})$ (or in $\mathrm{Ho}(\mathcal{S}_*)$, $\mathrm{Ho}(\mathcal{T})$, $\mathrm{Ho}(\mathcal{T}_*)$) is a sequence isomorphic in $\mathrm{Ho}(\mathcal{S})$ (resp. in $\mathrm{Ho}(\mathcal{S}_*)$ etc.) to a sequence

$$\mathrm{Fibre}(f)\longrightarrow X\xrightarrow{f}Y\ ,$$

with f a fibration, so fibre sequences induce long exact homotopy sequences.

(1.17) As Dold and Puppe discovered, the category of simplicial *abelian groups* is equivalent to the category $C^+(Ab)$ [D–P] [May, 22.4]. The functor from the first to the second category sends a simplicial abelian group A_* to the complex

$$\cdots\longrightarrow A_2/\mathrm{Im}s_0+\mathrm{Im}s_1 \overset{d_0-d_1+d_2}{\longrightarrow} A_1/\mathrm{Im}s_0 \overset{d_0-d_1}{\longrightarrow} A_0\ .$$

Under this equivalence, homotopy groups correspond to homology groups and the weak equivalences correspond to quasi–isomorphisms, so $\mathrm{Ho}(Ab^{\Delta^{\mathrm{opp}}})$ is equivalent to Verdier's derived category $D^+(Ab)$. Fibre sequences in $\mathrm{Ho}(Ab^{\Delta^{\mathrm{opp}}})$ correspond to triangles in $D^+(Ab)$.

(1.18) *Spectra.* For $n\geq 1$, let S^n be the simplicial n–sphere [B–K, Ch. VIII 2.12]. For $X\in\mathcal{S}_*$, define ΩX as the pointed simplicial function space

$$\mathrm{Hom}_*(S^1,X)\quad\in\mathcal{S}_*$$

[B–K, Ch. VIII 4.8]. For $X\in\mathcal{S}_*$, define $\Sigma X\in\mathcal{S}_*$ as the smash product

$$S^1\wedge X$$

[B–K, Ch. VIII 4.8(iii)]. The functor Σ is left adjoint to Ω.

A *prespectrum* X is a sequence of pointed simplicial sets X_n for $n \geq 0$, together with structure maps

$$\Sigma X_n \to X_{n+1}$$

(the structure maps can also be described by their adjoints $X_n \to \Omega X_{n+1}$) [Th 1, 5.1].

A *spectrum* (also called *fibrant spectrum*) is a prespectrum such that each X_n is fibrant, and the structure maps are weak equivalences [Th 1, 5.2].

The homotopy groups of a prespectrum are defined for any $i \in \mathbb{Z}$ as

$$\pi_i(X) := \lim_{\substack{\to \\ n}} \pi_{i+n}(X_{i+n}) \ .$$

If X is a spectrum, then $\pi_i(X)$ is isomorphic to the homotopy of the 0–th space $\pi_i(X_0)$ for $i \geq 0$. For negative i, it turns out that $\pi_i(X)$ is $\pi_{i+m}(X_m)$ for any $m \geq -i$, again provided X is a spectrum [Th 1, 5.3].

Weak equivalences of (pre)spectra are defined as maps inducing isomorphisms on all homotopy groups; one can also define (co)fibrations, prove that this gives a closed model category structure on the category of (pre)spectra and localize to get $\mathrm{Ho}(\mathcal{PRESP})$ and $\mathrm{Ho}(\mathcal{SP})$, the homotopy category of prespectra resp. spectra.

(1.19) The category of spectra of simplicial *abelian groups* is equivalent to the category $C(Ab)$ of complexes that are not necessarily 0 in negative degree (compare the Dold–Puppe equivalence (1.17)). The homotopy category of such spectra is equivalent to the derived category $D(Ab)$; cf. Thomason's "Scholium of great enlightenment" [Th 1, 5.32].

Under this equivalence, the "looping" and "delooping" functors Ω and Σ on spectra correspond to shifting the complex up and down (i.e. to the functors [1] and [−1] on complexes that send a complex A_* to A_{*+1} resp. A_{*-1}).

A non–abelian version of descent can be obtained by using homotopy (co)limits of simplicial sets or spectra:

(1.20) *Holim and Hocolim.* For a small category I and a category $\mathcal{C}$, let $\mathcal{C}^I$ denote the category of I–diagrams over $\mathcal{C}$ (i.e. covariant functors $I \to \mathcal{C}$). The homotopy limit (called "homotopy inverse limit" in [B–K, Ch. XI]) is a functor

$$\mathrm{Holim} : \mathcal{S}_*^I \longrightarrow \mathcal{S}_* \ ,$$

covariant in $\mathcal{S}_*^I$ and contravariant in I, preserving weak equivalences between fibrant objects [B–K, Ch. XI §9].

One has a spectral sequence, relating $\pi_j(X_i)$ to $\pi_{i+j}(\mathrm{Holim}\ X)$ [B–K, Ch. XI §7].

There is a total right derived functor (in the sense of [Qu 1])

$$\mathbb{R}\mathrm{Holim}\colon \mathrm{Ho}(\mathcal{S}_*^I) \longrightarrow \mathrm{Ho}(\mathcal{S}_*) \ ,$$

and if $X \in \mathcal{S}^I$ is such that $X_i \in \mathcal{S}_*$ is fibrant for each $i \in I$, then $\mathrm{Holim}\ X \in \mathcal{S}_*$ represents $\mathbb{R}\mathrm{Holim}\ X \in \mathrm{Ho}(\mathcal{S}_*)$ [B–K, Ch. XI §8].

Dual to Holim is the homotopy colimit (called "homotopy direct limit" in [B–K, Ch. XII]). This is a functor

$$\mathrm{Hocolim} : \mathcal{S}_*^I \longrightarrow \mathcal{S}_* \ ,$$

covariant in $\mathcal{S}_*^I$ and in I.

Again there is a spectral sequence [B–K, Ch. XII §4], and there is a total left derived functor

$$\mathbb{L}\mathrm{Hocolim} : \mathrm{Ho}(\mathcal{S}_*^I) \longrightarrow \mathrm{Ho}(\mathcal{S}_*) \ ,$$

represented by Hocolim [B–K, Ch. XII, 2.4].

Thomason [Th 1, §5] has extended the theory of homotopy limits resp. colimits to the category of spectra resp. prespectra, in such a way that all of the above statements continue to hold.

(1.21) **Lemma:** Let F be a contravariant resp. covariant functor from $\mathcal{SCH}_S$ to the category of spectra, satisfying $F(X \amalg Y) = F(X) \amalg F(Y)$ for all schemes X, Y. Given an M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

in $\mathcal{SCH}_S$, let X_* be the cubical scheme obtained from $\widetilde{X}$, Y and some cubical scheme $\widetilde{Y}_*$ lying over $\widetilde{Y}$. Then there is a fibre sequence

$$\mathrm{Holim}_i\, F(X_i) \longrightarrow F(\widetilde{X} \amalg Y) \longrightarrow \mathrm{Holim}_i\, F(\widetilde{Y}_j)$$

in $\mathrm{Ho}(\mathcal{SP})$, resp. a fibre sequence

$$\mathrm{Hocolim}_i F(\widetilde{Y}_j) \longrightarrow F(\widetilde{X} \amalg Y) \longrightarrow \mathrm{Hocolim}_i F(X_i)$$

in $\mathrm{Ho}(\mathcal{SP})$.

Proof: This is analogous to (1.9).2. One can replace the spectrum $F(\widetilde{X} \amalg Y)$ by the spectrum

$$\mathrm{Holim}_i\, F(Z_i) \ ,$$

where the cubical scheme Z_i is defined as follows: $Z_k = \widetilde{X}$ for the cubical index $k = (0,1,0,\ldots,0)$, $Z_k = Y$ for $k = (1,0,\ldots,0)$, and for $k = (0,j)$ or $(1,j)$ for some cubical index j, Z_k equals $\widetilde{Y}_j$; the maps from $X_{(1,j)}$ to $X_{(0,j)}$ are the identity maps.

Since for each cubical index, one trivially has a fibre sequence

$$F(X_i) \longrightarrow F(Z_i) \longrightarrow F(\widetilde{Y}_i) \ ,$$

and since Holim preserves fibre sequences [Th 1, 5.12], there is a fibre sequence

$$\mathrm{Holim}_i\, F(X_i) \longrightarrow \mathrm{Holim}_i\, F(Z_i) \longrightarrow \mathrm{Holim}\, F(\widetilde{Y}_i) \ .$$

On the other hand, the natural map $F(\widetilde{X} \amalg Y) \to F(Z_*)$ induces a weak equivalence in the homotopy limit, as can be seen from the spectral sequence [Th 1, 5.13].

The proof for Hocolim is similar, now referring to [Th 1, 5.19] and [Th 1, 5.17 and 5.21]. $\square$

(1.22) *Remark.* An alternative to the homotopy colimit for cubical spectra is given in [G–N–P–P, Exposé VI] in treating descent for K_*–theory; it is the "simple of a cubical spectrum". In light of the accordance with Hocolim [G–N–P–P, Exposé VI 2.8] and the descent property for the simple [loc. cit. §2], this reproves the Hocolim part of lemma (1.21).

Chapter 2: Bigraded Chow homology

In this chapter, a "reconstruction" of Bloch's higher Chow groups is given; that is, I define groups $A_*(-,*)$, called "bigraded Chow homology" that partially coincide with Bloch's higher Chow groups. The main reason for this reconstruction is to rid Bloch's results on higher Chow groups ([Bl 1], [Bl 2], [Bl 3], [Bl 4]) of quasi–projective hypotheses.

The chapter is largely self–contained, in the sense that I don't rely on more than Bloch's original article [Bl 1]—in particular, the localization sequence is here constructed directly following the ideas of [Bl 1], avoiding reference to the highly non–trivial proof in [Bl 3]. We use cubical descent and Chow's lemma to extend this localization property also to the non quasi–projective case (2.20).

Having localization without the quasi–projectivity hypothesis enables one to extend the relation with K–theory. That is, we have for any variety X

$$K_jX\otimes\mathbb{Q}\xrightarrow{\sim}\bigoplus_i A_i(X,j)\otimes\mathbb{Q}$$

(2.24); the case X quasi–projective was proven by Bloch [Bl 1].

In treating bigraded Chow homology, I closely follow Fulton's treatment of Chow groups [Fu 3, Chapters 1–6]; in particular, in §2.3 refined Gysin maps will be constructed for $A_*(-,*)$.

Both these innovations (removing quasi–projectivity hypotheses and introducing refined Gysin maps) will be exploited in the construction (in chapter 3) of a bigraded operational cohomology theory $A^*_{\mathrm{op}}(-,*)$ to pair with $A_*(-,*)$.

(2.1) *History.* Higher Chow groups were first defined by Bloch in [Bl 1], in order to give a cycle–theoretic description of higher K–theory. In proving the relation with K-theory, one needs a localization sequence for the higher Chow groups (2.14). However, as Suslin discovered, the proof of this localization sequence given in [Bl 1] was incorrect.

One way out of this difficulty was found by Levine [Le 1], who establishes the desired relation with K–theory by further developing the multi–relative K–theory introduced by Bloch, and without appealing to the localization sequence for higher Chow groups. Another way out is [Bl 4], where a correct proof of the localization sequence is given (using Spivakovsky's solution to Hironaka's polyhedra game).

In this chapter I present a third solution: I modify (very slightly) the definition of higher Chow groups, in such a way that Bloch's original proof of the localization sequence [Bl 1] becomes correct for the modified groups.

§2.1. Definition and first properties

(2.2) *Notation.* For any scheme X, $z_r(X)$ denotes the free abelian group on r–dimensional irreducible closed subvarieties of X. Likewise, $Z_r(X)$ will denote the free abelian group on r–dimensional irreducible locally closed subvarieties. The groups $z_r(X)$ and $Z_r(X)$ are covariantly functorial (resp. contravariantly, with a shift in the grading) for proper (resp. flat) morphisms; for instance if $f\colon X' \to X$ is proper and $\alpha \in Z_r(X')$ is irreducible,

$$f_*(\alpha) := \begin{cases} \deg(\alpha/f(\alpha)) \cdot f(\alpha) & \text{if } \dim \alpha = r; \\ 0 & \text{otherwise.} \end{cases}$$

As a notational matter, writing an element $\alpha \in z_r(X)$ or $Z_r(X)$ as a sum $\alpha = \sum m_i \alpha_i$, it will always be understood that the α_i are irreducible, reduced and that $\alpha_i \neq \alpha_j$ for $i \neq j$.

(2.3) *Simplices.* For any $m \in \mathbb{N}$, let Δ^m denote $\mathrm{Spec}\Big(k[t_0, \ldots, t_m]/(\sum t_i - 1)\Big) \cong \mathbb{A}^m_k$. One can give Δ^m a simplicial structure, with face maps $\Delta^{m-1} \to \Delta^m$ defined by $t_i = 0$, and degeneracies $\Delta^m \to \Delta^{m-1}$ defined by $t_i \mapsto t_i + t_{i+1}$.

Any $\Delta^l \subset \Delta^m$, $0 \leq l \leq m$, given by the vanishing of coordinates t_i is called a *face* of Δ^m; thus Δ^m has $m+1$ codimension 1 faces $\Delta^{m-1}_0, \ldots, \Delta^{m-1}_m$. Likewise, $X \times \Delta^l \subset X \times \Delta^m$, $0 \leq l \leq m$, is called a *face* of $X \times \Delta^m$. The relation "F' is a face of F" will be denoted $F' \prec F$. For any $F = X \times \Delta^m$, the subvariety $\delta F \subset F$ is defined as

$$\delta F := \bigcup_{\substack{F' \prec F, \\ F' \neq F}} F' \,.$$

(2.4) **Definition:** For any variety X and any $(i,j) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$, the group $Z_i(X,j) \subset Z_{i+j}(X \times \Delta^j)$ is defined as the group on those $\psi \in Z_{i+j}(X \times \Delta^j)$ which are not contained in any codimension 1 face $X \times \Delta^{j-1}_k$.

For X, i, j as above, the group $z_i^{\mathrm{Bl}}(X,*)$ ("Bl" for Bloch) is defined as the group on those $\psi \in z_{i+j}(X \times \Delta^j)$ satisfying $\dim \psi \cap F \leq i+j-k$ for all faces $F = X \times \Delta^{j-k} \prec X \times \Delta^j$. (Equivalently, $\dim \psi \cap F = i+j-k$ or -1 where we set $\dim \emptyset = -1$.)

To make the $Z_i(-,j)$ and the $z_i^{\mathrm{Bl}}(-,j)$ into a complex, we use the following operation of intersecting with principal divisors:

(2.5) Suppose D is a pseudo–divisor [Fu 3, 2.2] on a variety X, such that the restriction of $\mathcal{O}_X(D)$ to the associated Weil divisor $|D|$ is trivial. Then according to [Fu 3, 2.3 Remark 2.3] D determines a homomorphism

$$z_i(X) \longrightarrow z_{i-1}(|D|)$$

denoted $\alpha \mapsto D\cdot\alpha$, as follows: Suppose α is a generator of $z_i(X)$, and let $g\colon\alpha\hookrightarrow X$ denote the inclusion morphism. Then

$$D\cdot\alpha := \begin{cases} |g^*D| & \text{if } \alpha\not\subset |D|; \\ 0 & \text{if } \alpha\subset|D|. \end{cases}$$

This operation can be extended to a homomorphism

$$Z_i(X) \longrightarrow Z_{i-1}(|D|)$$

as follows: Supposing α is a generator of $Z_i(X)$, there is an open $\tau\colon U\hookrightarrow X$ such that $\alpha\subset U$ is a closed subvariety. We define

$$D\cdot\alpha := (\tau^D)_!(\tau^*D\cdot\alpha)\ ,$$

where $(\tau^D)_!$ means: consider a closed subvariety of $|\tau^*D|$ as a locally closed subvariety of $|D|$.

(2.6) **Proposition:** Let $f\colon X'\to X$ be a morphism, D an effective Cartier divisor on X with trivial normal bundle such that $f(X')\not\subset|D|$. Let $f_D\colon|f^*D|\to|D|$ denote the morphism induced by f.

(i) If f is proper, the diagram

$$\begin{array}{ccc} Z_i(X') & \xrightarrow{\cdot f^*D} & Z_{i-1}(|f^*D|) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle (f_D)_*} \\ Z_i(X) & \xrightarrow{\cdot D} & Z_{i-1}(D) \end{array}$$

commutes;

(ii) If f is flat of relative dimension m, the diagram

$$\begin{array}{ccc} Z_i(X) & \xrightarrow{\cdot D} & Z_{i-1}(|D|) \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle (f_D)^*} \\ Z_{i+m}(X') & \xrightarrow{\cdot f^*D} & Z_{i+m-1}(|f^*D|) \end{array}$$

commutes.

Proof:

(i) Let $\alpha\in Z_i(X')$ be a generator. Restricting to an open subset containing α as a closed subvariety, we may replace Z by z.

Clearly if $\alpha\in z_i(X')$ is contained in $|f^*D|$, then $f_*\alpha$ is (0 or) supported on a subvariety contained in $|D|$. It remains to check commutativity in case $\alpha\not\subset|f^*D|$. This case follows from (the proof of) [Fu 3, 2.3.c].

(ii) The proof is similar to that of (i), invoking [Fu 3, 2.3.d]. □

(2.7) **Lemma:** Suppose D_1, $D_2\subset X$ are two effective Cartier divisors on X with trivial normal bundle and intersecting properly. Then for any $\alpha\in Z_i(X)$ such that the intersection scheme $|D_1|\cap|D_2|\cap\alpha$ is purely $i-2$–dimensional,

$$(D_1|_{|D_2|})\cdot D_2\cdot\alpha = (D_2|_{|D_1|})\cdot D_1\cdot\alpha \quad \in Z_{i-2}\big(|D_1|\cap|D_2|\big)$$

(here the notation $D_k\big|_{|D_{k\pm1}|}$ means restriction of the Cartier divisor D_k to the support of $D_{k\pm1}$).

Proof: Using (2.6)(i), we may suppose $\alpha = X$, so the intersection $|D_1| \cap |D_2|$ is purely $i-2$–dimensional. Then

$$z_{i-2}\big(|D_1| \cap |D_2|\big) = A_{i-2}\big(|D_1| \cap |D_2|\big) \ ,$$

and the assertion on the level of A_{i-2} is proven in [Fu 3, Theorem 2.4 Case 1]. $\square$

(2.8) **Definition:** The (homological) complex $Z_i(X,*)$, concentrated in positive degree, is defined by

$$\delta_j := \sum_{k=0}^{j} (-1)^k \delta_j^k \ : \ Z_i(X,j) \longrightarrow Z_i(X,j-1) \ ,$$

where

$$\delta_j^k \ : \ Z_i(X,j) \longrightarrow Z_{i+j-1}(F)$$

(F being shorthand for the face $X \times \Delta_k^{j-1}$) is defined as

$$\delta_j^k(\psi) := F \cdot \psi \setminus \{\text{irreducible components of } F \cdot \psi \text{ contained in } \delta F\}$$

(The notation $(\sum r_i \alpha_i) \setminus \beta$, for subvarieties α_i and β is taken to mean $\sum r_i\big(\alpha_i \setminus (\alpha_i \cap \beta)\big)$).

Likewise, the (homological) complex $z_i^{\mathrm{Bl}}(X,*)$ is defined by

$$\delta_j := \sum_{k=0}^{j} (-1)^k \delta_j^k \ ,$$

where

$$\delta_j^k(\psi) := F \cdot \psi$$

(F denoting again the face $X \times \Delta_k^{j-1}$).

Note that lemma (2.7) ensures that $Z_i(X,*)$ is a complex, and note that $z_i^{\mathrm{Bl}}(X,*) \subset Z_i(X,*)$ is a subcomplex.

The complex $Z_i(X,*)$ is not the right one for our purposes (for instance, the homology $H_0(Z_i(X,*)) = 0$, whereas we want it to be the Chow group $A_i X$) so in it we define two (quasi–isomorphic) subcomplexes $z_i(X,*)$ and $z_i'(X,*) \subset Z_i(X,*)$:

(2.9) **Definition:** Let X be a quasi–projective variety. The group $w_i(X,m) \subset Z_i(X,m)$ is defined as follows: $w_i(X,m)$ is the group on *closed* irreducible subvarieties $\alpha \in Z_i(X,m)$ that meet faces up to codimension 2 properly. That is, α satisfies:

$$(*) \qquad \dim(\alpha \cap F) \leq \dim \alpha - k \ \text{ for any face } \ F = X \times \Delta^{m-k} \prec X \times \Delta^m, \ k = 0,1,2.$$

The complex $z'_i(X,*) \subset Z_i(X,*)$ is defined as follows: $z'_i(X,j)$ is the subgroup of $Z_i(X,j)$ generated by $w_i(X,j)$ and boundaries $\delta_{j+1}(\psi')$ for $\psi' \in w_i(X,j+1)$.

The complex $z_i(X,*) \subset Z_i(X,*)$ is defined as follows: $z_i(X,j)$ consists of (not necessarily irreducible) cycles $\psi \in w_i(X,j)$ for which $\delta_j(\psi) \in w_i(X,j-1)$. That is, $z_i(X,j)$ consists of cycles ψ, such that irreducible components of both ψ and $\delta_j(\psi)$ meet faces of codimension at most 2 properly.

Clearly the inclusion $z_i(X,*) \subset z'_i(X,*)$ is a quasi–isomorphism, and we define *bigraded Chow homology* $A_i(X,j)$, for any $(i,j) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$, as

$$A_i(X,j) := H_j\big(z_i(X,*)\big) = H_j\big(z'_i(X,*)\big) \ .$$

Bloch's higher Chow groups are defined as

$$A_i^{\mathrm{Bl}}(X,j) := H_j\big(z_i^{\mathrm{Bl}}(X,*)\big) \ .$$

(2.10) Note that $A_i(X,0) = A_i^{\mathrm{Bl}}(X,0)$ is $z_i(X)$ modulo the group on cycles $i_0^*\psi - i_1^*\psi$ for $\psi \in z_{i+1}(X \times \mathbb{A}^1)$ a subvariety not contained in $X \times \{0\}$ or $X \times \{1\}$. From the alternative definition of rational equivalence [Fu 3, §1.6], we see that $A_i(X,0) = A_i^{\mathrm{Bl}}(X,0) = A_iX$, the usual Chow group of X.

Functorial properties:

(2.11) **Proposition:**

(i) On the category of quasi–projective varieties, the complexes $Z_i(-,*)$, $z_i(-,*)$ and $z'_i(-,*)$ are covariant functorial with respect to closed immersions, and contravariant (with a shift in the i grading) with respect to flat morphisms;

(ii) The complexes $z_i(-,*)$ and $z'_i(-,*)$ are covariant functorial with respect to proper morphisms;

(iii) Push–forward and pull–back commute in the usual way: for a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with f, f' proper and g, g' flat morphisms of relative dimension d,

$$(f')_*(g')^*\alpha = g^*f_*\alpha \quad \in z_{i+d}(Y',j)$$

for any $\alpha \in z_i(X,j)$.

Proof:

(i) Let's first consider the complexes $Z_i(-,*)$. Let $\pi: Y \to X$ be a closed immersion, and $(\pi \times \mathrm{id})_*: Z_{i+j}(Y \times \Delta^j) \to Z_{i+j}(X \times \Delta^j)$ the proper push–forward of cycles.

In the following, we will denote this push–forward on $Z_i(-,j)$ simply by π_*. One checks next that this push–forward induces a push–forward on complexes, i.e. that the diagram

$$\begin{array}{ccc} Z_i(Y,j) & \xrightarrow{\pi_*} & Z_i(X,j) \\ \downarrow{\scriptstyle \delta_j} & & \downarrow{\scriptstyle \delta_j} \\ Z_i(Y,j-1) & \xrightarrow{\pi_*} & Z_i(X,j-1) \end{array}$$

commutes. First, if the vertical maps are given simply by $\sum_k(-1)^k F_k\cdot$ (without cutting out components), this diagram commutes for *any proper morphism* π (2.6)(i). The assertion about a closed immersion π is now clear from the fact that for any $\alpha \in Z_i(Y,j)$ and any face $F = \Delta_k^{j-1}$, the irreducible components of $\pi_*\big((Y\times F)\cdot\alpha\big)$ contained in $\delta(X\times F)$ correspond via π_* to the irreducible components of $(Y\times F)\cdot\alpha$ contained in $\delta(Y\times F)$.

Finally, the functoriality of this push–forward follows from the corresponding assertion for the push–forward of cycles. The assertion about flat pull–backs is similar (using now (2.6)(ii)).

Functoriality of the subcomplexes $z_i(-,*)$ and $z_i'(-,*)$ follows from the fact that push–forward and pull–back on $Z_i(-,*)$ send closed subvarieties to closed subvarieties, and respect the face condition $(*)$.

(ii) Let $\pi\colon Y\to X$ be a proper morphism. Defining π_* by push-forward of cycles as above, one needs to check commutativity of

$$\begin{array}{ccc} z_i(Y,j) & \xrightarrow{\pi_*} & Z_i(X,j) \\ \downarrow{\scriptstyle \delta_j} & & \downarrow{\scriptstyle \delta_j} \\ z_i(Y,j-1) & \xrightarrow{\pi_*} & Z_i(X,j-1) \end{array}$$

(resp. a similar diagram for $z_i'(-,*)$). Once one has this commutativity, it follows that for $\alpha\in z_i(Y,j)$,

$$\delta_j(\pi_*\alpha) = \pi_*\delta_j\alpha$$

meets codimension 1 and 2 faces properly, so $\pi_*\alpha\in z_i(X,j)$, and we may replace Z by z in the above diagram.

To prove the desired commutativity, note that by definition of the complexes $z_i(-,*)$ and $z_i(-,*)$, the boundary δ_j is either 0 or given by $\sum_k(-1)^k F_k\cdot$ (without cutting out components), so this follows from the result (2.6)(i) invoked above.

(iii) This is again straightforward from the corresponding assertion on the level of cycles [Fu 3, Proposition 1.7]. $\square$

Of fundamental importance is the homotopy property of bigraded Chow homology:

(2.12) **Theorem** *(Homotopy):* For any quasi–projective variety X, and any $(i,j)\in\mathbb{Z}\times\mathbb{Z}_{\geq 0}$, pull–back along the projection $p\colon X\times\mathbb{A}^m\to X$ induces isomorphisms

$$p^*:\quad A_i(X,j)\xrightarrow{\sim}A_{i+m}(X\times\mathbb{A}^m,j)\ .$$

Proof: The proof is really just the same as Bloch's proof for the $A_i^{\mathrm{Bl}}(X,j)$ [Bl 1, §2], but having changed Bloch's original definition it seems prudent to go over the proof. All the notation and definitions that follow come from [Bl 1, §2].

Let T_* be a triangulation of $\Delta^* \times \mathbb{A}^1$, i.e. a collection of maps

$$T_j \,:\, z_m(X \times \mathbb{A}^1 \times \Delta^j) \longrightarrow z_m(X \times \Delta^{j+1})$$

[Bl 1, §2]. Let G be a connected algebraic k-group acting on X, and $\psi\colon \mathbb{A}^1 \to G$ a morphism defined over some field extension K of k. Denote by h_j the composition

$$z_i(X,j) \xrightarrow{\pi^*} z_i(X_K,j) \xrightarrow{\mathrm{pr}_1^*} z_{i+1}(X_K \times \mathbb{A}^1_K, j) \xrightarrow{\phi^*} z_{i+1}(X_K \times \mathbb{A}^1_K, j) \xrightarrow{T_j} z_{i+j+1}(X_K \times \Delta^{j+1})$$

(where $\phi(x,y) := (\psi(y)\cdot x, y)$). We would like h_j to land in $z_i(X_K, j+1)$ so that it gives a homotopy; this will be the content of lemma 1.

First some more notation: Given a finite collection $y = \{Y_l\}$ of locally closed subvarieties $Y_l \subset X$, let $z_i^y(X,*) \subset z_i(X,*)$ denote the subcomplex consisting of cycles ψ on $X \times \Delta^j$ that lie in general position with respect to faces of type $Y_l \times \Delta^m$, $m \le j$. More precisely, $z_i^y(X,j)$ consists of elements ψ of $z_i(X,j)$ such that $\psi \cap (Y \times \Delta^j) \in z_{i-r}(Y,j)$, i.e. for every irreducible component α of ψ:

$$\alpha \cap (Y \times \Delta^{j-k}) \text{ intersects } Y \times \Delta^{j-k} \ \text{ properly, } k = 0,1,2,$$

and $\delta_j(\psi)$ satisfies the same requirement, with j replaced by $j-1$.

If $Y \in y$ is a Cartier divisor, restriction gives a pull–back map

$$z_i^y(X,*) \longrightarrow z_{i-1}(Y,*) \ ,$$

which is a homomorphism of complexes.

Given a locally closed $Y \subset X$ of codimension d and an $A \subset Y$ closed, one moreover defines a subcomplex $z_i^{(Y,A)}(X,*) \subset z_i(X,*)$: $z_i^{(Y,A)}(X,j)$ is generated by $\psi \in z_i^{Y-A}(X,j)$, such that the intersection scheme $\psi \cdot (Y \times \Delta^j)$ is a sum

$$\psi \cdot (Y \times \Delta^j) = \psi_1 + \psi_2 \ ,$$

with ψ_1 being in $z_{i-r}(Y,j)$, and ψ_2 being supported on $A \times \Delta^j$.

With these definitions, one has inclusions of complexes

$$z_i^Y(X,*) \subset z_i^{(Y,A)}(X,*) \subset z_i^{Y-A}(X,*) \subset z_i(X,*) \ ,$$

and sending ψ to ψ_1 gives a homomorphism of complexes

$$z_i^{(Y,A)}(X,*) \longrightarrow z_{i-r}(Y,*)/z_{i-r}(A,*) \ .$$

Lemma 1: With the above notations, suppose $\psi(0) = \mathrm{id} \in G$ and $\psi(a)$ is k-generic for all $a \in \mathbb{A}^1(\bar{k})$, $a \neq 0$. Let $y = \{Y_l\}$ be such that $G \cdot Y_l = X$ for all l. Let $A \subset Y \subset X$ with A closed in Y. Then h_j defines maps

$$\begin{aligned} z_i^y(X, j) &\longrightarrow z_i^y(X_K, j+1) \ , \\ z_i^{(Y,A)} &\longrightarrow z_i^{(Y,A)}(X_K, j+1) \ . \end{aligned}$$

Moreover, if $\theta(a)\colon X_K \longrightarrow X$ denotes $x \mapsto \psi(a) \cdot x$ followed by π, then

$$\theta(1)^* z_i(X, *) \subset z_i^y(X_K, *) \ .$$

proof: (cf. [Bl 1, Lemma 2.2]) Let's first do the case $y = X$, i.e. let's prove that for $\alpha \in z_i^X(X, j) = z_i(X, j)$, the image $h_j(\alpha) \in z_i(X_K, j+1)$.

We need to prove that the intersection

$$\phi^* \mathrm{pr}_1^* \pi^*(\alpha) \cap (X \times F)$$

is proper, for all faces $F \subset \Delta^j \times \mathbb{A}^1$ of codimension at most 2 appearing in the triangulation. Replacing ψ by its inverse, we may also consider the intersection

$$\mathrm{pr}_1^* \pi^*(\alpha) \cap \phi^*(X \times F) \ .$$

For F contained in $\Delta^j \times \{0\}$, this intersection is proper since $\alpha \in z_i(X, j)$. For the other F, we apply Bloch's "elementary moving lemma" [Bl 1, Lemma 1.2] to the subvarieties $A = \mathrm{pr}_1^* \pi^*(\alpha) \cap (X \times F)$ (this intersection is proper by virtue of α being in $z_i(X, j)$, so A has the expected dimension) and $B = X \times F$.

To prove $h_j(\alpha) \in z_i(X_K, j+1)$, we also need to prove that $\delta_{j+1} h_j(\alpha)$ meets faces of codimension up to 2 properly. By construction of the triangulation

$$\delta_{j+1} h_j(\alpha) = h_{j-1} \delta_j(\alpha) \ + \ \alpha \ - \ \phi^* \mathrm{pr}_1^* \pi^*(\alpha) \cap (X_K \times \Delta^j \times \{1\}) \ ;$$

By the above argument, the first and last summand satisfy condition $(*)$, so we conclude that $h_j(\alpha) \in z_i(X_K, j+1)$.

Now suppose that the set y also contains a locally closed subvariety $Y \neq X$. The proof is much the same, applying [Bl 1, Lemma 1.2] with the same A and $B = Y \times F$.

The assertion about $\theta(1)'$ is proven by another reference to [Bl 1, Lemma 1.2], this time applied to the constant map $\theta(1)$.

The assertion about $z_i^{(Y,A)}(X, j)$ is proven in a similar way.

Lemma 2: With hypotheses of lemma 1, the map

$$\pi^* \colon \ z_i(X, *)/z_i^?(X, *) \longrightarrow z_i(X_K, *)/z_i^?(X_K, *)$$

is null–homotopic, where ? indicates y or (Y, A). If K over k is purely transcendental, the inclusion

$$z_i^?(X, *) \longrightarrow z_i(X, *)$$

is a quasi–isomorphism.
proof: The first assertion follows from the first lemma; the h_j give a homotopy from $\theta(0)^* = \pi^*$ to $\theta(1)^*$ which lands in $z_i^y(X_K, *) \subset z_i^{(Y,A)}(X_K, *)$.

For the second assertion, we need to show that $z_i(X, *)/z_i^y(X, *)$ is acyclic. Using the norm map as in [Bl 1, Lemma 2.3], we can reduce to the case k infinite, $K = k[t]$. Using the first assertion of lemma 2, it suffices to prove π^* is injective; this is a specialization argument as in [Bl 1, Lemma 2.3].

Corollary 1: The inclusion

$$z_i^{\{X\times 0, X\times 1\}}(X \times \mathbb{A}^1, *) \longrightarrow z_i(X \times \mathbb{A}^1, *)$$

is a quasi–isomorphism.
proof: (as [Bl 1, Cor. 2.4]) Take G to be the additive group $\mathbb{A}^1_k$. Let $K = k[t]$, and let $\psi \colon \mathbb{A}^1_K \to G_K$ be the map $a \mapsto a \cdot t$. Let G act on $X \times \mathbb{A}^1$ by additive translation on $\mathbb{A}^1$. Taking $y = \{X \times 0, X \times 1\}$, the hypotheses of lemmas 1 and 2 are satisfied.

Corollary 2: In the above situation, denote by i_0 (resp. i_1) the inclusion of $X_K \times 0$ (resp. $X_K \times 1$) in $X_K \times \mathbb{A}^1_K$. The restriction maps i_0^* and i_1^* :

$$z_i^{\{X\times 0, X\times 1\}}(X_K \times \mathbb{A}^1_K, *) \longrightarrow z_i(X_K, *)$$

induce the same map on homology.
proof: (as [Bl 1, Cor. 2.5]) By a specialization argument, it suffices to prove that $i_0^* \circ \pi^* = i_1^* \circ \pi^*$ on homology. Let $\theta \colon X_K \times \mathbb{A}^1_K \to X \times \mathbb{A}^1_k$ be translation on $\mathbb{A}^1_K$ followed by π. From lemma 1, it follows that $\pi^* = \theta^*$ on homology, so it suffices to show $i_0^* \circ \theta^* = i_1^* \circ \theta^*$.

As in the proof of lemma 1, the composition

$$z_i^{\{X\times 0, X\times 1\}}(X \times \mathbb{A}^1_k, *) \xrightarrow{\theta^*} z_i^{\{X\times 0, X\times 1\}}(X_K \times \mathbb{A}^1_K, *) \xrightarrow{T_*} z_i(X_K, * + 1)$$

is defined. Since T_* is a triangulation, we have

$$\delta(T_* \circ \theta^*) - (T_* \circ \theta^*)\delta = i_0^* \circ \theta^* - i_1^* \circ \theta^* \ ,$$

which proves corollary 2.

These corollaries suffice to prove the theorem: reasoning in a diagram

$$\begin{array}{ccccc}
z_i(X,*) & \xrightarrow{p^*} & z_{i+1}(X \times \mathbb{A}^1, *) & \xrightarrow{\tau^*} & z_{i+1}(X \times \mathbb{A}^1 \times \mathbb{A}^1, *) \\
 & & \Big\uparrow{\scriptstyle q.-iso} & & \Big\uparrow{\scriptstyle q.-iso} \\
 & & z_{i+1}^{X\times\{0,1\}}(X \times \mathbb{A}^1, *) & & z_{i+1}^{X\times\mathbb{A}^1\times\{0,1\}}(X \times \mathbb{A}^1 \times \mathbb{A}^1, *) \\
 & & & & \Big\downarrow{\scriptstyle i_0^*} \ \Big\downarrow{\scriptstyle i_1^*} \\
 & & & & z_{i+1}(X \times \mathbb{A}^1, *)
\end{array}$$

(where $\tau \colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is multiplication, which is a flat morphism so has a pull–back) proves the result as in [Bl 1, §2]. □

(2.13) *Remarks:*

1. In case the ground field k is algebraically closed, the reference to Bloch's "elementary moving lemma" [Bl 1, Lemma 1.2] in the proof of (2.12, Lemma 1) can be replaced by a reference to [Kl].

2. The lemmas 1 and 2 in the proof of (2.12) yield the following corollary (cf. [Bl 1, 2.5]): If $X \subset \mathbb{P}^N$ is a quasi–projective variety, and $A \subset X$ is closed, the inclusion

$$z_i^{(Y,A)}(\mathbb{P}^N, *) \subset z_i(\mathbb{P}^N, *)$$

is a quasi-isomorphism.

To prove this, apply the lemmas to $G = SL_{N+1}$, $K = k(G)$, and $\psi: \mathbb{A}^1_K \to G_K$ with $\psi(0) = \mathrm{id}$, $\psi(1) = \nu$ for $\nu \to G$ a generic point (Such ψ exists since G is generated by transvections).

Very useful is the following localization sequence:

(2.14) **Theorem** *(localization):* Let X be a quasi–projective variety, $\tau: Y \hookrightarrow X$ a closed immersion, and $\phi: U \hookrightarrow X$ the inclusion of the complement $U := X \setminus Y$. Then the sequence

$$z_i(Y, *) \xrightarrow{\tau_*} z_i(X, *) \xrightarrow{\phi^*} z_i(U, *)$$

forms a triangle in $D^+(Ab)$.

Proof: Since the sequence is clearly left–exact, it suffices to prove that ϕ^* induces a quasi–isomorphism

$$z_i(X, *)/z_i(Y, *) \longrightarrow z_i(U, *) \ .$$

To prove this assertion, embed X as locally closed subvariety of codimension r in $\mathbb{P}^N$, and consider the exact sequence

$$0 \longrightarrow z_{i+r}^{(X,Y)}(\mathbb{P}^N, *) \longrightarrow z_{i+r}^U(\mathbb{P}^N, *) \oplus \Big(z_i(X, *)/z_i(Y, *)\Big) \xrightarrow{j^* - \phi^*} z_i(U, *) \ ,$$

where j^* denotes restriction from $z_{i+r}^U(\mathbb{P}^N, *)$ to $z_i(U, *)$ as in (2.12). Since the inclusion

$$z_{i+r}^{(X,Y)}(\mathbb{P}^N, *) \subset z_{i+r}^U(\mathbb{P}^N *)$$

is a quasi–isomorphism (2.13.2), the theorem follows if we can prove that $j^* - \phi^*$ is a surjection.

I claim that this is the case if the codimension r is ≥ 2. To prove the claim, we follow [Bl 1, Lemma 3.5]:

Let $\alpha \in z_i(U, j)$, and let $\bar{\alpha} \in z_{i+j}(X \times \Delta^j)$ be the closure of α in X. By the classical moving lemma [Ro], [Al], there exists a cycle $W \in z_{i+j+r}(\mathbb{P}^N \times \Delta^j)$ such that

$$W \cap (X \times \Delta^j) = \bar{\alpha} + \bar{\alpha}'$$

where $\bar{\alpha}'$ meets faces $X \times \Delta^{j-k} \prec X \times \Delta^j$ properly so $\bar{\alpha}' \in z_i(X,j)$.

We need $W \in z^U_{i+r}(\mathbb{P}^N, *)$, which is certainly the case if the following holds:

(i) W meets faces $\mathbb{P}^N \times \Delta^{j-k} \prec \mathbb{P}^N \times \Delta^j$ properly, $k = 0,1,2,3$;

(ii) $W \cap (\mathbb{P}^N \times F)$ meets $U \times F$ properly, for all $F = \Delta^{j-k} \prec \Delta^j$, $k = 0,1,2$;

(iii) $\delta_j(W)$ meets $U \times F$ properly, for $F = \Delta^{j-1-k} \prec \Delta^{j-1}$, $k = 0,1,2$.

Using the usual norm argument, we reduce to the case k infinite.

The cycle W in the moving lemma is constructed as a sum of cones $C(L_m, Z_m)$, where the L_m are general linear spaces and

$$C(L_m, Z_m) \cap (X \times \Delta^j) = Z_m + Z_{m+1} \ .$$

The L_m can be chosen such that Z_{m+1} has better intersection properties with all the $U \times \Delta^{j-k}$ than Z_m. So since by assumption $\bar{\alpha} = Z_0$ meets all $U \times \Delta^{j-k}$ for $k = 0,1,2$ properly, the $C(L_m, Z_m)$ can be chosen to do the same; this proves (ii).

Since $C(L_m, Z_m) \cap F = C(L_m \cap F, Z_m \cap F)$, the same reasoning proves (iii).

To prove (i), it suffices in view of [Bl 1, Sublemma 3.6] to prove that

$$\dim\bigl(\bar{\alpha} \cap (\mathbb{P}^N \times \Delta^{j-k})\bigr) \le i + j + r - k$$

(the expected dimension of $C(L_m, Z_m) \cap (\mathbb{P}^N \times \Delta^{j-k})$), for $k = 0, \ldots, 3$. Since $\bar{\alpha}$ is not contained in any codimension 1 face, we know that

$$\dim\bigl(\bar{\alpha} \cap (\mathbb{P}^N \times \Delta^{j-k})\bigr) \le i + j - 1 \ ,$$

so the above inequality is true for $r \ge 2$. $\square$

(2.15) *Remarks:*

1. The problem in proving localization for Bloch's higher Chow groups is to prove in the last step that $W \in z_i^{\mathrm{Bl}}(\mathbb{P}^N, j)$; here the face condition has to be checked for *all* faces $\mathbb{P}^N \times \Delta^{j-k}$, $k = 0, \ldots, j$, so N has to grow along with j. This was the reason for changing the definition of $z_i^{\mathrm{Bl}}(-,*)$.

In fact, for any $n \in \mathbb{N}_{\ge 2}$ we could define a complex $z_i^{(n)}(-,*)$ by imposing the condition (2.9) for faces of codimension at most n. Then the above proof works for any $n \in \mathbb{N}$. There is a chain of inclusions

$$z_i^{\mathrm{Bl}}(X,*) = \cap_{n=2}^{\infty} z_i^{(n)}(X,*) \subset z_i^{(N)}(X,*) \subset z_i^{(N-1)}(X,*) \subset \cdots \subset z_i^{(2)}(X,*) = z_i(X,*)$$

for any $N >> 0$, and the conjecture is that all these inclusions are quasi–isomorphisms; that this is true after tensoring with $\mathbb{Q}$ will follow from (2.24).

2. Another bigraded theory extending the Chow groups is formed by the "left derived functor" Chow groups L_pCH_q of Gillet and Soulé [G–S, 3.1.1 and proposition 4]. These groups are defined using descent theory (so one needs ROS), and they also satisfy homotopy and localization. I don't know in how far L_*A_* coincides with $A_*(-,*)$.

3. Yet another bigraded theory extending the Chow groups is introduced in [Gi 1, §8], as the E_2 terms in the Brown–Gersten–Quillen (or "coniveau") spectral sequence of K–theory. This theory satisfies homotopy [Gi 1, Theorem 8.3] and localization (trivial). In general, this theory is different from $A_i(-,j)$, but they coincide rationally for $j \leq 2$, see (2.27) below.

(2.16) **Corollary:** For any M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \stackrel{\widetilde{\tau}}{\longrightarrow} & \widetilde{X} \\ \downarrow \pi_Y & & \downarrow \pi \\ Y & \stackrel{\tau}{\longrightarrow} & X \end{array}$$

with $\widetilde{X}$ and X quasi–projective, the sequence

$$z_i(\widetilde{Y},*) \stackrel{(\widetilde{\tau}_*,\pi_{Y\,*})}{\longrightarrow} z_i(\widetilde{X},*) \oplus z_i(Y,*) \stackrel{\pi_* - \tau_*}{\longrightarrow} z_i(X,*)$$

forms a triangle in $D^+(Ab)$.

Proof: In view of (2.14) and (2.11), there is a commutative diagram

$$\begin{array}{ccccccc} z_i(\widetilde{Y},*) & \longrightarrow & z_i(\widetilde{X},*) & \longrightarrow & z_i(\widetilde{X} \setminus \widetilde{Y},*) & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow \wr & \\ z_i(Y,*) & \longrightarrow & z_i(X,*) & \longrightarrow & z_i(X \setminus Y,*) & \longrightarrow \end{array}$$

whose rows are triangles in $D^+(Ab)$. □

(2.17) *Chow homology as hypercohomology.* Let X be quasi–projective, and let $\mathcal{Z}_i^X(*)$ denote the cohomological complex of Zariski sheaves on X (concentrated in negative degrees) associated to the presheaves $U \mapsto z_i(U,-j)$. Then one has

$$A_i(X,j) = \mathbb{H}^{-j}\left(X, \mathcal{Z}_i^X(*)\right) .$$

Hence if $\mathcal{A}_i^X(-j)$ denotes the Zariski sheaf associated to $U \mapsto A_i(U,-j)$, there is a "local to global" spectral sequence

$$E_2^{pq} = H^p\left(X, \mathcal{A}_i^X(-q)\right) \;\Rightarrow\; A_i(X,-p-q)$$

—for the higher Chow groups $A_i^{\mathrm{Bl}}(X,j)$, the corresponding statements are proven in [Bl 1, §3].

This follows from the facts that the Zariski presheaves on X

$$U \mapsto s_i(X,-j) := z_i(X,-j)/z_i(X \setminus U,-j)$$

(which form a complex quasi–isomorphic to $\mathcal{Z}_i^X(*)$ by the localization sequence (2.14)) are sheaves, and that they are flasque. To prove these two facts, it suffices to prove exactness of the sequence

$$0 \longrightarrow s_i(U \cup V,*) \longrightarrow s_i(U \amalg V,*) \longrightarrow s_i(U \cap V,*)$$

for U, V open in X; flasqueness is then obvious from the definition of s_i.

This exact sequence is deduced from the following Mayer–Vietoris property: for any two closed subschemes Y_1, $Y_2 \subset X$, one has an exact sequence of complexes

$$0 \longrightarrow z_i(Y_1 \cap Y_2, *) \longrightarrow z_i(Y_1 \amalg Y_2, *) \longrightarrow z_i(Y_1 \cup Y_2, *) \longrightarrow 0 \ .$$

This last property is easily checked directly.

The fact that Chow homology is hypercohomology for the Zariski site is of some importance, since it makes Chow homology a homology theory in the sense of (1.3). Later on, when we have defined bigraded Chow cohomology, we shall see that the pair

$$A_*(-,*) \ , \quad A^*(-,*)$$

(or the pair

$$A_*(-,*) \ , \quad \mathbb{H}^*_{\mathrm{Zar}}(-, \mathcal{Z}^i_{\mathrm{op}}(*))$$

for which we don't need to suppose ROS) is a Poincaré duality theory satisfying Gillet's axioms (1.5.1) [Gi 1, §1], from which it follows that $A_*(-,*) \otimes \mathbb{Q}$ is a receiver of Gillet's Riemann–Roch map τ_* from K_*; cf. (2.24).

The following result, which is a strengthening of Corollary 1 in the proof of (2.12), is proven by the same techniques as (2.14):

(2.18) **Proposition:** Let X be a quasi–projective variety, and let $y = \{Y_i\}$ be a finite collection of closed subvarieties. Define a complex $z_i^{[y]}(X,*) \subset z_i(X,*)$ by

$$z_i^{[y]}(X,j) := \Big\{ \psi \in z_i(X,j) \ \mid \ \psi = \psi' + \psi'', \ \psi' \in z_i^y(X,j), \ \mathrm{Supp}(\psi'') \subset y \Big\} \ .$$

Then the inclusion

$$z_i^{[y]}(X,*) \subset z_i(X,*)$$

is a quasi–isomorphism.

Proof: (cf. [Bl 1, Lemma 4.2]) Embed X as locally closed subset of codimension r of $\mathbb{P}^N$, and define a complex $z_i^{X;(y)}(\mathbb{P}^N,*) \subset z_i^X(\mathbb{P}^N,*)$ by

$$z_i^{X;(y)}(\mathbb{P}^N,j) := \Big\{ \alpha \in z_i^X(\mathbb{P}^N,j) \ \mid \ \alpha \cdot (X \times \Delta^j) \in z_{i-r}^{[y]}(X,j) \Big\} \ .$$

The same argument as (2.13).2 proves that the inclusion

$$z_i^{X;(y)}(\mathbb{P}^N,*) \subset z_i^X(\mathbb{P}^N,*)$$

is a quasi–isomorphism. As in the proof of (2.14), there is an exact sequence

$$0 \longrightarrow z_{i+r}^{X;(y)}(\mathbb{P}^N,*) \longrightarrow z_{i+r}^X(\mathbb{P}^N,*) \oplus z_i^{[y]}(X,*) \longrightarrow z_i(X,*) \ ,$$

and we are done if we can prove this is also right–exact. This is similar to the proof of (2.14). $\square$

(2.19) **Corollary:** Let X, Y, $\phi\colon U \hookrightarrow X$ be as in (2.18). Then ϕ^* induces a quasi–isomorphism

$$z_i^{[Y]}(X,*)/z_i(Y,*) \longrightarrow z_i(U,*) \ .$$

Proof: This is just a combination of (2.14) and (2.18), since ϕ^* on $z_i^{[Y]}(X,*)$ factors as

$$z_i^{[Y]}(X,*) \hookrightarrow z_i(X,*) \xrightarrow{\phi^*} z_i(U,*) \ .$$

$\square$

To extend bigraded Chow homology to non–quasi–projective varieties, we use a descent trick:

(2.20) **Definition:** Let X be an arbitrary variety. For any $i \in \mathbb{Z}$, the homological complex $z_i(X,*)$, concentrated in positive degree, is defined as

$$z_i(X,*) := s\big(z_i(X_*,*)\big) \ ,$$

where $X_* \to X$ is a cubical Chow envelope (1.10), and sK_{**} denotes single complex associated to a cubical complex (1.9).

Equivalently (and more naively), the complex $z_i(X,*)$ is defined inductively by taking an M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\widetilde{\tau}} & \widetilde{X} \\ \big\downarrow{\scriptstyle \pi_Y} & & \big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\tau} & X \end{array}$$

with $\widetilde{X}$ quasi–projective, and defining

$$z_i(X,*) := \mathrm{Cokernel}\Big(z_i(\widetilde{Y},*) \longrightarrow z_i(\widetilde{X},*) \oplus z_i(Y,*)\Big) \ .$$

It follows from (2.16) that $z_i(X,*)$ is well–defined in $D^+(Ab)$ (and in case X quasi–projective, coincides with the former definition in $D^+(Ab)$).

So for $(i,j) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$, the *bigraded Chow homology groups*

$$A_i(X,j) := H_j\big(z_i(X,j)\big)$$

are well–defined. Note that from descent for Chow groups [Fu 3, Example 1.8.1] it follows that one still has $A_i(X,0) = A_i X$.

Now we can easily extend properties of $A_*(-,*)$ from the quasi–projective to the general case:

(2.21) **Proposition:**

(i) On the category of varieties, the assignment $X \mapsto z_i(X,*)$ is a functor to $D^+(Ab)$, covariant with respect to proper morphisms, and contravariant (with a shift in the i–grading) with respect to open immersions and structure morphisms of vector bundles; pull–back and push–forward commute in the usual way;

(ii) The localization sequence (2.14) and its corollary (2.16) are true for any X (not necessarily quasi–projective);

(iii) For any rank n vector bundle $p\colon E \to X$, the map

$$p^* \,:\, z_i(X,*) \longrightarrow z_{i+n}(E,*)$$

is a quasi–isomorphism;

(iv) The relation

$$A_i(X,j) = \mathbb{H}^{-j}\left(X, \mathcal{Z}_i^X(*)\right)$$

(2.17) is true for any X;

(v) The quasi–isomorphisms (2.18), (2.19) are true without quasi–projectivity assumption.

Proof:
(i) Let $f\colon X_1 \to X_2$ be a proper morphism. Using Chow's lemma, we can find M–V diagrams

$$(\text{M–V})_l \qquad \begin{array}{ccc} \widetilde{Y}_l & \longrightarrow & \widetilde{X}_l \\ \big\downarrow & & \big\downarrow \\ Y_i & \longrightarrow & X_i \end{array}$$

($l = 1,2$) with $\widetilde{X}_l$ quasi–projective, and such that these two diagrams commute with the morphism f. Then the push–forward in the quasi–projective case induces f_* by noetherian induction, according to a diagram

$$\begin{array}{ccccccc} z_i(\widetilde{Y}_1,*) & \longrightarrow & z_i(\widetilde{X}_1 \amalg Y_1,*) & \longrightarrow & z_i(X_1,*) & \longrightarrow 0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f_*} & \\ z_i(\widetilde{Y}_2,*) & \longrightarrow & z_i(\widetilde{X}_2 \amalg Y_2,*) & \longrightarrow & z_i(X_2,*) & \longrightarrow 0\ . \end{array}$$

To prove that this f_* does not depend on choice of $\widetilde{X}_l$, it suffices to prove that f_* makes $z_i(-,*)$ into a covariant functor. So defining f_* by a fixed choice of $\widetilde{X}_l$ as above, and given a commutative diagram

$$(\#) \qquad \begin{array}{ccc} X_1' & \longrightarrow & X_1 \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ X_2' & \longrightarrow & X_2 \end{array}$$

where all arrows are proper morphisms, we must prove commutativity up to homotopy of

$$\begin{array}{ccc} z_i(X_1',*) & \longrightarrow & z_i(X_1,*) \\ \big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ z_i(X_2',*) & \longrightarrow & z_i(X_2,*)\ . \end{array}$$

Again using Chow's lemma, we can find M–V diagrams

$$\begin{array}{ccc} \widetilde{Y}'_l & \longrightarrow & \widetilde{X}'_l \\ \big\downarrow & & \big\downarrow \\ Y'_l & \longrightarrow & X'_l \end{array}$$

($l = 1,2$) with $\widetilde{X}'_l$ quasi–projective, commuting with $(\text{M–V})_l$ and with (#). By (2.11)(i) and noetherian induction, the diagram

$$\begin{array}{ccc} z_i(\widetilde{X}'_1 \amalg Y'_1, *) & \longrightarrow & z_i(\widetilde{X}_1 \amalg Y_1, *) \\ \big\downarrow & & \big\downarrow \\ z_i(\widetilde{X}'_2 \amalg Y'_2, *) & \longrightarrow & z_i(\widetilde{X}_2 \amalg Y_2, *) \end{array}$$

commutes, and by a diagram chase using (2.16) we are done.

With respect to flat morphisms the situation is similar, but to make the above construction work we need to assume that a flat morphism $f\colon X_1 \to X_2$ fits into a commutative diagram

$$\begin{array}{ccc} \widetilde{X}_{1*} & \longrightarrow & X_1 \\ \big\downarrow{\scriptstyle \widetilde{f}_*} & & \big\downarrow{\scriptstyle f} \\ \widetilde{X}_{2*} & \longrightarrow & X_2 \ , \end{array}$$

with the horizontal maps being cubical Chow envelopes (1.10) and $\widetilde{f}_*$ a flat morphism of cubical varieties. For later reference, let's call such flat morphisms *Chow flat*. Obvious exemples of Chow flat morphisms are open immersions, and structure morphisms for vector bundles.

(ii) Let $Z \subset X$ be closed with complement U. Given an M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X \end{array}$$

with $\widetilde{X}$ quasi–projective, we may fiber this diagram with $\times_X Z$ resp. $\times_X U$ to obtain quasi–projective M–V diagrams for Z resp. U. The result now follows from an easy diagram chase.

(iii) Using (ii), one reduces to the case where $E = X \times \mathbb{A}^m$ and X quasi–projective, which is theorem (2.12).

(iv) The same proof as (2.17): the presheaves

$$U \mapsto s_i(U,j) := z_i(X,j)/z_i(X \setminus U, j)$$

(which form a complex quasi–isomorphic to $\mathcal{Z}_i^X(*)$ by localization (ii)) are sheaves, and they are flasque.

(v) In the general (i.e. possibly non quasi–projective) case, the complex $z_i^y(X,*)$ is defined as

$$\left\{\psi \in z_i(X,j) \ \middle|\ \psi \cdot (Y_k \times \Delta^j) \in z_{i-r_k}(Y_k, j) \ \ \forall Y_k \in y\right\}$$

(where $r_k := \mathrm{codim} Y_k$). Suppose now that all $Y_k \in y$ are closed subvarieties. There exists an M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\widetilde{\tau}} & \widetilde{X} \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\tau} & X \end{array}$$

with $\widetilde{X}$ quasi-projective and such that Y does not contain any $Y_k \in y$, so that taking the fibre product $\times_X y$ gives an M–V diagram for $y = \cup Y_k$. Choosing this diagram to define $z_i(X,*)$ and $z_*(y,*)$, there is (by definition) an exact sequence

$$0 \longrightarrow z_i^{\widetilde{Y}\times_X y}(\widetilde{Y},*) \longrightarrow z_i^{\widetilde{X}\times_X y}(\widetilde{X},*) \oplus z_i^{Y\times_X y}(Y,*) \longrightarrow z_i^{y}(X,*) \longrightarrow 0 \ .$$

There is a similar exact sequence for the complex $z_i^{[y]}(X,*)$, and noetherian induction ends the proof. □

(2.22) *Remarks.*
1. Remark that we have in fact constructed a push–forward (resp. pull–back) from some complex in $\{z_i(X_1,*)\}$ to *some* complex in $\{z_i(X_2,*)\}$ (in the notation of (1.12)); using (1.14) this generates a set of homomorphisms $\{f_*\}$: $\{z_i(X_1,*)\} \to \{z_i(X_2,*)\}$ (resp. $\{f^*\}$), all inducing the same map on homology.

2. The loss in contravariant functoriality is somewhat annoying; for instance in the non quasi–projective case the $z_i(X,j)$ are not étale presheaves, so in the non quasi–projective case one cannot speculate on a connection with Lichtenbaum's axioms for motivic complexes, cf. (2.42). The same problem comes up again in chapter 4, where in constructing Gysin homomorphisms for bigraded Chow cohomology we need to make the artificial "Chow flatness" restriction, cf. (4.5).

§2.2. Enter K_*–theory

Before proving the relation between Chow homology and K–theory (2.24), we pause to recall a few facts about K–theory:

(2.23) For any noetherian scheme X, let K_*X (resp. K^*X) denote the K–theory associated to the category of coherent sheaves on X (resp. of locally free sheaves on X).

In case X is quasi–projective over a regular base scheme, there exists a γ–filtration on $K_*X_{\mathbb{Q}}$, and a direct sum decomposition

$$K_j X_{\mathbb{Q}} \cong \oplus_i \mathrm{Gr}_i^{\gamma} K_j X_{\mathbb{Q}}$$

[Kr], [So 2], [Gra].

For $j = 0$, and X quasi–projective over a field k, the γ–filtration on $K_0X_{\mathbb{Q}}$ coincides with the topological filtration (graded by dimension of support), and Grothendieck proved a Riemann–Roch type relation with the Chow groups: there exist natural isomorphisms

$$\tau\colon \oplus_i \mathrm{Gr}_i^\gamma K_0 X_{\mathbb{Q}} \xrightarrow{\sim} \oplus_i A_i X_{\mathbb{Q}}$$

[Bo–Se], [SGA 6], [B–F–M], with extension to the non quasi–projective case in [F–G].

For arbitrary j, Gillet's Riemann–Roch theorem [Gi 1] asserts the existence of maps

$$\tau_j\colon K_jX \longrightarrow H_*\big(X, \Gamma(*)\big)_{\mathbb{Q}}$$

compatible with push–forward for projective maps, pull–back for open immersions or l.c.i. maps. Here the right–hand–side denotes any homology theory that satisfies the axioms of [Gi 1, §1], cf. (1.5).1. For $j = 0$, we get back the map τ of [SGA 6] and [B–F–M].

(2.24) **Theorem:** Let X be a variety over k. Then there are natural isomorphisms

$$\tau_j\,:\, K_jX_{\mathbb{Q}} \xrightarrow{\sim} \oplus_i A_i(X, j)_{\mathbb{Q}}\ .$$

In case X is quasi–projective, these isomorphisms are compatible with the γ–filtration:

$$\oplus_i \mathrm{Gr}_i^\gamma K_jX_{\mathbb{Q}} \xrightarrow{\sim} K_jX_{\mathbb{Q}} \xrightarrow{\sim} \oplus_i A_i(X, j)_{\mathbb{Q}}\ .$$

Proof:
Step 1: the quasi–projective case. In case X quasi–projective, the proof of the theorem is essentially the same as [Bl 1, §7—9], with the following simplification in constructing the map τ_j: Taking $\Gamma(i)$ to be the complex of (Zariski) sheaves $\mathcal{Z}^i_{\mathrm{op}}(-,*)$ of chapter 3, we get a cohomology theory $H^*(-, \Gamma(*))$ such that the couple

$$H^*(-,*),\quad A_*(-,*)$$

satisfies all the axioms of [Gi 1, §1], hence Gillet's Riemann–Roch theorem immediately furnishes the map $\tau_\bullet$ into Chow homology.

(This is an advantage of the operational frame–work developed in chapter 3: In Bloch's set–up, where there is no operational cohomology, it is hard to find *contravariant* complexes $\Gamma(i)$; this is why Bloch needs to pass to a field extension L in [Bl 1, §7] to get the necesary contravariant functoriality.)

Once one has a map τ_j, the proof is as in [Bl 1, §8–9]: Using localization, one reduces to the case X smooth affine; one introduces multi–relative Chow and K–theory and defines the γ–filtration on multirelative K–theory; then one constructs a "cycle map" c that's going to be inverse to τ_j. To check that τ_j and c are inverse, one reduces to the case $j = 0$ using the fact that for X smooth affine

$$K_jX \cong K_0(X \times S^j)/K_0X\ ,$$

where S^j is the simplicial j–sphere [D–W], cf. also [Bl 5].

Step 2: Reduction to quasi–projective case. Suppose that the τ_j of Gillet's Riemann–Roch theorem can be extended to a covariant functor on the category of (abstract) varieties with proper morphisms. Then taking an M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X \end{array}$$

with $\widetilde{X}$ quasi–projective (which exists by Chow's lemma), we get a diagram

$$\begin{array}{ccccccccc} \longrightarrow & K_j\widetilde{X}_{\mathbb{Q}} \oplus K_jY_{\mathbb{Q}} & \longrightarrow & K_jX_{\mathbb{Q}} & \longrightarrow & K_{j-1}\widetilde{Y}_{\mathbb{Q}} & \longrightarrow \\ & \big\downarrow{\scriptstyle \tau_j \oplus \tau_j} & & \big\downarrow{\scriptstyle \tau_j} & & \big\downarrow{\scriptstyle \tau_{j-1}} & \\ \longrightarrow & \oplus_i A_i(\widetilde{X}, j)_{\mathbb{Q}} \oplus A_i(Y, j)_{\mathbb{Q}} & \longrightarrow & \oplus_i A_i(X, j)_{\mathbb{Q}} & \longrightarrow & \oplus_i A_i(\widetilde{Y}, j-1)_{\mathbb{Q}} & \longrightarrow \end{array}$$

which we hope can be proven commutative, and we reduce by noetherian induction to the quasi–projective case.

The difficulty is thus to extend the map $\tau_\bullet$ from quasi–projective to arbitrary varieties (This is noted as an open problem in [F–G, §3 Remark 4]). We will actually do this for any homology theory $\Gamma_h(*)$ (for the Zariski topology) that is part of a Poincaré duality theory (1.3). Before doing this, we first need some preparations:

For a given noetherian scheme Z, let $M(Z)$ (resp. $P(Z)$) denote the exact category of coherent $\mathcal{O}_Z$–module sheaves (resp. locally free sheaves), and let

$$\mathbb{K}(Z) = \mathbb{K}(P(Z)), \quad \mathbb{K}'(Z) = \mathbb{K}(M(Z))$$

denote the K–theory spectra (in the sense of [Th 1, Appendix A]) which have the looped classifying space ΩBQP_Z (resp. ΩBQM_Z) as 0–th space. By definition, we get Quillen's K–theory groups as homotopy of these spectra:

$$K^jZ = \pi_j(\mathbb{K}(Z)), \quad K_jZ = \pi_j(\mathbb{K}'(Z)) .$$

These groups are functorial in Z: contravariantly for arbitrary morphisms, resp. covariantly for proper morphisms, but the spectra $\mathbb{K}$ and $\mathbb{K}'$ themselves are not strictly functorial (only up to homotopy). To remedy this problem, several "rigidifications" have been proposed [T–T], [G–S, §5]. We will use here the solution of Gillet and Soulé.

To make $\mathbb{K}$ strictly functorial, Gillet and Soulé follow Street's "second construction" [Str]: They replace $\mathbb{K}$ by the strictly functorial spectrum

$$\mathbb{K}^{\mathrm{big}}(Z) := \mathbb{K}(P^{\mathrm{big}}(Z)) ,$$

where $P^{\mathrm{big}}(Z)$ is the category of locally free sheaves on the big Zariski site over Z, and prove that the map $\mathbb{K}^{\mathrm{big}}(Z) \to \mathbb{K}(Z)$ is a homotopy equivalence [G–S, 5.1.2]. For $Y \subset Z$ closed, we can also define a relative K–theory spectrum

$$\mathbb{K}^{\mathrm{big}}_Y(Z) := \mathrm{Fibre}\big(\mathbb{K}^{\mathrm{big}}(Z) \longrightarrow \mathbb{K}^{\mathrm{big}}(Z \setminus Y)\big) .$$

To get a $\mathbb{K}'$ that is covariantly functorial and has further good properties, Gillet and Soulé [G–S, 5.1.3] further rigidify Thomason's $\mathbb{K}'$ spectrum, that was given by the category of complexes of flasque quasi–coherent sheaves with bounded and coherent cohomology [T–T, §3]. Let's denote the resulting spectrum $\mathbb{K}'$.

(A nice result of Gillet and Soulé is that with these definitions, the projection formula is true on the level of spectra [G–S, 5.1.3]; this improves on [T-T, 3.17], where such a projection formula was established up to homotopy.)

Suppose given a cohomology theory Γ^* satisfying the axioms in [Gi 1, §1], cf. remark (1.5).1. We suppose, in addition to Gillet's axioms, that the associated homology theory can be computed from complexes of Zariski sheaves $\Gamma_h(*)$ that are strictly functorial (as in (1.3)) at least in the quasi–projective case (In our case, this assumption is fulfilled by the complexes $\mathcal{Z}_i^X(*)$, with $\mathcal{Z}^j_{\mathrm{op}}(*)$ of chapter 3 playing the role of cohomology).

Given an arbitrary variety X, there exists a cubical variety $X_* \to X$, with each X_α being embedded as a closed subset in a smooth variety Z_α (Chow's lemma, cf. (1.10)). For each cubical index α, we have Gillet's Riemann–Roch map [Gi 1, §4]

$$\tau_j \,:\, K_j X_\alpha \longrightarrow \amalg_{k\geq 0} H_{k+j}\big(X_\alpha, \Gamma(k)\big) \otimes \mathbb{Q} \;,$$

constructed as follows:

The 0–th space of each spectrum $\mathbb{K}^{\mathrm{big}}_{X_\alpha}(Z_\alpha)$ maps to the 0–th space of $\mathbb{K}_{X_\alpha}(Z_\alpha)$, which is (by definition)

$$\mathrm{Fibre}\big(\Omega BQP_{Z_\alpha} \longrightarrow \Omega BQP_{Z_\alpha \setminus X_\alpha}\big) \;,$$

which maps in its turn to

$$F_\alpha := \mathrm{Fibre}\Big(R\Gamma(Z_\alpha, \Omega B_\bullet QP_{Z_\alpha}) \longrightarrow R\Gamma(Z_\alpha \setminus X_\alpha, \Omega B_\bullet QP_{Z_\alpha\setminus X_\alpha})\Big)$$

(notation is as in [Gi 1] and [B–G]; the $B_\bullet QP$ are simplicial sheaves so F_α is a simplicial set). The crucial observation is that Gillet's chern character $ch_\bullet$ is defined as a map on the level of simplicial sheaves

$$ch_\bullet \,:\, \Omega B_\bullet QP_{Z_\alpha} \longrightarrow \amalg_{k\geq 0} \mathcal{K}(dk, \Gamma(k) \otimes \mathbb{Q})$$

(here $\mathcal{K}()$ denotes the sheaf version of the Dold–Puppe construction, associating to a complex of sheaves a simplicial sheaf [Gi 1, Theorem 1.12]). This map is compatible with open immersions (being a map of sheaves), so induces maps of simplicial sets

$$\begin{aligned} ch_\bullet \,:\, F_\alpha &\longrightarrow \amalg_{k\geq 0} R\Gamma_{X_\alpha}\big(Z_\alpha, \mathcal{K}(dk, \Gamma(k)\otimes\mathbb{Q})\big) \\ &:= \amalg_{k\geq 0} \mathrm{Fibre}\Big(R\Gamma\big(Z_\alpha, \mathcal{K}(dk,\Gamma(k)\otimes\mathbb{Q})\big) \;\to R\Gamma\big(Z_\alpha\setminus X_\alpha, \mathcal{K}(dk,\Gamma(k)\otimes\mathbb{Q})\big)\Big). \end{aligned}$$

Now Z_α being by assumption a smooth variety, the canonical map of spectra

$$\mathbb{K}^{\mathrm{big}}_{X_\alpha}(Z_\alpha) \longrightarrow \mathbb{K}'(X_\alpha)$$

is a homotopy equivalence [Qu 2, §7.1], so $ch_\bullet$ induces a map in $\mathrm{Ho}(\mathcal{S}_*)$, that we will also call $\tau_\bullet$:

$$\tau_\bullet \,:\, \mathbb{K}'(X_\alpha)_0 \longrightarrow \amalg_{k\geq 0} R\Gamma\big(X_\alpha, \mathcal{K}(dk, \Gamma_h^{X_\alpha}(k)\otimes\mathbb{Q})\big) \ ;$$

the map τ_j is now defined as $\pi_j(\tau_\bullet)$. Summarizing, we get τ_j after taking homotopy of the composed arrows in a diagram

$$\begin{array}{ccc}
\mathbb{K}^{\mathrm{big}}_{X_\alpha}(Z_\alpha)_0 & \xrightarrow{ch_\bullet} & \amalg_{k\geq 0} R\Gamma_{X_\alpha}\big(Z_\alpha, \mathcal{K}(dk, \Gamma(k)\otimes\mathbb{Q})\big) \\
\big\downarrow\wr & & \big\downarrow\wr\ \cap[X_\alpha] \\
\mathbb{K}'(X_\alpha)_0 & & \amalg_{k\geq 0} R\Gamma\big(X_\alpha, \mathcal{K}(dk, \Gamma_h^{X_\alpha}(k)\otimes\mathbb{Q})\big)
\end{array}$$

in $\mathrm{Ho}(\mathcal{S}_*)$.

Up till now the cubical index α has been fixed; the next step is to take into account that the various X_α fit together to form a cubical variety. A consequence of Gillet's Riemann–Roch [Gi 1, Theorem 4.1] is that the $\tau_\bullet$ commute with push–forward for projective morphisms, so that the maps for the various cubical indices α fit together to form a map

$$\tau_\bullet \,:\, \mathbb{K}'(X_*)_0 \longrightarrow \amalg_{k\geq 0} R\Gamma\big(X_*, \mathcal{K}(dk, \Gamma_h(k)\otimes\mathbb{Q})\big)$$

in $\mathrm{Ho}(\mathcal{S}_*^{\square^{\mathrm{opp}}})$, the homotopy category of *cubical* pointed simplicial sets.

Consider now the spectrum

$$\mathrm{Hocolim}_\alpha\ \mathbb{K}'(X_\alpha)\ ,$$

which has the simplicial set

$$\mathrm{Hocolim}_\alpha\ \big(\mathbb{K}'(X_\alpha)_0\big)$$

as its 0–th space (for basics on homotopy colimits of spectra and of simplicial sets, cf. (1.20) and the references given there).

Because of descent for K_*–theory ([Gi 2] for simplicial descent; [G–N–P–P, Exposé VI] for cubical descent, which is easier since it follows essentially from Quillen's localization theorem for K_* [Qu 2, §7 Prop. 3.2]), there is a homotopy equivalence of spectra

$$\mathrm{Hocolim}_\alpha\ \mathbb{K}'(X_\alpha) \longrightarrow \mathbb{K}'(X)\ .$$

Likewise, since the homology theory is assumed to satisfy localization, there is a homotopy equivalence of simplicial abelian groups

$$\mathrm{Hocolim}_\alpha\ R\Gamma\big(X_\alpha, \mathcal{K}(dk, \Gamma_h^{X_\alpha}(k))\big) \longrightarrow R\Gamma\big(X, \mathcal{K}(dk, \Gamma_h^{X}(k))\big)$$

for any k.

On the other hand, the functor $\mathrm{Hocolim}\colon \mathcal{S}_*^{\square^{\mathrm{opp}}} \to \mathcal{S}_*$ represents the "total left derived functor" (in the sense of [Qu 1, 4.3])

$$\mathbb{L}\mathrm{Hocolim}\ :\ \mathrm{Ho}(\mathcal{S}_*^{\square^{\mathrm{opp}}}) \longrightarrow \mathrm{Ho}(\mathcal{S}_*)$$

[B–K, Chapter XII 2.4] (According to Thomason the analogous statement holds for the homotopy colimit of spectra [Th 1, 5.32], but we won't need this here). In particular, the map $\tau_\bullet$ in $\mathrm{Ho}(\mathcal{S}_*^{\square^{\mathrm{opp}}})$ constructed above induces a map in $\mathrm{Ho}(\mathcal{S}_*)$:

$$\tau_\bullet : \mathbb{L}\mathrm{Hocolim}_\alpha \ \mathbb{K}'(X_\alpha)_0 \longrightarrow \mathbb{L}\mathrm{Hocolim}_i \ \amalg_{k\geq 0} R\Gamma\big(X_\alpha, \mathcal{K}(dk, \Gamma_h^{X_\alpha}(k)\otimes\mathbb{Q})\big) \ .$$

Taking homotopy, we get the desired maps

$$\tau_j = \pi_j(\tau_\bullet) : K_jX \longrightarrow \amalg_{k\geq 0} H_{k+j}\big(X, \Gamma(k)\big)\otimes\mathbb{Q}$$

(where to compute π_* of the last homotopy colimit, we use the fact that the homotopy spectral sequence for hocolim of *abelian* simplicial groups coincides from E_2 on with the hypercohomology spectral sequence ([B–K, Chapter XII 5.6], Thomason's "Scholium of great enlightenment" [Th 1, 5.32]); alternatively, the identification of π_* of the last homotopy colimit follows from lemma (1.21) and the descent property of homology (1.5).2).

Note that this $\tau_\bullet$ indeed makes the diagram mentioned at the beginning of step 2 commutative; this follows from the observation that $\tau_\bullet$ comes from an underlying map of simplicial sets. This commutative diagram shows in particular that τ_0 coincides with the Riemann–Roch map for general algebraic varieties constructed in [F–G] [Fu 3, §18.3]; indeed, Fulton–Gillet's map is defined by such a diagram.

It is left to check that this $\tau_\bullet$ is well–defined: first, for a given cubical Chow envelope $X_* \to X$, $\tau_\bullet$ does not depend on choice of closed imbeddings $X_* \subset Z_*$. Indeed, this is true for each cubical index α separately on the level of simplicial sheaves, as is proven by Gillet's Riemann–Roch [Gi 1, Theorem 4.1].

Second, to prove that $\tau_\bullet$ does not depend on choice of X_*, it suffices to prove $\tau_\bullet$ is compatible with push–forward for projective morphisms. This may be checked again for each cubical index separately on the level of simplicial sheaves, which is again [Gi 1, Theorem 4.1].

All the other properties of a Riemann–Roch map can be checked likewise. □

(2.25) *Remarks:*

1. As Bloch repeatedly stresses (in particular in [Bl 2]), in view of possible arithmetic applications, it would be very interesting to extend theorem (2.24) to general schemes. Bigraded Chow homology can be defined for separated schemes of finite type over a regular noetherian base scheme, using the notion of dimension of cycles appearing in [Fu 3, Chapter 20]. One still has an isomorphism

$$K_0X_{\mathbb{Q}} \cong \bigoplus_i A_iX_{\mathbb{Q}}$$

for such a scheme X [Fu 3, p. 395], but theorem (2.24) with $j > 0$ is not yet known for such schemes.

The main difficulty is to prove homotopy and localization for the bigraded Chow homology in this generality.

2. This cycle–theoretic interpretation of K–theory is very fruitful. For instance, Suslin and Nesterenko [N–S] and, independently, Totaro [To 1] have proven that for fields k, $A_{-j}(\operatorname{Spec} k, j)$ is *integrally* isomorphic to Milnor K–theory $K_j^M k$, which is more precise than the statement

$$\mathrm{Gr}^{\gamma}_{-j} K_j k_{\mathbb{Q}} \cong K_j^M k_{\mathbb{Q}}$$

proven in [So 2, Théorème 2].

In the same vein, we find as an immediate corollary of (2.24) that for X quasi–projective,

$$\mathrm{Gr}^{\gamma}_i K_j X \otimes \mathbb{Q} = 0 \quad \text{for } i + j < 0$$

(since clearly $z_i(X, j) = 0$ for $i + j < 0$), in particular if X is smooth quasi–projective of dimension n:

$$\mathrm{Gr}^i_{\gamma} K^j X \otimes \mathbb{Q} = 0 \quad \text{for } i > n + j \ .$$

This is non–trivial: Soulé [So 2, Corollaire 1] proves the last vanishing for affine (not necessarily regular) schemes X using stability results of Suslin.

It might moreover be hoped that the isomorphism (2.24) can be used to solve (part of) the Beilinson–Soulé conjecture on the length of the γ–filtration [So 2, 2.9]. In terms of Chow homology, this conjecture reads

$$A_i(X, j)_{\mathbb{Q}} = 0 \quad \text{if } j > 2n - 2i \ ,$$

where $n = \dim X$. As yet this conjecture is unproven for $j > 2$, even for $X = \operatorname{Spec} k$.

3. It is not necessary to discard all the torsion in (2.24); Levine [Le 2] has recently proven that for smooth quasi–projective X of dimension n, the isomorphism (2.24) holds after inverting $(n + j - 1)!$. This generalizes classical results on the relation between $K_0 X$ and $A_* X$ involving factorials [SGA 6], [Fu 3, Example 15.36].

Levine's result gives in particular that for fields k, there are integral isomorphisms

$$K_j k \xrightarrow{\sim} A_{-j}(\operatorname{Spec} k, j) \quad \text{for } j \leq 2,$$

as was also implied by Totaro's result since $K^j k = K^j_M k$ for $j \leq 2$.

4. In the non quasi–projective case, it seems reasonable to *define* gradeds of the γ–filtration on $K_j X_{\mathbb{Q}}$ by the relation (2.24). These gradeds are functorial, and form exact functors on the usual exact sequences for K–theory tensored with $\mathbb{Q}$, which is what we expect from a γ–filtration.

The cycle–theoretic interpretation of K_*–theory (2.24) thus provides a nice way of getting around the difficulty that as yet, one cannot define the γ–filtration on the level of spectra (this is noted as an open problem in [Gra, Section 18]), so that a priori descent methods do not work to extend the γ–filtration to the non quasi–projective case.

(2.26) *B–G–Q spectral sequence.* (cf. [Bl 1, §10]) The complexes $z_i(-, *)$ can be filtered by dimension of support:

$$F_r z_i(X, *) := \{\alpha \in z_i(X, *) \ \mid \ \dim p(\alpha) \leq r, \ \text{for } p\colon X \times \Delta^* \to X\}$$

(Note that since F_r preserves triangles (2.14) and (2.16), the complexes $F_r z_i(X,*)$ are well–defined in the derived category).

Using localization one can prove that there exist quasi–isomorphisms

$$F_r z_i(X,*)/F_{r-1}z_i(X,*) \xrightarrow{\sim} \bigoplus_{x\in X_{(r)}} z_{i-r}(\operatorname{Spec} k(x),*) \ ,$$

where $X_{(r)} := \{x \in X \mid \dim \bar{x} = r\}$. The spectral sequence associated to the filtration F_r is

$$E^1_{pq}(i) = \bigoplus_{x\in X_{(p)}} A_{i-p}(k(x), p+q) \ \Rightarrow \ A_i(X, p+q) \ ,$$

where for a field K, $A_*(K,*)$ is short for $A_*(\operatorname{Spec} K,*)$. (This kind of spectral sequence is sometimes called "Brown–Gersten–Quillen" (or B–G–Q) spectral sequence, after the persons who introduced such a spectral sequence for K–theory [B–G], [Ge 1], [Qu 2]. It was Grothendieck, however, who first wrote down this spectral sequence for De Rham cohomology [Gro 1], [Gro 2].)

The filtration on $A_i(X,p+q)$ induced by the spectral sequence is called the arithmetic, or (co–)niveau filtration. This filtration is studied for a general Poincaré duality theory by Bloch and Ogus in [B–O], and their results are extended in [C–H–K].

Following Gersten, let $\mathcal{R}_{*q}(i)$ be the E^1 complex considered as Zariski sheaves on X:

$$\begin{aligned} \mathcal{R}_{*q}(i)\colon \quad & \bigoplus_{x\in X_{(n)}} (i_x)_* A_{i-n}(k(x), q+n) \longrightarrow \bigoplus_{x\in X_{(n-1)}} (i_x)_* A_{i-n+1}(k(x), q+n-1) \longrightarrow \\ & \cdots \bigoplus_{x\in X_{(-q+1)}} (i_x)_* A_{i+q-1}(k(x), 1) \longrightarrow \bigoplus_{x\in X_{(-q)}} (i_x)_* A_{i+q}(k(x), 0) \end{aligned}$$

(here $n := \dim X$, and $(i_x)_*A$ denotes the sheaf on X which is A in x and 0 outside of x). There is an obvious augmentation of Zariski sheaves

$$\mathcal{A}^X_i(q+n) \longrightarrow \mathcal{R}_{*q}(i) \ ,$$

and the analogue of Gersten's conjecture [Ge 1], [Ge 2] is that for X regular, the complex $\mathcal{R}_{*q}(i)$ is a resolution of the sheaf $\mathcal{A}^X_i(q+n)$. This is proven by Bloch in [Bl 1, §10] for the $A^{\mathrm{Bl}}_*(-,*)$ for X smooth quasi–projective over k, and the same proof also works for the $A_*(-,*)$ (without quasi–projectivity assumption, since this was only needed for localization).

(Alternatively (and less precisely), the fact that after tensoring with $\mathbb{Q}$ the complex $\mathcal{R}_{*q}(i)$ is an exact complex of flasque sheaves follows from Quillen's proof of Gersten's conjecture for K–theory [Qu 2, §7 Prop. 5.14], plus the fact that Gillet's Riemann–Roch map $ch_\bullet \otimes \mathbb{Q}$ is compatible with B–G–Q spectral sequences [Gi 1, Theorem 3.9], plus relation (2.24), plus the fact that the spectral sequence for K–theory can be graded for the γ–filtration [So 2, Théorème 4(iii)].)

As a corollary, we find the following formula for the E^2 terms of the B–G–Q spectral sequence:

$$E^2_{pq}(i) = H^{n-p}\big(X, \mathcal{A}^X_i(q+n)\big) \ ,$$

and the B–G–Q spectral sequence coincides (after a reindexing) from E_2 onwards with the hypercohomology spectral sequence of (2.17). In particular, after the usual identification $E^2_{p,-p}(p) = A_pX$ as in the proof of Bloch's formula [Qu 2, §7 Prop. 5.14], [Ge 1], [Ge 2], we find a new version of Bloch's formula:

$$A_pX = H^{n-p}\big(X, \mathcal{A}^X_p(n-p)\big) \ ,$$

which actually looks better if we grade by codimension as in [Bl 1, §10, Corollary].

(2.27) **Proposition (Bloch formulas):** Let X be a smooth variety of dimension n over k. Let $\mathcal{K}^m$ (resp. $\mathcal{K}^m(l)$ resp. $\mathcal{K}^m_M$) denote the Zariski sheaf on X associated to $U \mapsto K^mU$ (resp. $U \mapsto \mathrm{Gr}^l_\gamma K^mU$ resp. $U \mapsto K^m_MU$).

Then

$$\begin{aligned} A_i(X,j)_{\mathbb{Q}} &= H^{n-i-j}\big(X, \mathcal{A}^X_i(n-i)\big)_{\mathbb{Q}} = H^{n-i-j}\big(X, \mathcal{K}^{n-i}(n-i)\big)_{\mathbb{Q}} \\ &= H^{n-i-j}\big(X, \mathcal{K}^{n-i}\big)_{\mathbb{Q}} = H^{n-i-j}\big(X, \mathcal{K}^{n-i}_M\big)_{\mathbb{Q}} \ , \end{aligned}$$

for $j \leq 2$, where for the last equality it is supposed that k is infinite.

Proof: The B–G–Q spectral sequence for K–theory is known to degenerate modulo torsion for $p+q \leq 2$ [So 2, Théorème 4(iv)]; the compatibility with the corresponding spectral sequence for Chow theory mentioned above allows one to conclude the same degeneration for this last spectral sequence:

$$A_i(X,j)_{\mathbb{Q}} = \bigoplus_{p+q=j} E^2_{pq}(i)_{\mathbb{Q}} \ , \qquad j \leq 2 \ .$$

On the other hand, one easily proves that for $p+q \leq 2$,

$$E^1_{pq}(i) = 0 \text{ if } i \neq -q$$

(this corresponds to the fact that $K_{p+q}k = K^M_{p+q}k$ for $p+q \leq 2$), so

$$A_i(X,j)_{\mathbb{Q}} = E^2_{i+j,-i}(i)_{\mathbb{Q}} = H^{n-i-j}\big(X, \mathcal{A}^X_i(n-i)\big)_{\mathbb{Q}} \ .$$

The second equality of the proposition follows from (2.24): one has an isomorphism of sheaves $\mathcal{A}^X_i(n-i)_{\mathbb{Q}} \cong \mathcal{K}^{n-i}(n-i)_{\mathbb{Q}}$ (there is a shift in the grading since by convention on the γ–filtration the isomorphism $K^mX \xrightarrow{\sim} K_mX$ sends $\mathrm{Gr}^l_\gamma K^mX$ to $\mathrm{Gr}^\gamma_{n-l}K_mX$ [So 2, Théorème 7(vi)]).

For the third equality of the proposition, it suffices to note that

$$H^{n-i-j}\big(X, \mathcal{K}^{n-i}(l)\big)_{\mathbb{Q}} = 0 \quad \text{if } l \neq n-i \text{ and } j \leq 2 \ \ ;$$

this follows from the vanishing of E^1 terms in the Chow spectral sequence mentioned above.

Finally the last equality (for infinite k) follows from the isomorphism of sheaves $\mathcal{K}^m(m)_{\mathbb{Q}} \cong \mathcal{K}^m_M \otimes \mathbb{Q}$ [So 2, Théorème 5(ii)]. $\square$

(2.28) *Remarks:*
1. Proposition (2.27) partially answers the question, posed by Murre [Mur 2, Remark 3.9], of finding a geometric interpretation of $H^p(X, \mathcal{K}^q)$ for $p < q$. The isomorphism

$$A_i X_{\mathbb{Q}} \cong H^{n-i}(X, \mathcal{K}^{n-i}_M)_{\mathbb{Q}}$$

for k infinite is proven by Soulé [So 2, Théorème 5(iii)]. The isomorphism

$$A_i(X, 1) \cong H^{n-i-1}(X, \mathcal{K}^{n-i})$$

(without tensoring by $\mathbb{Q}$!) is proven differently by Müller–Stach [Mul, Section 2.1], and is also stated and explained by Voisin [Voi, 0.5].
2. It would be interesting to prove degeneration of the B–G–Q spectral sequence for Chow theory directly (and maybe it is not necessary to tensor with $\mathbb{Q}$; this would give integral versions of (2.27) ?).

§2.3. Refined Gysin homomorphisms

The goal of this section is to apply Fulton's program for Chow groups [Fu 3, Chapters 1–6], to bigraded Chow homology. This program starts with the three fundamental operations on Chow groups: proper push–forward, flat pull–back, and intersecting with principal divisors. These three operations are then used as the building–blocks in constructing refined Gysin homomorphisms, Chern classes and (in the smooth case) an intersection product.

(2.29) *Intersecting with principal divisors.* Suppose D is a pseudo–divisor [Fu 3, 2.2] on the scheme X, such that the restriction of $\mathcal{O}_X(D)$ to $|D|$ is trivial. Analogous to [Fu 3, 2.3] (mentioned in (2.5)), we want to define homomorphisms

$$h: \quad A_i(X, j) \longrightarrow A_{i-1}(|D|, j) .$$

In view of the quasi–isomorphism (2.19), we get such h from a homomorphism of complexes

$$h\colon\ z_i^{[D]}(X, *) \longrightarrow z_{i-1}(D, *) ,$$

which is defined as follows:

For any generator ψ of $z_i^{[D]}(X,j)$, let g denote the inclusion $\psi \hookrightarrow X\times\Delta^j$. Let $D\times\Delta^j$ be the pseudo–divisor on $X\times\Delta^j$ determined by D. The homomorphism h is now defined as

$$h(\psi) := \begin{cases} [g^*(D\times\Delta^j)], & \text{if } \psi \not\subset |D\times\Delta^j|;\\ 0, & \text{otherwise.}\end{cases}$$

Note that if $\psi \in z_i^{[D]}(X,j)$, then $h(\psi)\in Z_{i-1}(D,j)$ meets faces of codimension at most 2 properly. Also, it follows from (2.7) that h is a homomorphism of complexes $z_i^{[D]}(X,*)\to Z_{i-1}(D,*)$. Taken together, these two facts imply $h\big(z_i^{[D]}(X,j)\big)\subset z_{i-1}(D,j)$, so we get a homomorphism of complexes

$$h\,:\, z_i^{[D]}(X,*) \longrightarrow z_{i-1}(D,*)\ .$$

In case $\alpha\colon D\hookrightarrow X$ is the inclusion of an effective Cartier divisor with trivial normal bundle, the above constructed homomorphism will be denoted α^*, and the same notation will be used for the induced homomorphism on homology

$$\alpha^*\,:\, A_i(X,j)\longrightarrow A_{i-1}(D,j).$$

(2.30) **Proposition:** Let $f\colon X'\to X$ be a morphism, D an effective Cartier divisor on X with trivial normal bundle such that $f(X')\not\subset D$. Let $\alpha\colon D\hookrightarrow X$, $\alpha'\colon f^*D\hookrightarrow X'$ denote the inclusions, and let $f_D\colon f^*D\to D$ denote the morphism induced by f.

(i) If f is proper, the diagram

$$\begin{array}{ccc} z_i^{[f^*D]}(X',j) & \xrightarrow{\alpha'^*} & z_{i-1}(f^*D,j)\\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle (f_D)_*}\\ z_i^{[D]}(X,j) & \xrightarrow{\alpha^*} & z_{i-1}(D,j)\end{array}$$

commutes;

(ii) If f is Chow flat of relative dimension m, the diagram

$$\begin{array}{ccc} z_i^{[D]}(X,j) & \xrightarrow{\alpha^*} & z_{i-1}(D,j)\\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle (f_D)^*}\\ z_{i+m}^{[f^*D]}(X',j) & \xrightarrow{\alpha'^*} & z_{i+m-1}(f^*D,j)\end{array}$$

commutes.

Proof:
(i) This is immediate from commutativity of the corresponding diagram on the cycle level (see (the proof of) [Fu 3, 2.3.c]):

$$\begin{array}{ccc} z_{i+j}(X'\times\Delta^j) & \xrightarrow{(\alpha'\times\mathrm{id})^*} & z_{i+j-1}(f^*D\times\Delta^j)\\ \big\downarrow{\scriptstyle (f\times\mathrm{id})_*} & & \big\downarrow{\scriptstyle (f_D\times\mathrm{id})_*}\\ z_{i+j}(X\times\Delta^j) & \xrightarrow{(\alpha\times\mathrm{id})^*} & z_{i+j-1}(D\times\Delta^j)\ .\end{array}$$

(ii) The proof is similar to that of (i); the corresponding diagram on cycle level is now [Fu 3, 2.3.d]. □

(2.31) *Specialization homomorphisms.* Let $Y \subset X$ be a closed subscheme, and let $C = C_Y X$ be the normal cone to Y in X [Fu 3, Appendix B.6]. Following the example of specialization homomorphisms for Chow groups constructed by Verdier [Ve 4], [Fu 3, §5.2], we want to construct *specialization homomorphisms*

$$\sigma: \quad z_i(X,*) \longrightarrow z_i(C,*) \ .$$

The construction is based on Baum–Fulton–MacPherson's "deformation to the normal cone" [B–F–M], [Ve 4], [Fu 3, Chapter 5]. That is, if $\alpha: C \hookrightarrow M^\circ$ denotes the inclusion of C in the deformation space M° (notation as in [Fu 3, §5.1]), and ϕ denotes the inclusion of the complement to C, which is isomorphic to $X \times \mathbb{A}^1$, in M°, there is a diagram

$$\begin{array}{ccc} z_{i+1}^{[C]}(M^\circ,*)/z_{i+1}(C,*) & \xrightarrow{\phi^*} & z_{i+1}(X \times \mathbb{A}^1,*) \\ \downarrow{\scriptstyle \alpha^*} & & \uparrow{\scriptstyle \mathrm{pr}^*} \\ z_i(C,*) & \leftarrow - - & z_i(X,*) \end{array}$$

(Compare the similar diagram in [Fu 3, §5.2]). The map ϕ^* is a quasi–isomorphism by (2.21)(ii); the map α^* is as in (2.29) (note that C is a principal divisor on one of its neighborhoods in M°, so the normal bundle is indeed trivial; note also that since the composite $\alpha^*\alpha_*$ is 0 by definition of α^*, the map α^* passes to the indicated cokernel).

The diagram defines a homomorphism of complexes in $D^+(Ab)$ (the dotted arrow) from (the class of) $z_i(X*)$ to (the class of) $z_i(C,*)$; the induced homomorphism on homology will be denoted

$$\sigma : A_i(X,j) \longrightarrow A_i(C,j) \ .$$

(2.32) *Refined Gysin homomorphisms.* Let $k: Y \hookrightarrow X$ be a regular embedding of codimension d, and let $f: X' \to X$ be any morphism. Form the fibre square

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{k} & X \ . \end{array}$$

As in [Fu 3, 6.2], we want to define homomorphisms

$$k^! : \quad z_i(X',j) \to A_{i-d}(Y',j) \ .$$

Denote by $C' = C_{Y'}X'$ the normal cone to Y' in X'; it is a closed subcone of $N' := f^*N_Y X$. Following Fulton, we define $k^!$ as the composite

$$z_i(X',j) \xrightarrow{\sigma} z_i(C',j) \longrightarrow A_i(N',j) \xrightarrow{s^*} A_{i-d}(Y',j)$$

where σ is the specialization homomorphism of (2.31) which is well–defined in $D^+(Ab)$, the second map is induced by the inclusion of C' in N', and s^* is the inverse of the pull–back isomorphism $p^*: A_{i-d}(Y',j) \to A_i(N',j)$ (2.22)(iii).

Since σ and proper push–forward are homomorphisms in the derived category, $k^!$ is well–defined on homology; the induced homomorphisms

$$k^! : \quad A_i(X',j) \longrightarrow A_{i-d}(Y',j)$$

will be called *refined Gysin homomorphisms.*

Remark (for later use) that with the notation of (1.12), the above construction in fact gives a well–defined homomorphism from a certain complex in $\{z_i(X',*)\}$ (namely, the complex $z_{i+1}^{[C]}(M^\circ,*)/z_{i+1}(C,*)$) to a certain complex in $\{z_{i-d}(Y',*)\}$ (namely, the complex of (2.9) if Y' is quasi–projective, the single complex as in definition (2.20) for a fixed cubical Chow envelope of Y' otherwise). Then (1.14) there is also a set $\{k^!\}$ of homomorphisms of complexes, all inducing the same map on homology.

The following properties of $k^!$ are the obvious extension of [Fu 3, Theorem 6.2]:

(2.33) **Proposition:** Consider a fibre diagram

$$\begin{array}{ccc} Y'' & \longrightarrow & X'' \\ \big\downarrow{\scriptstyle g_Y} & & \big\downarrow{\scriptstyle g} \\ Y' & \xrightarrow{k'} & X' \\ \big\downarrow{\scriptstyle f_Y} & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{k} & X \end{array}$$

with k a regular embedding of codimension d.

(i) If g is proper, the diagram

$$\begin{array}{ccc} A_i(X'',j) & \xrightarrow{k^!} & A_{i-d}(Y'',j) \\ \big\downarrow{\scriptstyle g_*} & & \big\downarrow{\scriptstyle (g_Y)_*} \\ A_i(X',j) & \xrightarrow{k^!} & A_{i-d}(Y',j) \end{array}$$

commutes;

(ii) If g is Chow flat of relative dimension m, the diagram

$$\begin{array}{ccc} A_i(X',j) & \xrightarrow{k^!} & A_{i-d}(Y',j) \\ \big\downarrow{\scriptstyle g^*} & & \big\downarrow{\scriptstyle g_Y^*} \\ A_{i+m}(X'',j) & \xrightarrow{k^!} & A_{i+m-d}(Y'',j) \end{array}$$

commutes;

(iii) If k' is also a regular embedding of codimension d, and $\alpha \in z_i(X'',j)$, then

$$k^!\alpha = k'^!\alpha \qquad \in A_{i-d}(Y'',j)\,.$$

Proof:
(i) This will be checked using the definition of $k^!$ as a composite. Let C', $'M^\circ$ (resp. C'' and $''M^\circ$) denote the normal cone and the deformation space associated to the inclusion $Y' \longrightarrow X'$ (resp. $Y'' \longrightarrow X''$).

The morphism g induces a morphism

$$g_M\colon \quad ''M^\circ \longrightarrow {'M^\circ} \times_{(X' \times \mathbb{P}^1)} (X'' \times \mathbb{P}^1) \longrightarrow {'M^\circ} \ ,$$

proper when g is proper. Since g_M commutes with the projections to $\mathbb{P}^1$, the pull–back $(g_M)^* C'$ equals C''.

One knows from (2.21)(i) and (2.30)(i) that the diagram defining σ

$$\begin{array}{ccc} z_{i+1}^{[C'']}(''M^\circ, j)/z_{i+1}(C'', j) & \longrightarrow & z_{i+1}(X'' \times \mathbb{P}^1, j) \\ \downarrow & & \uparrow \\ z_i((g_M)^* C', j) & \leftarrow - - & z_i(X'', j) \end{array}$$

commutes up to homotopy with the similar diagram for $C' \hookrightarrow {'M^\circ}$.

Therefore the diagram

$$\begin{array}{ccc} A_i(X'', j) & \xrightarrow{\sigma} & A_i(C'', j) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle (g_Y)_*} \\ A_i(X', j) & \xrightarrow{\sigma} & A_i(C', j) \end{array}$$

commutes, and one concludes by functoriality of push–forward and compatibility of push–forward and pull–back (2.21)(i).
(ii) Define

$$\begin{aligned} g^* C' &:= C' \times_{X'} X'' \ ; \\ g^*({'M^\circ}) &:= {'M^\circ} \times_{(X' \times \mathbb{P}^1)} (X'' \times \mathbb{P}^1) \ ; \end{aligned}$$

then C'' and $''M^\circ$ are closed subschemes of $g^* C'$ resp. $g^*({'M^\circ})$. Let $g_{C'}$ denote the induced morphism $g^* C' \longrightarrow C'$, flat when g is flat. As in the proof of (i) above, one finds using (2.30)(ii) that the diagram

$$\begin{array}{ccc} A_i(X', j) & \xrightarrow{\sigma} & A_i(C', j) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle (g_{C'})^*} \\ A_{i+m}(X'', j) & \xrightarrow{\sigma_g} & A_{i+m}(g^* C', j) \end{array}$$

commutes up to homotopy, where σ_g is defined by a diagram

$$\begin{array}{ccc} z_{i+1}^{[g^* C']}(g^*({'M^\circ}), *)/z_{i+1}(g^* C', *) & \longrightarrow & z_{i+1}(X'' \times \mathbb{A}^1, *) \\ \downarrow & & \uparrow \\ z_i(g^* C', *) & \leftarrow - - & z_i(X'', *) \ . \end{array}$$

Comparing this diagram to the one defining σ, one finds that σ_g factorizes as

$$z_i(X'',*)\xrightarrow{\sigma}z_i(C'',*)\longrightarrow z_i(g^*C',*)\ ,$$

where the second map is proper push–forward.

Note that g^*C' is a closed subcone of $N'' := (f\circ g)^*N_Y X$; therefore by functoriality of push–forward one finds that

$$\begin{array}{ccccc} A_i(X',j) & \xrightarrow{\sigma} & A_i(C',j) & \longrightarrow & A_i(f^*N_Y X,j) \\ \downarrow{\scriptstyle g^*} & & & & \downarrow \\ A_{i+m}(X'',j) & \xrightarrow{\sigma} & A_{i+m}(C'',j) & \longrightarrow & A_{i+m}(N'',j) \end{array}$$

commutes. The conclusion now follows from (2.21)(i).
(iii) Just as [Fu 3, Theorem 6.2.c], this follows from the observation that $N_{Y'}X'$ equals $f^*N_Y X$. $\square$

(2.34) *Exterior product.* Let X, Y be two varieties. For later use, we note that there is a map in the derived category

$$s\big(z_i(X,*)\otimes z_k(Y,*)\big)\longrightarrow z_{i+k}(X\times Y,*)$$

($s :=$ associated single complex). (Using definition (2.20), we reduce to the case X and Y quasi–projective. Then we use either Bloch's construction of this map [Bl 1, §5], or we avoid Bloch's triangulations by using the cubical definition of higher Chow groups à la Levine [Le 1].)

As a corollary, we get an exterior product

$$A_i(X,j)\otimes A_k(Y,l)\xrightarrow{\times}A_{i+j}(X\times Y,j+l)\ .$$

This exterior product is compatible with push–forward, pull–back and refined Gysin homomorphisms (the proofs being the same as their analogues on cycle level [Fu 3, 1.10 and Example 6.5.2]).

(2.35) **Corollary:** If X is a smooth variety of dimension n, there is a product structure

$$A_i(X,j)\otimes A_k(X,l)\longrightarrow A_{i+k-n}(X,j+l)\ ,$$

compatible with push–forward, pull–back and making $A_*(X,*)$ an associative, commutative ring.

Proof: The diagonal $\Delta\colon X\to X\times X$ is a regular morphism, and one defines

$$\alpha\otimes\beta\mapsto\alpha\cdot\beta:=\Delta^!(\alpha\times\beta)$$

(compare [Fu 3, Chapter 8]). $\square$

§2.4. Motivic homology

(2.36) For what follows, we need to know that the homology theory H_* can be computed from a complex of abelian groups. For any scheme M and a fixed $i \in \mathbb{Z}$, let $C^*_M(i)$ denote the bounded complex of abelian groups, unique up to quasi–isomorphism, such that

$$H_j\big(M, \Gamma(i)\big) = H^{-j}(C^*_M(i))$$

(for instance if $H_j M = H^{\mathrm{et}}_j(M_{\bar{k}}, \mathbb{Z}_\ell(i))$ is geometric étale homology, we can take for C^*_M the complex of sections of an injective resolution of the dualizing complex $R\pi^!\mathbb{Z}_\ell^{\otimes -i}$ in the étale topology [B–O], [La]. Likewise, for singular homology with $\mathbb{Z}$ coefficients on a complex variety, we can take for C^*_M sections of an injective resolution of $R\pi^!\mathbb{Z}$ in the classical topology [Ve 1]).

Suppose now that homology is either singular or geometric étale. Analogous to (2.31), for any closed inclusion $Y \hookrightarrow X$ with normal cone $C = C_Y X$ one has specialization homomorphisms

$$\sigma\,:\; C^*_X \longrightarrow C^*_C$$

[Ve 4, §8]. Note furthermore that for any rank r vector–bundle $E \to M$, there is a quasi–isomorphism

$$C^*_M \xrightarrow{\sim} C^{*-2r}_E$$

(homotopy), and that a proper morphism $f\colon M \to N$ induces a push–forward homomorphism $f_! C^*_M \longrightarrow C^*_N$. Now we can repeat the normal cone construction (2.32), to obtain for each fibre square

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow f \\ Y & \xrightarrow{k} & X \end{array}$$

with k a codimension d regular embedding, refined Gysin homomorphisms

$$k^!\,:\; C^*_{X'} \longrightarrow C^{*+2d}_{Y'}\ .$$

The same notation is used also on homology level:

$$k^!\,:\; H_i X' \longrightarrow H_{i-2d} Y'\ .$$

The analogue of proposition (2.33)(i) and (ii), with A_* replaced by H_* follows from [Ve 4, 8.5 and 8.7].

(2.37) *Cycle class maps.* Let homology be singular or geometric étale as in (2.36). We know there exist cycle class maps

$$cl_i\,:\; A_i M \longrightarrow H_{2i}\big(M, \Gamma(i)\big)\,,$$

compatible with proper push–forward and flat pull–back.

It follows from [Ve 4, 8.10] that the cycle class maps are also compatible with refined Gysin homomorphisms, i.e. for each fibre square as in (2.32) the diagram

$$\begin{array}{ccc} A_iX' & \xrightarrow{k^!} & A_{i-d}Y' \\ \downarrow{\scriptstyle cl_i} & & \downarrow{\scriptstyle cl_i} \\ H_{2i}\big(X',\Gamma(i)\big) & \xrightarrow{k^!} & H_{2i-2d}\big(Y',\Gamma(i-d)\big) \end{array}$$

commutes.

More generally, for any homology theory (1.3) there exists a cycle class map

$$\mathrm{cl}_i\,:\; z_iM \longrightarrow H_{2i}\big(M,\Gamma(i)\big)\ ,$$

as follows from the existence of a fundamental class.

(2.38) **Proposition** *(higher cycle class maps)*: Let X be a variety, and let $H_*\big(-,\Gamma(*)\big)$ be singular, geometric étale or Deligne homology. Then there exist homomorphisms

$$cl_i(j)\,:\; A_i(X,j) \longrightarrow H_{2i+j}\big(X,\Gamma(i)\big)\ ,$$

for all $(i,j)\in\mathbb{Z}\times\mathbb{Z}_{\geq 0}$, satisfying

(i) $cl_i(0)=cl_i$;
(ii) $cl_i(j)$ is compatible with proper push–forward and pull–back for open immersions (if H_* is singular or étale homology, $\mathrm{cl}_i(j)$ is even compatible with refined Gysin homomorphisms);
(iii) for any $Y\hookrightarrow X$ closed with complement U, one has a commutative diagram with exact rows

$$\begin{array}{cccccccc} A_i(X,j) & \longrightarrow & A_i(U,j) & \longrightarrow & A_i(Y,j-1) & \longrightarrow & \\ \downarrow{\scriptstyle cl_i(j)} & & \downarrow{\scriptstyle cl_i(j)} & & \downarrow{\scriptstyle cl_i(j-1)} & & \\ H_{2i+j}\big(X,\Gamma(i)\big) & \longrightarrow & H_{2i+j}\big(U,\Gamma(i)\big) & \longrightarrow & H_{2i+j-1}\big(Y,\Gamma(i)\big) & \longrightarrow & . \end{array}$$

Proof: The proof is an obvious extension of a construction of Bloch ([Bl 2, ¶4], also explained in [D–S, 2.8]), who supposing X smooth quasi–projective, constructed maps to cohomology instead of homology. The construction involves two (third quadrant cohomological) double complexes $'K^{p,q}$ and $K^{p,q}$, giving rise to two spectral sequences $'E$ and E.

If H_* is not singular or étale homology, we first need to construct certain Gysin homomorphisms in homology. For a given variety M, let C^*_M denote a complex of abelian groups such that

$$H_j\big(M,\Gamma(i)\big)=H^{-i}(C^*_M)$$

for a fixed i, and such that C^* is contravariantly functorial for the face maps

$$k\,:\; X\times\Delta^m\hookrightarrow X\times\Delta^{m+1}\ .$$

For X smooth, we can take C^* to be the complex of sections of a resolution of the complex of currents. For X singular, we choose a cubical hyperresolution $X_* \to X$, and take for C^* the single complex associated to the cubical complex (this works since Deligne homology is defined by descent, in particular the complex C^* is independent up to quasi–isomorphism of choice of X_*). We will also need Gysin homomorphisms for the inclusion

$$Z \cap (X \times \Delta^m) \subset Z \ ,$$

where $Z \in z_i(X, m+1)$; for this we take C^*_Z to be the complex of $X \times \Delta^{m+1}$ (as defined above) with support in Z.

We fix an integer $i \in \mathbb{Z}$. The first double complex is defined as

$${}'K^{p,*} := \lim_{\substack{\leftarrow \\ Z \in z_i(X,-p)}} C_Z^{*+2p} \ ,$$

the coboundary $d: {}'K^{p,q} \to {}'K^{p+1,q}$ being the alternating sum of

$$k^!\colon \ C_Z^{q+2p} \to C_{Z\cap F}^{q+2p+2}$$

over all codimension one faces $k: F \hookrightarrow X \times \Delta^{-p}$.

One has an associated spectral sequence

$${}'E_1^{p,q} = \lim_{\substack{\leftarrow \\ Z \in z_i(X,-p)}} H_{-q-2p}\big(Z, \Gamma(i)\big) \Rightarrow \ ?? \ .$$

To assure convergence, we cut off the complex at some large even p, i.e. we set ${}'K^{p,q} = 0$ for $p < 2N << 0$. Now we can write

$${}'E_1^{p,q} \ \Rightarrow \ {}'E^{p+q} \ .$$

The second double complex is defined as

$$K^{p,*} := \begin{cases} C^{*+2p}_{X\times\Delta^{-p}}, & \text{if } 2N \le p \le 0; \\ 0, & \text{otherwise,} \end{cases}$$

the coboundary $d: K^{p,q} \to K^{p+1,q}$ again being the alternating sum of

$$k^! \ : \ C^{q+2p}_{X\times\Delta^{-p}} \longrightarrow C_F^{q+2p+2}$$

over all codimension one faces $k: F \hookrightarrow X \times \Delta^{-p}$.

There is an associated spectral sequence

$$E_1^{p,q} = H_{-q-2p}\big(X \times \Delta^{-p}, \Gamma(i)\big) \ \Rightarrow \ E^{p+q} \ .$$

By the homotopy property for homology, the maps $d_1^{p,q}$ in this spectral sequence are either 0 or the identity, depending on whether p is even or odd. It follows that $E_2^{p,q} = 0$ for $p \neq 0$, and $E_2^{0,q} = H_{-q}\big(X, \Gamma(i)\big)$. Therefore one has

$$E_1^{p,q} \;\Rightarrow\; E^{p+q} = H_{-q-p}\big(X, \Gamma(i)\big)\ .$$

Note that proper push–forward induces a map of double complexes $'K^{*,*} \to K^{*,*}$, hence also a map of spectral sequences $'E_n \to E_n$.

By (2.37), the cycle class map induces a homomorphism of (cohomological) complexes

$$\sigma_{\geq 2N}\, z_i(X, -*) \longrightarrow \big({}'E_1^{*,-2i}, d_1\big)$$

($\sigma_{\geq 2N}$ means cut off at degree $2N$), hence a homomorphism

$$A_i(X, j) \longrightarrow {}'E_2^{-j,-2i}$$

for $j < -2N$. Since one clearly has $'E_r^{-j+r,-2i-r+1} = 0$ for $r > 1$ (this follows from the vanishing $H_i M = 0$ if $i > 2 \cdot \dim M$), the term $'E_2^{-j,-2i}$ maps to the limit $'E^{-j-2i}$, which maps in its turn to $E^{-j-2i} = H_{2i+j}X$.

The composition of these maps will be denoted

$$cl_i(j) :\ A_i(X, j) \longrightarrow H_{2i+j}X\ .$$

The enumerated properties are easy to prove.

(For instance, to prove the important property (iii), suppose the complexes C_M^* are chosen in such a way that $C_M^i = 0$ for $i < -2 \cdot \dim M$, so $'K^{*,q} = 0$ for $q < -2i$. In that case there is a commutative diagram of complexes

$$\begin{array}{ccccc}
\sigma_{\geq 2N} z_i(Y, -*) & \longrightarrow & \sigma_{\geq 2N} z_i(X, -*) & \longrightarrow & \sigma_{\geq 2N} z_i(U, -*) \\
\downarrow & & \downarrow & & \downarrow \\
{}'E_1^{*,-2i}(Y) & \longrightarrow & {}'E_1^{*,-2i}(X) & \longrightarrow & {}'E_1^{*,-2i}(U) \\
\downarrow & & \downarrow & & \downarrow \\
{}'K^{*,-2i}(Y) & \longrightarrow & {}'K^{*,-2i}(X) & \longrightarrow & {}'K^{*,-2i}(U) \\
\downarrow & & \downarrow & & \downarrow \\
s'K^{**}(Y) & \longrightarrow & s'K^{**}(X) & \longrightarrow & s'K^{**}(U) \\
\downarrow & & \downarrow & & \downarrow \\
sK^{**}(Y) & \longrightarrow & sK^{**}(X) & \longrightarrow & sK^{**}(U)\ ;
\end{array}$$

all lines form triangles in $D(Ab)$, and the diagram (iii) is obtained after taking cohomology.)
□

(2.39) *Remarks:*
1. Clearly the above proof also works for the higher Chow groups $A_i^{\mathrm{Bl}}(-, j)$. In fact, a number of more direct constructions of higher cycle class maps from $A_i^{\mathrm{Bl}}(-, j)$ to Deligne

homology have been proposed [Go] [Mul, Section 2.3] [Voi, 0.5], but (as far as I know) only for smooth varieties. It would be interesting to extend these explicit constructions to the singular case.

2. Bloch [Bl 2] has used the maps $cl_i(j)$ to give a reformulation of the Beilinson conjectures (this reformulation appears more extensively in [D–S]):

Let X be a smooth projective variety of dimension n over a number field k. Define

$$H^i_{\mathcal{D}}\big(X_{\mathbb{R}}, \mathbb{R}(j)\big) := \Pi_{\substack{w:\,\text{complex}\\ \text{place of } k}} H^i_{\mathcal{D}}\big(X(\mathbb{C}), \mathbb{R}(j)\big) \times \Pi_{\substack{w:\,\text{real}\\ \text{place of } k}} H^i_{\mathcal{D}}\big(X(\mathbb{C}), \mathbb{R}(j)\big)^+ \ ,$$

where $H^i_{\mathcal{D}}$ is Deligne cohomology, and $+$ denotes invariant under real conjugation. The above gives cycle class maps

$$c_{i,j} \,:\, A_{n-i}(X, j) \longrightarrow H^{2i-j}_{\mathcal{D}}\big(X_{\mathbb{R}}, \mathbb{R}(i)\big) \ .$$

Assuming there exists a global regular model $\mathcal{X}$ of X over the ring of integers of k, one can also consider the composition

$$r = r_{i,j} \,:\, A_{n-i}(\mathcal{X}, j) \longrightarrow A_{n-i}(X, j) \xrightarrow{c_{i,j}} H^{2i-j}_{\mathcal{D}}\big(X_{\mathbb{R}}, \mathbb{R}(i)\big) \ ;$$

this should be a *regulator* in the sense of [Be].

The conjecture is that for $j \geq 2$:

(i) $r\big(A_i(\mathcal{X}, j)\big) \subset H^{2i-j}_{\mathcal{D}}\big(X_{\mathbb{R}}, \mathbb{R}(i)\big)$ is a $\mathbb{Q}$–lattice of maximal rank;

(ii)
$$\mathrm{vol}\Big(r\big(A_i(\mathcal{X}, j)\big)\Big) = q \cdot \lim_{s \to i-j} s^{-d_{i,j}} L^{(2i-j-1)}(X, s) \ , \qquad q \in \mathbb{Q}^* \ ,$$

where $L^{(m)}(X, s)$ is the global L–function associated to the $\mathrm{Gal}(\bar{k}/k)$–representation on $H^m_{\mathrm{et}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ [Ra, §5.5], and $d_{i,j} := \mathrm{ord}_{s=i-j} L^{(2i-j-1)}(X, s)$.

Usually, Beilinson's conjectures ([Be, §5], [So 1], [Schn], [Ra, §6.4], [D–S]) are formulated using graded pieces $\mathrm{Gr}^\gamma_* K_* \otimes \mathbb{Q}$ instead of $A_*(-,*)$, and with Gillet's Chern classes [Gi 1] from K_* into $H^*_{\mathcal{D}}$ instead of the maps $cl_i(j)$. To get compatibility between this form of the conjecture and the above reformulation in terms of $A_*(-,*)$, we have to assume that the diagram

$$\begin{tikzcd} K_j X \arrow[r, "ch_j"] \arrow[dr, "ch_j"'] & \Pi_i A_i(X, j)_{\mathbb{Q}} \arrow[d, "\Pi_i cl_i(j)"] \\ & \Pi_i H^{2n-2i-j}_{\mathcal{D}}\big(X, \mathbb{R}(n-i)\big) \end{tikzcd}$$

commutes (where ch_j is Gillet's Chern character); however, I cannot prove this.

3. Unfortunately, the construction used in (2.38) does not produce maps into graded pieces of K–theory

$$A_i(X, j) \longrightarrow \mathrm{Gr}^?_i K_j X \ ,$$

where ? is some unknown filtration coinciding rationally with the γ–filtration, and which is the topological filtration for $j = 0$. The obstructions are first that the filtration should already be defined on the level of spectra, and second that one needs a vanishing for graded pieces analogous to $H_i M = 0$ for $i > 2 \cdot \dim M$ (this seems difficult, since it would imply the Beilinson–Soulé conjecture).

(2.40) *Chow homology with coefficients.* For any ring R, one can define bigraded Chow homology with coefficients in R: One defines

$$A_i(X, j; R) := H_j\big(z_i(X, *) \otimes R\big) \ .$$

Since tensoring with R is right exact, one has $A_i(X, 0; R) \cong A_i X \otimes R$.

(2.41) *Motivic homology.* Let X be a variety of dimension n. It is conjectured by Beilinson that there exist for each integer $r \leq n$, objects $\mathbb{Z}_h(r)$ in the derived category of sheaves of abelian groups on the small Zariski site of X, satisfying (at least):

(1) $\mathbb{Z}_h(n) = \mathbb{Z}[-2n]$ (the constant sheaf $\mathbb{Z}$ in degree $-2n$);
(2) If X is regular, $\mathbb{Z}_h(n-1) = \mathcal{O}^*[1-2n]$; more generally, the cohomology sheaf $\mathcal{H}^{q-2n}\big(\mathbb{Z}_h(n-q)\big)$ coincides with the sheaf of Milnor K–theory $\mathcal{K}^q_M$;
(3) *Vanishing.* For $r < n$, the complex $\mathbb{Z}_h(r)$ has cohomology sheaves concentrated in $[1-2n, r-2n]$;
(4) *Relation to K–theory.* There exists a spectral sequence

$$E_2^{pq} = H^{\mathcal{M}}_{-p+q}\big(X, \mathbb{Z}(q)\big) \ \Rightarrow \ K_{-p-q}(X) \ ,$$

degenerate at E_2 after tensoring with $\mathbb{Q}$. The induced filtration on the limit coincides (at least after tensoring with $\mathbb{Q}$) with the γ–filtration. Here the "motivic homology" is defined as

$$H^{\mathcal{M}}_i\big(X, \mathbb{Z}(j)\big) := \mathbb{H}^{-i}\big(X, \mathbb{Z}_h(j)\big) \ ;$$

(5) *Relation with Milnor K–theory.* If X is the spectrum of a regular local ring of dimension n, there is an isomorphism

$$H^{\mathcal{M}}_{2n-j}\big(X, \mathbb{Z}(n-j)\big) \cong K^j_M X \ ,$$

where K^*_M denotes Milnor K–theory;
(6) *Relation with étale homology.* There is a functorial quasi–isomorphism

$$\mathbb{Z}_h(r) \otimes^{\mathbb{L}} \mathbb{Z}/\ell \cong \tau_{\leq r-2n} R\pi_* Rs^!(\mu_\ell^{\otimes n-r}) \ ,$$

where $\tau_{\leq p}$ is what is called the "canonical filtration" in [De 1], π is the canonical morphism from the étale to the Zariski site, and $s\colon X \to \operatorname{Spec} k$ is the structure morphism.

(2.42) *Remarks, and proposed solutions to (2.41).*
1. Bloch [Bl 1, Introduction] [Bl 2] has conjectured that the complexes $\mathcal{Z}_r^X(*)[-2r]$ (2.17), (2.21)(iv) have all the properties of $\mathbb{Z}_h(r)$; this would justify thinking of bigraded Chow homology as motivic homology, after the reindexing

$$H_i^{\mathcal{M}}(X,\mathbb{Z}(j)) = A_j(X, i-2j)\ .$$

The vanishing property is still unproven (and the non–trivial part, i.e.

$$H_i^{\mathcal{M}}(X,\mathbb{Z}(j)) = 0 \ \text{ for } i > 2{\cdot}\dim X$$

corresponds to the Beilinson–Soulé conjecture mentioned in (2.25).2); the spectral sequence to K–theory exists at least for $X = \operatorname{Spec} k$ by work of Bloch and Lichtenbaum [B–L]; the relation with Milnor K–theory is also true for $X = \operatorname{Spec} k$ [N–S] [To 1] (and more generally, for X the spectrum of a regular local ring it is true modulo torsion as follows from [So 2, Théorème 5(ii)] and (2.24)); the relation with étale homology has been proven recently by Suslin and Voevodsky [Su], [S–V], [Voe], at least for X smooth.

There are several other candidates for motivic complexes; see [B–M–S] and [Su] for nice overviews. There is for instance the "Grassmanian homology" of [B–M–S, Chapter 1], which has the merit of being closely connected (through polylogarithms) with the arithmetic conjectures (2.39).2. The Grassmanian homology is a lot easier to handle than bigraded Chow homology (to quote [B–M–S, Introduction]: "The cochains for higher Chow groups are of the complexity of all algebraic subvarieties of projective space, rather than all linear ones"), but alas it is now known that Grassmanian homology does not have the desired relation with higher K–theory [Gerd].

2. Usually, the conjecture is stated for regular schemes only, and in terms of *co*homology complexes $\mathbb{Z}(j)$; the above is the natural homology version, with indexing chosen in accordance with Poincaré duality theories (1.3). For instance, in the regular case, Poincaré duality should be a quasi–isomorphism

$$\mathbb{Z}(j) \xrightarrow{\sim} \mathbb{Z}_h(n-j)[-2n]\ .$$

3. Lichtenbaum [Li 1] [Li 4] has made a similar conjecture for sheaves on the étale site; in the quasi–projective case the $z_i(X,j)$ are étale presheaves, and the conjecture [Bl 1, Introduction] [Bl 2] is then that the sheafified complexes for the étale topology satisfy Lichtenbaum's axioms.

Lichtenbaum [Li 2] [Li 3] has also constructed a candidate $\mathbb{Z}^{\text{et}}(2)$ for the first non–trivial complex, the so–called *étale complex of weight two motivic cohomology*. The conjectural relation

$$\mathbb{Z}(j) = \tau_{\leq j} R\alpha_* \mathbb{Z}^{\text{et}}(j)$$

for going from étale to Zariski complexes [Li 1, §5], where α is the canonical morphism of sites, also gives a candidate for the Zariski complex $\mathbb{Z}(2)$. For X smooth, the hypercohomology of this complex $\mathbb{Z}(2)$ has been calculated [Li 3] [Ka, Theorem 1.6]; part of the calculation is that

$$\mathbb{H}^i(X,\mathbb{Z}(2)) = H^{i-2}_{\text{Zar}}(X,\mathcal{K}^2) \ \text{ if } 2 \leq i \leq 4,$$

which nicely agrees with proposition (2.27) (and suggests that (2.27) should already hold without tensoring by $\mathbb{Q}$).

4. Of course, property (4) of (2.41) was motivated by the Atiyah–Hirzebruch spectral sequence relating singular cohomology and topological K–theory [A–H, Section 2].

Thomason [Th 1, 2.48, 4.15–16] has given important evidence for conjecture (2.41) by constructing a spectral sequence relating étale homology and a "localized" form of K_* with finite coefficients (the existence of such a spectral sequence, preferably for the non–localized K–theory, is already an old problem in algebraic K–theory, mentioned in [Qu 3, Section 9]); something like this (but without the localization) is also predicted by the conjunction of (2.41)(4) and (5).

Building on his work [Th 1], Thomason has moreover proposed what is (I think) the only non–cycle–theoretic candidate for the motivic cohomology complexes $\mathbb{Z}(r)$ [Th 2, §11].

5. Guided by (2.41)(4) one sometimes takes graded pieces of K–theory as a *definition* of motivic (co)homology, that is one writes

$$H^i_{\mathcal{M}}\big(X, \mathbb{Q}(j)\big) := \mathrm{Gr}^j_\gamma K^{2j-i}(X) \otimes \mathbb{Q} \; ;$$
$$H_i^{\mathcal{M}}\big(X, \mathbb{Q}(j)\big) := \mathrm{Gr}_j^\gamma K_{i-2j}(X) \otimes \mathbb{Q} \; .$$

Though of course these groups are not known to have all the properties that (2.41) predicts (for instance, the vanishing), Soulé has proven at least that this gives a Poincaré duality theory in the sense of [B–O] (but not in the sense of (1.3) !), when restricted to smooth quasi–projective schemes [So 2, Théorème 9].

An important advantage of bigraded Chow homology with respect to these gradeds of K–theory is that the first theory has a natural *integral* structure; this is precisely the advantage that Grothendieck already stressed when comparing A_* and $\mathrm{Gr}^\gamma_* K_0 \otimes \mathbb{Q}$ [SGA 6, Exposé 0].

Chapter 3: An operational theory

Starting with the usual Chow groups, one can introduce—as above, following Bloch—a second grading, to get bigraded Chow homology. Starting with the usual Chow groups, one can also define—as Fulton and MacPherson did—operational Chow cohomology.

The bigraded operational theory $A^*_{\mathrm{op}}(-,*)$ defined in this chapter, is an obvious extrapolation of these two constructions: an element in $A^i_{\mathrm{op}}(X,j)$ is a collection of operations

$$A_*(X',*) \longrightarrow A_{*-i}(X',*+j)$$

for every X' mapping to X, satisfying certain compatibility conditions.

As is to be expected, $A^*_{\mathrm{op}}(-,*)$ is a (bigraded) ring and $A_*(-,*)$ is a (bigraded) module over $A^*(-,*)$. Moreover, for X smooth there is a "Poincaré duality" isomorphism from $A^*_{\mathrm{op}}(X,*)$ to $A_*(X,*)$.

Interestingly, $A^*_{\mathrm{op}}(-,0)$ is not (necessarily) equal to the Fulton–MacPherson operational Chow cohomology A^*; the first group might be larger (3.15.1). For this reason, "reduced" cohomology groups $\widetilde{A}^i_{\mathrm{op}}(-,j)$ are introduced such that $\widetilde{A}^i_{\mathrm{op}}(-,0) = A^i$. Formal properties of this theory $\widetilde{A}^*_{\mathrm{op}}(-,*)$ will be exploited in chapter 5.

Note that I do not want to call the $A^*_{\mathrm{op}}(-,*)$ "bigraded Chow cohomology"; this term will be reserved for the theory $A^*(-,*)$ of chapter 4. This theory $A^*(-,*)$ has better formal properties (a long exact sequence; a relation with K–theory), and its definition uses $A^*_{\mathrm{op}}(-,*)$. So actually $A^*_{\mathrm{op}}(-,*)$ is only an intermediate step, necessary in defining $A^*(-,*)$.

Because of the application in chapter 4, it is (unfortunately) necessary to define the operational theory on the level of complexes z, rather than on the level of their homology A.

(3.0) *Remarks:*

1. The operational Chow cohomology, and the bivariant theory of which it is part, are introduced by Fulton and MacPherson in [F–M]. A detailed exposition to operational Chow cohomology can be found in [Fu 3, Chapter 17].

2. In [Bl 1, §5], Bloch also introduces a cohomology theory to pair with his "homology " higher Chow groups. This theory, denoted $OPCH^i(-,j)$ by Bloch, is based on the Chow cohomology introduced in [Fu 1] (that is, the theory I denoted CH^* in the introduction), instead of on the newer, operational theory A^* of [F–M] and [Fu 3].

For quasi–projective X, the theory $OPCH^*(X,j)$ coincides rationally with K^jX (the K–theory associated to the category of vector bundles on X) for $j=0$ [Fu 1, §3.3 Corollary], and for $j>0$ if the proof of [Gi 1, Lemma 4.5] can be repaired (cf. [G–S, Remark 5.1.4]).

3. The cohomology theory that Gillet introduces to pair with his bigraded homology mentioned in (2.15).3 is defined as $H^p(X,\mathcal{K}^q)$, where $\mathcal{K}^q$ is the Zariski sheaf associated to the K–theory of locally free sheaves [Gi 1, §8].

§3.1. Definition and first properties

(3.1) Recall that in (2.21), (2.32) we have constructed push–forward (for proper morphisms), pull–back (for Chow flat morphisms) and refined Gysin homomorphisms (for regular immersions) in $A_*(-,*)$. We have seen that in fact the push–forward

$$f_* : A_i(X,*) \longrightarrow A_i(Y,*)$$

comes from a whole collection of homomorphisms of complexes, denoted

$$\{f_*] : \{z_i(X,*)\} \longrightarrow \{z_i(Y,*)\}$$

in the notation of (1.14), all inducing f_* after taking homology (and likewise for the pull–back and Gysin homomorphism).

The next two definitions associate to a given morphism $f\colon X \to Y$, and any $i \in \mathbb{Z}$, a complex

$$z^i(X \xrightarrow{f} Y, *) \ .$$

The construction is directly adapted from [Fu 3, Definition 17.1].

(3.2) **Definition:** Let $f\colon X \to Y$ be a morphism. For each morphism $g\colon Y' \to Y$, form the fibre square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \big\downarrow{\scriptstyle g'} & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \ . \end{array}$$

Let $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. An element c in $z^i(X \xrightarrow{f} Y, j)$ is a collection of homomorphisms c_g^k of type

$$c_g^k : \quad s_k(Y',l) \longrightarrow s_{k-i}(X', l+j)$$

for all $g\colon Y' \to Y$, all $s_k(Y',*) \in \{z_k(Y',*)\}$, all $(k,l) \in \mathbb{Z} \times \mathbb{Z}$ and some $s_{k-i}(X',*) \in \{z_{k-i}(X',*)\}$. satisfying:

(C_{-1}) If a homomorphism $q\colon s_{k-i}(X',*) \longrightarrow s'_{k-i}(X',*)$ is in $\{z_{k-i}(X',*)\}$, and

$$c_g^k\colon s_k(Y',l) \longrightarrow s_{k-i}(X', l+j)$$

is in the collection c, then also

$$s_k(Y',l) \xrightarrow{c_g^k} s_{k-i}(X', l+j) \xrightarrow{q} s'_{k-i}(X', l+j)$$

is in the collection c.

If a homomorphism $q: s_k(Y',*) \to s'_k(Y',*)$ is in $\{z_k(Y',*)\}$, then the diagram

$$\begin{array}{ccc} s_k(Y',l) & \xrightarrow{c} & A_{k-i}(X',l+j) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \mathrm{id}} \\ s'_k(Y',l) & \xrightarrow{c} & A_{k-i}(X',l+j) \end{array}$$

commutes.

(C_0) For all $g: Y' \to Y$, all $s_k(Y',*) \in \{z_k(Y',*)\}$ and all $(k,l) \in \mathbb{Z} \times \mathbb{Z}$, there exists a splitting $s_k(Y',l) \cong \mathrm{Ker}\delta_l \oplus \mathrm{Compl}_l$ such that the induced homomorphism

$$\mathrm{Im}\delta_l \xrightarrow{\sim} \mathrm{Compl}_l \xrightarrow{c_g^k} s_{k-i}(X',l+j)$$

can be extended to a homomorphism

$$c_g'^k: \mathrm{Ker}\delta_{l-1} \longrightarrow s_{k-i}(X',l+j) \ ,$$

and the collection $c_g'^k$ for the various g and k satisfies (C_{-1}), (C_1)—(C_5).

(C_1) If $\alpha \in s_k(Y',l)$ with $l < -j$, then

$$c_g^k(\alpha) = 0 \ .$$

(C_2) If $h: Y'' \to Y'$ is proper, $g: Y' \to Y$ arbitrary, and one forms the fibre diagram

$$\binom{*}{*} \qquad \begin{array}{ccc} X'' & \longrightarrow & Y'' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ X' & \longrightarrow & Y' \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} ,$$

then any diagram

$$\begin{array}{ccc} s_{k-i}(X'',l+j) & \xleftarrow{c_{g\circ h}^k} & s_k(Y'',l) \\ & & \downarrow{\scriptstyle h_*} \\ s_{k-i}(X',l+j) & \xleftarrow{c_g^k} & s_k(Y',l) \end{array}$$

(with $h_* \in \{h_*\}$) can be completed to a commutative diagram by a map in $\{h'_*\}$.

(C_3) If $h: Y'' \to Y'$ is Chow flat (h may be any flat morphism if j and l are 0) of relative dimension m, and $g: Y' \to Y$ is arbitrary, and one forms the fibre diagram $\binom{*}{*}$, then any diagram

$$\begin{array}{ccc} s_{k-i}(X',l+j) & \xleftarrow{c_g^k} & s_k(Y',l) \\ & & \downarrow{\scriptstyle h^*} \\ s_{k+m-i}(X'',l+j) & \xleftarrow{c_{g\circ h}} & s_{k+m}(Y'',l) \end{array}$$

can be commutatively completed by a map in $\{h'^*\}$.

(C_4) If $g\colon Y' \to Y$, $h\colon Y' \to Z'$ are morphisms, and $\tau\colon Z'' \to Z'$ is a regular embedding of codimension e, and one forms the fibre diagram

$$\begin{array}{ccccc} X'' & \longrightarrow & Y'' & \longrightarrow & Z'' \\ \downarrow & & \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow{\scriptstyle g} & & \\ X & \xrightarrow{f} & Y & & \end{array}$$

then any diagram

$$\begin{array}{ccc} s_{k-i}(X', l+j) & \xleftarrow{c_g^k} & s_k(Y', l) \\ & & \downarrow{\scriptstyle \tau^!} \\ s_{k-e-i}(X'', l+j) & \xleftarrow{c_{g\circ\tau'}^{k-e}} & s_{k-e}(Y'', l) \end{array}$$

can be commutatively completed by a map in $\{\tau^!\}$.

(C_5) If $g\colon Y' \to Y$ is any morphism, $h\colon Y' \times Y'' \to Y'$ the projection from a Cartesian product, and one forms the fibre diagram

$$\begin{array}{ccc} X' \times Y'' & \longrightarrow & Y' \times Y'' \\ \downarrow & & \downarrow{\scriptstyle h} \\ X' & \longrightarrow & Y' \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} ,$$

then for all $\alpha \in s_k(Y', l)$, $\beta \in s_m(Y'', n)$ with $l \geq -j$ and $\delta_l(\alpha) = \delta_n(\beta) = 0$,

$$c_{g\circ h}^{k+m}(\alpha \times \beta) = c_g^k(\alpha) \times \beta \qquad \in A_{k+m-i}(X' \times Y'', l+n+j) .$$

(3.3) *Notation.* I follow Fulton in abolishing sub- and superscripts on elements $c \in z^*(X \xrightarrow{f} Y, *)$ whenever possible. That is, for $g\colon Y' \to Y$ and $\alpha \in z_k(Y', l)$ we often write simply $c(\alpha)$ instead of $c_g^k(\alpha)$.

Also, the analogy with topology is emphasized by using interchangeably the notations $c(\alpha)$ and $c \cap \alpha$.

(3.4) **Definition:** The set $z^i(X \xrightarrow{f} Y, j)$ can be given an abelian group structure: To define $(c + c')$ on a given $Y' \to Y$, $s_k(Y', *) \in \{z_k(Y', *)\}$, consider homomorphisms on $s_k(Y', *)$ in the collections c resp. c' that have the same target (these exist by (C_{-1}) and (1.12)), and consider the sum homomorphism; the collection $c + c'$ is defined as the collection of all homomorphisms obtained in this way.

The groups $z^i(X \xrightarrow{f} Y, j)$ are made into a homological complex, defining a boundary

$$\delta_j : \quad z^i(X \xrightarrow{f} Y, j) \longrightarrow z^i(X \xrightarrow{f} Y, j-1)$$

by the rule

$$\big(\delta_j(c)\big)(\alpha) := c\big(\delta_l(\alpha)\big) - \delta_{l+j}\big(c(\alpha)\big) \qquad \in s_{k-i}(X', l+j-1)$$

for any $\alpha \in s_k(Y', l)$.

The *bigraded bivariant theory* is defined as the homology of this complex:

$$A^i(X \xrightarrow{f} Y, j) := H_j\left(z^i(X \xrightarrow{f} Y, *)\right) .$$

(3.5) **Lemma:** For a given morphism $f: X \to Y$, let $B^i(X \xrightarrow{f} Y, j)$ denote the group of collections of homomorphisms

$$A_k(Y', l) \longrightarrow A_{k-i}(X', l+j)$$

for all $Y' \to Y$, and all $(k, l) \in \mathbb{Z} \times \mathbb{Z}$, satisfying the analogues on $A_*(-, *)$–level of conditions (C_1) up to (C_5). Then there is a canonical isomorphism

$$A^i(X \xrightarrow{f} Y, j) \xrightarrow{\sim} B^i(X \xrightarrow{f} Y, j) ,$$

for any $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.

Proof: The map Ψ from left to right is given as follows: Let $c \in z^i(X \xrightarrow{f} Y, j)$ with $\delta_j(c) = 0$, i.e. c consists of homomorphisms of complexes.

The collection of operations $\Psi(c)$ is given as the induced homomorphism on homology, for some choice of complexes $s_k(Y', *)$ and $s_{k-i}(X', *)$.

This does not depend on choice of complexes by condition (C_{-1}); also it is clear that $\Psi(c)$ satisfies (C_1)—(C_5), and that Ψ sends elements $\delta_{j+1}(c') \in z^i(X \xrightarrow{f} Y, *)$ to 0.

An inverse L to Ψ is given as follows: Let $d \in B^i(X \xrightarrow{f} Y, j)$, let $Y' \to Y$ and $s_k(Y', *) \in \{z_k(Y', *)\}$ be as in (3.2). Then for $d \in B^i(X \xrightarrow{f} Y, j)$, the element $L(d) \in A^i(X \xrightarrow{f} Y, j)$ is given by choosing a splitting $s_k(Y', l) \cong \mathrm{Ker}\delta_l \oplus \mathrm{Compl}_l$ (this is possible since $s_k(Y', *)$ is a complex of *free* abelian groups [Iv, Chapter I, Corollary 10.2]), and defining $L(d)$ as the composition

$$L(d) \,:\, s_k(Y', l) \longrightarrow \mathrm{Ker}\delta_l \longrightarrow A_k(Y', l) \xrightarrow{d} A_{k-i}(X', l+j) .$$

Note that $L(d)$ satisfies conditions (C_{-1})—(C_5) of definition (3.2), and that $L(d) \in \mathrm{Ker}\left(z_i(X \xrightarrow{f} Y, j) \xrightarrow{\delta_j} z_i(X \xrightarrow{f} Y, j-1)\right)$.

It is immediate that $\Psi \circ L$ is the identity. As for the composition $L \circ \Psi$, the difference

$$L \circ \Psi(c) - c$$

is a collection of operations e which send $\mathrm{Ker}\delta_l$ to $\mathrm{Im}\delta_{l+j+1}$. Since the composition

$$\mathrm{Ker}\delta_l \xrightarrow{e} \mathrm{Im}\delta_{l+j+1} \xrightarrow{\sim} \mathrm{Compl}_{l+j+1} \subset s_{k-i}(X', l+j+1)$$

gives an element $e' \in z^i(X \xrightarrow{f} Y, j+1)$ with $\delta_{j+1}(e')$ equal to e when restricted to $\mathrm{Ker}\delta_l$, we may suppose e sends $\mathrm{Ker}\delta_l$ to 0, i.e. the operations in e are concentrated on $\mathrm{Compl}_l \subset s_k(Y', l)$.

Consider now the composition

$$\mathrm{Im}\delta_l \xrightarrow{\sim} \mathrm{Compl}_l \xrightarrow{e} A_{k-i}(X', l+j) \ .$$

Using (C_1), these homomorphisms can be extended (for some splitting) to $\mathrm{Ker}\delta_{l-1}$, and hence to $s_k(Y', l-1)$ to give a collection of operations $e' \in z^i(X \xrightarrow{f} Y, j+1)$. Since e is 0 on each $\mathrm{Ker}\delta_l$, the boundary $\delta_{j+1}(e')$ equals e, so $L \circ \Psi$ is the identity on $A^i(X \xrightarrow{f} Y, j)$. $\square$

(3.6) *Remark:* The only new compatibility condition in definition (3.2) with respect to Fulton's definition [Fu 3, Definition 17.1] is condition (C_5) on compatibility with exterior products.

However, in case one leaves out the second grading (i.e. $j = l = n = 0$, which is Fulton's situation), condition (C_5) is a consequence of (C_2) and (C_3) (cf. [Fu 3, proof of 17.3.1]). For $j \neq 0$, this is not the case.

(3.7) *Operations.* The $z^i(X \xrightarrow{f} Y, j)$ have operations as detailed in [Fu 3, 17.2]:

(P_1) For all morphisms $f: X \to Y$, $g: Y \to Z$, there is a product

$$z^i(X \xrightarrow{f} Y, j) \otimes z^k(Y \xrightarrow{g} Z, l) \longrightarrow z^{i+k}(X \xrightarrow{g \circ f} Z, j+l) \ ,$$

given by composing operations;

(P_2) If $f: X \to Y$ is a proper morphism, $g: Y \to Z$ any morphism, there is a push–forward homomorphism

$$f_* : \quad z^i(X \xrightarrow{g \circ f} Z, j) \longrightarrow z^i(Y \xrightarrow{g} Z, j) \ ,$$

given on $\alpha \in s_k(Z', l)$ by

$$(f_* c)(\alpha) := (f')_* c(\alpha) \in A_{k-i}(Y', l+j)$$

($f': X' \to Y'$ is obtained after base change);

(P_3) Given $f: X \to Y$, $g: Y' \to Y$, form the fibre square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \ . \end{array}$$

Then there is a pull–back homomorphism

$$g^* : \quad z^i(X \xrightarrow{f} Y, j) \longrightarrow z^i(X' \xrightarrow{f'} Y', j) \ ,$$

given by considering a variety Y'' mapping to Y' as a variety mapping to Y by composing with g.

These three operations satisfy compatibility properties as in [Fu 3, 17.2], in particular there is a *projection formula:* If $p: X \to Y$ is a proper morphism, $Y' \to Y$ arbitrary and $p': X' = X \times_Y Y'$ the base change, then for any $\alpha_i n s_k(X', l)$,

$$p'_*(p^* c \cap \alpha) = c \cap p'_* \alpha \in s_{k-l}(Y', l+j) \ .$$

Since operations (P_1)—(P_3) are compatible with boundaries (for (P_2) this follows from (2.24)(i), for (P_1) and (P_3) it is a mere tautology), the same operations, with the same compatibility properties, can also be constructed on $A^*(-,*)$ level.

(3.8) **Proposition:** Let $S := \mathrm{Spec}(k)$, k the ground field. Let X be a variety over k, and $(i,j) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. The homomorphism

$$\varphi\colon\ A^{-i}(X \longrightarrow S, -j) \longrightarrow A_i(X,j)$$

sending c to $c([S])$, is an isomorphism.

Proof: (following [Fu 3, Proposition 17.3.1]) An inverse ψ to φ can be constructed as follows: Let $a \in A_i(X,j)$. For any $Y \to S$, and $\alpha \in A_k(Y,l)$, define

$$\psi(a)(\alpha) := a \times \alpha \qquad \in A_{i+k}(X \times Y, j+l)\ .$$

Since exterior product is associative and compatible with push–forward, pull–back and intersection product, $\psi(a)$ gives indeed an element in $B^{-i}(X \longrightarrow S, j) = A^{-i}(X \longrightarrow S, j)$. Clearly $\varphi \circ \psi$ is the identity. To check that $\psi \circ \varphi$ is the identity, it remains to see that

$$c(\alpha) = \varphi(c) \times \alpha \qquad \in A_{i+k}(X \times Y, j+l)\ .$$

This follows from condition (C_5) in definition (3.2):

$$c(\alpha) = c(S \times \alpha) = c(S) \times \alpha\ .$$

□

(3.9) **Definition:** For any scheme X, and any $(i,j) \in \mathbb{Z}\times\mathbb{Z}$, define the *bigraded operational Chow cohomology* as

$$A^i_{\mathrm{op}}(X,j) := A^i(X \xrightarrow{\mathrm{id}} X, j)\ .$$

(Likewise, the shorthand $z^i_{\mathrm{op}}(X,j)$ will be used for the complex $z^i(X \xrightarrow{\mathrm{id}} X, *)$.) Then $A^*_{\mathrm{op}}(-,*)$ is an associative, bigraded ring; pull–backs (3.6) are ring–homomorphisms, and there is a “cap” product making $A_*(-,*)$ a module over $A^*(-,*)$.

(3.10) *Orientations* (after [Fu 3, §17.4]). Certain morphisms $f\colon X \to Y$ determine canonical elements $[f] \in z^*(X \xrightarrow{f} Y, 0)$ (or in $A^*(X \xrightarrow{f} Y, 0)$), called *canonical orientations*:
(i) If $f\colon X \to S := \mathrm{Spec}(k)$ and X has dimension n, then $[f] \in A^{-n}(X \xrightarrow{f} S, 0)$ is given by pull–back (note that f is Chow flat): For any morphism $g\colon Y \to S$, the induced morphism $f'\colon X \times_S Y \to Y$ is Chow flat, and for $\alpha \in A_k(Y,l)$ we define

$$[f](\alpha) := f'^*(\alpha) \quad \in A_{k+n}(X \times_S Y, l)$$

(ii) If $f\colon X \to Y$ is a regular embedding of codimension d, then $[f] \in A^d(X \xrightarrow{f} Y, 0)$ is given by the refined Gysin homomorphism: For any morphism $g\colon Y' \to Y$, and $\alpha \in A_k(Y',l)$, we define

$$[f](\alpha) := f^!(\alpha) \quad \in A_{k-d}(X', l)$$

(X' being as before the fibre product $X \times_Y Y'$).

(3.11) **Proposition** *(Poincaré duality)*: Let X be a smooth n-dimensional scheme, and $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. Then the canonical homomorphisms

$$A^i_{\mathrm{op}}(X,j) \xrightarrow{\cap[X]} A_{n-i}(X,j)$$

are isomorphisms.

Proof: Let $f\colon X \to S$ be the structural morphism to $S := \mathrm{Spec}(k)$, and denote by $[f] \in A^{-n}(X \xrightarrow{f} S, 0)$ the canonical orientation class. We prove that for all $i \in \mathbb{Z}$, $[f]$ induces isomorphisms

$$A^i(X \xrightarrow{\mathrm{id}} X, *) \xrightarrow{\cdot[f]} A^{i-n}(X \xrightarrow{f} S, *) \ ,$$

then we are done by (3.7). The proof of this last isomorphism follows from formal properties of the bivariant theory, as in [Fu 3, Proposition 17.4.2]. □

(3.12) *Digression.* It follows that for a smooth scheme X of dimension n, bigraded Chow homology has a ring structure

$$A_i(X,j) \otimes A_k(X,l) \longrightarrow A_{i+k-n}(X, j+l)$$

coming from (3.11). One can verify that this ring structure coincides with the one constructed in (2.37), cf. [Fu 3, Corollary 17.4].

This verification shows in particular that in the smooth case, the ring structure on the Chow groups coincides with the ring structure on $A^*(-,0)$, i.e. $A^*(-,0)$ is an intersection ring in the sense of the introduction.

(3.13) *Remarks:*
1. A note on the indexing of $A^*_{\mathrm{op}}(-,*)$: The choice of indexing is inspired by K–theory. That is, the pairing

$$A^i_{\mathrm{op}}(X,j) \otimes A_k(X,l) \longrightarrow A_{k-i}(X, l+j)$$

corresponds to the pairing in K–theory

$$\mathrm{Gr}^i_\gamma K^j X \otimes \mathrm{Gr}^\gamma_k K_l X \longrightarrow \mathrm{Gr}^\gamma_{k-i} K_{l+j} X$$

[So 2, Théorème 7(vi)].

2. The complicated definition (3.2) now shows its profit in the following way: on the category of smooth varieties, we have a well–defined contravariant functor $z^i_{\mathrm{op}}(X,*)$ to the category of complexes, and this functor computes Chow homology $A_*(-,*)$. The *well–definedness* solves the nuisance that $z_i(-,*)$ was only well–defined in the derived category; the *contravariance* solves the problem that $z_i(-,*)$ is not naturally contravariant.

(3.14) **Proposition:** For any M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\widetilde{\tau}} & \widetilde{X} \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\tau} & X \end{array}$$

there is a short exact sequence of complexes

$$0 \longrightarrow z^i_{\mathrm{op}}(X,*) \longrightarrow z^i_{\mathrm{op}}(\widetilde{X},*) \oplus z^i_{\mathrm{op}}(Y,*) \longrightarrow z^i_{\mathrm{op}}(\widetilde{Y},*)$$

(with arrows given by pull–back (3.6)).

Proof: This follows from the reverse property of the complexes $z_i(-,*)$ (2.21)(ii), analogous to the case of Chow cohomology [Ki], [B–G–S, Appendix].

For instance, to prove injectivity, suppose $c \in z^i_{\mathrm{op}}(X,*)$ with $\pi^*c = \tau^*c = 0$. For a given $X' \to X$, let

$$\begin{array}{ccc} \widetilde{Y}' & \longrightarrow & \widetilde{X}' \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ Y' & \xrightarrow{\tau'} & X' \end{array}$$

be the M–V diagram obtained by fibering the given one with $\times_X X'$. For a given $s_k(X',*) \in \{z_k(X',*)\}$, there exist $s_k(\widetilde{X}' \amalg Y',*) \in \{z_k(\widetilde{X}' \amalg Y')\}$ such that

$$\pi'_* - \tau'_*\colon\ s_k(\widetilde{X}' \amalg Y',*) \longrightarrow s_k(X',*)$$

is surjective (1.14). So any $\alpha \in s_k(X',l)$ is an image $\pi'_*\beta - \tau'_*\gamma$. But then

$$c(\alpha) = c(\pi'_*\beta - \tau'_*\gamma) = \pi'_*(\pi^*c)(\beta) - \tau'_*(\tau^*c)(\gamma) = 0$$

by the projection formula.

Exactness in the middle is proven similarly using (2.21)(ii) (and is not necessary for the corollary in remark (3.15).1 below). □

(3.15) *Remarks.*

1. A corollary is that $A^*_{\mathrm{op}}(X,*)$ has "cycle class maps" into cohomology theories, provided X has ROS and the cohomology in question satisfies descent and weak purity. Indeed, let $\Gamma(i)$ be the complexes of Zariski sheaves computing the cohomology $H^*(-,\Gamma(i))$, and for a smooth variety $\widetilde{X}$ of dimension n, define a double complex $K^{**}(\widetilde{X})$ as

$$K^{p*}(\widetilde{X}) := \begin{cases} R\Gamma\big(\widetilde{X} \times \Delta^{-p}, \Gamma(i)\big) & \text{if } 2N \le p \le 0; \\ 0 & \text{otherwise,} \end{cases}$$

for some fixed $N << 0$. Because of homotopy, the associated single complex has

$$H^m(sK^*) = H^m\big(\widetilde{X}, \Gamma(i)\big)\ ,$$

compare (2.38). Passing through an auxiliary double complex

$$'K^{p*}(\widetilde{X}) := \begin{cases} \varinjlim\limits_{Z \in z_{n-i}(\widetilde{X},-p)} R\Gamma_Z\big(\widetilde{X} \times \Delta^{-p}, \Gamma(i)\big) & \text{if } 2N \le p \le 0; \\ 0 & \text{otherwise,} \end{cases}$$

defines a non–trivial map in $D^+(Ab)$:

$$\sigma_{\geq 2N}\, z^i_{\mathrm{op}}(\widetilde{X},-*) \longrightarrow sK^{2i+*}\ ;$$

this is similar (but dual) to the homology construction in (2.38).

Now taking an M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

with $\widetilde{X}$ smooth, we know using the above map (for $\widetilde{X}$) and noetherian induction (for Y and $\widetilde{Y}$) there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0\longrightarrow & z^i_{\mathrm{op}}(X,-*) & \longrightarrow & z^i_{\mathrm{op}}(\widetilde{X}\amalg Y,-*) & \longrightarrow & z^i_{\mathrm{op}}(\widetilde{Y},-*) \\ & & & \downarrow & & \downarrow \\ 0\longrightarrow & \mathrm{Ker} & \longrightarrow & sK^{2i+*}(\widetilde{X}\amalg Y) & \longrightarrow & sK^{2i+*}(\widetilde{Y}). \end{array}$$

Since $\Gamma(i)$ is supposed to have descent, the kernel is quasi–isomorphic to $R\Gamma\big(X,\Gamma(i)\big)[2i]$, so this results in a map

$$A^i_{\mathrm{op}}(X,j)\longrightarrow H^{2i-j}\big(X,\Gamma(i)\big)$$

(using the usual descent argument one shows this map is independent of choice of resolution).

In particular, since Fulton–MacPherson Chow cohomology A^*X does not in general have such maps [To 2, Theorem 7], it follows that $A^*_{\mathrm{op}}(-,0)$ is a strictly finer intersection ring than A^*.

2. To prove (3.14) (even for quasi–projective X), it was necessary to dispose of the homological descent result (2.16) for *all* schemes mapping to X and $\widetilde{X}$, and not only the quasi–projective ones. This was one of the reasons for the descent trick (2.20) used to extend Chow homology to the non quasi–projective case.

3. I ignore whether the sequence (3.14) is also right–exact, or even whether it gives a triangle in $D^+(Ab)$ (though we'll see in (4.3) that this is the case if X is smooth). For this reason, a descent style definition of Chow cohomology will be given in chapter 4, so that long exact M–V sequences are built into the theory.

(3.16) *Operational Chow cohomology as hypercohomology.* For any variety X, let $\mathcal{Z}^i_{\mathrm{op},X}$ denote the cohomological complex of Zariski sheaves on X (concentrated in negative degrees) associated to presheaves $U\mapsto z^i_{\mathrm{op}}(U,-j)$ if $j\leq 0$ (and 0 if $j>0$). If X is smooth of dimension n, Poincaré duality (3.11) plus the fact that sheafification is an exact functor imply that the homomorphism of complexes (well–defined in $D^+(Ab)$) given by $\cap[X]$

$$\mathcal{Z}^i_{\mathrm{op},X}(*)\longrightarrow\mathcal{Z}^X_{n-i}(*)$$

is a quasi–isomorphism, hence by (2.21)(iv)

$$A^i_{\mathrm{op}}(X,j) = \mathbb{H}^{-j}\big(X, \mathcal{Z}^i_{\mathrm{op},X}(*)\big) \qquad (*)$$

for $j \geq 0$.

In fact, supposing ROS we can prove $(*)$ for arbitrary varieties, as we will show below (except for (3.17), this will not be used in the sequel).

Let U, V be opens in X. Resolve singularities of X by an M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X \end{array}$$

and define diagrams

$$\begin{array}{ccc} \widetilde{Y}_U & \longrightarrow & \widetilde{U} \\ \big\downarrow & & \big\downarrow \\ Y_U & \longrightarrow & U \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{Y}_V & \longrightarrow & \widetilde{V} \\ \big\downarrow & & \big\downarrow \\ Y_V & \longrightarrow & V \end{array}$$

by fibering the first diagram with $\times_X U$ resp. $\times_X V$.

Define a complex of abelian groups $'z^i(U,*)$ concentrated in negative degrees by exactness of

$$0 \longrightarrow {'z^i}(U,*) \longrightarrow z_{n-i}(\widetilde{U},-*) \oplus z^i_{\mathrm{op}}(Y_U,-*) \longrightarrow {}^{\amalg}z^i(\widetilde{Y}_U,-*)\ ,$$

where ${}^{\amalg}z^i(\widetilde{Y}_U,*)$ is defined as the fibre coproduct

$$z_{n-i}(\widetilde{U},*) \amalg_{z^i_{\mathrm{op}}(\widetilde{U},*)} z^i_{\mathrm{op}}(\widetilde{Y}_U,*)$$

(so $'z^i(U,*)$ is quasi–isomorphic to $z^i_{\mathrm{op}}(U,-*)$). Let $'\mathcal{Z}^i_X(*)$ denote the complex of associated Zariski sheaves (so $'\mathcal{Z}^i_X(*)$ is quasi–isomorphic to $\mathcal{Z}^i_{\mathrm{op},X}(*)$).

We will prove that $U \mapsto {'\mathcal{Z}^i}(U,j)$ is a sheaf, and that it is flasque. This suffices to prove $(*)$.

For the first assertion, note that one has a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
0 \longrightarrow & & {'z^i}(U \cup V,*) & \longrightarrow & {'z^i}(U \amalg V,*) & \longrightarrow & {'z^i}(U \cap V,*) \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
 & & z_{n-i}(\widetilde{U} \cup \widetilde{V},-*) & & z_{n-i}(\widetilde{U} \amalg \widetilde{V},-*) & & z_{n-i}(\widetilde{U} \cap \widetilde{V},-*) \\
0 \longrightarrow & & \oplus & \longrightarrow & \oplus & \longrightarrow & \oplus \\
 & & z^i_{\mathrm{op}}(Y_U \cup Y_V,-*) & & z^i_{\mathrm{op}}(Y_U \amalg Y_V,-*) & & z^i_{\mathrm{op}}(Y_U \cap Y_V,-*) \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
0 \longrightarrow & & {}^{\amalg}z^i(\widetilde{Y}_U \cup \widetilde{Y}_V,-*) & \longrightarrow & {}^{\amalg}z^i(\widetilde{Y}_U \amalg \widetilde{Y}_V,-*) & \longrightarrow & {}^{\amalg}z^i(\widetilde{Y}_U \cap \widetilde{Y}_V,-*)
\end{array}$$

with exact columns. Note that Y_U, Y_V (resp. $\widetilde{Y}_U$, $\widetilde{Y}_V$) are opens in $Y_U \cup Y_V$ (resp. in $\widetilde{Y}_U \cup \widetilde{Y}_V$), so by noetherian induction we may assume the second and third row are exact. A diagram chase then shows that the first row is exact, i.e. $U \mapsto {}'z^i(U,j)$ is a sheaf.

Flasqueness of ${}'\mathcal{Z}^i_X(j)$ is proven by a similar diagram chase, and we conclude that the relation $(*)$ holds for arbitrary X (supposing ROS).

As a corollary, one has the usual "local to global" spectral sequence

$$E_2^{pq} = H^p\big(X, \mathcal{A}^i_{\mathrm{op},X}(-q)\big) \ \Rightarrow \ A^i_{\mathrm{op}}(X, -p-q) \ ,$$

where $\mathcal{A}^i_{\mathrm{op},X}(-q)$ is the Zariski sheaf on X associated to $U \mapsto A^i_{\mathrm{op}}(U,-q)$.

Now we can also introduce bigraded operational Chow cohomology with supports: Suppose ROS, and let $j \geq 0$. Then for any closed subvariety $Y \subset X$, we define

$$A^i_{\mathrm{op},Y}(X,j) := \mathbb{H}^j_Y\left(X, \mathcal{Z}^i_{\mathrm{op},X}(*)\right) \ .$$

(3.17) *Corollary:* It follows that on the category of varieties with ROS, there is a natural transformation of contravariant functors

$$ch: \ K^0 \longrightarrow A^* \otimes \mathbb{Q}$$

coinciding with the Chern character of [Bo–Se] on smooth varieties (as before, K^0 denotes K–theory associated to the category of vector bundles; A^* is Fulton–MacPherson Chow cohomology).

Indeed, since $A^*_{\mathrm{op}}(-,*)$ comes from a complex of Zariski sheaves (3.16) satisfying Gillet's axioms [Gi 1, §1], Gillet's work furnishes a natural transformation

$$K^j \longrightarrow A^*_{\mathrm{op}}(-,j) \otimes \mathbb{Q} \ .$$

Restricting to $j = 0$ and composing with the forgetful map $A^*_{\mathrm{op}}(-,0) \to A^*$ proves the claim.

This is not exactly a new result (indeed, at least in the quasi–projective case it is part of Fulton's bivariant Riemann–Roch [Fu 3, Example 18.3.16], which does not need ROS), but it does indicate how nice it is to have your cohomology coming from a complex of sheaves.

Note that in the singular case, $ch \otimes \mathbb{Q}$ will not in general be an isomorphism. (Indeed, for a given singular X, let

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

be an M–V diagram with $\widetilde{X}$ smooth; then $K^0 X_{\mathbb{Q}} \to K^0(\widetilde{X} \amalg Y)_{\mathbb{Q}}$ will not, in general, be an injection, whereas $A^* X_{\mathbb{Q}} \to A^*(\widetilde{X} \amalg Y)_{\mathbb{Q}}$ is always an injection, cf. [Ki] or (3.20).)

§3.2. A reduced theory

A *reduced* bigraded operational theory $\widetilde{A}^*(-,*)$ is introduced to get a relation with Fulton–MacPherson operational Chow cohomology:

(3.18) **Definition:** Let $f\colon X \to Y$ be a morphism of varieties. For any morphism $g\colon Y' \to Y$, form the fibre diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} .$$

For any $(i,j) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$, the group $\widetilde{A}^i(X \xrightarrow{f} Y, j)$ is defined as the group of collections of operations

$$c_g^k \,:\, A_i(Y',0) \longrightarrow A_{i-k}(X',j)$$

for all $g\colon Y' \to Y$ and all $k \in \mathbb{Z}$, satisfying (analogues on homology level of) conditions (C_1)—(C_5) of definition (3.2).

One defines

$$\widetilde{A}^i(X,j) := \widetilde{A}^i(X \xrightarrow{\mathrm{id}} X, j) \ .$$

(3.19) The reduced theory $\widetilde{A}$ has products, pull–backs, push–forwards and orientations, just as (3.6). Copying the proof of (3.11), we find that for smooth X of dimension n, the map $c \mapsto c([X])$ induces isomorphisms

$$\widetilde{A}^i(X,j) \xrightarrow{\sim} A_{n-i}(X,j) \ .$$

In the singular case, however, $A^i_{\mathrm{op}}(X,j)$ surjects onto $\widetilde{A}^i(X,j)$, but will in general be different; the first is a *finer* theory.

It is clear from the definition that $\widetilde{A}^i(X,0) = A^iX$, the Chow cohomology of Fulton–MacPherson [F–M], [Fu 3, Chapter 17].

(3.20) **Proposition:** For any M–V diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\widetilde{\tau}"] \arrow[d, "\pi_Y"] & \widetilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "\tau"] & X \end{tikzcd}$$

there are short exact sequences

$$0 \longrightarrow \widetilde{A}^i(X,j) \longrightarrow \widetilde{A}^i(\widetilde{X} \amalg Y, j) \longrightarrow \widetilde{A}^i(\widetilde{Y},j)$$

(arrows are pull–back) for any $(i,j) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$.

Proof: This follows from the reverse property for the Chow groups; cf. [Ki] or [B–G–S, Appendix], where the $j = 0$ case is proven. The argument is formally similar to the proof of (3.14).

For instance, injectivity is proven as follows: let $c \in \widetilde{A}^i(X, j)$ with $\pi^* c = \tau^* c = 0$. Given a morphism $X' \to X$ and a cycle $\alpha \in A_k(X', 0) = A_k X'$, we need to prove $c(\alpha) = 0 \in A_{k-i}(X', j)$. Let $\widetilde{X}'$ resp. Y' be $\widetilde{X} \times_X X'$ resp. $Y \times_X X'$, and let π' resp. τ' be the induced morphisms to X'. Since $\widetilde{X}' \amalg Y' \to X'$ is an envelope, $\alpha = \pi'_* \beta + \tau'_* \gamma$ for some cycles $(\alpha, \gamma) \in A_k(\widetilde{X}' \amalg Y')$. But then we are done by the projection formula:

$$\begin{aligned} c(\alpha) &= c(\pi'_* \beta + \tau'_* \gamma) \\ &= \pi'_* \big((\pi^* c)(\beta) \big) + \tau'_* \big((\tau^* c)(\gamma) \big) = 0 \ . \end{aligned}$$

□

(3.21) *Compact supports.* Given a variety X, define the compactly supported reduced theory as

$$\widetilde{A}^i_c(X, j) := \operatorname{Ker}\big(\widetilde{A}^i(\bar{X}, j) \longrightarrow \widetilde{A}^i(D, j) \big) \ ,$$

where $\bar{X}$ is a compactification of X (which exists by Nagata), and D is the complement $\bar{X} \setminus X$. It follows from (3.20) that this definition is independent of choice of compactification, and it follows from an easy diagram chase that for any Y in X closed with complement U, there is a short exact sequence

$$0 \longrightarrow \widetilde{A}^i_c(U, j) \longrightarrow \widetilde{A}^i_c(X, j) \longrightarrow \widetilde{A}^i(Y, j) \ .$$

As a matter of notation, the groups $\widetilde{A}^i_c(X, 0)$ will also be written as $A^i_c X$, since they coincide with Fulton–MacPherson Chow cohomology for X complete.

Chapter 4: Bigraded Chow cohomology

The work done in chapter 3 can now be used to give a descent style definition of bigraded Chow cohomology $A^*(-,*)$. To show this is well–defined, we need to rely on a result of Gillet and Soulé (4.3).

A similar descent style definition gives a new, contravariant K–theory, called K_f^* ("formal K–theory"), and there is a Riemann–Roch type isomorphism with rational coefficients between K_f^* and $A^*(-,*)$, coinciding with the Chern character in the smooth case (4.9).

One also introduces compactly supported theories $A_c^*(-,*)$ (4.6) and K_c^* (4.8); the latter theory coincides with the compactly supported K–theory that Gillet and Soulé define using simplicial descent [G–S, §5]. There is once more a Riemann–Roch–like isomorphism with rational coefficients between K_c^* and $A_c^*(-,*)$ (4.9).

Working with descent obliges us to make the hypothesis ROS (resolution of singularities, cf. (1.2)) for the whole of this chapter.

§4.1. Definition and first properties

(4.1) **Definition:** Let $X \in \mathcal{VAR}_k$, and suppose $\mathcal{VAR}_k$ has ROS. For any $i \in \mathbb{Z}$, the homological complex $z^i(X,*)$ is defined by

$$z^i(X,-*) := s\big(z^i_{\mathrm{op}}(X_*,-*)\big) \ ,$$

where $X_* \to X$ is a cubical hyperresolution, and s is as in (1.9).

The bigraded Chow cohomology is defined as

$$A^i(X,j) := H_j\big(z^i(X,*)\big) \ ,$$

for $(i,j) \in \mathbb{Z} \times \mathbb{Z}$.

(4.2) *Remarks:* We have seen (3.7) that $z^i_{\mathrm{op}}(-,*)$ is a contravariant functor, so that $z^i_{\mathrm{op}}(X_*,*)$ forms a cubical complex, so the above definition makes sense. We need to grade in negative degree because $z^i_{\mathrm{op}}(-,*)$ is contravariant, so the associated single complex at the right–hand–side of (4.1) becomes a cohomological complex. Note that $z^i(X,*)$ can be non–zero also in negative degrees, but only in degree $\geq -\dim X - 1$.

Of course, it is left to prove $z_i(X,*)$ is independent of choices in $D(Ab)$. As in (1.9.2), this follows from the following proposition:

(4.3) **Proposition:** For any M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \stackrel{\widetilde{\tau}}{\longrightarrow} & \widetilde{X} \\ \Big\downarrow{\scriptstyle \pi_Y} & & \Big\downarrow{\scriptstyle \pi} \\ Y & \stackrel{\tau}{\longrightarrow} & X \end{array}$$

of smooth varieties, there are exact sequences

$$0 \longrightarrow A^i_{\mathrm{op}}(X,j) \longrightarrow A^i_{\mathrm{op}}(\widetilde{X} \amalg Y, j) \longrightarrow A^i_{\mathrm{op}}(\widetilde{Y}, j) \longrightarrow 0$$

(arrows are given by pull–back).

Proof: Left–exactness is easy, cf. (3.20); but surjectivity is harder to prove (even for $j = 0$, when these are just the Chow groups !). The proof relies on a beautiful result of Gillet and Soulé.

1. The complete case: Suppose X, and hence every variety in the M–V diagram, is complete. Then (4.2) follows from the following more general result of [G–S]:

Theorem (Gillet–Soulé): Suppose ROS. Let Γ be a contravariant functor from the category of smooth complete varieties to some abelian category $\mathcal{A}$, and suppose Γ factorizes over the category $\mathcal{M}$ of (pure effective) Chow motives. Then for any M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \Big\downarrow & & \Big\downarrow \\ Y & \longrightarrow & X \end{array}$$

of smooth complete varieties, there are short exact sequences

$$0 \longrightarrow \Gamma(X) \longrightarrow \Gamma(\widetilde{X}) \amalg \Gamma(Y) \longrightarrow \Gamma(\widetilde{Y}) \longrightarrow 0 \ .$$

Indication of proof of theorem: (For a definition of the category $\mathcal{M}$, cf. [Scho] and the references given there.) Gillet and Soulé [G–S, 3.1.1] have defined "right derived functors" to Γ as

$$R^i\Gamma(X) := H^i\Big(\Gamma\big(W(X)\big)\Big)$$

for any (not necessarily smooth and complete) scheme X, where $W(X)$ is the *weight complex* of X, which they show to be well–defined in $\mathrm{Hot}(\mathcal{M})$, the homotopy category of cochain complexes in $\mathcal{M}$ ([G–S, Theorem 2]; this is a kind of "motivic descent").

By construction, for a smooth complete X, $R^0\Gamma(X) = \Gamma(X)$ and $R^i\Gamma(X) = 0$ for $i > 0$. Also, if $Y \subset X$ is closed with complement U, one has a long exact sequence

$$R^0\Gamma(U) \longrightarrow R^0\Gamma(X) \longrightarrow R^0\Gamma(Y) \longrightarrow R^1\Gamma(U) \longrightarrow \cdots \ ,$$

which is natural in (X, Y), and an easy diagram chase ends the proof.

This theorem applies to $A^*_{\mathrm{op}}(-,*)$, since the functor on smooth complete schemes

$$X \longrightarrow A^i_{\mathrm{op}}(X,j) = A_{\dim X - i}(X,j)$$

certainly factorizes over $\mathcal{M}$; to a motive (X,p) we associate $p \circ A^i_{\mathrm{op}}(X,j)$ (where $\circ$ denotes composition in the sense of correspondences), and to a correspondence $\alpha \in A^{\dim X}_{\mathrm{op}}(X \times Y)$ we associate the homomorphism

$$\begin{aligned} A^i_{\mathrm{op}}(X,j) &\longrightarrow A^i_{\mathrm{op}}(Y,j) \\ \beta &\mapsto (p_Y)_*\big((p_X)^*\beta \cdot \alpha\big) \end{aligned}$$

(p_X, p_Y being the projections from $X \times Y$ to X resp. Y). It follows from the compatibilities established in §3 that this is indeed a (covariant) functor, and that $A^i_{\mathrm{op}}(-,j)$ factorizes as

$$\mathcal{SC} \xrightarrow{h} \mathcal{M} \longrightarrow \{\mathrm{Ab}\} ,$$

where $\mathcal{SC} \subset \mathcal{VAR}$ denotes the full subcategory of smooth and complete varieties.

(In fact, any cohomology theory on $\mathcal{SC}$ that is part of a Poincaré duality theory in the sense of [B–O] or of (1.3) factorizes over $\mathcal{M}$, hence has short exact sequences as in the above theorem.)

2. The general case. Suppose now X is not complete. We already know (3.20) the sequence is left–exact; only surjectivity remains to be proven. By Nagata's compactification result and ROS, we know there exists an M–V diagram

$$\begin{array}{ccc} \widetilde{\bar{Y}} & \longrightarrow & \widetilde{\bar{X}} \\ \big\downarrow & & \big\downarrow \bar{\pi} \\ \bar{Y} & \xrightarrow{\bar{\tau}} & \bar{X} \end{array}$$

of smooth varieties, and such that X (resp. Y, $\widetilde{X}$, $\widetilde{Y}$) is open in $\bar{X}$ (resp. in $\bar{Y}$, $\widetilde{\bar{X}}$, $\widetilde{\bar{Y}}$) with complement a normal crossing divisor D (resp. D_Y, $\bar{D}$, $\widetilde{D}_Y$).

The desired surjectivity for $j = 0$ is now immediate (since $A^i(\bar{X},0)$ surjects onto $A^i(X,0)$); for $j > 0$ we use an inductive argument.

Write D as a union of irreducible components $D_1, \ldots, D_k$, and set

$$\begin{aligned} D_{(m)} &:= \bigcup_{j \le m} D_j \ ; \\ \widetilde{D}_{(m)} &:= \bar{\pi}^{-1}(D_{(m)}) \ ; \\ X_{(m)} &:= \bar{X} \setminus D_{(m)} \ ; \\ \widetilde{X}_{(m)} &:= \bar{\pi}^{-1}(X_{(m)}) \ ; \\ Y_{(m)} &:= \bar{\tau}^{-1}(X_{(m)}) \end{aligned}$$

etc. (so $X_{(0)} = \bar{X}$, $X_{(k)} = X$, etc.). Set $D^m = D_{(m)} \setminus D_{(m-1)}$, and $\widetilde{D}^m = \bar{\pi}^{-1}(D^m)$.

By an induction on the number of irreducible components of $\widetilde{D}^m$, one proves there are long exact sequences

$$A_{n-1-i}(D^m, j) \longrightarrow A_{n-1-i}(\widetilde{D}^m, j) \oplus A_{d_Y - i}(D^m_Y, j) \longrightarrow A_{d_{\widetilde{Y}} - i}(\widetilde{D}^m_Y, j) \longrightarrow ,$$

where d_Y resp. $d_{\widetilde{Y}}$ is the dimension of D^m_Y resp. of $\widetilde{D}^m_Y$. This long exact sequence is split since D^m is smooth (use Poincaré duality and the projection formula).

From the diagram whose rows are exact by induction

$$\begin{array}{ccccccc}
0 \to & A_{n-1-i}(D^m, j) & \to & A_{n-1-i}(\widetilde{D}^m, j) \oplus A_{d_Y-i}(D^m_Y, j) & \to & A_{d_{\widetilde{Y}}-i}(\widetilde{D}^m_Y, j) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & A^i(X_{(m-1)}, j) & \to & A^i(\widetilde{X}_{(m-1)} \amalg Y_{(m-1)}, j) & \to & A^i(\widetilde{Y}_{(m-1)}, j) & \to 0
\end{array}$$

we get a long exact sequence relating kernels and cokernels. Since the vertical maps of the above diagram fit into long exact sequences, this kernel–cokernel sequence fits into a commutative diagram with exact rows and columns

$$\begin{tikzcd}
{} \arrow[r] & A^i(\widetilde{X}_{(m-1)} \amalg Y_{(m-1)}, j+1) \arrow[r] \arrow[d] & A^i(\widetilde{Y}_{(m-1)}, j+1) \arrow[r] \arrow[d] & 0 & \\
{} \arrow[r] & A^i(\widetilde{X}_{(m)} \amalg Y_{(m)}, j+1) \arrow[r] \arrow[d] & A^i(\widetilde{Y}_{(m)}, j+1) \arrow[d] & & \\
{} \arrow[r] & \mathrm{Ker} \arrow[r] \arrow[d] & \mathrm{Ker} \arrow[r] \arrow[d] & \mathrm{Coker} \arrow[r] \arrow[d] & \mathrm{Coker} \arrow[d] \\
 & 0 & 0 \arrow[r] & A^i(X_{(m)}, j) \arrow[r] \arrow[d] & A^i(\widetilde{X}_{(m)} \amalg Y_{(m)}, j) \arrow[d] \\
 & & 0 \arrow[r] & A^i(D^m, j-1) \arrow[r] & A^i(\widetilde{D}^m \amalg D^m_Y, j-1)
\end{tikzcd}$$

The point is that we know by (3.20) that $A^i(X_{(m)}, j) \to A^i(\widetilde{X}_{(m)} \amalg Y_{(m)}, j)$ is an injection, so the kernel–cokernel sequence splits in the middle. But the surjectivity of Ker to Ker implies the surjectivity of

$$A^i(\widetilde{X}_{(m)} \amalg Y_{(m)}, j+1) \longrightarrow A^i(\widetilde{Y}_{(m)}, j+1) ,$$

and we are done.

□

(4.4) **Theorem:** Let $\mathcal{VAR}$ be a category of varieties which has ROS. Bigraded Chow cohomology has the following properties:

(i) $X \mapsto A^*(X, *)$ is a contravariant functor from $\mathcal{VAR}$ to the category of associative, bigraded, commutative rings;

(ii) There are cap–products

$$A^i(X, j) \otimes A_k(X, l) \xrightarrow{\cap} A_{k-i}(X, l + j)$$

for any $X \in \mathcal{VAR}$;

(iii) If $X \in \mathcal{VAR}$ is smooth of dimension n, cap–product with the fundamental class induces isomorphisms

$$A^i(X,j) \xrightarrow{\sim} A_{n-i}(X,j) \ ;$$

(iv) There is a projection formula for proper morphisms; there are functorial Gysin homomorphisms $f_*: A^i(X,j) \to A^{i+d}(Y,j)$ for a regular imbedding $f: X \to Y$ of codimension d;

(v) For any M–V diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\widetilde{\tau}"] \arrow[d, "\pi_Y"] & \widetilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "\tau"] & X \end{tikzcd}$$

in $\mathcal{VAR}$, there are long exact sequences

$$\longrightarrow A^i(X,j) \longrightarrow A^i(\widetilde{X} \amalg Y, j) \longrightarrow A^i(\widetilde{Y}, j) \longrightarrow A^i(X, j-1) \longrightarrow \cdots$$

(with arrows given by pull–back);

(vi) There is a "local to global" spectral sequence

$$E_2^{pq} = H^p\big(X, \mathcal{A}^i_X(-q)\big) \Rightarrow A^i(X, -p-q) \ ,$$

where $\mathcal{A}^i_X(-q)$ denotes the Zariski sheaf associated to $U \mapsto A^i(U,-q)$;

(vii) *(Homotopy)* The projection $p: \mathbb{A}^1_X \to X$ induces isomorphisms

$$p^*: \ A^i(X,j) \xrightarrow{\sim} A^i(\mathbb{A}^1_X, j) \ ;$$

(viii) *(Codimension 1)* The groups $A^1(X,j)$ are 0 for $j \geq 2$.

Proof: Most of this is standard descent theory (and similar to what we did for homology in (2.21)).

(i) To see $A^*(X,*)$ is a bigraded ring, let $X_* \to X$ be a cubical hyperresolution. For each cubical index α, there is a product

$$z^i_{\mathrm{op}}(X_\alpha, j) \otimes z^k_{\mathrm{op}}(X_\alpha, l) \longrightarrow z^{i+k}_{\mathrm{op}}(X, j+l) \ ,$$

compatible with boundaries and with maps $X_\alpha \to X_\beta$ (3.6). This results in a product

$$z^i(X,j) \otimes z^k(X,l) \longrightarrow z^{i+k}(X, j+l) \ ,$$

compatible with boundaries of the complex $z^i(X,*)$ (these are given as combinations of boundaries of $z^i_{\mathrm{op}}(X_\alpha,*)$ and maps $z^i_{\mathrm{op}}(X_\alpha,*) \to z^i_{\mathrm{op}}(X_\beta,*)$); this gives the ring structure on $A^*(X,*)$.

To show this is well–defined, if one has two cubical hyperresolutions X_* and X'_*, coming from M–V diagrams

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\widetilde{\tau}"] \arrow[d, "\pi_Y"] & \widetilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "\tau"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{Y}' \arrow[r] \arrow[d] & \widetilde{X}' \arrow[d] \\ Y \arrow[r] & X \end{tikzcd} ,$$

then we may suppose one maps to the other, and we have seen there are short exact sequences

$$0 \longrightarrow A^i(\widetilde{X}, j) \longrightarrow A^i(\widetilde{X}' \amalg \widetilde{Y}, j) \longrightarrow A^i(\widetilde{Y}', j) \longrightarrow 0 \ ,$$

which implies

$$\operatorname{Ker}\Big(A^i(\widetilde{X} \amalg Y, j) \to A^i(\widetilde{Y}, j)\Big) \xrightarrow{\sim} \operatorname{Ker}\Big(A^i(\widetilde{X}' \amalg Y, j) \to A^i(\widetilde{Y}', j)\Big) \ ;$$
$$\operatorname{Coker}\Big(A^i(\widetilde{X} \amalg Y, j) \to A^i(\widetilde{Y}, j)\Big) \xrightarrow{\sim} \operatorname{Coker}\Big(A^i(\widetilde{X}' \amalg Y, j) \to A^i(\widetilde{Y}', j)\Big) \ .$$

The arrows in the short exact sequence are compatible with products (since this was true for A^*_{op}, and by noetherian induction), so X_* and X'_* induce the same product structure on kernels and cokernels; this shows independence.

Commutativity of $A^*(X, *)$ follows from the fact that the ring structure on each $z^*_{\mathrm{op}}(X_\alpha, *)$ is commutative up to homotopy, since X_α is smooth (3.11) and (3.12).

(ii) The argument is similar to (i), using the pairing

$$s\Big(z^i_{\mathrm{op}}(X_\alpha, *) \otimes z_k(X_\alpha, *)\Big) \longrightarrow z_{k-i}(X_\alpha, *)$$

in $D^+(Ab)$, and descent for the Chow homology.

(iii) Taking X_* equal to X, we see that $A^i(X, j) = A^i_{\mathrm{op}}(X, j)$, for which Poincaré duality was proven in (3.11).

(iv) Let $p\colon X_1 \to X_2$ be a proper morphism, let $c \in z^i(X_2, j)$ and consider M–V diagrams

$$\begin{array}{ccc} \widetilde{Y}_m & \longrightarrow & \widetilde{X}_m \\ \big\downarrow & & \big\downarrow \\ Y_m & \longrightarrow & X_m \end{array}$$

$(m = 1, 2)$ with $\widetilde{X}_m$ smooth, compatible with p. By construction of the cap–product and descent for Chow homology, there is a commuting diagram of triangles

$$\begin{array}{ccccc} z_k(\widetilde{Y}_1, *) & \longrightarrow & z_k(\widetilde{X}_1 \amalg Y_1, *) & \longrightarrow & z_k(X_1, *) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ z_{k-i}(\widetilde{Y}_2, * + j) & \longrightarrow & z_{k-i}(\widetilde{X}_2 \amalg Y_2, * + j) & \longrightarrow & z_{k-i}(X_2, * + j) \ , \end{array}$$

where vertical arrows are given as $p_*(p^* c \cap \ ) - c \cap p_*(\ )$. The first two vertical arrows are 0 by the projection formula of (3.6), so the last vertical arrow must be 0 up to homotopy, i.e. for any $\alpha \in A_k(X_1, l)$ one has

$$p_*(p^* c \cap \alpha) = c \cap p_* \alpha \qquad \in A_{k-i}(X_2, l + j) \ .$$

The Gysin homomorphisms are constructed as follows: The regular imbedding f determines a canonical orientation

$$[f] \ \in \ z^d_{\mathrm{op}}(X \xrightarrow{f} Y, 0) \ ,$$

cf. (3.10). Consider a cubical hyperresolution $Y_* \to Y$ such that $X_* := Y_* \times_Y X$ is a cubical hyperresolution of X. Then the induced morphisms $f_\alpha : X_\alpha \to Y_\alpha$ (α is a cubical index) are not necessarily regular, but pull–back of the element $[f]$ induces orientations

$$[f_\alpha] \in z^d_{\text{op}}(X_\alpha \xrightarrow{f_\alpha} Y_\alpha, 0)$$

for each α.

Define a Gysin homomorphism

$$(f_\alpha)_* : z^i_{\text{op}}(X_\alpha, j) \longrightarrow z^{i+d}_{\text{op}}(Y_\alpha, j)$$

as

$$b \mapsto (f_\alpha)_*(b \cdot [f_\alpha]) ,$$

using the product and push–forward of the bivariant operational theory (3.7); cf. [Fu 3, p. 328]. By general properties of bivariant theory [F–M, §9.2.1], these Gysin homomorphisms commute with pull–back, so the various $(f_\alpha)_*$ fit together into a map of cubical complexes, and the result is a map

$$f_* : A^i(X, j) \longrightarrow A^i(Y, j) .$$

To prove this is well–defined, it suffices to prove compatibility with pull–backs; this follows from the analogous property for the $(f_\alpha)_*$ as above.

(v) The usual diagram chase.

(vi) As (3.16).

(vii) A diagram chase reduces to the smooth case, which is (2.12).

(viii) A diagram chase using (4.4)(v) reduces to the smooth case, where it suffices, in view of (4.4)(iii), to prove

$$A_{\dim X - 1}(X, j) = 0 \quad \text{for } j \geq 2 .$$

This is proven for $A^{\text{Bl}}_*(-, *)$ in [Bl 1, Theorem 6.1]; the same proof works for $A_*(-, *)$. □

(4.5) *Remark:* Unfortunately, we cannot construct Gysin homomorphisms for arbitrary l.c.i. morphisms, because of the problem that not all smooth morphisms have a pull–back in Chow homology (only those that are "Chow smooth", cf. (2.21), proof of (i) and remark (2.22).2).

However, if we consider only those l.c.i. morphisms that factor as $p \circ f$, with f a regular imbedding and p a Chow smooth morphism (e.g. $p : Y \times Z \to Z$ a projection with Y smooth), then there are Gysin homomorphisms, defined as in [Fu 3, p. 328].

(4.6) *Compact supports.* For any $X \in \mathcal{VAR}$ we can define compactly supported Chow cohomology $A^*_c(X, *)$ as follows: Set

$$z^i_c(X, *) := \text{Cone}\Big(z^i(\bar{X}, *) \longrightarrow z^i(D, *)\Big) ,$$

where $\bar{X}$ is a compactification of X (which exists by Nagata), and D is the complement $\bar{X} \setminus X$. Then $A^*_c(X, j)$ is defined as

$$A^i_c(X, j) := H_j\big(z^i_c(X, j)\big) ;$$

this definition is independent of choices by (4.4)(v). The theory $A^*_c(-, *)$ is covariant functorial for open immersions, and contravariantly functorial for proper morphisms (by Nagata, any proper morphism is compactifiable). For any $Y \subset X$ closed with complement $U = X \setminus Y$, there is a long exact sequence

$$\longrightarrow A^i_c(U, j) \longrightarrow A^i_c(X, j) \longrightarrow A^i_c(Y, j) \longrightarrow A^i_c(U, j-1) \longrightarrow \cdots .$$

(4.7) **Proposition:** Let $H^*\big(-, \Gamma(*)\big)$ be any of the cohomology theories (1.4). Then there exist natural transformations of contravariant functors on $\mathcal{VAR}$ resp. on $\mathcal{VAR}_*$ (the category of varieties with proper morphisms as arrows):

$$\mathrm{cl}^i(j) : A^i(-, j) \longrightarrow H^{2i-j}\big(-, \Gamma(i)\big)$$

resp.

$$\mathrm{cl}^i_c(j) : A^i_c(-, j) \longrightarrow H^{2i-j}_c\big(-, \Gamma(i)\big) ,$$

such that $\mathrm{cl}^i(j)$ coincides with $\mathrm{cl}_i(j)$ (2.38) on smooth varieties, and $\mathrm{cl}^i(j)$ and $\mathrm{cl}^i_c(j)$ coincide on complete varieties. The map $\mathrm{cl}^i_c(j)$ is also compatible with push–forward for open immersions, and $\mathrm{cl}^i(j)$ resp. $\mathrm{cl}^i_c(j)$ induces maps of long exact sequences, for long exact sequences of type (4.4)(v) resp. (4.6).

Proof: The proof uses Bloch's construction [Bl 2, ¶4] that was already used in (2.38) to get a natural transformation from bigraded Chow homology to homology.

For $X \in \mathcal{VAR}$, let $X_* \to X$ be a cubical hyperresolution. Fix an integer i, and for each X_α of dimension n_α consider the cohomological double complex

$$'K^{p*}(X_\alpha) := \begin{cases} \displaystyle\varinjlim_{Z \in z_{n_\alpha - i}(X_\alpha, -p)} R\Gamma_Z\big(X_\alpha \times \Delta^{-p}, \Gamma(i)\big) & \text{if } 2N \le p \le 0; \\ 0 & \text{otherwise,} \end{cases}$$

for some fixed $N << 0$. The maps $'K^{p*} \to {}'K^{p+1,*}$ are given by pull–backs along faces, cf. (2.38). Let $'E_r(X_\alpha)$ denote the spectral sequence defined by this double complex.

Consider another cohomological double complex K^{**}, which is defined by leaving out the "support in Z" condition in the definition of $'K^{**}$. This second double complex defines a spectral sequence

$$E^{pq}_1(X_\alpha) = H^q\big(X_\alpha \times \Delta^{-p}, \Gamma(i)\big) \;\Rightarrow\; E^n(X_\alpha) = H^n\big(X_\alpha, \Gamma(i)\big)$$

(because of homotopy for $H^*\big(-, \Gamma(*)\big)$, one has degeneration at E_2, cf. (2.38)). The cycle class map gives a map of complexes

$$\tau_{\ge 2N} z^i(X_\alpha, -*) \longrightarrow {}'E^{*,2i}_1(X_\alpha) .$$

This map is compatible with pull–backs (as one easily checks directly), so actually there is a map of *cubical* complexes

$$\tau_{\geq 2N} z^i(X_*, -*) \longrightarrow {}'E_1^{*,2i}(X_*) \ .$$

Using the arguments of (2.38), it follows that for $j < -2N$, there is a map from $A^i(X, j)$ to the limit in degree $2i - j$ of the spectral sequence $E_r(X)$ associated to the double complex given by

$$K^{p*}(X) := sK^{p*}(X_*)$$

(s denotes the single of a cubical complex as in (1.9)). Since $H^*\big(-, \Gamma(*)\big)$ has descent (1.5).2, and since for each X_α, the limit of $E_r(X_\alpha)$ is $H^*\big(X_\alpha, \Gamma(i)\big)$, an easy diagram chase shows that the limit of $E_r(X)$ is $H^*\big(X, \Gamma(i)\big)$.

Call the resulting map $\mathrm{cl}^i(j)$; well–definedness follows from compatibility with pull–backs, which can be reduced to the smooth case.

The construction of $\mathrm{cl}_c^i(j)$ is similar. □

§4.2. Enter K_f^*–theory

(4.8) *Formal K–theory.* Similar to the above descent construction for Chow theory, the formal K–theory K_f^* is defined as follows: Take any strictly contravariant spectrum $\mathbb{K}$ computing K^* (for instance, the spectrum defined using perfect complexes of [T–T, §3], or the Gillet–Soulé spectrum $\mathbb{K}$ mentioned in the proof of (2.24)). For a given variety X, define a new spectrum $\mathbb{K}_f(X)$ as

$$\mathbb{K}_f(X) := \mathrm{Holim}_\alpha \ \mathbb{K}(X_\alpha) \ ,$$

for a cubical hyperresolution $X_* \to X$. The formal K–groups K_f^* are then defined as

$$K_f^j(X) := \pi_j\big(\mathbb{K}_f(X)\big)$$

(Note these groups can be non–zero for negative j, but only for $j \geq -\dim X - 1$; note also that the corresponding descent spectral sequence is strongly convergent because the length of X_* is at most $\dim X + 1$)

To prove this is well–defined, one can proceed as in (4.3). A cheaper method, only proving that $K_f^j(X) \otimes \mathbb{Q}$ is well–defined (which is all we need), is given by the Riemann–Roch result (4.9) below.

One can also define compactly supported K–theory:

$$\mathbb{K}_c(X) := \mathrm{Fibre}\big(\mathbb{K}_f(\bar{X}) \longrightarrow \mathbb{K}_f(D)\big) \ ;$$
$$K_c^j(X) := \pi_j\big(\mathbb{K}_c(X)\big)$$

(here as before $\bar{X}$ denotes a compactification of X with complement $D = \bar{X} \setminus X$). Again the fact that $K_c^*(X) \otimes \mathbb{Q}$ is well–defined will follow from (4.9) below.

(4.9) **Theorem:** Let X be a variety in $\mathcal{VAR}_k$. Then there are functorial isomorphisms

$$ch^j \,:\, K_f^j(X) \otimes \mathbb{Q} \xrightarrow{\sim} \bigoplus_i A^i(X, j) \otimes \mathbb{Q} \,;$$
$$ch_c^j \,:\, K_c^j(X) \otimes \mathbb{Q} \xrightarrow{\sim} \bigoplus_i A_c^i(X, j) \otimes \mathbb{Q} \,.$$

Proof: For the first (resp. second) statement, it suffices to construct a functorial map ch^j (resp. ch_c^j) which coincides with Gillet's Chern character on K^* in the smooth case (resp. in the smooth complete case). Indeed, a diagram chase using (1.21) and (4.4)(v) then reduces to the smooth quasi–projective case, and in view of the commutative diagram

$$\begin{array}{ccc}
K^j\widetilde{X} & \xrightarrow{ch^j} & \bigoplus_i A^i(\widetilde{X}, j)_{\mathbb{Q}} \\
\big\downarrow \wr & & \big\downarrow \wr \ \cap[\widetilde{X}] \\
K_j\widetilde{X} & \xrightarrow{\tau_j} & \bigoplus_i A_i(\widetilde{X}, j)_{\mathbb{Q}}
\end{array}$$

for smooth quasi–projective $\widetilde{X}$ (which is actually the definition of τ_j [Gi 1, §4]), this case follows from (2.24).

Such a map ch^j can actually be constructed with target any cohomology theory satisfying Gillet's axioms [Gi 1, §1] *plus* the descent axiom (1.5).2; the construction is similar to the construction of τ_j in the proof of (2.24).

That is, fix a cubical hyperresolution $X_* \to X$. For each cubical index α, Gillet's Chern character gives a map of simplicial sets

$$ch_\bullet \,:\, \mathbb{K}(X_\alpha)_0 \longrightarrow \amalg_{k \geq 0} R\Gamma\big(X_\alpha, \mathcal{K}(dk, \Gamma(k) \otimes \mathbb{Q})\big) \,.$$

The space $\mathbb{K}(X_\alpha)_m$ for $m > 0$ is weakly equivalent to

$$\Sigma^m \mathbb{K}(X_\alpha)_0$$

(it is supposed that $\mathbb{K}$ is a (fibrant) spectrum), which maps to

$$\amalg_{k \geq 0} \, \Sigma^m R\Gamma\big(X_\alpha, \mathcal{K}(dk, \Gamma(k) \otimes \mathbb{Q})\big) \,,$$

which in its turn is weakly equivalent to

$$\amalg_{k \geq 0} R\Gamma\big(X_\alpha, \mathcal{K}(dk + m, \Gamma(k) \otimes \mathbb{Q})\big)$$

[Th 1, 5.29]. So the result is a map from the spectrum $\mathbb{K}(X_\alpha)$ to a disjoint union of spectra computing the cohomology in question $H^*\big(-, \Gamma(*)\big) \otimes \mathbb{Q}$, well–defined at least in $\mathrm{Ho}(\mathcal{SP})$. Since Gillet's Chern character is a natural transformation of contravariant functors [Gi 1, Lemma 2.23], the maps for the various cubical indices fit together to form a map of cubical spectra (well–defined in $\mathrm{Ho}(\mathcal{SP}^{\square^{\mathrm{opp}}})$). By functoriality of Holim for spectra [Th 1, 5.6], this gives a map from the spectrum

$$\mathrm{Holim}_\alpha\ \mathbb{K}(X_\alpha) =: \mathbb{K}_f(X)$$

to Holim of the second cubical spectrum, whose homotopy groups are the $H^*\big(X, \Gamma(*)\big)$ since we assumed the cohomology satisfied descent (use lemma (1.21)).

The result is thus a map

$$ch^j : K_f^j(X) \longrightarrow \amalg_{k\geq 0} H^{dk-j}\big(X, \Gamma(k)\otimes\mathbb{Q}\big)\ ,$$

and it is obvious from the construction that these maps are compatible with long exact M–V sequences. $\square$

(4.10) *Remarks.*
1. The definition of K_f^* and K_c^* was directly inspired by work of Gillet and Soulé [G–S]. In fact, they define the compactly supported theory K_c^* and show that it is (integrally) well–defined using their weight complex in the homotopy category of motives.

It follows from the Riemann–Roch isomorphism (4.9) that the rational groups $K_f^*\otimes\mathbb{Q}$ (and $K_c^*\otimes\mathbb{Q}$) have all the functorial properties that the $A^*(-,*)\otimes\mathbb{Q}$ (resp. $A_c^*(-,*)\otimes\mathbb{Q}$) have, and that are detailed in (4.4) (resp. (4.6)).

With some harder work in the homotopy category of spectra, most of these properties can also be proven directly (and integrally) for the K_f^* (and K_c^*), as Gillet and Soulé have shown; for instance, cup – and cap product come from a pairing of spectra [G–S, Appendix], there is a projection formula on the level of spectra [G–S, 5.1.2]. Note that since ch^* is a Riemann–Roch map, so compatible with cup – and cap products etcetera, these two ways of creating structures on K_f^* and K_c^* should be compatible.

(Note, by the way, that the approach taken here is inverse to the approach of [SGA 6], where Riemann–Roch was used to transport properties *from* K_0 *to* the rational Chow groups $A_*\otimes\mathbb{Q}$.)

However, (4.9) has at least one corollary that seems difficult to establish directly; that is that one can define "gradeds for the γ–filtration" on $K_f^*\otimes\mathbb{Q}$ and $K_c^*\otimes\mathbb{Q}$ using the isomorphism (4.9). These gradeds coincide with the real γ–filtration in the smooth (resp. the smooth and complete) case, and they are exact functors on M–V and localization long exact sequences with $\mathbb{Q}$–coefficients (as one expects from a γ–filtration).

2. Contrary to formal K–theory, the "honest" K^*–theory, i.e. the one defined by the exact category of vector bundles, does not always satisfy homotopy in the singular case [C–S]. In fact, Weibel [We] has introduced a different contravariant K–theory, called "homotopy K–theory" which *does* satisfy homotopy. I have no idea about the relation between Weibel's theory and K_f^*.

3. The "local to global" spectral sequence was proven for K_*–theory, and hence for K^* of regular schemes, by Brown and Gersten [B–G, §3 Corollary]. The corresponding result for

K^* of singular schemes is also true, but is a far deeper theorem, cf. [T–T, 10.3, 10.8], [Th 3, §2].

§4.3. A^*(-,0) versus A^*

(4.11) *Remark:* The subring $A^*(X,0) \subset A^*(X,*)$ is an intersection ring in the sense of the introduction. It is finer than Fulton–MacPherson Chow cohomology A^*, because the last theory does not, in general, have a non–trivial natural transformation into cohomology [To 2, Theorem 7] (see also (6.3) for an explicit example of a singular surface with $A^1(S,0)\otimes\mathbb{Q}$ different from $A^1S\otimes\mathbb{Q}$).

Note that for an M–V diagram

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

Kimura's theorem (3.19), [Ki] gives a short exact sequence for Chow cohomology

$$0\longrightarrow A^iX\longrightarrow A^i\widetilde{X}\oplus A^iY\longrightarrow A^i\widetilde{Y}\ ,$$

whereas for bigraded Chow cohomology, there is a long exact sequence

$$\cdots\longrightarrow A^i(\widetilde{Y},1)\longrightarrow A^i(X,0)\longrightarrow A^i(\widetilde{X},0)\oplus A^i(Y,0)\longrightarrow A^i(\widetilde{Y},0)$$

(4.4)(v), which in general does not split at the second arrow for singular X. Roughly speaking, A^i behaves (at least for complete varieties) like a graded piece of cohomology

$$\mathrm{Gr}^W_{2i}H^{2i}(-,\Gamma(i))$$

with respect to the weight filtration, whereas $A^i(-,0)$ behaves rather like the whole cohomology group

$$H^{2i}(-,\Gamma(i))\ .$$

This also helps to "explain" Totaro's result about the non–existence of a cycle class map from A^* into cohomology: the exact sequence shows that A^i maps to $\mathrm{Gr}^W_{2i}H^{2i}$ (this is the argument in [B–G–S, Appendix]), and this map doesn't always lift to H^{2i} since the weight filtration on the cohomology of singular varieties is generally non–split.

Chapter 5: Linear varieties

To get some interesting examples, we consider the class of linear varieties, as introduced by Jannsen [Ja 2] and studied by Totaro [To 2]; these are the among the simplest singular varieties.

The theme of the results in this section (which continue results of [Ja 2] and [To 2]) is that for linear varieties, the different cohomology theories are as related as they can possibly be; for instance rational equivalence (which is the finest equivalence relation) equals numerical equivalence (which is the coarsest equivalence relation) for cycles on smooth linear varieties.

All the proofs use localization and homotopy to reduce to the case of a point, where the statement to be proven is usually trivial.

At several (explicitly indicated) places in this chapter, the use of constructions involving descent will force us to suppose ROS (resolution of singularities, cf. (1.2)).

The following definition was made by Totaro [To 2], as a variant of a definition of Jannsen [Ja 2, §14]:

(5.1) **Definition:** The class of *linear varieties* over a field k is defined by the following inductive procedure:

(i) Any variety isomorphic to a linear variety is linear;
(ii) Affine space $\mathbb{A}_k^m$ is a linear variety for any $m \in \mathbb{Z}_{\geq 0}$;
(iii) The complement of a linear variety, closed in affine space, is a linear variety;
(iv) A variety that has a closed subvariety Y such that Y and its complement are both linear, is a linear variety.

(5.2) *Examples:* A variety that can be stratified in a finite number of strata isomorphic to $\mathbb{A}^a \times (\mathbb{G}_m)^b$ is a linear variety—in particular, as Totaro points out [To 2], any variety (over an algebraically closed field) that admits an action of a connected solvable linear algebraic group with finitely many orbits, is linear. This includes spherical varieties in the sense of [F–M–S–S] and the literature cited there; in particular toric varieties [Fu 4] are linear.

For examples of linear varieties that do not have a stratification with strata isomorphic to $\mathbb{A}^a \times (\mathbb{G}_m)^b$, cf. [To 2].

§5.1. Chow theory of linear varieties

(5.3) **Theorem:** Let X be a linear variety over k, and suppose one of the following:

(1) $k = \mathbb{C}$ and $H_*(X, \Gamma(*))$ resp. $H_c^*(X, \Gamma(*))$ is singular or Deligne homology resp. cohomology with compact supports;

(2) k is algebraically closed, X has ROS and H_* resp. H_c^* is étale homology resp. cohomology.

Then there are isomorphisms

$$\mathrm{cl}_i(0) : A_iX \otimes R \xrightarrow{\sim} \mathrm{Gr}^0_W H_{2i}\big(X, \Gamma(i)\big) = W^0 H_{2i}\big(X, \Gamma(i)\big) ,$$

where $R := H^0\big(\mathrm{Spec}\, k, \Gamma(0)\big)$ and W is the weight filtration (1.9). In case (1) there are moreover isomorphisms

$$\mathrm{cl}^i_c(0) : A^i_c X \xrightarrow{\sim} \mathrm{Gr}^W_0 H^{2i}_c(X, \Gamma(i)) .$$

Proof: Let $U \subset X$ be open, such that $U \cong \mathbb{A}^n \setminus Z$ for some $Z \subset \mathbb{A}^n$ closed which is a linear variety, and such that the complement $Y := X \setminus U$ is also a linear variety. By an induction on the dimension and on the number of irreducible components, we may suppose the statement holds for Z and Y.

For the first isomorphism, consider the commutative diagram (2.38) with exact rows

$$\begin{array}{ccccccccc}
A_i(U,1) & \to & A_iY & \to & A_iX & \to & A_iU & \to 0 \\
\downarrow{\scriptstyle \mathrm{cl}_i(1)} & & \wr\downarrow{\scriptstyle \mathrm{cl}_i(0)} & & \downarrow{\scriptstyle \mathrm{cl}_i(0)} & & \wr\downarrow{\scriptstyle \mathrm{cl}_i(0)} & \\
\mathrm{Gr}^0_W H_{2i+1}U & \to & \mathrm{Gr}^0_W H_{2i}Y & \to & \mathrm{Gr}^0_W H_{2i}X & \to & \mathrm{Gr}^0_W H_{2i}U & \to 0
\end{array}$$

(where we have abbreviated $H_*\big(-, \Gamma(i)\big)$ to H_*, and $A_i(-, j; R)$ to $A_i(-, j)$). We are done if we can prove the left vertical arrow is a surjection; this follows from a similar diagram

$$\begin{array}{ccccccccc}
A_i(\mathbb{A}^n,1) & \to & A_i(U,1) & \to & A_iZ & \to & A_i\mathbb{A}^n & \to \\
\downarrow & & \downarrow{\scriptstyle \mathrm{cl}_i(1)} & & \wr\downarrow{\scriptstyle \mathrm{cl}_i(0)} & & \wr\downarrow{\scriptstyle \mathrm{cl}_i(0)} & \\
0 & \to & \mathrm{Gr}^0_W H_{2i+1}U & \to & \mathrm{Gr}^0_W H_{2i}Z & \to & \mathrm{Gr}^0_W H_{2i}\mathbb{A}^n & \to ,
\end{array}$$

in which the lower left corner is $\mathrm{Gr}^0_W H_{2i+1}\big(\mathbb{A}^n, \Gamma(i)\big)$ which is 0 by homotopy, and the two right vertical arrows are isomorphisms by induction and by homotopy (in fact, for the far right arrow source and target are 0 except for $i = n$).

For the second statement, note that it follows from descent that the bigraded Chow cohomology $A^*_c(X, *)$ has a weight filtration, and $A^*_c X$ identifies to the graded $\mathrm{Gr}^W_0 A^*_c(X, 0)$ by comparing (3.21) and (4.4)(v). So we might as well prove that $cl^i_c(0)$ induces isomorphisms

$$\mathrm{Gr}^W_0 A^i_c(X, 0) \xrightarrow{\sim} \mathrm{Gr}^W_0 H^{2i}_c\big(X, \Gamma(i)\big) .$$

This is completely dual to the above proof, since we have the same kind of exact sequences but with arrows reversed. □

(5.4) *Remarks:*

1. In the case of singular (co)homology, usually $H_*(-, \mathbb{Z})$ is identified with $H_*(-, \mathbb{Z}(0))$ (instead of with $H_*(-, \mathbb{Z}(i))$ as above); under this identification (5.3) becomes

$$A_iX \xrightarrow{\sim} \mathrm{Gr}^{-2i}_W H_{2i}(X, \mathbb{Z}) = W^{-2i} H_{2i}(X, \mathbb{Z}) ;$$
$$A^i_c X \xrightarrow{\sim} \mathrm{Gr}^W_{2i} H^{2i}_c(X, \mathbb{Z}) ,$$

where $W \otimes \mathbb{Q}$ is now the weight filtration of Deligne [De 1].

The first isomorphism was proven with rational coefficients by Totaro [To 2, Theorem 3]. Surjectivity of the first map with rational coefficients, which implies the singular version of the Hodge and Tate conjectures (as formulated by Jannsen [Ja 2]) was proven by Jannsen [Ja 2, Theorem 14.7].

2. A subclass of the class of linear varieties is formed by the so–called cellular varieties [Fu 3, Example 1.9.1], i.e. varieties X admitting a filtration by closed subvarieties

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_{-1} = \emptyset$$

with each $X_i - X_{i-1}$ a disjoint union of varieties isomorphic to affine spaces. For cellular varieties, one easily proves by induction that H_i and H_c^i are pure of the expected weight (using that this is true for affine space), so the cycle class maps induce isomorphisms

$$\begin{aligned} A_i X &\xrightarrow{\sim} H_{2i}\big(X, \Gamma(i)\big) \ ; \\ A_c^i X &\xrightarrow{\sim} H_c^{2i}\big(X, \Gamma(i)\big) \ . \end{aligned}$$

This can actually be proven without a higher Chow group argument, cf. [Fu 3, Example 19.1.11].

3. Let X be a complete toric variety of dimension n. Using (5.3), one can recover a result of Barthel, Brasselet, Fieseler and Kaup, namely the existence of a commutative diagram

$$\begin{array}{ccc} \operatorname{Pic} X & \longrightarrow & A_{n-1}X \\ \big\downarrow \wr & & \big\downarrow \wr \\ H^2\big(X, \Gamma(1)\big) & \longrightarrow & H_{2n-2}\big(X, \Gamma(n-1)\big) \end{array}$$

in which the arrows are the obvious maps [B–B–F–K]. This follows from (5.3) after proving that the indicated H^2 and H_{2n-2} are all in weight 0, and that there is a natural isomorphism $\operatorname{Pic} X \xrightarrow{\sim} A^1 X$ [F–S, Corollary 2.4], [Br].

In particular, the bottom map in the diagram is an injection.

(5.5) **Theorem:** Let X be a linear variety over a field k, let V be an arbitrary variety over k and suppose $\mathcal{VAR}_k$ has ROS. Then the natural map induces isomorphisms

$$\bigoplus_{l+m=i} A_l X \otimes \mathrm{Gr}_W^{-j} A_m(V, j) \xrightarrow{\sim} \mathrm{Gr}_W^{-j} A_i(X \times V, j)$$

for each $j \in \mathbb{N}$, where W denotes the weight filtration (1.9). In particular,

$$\bigoplus_{l+m=i} A_l X \otimes A_m V \xrightarrow{\sim} A_i(X \times V)$$

(for this statement, ROS is not actually needed), and if X is smooth and complete,

$$\bigoplus_{l+m=i} A_l X \otimes A_m(k, j) \xrightarrow{\sim} A_i(X, j) \ .$$

Proof: Continuing with the notation Y, U and Z of (5.3), let M be the scheme obtained by pasting X and $\mathbb{A}^n$ together along the open U. The scheme M might be non–separated, but we shall see that we can still make sense of $A_*(M,*)$ and, more generally, of $A_*(M\times V,*)$.

We define $A_i(M\times V,j)$ as the jth homology group of the complex

$$z_i(M\times V,*) := \mathrm{Cone}\Big(z_i\big((X\times V)\amalg(\mathbb{A}^n\times V),*\big)\xrightarrow{f} z_i(U\times V,*)\Big)\ ,$$

where the map f is the difference of pull–back along the two inclusion maps (note that since the inclusion of the image of f in $z_i(U\times V,*)$ is a quasi–isomorphism (2.21)(ii), we get the same $A_i(M\times V,j)$ if we replace Cone by Kernel in this definition). We moreover define

$$\mathrm{Gr}_W^{-j}A_i(M\times V,j) = W^{-j}A_i(M\times V,j)$$

as the subspace of elements for which the image under f is in $\mathrm{Gr}_W^{-j}=W^{-j}$.

It is immediate from this definition that one has the expected exact sequences

$$W^{-j}A_i(Y\times V,j)\longrightarrow W^{-j}A_i(M\times V,j)\longrightarrow W^{-j}A_i(\mathbb{A}^n\times V,j)\longrightarrow 0$$

and

$$W^{-j}A_i(Z\times V,j)\longrightarrow W^{-j}A_i(M\times V,j)\longrightarrow W^{-j}A_i(X\times V,j)\longrightarrow 0\ .$$

Using this definition and (2.34), one can also extend the exterior product map

$$A_lM\otimes W^{-j}A_m(V,j)\longrightarrow W^{-j}A_{l+m}(M\times V,j)$$

to M.

Now for any variety N, let B_i denote either $\oplus_{l+m=i}A_lN\otimes W^{-j}A_m(V,j)$ or $W^{-j}A_i(N\times V,j)$. Consider the commutative diagram

$$(\sharp)\qquad \begin{array}{ccccccccc}
 & & & & & & B_iZ & & \\
 & & & & & & \downarrow & & \\
 & & & & B_iZ & \xrightarrow{\sim} & B_iZ & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_iY & \longrightarrow & B_iM & \longrightarrow & B_i\mathbb{A}^n & & \\
 & & \downarrow\wr & & \downarrow & & \downarrow & & \\
 & & B_iY & \longrightarrow & B_iX & \longrightarrow & B_iU & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0\ , & &
\end{array}$$

in which all arrows are given by push–forward resp. pull–back along a closed resp. an open immersion. The idea is to show that all rows and columns in ($\sharp$) are exact for both choices of B_i; then since the diagram for

$$B_i = \bigoplus_{l+m=i} A_l(-)\otimes W^{-j}A_m(V,j)$$

maps commutatively to the diagram for

$$B_i = W^{-j}A_i(- \times V, j)\ ,$$

the snake lemma gives a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\oplus A_l Z \otimes W^{-j}A_m(V,j) & \to & \oplus A_l Y \otimes W^{-j}A_m(V,j) & \to & & & \\
\downarrow & & \downarrow & & & & \\
W^{-j}A_i(Z\times V,j) & \to & W^{-j}A_i(Y\times V,j) & \to & & & \\
& & \oplus A_l X \otimes W^{-j}A_m(V,j) & \to & \oplus A_l U \otimes W^{-j}A_m(V,j) & \to 0 & \\
& & \downarrow & & \downarrow & & \\
& & W^{-j}A_i(X\times V,j) & \to & W^{-j}A_i(U\times V,j) & \to 0\ , &
\end{array}$$

and we are done by noetherian induction.

The exactness of ($\sharp$) for the second choice of B_i is a corollary of (2.21)(ii) and exactness properties of Gr^*_W. Since tensoring is right–exact, the same argument gives exactness of ($\sharp$) for the first choice of B_i, except for the injectivity of $B_iY \to B_iM$. However, by induction we may suppose (5.5) is true for Y, and the desired injectivity then follows from injectivity in the diagram for the second choice of B_i.

The second statement of (5.5) follows upon remarking that the Chow groups are always pure of weight 0 (and going through the above proof for $j = 0$ using the standard short exact sequence for Chow groups shows that the existence of a weight filtration, and hence the hypothesis ROS, is not needed to prove this second statement).

For the last statement, take $V = \mathrm{Spec}\, k$ and use that $A_i(X, j) = W^{-j}A_i(X, j)$ if X is smooth and complete. $\square$

(5.6) **Theorem** (Totaro): Let X be a complete linear variety over a field k. Then the canonical evaluation homomorphism induces isomorphisms

$$A^iX \xrightarrow{\sim} \mathrm{Hom}(A_iX, \mathbb{Z})$$

(A^* denotes, as before, operational Chow cohomology [F–M], [Fu 3, Chapter 17]; the evaluation homomorphism is given by $c \mapsto \deg(c \cap -)$).

Proof: Totaro [To 2, Theorem 2] proved this using the "Chow–Künneth criterion" of [F–M–S–S]. The Chow–Künneth criterion says: If X is a complete variety such that the obvious map

$$\bigoplus_{j+k=i} A_jX \otimes A_kY \longrightarrow A_i(X \times Y)$$

is an isomorphism for all varieties Y and all $i \in \mathbb{Z}_{\geq 0}$, then the evaluation homomorphisms are isomorphisms.

To check the Chow–Künneth criterion for linear varieties, one can either use higher Chow groups as in [To 2], or one can proceed as in (5.5). $\square$

(5.7) *Remarks:*
1. If a linear variety X can be embedded in a complete linear variety such that the complement is again linear, then it follows from (5.6) and (3.21) that there are isomorphisms

$$A^i_c X \xrightarrow{\sim} \mathrm{Hom}(A_i X, \mathbb{Z}) \ .$$

This conclusion applies for instance to toric varieties.

2. Let X be a smooth complete linear variety over k, and suppose ROS for $\mathcal{VAR}_k$. Then $A_*(X,*)$ is a free finitely generated $A_*(k,*)$–module. (Combine (5.6) and the last statement of (5.5).)

§5.2 Algebraic and topological K–theory of linear varieties

(5.8) Let $\mathrm{Gr}^{\mathrm{top}}_* K_0$ denote the gradeds of the topological filtration on K_0, i.e. graded by dimension of support of coherent sheaves (the notation $\mathrm{Gr}^*_{\mathrm{top}}$ will indicate grading by codimension). There is a homomorphism

$$\varphi : \ A_i X \longrightarrow \mathrm{Gr}^{\mathrm{top}}_i K_0 X \ ,$$

defined by sending $[V]$ to $[\mathcal{O}_V]$ [SGA 6, pp. 58—59], [Fu 3, 15.1.5]. The homomorphism φ is surjective, and $\varphi \otimes \mathbb{Q}$ is an isomorphism. φ is compatible with proper push–forward [Fu 3, 15.1.5], and with pull–back for open immersions [SGA 6, p. 52 2.22].

(5.9) **Theorem:** Let X be a linear variety over a field k. Then there are isomorphisms

$$\varphi : \ A_i X \xrightarrow{\sim} \mathrm{Gr}^{\mathrm{top}}_i K_0 X \ .$$

Proof: We will use the same strategy as in the proof of (5.5). So let U, Y, Z be as before, and let M be the scheme obtained by pasting together X and $\mathbb{A}^n$ along the open subset U. The scheme M might be non–separated, but we shall see that we can still make sense of the Chow groups and K_0 of M.

The Chow groups of M are as defined in the proof of (5.5). As for K–theory, we define $K_0 M$ as π_0 of the spectrum

$$\mathbb{K}'(M) := \mathrm{Fibre}\Big(\mathbb{K}'(X \amalg \mathbb{A}^n) \xrightarrow{f} \mathbb{K}'(U)\Big) \ ,$$

the map f being as before the difference of the two pull–backs along the inclusion maps. From this definition, one can deduce exact sequences for K_0 similar to the ones in (5.5) for Chow theory. In fact, one knows from [T–T] that $K_0 M$ is isomorphic to the Grothendieck

K^0 of the exact category of coherent sheaves on M (Thomason and Trobaugh have extended the K_*–theory that Quillen had defined for noetherian separated schemes to quasi–compact quasi–separated schemes, in such a way that one still has the localization sequence). Defining the topological filtration on K_0M as in (5.6), we need to know exactness of

$$\mathrm{Gr}_i^{\mathrm{top}} K_0 M \longrightarrow \mathrm{Gr}_i^{\mathrm{top}} K_0(X \amalg \mathbb{A}^n) \longrightarrow \mathrm{Gr}_i^{\mathrm{top}} K_0 U \longrightarrow 0 \ .$$

In the case of varieties (i.e. *separated* schemes over a field), Fulton and Lang [F–L, Ch. VI Prop. 6.5] show how this exactness property of $\mathrm{Gr}_*^{\mathrm{top}}$ follows from the Riemann–Roch theorem of [B–F–M], answering a question that was open at the time of SGA 6 [SGA 6, p. 458]. In our case, we could extend the Riemann–Roch theorem (and hence the exactness property) to all schemes obtained by pasting together varieties (using Gillet's result that the map τ comes from an underlying map of simplicial sheaves). But in this special case, the desired exactness is easily proven directly, since $\mathrm{Gr}_i^{\mathrm{top}} K_0 \mathbb{A}^n$ and $\mathrm{Gr}_i^{\mathrm{top}} K_0 U$ are 0 for $i < n$, and for $i = n$ the exactness is immediate.

From the exact sequences for Chow and K–theory, one can extend the map φ to

$$\varphi : \ z_i(M, 0) \longrightarrow \mathrm{Gr}_i^{\mathrm{top}} K_0 M \ ,$$

and the usual homotopy argument shows φ descends to $A_i M$.

Let B_i denote either A_i or $\mathrm{Gr}_i^{\mathrm{top}} K_0$, and consider (as in the proof of (5.5)) the commutative diagram

$$(\sharp) \qquad \begin{tikzcd}
 & & & B_i Z \arrow[d] & \\
 & & B_i Z \arrow[r, "\sim"] \arrow[d] & B_i Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_i Y \arrow[r] \arrow[d, "\wr"] & B_i M \arrow[r] \arrow[d] & B_i \mathbb{A}^n \arrow[d] & \\
 & B_i Y \arrow[r] \arrow[d] & B_i X \arrow[r] \arrow[d] & B_i U \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 \ , &
\end{tikzcd}$$

in which all arrows are given by push–forward resp. pull–back along a closed resp. an open immersion. Once more, we'll check that all rows and columns in ($\sharp$) are exact and then apply the snake lemma.

The middle row is exact; since $B_i \mathbb{A}^n = 0$ for $i < n$, we only need to prove $B_i Y \xrightarrow{\sim} B_i M$ for $i < n$. For Chow theory this is immediate since

$$A_i(M, 1) \longrightarrow A_i(\mathbb{A}^n, 1)$$

is surjective by homotopy; the same argument gives

$$K_0 Y \longrightarrow K_0 M$$

is an injection, so $F_i^{\mathrm{top}}K_0Y \xrightarrow{\sim} F_i^{\mathrm{top}}K_0M$ for $i < n$, so

$$\mathrm{Gr}_i^{\mathrm{top}}K_0Y \xrightarrow{\sim} \mathrm{Gr}_i^{\mathrm{top}}K_0M$$

for $i < n$.

The middle column is exact as well; for Chow theory this was already established above, for K–theory we need the graded analogue of

$$K_0Z \longrightarrow K_0M \longrightarrow K_0X \longrightarrow 0 \ .$$

This can be directly established from the diagram with exact rows and columns

$$\begin{tikzcd}
B_iZ \arrow[r, "\sim"] \arrow[d] & B_iZ \arrow[d] & & \\
B_iM \arrow[r] \arrow[d] & B_i(X \amalg \mathbb{A}^n) \arrow[r] \arrow[d] & B_iU \arrow[r] \arrow[d, "\wr"] & 0 \\
B_iX \arrow[r] & B_i(X \amalg U) \arrow[r] & B_i(U) \arrow[r] & 0 \ .
\end{tikzcd}$$

The left and right columns of ($\sharp$) are trivially exact.

Now considering the diagrams ($\sharp$) for A_i resp. for $\mathrm{Gr}_i^{\mathrm{top}}K_0$, the map φ maps the first diagram to the second, and by the snake lemma we get a commutative diagram with exact rows

$$\begin{tikzcd}
A_iZ \arrow[r] \arrow[d, "\varphi"] & A_iY \arrow[r] \arrow[d, "\varphi"] & A_iX \arrow[r] \arrow[d, "\varphi"] & A_iU \arrow[r] \arrow[d, "\varphi"] & 0 \\
\mathrm{Gr}_iK_0Z \arrow[r] & \mathrm{Gr}_iK_0Y \arrow[r] & \mathrm{Gr}_iK_0X \arrow[r] & \mathrm{Gr}_iK_0U \arrow[r] & 0 \ ,
\end{tikzcd}$$

and we are done by induction. □

(5.10) *Remarks:*

1. Theorem (5.9) does not hold for all varieties; in [SGA 6, pp. 679—680] Grothendieck gives an example where A_* has more torsion than $\mathrm{Gr}_*^{\mathrm{top}}K_0$.

2. The special case of (5.9) where X is a flag variety G/B (G a connected semi–simple linear algebraic group, B a Borel subgroup) was proven by Marlin [Mar, Section 3]; the case where X is a weighted projective space was proven by Al Amrani [AA, 3.1 Corollaire]—in both these cases, there is no torsion in A_*X and $\mathrm{Gr}_*^{\mathrm{top}}K_0X$.

In the complex case, one can compare algebraic and topological K–theory:

(5.11) **Theorem:** Let X be a smooth projective linear variety over $\mathbb{C}$. Then the forgetful map induces an isomorphism

$$K^0X \xrightarrow{\sim} K^0_{\mathrm{top}}X \ .$$

Proof: In fact, we will prove a more general result formulated in terms of homology, namely that for any quasi–projective linear variety X, there are isomorphisms

$$K_0X \xrightarrow{\sim} \mathrm{Gr}^0_W K_0^{\mathrm{top}}X \ .$$

Here $K_0^{\rm top}$ is the homology version of $K^0_{\rm top}$, as defined in [B–F–M 2] (the same theory can also be constructed in a somewhat different way as part of a bivariant theory, as in [F–M, 3.1]), and Gr^0_W is a graded of the weight filtration, which can be defined ad hoc as

$$\mathrm{Gr}^0_W K_0^{\rm top} X := \mathrm{Im}\Big(K_0^{\rm top} X' \longrightarrow K_0^{\rm top} X\Big) \ ,$$

for some smooth complete X' having an open $\widetilde{X}$ that maps properly and surjectively to X, and the map on K_0 is composition of pull–back and push–forward (One can show the result is independent of choices).

The theorem follows from this homology result, since for smooth X, there are isomorphisms from K^0 resp. $K^0_{\rm top}$ to K_0 resp. $K_0^{\rm top}$, and for smooth complete X, $K_0^{\rm top} X$ is all in weight 0; the construction will show that the induced isomorphism is indeed the forgetful map.

The proof of the homology statement is similar to the proofs of (5.5) and (5.9). That is, we again consider an M which is possibly non–separated, show that one can make sense of K_0 and $K_0^{\rm top}$ of M, and get from the snake lemma a commutative diagram whose rows are 4–term exact sequences.

We have already discussed $K_0 M$; topological K–homology of M is defined as the homotopy groups of the spectrum $\mathbb{K}^{\rm top}(M)$, which is

$$\mathbb{K}^{\rm top}(M) := \mathrm{Fibre}\Big(\mathbb{K}^{\rm top}(X \amalg \mathbb{A}^n) \longrightarrow \mathbb{K}^{\rm top}(U)\Big) \ ,$$

where for a *variety* W, the spectrum $\mathbb{K}^{\rm top}(W)$ is the spectrum computing $K_*^{\rm top}$, cf. [Gi 1, Section 5].

We define the graded

$$\mathrm{Gr}^0_W K_0^{\rm top} M$$

as those elements for which the image is in

$$\mathrm{Gr}^0_W K_0^{\rm top}(X \amalg \mathbb{A}^n) \ .$$

With this definition, we get short exact sequences

$$\mathrm{Gr}^0_W K_0^{\rm top} Y \longrightarrow \mathrm{Gr}^0_W K_0^{\rm top} M \longrightarrow \mathrm{Gr}^0_W K_0^{\rm top} \mathbb{A}^n \longrightarrow 0$$

and

$$\mathrm{Gr}^0_W K_0^{\rm top} Z \longrightarrow \mathrm{Gr}^0_W K_0^{\rm top} M \longrightarrow \mathrm{Gr}^0_W K_0^{\rm top} X \longrightarrow 0 \ .$$

As in the proof of (5.7), we get two diagrams; the first for K_0, the second for $\mathrm{Gr}^0_W K_0^{\rm top}$. The map from the first to the second diagram is given by Baum–Fulton–MacPherson's topological Riemann–Roch theorem; this map is compatible with push–forward resp. pull–back for closed resp. open immersions, and we conclude as in (5.5) and (5.9).

(Alternatively, the map from algebraic K_0 to $K_0^{\rm top}$ appears as part of a Grothendieck transformation between bivariant theories [F–M, 3.2], which also implies this map commutes with push–forward and pull–back). $\square$

(5.12) *Remark:* Let X be as in (5.11). Then one has a diagram of isomorphisms

$$\begin{array}{ccc} A^iX & \xrightarrow{\sim} & H^{2i}(X,\mathbb{Z}) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Gr}^i_{\mathrm{top}}K^0X & \xrightarrow{\sim} & \mathrm{Gr}^iK^0_{\mathrm{top}}X \end{array} ,$$

in which the two vertical resp. the two horizontal arrows are given by (5.9) and degeneration of the Atiyah-Hirzebruch spectral sequence ($H^*(X,\mathbb{Z})$ being torsion–free by (5.3) and (5.5)), resp. by (5.3) and (5.11). This diagram commutes (because of torsion freeness, it suffices to prove this after tensoring with $\mathbb{Q}$, and then it follows from Riemann–Roch).

Note that whereas the second vertical arrow comes from degeneration of the Atiyah–Hirzebruch spectral sequence, there is no spectral sequence explaining the first vertical arrow. In fact, the isomorphism induced by the first vertical arrow should follow from degeneration of the conjectural motivic spectral sequence (2.41).4.

For higher K–theory, theorem (5.11) cannot possibly be true (for instance, the complex point $\mathrm{Spec}\,\mathbb{C}$ has enormous higher algebraic K–groups, whereas $K^{2j+1}_{\mathrm{top}}(\mathrm{Spec}\,\mathbb{C}) = 0$ for all j). Instead, we shall see (5.14) that a version of (5.11) with finite coefficients still holds.

(5.13) *Notation.* Let X be a variety over an arbitrary field k. For ℓ^r a prime power, one can consider K–theory with coefficients

$$(K/\ell^r)^j(X) := \pi_j(\mathbb{K}/\ell^r) ,$$

where $\mathbb{K}/\ell^r$ is the mod ℓ^r reduction of the K–theory spectrum, defined by the homotopy fibre sequence

$$\mathbb{K} \xrightarrow{\cdot\ell^r} \mathbb{K} \longrightarrow \mathbb{K}/\ell^r$$

[Th 1, Appendix A], [Th 2, 3.0].

Replacing K^0_{top} (which only exists if $k = \mathbb{C}$), there is the étale topological K–theory

$$(K_{\mathrm{top}}/\ell^r)^*(X)$$

of Dwyer and Friedlander [D–F], related to étale cohomology in much the same way that K^0_{top} in the complex case was related to singular cohomology.

As in the complex case, there is again a map

$$(K/\ell^r)^j(X) \longrightarrow (K_{\mathrm{top}}/\ell^r)^j(X) ,$$

intensively studied by Suslin, Gabber, Gillet, Thomason; cf. [Th 2] and the references given there. The conjecture is that for ℓ prime to the characteristic of k, this map is an isomorphism for $j > 2 \cdot \dim X$.

(5.14) **Proposition:** Let X be a smooth complete linear variety over an algebraically closed field k, let ℓ be prime to $\mathrm{char}\,k$. Then there are isomorphisms

$$(K/\ell^r)^j(X) \xrightarrow{\sim} (K_{\mathrm{top}}/\ell^r)^j(X)$$

for all $j \geq 0$.

Proof: Again, we actually prove a more general homological result, which is that for any linear variety X, there are isomorphisms in homology

$$(K/\ell^r)_j(X) \xrightarrow{\sim} (K_{\text{top}}/\ell^r)_j(X) \ .$$

Here the right–hand–side is defined as homotopy groups of a spectrum

$$(\mathbb{K}'/\ell^r)(X)[\beta^{-1}]$$

obtained by inverting β mapping to the Bott element; cf. [Th 1, Remark 4.17], where this definition is justified by the fact that these homotopy groups equal the étale topological K–theory for regular schemes.

The homology result is proven inductively from a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\longrightarrow & (K/\ell^r)_j(X) & \longrightarrow & (K/\ell^r)_j(U) & \longrightarrow & (K/\ell^r)_{j-1}(Y) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & (K_{\text{top}}/\ell^r)_j(X) & \longrightarrow & (K_{\text{top}}/\ell^r)_j(U) & \longrightarrow & (K_{\text{top}}/\ell^r)_{j-1}(Y) & \longrightarrow
\end{array}$$

and the case of a point, which is the result of Gabber and Suslin [Th 2, Theorem 7.1] (Note that contrary to the other results in this section, here the case of a point is not trivial but a very deep result !) $\square$

(5.15) *Remark:* The result (5.14) was proven by Friedlander for cellular varieties [Fr], cf. also [Th 2, Proposition 10.3].

§5.3. Comments

(5.16) *Remarks:*
1. For a smooth complete linear variety X, many of the results in this chapter also follow from motivic considerations. For instance, using the Chow–Künneth formula (5.5) and Manin's identity principle, it follows that the Chow motive of X is a direct sum of twisted Lefschetz motives. This implies that for any Weil cohomology theory, the cycle class map is an isomorphism—see [Ja 3, Theorem 3.8] for more implications along these lines.

In this light, linear varieties should perhaps be considered a (very simple) prototype of what is expected to be a general principle: the behaviour of algebraic cycles somehow "determines" the various cohomology theories.

2. The reader will have noticed, that in order for the inductions in this section to work, it was essential to have a *higher* theory at our disposal (i.e. K_1, $A_*(-,1)$), extending the standard short exact sequences into long exact sequences. For this reason Grothendieck, writing before the birth of higher K–theory, could only establish surjectivity in the Künneth formula for K–theory of cellular varieties [SGA 6, p. 60 Prop. 2.13] (this is the same formula as the Chow–Künneth formula of (5.5), but with K_0 instead of A_*).

Chapter 6: On A^1(-,0)

In general, it seems difficult to relate the codimension 1 part of Chow cohomology to the Picard group. In the complex case, this is made easier by the existence of Deligne cohomology. In fact, for a complex variety X it turns out that $A^1(X,0)$ coincides with a certain Deligne cohomology group (which actually is not really surprising, both these cohomology theories being defined by descent). As a corollary, the question whether $A^1(X,0)$ coincides with the Picard group of X becomes a question about the Hodge filtration on singular cohomology of X (6.2).

(6.0) For X a variety over $\mathbb{C}$, $H^*_{\mathcal{D}}(X,\mathbb{Z}(*))$ will denote the Deligne cohomology groups, as defined in (1.4).4 and the references given there.

(6.1) **Proposition:** Let X be a variety over $\mathbb{C}$. Then the map $\mathrm{cl}^1(0)$ of (4.7) induces an isomorphism

$$\mathrm{cl}^1(0):\ A^1(X,0)\xrightarrow{\sim}H^2_{\mathcal{D}}(X,\mathbb{Z}(1))\ .$$

Proof: If X is smooth, the left–hand–side is naturally isomorphic to $A_{\dim X-1}X\cong\mathrm{Pic}\,X$, and by construction the map $\mathrm{cl}^1(0)$ identifies to the cycle class map of (2.37), which is well–known to be an isomorphism (actually, this is equivalent to Jacobi's theorem, since Deligne cohomology forms an extension

$$0\longrightarrow\mathrm{Pic}^0(X)\longrightarrow H^2_{\mathcal{D}}(X,\mathbb{Z}(1))\longrightarrow H^{1,1}(X,\mathbb{Z})\longrightarrow 0\ ,$$

and the cycle class map into Deligne cohomology, restricted to cohomologically trivial cycles, is known to be the Abel–Jacobi map. In more modern language, the conjectural filtration on the Chow groups [Mur 1] is well–understood for codimension 1, where it is a two step filtration involving only homological and Abel–Jacobi equivalence.)

In the general case, consider an M–V diagram

$$\begin{array}{ccc}\widetilde{Y} & \xrightarrow{\widetilde{\tau}} & \widetilde{X}\\ \big\downarrow{\scriptstyle\pi_Y} & & \big\downarrow{\scriptstyle\pi}\\ Y & \xrightarrow{\tau} & X\end{array}$$

with $\widetilde{X}$ smooth, and the commutative diagram with exact rows

$$\begin{array}{ccccccccc}\longrightarrow & A^1(\widetilde{X}\amalg Y,1) & \longrightarrow & A^1(\widetilde{Y},1) & \longrightarrow & A^1(X,0) & \longrightarrow & A^1(\widetilde{X}\amalg Y,0)\\ & \big\downarrow{\scriptstyle\mathrm{cl}^1(1)} & & \big\downarrow{\scriptstyle\mathrm{cl}^1(1)} & & \big\downarrow{\scriptstyle\mathrm{cl}^1(0)} & & \big\downarrow{\scriptstyle\mathrm{cl}^1(0)}\\ \longrightarrow & H^1_{\mathcal{D}}(\widetilde{X}\amalg Y) & \longrightarrow & H^1_{\mathcal{D}}\widetilde{Y} & \longrightarrow & H^2_{\mathcal{D}}X & \longrightarrow & H^2_{\mathcal{D}}(\widetilde{X}\amalg Y)\end{array}$$

which exists by (4.7) (here $H^*_{\mathcal{D}}(-)$ abbreviates $H^*_{\mathcal{D}}(-,\mathbb{Z}(1))$).

By noetherian induction, we are done if we can prove that $\mathrm{cl}^1(1)$ is an isomorphism. In the smooth case, of course we know that both $A^1(-,1)$ and $H^1_{\mathcal{D}}$ equal $\mathbb{C}^*$, but it is left to prove that the map $\mathrm{cl}^1(1)$ is an isomorphism. This follows (in the smooth case) from the fact that $\mathrm{cl}^1(1)$ equals the composition

$$A^1(M,1) = H^0_{\mathrm{Zar}}(M,\mathcal{K}^1) \longrightarrow H^0_{\mathrm{Zar}}\big(M,\mathcal{H}^1_{\mathcal{D}}(1)\big) = H^1_{\mathcal{D}}\big(M,\mathbb{Z}(1)\big)\ ,$$

where $\mathcal{H}^1_{\mathcal{D}}(1)$ is the Zariski sheaf associated to $U \mapsto H^1_{\mathcal{D}}\big(U,\mathbb{Z}(1)\big)$ [Mul, Section 3], and the middle arrow comes from a "local regulator". This proves that $\mathrm{cl}^1(1)$ is an isomorphism, since both sheaves are equal to $\mathcal{O}^*_M$ and the local regulator is an isomorphism [Es], [Ja 1, 3.2].

The fact that $\mathrm{cl}^1(1)$ is an isomorphism also in the singular case follows from a similar diagram and the fact that $A^1(-,2) = H^0_{\mathcal{D}}\big(-,\mathbb{Z}(1)\big)$ are always 0 (for Chow cohomology, cf. (4.4)(viii); for Deligne cohomology, cf. [E–V, Proposition (2.12)(i)]). □

(6.2) **Corollary:** Let X be a complete variety over $\mathbb{C}$. There is an isomorphism from $\mathrm{Pic}\,X$ to $A^1(X,0)$ if and only if the following condition holds:

$$\mathrm{Gr}^0_F H^i(X,\mathbb{C}) \cong H^i(X,\mathcal{O}_X) \quad \text{for } i = 1,2$$

(here F denotes the Hodge filtration of Deligne [De 1]; note that in general there is an inclusion

$$\mathrm{Ker}\Big(H^i(X,\mathbb{C}) \to H^i(X,\mathcal{O}_X)\Big) \subset F^1H^i(X,\mathbb{C})\ ,$$

hence a surjection

$$H^i(X,\mathcal{O}_X) \longrightarrow \mathrm{Gr}^0_F H^i(X,\mathbb{C})\ .)$$

Proof: X being complete, the Picard group $\mathrm{Pic}\,X$ is isomorphic to the analytic Picard group $\mathrm{Pic}\,(X_{\mathrm{an}})$ [SGA 1, Exposé XII 4.4], so by the exponential sequence $\mathrm{Pic}\,X$ fits into a short exact sequence

$$0 \longrightarrow H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) \longrightarrow \mathrm{Pic}\,X \longrightarrow \mathrm{Ker}\Big(H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X)\Big) \longrightarrow 0$$

(here all cohomology groups are with respect to the classical topology on X).

Let $X_* \to X$ denote a cubical hyperresolution. The cohomology groups $H^i(X,\mathcal{O}_X)$ resp. $H^1(X,\mathcal{O}^*_X) = \mathrm{Pic}\,X$ map to

$$H^i(X_*,\mathcal{O}_{X_*}) = \mathrm{Gr}^0_F H^i(X,\mathbb{C})$$

resp. to

$$H^1(X_*,\mathcal{O}^*_{X_*}) = H^2_{\mathcal{D}}\big(X,\mathbb{Z}(1)\big)\ ,$$

where the identifications follow from the fact that both the Hodge filtration and Deligne cohomology are defined by descent.

These maps fit into a commutative diagram with exact rows

$$\begin{array}{ccccc}
0 & \longrightarrow & H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) & \longrightarrow & \operatorname{Pic} X \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Gr}_F^0 H^1(X,\mathbb{C})/H^1(X,\mathbb{Z}) & \longrightarrow & H^2_{\mathcal{D}}\big(X,\mathbb{Z}(1)\big)
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow & \operatorname{Ker}\Big(H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)\Big) & \longrightarrow 0 \\
 & \downarrow & \\
\longrightarrow & \operatorname{Ker}\Big(H^2(X,\mathbb{Z}) \to \mathrm{Gr}_F^0 H^2(X,\mathbb{C})\Big) & \longrightarrow 0\ ,
\end{array}$$

where the vertical maps from $H^i(X,\mathbb{Z})$ to $H^i(X,\mathbb{Z})$ are the identity (we can send $H^i(X,\mathbb{Z})$ to $H^i(X_*,\mathbb{Z}_{X_*})$ which identifies again to $H^i(X,\mathbb{Z})$ by descent for singular cohomology).

Observing that the left vertical map is always a surjection, as noted above, proves the corollary. □

(6.3) The condition of (6.2) is satisfied by all varieties that have only DuBois singularities [Ste, Section 3] [Ko, Chapter 12]. The class of DuBois singularities includes normal crossings singularities and quotient singularities. More generally, a complex projective variety with only rational singularities has only DuBois singularities [Ko, 12.9] (and it is expected this also holds for quasi–projective varieties, cf. [Ko, Chapter 12]).

The condition on $\mathrm{Gr}_F^0 H^2$ is moreover related to the "singular Hodge conjecture for line bundles" studied in [Ba–Sr]: the condition on $\mathrm{Gr}_F^0 H^2$ is satisfied if and only if

$$\operatorname{Im}\Big(\operatorname{Pic} X \to H^2(X,\mathbb{Z})\Big) = \{x \in H^2(X,\mathbb{Z}) \mid x_{\mathbb{C}} \in F^1 H^2(X,\mathbb{C})\}\ .$$

In particular, we find that $A^1(S,0) \cong \operatorname{Pic} S$ if S is the singular surface of [To 2, Theorem 7], for which $A^1 S$ did not have a reasonable map to $H^2(S,\mathbb{Q})$—so here we have an explicit case where A^i is different from $A^i(-,0)$.

On the other hand, in [Ba–Sr] examples are given of singular surfaces which do *not* satisfy the above form of the "singular Hodge conjecture", so we conclude that

$$A^1(X,0) \cong \operatorname{Pic} X$$

does not always hold; in fact a dimension count shows this does not always hold even after tensoring with $\mathbb{Q}$.

As a consequence, since $\operatorname{Pic} X \otimes \mathbb{Q}$ is a graded piece of $K^0 X_{\mathbb{Q}}$, we find that the formal K–theory

$$K_f^0 X_{\mathbb{Q}} \;\cong\; \bigoplus_i A^i(X,0)_{\mathbb{Q}}$$

does *not* always coincide with the vector bundle K–theory $K^0 X_{\mathbb{Q}}$.

(6.4) *Remark.* It follows from (6.2) and (6.3) that if X is a projective variety with at most rational singularities, then

$$\operatorname{Pic} X \xrightarrow{\sim} A^1(X,0)\ .$$

As for the Fulton–MacPherson Chow cohomology, for any X having at most rational singularities, one has an isomorphism

$$\operatorname{Pic} X \otimes \mathbb{Q} \xrightarrow{\sim} A^1 X \otimes \mathbb{Q}$$

(this follows from [K–M] as noted in [F–M–S–S, Introduction] and [To 2, Section 8]). If $\dim X \geq 3$, this isomorphism does not hold without tensoring by $\mathbb{Q}$ (as is shown by Corti, and mentioned in [F–M–S–S, Introduction] and [To 2, Section 8]).

I wonder if one can also prove directly that the map $A^1(X, 0) \to A^1 X$ is an isomorphism over $\mathbb{Q}$?

(6.5) *Open question.* The above counterexample (6.3) to the property

$$\operatorname{Pic} X \otimes \mathbb{Q} \xrightarrow{\sim} A^1(X, 0) \otimes \mathbb{Q}$$

can also be understood from the fact that Pic and K^0 do not always satisfy descent in the singular case. Since we have seen (3.15.3) that it is doubtful whether the operational theory $A^*_{\mathrm{op}}(-, *)$ of chapter 3 satisfies descent, it seems natural to ask: Is the map

$$\operatorname{Pic} X \longrightarrow A^1_{\mathrm{op}}(X, 0)$$

always an isomorphism ? That is, does every well–formed collection of operations

$$A_*(X', *) \longrightarrow A_{*-1}(X', *)$$

for all $X' \to X$ determine one and only one line bundle ?

More generally, it would be interesting to understand the relation between K^0 and $A^*_{\mathrm{op}}(-, 0)$ in the singular case.

References:

[AA] A. Al Amrani, Groupes de Chow et K–théorie cohérente des espaces projectifs tordus, K–Theory 2 (1989) 579—602,

[A–H] M. Atiyah, F. Hirzebruch, Vector bundles and homogeneous spaces, in: Differential geometry, Proceedings of Symposia in Pure Mathematics Vol. 3, Amer. Math. Soc 1961, 7—38,

[Al] D. Alibert, Lemme de déplacement de Chow, in: Séminaire de Géométrie Analytique de l'Ecole Normale Supérieure 1974/75, Astérisque 36-37 (1976), 21—34

[Ba–Sr] L. Barbieri–Viale, V. Srinivas, The Néron–Severi group and the mixed Hodge structure on H^2, J. reine angew. Math. 450 (1994) 37—42,

[B–B–F–K] G. Barthel, J.-P. Brasselet, K.-H. Fieseler, L. Kaup, Diviseurs invariants et homomorphisme de Poincaré de variétés toriques complexes, Tôhôku Math. J. 48 (1996) 363—390,

[Be] A. Beilinson, Higher regulators and values of L–functions, Journal of Soviet Math. 30 no. 2 (1985) 2036—2070,

[B–F–M] P. Baum, W. Fulton, R. MacPherson, Riemann–Roch for singular varieties, Publ. Math. IHES 45 (1975) 101—145,

[B–F–M 2] P. Baum, W. Fulton, R. MacPherson, Riemann–Roch and topological K–theory for singular varieties, Acta Math. 143 (1979) 193—211,

[B–G] K. Brown, S. Gersten, Algebraic K–theory as generalized sheaf cohomology, in: Proc. Batelle conference on algebraic K–theory, I, Springer–Verlag Lecture Notes in Mathematics 343, Berlin Heidelberg New York 1973, 266—292,

[B–G–S] S. Bloch, H. Gillet, C. Soulé, Non–archimedean Arakelov theory, J. Alg. Geom. 4 (1995) 427—487,

[B–K] A. Bousfield, D. Kan, Homotopy limits, completions and localization, Springer–Verlag Lecture Notes in Mathematics 304, Berlin Heidelberg New York 1972,

[Bl 1] S. Bloch, Algebraic cycles and higher K–theory, Advances in Math. 61 (1986) 267—304,

[Bl 2] S. Bloch, Algebraic cycles and the Beilinson conjectures, in: The Lefschetz centennial conference Part I: Proceedings in algebraic geometry, Contemporary Math. 58, A.M.S. Providence 1986, 65—79,

[Bl 3] S. Bloch, Algebraic K–theory, motives and algebraic cycles, in: Proceedings of the International Congress of Mathematicians Kyoto 1990, Springer–Verlag Tokyo Berlin Heidelberg New York 1991, 43—54,

[Bl 4] S. Bloch, The moving lemma for higher Chow groups, J. Alg. Geom. 3 (1994) 537—568,

[Bl 5] S. Bloch, An elementary presentation for K–groups and motivic cohomology, in: Motives (U. Jannsen, S. Kleiman, J.-P. Serre, eds.), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994) Part 1, Amer. Math. Soc. Providence 1994, 239—244,

[B–L] S. Bloch, S. Lichtenbaum, A spectral sequence for motivic cohomology, preprint 1994,

[B–M] A. Borel, J. Moore, Homology theory for locally compact spaces, Michigan Math. J. 7 (1960) 137—159, [B–M–S] A. Beilinson, R. MacPherson, V. Schechtman, Notes on motivic cohomology, Duke Math. J. 54 (1987) 679—710,

[B–O] S. Bloch, A. Ogus, Gersten's conjecture and the homology of schemes, Ann. Sci. Ecole Norm. Sup. 4^e série 7 (1974) 181—201,

[Bo–Se] A. Borel, J.–P. Serre, Le théorème de Riemann–Roch (d'après Grothendieck), Bull. Soc. Math. France 86 (1958) 97—136,

[Br] M. Brion, Curves and divisors on spherical varieties, preprint,

[C–H–K] J.-L. Colliot-Thélène, R. Hoobler, B. Kahn, The Bloch–Ogus–Gabber theorem, preprint 1996,

[C–S] K. Coombes, V. Srinivas, Relative K–theory and vector bundles on certain singular varieties, Invent. math. 70 (1982) 1—12,

[De 1] P. Deligne, Théorie de Hodge II, Publ. Math. IHES 40 (1971) 5—58; Théorie de Hodge III, Publ. Math. IHES 44 (1974) 5—77,

[De 2] P. Deligne, Cohomologie étale (SGA $4\frac{1}{2}$), Springer–Verlag Lecture Notes in Mathematics 569, Berlin Heidelberg New York 1977,

[De 3] P. Deligne, A quoi servent les motifs ?, in: Motives (U. Jannsen, S. Kleiman, J.-P. Serre, eds.), Proceedings of Symposia in Pure Mathematics Volume 55 (1994) Part 1, Amer. Math. Soc. Providence 1994, 143—161,

[D–F] W. Dwyer, F. Friedlander, Algebraic and étale K–theory, Trans. A.M.S. 292 (1985) 247—280,

[dJ] A.J. de Jong, Smoothness, semistability and alterations, preprint 1995,

[D-P] A. Dold, D. Puppe, Homologie nicht–additiver Functoren. Anwendungen, Ann. Inst. Fourier 11 (1961) 201—312,

[D–S] C. Deninger, A. Scholl, The Beilinson conjectures, in: L–functions and Arithmetic, Proceedings of the Durham Symposium 1989 (J. Coates, M. Taylor, eds.), London Math. Society Lecture Note Series 153, Cambridge University Press 1991,

[D–W] B. Dayton, C. Weibel, A spectral sequence for the K–theory of affine glued schemes, in: Algebraic K–theory, Proceedings Evanston 1980 (E. Friedlander, M. Stein, eds.), Springer–Verlag Lecture Notes in Mathematics 854, Berlin Heidelberg New York 1981, 24—92,

[Es] H. Esnault, Une remarque sur la cohomologie du faisceau de Zariski de la K–théorie de Milnor sur une variété lisse complexe, Math. Z. 205 (1990) 373—378,

[E–V] H. Esnault, E. Viehweg, Deligne–Beilinson cohomology, in: Beilinson's conjectures on special values of L–functions (M. Rapoport, N. Schappacher, P. Schneider, eds.), Academic Press Perspectives in Mathematics 4 (1988), 43—92,

[F–G] W. Fulton, H. Gillet, Riemann–Roch for general algebraic varieties, Bull. Soc. Math. France 111 (1983) 287—300,

[F–L] W. Fulton, S. Lang, Riemann–Roch algebra, Springer–Verlag New York Berlin Heidelberg Tokyo 1985,

[F–M] W. Fulton, R. MacPherson, Categorical framework for the study of singular spaces, Mem. Amer. Math. Soc. 243 (1981),

[F–M–S–S] W. Fulton, R. MacPherson, F. Sottile, B. Sturmfels, Intersection theory on spherical varieties, J. Alg. Geom. 4 (1995) 181—193,

[Fr] E. Friedlander, Etale K–theory II: Connections with algebraic K–theory, Ann. Sci. Ecole Norm. Sup. 4^e série 15 (1982) 231—256,

[F–S] W. Fulton, B. Sturmfels, Intersection theory on toric varieties, Topology Vol. 36 No. 2 (1997) 335—353,

[Fu 1] W. Fulton, Rational equivalence on singular varieties, Publ. Math. I.H.E.S. 45 (1975) 147—167,

[Fu 2] W. Fulton, Introduction to intersection theory in algebraic geometry, Expository lectures from the CBMS Regional Conference 1983, Amer. Math. Soc. 1984,

[Fu 3] W. Fulton, Intersection theory, Springer–Verlag Berlin Heidelberg New York Tokyo 1984,

[Fu 4] W. Fulton, Introduction to toric varieties, Ann. of Math. Studies, Princeton University Press 1993,

[Ge 1] S. Gersten, Some exact sequences in the higher K–theory of rings, in: Proc. Batelle Conference on algebraic K–theory, I, Springer–Verlag Lecture Notes in Mathematics 343, Berlin Heidelberg New York 1973, 211—243,

[Ge 2] S. Gersten, K–theory and algebraic cycles, in: Proceedings of the International Congress of Mathematicians Vancouver 1974, Canad. Math. Congress 1975, 309—314,

[Gerd] W. Gerdes, The linearization of higher Chow cycles of dimension one, Duke Math. J. 62 (1991) 105—129,

[Gi 1] H. Gillet, Riemann–Roch theorems for higher algebraic K–theory, Advances in Math. 40 (1981) 203—289,

[Gi 2] H. Gillet, Homological descent for the K–theory of coherent sheaves, in: Algebraic K–theory, number theory, geometry and analysis, Proceedings Bielefeld 1982 (A. Bak, ed.), Springer–Verlag Lecture Notes in Mathematics 1046, Berlin Heidelberg New York 1984, 80—103,

[Gi 3] H. Gillet, Deligne homology and Abel–Jacobi maps, Bulletin of the American Mathematical Society Vol. 10 No. 2 (1984) 285—288,

[G–N] F. Guillén, V. Navarro Aznar, Un critère d'extension d'un foncteur défini sur les schémas lisses, preprint 1995,

[G–N–P–P] F. Guillén, V. Navarro Aznar, P. Pascual–Gainza, F. Puerta, Hyperrésolutions cubiques et descente cohomologique, Springer–Verlag Lecture Notes in Mathematics 1335, Berlin Heidelberg New York London Paris Tokyo 1988,

[Go] A. Goncharov, Chow polylogarithms and regulators, Mathematical Research Letters 2 (1995) 95—112,

[G–P] F. Guillén, F. Puerta, Hyperrésolutions cubiques et applications à la théorie de Hodge–Deligne, in: Hodge theory, Proceedings, Sant Cugat, Spain, 1985 (E. Cattani, F. Guillén, A. Kaplan, F. Puerta, eds.), Springer–Verlag Lecture Notes in Mathematics 1246, Berlin Heidelberg 1987, 49—74,

[Gra] D. Grayson, Weight filtrations in algebraic K–theory, in: Motives (U. Jannsen, S. Kleiman, J.–P. Serre, eds.), Proceedings of Symposia in Pure Mathematics Volume 55 (1994) Part 1, Amer. Math. Soc. Providence 1994, 207—237,

[Gro 1] A. Grothendieck, On the De Rham cohomology of algebraic varieties, Publ. Math. IHES 29 (1966) 95—103, [Gro 2] A. Grothendieck, Le groupe de Brauer, in: Dix exposés sur la cohomologie des schémas, Masson North–Holland Paris Amsterdam 1968,

[G–S] H. Gillet, C. Soulé, Descent, motives and K–theory, J. reine angew. Math. 478 (1996) 127—176,

[Ha 1] R. Hartshorne, On the De Rham cohomology of algebraic varieties, Publ. Math. IHES 45 (1976) 5—99,

[Ha 2] R. Hartshorne, Algebraic geometry, Springer–Verlag New York 1977,

[Hi] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math. 79 (1964) 109—326,

[Iv] B. Iversen, Cohomology of sheaves, Springer–Verlag Berlin Heidelberg 1986,

[Ja 1] U. Jannsen, Deligne homology, Hodge D–conjecture and motives, in: Beilinson's conjectures on special values of L–functions (M. Rapoport, N. Schappacher, P. Schneider, eds.), Academic Press Perspectives in Math. 4 (1988) 305—372,

[Ja 2] U. Jannsen, Mixed motives and algebraic K–theory, Springer–Verlag Lecture Notes in Mathematics 1400, Berlin Heidelberg New York London Paris Tokyo Hong Kong 1990,

[Ja 3] U. Jannsen, Mixed motives, motivic cohomology and Ext–groups, in: Proceedings of the International Congress of Mathematicians Zürich 1994, Birkhäuser Verlag Basel 1995, 667—679,

[Ka] B. Kahn, Applications of weight–two motivic cohomology, preprint 1996,

[Ki] S. Kimura, Fractional intersection and bivariant theory, Comm. Algebra 20 (1992) 285—302,

[Kl] S. Kleiman, The transversality of a general translate, Comp. Math. 38 (1974) 287—297,

[K–M] J. Kollár, S. Mori, Classification of three–dimensional flips, Journal of the American Mathematical Society 5 (1992) 533—703,

[Ko] J. Kollár, Shafarevich maps and automorphic forms, Princeton University Press 1995,

[Kr] C. Kratzer, λ–structure en K–théorie algébrique, Comment. Math. Helv. 55 (1980) 233—254,

[La] G. Laumon, Homologie étale, in: Séminaire de Géométrie Analytique de l'Ecole Normale Supérieure 1974/75, Astérisque 36–37 (1976), 163—188,

[Le 1] M. Levine, Bloch's higher Chow groups revisited, in: K–theory, Strasbourg 1992, Astérisque 226 (1994),

[Le 2] M. Levine, Lambda operations, K–theory and motivic cohomology, preprint (1996),

[Li 1] S. Lichtenbaum, Values of zeta–functions at non–negative integers, in: Number theory, Noordwijkerhout 1983 (H. Jager, ed.), Springer–Verlag Lecture Notes in Mathematics 1068, Berlin Heidelberg New York Tokyo 1984, 127—138,

[Li 2] S. Lichtenbaum, The construction of weight–two arithmetic cohomology, Invent. Math. 88 (1987) 183—215,

[Li 3] S. Lichtenbaum, New results on weight two motivic cohomology, in: The Grothendieck Festschrift Volume III, Birkhäuser Boston Basel Berlin 1990,

[Li 4] S. Lichtenbaum, Motivic complexes, in: Motives (U. Jannsen, S. Kleiman, J.-P. Serre, eds.), Proceedings of Symposia in Pure Mathematics Volume 55 (1994) Part 1, Amer. Math. Soc. Providence 1994, 303—314,

[Mar] R. Marlin, Comparaison de l'anneau de Chow et de l'anneau de Grothendieck, in: Séminaire de Géométrie Analytique de l'Ecole Normale Supérieure 1974/75, Astérisque 36—37 (1976), 229—240,

[May] J. May, Simplicial objects in algebraic topology, Van Nostrand New York 1967,

[Mul] S. Müller–Stach, Constructing indecomposable motivic cohomology classes on an algebraic surface, preprint 1995,

[Mur 1] J. Murre, On a conjectural filtration on the Chow groups of an algebraic variety, Indag. Mathem., N.S., 4 (2) (1993) 177—188,

[Mur 2] J. Murre, Algebraic cycles and algebraic aspects of cohomology and K–theory, in: Algebraic cycles and Hodge theory, Torino 1993 (A. Albano, F. Bardelli, eds.), Springer–Verlag Lecture Notes in Mathematics 1594, Berlin Heidelberg New York 1994,

[N–S] Y. Nesterenko, A. Suslin, Homology of the full linear group over a local ring and Milnor's K–theory, Math. USSR Izv. 34 (1990) 121—146,

[Qu 1] D. Quillen, Homotopical algebra, Springer–Verlag Lecture Notes in Mathematics 43, Berlin 1967,

[Qu 2] D. Quillen, Higher algebraic K–theory I, in: Proc. Batelle conference on algebraic K–theory, I, Springer–Verlag Lecture Notes in Mathematics 343, Berlin Heidelberg New York 1973, 77—147,

[Qu 3] D. Quillen, Higher Algebraic K–theory, in: Proceedings of the International Congress of Mathematicians Vancouver 1974, Canad. Math. Congress 1975, 171—176,

[Ra] D. Ramakrishnan, Regulators, algebraic cycles and values of L–functions, in: Algebraic K–theory and algebraic number theory, Contemporary Mathematics Vol. 83 (1989),

[Ro] J. Roberts, Chow's moving lemma, in: Algebraic geometry Oslo 1970 (F. Oort, ed.), Wolters–Noordhoff Publ. Groningen 1972,

[Schn] P. Schneider, Introduction to the Beilinson conjectures, in: Beilinson's conjectures on special values of L–functions (M. Rapoport, N. Schappacher, P. Schneider, eds.), Academic Press Perspectives in Math. 4 (1988), 1—36,

[Scho] A. Scholl, Classical motives, in: Motives (U. Jannsen, S. Kleiman, J.-P. Serre, eds.), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994) Part 1, Amer. Math. Soc. Providence 1994, 163—187,

[SGA 1] A. Grothendieck et al., Revêtements étales et groupe fondamental (SGA 1), Springer–Verlag Lecture Notes in Mathematics 224, Berlin Heidelberg New York 1971,

[SGA 6] A. Grothendieck et al., Théorie des intersections et théorème de Riemann–Roch (SGA 6), Springer–Verlag Lecture Notes in Mathematics 225, Berlin Heidelberg New York 1971,

[So 1] C. Soulé, Régulateurs, Séminaire Bourbaki 1984/85 exposé no. 644, in: Astérisque 133–134 (1986),

[So 2] C. Soulé, Opérations en K–théorie algébrique, Can. J. Math. Vol. XXXVII (1985) 488—550,

[Ste] J. Steenbrink, Mixed Hodge structures associated with isolated singularities, in: Singularities, Proceedings of Symposia in Pure Mathematics Volume 40 (1983) Part 2, Amer. Math. Soc. Providence 1983,

[Str] R. Street, Two constructions on lax functors, Cahiers Top. Geom. Diff. XIII (1972) 217—264,

[Su] A. Suslin, Algebraic K–theory and motivic cohomology, in: Proceedings of the International Congress of Mathematicians Zürich 1994, Birkhäuser Verlag Basel 1995, 342—351,

[S–V] A. Suslin, V. Voevodsky, Singular homology of abstract algebraic varieties, Invent. Math. 123 (1996) 61—94,

[Th 1] R. Thomason, Algebraic K–theory and étale cohomology, Ann. Sci. Ecole Norm. Sup. 4^e série 18 (1985) 437—552,

[Th 2] R. Thomason, Survey of algebraic vs. étale topological K–theory, in: Algebraic K–theory and algebraic number theory, Contemporary Mathematics Vol. 83 (1989), 393—443,

[Th 3] R. Thomason, The local to global principle in algebraic K–theory, in: Proceedings of the International Congress of Mathematicians, Kyoto, Japan, 1990, Springer–Verlag Tokyo Berlin Heidelberg New York 1991,

[Th 4] R. Thomason, Les K–groupes d'un schéma éclaté et une formule d'intersection excédentaire, Invent. Math. 112 (1993) 195—215,

[To 1] B. Totaro, Milnor K–theory is the simplest part of algebraic K–theory, K–theory 6 (1992) 177—189,

[To 2] B. Totaro, Chow groups, Chow cohomology and linear varieties, to appear in J. Alg. Geom.,

[T–T] R. Thomason, T. Trobaugh, Higher algebraic K–theory of schemes and of derived categories, in: The Grothendieck Festschrift, Volume III, Birkhäuser Boston Basel Berlin 1990,

[Ve 1] J.-L. Verdier, Dualité dans la cohomologie des espaces localement compacts, Séminaire Bourbaki No. 300 1965/66,

[Ve 2] J.-L. Verdier, Catégories dérivées, in: Cohomologie étale (SGA $4\frac{1}{2}$), Springer–Verlag Lecture Notes in Mathematics 569, Berlin Heidelberg New York 1977,

[Ve 3] J.-L. Verdier, Classe d'homologie associée à un cycle, in: Séminaire de Géométrie Analytique de l'Ecole Normale Supérieure 1974/75, Astérisque 36–37 (1976), 101—151,

[Ve 4] J.-L. Verdier, Le théorème de Riemann–Roch pour les intersections complètes, in: Séminaire de Géométrie Analytique de l'Ecole Normale Supérieure 1974/75, Astérisque 36–37 (1976), 189—228,

[Voe] V. Voevodsky, Homology of schemes I, preprint 1993; Homology of schemes II, preprint 1994,

[Voi] C. Voisin, Variations of Hodge structure and algebraic cycles, in: Proceedings of the International Congress of Mathematicians Zürich 1994, Birkhäuser Verlag Basel 1995, 706—715,

[We] C. Weibel, Homotopy algebraic K–theory, in: Algebraic K–theory and algebraic number theory, Contemporary Mathematics 83 (1989), 461—488.