Symplectique vs Projective

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Dragilor mei părinți

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Resumé de la thèse

Cette thèse est un mélange des idées provenant de la géométrie symplectique et de la géométrie algébrique. Motivé par les resultats de S.K.Donaldson, j'ai essayé de regarder les variétés symplectiques compactes commes des "mauvaises" variétés projectives et réciproquement, j'ai voulu clarifier l'analogue symplectique de certaines constructions algébriques. La thèse est divisée en quatre parties, largement indépendantes.

La première partie de la thèse est une introduction dont le but est de rappeler brièvement des définitions et des résultats elementaires de la géométrie symplectique ainsi que l'énoncé des résultats de S.K.Donaldson sur l'existence des sous-variétés symplectiques des variétés symplectiques. Il est tout a fait evident que n'importe quelle variété projective admet des sous-variétés (projectives) de codimension un qui sont obtenus, par exemple, comme sections hyperplanes. Le résultat de Donaldson dit que ceci reste valable dans le contexte symplectique, c'est à dire, n'importe quelle variété symplectique compacte a des sous-variétés symplectiques lisses de codimension réelle deux. C'est précisément ce résultat qui m'a motivé à regarder les variétés symplectique "avec un œil projectif".

Dans la deuxieme partie je me suis intéréssé à la géographie des variétés symplectiques compactes de dimension réelle six. L'intérêt pour une telle étude reside dans le problème à trouver des obstructions pour l'existence des structures symplectiques sur les variétés compactes lisses admettant une structure presque complexe. Je prouve (théorème 2.5.1) que chaque triplet de la forme $(2a, 24b, 2c)$, avec a, b, c des entiers, apparaît comme le triplet (c_1^3, c_1c_2, c_3) des nombres de Chern d'une variété symplectique compacte, connexe et simplement connexe de dimension réelle six. D'autre part, les nombres de Chern de toute variété symplectique (même presque complexe) de dimension six satisfont ces restrictions arithmétiques. Ceci montre que les nombres de Chern ne représentent pas une obstruction pour l'existence des structures symplectiques. Je construis mes exemples à partir de certaines variétés de base, en utilisant la somme connexe symplectique et les eclatements.

"Symplectic bend-and-break: Possible ?", présente une extension au cas symplectique de la construction correspondante en géométrie algébrique dans le cas symplectique (au moins en dimensions réels quatre et six). Ce procédé, appellé bend-and-break, a été découvert par S.Mori et permet d'établir l'existence des courbes rationnelles (non-constantes) sur certaines variétés projectives. Au-delà de l'intérét en soit même de ce problème, cette méthode occupe une position clef dans la classification des variétés projectives. Je démontre (théorème 3.9.9) que certains revêtements finis de l'espace des applications pseudo-holomorphes stables dans une variété symplectique admet "génériquement" une structure presque complexe. Ensuite j'essaye de combiner, dans la section 3.11, ce résultat avec certaines propriétés topologiques de l'espace Deligne-Mumford des courbes stables pour pouvoir en déduire que vers le bord de l'espace des applications stables il apparaît soit des courbes rationnelles, soit des courbes de genre strictement plus petit. En particulier, ces idées suggèrent (proposition 3.11.7) qu'il n'est pas possible d'avoir, en dimension fixée, des variétés symplectiques dont la première classe de Chern est un multiple arbitrairement grand d'une forme symplectique entière.

Dans la quatrième partie de la thèse j'ai étudié le lien entre les invariants de Gromov-Witten d'une variété algébrique complèxe et les invariants de Gromov-Witten de son quotient invariant sous l'action d'un groupe algébrique linéairement réductif. Les invariants de Gromov-Witten sont des invariants puissants de la structure symplectique, mais ils sont très difficiles à calculer. Il est donc naturel d'essayer établir des liens entre les invariants d'une variété et ceux d'une autre qui est obtenue à partir de la variété de départ par un procédé simple, en faisant un quotient dans le cas que j'ai étudié. Je construis d'abord le quotient invariant, sous l'action induite du groupe, de l'espace des modules des applications stables en décrivant dans la section 4.6 (proposition 4.6.1 et 4.6.7) les applications stables dans la variété qui définissent des points semi-stables dans l'espace des modules correspondant. Dans la section 4.8 je démontre que, en général, on peut s'attendre d'avoir des bonnes rélations entre les invariants de Gromov-Witten seulement dans le cas des courbes de genre zero. A cause des difficultés techniques, je démontre ces liens seulement dans le cas des variétés des drapeaux (théorème 4.8.3). Dans la dernière section 4.9 je calcule (proposition 4.9.5) de manière explicite la forme de Kähler, induite par le plongement projectif, de l'espace des modules des application stables ayant le domaine de définition une courbe fixée. A l'aide de la fonction moment induite (proposition 4.9.6) je donne (théorème 4.9.8) l'interpretation symplectique du théorème 4.6.7. En fait l'etude que je fais dans cette dernière section est motivée par la question suivante: puisque les invariants de Gromov-Witten sont de nature symplectique et parce que le quotient invariant en géométrie algébrique a son analogue symplectique, le quotient de Marsden-Weinstein, est-il possible de trouver une relation entre les invariants d'une variété symplectique et ceux de son quotient de Marsden-Weinstein?

Summary of the thesis

This thesis is a mixture of ideas coming from symplectic and algebraic geometry. Motivated by some results of S.K.Donaldson, I have considered compact symplectic manifolds as some kind of ill-behaved projective varieties; conversely, I wanted to clarify the symplectic counterpart of some algebraic constructions. The thesis is divided into four parts.

The first part serves as an introduction. Its purpose is to review some elementary definitions and results in symplectic geometry as well as to recall the statement of the results of S.K.Donaldson about the existence of symplectic submanifolds in symplectic manifolds. It is an obvious thing that any projective variety has codimension one subvarieties which can be obtained as hyperplane sections for instance. Donaldson proves that this still holds for closed symplectic manifolds, namely any such manifold has real codimension two closed symplectic submanifolds. It is precisely this result which motivated me to look at closed syplectic varieties with a "projective eye".

In the second part I have studied the geography of connected and simply connected, compact symplectic manifolds. The interest for studying such a problem is finding obstructions for the existence of symplectic structures on almost complex, compact manifolds. I prove (theorem 2.5.1) that each triple of the form $(2a, 24b, 2c)$, with a, b, c integers, occurs as the triple (c_1^3, c_1c_2, c_3) of some connected and simply connected, compact symplectic manifold. On the other hand it is well known that the Chern numbers of any symplectic variety (even almost complex) satisfy these arithmetic restrictions. The conclusion is that Chern numbers do not represent any obstruction for the existence of symplectic structures. I construct the examples from some basic buiding blocks using the symplectic connected sum and the symplectic blow-up.

"Symplectic bend-and-break: Possible?" presents an attempt at extending the bend-and-break construction in algebraic geometry to the symplectic category, at least for real dimension four and six. The bend-and-break procedure, invented by S.Mori, is a poverful tool in finding (non-trivial) rational curves on certain projective varieties. Apart from its own interest, this procedure plays a key rôle in the classification of projective varieties. I prove (theorem 3.9.9) that some finite global covers of the space of pseudo-holomorphic stable maps into a symplectic variety "generically" admits an almost complex structure. After I try to combine this result with some topological properties of the Deligne-Mumford space of stable curves in order to deduce that in the boundary of the space of stable maps either bubble components or curves of strictly smaller genus appear. In particular, these ideas suggest (proposition 3.11.7) that in fixed dimension it is not possible to have symplectic varieties whose Chern class is an arbitrarily large multiple of an integral symplectic form.

In the fourth part of the thesis I have studied the link between the Gromov-Witten invariants of a complex projective variety and those of its geometric invariant quotient under the action of a linearly reductive complex algebraic group. The Gromov-Witten invariants are powerful symplectic invariants, but they are very difficult to compute. It is therefore natural to try to relate the Gromov-Witten invariants of a variety with those of a variety which is obtained by a simple procedure from the starting one, a quotient in my case. First, I construct the geometric invariant quotient of the moduli space of stable maps, under the induced group action, describing in section 4.6 (proposition 4.6.1 and theorem 4.6.7) the stable maps in the variety which define semi-stable points in the corresponding moduli space. In section 4.8 I show that one can expect to have nice relations only between the rational Gromov-Witten invariants. Because of technical difficulties, I prove these links only for homogeneous manifolds (theorem 4.8.3). In the last section 4.9 I explicitely compute (proposition 4.9.5) the Kähler form, induced by the projective embedding, on the moduli space of stable maps whose domain of definition is a fixed

curve. Using the induced moment map (proposition 4.9.6) I give in theorem 4.9.8 the symplectic counterpart of theorem 4.6.7. In fact, my study in this last section is motivated by the following problem: the Gromov-Witten invariants being symplectic in nature and because the geometric invariant quotient in algebraic geometry has its symplectic counterpart, the Marsden-Weinstein quotient, is it possible to relate the Gromov-Witten invariants of a symplectic manifold to those of its Marsden-Weinstein quotient?

Contents

Part I

Basic notions

1.2 What is a symplectic manifold?

In this section I shall give the basic definitions and elementary results concerning the symplectic manifolds. A very good reference is the book [McDS1] of D.McDuff and D.Salamon.

Definition 1.2.1 *Let* $\mathbf{V}$ *be a (not necessarily compact) manifold ; it is called symplectic if there is a two-form* $\omega \in \bigwedge^2 T^*\mathbf{V}$ *which is closed and non-degenerate.*

The non-degeneracy means that at each point $p \in \mathbf{V}$ the corresponding skew-symmetric two-form ω_p on $T_p\mathbf{V}$ has non-vanishing determinant.

Example 1.2.2 The standard example of a symplectic manifold is the phase space of a manifold. More precisely, we take a (smooth) variety $\mathbf{N}$ of dimension $\dim\mathbf{N} = n$ and let $\mathbf{V} := T^*\mathbf{N}$. If $(x^1, \ldots, x^n)$ are local coordinates on $\mathbf{N}$, we get the local coordinates $(x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$ on $\mathbf{V}$ where $\xi_j := \mathrm{d}x^j$. In these local coordinates, the symplectic form is given by

$$\omega := \sum_{j=1}^{n} \mathrm{d}x^j \wedge \mathrm{d}\xi_j$$

It can be checked easily that this two-form is globally defined. From its local description it is obvious that it is both closed and non-degenerate. Let me remark that in this case ω is also exact: $\omega = -\mathrm{d}\lambda$ where the one-form λ is defined by

$$\lambda := \sum_{j=1}^{n} \xi_j \mathrm{d}x^j$$

When $\mathbf{N} = \mathbb{R}^n$, we get the canonical (global) symplectic form ω_0 on $\mathbb{R}^{2n}$. $\diamond$

Remark 1.2.3 In this example the manifold $\mathbf{V}$ is non-compact and this is the reason why the symplectic form can be exact. On compact manifolds this never happens by the following reason: ω being non-degenerate gives rise to the volume form ω^n (here $\dim\mathbf{V} = 2n$) and therefore $\int_{\mathbf{V}} \omega^n > 0$. If ω was exact, the integral vanished. Because the symplectic form is closed, it also determines a cohomology class $[\omega] \in H^2(\mathbf{V}, \mathbb{R})$. The very same argument as the the previous one shows that all the cohomology classes $[\omega]^k, 1 \leq k \leq n$ are non-zero. $\diamond$

Example 1.2.4 A lot of examples come from complex geometry: projective algebraic varieties $(\mathbf{X}, \omega)$ with their integral Kähler form are symplectic or more generally, any Kähler variety. $\diamond$

Definition 1.2.5 *Two symplectic varieties* $(\mathbf{V}_1, \omega_1)$ *and* $(\mathbf{V}_2, \omega_2)$ *are called symplectomorphic if there is a diffeomorphism* $\varphi : \mathbf{V}_1 \to \mathbf{V}_2$ *such that* $\varphi^*\omega_2 = \omega_1$*. The map* φ *is called symplectomorphism.*

The problem of distinguishing between two symplectic manifolds relies in the following result

Theorem 1.2.6 (Darboux) *Let* $(\mathbf{V}, \omega)$ *be a* $2n$*-dimensional symplectic manifold. Around each point* $p \in \mathbf{V}$ *there is a neighborhood* $V_p \overset{\iota}{\hookrightarrow} \mathbf{V}$ *such that* $(V_p, \iota^*\omega)$ *is symplectomorphic to an open disc* $\mathbf{D} \hookrightarrow \mathbb{R}^{2n}$ *endowed with the standard symplectic form* ω_0 *coming from* $\mathbb{R}^{2n}$*.*

Proof A proof can be found in [McDS1], page 93. □

This shows that locally all symplectic varieties (of equal dimensions) look the same. It is a good time to explain why the dimension of any symplectic variety is even. In fact any such variety carries "a lot of" almost complex structures and therefore the dimension of the tangent space at any point must be even; the precise result is stated in the proposition below, but before stating it I need to introduce some definitions and notations:

Definition 1.2.7 *(i) An almost complex structure J on $\mathbf{V}$ is called* compatible *with the symplectic form ω if*

$$\omega(Jv, Jw) = \omega(v, w) \qquad \forall v, w \in T\mathbf{V}$$

and

$$\omega(v, Jv) > 0 \qquad \forall v \in T\mathbf{V}, v \neq 0$$

(ii) An almost complex structure J is called ω-tame if it satisfies only the second condition in the previous definition.

The set of all ω-compatible almost complex structures on $\mathbf{V}$ is denoted by $\mathcal{J}(\mathbf{V}, \omega)$, the set of the ω-tame ones is denoted by $\mathcal{J}_\tau(\mathbf{V}, \omega)$ and the set of metrics on $\mathbf{V}$ is denoted by $\mathcal{M}et(\mathbf{V})$. For $J \in \mathcal{J}(\mathbf{V}, \omega)$,

$$g_J(\cdot, \cdot) := \omega(\cdot, J\cdot)$$

defines a metric on $\mathbf{V}$. Now I am in position to state the following result ([McDS1], prop. 2.48, page 61 and prop.2.49, page 63):

Proposition 1.2.8 *(i) There is a continuous map $r : \mathcal{M}et(\mathbf{V}) \to \mathcal{J}(\mathbf{V}, \omega)$ such that $r(g_J) = J$ for all $J \in \mathcal{J}(\mathbf{V}, \omega)$.*
(ii) $\mathcal{J}(\mathbf{V}, \omega)$ is contractile;
(iii) $\mathcal{J}_\tau(\mathbf{V}, \omega)$ is contractile. (iv) Assume that there is an action of a compact group K on $(\mathbf{X}, \omega)$ which preserves the symplectic form. Then there are K-invariant almost complex structures on $\mathbf{X}$ compatible with ω.

Proof The complete proof can be found in the given reference. Here I just want to sketch the proof of the first statement: let us consider a metric $g \in \mathcal{M}et(\mathbf{V})$. Then the symplectic form can be expressed by means of a section $\mathbf{V} \to Aut(T\mathbf{V})$

$$\omega(\cdot, \cdot) = g(A\cdot, \cdot)$$

and the condition of ω to be skew-symmetric can be translated into the equality $A^* = -A$, where the adjoint is taken with respect to the metric g. This equality shows that A is normal, i.e. $A^*A = AA^*$; then there is an automorphism of $T\mathbf{V}$ denoted by $|A|$, "the absolute value of A", having the properties that it commutes with A and $|A|^2 = A^*A$. The equality

$$(|A|A^{-1})^2 = |A|^2A^{-2} = A^*AA^{-2} = (-A)A^{-1} = -Id$$

shows that $J := |A|A^{-1}$ is an almost complex structure and an easy computation proves that it is also compatible with ω.

Suppose now that there is an action of a compact group K on $\mathbf{X}$ preserving the symplectic form ω. Then there is a unique left and right-invariant measure $\mathscr{H}$ on K with total mass equal one ; it is called the Haar measure of K. For any metric $g \in \mathcal{M}et(\mathbf{X})$ let g_K the metric on $\mathbf{X}$ which, at a point $x \in \mathbf{X}$, is defined as follows

$$(g_K)_x := \int_K (k^* g)_x \mathrm{d}\mathscr{H}(k)$$

Because $\mathscr{H}$ is bi-invariant, g_K is an invariant metric on $\mathbf{X}$. The polar decomposition for g_K and ω yields now a K-invariant almost complex structure on $\mathbf{X}$. □

Remark 1.2.9 (i) One should be aware that in general these almost complex structures are not integrable, their Nijenhuis tensor do not vanish. In fact, there are examples of varieties carrying symplectic structures but no Kähler ones (see [McD1] for details).

(ii) Let us notice that for performing our polar decomposition we did not use that the form ω is closed but only its non-degeneracy. Consequently, as soon as on a vector space we have a non-degenerate two-form and a metric we automatically get a compatible almost complex structure. ⋄

The results I gave until now are all valid for arbitrary symplectic manifolds, compact or not. From now on I shall always suppose that the symplectic manifolds I am working with are compact and to emphasize this I shall denote them by $\mathbf{X}$. I mentioned already that in this case the symplectic form gives rise to a cohomology class $[\omega] \in H^2(\mathbf{X}, \mathbb{R})$. Trying to find some links with the projective case, one may wonder if there are integral symplectic forms. The following lemma gives a positive answer to this question:

Lemma 1.2.10 *Let $(\mathbf{X}, \omega)$ a compact symplectic manifold. Then for a given $\varepsilon > 0$ there is a positive integer k with the property that there is another integral symplectic form ω'_k on $\mathbf{X}$ satisfying:*

$$\|\omega'_k - k \cdot \omega\| \leq \varepsilon$$

Proof First of all I have to clarify which is the norm in the statement of the lemma: I chose any metric $g \in \mathcal{M}et(\mathbf{X})$ and I define the following norm on the space of forms

$$\|u\| := sup_{p \in \mathbf{X}} |u_p|_{g_p}$$

With respect to this metric g there is the corresponding Hodge decomposition and accordingly I may write

$$\omega = \overset{\circ}{\omega} + \mathrm{d}\alpha$$

where $\overset{\circ}{\omega}$ is the g-harmonic part of ω. Now the space $\mathcal{H}^2(\mathbf{X}, \mathbb{R})$ is a finite dimensional vector space in which $\mathcal{H}^2(\mathbf{X}, \mathbb{Z})$ is a lattice. The norm defined above induces a norm on $\mathcal{H}^2(\mathbf{X}, \mathbb{R})$. For $\varepsilon > 0$ I will find an integral harmonic form $\overset{\circ}{\omega}_k{}'$ and $k > 0$ such that

$$\| \overset{\circ}{\omega}_k{}' - k \cdot \overset{\circ}{\omega} \| \leq \varepsilon$$

Now I define $\omega'_k := \overset{\circ}{\omega}_k{}' + k \cdot \mathrm{d}\alpha$ and one may see in a moment that $\|\omega'_k - k \cdot \omega\| \leq \varepsilon$; it will be an integral form by construction. Eventually taking a smaller $\varepsilon > 0$, the non-degeneracy of ω will imply that of ω'_k. □

This lemma combined with the result on the existence of compatible almost complex structures shows that symplectic manifolds are some kind of "bad" projective varieties. Let me explain why: Kodaira embedding theorem says that if the Kähler form of a compact complex Kähler manifold is *integral*, then it is projective. The embedding is given by means of holomorphic sections in a high enough power of the line bundle associated to the Kähler form. We see that in the symplectic setting we have integral two-forms but, of course, there are in general no *integrable* compatible almost complex structures. However, S.Donaldson found some very strong analogies with the projective case. A brief presentation of his results in this direction makes the object of the next section.

1.3 Some results of S.Donaldson

All the results which are stated here appear in the paper [Do] which is the main reference for this section. An obvious property of an algebraic variety is that it has divisors, complex codimension-one subvarieties which can be obtained intersecting them with some hypersurfaces of the projective space where the variety is immersed. The symplectic analogue is the following:

Theorem 1.3.1 (S.Donaldson) *Let $(\mathbf{X}, \omega)$ be a compact symplectic manifold of dimension $\dim\mathbf{X} = 2n$ such that the DeRham cohomology class $[\omega]$ is integral. Then for sufficiently large integers k the Poincaré-dual of $k[\omega]$ can be realized by a real codimension-two symplectic submanifold $\mathbf{Z}_k \hookrightarrow \mathbf{X}$. If $\mathbf{X}$ is connected, then $\mathbf{Z}_k$ can be chosen to be connected too.*

Proof The proof is technical and I won't give any details, but the idea of it. First one chooses an almost complex structure J compatible with the symplectic form. This form being integral, one may associate a complex line bundle $L \to \mathbf{X}$ endowed with a unitary connection whose curvature form is $2\pi i\omega$; the first Chern class of it will be $[\omega]$. Now the situation is similar to the Kähler case, but the non-integrability of J will be an obstruction in finding pseudo-holomorphic sections in L (or its powers). The way-out is the following: the aim is to find the subvarieties as the zero set of some section $s : \mathbf{X} \to L^{\otimes k}$ in a power of the line bundle; therefore one is interested in the behavior of s only along its zero set. Let me recall that the choice of the almost complex structure J on $\mathbf{X}$ determines a splitting $d = \partial + \overline{\partial}$ of the exterior derivative; the splitting extends to a similar one which acts on the sections of a complex line bundle. If s was pseudo-holomorphic, i.e. $\overline{\partial}s = 0$, along its zero set then we were done, but this is still not possible. The important remark is that the failure of the tangent space to $Zero(s)$ to be invariant under J, at a certain point p, is measured by the quantity $\overline{\partial}s(p)/\partial s(p)$; if it was zero, the tangent space in that point was J-invariant. If this quotient is non-zero but small, one looses the invariance; however the symplectic form is still non-degenerated on it. Therefore the proof is reduced to the following (stronger) statement:

Theorem 1.3.2 (S.Donaldson) *Let $L \to \mathbf{X}$ be a complex line bundle over a compact symplectic manifold $\mathbf{X}$ with compatible almost complex structure and with $c_1(L) = [\omega]$. Then there is a constant C such that for all large k there is a section $s : \mathbf{X} \to L^{\otimes k}$ with*

$$|\overline{\partial}s| \leq \frac{C}{\sqrt{k}}|\partial s|$$

on the zero set of s.

This statement is stronger that the previous theorem because it proves that for a given almost complex structure J compatible with the symplectic form there are "hyperplane sections" in $\mathbf{X}$ which are as close to be J-holomorphic as one wants to.

What happens in the case when the symplectic form ω is not any more integral? Lemma 1.2.10 says that there are integral symplectic forms ω'_k as close as wanted to some multiples $k\omega$, $k \gg 0$. If I fix a ω-compatible almost complex structure J, I may use the (fixed) metric $g_{\omega,J}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ to measure the closeness. Because J is compatible with $k\omega$ for any k, it will tame ω'_k for k big enough. Let us fix $\omega' := \omega'_k$ a "close enough" integral symplectic form. This means that

$$\|\omega'_k - k\omega\|_{g_{\omega,J}} \leq \varepsilon$$

for $\varepsilon > 0$ (very) small. Therefore one may write

$$\omega'(\cdot,\cdot) = (k\omega)(\cdot, A\cdot)$$

where $A \in Aut(T\mathbf{X})$ is close to the identity, $\|A - Id\| = o(\varepsilon)$. The aim is to find an almost complex structure J' compatible with ω' which is close to J. One may find it as follows: consider the metric $kg_{\omega,J} = g_{k\omega,J}$. Let us recall (proposition 1.2.8(i)) that there is a polar decomposition on the space of metrics with respect to the symplectic form ω'. I apply this decomposition to the metric $kg_{\omega,J} = g_{k\omega,J}$

$$\omega'(\cdot,\cdot) = g_{k\omega,J}(\cdot, -JA\cdot)$$

The continuity of the polar decomposition shows that the ω'-compatible almost complex structure we get will be close to J.

Now we may apply Donaldson's result to deduce the existence of ω'-symplectic connected, codimension-two subvarieties $\mathbf{H}_m$ of $\mathbf{X}$ representing the Poincaré-dual of $m\omega'$; they tend to be J'-holomorphic as m grows. Therefore, if ε is chosen small enough and after m is also taken big enough, the resulting ω'-symplectic subvarieties will be also ω-symplectic. This proves the following:

Lemma 1.3.3 *If $(\mathbf{X}, \omega)$ is a compact, connected symplectic manifold, then there are connected real codimension-two symplectic subvarieties of $\mathbf{X}$. Moreover, if one fixes a ω-compatible almost complex structure J on $\mathbf{X}$, then these subvarieties can be chosen to be as close to be J-holomorphic as wanted.*

Applying this procedure $\frac{1}{2}dim_{\mathbb{R}}\mathbf{X} - 1$ times we obtain that in any symplectic manifold $(\mathbf{X}, \omega, J)$ there are embedded connected symplectic Riemann surfaces $\mathbf{C} \overset{\iota}{\hookrightarrow} \mathbf{X}$ (of very high genus in general), whose tangent bundle fail to be J-invariant by an amount as small as wanted. If we consider on $\mathbf{C}$ the (non-degenerated) two-form $\iota^*\omega$ and the metric $\iota^* g_J$, the polar decomposition provides us an (integrable) complex structure $j \in Aut\, T\mathbf{C}$. This way $(\mathbf{C}, j)$ becomes a complex algebraic curve and the inclusion $\iota : (\mathbf{C}, j) \to (\mathbf{X}, J)$ can be viewed as a perturbed jJ-holomorphic curve; moreover,

$$\sup_{x \in \mathbf{C}} \|\overline{\partial}_{jJ}\iota\| = O\left(\text{dist}(T\mathbf{C}, JT\mathbf{C})\right)$$

where $\text{dist}(T\mathbf{C}, JT\mathbf{C})$ denotes the distance between the corresponding two-plane bundles along $\mathbf{C}$.

Remark 1.3.4 It is easy to construct almost complex structures J' on $\mathbf{X}$ with the properties that our curve $\mathbf{C}$ is J'-holomorphic and that J' is $\mathcal{C}^0$-close to J: first one works out such a $J'_{|\mathcal{U}}$

in a tubular neighborhood $\mathcal{U} \supset \mathbf{C}$. There is $A \in Aut\, T\mathcal{U}$ with the property that $J'_{|\mathcal{U}} = \mathrm{e}^{A} J \mathrm{e}^{-A}$. For a cut-off function f supported in $\mathcal{U}$ and which equals one on $\mathbf{C}$, define $J' := \mathrm{e}^{fA} J \mathrm{e}^{-fA}$; the problem is that A will be only $\mathcal{C}^0$-close to zero, f having big slope in general. Consequently $\|N_{J'} - N_J\|$ can hardly be expected to be close to zero. $\diamond$

Part II

On the geography of six-dimensional compact symplectic manifolds

2.1 Some generalities

Given a complex vector bundle $(E,j) \to \mathbf{X}$ over a compact manifold $\mathbf{X}$ one can define its Chern classes $c_k(E,j) \in H^{2k}(\mathbf{X},\mathbb{Z})$. If one has a continuous family $(j_t)_{t\in T}$ (T connected) of almost complex structures in E, one may associate to each element of this family the corresponding Chern classes $c_k(E,j_t) \in H^{2k}(\mathbf{X},\mathbb{Z})$ obtaining continuous maps $T \to H^{2k}(\mathbf{X},\mathbb{Z})$; such maps must be constant. This proves that the Chern classes of a complex vector bundle are invariant under continuous deformations of the complex structure.

If $(\mathbf{X},J)$ is an almost complex manifold, i.e. there are almost complex structures on its tangent space, of dimension $\dim\mathbf{X} = 2n$ the previous discution shows that the Chern classes of $\mathbf{X}$ (which are by definition the Chern classes of its tangent bundle) depend only on the deformation type of *the given* J. The Chern numbers are defined to be

$$c_1^n := \langle c_1^n(\mathbf{X}), [\mathbf{X}]\rangle,\ c_1^{n-1}c_2 := \langle c_1^{n-1}(\mathbf{X})c_2(\mathbf{X}), [\mathbf{X}]\rangle, \ldots, c_n := \langle c_n(\mathbf{X}), [\mathbf{X}]\rangle.$$

Remark 2.1.1 On the same differentiable variety $\mathbf{X}$ there can be almost complex structures having different Chern classes; for details in the six-dimensional case one may consult [OV].

We have seen in section 1.2 that any symplectic manifold $(\mathbf{X},\omega)$ is almost complex, the space $\mathcal{J}(\mathbf{X},\omega)$ being contractile. Therefore the Chern classes are well defined and it can be easily seen that they depend only on the deformation type of the symplectic form.

If $\mathbf{X}$ is a six-dimensional *almost complex* manifold, its Chern numbers obey some arithmetical restrictions. Namely

Lemma 2.1.2 *Let $(\mathbf{X},J)$ be a connected almost complex six-manifold. Then its Chern numbers satisfy the followings*

$$c_1^3 \equiv c_3 \equiv 0 \bmod 2,\ c_1c_2 \equiv 0 \bmod 24$$

Proof The third Chern number is just the Euler characteristic of $\mathbf{X}$ and it can be expressed using the Betti numbers of $\mathbf{X}$

$$c_3 = 1 - b_1 + b_2 - b_3 + b_4 - b_5 + 1 = 2 - 2b_1 + 2b_2 - b_3$$

The cup-product on $H^3(\mathbf{X},\mathbb{Z})$ is skew symmetric and non-degenerate (by Poincaré-duality); therefore its dimension b_3 must be even. This proves that c_3 is an even integer.

The restrictions on c_1^3 and c_1c_2 are less obvious, they require the use of the Atiyah-Singer index theorem. The complex structure on the tangent space to $\mathbf{X}$ determines a splitting

$$T^*\mathbf{X} \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}$$

of the complexified cotangent space. The *canonical complex spinor bundle* $\$^c$ can be identified with $\Lambda^{0,*} := \bigwedge \Lambda^{0,1}$ and it has a decomposition into the positive and negative spinors:

$$\$^c_+ := \Lambda^{0,even},\quad \$^c_- := \Lambda^{0,odd}$$

The exterior derivative splits as $\mathrm{d} = \partial + \overline{\partial}$ with $\overline{\partial}$ acting on the complex spinors. Consider now the formal adjoint $\overline{\partial}^*$ of $\overline{\partial}$ with respect to the hermitian metric determined by the complex structure; this way we get *the Dirac operator*

$$\not{D}^c_+ := \overline{\partial} + \overline{\partial}^* : \not{S}^c_+ \longrightarrow \not{S}^c_-$$

which is elliptic and its index is given by

$$index\, \not{D}^c_+ = \int_{\mathbf{X}} \hat{\mathbf{A}}(\mathbf{X}) e^{\frac{1}{2}c_1(\mathbf{X})}$$

In the six-dimensional case

$$\hat{\mathbf{A}}(\mathbf{X}) = 1 - \frac{1}{24} p_1(\mathbf{X}) = 1 - \frac{1}{24}(c_1^2(\mathbf{X}) - 2c_2(\mathbf{X}))$$

Doing the computations in the previous formula we get that

$$index\, \not{D}^c_+ = \frac{1}{24} c_1 c_2$$

and this proves that $c_1 c_2 \equiv 0 \bmod 24$.

In order to get informations on c_1^3 we consider the twisted Dirac operator (which is in fact the Dirac operator associates to another choice of the $Spin^c$-structure)

$$\not{D}^c_+(K_{\mathbf{X}}^{-1}) : \not{S}^c_+ \otimes K_{\mathbf{X}}^{-1} \longrightarrow \not{S}^c_- \otimes K_{\mathbf{X}}^{-1}$$

whose index is given by

$$index\, \not{D}^c_+(K_{\mathbf{X}}^{-1}) = \int_{\mathbf{X}} \hat{\mathbf{A}}(\mathbf{X}) e^{\frac{3}{2}c_1(\mathbf{X})} = \frac{1}{2} c_1^3 + \frac{1}{8} c_1 c_2$$

Because we know already that $\frac{1}{8} c_1 c_2 \in \mathbb{Z}$, we deduce that $c_1^3 \equiv 0 \bmod 2$. $\square$

A detailed presentation of the Dirac operators can be found in [LM] where are also proved these index theorems: corollary D.18, page 400.

2.2 Basic tools

In the whole paper, if $\mathbf{Y}$ is a submanifold of $\mathbf{X}$, $\mathcal{N}_{\mathbf{Y}|\mathbf{X}}$ will denote the normal bundle of $\mathbf{Y}$ in $\mathbf{X}$. I shall recall some results used in the article in order to produce new examples of symplectic manifolds from given ones. The method I shall use extensively is that of the symplectic connected sum of two symplectic manifolds. The details of this construction can be found in [G].

Proposition 2.2.1 *Let $\mathbf{X}_j$, $j = 1, 2$ denote two symplectic manifolds of dimension $dim(\mathbf{X}_j) = n$ and let $\mathbf{Y}$ be another symplectic manifold of dimension $dim(\mathbf{Y}) = n - 2$ such that there are two symplectic embeddings $i_j : \mathbf{Y} \longrightarrow i_j(\mathbf{Y}) = \mathbf{Y}_j \hookrightarrow \mathbf{X}_j$ with the property:*

$$c_1(\mathcal{N}_{\mathbf{Y}_1|\mathbf{X}_1}) + c_1(\mathcal{N}_{\mathbf{Y}_2|\mathbf{X}_2}) = 0.$$

Then one can make the symplectic connected sum of $\mathbf{X}_1$ and $\mathbf{X}_2$ along $\mathbf{Y}$; the result will be a new symplectic manifold.

Suppose now that there are two 2-spheres $\mathbf{L}_j \hookrightarrow \mathbf{X}_j$ which meet $\mathbf{Y}_j$ transversally in one point. Then we can form the symplectic sum of the pair $(\mathbf{X}_1, \mathbf{Y}_1, \mathbf{L}_1)$, $(\mathbf{X}_2, \mathbf{Y}_2, \mathbf{L}_2)$. It will also contain an embedded 2-sphere $\mathbf{L}$ which is the (usual) connected sum of $\mathbf{L}_1$ and $\mathbf{L}_2$.

Proof See [G], page 538. □

Now we are interested in deciding whether such a symplectic sum is or is not simply connected. This criterion is also known and used by Gompf in his article.

Proposition 2.2.2 *Let* $\mathbf{X}$ *be a symplectic manifold and suppose that there is a symplectic codimension two submanifold* $\mathbf{Y}$ *which has the properties:*

- *its normal bundle is trivial, i.e.* $\mathcal{N}_{\mathbf{Y}|\mathbf{X}} = \mathcal{O}_{\mathbf{Y}}$;
- *there is an embedded projective line* $\mathbf{L} \hookrightarrow \mathbf{X}$ *which cuts* $\mathbf{Y}$ *transversally in exactly one point.*

Then the homomorphism $i_* : \pi_1(\mathbf{X} - \mathbf{Y}) \longrightarrow \pi_1(\mathbf{X})$ *induced by inclusion is in fact an isomorphism. In particular, if* $\mathbf{X}$ *if 1-connected,* $\mathbf{X} - \mathbf{Y}$ *will be also.*

Proof The proof uses Weinstein's theorem which tells us that in our situation there is a symplectically embedded neighborhood $\mathbf{D}_\epsilon \times \mathbf{Y} \hookrightarrow \mathbf{X}$ of $\mathbf{Y}$ (see for instance [McDS], page 98), where $\mathbf{D}_\epsilon$ is the disc of radius ϵ in $\mathbb{R}^2$ with the induced symplectic structure. Because $codim_{\mathbf{X}}\mathbf{Y} = 2$, any loop in $\mathbf{X}$ can be moved away from $\mathbf{Y}$ and so i_* is surjective. Let us now consider a loop $\gamma \hookrightarrow \mathbf{X} - \mathbf{Y}$ whose image $i_*[\gamma] = 0$; this means that there is a homotopy $\Gamma : I^2 \longrightarrow \mathbf{X}$ of γ to a constant map. This homotopy can be taken in such a way that it meets $S(\epsilon) \times \mathbf{Y}$ transversally in a finite number of circles γ_k, each of them homotopic to a meridian $S^1(\epsilon) \times \text{pt.}$ of $S^1(\epsilon) \times \mathbf{Y}$. We deduce that $[\gamma] = \prod_k [\gamma_k]$ in $\mathbf{X} - \mathbf{Y}$. In order to prove that $[\gamma] = 0$ in $\mathbf{X} - \mathbf{Y}$ it suffices to prove that each class $[\gamma_k]$ is so. We may move each circle γ_k until we reach the intersection circle $c := \mathbf{L} \cap (S(\epsilon) \times \mathbf{Y})$; but now c is contractible in $\mathbf{L} - \{point\} = \mathbb{C}$. □

Proposition 2.2.3 *(i) Suppose* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are symplectic 4-manifolds which contain isomorphic symplectically embedded curves* $\mathbf{Y}_1 \cong \mathbf{Y}_2 \cong \mathbf{Y}$ *with trivial normal bundle. Then the Chern numbers of the symplectic connected sum* $\mathbf{X} := \mathbf{X}_1 \sharp_{\mathbf{Y}} \mathbf{X}_2$ *are the following:*

$$\begin{aligned} c_1^2(\mathbf{X}) &= c_1^2(\mathbf{X}_1) + c_1^2(\mathbf{X}_2) - 4c_1(\mathbf{Y}) \\ c_2(\mathbf{X}) &= c_2(\mathbf{X}_1) + c_2(\mathbf{X}_2) - 2c_1(\mathbf{Y}) \end{aligned}$$

(ii) Suppose $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are symplectic 6-manifolds which contain isomorphic symplectically embedded 4-folds* $\mathbf{Y}_1 \cong \mathbf{Y}_2 \cong \mathbf{Y}$ *with trivial normal bundle. Then the Chern numbers of the symplectic connected sum* $\mathbf{X} := \mathbf{X}_1 \sharp_{\mathbf{Y}} \mathbf{X}_2$ *are the following:*

$$\begin{aligned} c_1^3(\mathbf{X}) &= c_1^3(\mathbf{X}_1) + c_1^3(\mathbf{X}_2) - 6c_1^2(\mathbf{Y}), \\ c_1c_2(\mathbf{X}) &= c_1c_2(\mathbf{X}_1) + c_1c_2(\mathbf{X}_2) - 2(c_1^2(\mathbf{Y}) + c_2(\mathbf{Y})), \\ c_3(\mathbf{X}) &= c_3(\mathbf{X}_1) + c_3(\mathbf{X}_2) - 2c_2(\mathbf{Y}). \end{aligned}$$

Proof Regardless of the dimension of the data, we may again apply Weinstein's theorem and deduce that there are the symplectic embeddings $\varphi_j : \mathbf{Y} \times \mathbf{D}_\epsilon \longrightarrow \mathbf{X}_j$ such that the restrictions $\varphi_{j|\mathbf{Y}\times 0} = i_j$. Take the involution ψ of $\mathbf{D}_\epsilon - 0$ given in polar coordinates by: $\psi(r, \theta) := (\sqrt{\epsilon^2 - r^2}, -\theta)$. This is a symplectomorphism of the punctured disc. We can use it to identify (symplectically by $id_{\mathbf{Y}} \times \psi$) the two tubular neighborhoods $\mathbf{Y}_j \hookrightarrow \mathbf{X}_j$ in order to obtain the symplectic sum $\mathbf{X}$. Topologically, $\mathbf{X}$ is obtained by identifying the cylinders $\mathbf{Y}_j \times S(r_0)$, where $S(r_0)$ denotes the circle of radius $r_0 := \epsilon/\sqrt{2}$. Let me consider now a top-dimensional form η on $\mathbf{X}$. The restrictions $\eta_{|\mathbf{X}_1 - (\mathbf{D}_{r_0} \times \mathbf{Y}_1)}$ and $\eta_{|\mathbf{X}_2 - (\mathbf{D}_{r_0} \times \mathbf{Y}_2)}$ can always be extended (not uniquely, of course) to $\mathbf{X}_1$ and $\mathbf{X}_2$; let us chose such extensions and denote them by η_1 and η_2. Remark now that the map $id_{\mathbf{Y}} \times \psi$ gives us a form θ on $\mathbf{Y} \times S^2$ obtained from η_1 and η_2 and identifying them on $\mathbf{Y} \times S(r_0)$ where they agree with η. With these choices, let us integrate η on $\mathbf{X}$:

$$\begin{aligned}\int_{\mathbf{X}} \eta &= \int_{\mathbf{X}_1-(\mathbf{D}_{r_0}\times\mathbf{Y}_1)} \eta + \int_{\mathbf{X}_2-(\mathbf{D}_{r_0}\times\mathbf{Y}_2)} \eta \\ &= \int_{\mathbf{X}_1} \eta_1 + \int_{\mathbf{X}_2} \eta_2 - \int_{\varphi_1(\mathbf{D}_{r_0}\times\mathbf{Y})\cup_\psi \varphi_2(\mathbf{D}_{r_0}\times\mathbf{Y})} \eta \\ &= \int_{\mathbf{X}_1} \eta_1 + \int_{\mathbf{X}_2} \eta_2 - \int_{\mathbf{Y}\times S^2} \theta\end{aligned}$$

In order to compute the Chern numbers of the connected sum we may take connections on $\mathbf{X}_1$ and $\mathbf{X}_2$ having the property that they agree on $\mathbf{Y}_1 \times \mathbf{D}_\epsilon$ and $\mathbf{Y}_2 \times \mathbf{D}_\epsilon$ with a product connection on $\mathbf{Y} \times \mathbf{D}_\epsilon$. Let η_1 and η_2 be the forms representing the (same) Chern classes on $\mathbf{X}_1$ and $\mathbf{X}_2$ respectively, corresponding to these connections. By our choice we may glue together η_1 and η_2 obtaining a form η on $\mathbf{X}$ which will represent the Chern class of $\mathbf{X}$. Now, when we want to integrate η on $\mathbf{X}$, we have already at our disposal the forms η_1 and η_2, by the very construction. The form θ which appears in the formula above will represent some Chern class on $\mathbf{Y} \times S^2$. $\square$

2.3 Behaviour of the Chern numbers

In this paragraph I shall study the behaviour of the Chern numbers under blow-ups. Let us note in passing that the blow-up procedure preserves the fundamental group as can be seen using the Seifert-van Kampen theorem. In [McD] is presented the blowing-up procedure of a symplectic submanifold of a symplectic manifold; I shall recall it for the sake of completeness. Let $\mathbf{Y} \hookrightarrow \mathbf{X}$ be a symplectic submanifold and denote by $\mathcal{N}_{\mathbf{Y}|\mathbf{X}}$ its normal bundle. On the projectivized bundle $\tilde{\mathbf{Y}} := \mathbb{P}(\mathcal{N}_{\mathbf{Y}|\mathbf{X}}) \longrightarrow \mathbf{Y}$ (with respect to a complex stucture which is compatible with the symplectic form ω), there is a tautological line bundle T over $\tilde{\mathbf{Y}}$ whose fiber over a point $(y, l_v) \in \tilde{\mathbf{Y}}$ is $\{(y, \lambda v), \lambda \in \mathbb{C}\}$, the complex line corresponding to l_v. There is the commuting diagram:

$$\begin{array}{ccc} T & \longrightarrow & \tilde{\mathbf{Y}} \\ \varphi \downarrow & & \downarrow f \\ \mathcal{N}_{\mathbf{Y}|\mathbf{X}} & \longrightarrow & \mathbf{Y} \end{array}$$

Moreover, φ is a diffeomorphism if restricted to non-zero vectors of T. Therefore one can take a small disc sub-bundle of T around the zero section which is identified with a tubular neighborhood V of $\mathbf{Y} \hookrightarrow \mathbf{X}$. Then one takes

$$\tilde{\mathbf{X}} := (\mathbf{X} - V) \cup_{\partial V} \varphi^{-1}(V)$$

to be the symplectic blow-up of $\mathbf{X}$ along $\mathbf{Y}$. What is still to be shown is that $\tilde{\mathbf{X}}$ is really a symplectic manifold. This is done in §3 of McDuff's article and (if $h : \tilde{\mathbf{X}} \longrightarrow \mathbf{X}$ is the projection) the new symplectic form is essentially $\tilde{\omega} = h^*\omega + \epsilon a$ with $a \in H^2(\tilde{\mathbf{X}})$ a form which is non-degenerate along the "exceptional divisor". We may notice that the Kähler form on the blow-up of a Kähler manifold is constructed in a similar way.

Lemma 2.3.1 *In the following, Σ will always denote a symplectic manifold of dimension six.*

• **Blow-up of a point** *If Σ' is the blow-up at a point of Σ, then the Chern numbers transform as follows:*

$$\begin{aligned} c_1^3(\Sigma') &= c_1^3(\Sigma) - 8, \\ c_1c_2(\Sigma') &= c_1c_2(\Sigma), \\ c_3(\Sigma') &= c_3(\Sigma) + 2. \end{aligned}$$

• **Blow-up of a curve** *Let $C \hookrightarrow \Sigma$ be an algebraic curve of genus g and consider the blow-up Σ' of Σ along C. Then we have the relations:*

$$\begin{aligned} c_1^3(\Sigma') &= c_1^3(\Sigma) + 6(g-1) - 2\langle c_1(\mathcal{N}_{C|\Sigma}), [C]\rangle, \\ c_1c_2(\Sigma') &= c_1c_2(\Sigma), \\ c_3(\Sigma') &= c_3(\Sigma) - 2(g-1). \end{aligned}$$

Proof Since c_3 represents the Euler number, one can immediately deduce the corresponding formulae for it.

Let me compute now the transformation laws for c_1^3 and c_1c_2 in the case when a point is blown-up. According to [GH, p.608-609], $c_1' = h^*c_1 - 2\eta$ and $c_2' = h^*c_2$, where $h : \Sigma' \longrightarrow \Sigma$ denotes the natural projection and η represents the Poincaré dual of the exceptional divisor $\mathcal{E} \hookrightarrow \Sigma'$. Using these relations we deduce that

$$(c_1')^3 = c_1^3 - 8\eta^3 = c_1^3 - 8\int_{\mathcal{E}} \eta^2 = c_1^3 - 8,$$

$$c_1'c_2' = (h^*c_1 - 2\eta)\cdot h^*c_2 = c_1c_2 - 2\int_{\mathcal{E}} j^*h^*c_2 = c_1c_2 - 2\int_{\mathcal{E}} f^*i^*c_2 = c_1c_2.$$

I shall compute now $(c_1')^3$ and $c_1'c_2'$ when the curve $C \hookrightarrow \Sigma$ of genus g is blown-up. In this case we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{E} & \overset{j}{\hookrightarrow} & \Sigma' \\ f\downarrow & & \downarrow h \\ C & \overset{i}{\hookrightarrow} & \Sigma \end{array}$$

It is proved in [GH, p.609-610] that $c_1' = h^*c_1 - \eta$ and $c_2' = h^*c_2 + h^*\mathrm{PD}_{\mathbf{X}}C - h^*c_1(\mathbf{X})\cdot\eta$. Consequently,

$$(c_1')^3 = c_1^3 - 3h^*c_1^2\cdot\eta + 3h^*c_1\cdot\eta^2 - \eta^3.$$

Let me compute the righthandside of the previous equality:

$$\begin{aligned} h^*c_1^2\cdot\eta &= \int_{\mathcal{E}} j^*h^*c_1^2 = \int_{\mathcal{E}} f^*i^*c_1^2 = 0 \quad \text{ since } i^*c_1^2 \in H^4(C) \text{ is zero,} \\ h^*c_1\cdot\eta^2 &= \int_{\mathcal{E}} j^*h^*c_1\cdot j^*\eta = \int_{\mathcal{E}} f^*i^*c_1\cdot j^*\eta \\ &= \int_{\mathcal{E}} \big(2(1-g) + \langle c_1(\mathcal{N}_{C|\Sigma}), [C]\rangle\big)\, f^*vol_C\cdot j^*\eta \\ &\overset{\star}{=} 2(g-1) - \langle c_1(\mathcal{N}_{C|\Sigma}), [C]\rangle, \\ \eta^3 &= \int_{\mathcal{E}} j^*\eta^2 \overset{\star\star}{=} \int_{\mathcal{E}} \zeta^2 \overset{\star\star\star}{=} \int_{\mathcal{E}} f^*c_1(\mathcal{N}_{C|\Sigma})\cdot\zeta = -\langle c_1(\mathcal{N}_{C|\Sigma}), [C]\rangle. \end{aligned}$$

For writing the equality $(\star)$, I use that the restriction of $j^*\eta$ to the fibres of $\mathcal{E} \overset{f}{\to} C$ (which are all isomorphic to $\mathbb{P}^1$) is *minus* the class of the volume form of the fibres, and therefore $f^*vol_C \cdot j^*\eta$

is minus the class of the volume form of $\mathcal{E}$. The form ζ is the first Chern class of the tautological bundle $T \to \mathcal{E} = \mathbb{P}(\mathcal{N}_{C|\Sigma})$. It is known that $\zeta = j^*\eta$ (see [GH, p.607]) and that the following equality holds (see [GH, p.606]):

$$\zeta^2 - f^*\left(c_1(\mathcal{N}_{C|\Sigma})\right) \cdot \zeta = 0.$$

These considerations justify ($\star\star$) and ($\star\star\star$). It remains to prove that $c_1 c_2$ is invariant under the blow-up. Indeed

$$\begin{aligned} c_1' c_2' &= (h^*c_1 - \eta) \cdot (h^*c_2 + h^*\mathrm{PD}_{\mathbf{X}}C - h^*c_1 \cdot \eta) \\ &= c_1c_2 + c_1 \cdot \mathrm{PD}_{\mathbf{X}}C - h^*c_1^2 \cdot \eta - h^*c_2 \cdot \eta - h^*\mathrm{PD}_{\mathbf{X}}C \cdot \eta + h^*c_1 \cdot \eta^2 \\ &= c_1c_2 \end{aligned}$$

To obtain the last equality, I use that:

$$\begin{aligned} &h^*c_1^2 \cdot \eta = \int_{\mathcal{E}} j^*h^*c_1^2 = \int_{\mathcal{E}} f^*i^*c_1^2 = 0, \\ &h^*\mathrm{PD}_{\mathbf{X}}C \cdot \eta = \int_{\mathcal{E}} j^*h^*\mathrm{PD}_{\mathbf{X}}C = \int_{\mathcal{E}} f^*i^*\mathrm{PD}_{\mathbf{X}}C = 0, \\ &h^*c_2 \cdot \eta = \int_{\mathcal{E}} j^*h^*c_2 = \int_{\mathcal{E}} f^*i^*c_2 = 0, \\ &c_1 \cdot \mathrm{PD}_{\mathbf{X}}C = \int_C i^*c_1 = 2(1-g) + \langle c_1(\mathcal{N}_{C|\Sigma}), C\rangle. \end{aligned}$$

The remaining term was already computed and it is $h^*c_1 \cdot \eta^2 = 2(g-1) - \langle c_1(\mathcal{N}), C\rangle$. □

2.4 The building blocks of the construction

The following examples can be found in [G]. I shall recall the construction of those which will be used here.

A Consider $W_1 := \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2$. Supposing that the nine points are in general position, we can think them of as the intersection points of two smooth cubics $\mathcal{C}$ and $\mathcal{C}'$ in $\mathbb{P}^2$. We may consider the linear system $\lambda\mathcal{C}+\mu\mathcal{C}'$, $[\lambda : \mu] \in \mathbb{P}^1$, which has base points exactly the nine given points. Blowing-up $\mathbb{P}^2$ in these points we separate the directions and obtain an elliptic fibration structure on W_1 over $\mathbb{P}^1$. Any of the exceptional divisors provides a symplectically embedded projective line which cuts in only one point the generic fiber which is, topologically, a torus of self-intersection 0. We are in the case of propositions 1.1 and 1.2 and we can do the symplectic sum of m pairs $(W_1, fibre, E)$, where E denotes an exceptional divisor. The resulting (elliptic) fibration W_m contains a projective line which cuts transversally in one point the generic elliptic fibre. Let now W_m^* denote the blow-up of W_m in one point and let e_m be the corresponding exceptional divisor. Obviously W_m^* is 1-connected; the Chern numbers are:

$$\begin{aligned} c_1^2(W_m^*) &= -1, \\ c_2(W_m^*) &= 12m + 1. \end{aligned}$$

B This building block is the 4-manifold $S_{1,1}$ in [G], page 566-568. I won't describe its construction but I shall mention its properties:

- it is symplectic and 1-connected;

• it contains symplectically embedded curves F_1, F_2 of genus 1 and 2 respectivelly, with trivial normal bundle; in addition, each of these curves are cut by (disjoint) spheres transversally, in one point;

• its Chern numbers are:

$$c_1^2(S_{1,1}) = 1,$$
$$c_2(S_{1,1}) = 23.$$

C Consider a quartic curve in $\mathbb{P}^2$ having a single node. Blow it up to obtain a non-singular projective curve of degree 4 and genus 2 in $\mathbb{P}^2\sharp\overline{\mathbb{P}}^2$ having self-intersection 2. Blowing-up twelve more times one has a genus two curve F_2 of self-intersection 0 in $P_1 := \mathbb{P}^2\sharp 13\overline{\mathbb{P}}^2$. The 13^{th} exceptional divisor will be a projective line which intersects our curve F_2 transversally in only one point. The Chern numbers are: $c_1^2(P_1) = 9 - 13 = -4$, $c_2(P_1) = 3 + 13 = 16$. We can make the symplectic sum of l pairs (P_1, F_2, E), the result being the 1-connected symplectic manifold P_l containing a curve F_2 of self-intersection 0 and a projective line which cut it transversally in one point. We may compute the Chern numbers for this manifold using proposition 2.2.3:

$$c_1^2(P_l) = 4l - 8,$$
$$c_2(P_l) = 20l - 4.$$

2.5 The construction of the examples

In this section I prove the main result of this part, namely:

Theorem 2.5.1 *Let* $\mathbf{X}$ *be a symplectic* 6*-manifold. Then*

$$c_1^3(\mathbf{X}) \equiv c_3(\mathbf{X}) \equiv 0 \bmod 2, \ \ c_1c_2(\mathbf{X}) \equiv 0 \bmod 24$$

Conversely, any triple (a, b, c) *with* $a \equiv c \equiv 0 \bmod 2$ *and* $b \equiv 0 \bmod 24$ *occurs as a triple* (c_1^3, c_1c_2, c_3) *of Chern numbers of some connected and 1-connected, compact symplectic 6-manifold.*

The basic tool in constructing the examples is the symplectic connected sum of pairs. I shall use the notation $\mathbf{X}_1\sharp_{\mathbf{Y}}\mathbf{X}_2$ to denote the symplectic connected sum of $\mathbf{X}_1$ and $\mathbf{X}_2$ along $\mathbf{Y}$; F_g will denote always a projective curve of genus g. Consider now first the symplectic four manifold

$$\mathbf{X} := W_m^*\sharp_{F_1}S_{1,1},$$

where the connected sum is done away from the exceptional divisor e_m. This manifold is symplectic but also 1-connected. Indeed, as mentioned, W_m^* contains a 2-sphere which meets the generic fiber F_1 transversally in one point; the curve $F_1 \hookrightarrow S_{1,1}$ is cut also transversally by such a 2-sphere. Now we can apply proposition 1.2 and deduce that $\mathbf{X}$ is 1-connected. It must be mentioned at this point that we have done again a connected sum of pairs: because in both W_m^* and $S_{1,1}$ there are 2-spheres cutting F_1 transversally, we may do the identifications in such a way that $F_1 \hookrightarrow \mathbf{X}$ is cut again by a 2-sphere transversally in one point, according to proposition 2.2.2. We will use this property to show the simply connectedness of some of our examples. The Chern numbers are:

$$c_1^2(\mathbf{X}) = 0,$$
$$c_2(\mathbf{X}) = 12m + 24.$$

The constuction will be broken into several steps, according to the values of c_1c_2. Let me begin with a lemma which gives us a simple method to vary c_1^3 and c_3 independently.

Lemma *Suppose that we have a six-dimensional, 1-connected symplectic manifold Σ having the Chern numbers $(2a, 24b, 2c)$. Suppose that we have the additional properties:*

- *Σ contains a symplectically embedded product $U \times \mathbf{D}$, where U is an open subset of a symplectic 4-manifold and $\mathbf{D}$ is a disc. Assume further that U contains a projective line E having the property that $-\alpha := \langle c_1(\mathcal{N}_{E|\Sigma}), [E]\rangle \leq -1$. This happens in the case where Σ is either a product $\mathbf{S} \times F$ ($\mathbf{S}$ is a surface containing the projective line E, and F is an algebraic curve) or is obtained by a symplectic connected sum of such a product, away from E. In this case one can move E in the direction given by F;*
- *there is a projective curve F_2 of genus two, disjoint from E, with trivial normal bundle.*

Then, just blowing up x points, r distinct copies of E and z distinct copies of F_2, one can obtain all triples of Chern numbers of the form $(2a', 24b, 2c')$, where a', c' are arbitrary integers.

Proof In fact, the formulas given in §2 show that c_1c_2 is invariant and the other two Chern numbers are:

$$c_1^3 = 2(a - (3-\alpha)\cdot r - 4x + 3z),$$
$$c_3 = 2(c + r + x - z).$$

Imposing the condition that $c_1^3 = 2a', c_3 = 2c'$, we obtain

$$x = \alpha \cdot r - (a' - a + 3(c' - c)),$$
$$z = (1+\alpha)r - (a' - a + 4(c' - c))$$

and one can see that we can chose r big enough to ensure the positivity of x and z. □

Now we can finally construct the examples. We will distinguish several cases, according to the value of c_1c_2. Let us begin with:

$\underline{c_1c_2/24 = 0}$ Consider $W_1^* \times F_1$; it is a symplectic manifold, but there are two problems: it is not 1-connected and there is no F_2 curve on it. We will see that considering the following

$$\Sigma := W_1^* \times F_1 \sharp_{F_1 \times F_1} S_{1,1} \times F_1,$$

where we identify in cross the two F_1's, these problems disappear. Making the connected sum with $S_{1,1} \times F_1$ has the effect of introducing the needed curve F_2. We have to show that this manifold is 1-connected.

In order to make the proof clear, let us denote the two F_1 factors by F_1' and F_1''. In each summand $F_1' \times F_1''$ has trivial normal bundle, so we may chose the (symplectically embedded) neighborhoods $F_1' \times F_1'' \times \mathbf{D}_\epsilon$ in each. For writing down the Seifert-van Kampen diagram, we consider the open sets

$U_1 := W_1^* \times F_1' - F_1'' \times F_1' \times \mathbf{D}_{3\epsilon/4} = (W_1^* - F_1'' \times \mathbf{D}_{3\epsilon/4}) \times F_1',$
$U_2 := S_{1,1} \times F_1'' - F_1' \times F_1'' \times \mathbf{D}_{3\epsilon/4} = (S_{1,1} - F_1' \times \mathbf{D}_{3\epsilon/4}) \times F_1'',$

the intersection being $U_1 \cap U_2 = F_1' \times F_1'' \times \mathbf{A}$ with $\mathbf{A} := \mathbf{D}(\epsilon/4, 3\epsilon/4)$ an annulus. There is the following commuting diagram:

$$\begin{array}{ccccc} & & \pi_1(U_1) & & \\ & i'_* \nearrow & & \searrow j'_* & \\ \pi_1(F'_1)\oplus\pi_1(F''_1)\oplus\pi_1(\mathbf{A}) & & & & \pi_1(\Sigma) \\ & i''_* \searrow & & \nearrow j''_* & \\ & & \pi_1(U_2) & & \end{array}$$

$\pi_1(\Sigma)$ is generated by the images of $\pi_1(U_1)$ and $\pi_1(U_2)$. We will prove that these images are trivial.

$\pi_1(U_1) = \pi_1(W_1^* - F''_1 \times \mathbf{D}_{3\epsilon/4}) \oplus \pi_1(F'_1) = 0 \oplus \pi_1(F'_1)$

$\pi_1(U_2) = \pi_1(S_{1,1} - F'_1 \times \mathbf{D}_{3\epsilon/4}) \oplus \pi_1(F''_1) = 0 \oplus \pi_1(F''_1)$

To write down these equalities we used the fact that in W_1^* and $S_{1,1}$ there are 2-spheres which cut transversally in one point F''_1 and F'_1 respectively; then we applied proposition 2.2.2. Let us prove that the images of $\pi_1(U_1)$ and $\pi_1(U_2)$ are trivial in $\pi_1(\Sigma)$:

$$j'_*\pi_1(U_1) = j'_*i'_*\pi_1(F'_1) = j''_*i''_*\pi_1(F'_1) = j''_*0 = 0.$$

An analogous reasoning shows the triviality of $j''_*\pi_1(U_2)$. Therefore Σ is 1-connected and has obviously $c_1c_2 = 0$. It satisfies the conditions of the lemma and we are done. □

$\underline{c_1c_2/24 = 1}$ Take $\Sigma := P_1 \times \mathbb{P}^1$. The generic projective line passing through the node of the quartic considered in the construction of P_1 meets it two more times. Blowing-up the node we will have a projective line (fix one) which meets the proper transform of the quartic in two points. Now, blow-up these two points and ten more to obtain in P_1 the F_2 curve of self-intersection 0 and the (-2) projective line E. We are again in the situation of the lemma with $\alpha = 2$. □

$\underline{c_1c_2/24 = -1}$ Just take

$$\Sigma := W_1^* \times F_2 \sharp_{F_1\times F_2} P_1 \times F_1.$$

The result is 1-connected and satisfies the conditions of the lemma with $\alpha = 1$. □

$\underline{c_1c_2/24 \geq 2}$ We are just considering the product $\Sigma := \mathbf{X} \times \mathbb{P}^1$; its Chern numbers are $(0, 24(m+2), 2(12m+24))$. This is a 1-connected symplectic manifold. In Σ there is the projective line e_m which lies in W_m^* and has the normal bundle $\mathcal{O} \oplus \mathcal{O}(-1)$. One can see that the α in the lemma above is -1. There is also the curve F_2 on Σ having trivial normal bundle. Blowing-up r times along e_m, x points and z times along F_2, we obtain a symplectic manifold having the Chern numbers:

$$\begin{aligned} c_1^3 &= 2(-2r - 4x + 3z), \\ c_1c_2 &= 24(m+2), \\ c_3 &= 2(12m + r + x - z + 24). \end{aligned}$$

Let us impose now that $(c_1^3, c_1c_2, c_3) = (2a, 24b, 2c)$. We will obtain the following restrictions on the parameters r, x, z:

- $r - x = a - 36b + 3c$,
- $m = b - 2$,
- $2r - z = a - 48b + 4c$.

One can see immediately that this system can be solved under the restriction that all parameters are positive, if and only if $b \geq 2$. □

$\underline{c_1c_2/24 \leq -2}$ Let us consider now the product $\mathbf{X} \times F_2$; the Chern numbers are: $(0, -24(m+2), -2(12m+24))$. The problem is that this symplectic manifold is not 1-connected; $\pi_1(\Sigma) = \mathbb{Z}^{\oplus 4}$ which comes from the F_2 factor. We cancel this fundamental group taking the symplectic connected sum (of pairs)

$$\Sigma := \mathbf{X} \times F_2 \sharp_{F_1 \times F_2} S_{1,1} \times F_1.$$

Σ is symplectic, and using proposition 2.2.3, we deduce that its Chern numbers are $(0, -24(m+2), -2(12m+24))$. A completely similar argument as in the $c_1c_2 = 0$ case shows that Σ is 1-connected. Doing again the blow-up of x points, r curves e_m and z curves F_2 we obtain a symplectic manifold having the Chern numbers:

$$\begin{aligned} c_1^3 &= 2(-2r - 4x + 3z), \\ c_1c_2 &= -24(m+2), \\ c_3 &= 2(-12m + r + x - z - 24). \end{aligned}$$

Imposing $(c_1^3, c_1c_2, c_3) = (2a, 24b, 2c)$ we obtain the following restrictions:

- $r - x = a - 36b + 3c$,
- $m = -b - 2$,
- $2r - z = a - 48b + 4c$.

This system is solvable if and only if $b \leq -2$. □

Part III

”Symplectic bend-and-break” Possible?

3.1 What am I looking for?

In this part I shall try to develop an alternative, differential geometric approach to the well-known bend-and-break technique in algebraic geometry which (at least theoretically) could be extended to the case of symplectic manifolds. The basic remark, which should be the guiding idea, is that the rational curve which appears when one "bends" an algebraic curve inside a projective manifold (keeping a point fixed), is nothing but a "bubble" which appears in the Gromov compactification of those morphisms.

A strong hint that positivity assumptions on the first Chern class of the tangent bundle of a symplectic manifold would imply the existence of symplectic rational curves on that manifold is given by the following:

Theorem (Taubes) *Let $(\mathbb{P}^2, \omega, J)$ be a symplectic projective plane such that the class of the symplectic form is the generator of the second cohomology class and $c_1(\mathbb{P}^2) = +3[\omega]$. In this case there is an embedded J-holomorphic sphere representing the generator of the second homology class.*

This theorem is very deep and it was proved by C.H.Taubes using the Seiberg-Witten invariants in [T1] and [T2]. However it has the shortcoming to be specific to the four-dimensional case and gives no hint if higher dimensional symplectic manifolds carry low-genus symplectic curves (or under which assumptions they would).

Even if the techniques I shall try to develop in the sequel are far from being strong enough to reprove this theorem they have no dimensional restrictions and (I think) suggest that if the anticanonical bundle of a symplectic manifold is suitably positive then there are symplectic curves of low-genus on it.

3.2 Bend-and-break, an overview

It is a method of producing rational curves (i.e. morphisms with source the projective line) on an algebraic variety. The result can be stated as:

Theorem (S.Mori) *Let $\mathbf{X}$ be an algebraic variety over a characteristic zero field having a curve $\mathbf{C}$ on it with the property that $-K_{\mathbf{X}} \cdot \mathbf{C} > 0$. Then through each point of $\mathbf{C}$ passes a non-trivial rational curve.*

The idea behind it is to raise the dimension of the space $Hom(\mathbf{C}, \mathbf{X})$ of morphisms from $\mathbf{C}$ to $\mathbf{X}$. This is possible only going through positive characteristics and making use of the Frobenius morphism. For this reason the proof of this theorem is strongly algebraic; but if it happens that the space $Hom(\mathbf{C}, \mathbf{X})$ has a sufficiently high dimension, the proof is much easier. Let me state the result in this weaker form as it can be found in [CKM], lecture #1:

Proposition 3.2.1 *Let $\mathbf{X}$ be an algebraic variety of dimension n having a curve $\mathbf{C}$ on it with the property that $\dim Hom(\mathbf{C}, \mathbf{X}) \geq n + 1$. Then through every point of $\mathbf{C}$ passes a non-trivial rational curve.*

Proof Obviously one may suppose that $\mathbf{C}$ is not the projective line. Let $\mathbf{C} \overset{u_0}{\to} \mathbf{X}$ be the given morphism, $0 \in \mathbf{C}$ a marked point and $*_0 := u_0(0) \in \mathbf{X}$. Because $\mathbf{C}$ is not the projective line, $(\mathbf{C}, 0)$ will be a stable curve. Let $Hom(\mathbf{C}, \mathbf{X})_{0 \to *_0} \hookrightarrow Hom(\mathbf{C}, \mathbf{X})$ be the subspace of the morphisms form $(\mathbf{C}, 0)$ to $(\mathbf{X}, *)$; it is a quasi-projective variety of dimension at least

$n+1-n=1$. Take a (non-complete) curve $\mathbf{D}$ on it and let $\bar{\mathbf{D}}$ be its completion. Evaluating the maps we get a morphism

$$f : \mathbf{D} \times \mathbf{C} \longrightarrow \mathbf{X}$$

having the property that $f(s,0) = *_0$ for all $s \in \mathbf{D}$. Suppose that it extends to a morphism

$$\bar{f} : \bar{\mathbf{D}} \times \mathbf{C} \longrightarrow \mathbf{X}$$

It will have the same property that $\bar{f}(s,0) = *_0$ for all $s \in \bar{\mathbf{D}}$. Because $(\mathbf{C}, 0)$ has finite automorphism group, the image of $\bar{f}$ (which is a closed subvariety of $\mathbf{X}$) is two dimensional. Define

$$A := \{p \in \mathbf{C} \mid \bar{f}(s,p) = *_p \in \mathbf{X} \ \forall s \in \bar{\mathbf{D}}\}$$

Here $*_p$ denotes a point of $\mathbf{X}$ depending on $p \in \mathbf{C}$. Obviously A is closed and non-empty because $0 \in A$. It is also open: for $p \in A$ consider an affine neighborhood U of $*_p$ in $\mathbf{X}$. There is a neighborhood V of p in $\mathbf{C}$ such that $\bar{f}(\pi_{\mathbf{C}}^{-1}(V)) \subset U$; here $\pi_{\mathbf{C}}$ denotes the projection $\bar{\mathbf{D}} \times \mathbf{C} \to \mathbf{C}$. For $p' \in V$ the map $\bar{\mathbf{D}} \ni s \mapsto \bar{f}(s,p') \in \mathbf{X}$ has its image in the affine U. Because $\bar{\mathbf{D}}$ is complete, it must be constant. This proves that $A = \mathbf{C}$. In this case the image of $\bar{f}$ is at most one dimensional, a contradiction with the previous statement.

The contradiction comes from the assumption that the evaluation map f extends from $\mathbf{D} \times \mathbf{C}$ to $\bar{\mathbf{D}} \times \mathbf{C}$. Therefore

$$\bar{f} : \bar{\mathbf{D}} \times \mathbf{C} - \to \mathbf{X}$$

is a rational map. By Hironaka, one may desingularize $\bar{f}$ blowing-up $\bar{\mathbf{D}} \times \mathbf{C}$ several times. At least one of the exceptional curves will be a non-trivial rational curve on $\mathbf{X}$ and will pass through $* \in \mathbf{X}$. $\square$

Remark 3.2.2 It is important to keep in mind that in order to satisfy the assumptions of proposition 3.2.1, Mori goes into positive characteristics where he makes use of the Frobenius morphism which allows him to raise the dimension of $Hom(\mathbf{C}, \mathbf{X})$. Simple index computation shows that in characteristic zero, e.g. when $\mathbf{X}$ is a complex projective variety, one can not fulfill, in general, the required assumptions even if multiple covered curves are allowed. $\diamond$

Even if, as I just explained, in the complex case the proposition above is almost vacuous (except the case of rational and elliptic curves), I shall give an alternative proof of it in this case i.e. $\mathbf{C}$ and $\mathbf{X}$ are complex algebraic varieties.

Alternative proof As before, I assume that there is an evaluation morphism

$$\bar{f} : \bar{\mathbf{D}} \times \mathbf{C} \longrightarrow \mathbf{X}$$

We saw that the image of $\bar{f}$ is two-dimensional, so there is $p \in \mathbf{C}$ such that

$$\bar{f}_p : \bar{\mathbf{D}} \to \mathbf{X} \qquad \bar{f}_p(u) := u(p)$$

is non-constant. If ω is the (integral) Kähler form on $\mathbf{X}$, then the evaluation $\langle \bar{f}_p^* \omega, \bar{\mathbf{D}} \rangle$ is a strictly positive integer by the simple fact that on any non-constant curve the Kähler form has positive degree. On the other hand, $\bar{f}_p$ is homotopic to $\bar{f}_0$ which is a constant map and consequently $\langle \bar{f}_0^* \omega, \bar{\mathbf{D}} \rangle = 0$. The contradiction proves that $\bar{f}$ can not be a morphism.

The fact that $\langle \bar{f}_p^* \omega, \bar{\mathbf{D}} \rangle > 0$ can be deduced also as follows: taking possibly a multiple, I may assume that the Poincaré-dual of ω is represented by a hypersurface $\mathbf{H} \hookrightarrow \mathbf{X}$ which does not contain $\bar{f}_p(\bar{\mathbf{D}})$ and which intersects it in at least one point (just take a hypersurface trough an arbitrary point of $\bar{f}_p(\bar{\mathbf{D}})$). Because we are dealing with complex varieties, at each intersection point the intersection number of $\mathbf{H}$ and $\bar{f}_p(\bar{\mathbf{D}})$ is $+1$ and therefore the evaluation above is strictly positive. □

Remark 3.2.3 Let me notice that this alternative proof doesn't use the assumption that $\mathbf{X}$ is a projective variety, but only the fact that $ev : Hom(\mathbf{C}, \mathbf{X}) \to \mathbf{X}$ is a holomorphic map. Consequently, such a proof works also for Kähler varieties and also, as we shall see, for varieties whose Nijenhuis tensor is small enough. The crucial assumption in this proposition is that on the dimension of the space of morphisms $\mathbf{C} \to \mathbf{X}$ and which is rearly satisfied in characteristic zero. ⋄

3.3 Some explicit examples

I want to leave for a moment this very abstract setting and to make an explicit computation which will show the link between the bend-and-break and bubbling phenomenon.

Example 3.3.1 I consider $\mathbf{X} = \mathbb{P}^2$ to be the standard projective plane with homogeneous coordinates $[z_0 : z_1 : z_2]$. Let me consider in this plane the point $* := [0 : 0 : 1]$ and the lines $\mathcal{L}_1 := \{z_1 - z_2 = 0\}$, $\mathcal{L}_\infty := \{z_1 + z_2 = 0\}$. Let us notice that $\mathcal{L}_1$ and $\mathcal{L}_\infty$ do not pass through $*$ and the intersection $\mathcal{L}_1 \cap \mathcal{L}_\infty = [1 : 0 : 0]$.

I want to study the space of holomorphic curves

$$u : (\mathbb{P}^1, 0, 1, \infty) \longrightarrow \mathbb{P}^2$$

of degree one such that $u(0) = *$, $u(1) \in \mathcal{L}_1$, $u_\infty \in \mathcal{L}_\infty$. The family of *unparameterized* curves which pass through $*$ is given by

$$L_\varepsilon := \{\varepsilon z_0 - z_1 = 0\}$$

where the slope $\varepsilon \in \mathbb{P}^1 = (\mathbb{P}^1 - \{0\}) \cup \{0\} = \mathbb{C} \cup \{0\}$. For $\varepsilon = 0$ we get the line $L_0 = \{z_1 = 0\}$ joining $*$ to $[1 : 0 : 0]$.

The family of parameterized holomorphic lines will be indexed by ε and I shall denote u_ε the holomorphic map whose image is L_ε. If ζ is the variable on $\mathbb{C}$, the form of the degree-one holomorphic maps u_ε will be

$$u_\varepsilon = [u_{\varepsilon,0}(\zeta) : u_{\varepsilon,1}(\zeta) : u_{\varepsilon,2}(\zeta)]$$

with $u_{\varepsilon,0}, u_{\varepsilon,1}, u_{\varepsilon,2}$ affine holomorphic maps. The coefficients of these maps can be determined using the restrictions

$$u_\varepsilon(0) = [0 : 0 : 1],\ u_\varepsilon(1) = [1 : \varepsilon : \varepsilon] = \mathcal{L}_1 \cap L_\varepsilon,\ u_\varepsilon(\infty) = [1 : \varepsilon : -\varepsilon] = \mathcal{L}_\infty \cap L_\varepsilon$$

The computations give, for $\varepsilon \neq 0$

$$u_\varepsilon(\zeta) = \left[\frac{1}{\varepsilon}\zeta : \zeta : -\zeta + 2\right]$$

A first remark is that for $\varepsilon = \infty$ this map is still well defined. In fact the problems come as ε approaches 0. Away from $\zeta = 0$ the map has the form

$$u_\varepsilon = \left[1 : \varepsilon : \varepsilon\left(-1 + \frac{2}{\zeta}\right)\right]$$

which is well defined even for $\varepsilon = 0$. On compact sets $K \subset \mathbb{P}^1 - \{0\}$ the differentials are uniformly bounded and in the limit we get a constant map. Let us turn our attention to the form of u_ε around $\zeta = 0$

$$u_\varepsilon = \left[\frac{1}{\varepsilon} \cdot \frac{\zeta}{2-\zeta} : \frac{\zeta}{2-\zeta} : 1\right]$$

It is clear that as $\varepsilon \to 0$, the differential of u_ε is unbounded in $\zeta = 0$. I shall analyze this situation from two points of view: the algebraic and the symplectic one.

The algebraic point of view The space of the considered morphisms is the non-complete curve $\mathbb{C} = \mathbb{P}^1 - \{0\}$ and we have the evaluation morphism

$$ev : \mathbb{C} \times \mathbb{P}^1 \longrightarrow \mathbb{P}^2$$

It extends to a rational map

$$\overline{ev} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$$

where the first $\mathbb{P}^1$-factor is $\overline{\mathbb{C}} = \mathbb{C} \cup \{0\}$. In order to make $\overline{ev}$ a morphism I need to consider the blow-up $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$ of $\mathbb{P}^1 \times \mathbb{P}^1$ in $[0:1] \times [0:1]$. This way I shall obtain a morphism

$$\widetilde{ev} : \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathbb{P}^2$$

For explicit computations, let me choose local coordinates (ε, ζ) in $\mathbb{P}^1 \times \mathbb{P}^1$ around $[0:1] \times [0:1]$. In these coordinates $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$ is nothing but the blow-up $Bl_{(0,0)}\mathbb{C}^2$ of $\mathbb{C}^2$ in the origin on which we have the local coordinates

$$\{(\varepsilon, \zeta) \times \mu \mid \zeta = \mu\varepsilon\}$$

The map $\widetilde{ev}$ will be given by

$$(\varepsilon, \zeta) \times \mu \overset{\widetilde{ev}}{\mapsto} \left(\frac{1}{\varepsilon} \cdot \frac{\mu\varepsilon}{2-\mu\varepsilon}, \frac{\mu\varepsilon}{2-\mu\varepsilon}\right) = \left(\frac{\mu}{2-\mu\varepsilon}, \frac{\mu\varepsilon}{2-\mu\varepsilon}\right)$$

For $\varepsilon = 0$ I get the restriction of $\widetilde{ev}$ to the exceptional divisor

$$(0,0) \times \mu \mapsto \left(\frac{\mu}{2}, 0\right)$$

whose image is the line $L_0 = \{z_1 = 0\}$.

The symplectic point of view Here the problem is formulated as follows: the differentials $du_\varepsilon(0)$ are unbounded as $\varepsilon \to 0$ and therefore the rescaling procedure will give us a tower of bubbles. In this concrete example there is only one bubble which appears. The limit *stable map* u_0 will have the domain

$$T = \mathbb{P}^1_{bubble} \sqcup_0 \mathbb{P}^1_{principal}$$

The image of restriction of u_0 to $\mathbb{P}^1_{bubble}$ will be the line L_0, while u_0 contracts the principal component $\mathbb{P}^1_{principal}$ to the point $[1:0:0]$. One may see that in this case the bubble component "absorbs" the whole energy of the principal component.

There is one more thing to notice, namely that the space of *unparameterized* holomorphic lines which pass through $* \in \mathbb{P}^2$ is $\mathbb{P}^1$ which is smooth and compact variety. But when one wants to parameterize these curves one gets a non-compact space. This happens because the point $1 \in \mathbb{P}^1$ will have to be "pushed" closer and closer to $\infty \in \mathbb{P}^1$ as we approach the line L_0. $\diamond$

As it will be explained in the sequel, in the higher genus case we have to allow the conformal structure to vary. The next two examples show that families of holomorphic curves with variable conformal structure may or may not give bubbles in the limit.

Example 3.3.2 The Veronese map

$$V : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
$$V[\zeta_0 : \zeta_1] := [\zeta_0^3 : \zeta_0^2\zeta_1 : \zeta_0\zeta_1^2 : \zeta_1^3]$$

gives an embedding of the projective line into the projective space whose image $V(\mathbb{P}^1) \hookrightarrow \mathbb{P}^3$ is a curve of degree three. Set theoretically $V(\mathbb{P}^1) = \mathbf{H}_2 \cap \mathbf{H}_3$ is the intersection of two hypersurfaces in $\mathbb{P}^3$ of degrees two and three respectively. Their equations are

$$\mathbf{H}_2 = \{z_1^2 - z_0 z_2 = 0\} \qquad \mathbf{H}_3 = \{z_0 z_3^2 + z_2^3 - 2 z_1 z_2 z_3 = 0\}$$

The intersection has degree six, so $\mathbf{H}_2 \cap \mathbf{H}_3$ double covers the map V. A simple computation shows that transverse intersection of a quadric and a cubic hypersurface in $\mathbb{P}^3$ is a smooth curve of genus four and degree six; therefore the above intersection is not transverse. But both $\mathbf{H}_2$ and $\mathbf{H}_3$ can be approximated by quadric and cubic hypersurfaces $\mathbf{H}_{2,\varepsilon}$ and $\mathbf{H}_{3,\varepsilon}$ whose intersection $\mathbf{C}_\varepsilon = \mathbf{H}_{2,\varepsilon} \cap \mathbf{H}_{3,\varepsilon}$ is smooth and transverse. We get this way a family $(\mathbf{C}_\varepsilon)_\varepsilon$ of curves of genus four and degree six inside $\mathbb{P}^3$ whose limit is a double covered rational curve. This can be explained as follows when we try to parameterize these genus-four curves, the differential of the maps will be unbounded and in the limit the bubble component absorbs the whole energy of the principal component (as in the previous example) yielding a double covered $\mathbb{P}^1$. $\diamond$

But it is much to optimistic to expect that bubbling appears when we let curves (with variable conformal structure) to move, keeping a marked point fixed. There is the following elementary counter-example even in the algebraic case:

Example 3.3.3 Consider the pencil $\{\lambda \mathbf{C}_1 + \mu \mathbf{C}_2 \mid [\lambda : \mu] \in \mathbb{P}^1\}$ generated by two cubics $\mathbf{C}_1$ and $\mathbf{C}_2$ in general position in $\mathbb{P}^2$. Blowing up the nine intersection points of these cubics we get the surface $\mathbf{S} = \mathbb{P}^2 \sharp 9 \overline{\mathbb{P}}^2$ which fibers over $\mathbb{P}^1$: the fiber over $[\lambda : \mu] \in \mathbb{P}^1$ is just the cubic $\lambda \mathbf{C}_1 + \mu \mathbf{C}_2$. Any of the exceptional divisors $\mathbf{E} \hookrightarrow \mathbf{S}$ can be thought of as a section of this fibration which gives a marked point on each fibre. The natural projection $\pi : \mathbf{S} \to \mathbb{P}^2$ is nothing but an evaluation map and it has the property that contracts the exceptional divisor $\mathbf{E}$. $\diamond$

This example shows that the classical bend-and-break falls short if the conformal structure of the curves is left to vary; but suggests also a way out: not all the fibers of $\mathbf{S} \to \mathbb{P}^1$ are smooth. There are also singular fibres with degenerated conformal structure (in this example we get rational curves) and these ones have *geometric genus* strictly smaller than that of the general fibre.

3.4 Problems in the symplectic case

The necessary ingredients needed to treat the symplectic case will be introduced in the next sections; for the moment I just want to give the motivations for the forthcoming constructions. The difficulties appear from the very beginning: for imitating the "bend-and-break" a natural choice would be the space of simple J-holomorphic curves $\mathbf{C} \stackrel{u}{\to} \mathbf{X}$, the analogue of $Hom(\mathbf{C}, \mathbf{X})$ when $\mathbf{X}$ was projective. But when the complex structure of $\mathbf{C}$ is fixed, the expected dimension of this moduli space is

$$2(n(1-g) - K_{\mathbf{X}} \cdot A)$$

where $A \in H_2(\mathbf{X}, \mathbb{Z})$ represents the homology class of the curves and $\dim_{\mathbb{R}} \mathbf{X} = 2n$. One may see that for $g \gg 0$ it is a negative number (this actually happens in the case of the standard projective plane) and therefore a symplectic analogue of proposition 3.2.1 wouldn't be realistic or it would apply only to a restricted class of manifolds. Consequently I am forced to allow the complex structure of $\mathbf{C}$ to vary, increasing this way the expected dimension of the space of curves by the dimension of the Deligne-Mumford space i.e. $6(g-1)$; it will be

$$2((3-n)(g-1) - K_{\mathbf{X}} \cdot A)$$

For this reason, the next section will be devoted to remind some elementary facts about Deligne-Mumford spaces.

3.5 Deligne-Mumford spaces

There is a wide literature discussing this subject: the algebraic point of view was attacked by Deligne, Grothendieck, Mumford, Knudson *et al*; a differential-geometric point of view is discussed in great detail by J.Jost in [J]. Here I adopted a less complete but (I find) more intuitive approach to the subject, modeled after the paper [EE].

Let Σ denote an oriented compact Riemann surface of genus g and consider the marked points $(x_1, \ldots, x_k)$, $k \geq 0$ on it; we require to be satisfied the following inequality: $k + 2g \geq 3$. First I give a few notations:

- $\mathcal{J}(\Sigma)$ will denote the space of complex structures on Σ;
- $\mathcal{D}_k(\Sigma)$ will stay for for the group of orientation-preserving diffeomorphisms of Σ which have $(x_1, \ldots, x_k)$ as fixed points;
- finally, $\mathcal{D}_k^0(\Sigma)$ will be the connected component of the identity in $\mathcal{D}_k(\Sigma)$.

There are natural actions

$$\mathcal{D}_k(\Sigma) \times \mathcal{J}(\Sigma) \to \mathcal{J}(\Sigma) \text{ and } \mathcal{D}_k^0(\Sigma) \times \mathcal{J}(\Sigma) \to \mathcal{J}(\Sigma)$$

Definition 3.5.1 *The Teichmüller space $\mathcal{T}_{g,k}$ of curves of genus g with k marked points is defined to be*

$$\mathcal{T}_g := \mathcal{J}(\Sigma)/\mathcal{D}_k^0(\Sigma)$$

Theorem *$\mathcal{T}_{g,k}$ is a Kähler manifold of complex dimension $3(g-1)+k$. More precisely, it is a bounded holomorphically convex domain in $\mathbb{C}^{3(g-1)+k}$.*

The problem with this space is to be non-compact and it can not be compactified in a nice way (i.e. it is not quasi-projective). One may fix this problem as follows: there is still an action

$$\mathcal{D}_k(\Sigma)/\mathcal{D}_k^0(\Sigma) \times \mathcal{T}_g \longrightarrow \mathcal{T}_g$$

where $\Gamma_{g,k} := \mathcal{D}_k(\Sigma)/\mathcal{D}_k^0(\Sigma)$ is the Teichmüller group. Taking the quotient

$$M_{g,k} := \mathcal{T}_{g,k}/\Gamma_{g,k}$$

we obtain the *open Deligne-Mumford space* $M_{g,k}$ of smooth curves of genus g with k marked points. It is a singular space, an orbifold, but has the advantage of being quasi-projective; the compactification is done by means of stable curves.

Definition 3.5.2 (stable curve) *A connected, non-necessarily irreducible, reduced algebraic curve $\mathbf{C}$ of arithmetic genus g and with k marked points is called* stable *if the following holds:*

- *the singularities of $\mathbf{C}$ are at worst ordinary double points;*
- *the marked points lie in the smooth locus of $\mathbf{C}$;*
- *for each irreducible component $\mathbf{C}_\nu$ of $\mathbf{C}$ of arithmetic genus g_ν it is satisfied the inequality*

$$m_\nu + 2g_\nu \geq 3$$

where m_ν is obtained summing the number of marked points on $\mathbf{C}_\nu$ with that of contact points with other irreducible components and that of singular points.

This second condition is equivalent saying that $\mathbf{C}$ has a finite automorphism group. It is a classical result that adding the classes of singular stable curves one obtains a projective compactification $\overline{M}_{g,k}$ of $M_{g,k}$ which is called the *Deligne-Mumford* space of curves of genus g and with k marked points.

The functorial properties of these spaces imply that there is a "forgetful" map

$$\pi_{g,k} : \overline{M}_{g,k+1} \longrightarrow \overline{M}_{g,k}$$

which was proved by Knudsen to be in fact a morphism of algebraic varieties. Corresponding to the k marked points there are also the k "marking" sections

$$\mathbf{x}_1, \ldots, \mathbf{x}_k : \overline{M}_{g,k} \longrightarrow \overline{M}_{g,k+1}$$

which are also proved to be morphisms.

When $g \geq 2$, so we may take $k = 0$, $\overline{M}_g$ is a parameter space for the (possibly degenerate) complex structures on a Riemann surface of genus g. For algebraic purposes, the spaces $\overline{M}_{g,k}$ are the good ones to use in further constructions of the moduli spaces of stable maps *à la* Kontsevich, because they coarsely represent certain functors. Their shortcoming, from differential geometric point of view, is to be not *fine* moduli spaces.

Take the example of $\pi : \overline{M}_{2,1} \to \overline{M}_2$: we would like it to define a "universal curve" over $\overline{M}_g$, i.e. the geometric fibres $\pi^{-1}(j)$ to represent the equivalence class of j for all geometric points $j \in \overline{M}_g$. But this property fails: $\pi^{-1}(j)$ is the quotient of the curve represented by j by its automorphism group. For this reason, in the case $g = 2$ the fibres of $\pi : \overline{M}_{2,1} \to \overline{M}_2$ are projective lines because all genus-two curves are hyperelliptic and they have the hyperelliptic involution.

A second "bad" property is that the Deligne-Mumford spaces have orbifold singularities. For these reason I shall present the construction of some smooth global covers of $\overline{M}_g$ which does have "universal curves" over it. Let me very briefly recall the definition of some ingredients which are important for the understanding of M_g, $g \geq 2$.

I shall begin with the Weil-Petersson metric: the tangent space of the Teichmüller space $\mathcal{T}_g$ at a point j is

$$T_j\mathcal{T}_g = H^0(\mathbf{C}_j, K_{\mathbf{C}_j}^{\otimes 2})$$

the complex vector space of holomorphic quadratic differentials on the curve $\mathbf{C}_j$ which represents the point $j \in \mathcal{T}_g$. There is the universal curve $\pi_g : \mathcal{T}_{g,1} \to \mathcal{T}_g$ which has the property that $\pi_g^{-1}(j) = \mathbf{C}_j$. If we take the relative dualizing sheaf $\Omega_{\mathcal{T}_{g,1}/\mathcal{T}_g} := (Ker\, \mathrm{d}\pi_g)^*$ of this fibration it is easy to see that the germ at $j \in \mathcal{T}_g$ of the direct image

$$(\pi_g)_* \left(\Omega^{\otimes 2}_{\mathcal{T}_{g,1}/\mathcal{T}_g}\right)_j = H^0(\mathbf{C}_j, K_{\mathbf{C}_j}^{\otimes 2})$$

The Poincaré metric on $\mathbf{C}_j$, $j \in \mathcal{T}_g$ can be used to put a metric on the space of holomorphic quadratic differentials, i.e. on the tangent space $T_j\mathcal{T}_g$. This way we get the Weil-Petersson metric ω_{WP} on $\mathcal{T}_g$; it is Kähler, invariant under the action of the modular group Γ_g and therefore induces a Kähler forms on each quotient $\mathcal{T}_g/\Gamma$ of the Teichmüller space by a subgroup $\Gamma \subset \Gamma_g$. In particular we get the Weil-Petersson metric on M_g and this one can be extended to a Kähler form $\overline{\omega}_{\mathrm{WP}}$ on the whole $\overline{M}_g$.

The other key ingredient is the canonical class κ_1. This is defined as follows: using the hermitian metric induced by the hyperbolic metrics on the fibres of $\mathcal{T}_{g,1} \to \mathcal{T}_g$, one may compute the $(1,1)$-form $\eta \in \Lambda^{1,1}\mathcal{T}_{g,1}$ representing the first Chern class of $\Omega_{\mathcal{T}_{g,1}/\mathcal{T}_g}$. The *first canonical class* κ_1 is defined

$$\kappa_1(j) := \int_{\mathbf{C}_j} \eta^2 \qquad j \in \mathcal{T}_g$$

η being invariant under the action of Γ_g, κ_1 is also and it induces a 2-form on any quotient of the Teichmüller space by a subgroup of Γ_g. It is proved by S.Wolpert in [W1] that κ_1, on M_g, can be extended to a form $[\overline{\kappa}_1] \in H^2(\overline{M}_g, \mathbb{Z})$ and $[\overline{\kappa}_1] = (1/2\pi^2)[\overline{\omega}_{\mathrm{WP}}]$. The much finer equality proved in [W2], which will be interesting, states that

$$\kappa_1 = \frac{1}{2\pi^2}\omega_{\mathrm{WP}} \qquad \text{on } \mathcal{T}_g$$

if they are computed as explained above. The result says that κ_1 can be represented by a strictly positive $(1,1)$-form.

In the sequel, following the ideas of E.Looijenga, I shall give the construction of some "good" finite covers of $\overline{M}_g$. As I already mentioned, there are at least two reasons for searching such covers: $\overline{M}_g$ is a singular variety and it do not have a universal curve.

Let me introduce the following definition which can be found in [RT2]:

Definition 3.5.3 *A finite, connected cover*

$$p_{\text{good}} : \overline{M}_{g,k}^{\text{good}} \to \overline{M}_{g,k}$$

is a good cover if $\overline{M}_{g,k}^{\text{good}}$ *is a normal projective variety with quotient singularities such that there is an universal curve*

$$\pi_{g,k}^{\text{good}} : \overline{M}_{g,k+1}^{\text{good}} \longrightarrow \overline{M}_{g,k}^{\text{good}}$$

i.e. for each $j \in \overline{M}_{g,k+1}^{\text{good}}$*,* $(\pi_{g,k}^{\text{good}})^{-1}(j)$ *is isomorphic to* $p_{\text{good}}(j)$*. Furthermore there is the commuting diagram*

$$\begin{array}{ccc} \overline{M}_{g,k+1}^{\text{good}} & \xrightarrow{p_{\text{good}}} & \overline{M}_{g,k+1} \\ \pi_{g,k}^{\text{good}} \downarrow & & \downarrow \pi_{g,k} \\ \overline{M}_{g,k}^{\text{good}} & \xrightarrow{p_{\text{good}}} & \overline{M}_{g,k} \end{array}$$

D.Mumford proves in [Mu1] that such good covers exist using level structures. This result won't be satisfactory except for the case of tori: $\dim_{\mathbb{C}} \overline{M}_{1,1} = 1$ and normal varieties of dimension one are smooth. Therefore $\overline{M}_{1,1}^{\text{good}}$ will fulfill all the requirements above. In fact we have already a description of a good Deligne-Mumford space for the elliptic curves: it is just $\mathbb{P}^2 \sharp 9 \overline{\mathbb{P}}^2 \to \mathbb{P}^1$ described in the example 3.3.3.

For the case $g \geq 2$ the rescue comes from E.Looijenga's result in [L1] which states that there are smooth good covers of the Deligne-Mumford space. I shall sketch his construction: as before, let Σ denote the smooth, compact, oriented Riemann surface underlying genus g curves and fix a point $x \in \Sigma$. Let us consider:

$$N := \langle l^2 \mid l \in \pi_1(\Sigma, x) \rangle \lhd \pi_1(\Sigma, x)$$

the normal subgroup generated by the squares in $\pi_1(\Sigma, x)$ and take the Galois covering $\widetilde{\Sigma} \to \Sigma$ corresponding to N, i.e. $\pi_1(\widetilde{\Sigma}, \widetilde{x}) = N$; it is called *the Prym cover* of Σ. The Galois group is

$$G = \pi_1(\Sigma, x)/N \cong H_1(\Sigma, \mathbb{Z}_2)$$

G acts by diffeomorphisms on $\widetilde{\Sigma}$ and it induces an action on $H_1(\Sigma, \mathbb{Z}_n)$ for each $n \geq 1$.

Any orientation-preserving diffeomorphism $f : (\Sigma, x) \to (\Sigma, f(x))$ lifts to a diffeomorphism $\widetilde{f} : (\widetilde{\Sigma}, \widetilde{x}) \to (\widetilde{\Sigma}, \widetilde{f(x)})$ which is uniquely determined up to a G-action; it depends on the choice of the lifting $\widetilde{f(x)}$ of $f(x)$. $\widetilde{f}$ defines an automorphism of $H_1(\widetilde{\Sigma}, \mathbb{Z}_n)$ and therefore one may talk about the action (modulo G) of the group $\mathcal{D}(\Sigma)$ of orientation preserving diffeomorphisms of Σ on $H_1(\widetilde{\Sigma}, \mathbb{Z}_n)$.

Define $\Gamma_{\Sigma,(n)}$ to be the subgroup of $\mathcal{D}(\Sigma)$ consisting of diffeomorphisms f whose lift $\widetilde{f}$ act on $H_1(\Sigma, \mathbb{Z}_n)$ like an element of G ; this condition is independent of the lift. It is immediate that $\Gamma_{\Sigma,(n)} \lhd \mathcal{D}(\Sigma)$ is a normal subgroup. Let me notice that if f is in $\mathcal{D}^0(\Sigma)$, the connected component of the identity in $\mathcal{D}(\Sigma)$, then $\widetilde{f}$ is homotopic to a G-diffeomorphism of $\widetilde{\Sigma}$. Therefore we have the following inclusions

$$\mathcal{D}^0(\Sigma) \hookrightarrow \Gamma_{\Sigma,(n)} \hookrightarrow \mathcal{D}(\Sigma)$$

which induce the exact sequence of groups

$$0 \longrightarrow \frac{\Gamma_{\Sigma,(n)}}{\mathcal{D}^0(\Sigma)} \longrightarrow \frac{\mathcal{D}(\Sigma)}{\mathcal{D}^0(\Sigma)} \longrightarrow \frac{\mathcal{D}(\Sigma)}{\Gamma_{\Sigma,(n)}} \longrightarrow 0$$

Let us notice that $\frac{\Gamma_{\Sigma,(n)}}{\mathcal{D}^0(\Sigma)}$ is a subgroup of the Teichmüller group and that $\Gamma_g[n] := \frac{\mathcal{D}(\Sigma)}{\Gamma_{\Sigma,(n)}} \hookrightarrow$ $Aut\, H_1(\widetilde{\Sigma}, \mathbb{Z}_n)/G$ is finite. Therefore, if we define

$$M_g[n] := \mathcal{T}_g/(\tfrac{\Gamma_{\Sigma,(n)}}{\mathcal{D}^0(\Sigma)})$$

we get a finite (branched) cover of M_g whose Galois group $\Gamma_g[n]$ is finite.

Definition 3.5.4 *$M_g[n]$ is called the space of Prym level-n structures on Σ.*

This space has a compactification $\overline{M}_g[n]$ which is by definition the normalization of $M_g[n]$ in $\overline{M}_g$. The main result in [L1] is:

Theorem 3.5.5 (E.Looijenga) *$\overline{M}_g[n]$ is smooth for all even integers $n \geq 6$.*

Remark 3.5.6 The fibre of $p_{[n]} : \overline{M}_g[n] \to \overline{M}_g$ over a point $j \in M_g$ consists of pairs $(\mathbf{C}_j, [f])$, where $\mathbf{C}_j$ is an algebraic curve representing j and $[f]$ is an equivalence class of orientation-preserving diffeomorphisms $f : \Sigma \to \mathbf{C}_j$; two such maps f, g are equivalent if and only if $g^{-1}f \in G$. $\diamond$

How can one obtain the universal curve over $\overline{M}_g[n]$? Instead of working with all the diffeomorphisms of Σ we may look only to those ones which have x as fixed point, i.e. we consider the group $\mathcal{D}_1(\Sigma)$, and to define $\Gamma_{\Sigma,1,(n)}$ to be its (normal) subgroup which acts on $\widetilde{\Sigma}$ as an element of G. There is again the exact sequence

$$0 \longrightarrow \frac{\Gamma_{\Sigma,1,(n)}}{\mathcal{D}_1^0(\Sigma)} \longrightarrow \frac{\mathcal{D}_1(\Sigma)}{\mathcal{D}_1^0(\Sigma)} \longrightarrow \frac{\mathcal{D}_1(\Sigma)}{\Gamma_{\Sigma,1,(n)}} \longrightarrow 0$$

and it is an immediate remark that the monomorphisms

$$0 \longrightarrow \frac{\Gamma_{\Sigma,1,(n)}}{\mathcal{D}_1^0(\Sigma)} \longrightarrow \frac{\Gamma_{\Sigma,(n)}}{\mathcal{D}^0(\Sigma)} \quad \text{and} \quad 0 \longrightarrow \frac{\mathcal{D}_1(\Sigma)}{\Gamma_{\Sigma,1,(n)}} \longrightarrow \frac{\mathcal{D}(\Sigma)}{\Gamma_{\Sigma,(n)}} = \Gamma_g[n]$$

are in fact isomorphisms. Therefore

$$M_{g,1}[n] := \mathcal{T}_{g,1}/(\tfrac{\Gamma_{\Sigma,1,(n)}}{\mathcal{D}_1^0(\Sigma)})$$

is again a $\Gamma_g[n]$-Galois covering of $M_{g,1}$ and there is also a natural map $\pi^{[n]} : M_{g,1}[n] \to M_g[n]$. As before, the compactification $\overline{M}_{g,1}[n]$ is defined to be the normalization of $M_{g,1}[n]$ in $\overline{M}_{g,1}$.

Remark 3.5.7 The fibre of $p_{[n]} : \overline{M}_{g,1}[n] \to \overline{M}_{g,1}$ over a point $(j, *) \in M_{g,1}$ consists of triples $(\mathbf{C}_j, x_0, [f])$ where $\mathbf{C}_j$ and $[f]$ are as in remark 3.5.6 and $x_0 \in \mathbf{C}_j$ is a point which is in the equivalence class of the point $*$. Two maps $f : (\Sigma, x) \to (\mathbf{C}_j, x_0)$ and $g : (\Sigma, x) \to (\mathbf{C}_j, y_0)$ are equivalent if and only if: $x_0 = y_0$ and $g^{-1} f \in G$. $\diamond$

One can see that $\overline{M}_{g,1}[n] \to M_g[n]$ is a universal curve and so far we have satisfied our requirements for a good Deligne-Mumford space.

In my constructions of moduli spaces of pseudo-holomorphic curves I shall use such a $\overline{M}_g[n]$ as parameter space for the complex structures which can be put on the Riemann surface Σ. In order to have a unitary package of notations, I shall denote, for some fixed n

$$\begin{array}{rlrl} \overline{N}_0 &:= \overline{M}_{0,3} & \overline{N}_{0,1} &:= \overline{M}_{0,4} \\ \overline{N}_1 &:= \overline{M}_{1,1}^{\text{good}} & \overline{N}_{1,1} &:= \overline{M}_{1,2}^{\text{good}} \\ \overline{N}_g &:= \overline{M}_g[n] & \overline{N}_{g,1} &:= \overline{M}_{g,1}[n] \end{array}$$

I fix once for all a projective embedding $\phi : \overline{N}_{g,1} \to \mathbb{P}^N$; on each fibre $\mathbf{C}_j$, $j \in \overline{N}_g$ we get a metric γ_j which is just the restriction of the Fubini-Study metric of $\mathbb{P}^N$. I shall use the notation $\pi : \overline{N}_{g,1} \to \overline{N}_g$ for the projection.

An important result concerning the Deligne-Mumford spaces is the following:

Theorem 3.5.8 (S.Diaz) *If* $\mathbf{Z} \hookrightarrow M_g$ $(g \geq 2)$ *is a complete subvariety, then* $\dim_{\mathbb{C}} \mathbf{Z} \leq g - 2$. *Equivalently, if* $\mathbf{Z} \hookrightarrow \overline{M}_g$ *is a complete subvariety of dimension* $\dim_{\mathbb{C}} \mathbf{Z} \geq g-1$, *then* $\mathbf{Z}$ *intersects the boundary divisor* $\overline{M}_g - M_g$.

Proof The original proof of this result can be found in [Di] and uses a stratification of M_g. Here I shall present the alternative approach of E.Looijenga to this result, which can be found in [L2]. The main theorem of [L2] implies that $\kappa_1^{d-1} = 0$ in $H^{2(g-1)}(M_g)$, where κ_1 denotes the first canonical class defined before. If $\mathbf{Y} \hookrightarrow M_g$ would be a complete subvariety of dimension at least $(d-1)$, then $\langle \kappa_1^{\dim \mathbf{Y}}, [\mathbf{Y}] \rangle > 0$ because κ_1 is a strictly positive form. This contradicts that κ_1^{d-1} represents a zero cohomology class. $\square$

Because $M_{0,3} = *$ and $M_{1,1} = \mathbb{P}^1 - 0, 1, \infty$, it is clear that the only complete subvarieties of these ones are points. The spaces $\overline{N}_g \to \overline{M}_g$ being a finite ramified cover, the conclusion of the theorem holds for them also.

In fact, because κ_1 can be represented as a positive $(1,1)$-form, the proof given above says more: if $\mathbf{Z} \hookrightarrow \overline{M}_g$ is a closed symplectic subvariety of dimension $\dim_{\mathbb{R}} \mathbf{Z} \geq 2(g-1)$ then $\mathbf{Z}$ must intersect the boundary divisor of $\overline{M}_g$. As we shall see, this remark will allow us to treat "good" $\mathcal{C}^1$-small deformations of Kähler manifolds on the same footing with Kähler manifolds.

3.6 Pseudo-holomorphic curves

This concept was introduced by M.Gromov in his article [Gr] and became a very powerful tool in the study of symplectic manifolds an even projective manifolds. A good introduction can be found in [McDS2].

Definition 3.6.1 *Let* $(\mathbf{X}, J)$ *be an almost complex manifold and let* $(\mathbf{C}, j)$ *be a complex curve. A map* $u : \mathbf{C} \to \mathbf{X}$ *is called* jJ*-holomorphic if*

$$\overline{\partial}_{jJ}u := \frac{1}{2}(\mathrm{d}u + J \cdot \mathrm{d}u \cdot j) = 0$$

Let us notice that when J is integrable the definition above is that of a holomorphic map.

Definition 3.6.2 *A jJ-holomorphic map $u : (\mathbf{C}, j) \to (\mathbf{X}, J)$ is called* simple *if there is no factorization $u = u' \circ \rho$ where $(\mathbf{C}', j')$ is another complex curve, $\rho : (\mathbf{C}, j) \to (\mathbf{C}', j')$ is a (ramified) covering (of degree at least two) and $u' : (\mathbf{C}', j') \to (\mathbf{X}, J)$ is a $j'J$-holomorphic map.*

From now on I shall always suppose that $\mathbf{X}$ is compact symplectic and the almost complex structure is ω-compatible (sometimes I shall allow it to be only ω-tame). If γ is a metric on $(\mathbf{C}, j)$ in the conformal class of j, then the energy of an arbitrary map $\varphi : (\mathbf{C}, j) \to (\mathbf{X}, \omega, J)$ is defined to be

$$E(\varphi) := \frac{1}{2}\int_{\mathbf{C}} |\mathrm{d}\varphi|^2_{\gamma, g_J}(p)\mathrm{d}\gamma(p)$$

This definition do not depend on the choice of the metric γ in the conformal class of j. It is an immediate remark that in the case when $\varphi = u$ is pseudo-holomorphic

$$E(u) = \int_{\mathbf{C}} u^*\omega$$

and if furthermore u is simple,

$$\int_{\mathbf{C}} u^*\omega = \langle [\omega], [u(\mathbf{C})] \rangle =: \omega(A)$$

where $A \in H_2(\mathbf{X}, \mathbb{Z})$ is the homology class represented by the map u. In particular we see that each non-constant pseudo-holomorphic map defines a non-zero two-homology class.

The purpose of this section is to prove that exists a moduli space for simple pseudo-holomorphic curves which represent a fixed class $A \in H_2(\mathbf{X}, \mathbb{Z})$; I shall adopt the (now classical) line of proof which can be found in [McDS2] or [RT2]. Here one may see the use of the global covers $\overline{N}_g \to \overline{M}_g$ described in section 3.5: the existence of a universal curves over them makes easier the construction of the moduli spaces of stable maps because it is not necessary anymore to take into account the automorphisms of the curves.

Let Σ denote a fixed compact, oriented Riemann surface. For a homology class $A \in H_2(\mathbf{X}, \mathbb{Z})$ define

$$Map(\Sigma, \mathbf{X}, A) := \{f : \Sigma \to \mathbf{X} \mid f \text{ is smooth and } f_*[\Sigma] = A\}$$

Let $W^{m,p}_j(Map(\Sigma, \mathbf{X}, A))$ denote its completion with respect to the metric γ_j on the fibre $\mathbf{C}_j$, $j \in N_g$. Let me notice that this space is the same as that of $W^{m,p}$-maps $\mathbf{C}_j \to \mathbf{X}$ with respect to the same metric. “Putting together” these spaces as j varies in N_g we get

$$\chi^{m,p}_A := \bigsqcup_{j \in N_g} W^{m,p}(Maps(\Sigma, \mathbf{X}, A)) \times \{j\}$$

Sobolev estimates show that for $m \geq 1$ and $p > 2$ any $W^{m,p}$ map is of class $\mathcal{C}^{m-2/p}$.

For $l \geq m$ let $\mathcal{J}^l$ denote the space of $\mathcal{C}^l$, ω-compatible almost complex structures on $\mathbf{X}$. The *universal moduli space* is defined to be

$$N_g^l(\mathbf{X}, A) := \{(u, J) \in \chi_A^{m,p} \times \mathcal{J}^l \mid u \text{ is a simple } J-\text{holomorphic map}\}$$

The elliptic regularity says that any such a map is of class $\mathcal{C}^l$.

Proposition 3.6.3 *The universal moduli space $N_g^l(\mathbf{X}, A)$ is a $\mathcal{C}^{l-1}$-Banach manifold.*

Proof There is a $W^{m-1,p}$-Banach bundle

$$\mathcal{E}^{m-1,p} \longrightarrow \chi_A^{m,p} \times \mathcal{J}^l$$

whose fibre over $(f, j, J) \in \chi_A^{m,p} \times \mathcal{J}^l$ is

$$\mathcal{E}^{m-1,p}_{(f,j,J)} := W^{m-1,p}\Omega^{0,1}_{jJ}(\Sigma, f^*T\mathbf{X})$$

the space of complex jJ-antilinear sections $\Sigma \to f^*T\mathbf{X}$. The $\overline{\partial}$-operator defines a section of this bundle

$$\mathcal{S}(f, j, J) := \mathrm{d}f + J(f) \cdot \mathrm{d}f \cdot j$$

One may see that $N_g^l(\mathbf{X}, A)$ is the zero set of this section. In order to show that it is smooth $\mathcal{C}^{l-1}$-Banach manifold, I have to prove that $\mathcal{S}$ is transverse to the zero section; in other words, I have to prove that the projection at a zero of the differential of $\mathcal{S}$ on the vertical space is surjective. For a zero (u, j, J) of $\mathcal{S}$, a tangent vector is a triple (ξ, y, Y) where $\xi \in \Gamma(\Sigma, f^*T\mathbf{X})$, $y \in T_j N_g$, $Y \in End_{\overline{J}}(T\mathbf{X})$ is J-antilinear endomorphism. The projection of the differential on the vertical space is given by

$$D\mathcal{S}(u, j, J)(\xi, y, Y) = D_u\xi + Y(u) \cdot \mathrm{d}u \cdot j + J(u) \cdot \mathrm{d}u \cdot \mathcal{I}_y$$

Here D_u denotes the linearization of the Cauchy-Riemann operator; $\mathcal{I}_y$ denotes the variation of the complex structure j in the fibre $\mathbf{C}_j$ corresponding to the variation y of $j \in N_g$. It is proved in [McDS2], pages 34-35, that whenever u is a simple J-holomorphic map

$$T_u\chi_A^{m,p} \times T_J\mathcal{J}^l \ni (\xi, Y) \mapsto D_u\xi + Y(u) \cdot \mathrm{d}u \cdot j \in \mathcal{E}^{m-1,p}_{(u,j,J)}$$

is surjective. Therefore in our case $D\mathcal{S}(u, j, J)$ will be surjective also. This proves that $N_g^l(\mathbf{X}, A)$ is $\mathcal{C}^{l-1}$-Banach manifold. $\square$

Its tangent space at a point (u, j, J) is

$$T_{(u,j,J)}N_g^l(\mathbf{X}, A) = \{(\xi, y, Y) \mid D_u\xi + Y(u) \cdot \mathrm{d}u \cdot j + J \cdot \mathrm{d}u \cdot \mathcal{I}_y = 0\}$$

For the canonical projection $\Pi : N_g^l(\mathbf{X}, A) \to \mathcal{J}^l$ the following holds:

Lemma 3.6.4 Π *is a Fredholm operator of index given by*

$$index(\Pi) = 2(n(1-g) - K_{\mathbf{X}} \cdot A + dim_{\mathbb{C}}\overline{N}_g)$$

Proof I shall compute the index of Π at a point (u, j, J). Let me consider the operator

$$\mathcal{D} : T_{(u,j)}\chi_A^{m,p} \longrightarrow \Omega^{0,1}_{jJ}(u^*T\mathbf{X})$$
$$\mathcal{D}(\xi, y) := D_u\xi + J \cdot \mathrm{d}u \cdot \mathcal{I}_y$$

$\mathcal{D}$ is a compact perturbation of D_u which in turn is a compact perturbation of the $\overline{\partial}$-operator

$$\overline{\partial} : T_{(u,j)}\chi_A^{m,p} \longrightarrow W^{m-1,p}\Omega^{0,1}_{jJ}(u^*T\mathbf{X})$$

Therefore $\mathcal{D}$ is Fredholm of the same index as $\overline{\partial}$,

$$index(\mathcal{D}) = 2(n(1-g) - K_{\mathbf{X}} \cdot A + dim_{\mathbb{C}}\overline{N}_g)$$

I shall begin the proof of the lemma: the kernel of $\mathrm{d}\Pi := \mathrm{d}\Pi(u, j, J)$ is

$$Ker(\mathrm{d}\Pi) = \{(\xi, y, 0) \mid \mathcal{D}(\xi, y) = 0\} \cong Ker(\mathcal{D})$$

and therefore closed and finite dimensional. Because the range of $\mathcal{D}$ is closed, the defining property of the tangent space shows that the range of $\mathrm{d}\Pi$ is also. It remains to prove the cokernel is of finite dimension. Let

$$H : T_J\mathcal{J}^l \ni Y \mapsto Y(u) \cdot \mathrm{d}u \cdot j \in W^{m-1,p}\Omega^{0,1}_{jJ}(u^*T\mathbf{X})$$

There are two remarks to be done:

- $Ker(H) \subset Im(\mathrm{d}\Pi)$. Indeed

$$Ker(H) = \{Y \in T_J\mathcal{J}^l \mid Y(u) \cdot \mathrm{d}u \cdot j = 0\} = \mathrm{d}\Pi\{(0, 0, Y) \in T_{(u,j,J)}N_g^l(\mathbf{X}, A)\}$$

- Because $Im(\mathrm{d}\Pi) = \{Y \in T_J\mathcal{J} \mid \exists(\xi, y) \text{ s.t. } \mathcal{D}(\xi, y) + H(Y) = 0\}$ we deduce that

$$H(Im(\mathrm{d}\Pi)) = Im(\mathcal{D}) \cap Im(H)$$

$$\begin{aligned} Coker(\mathrm{d}\Pi) &= \frac{T_{(J,\nu)}\mathcal{J}^l}{Im(\mathrm{d}\Pi)} \cong \frac{T_{(J,\nu)}\mathcal{J}^l/Ker(H)}{Im(\mathrm{d}\Pi)/Ker(H)} \cong \frac{H(T_{(J,\nu)}\mathcal{J}^l/Ker(H))}{H(Im(\mathrm{d}\Pi)/Ker(H))} \\ &= \frac{Im(H)}{H(Im(\mathrm{d}\Pi)/Ker(H))} = \frac{Im(H)}{Im(H) \cap Im(\mathcal{D})} \cong \frac{Im(H) + Im(\mathcal{D})}{Im(\mathcal{D})} \\ &= \frac{Im(\mathcal{S})}{Im(\mathcal{D})} = \frac{\Omega^{0,1}_{j,J}(u^*T\mathbf{X})}{Im(\mathcal{D})} = Coker(\mathcal{D}) \ \square \end{aligned}$$

The Sard-Smale theorem says that the set of regular values of Π is of second Baire category inside $\mathcal{J}^l$. A standard argument shows that the same is true for the space $\mathcal{J}$ of $\mathcal{C}^\infty$ almost complex structures. For a smooth J the elliptic regularity implies that each J-map is smooth and the space of such curves is also smooth. I shall denote by $N_g(\mathbf{X}, A, J)$ the fibre of Π over a (smooth) almost complex structure J.

We can see that $\mathrm{d}\Pi$ is surjective if and only if

$$\textbf{(surj)} \qquad T_{(u,j)}\chi_A^{m,p} \ni (\xi, y) \mapsto D_u\xi + J(u) \cdot \mathrm{d}u \cdot \mathcal{I}_y \in \Omega^{0,1}_{jJ}(u^*T\mathbf{X})$$

is surjective for each J-map.

Remark 3.6.5 It is very important to keep in mind that it is possible that the differential of the projection Π is never surjective and therefore *the fibre over each regular value of* Π *is the empty set.* $\diamond$

Example 3.6.6 I shall give an example when such a situation occurs: the double cover $\mathbf{X}$ of $\mathbb{P}^2$ along a smooth sextic $\mathbf{C}_6 \hookrightarrow \mathbb{P}^2$ is known to be a $K3$ surface. There exist always a finite, non-zero, number of embedded projective lines $\mathbf{L} \hookrightarrow \mathbb{P}^2$ which are tangent to $\mathbf{C}_6$ at two distinct points. Such a line will intersect it simply in two more points, $\mathbf{C}_6$ being by assumption of degree six. This statement is true for any smooth sextic but, since here I am giving an example, explicit computations can be readily done for $\mathbf{C}_6 := \{z_0^6 + z_1^6 + z_2^6 = 0\}$.

The image in $\mathbf{X}$ of such a line provides an embedded curve $\mathbf{D}$ of arithmetic genus equal two and having two double points corresponding to the tangency points of $\mathbf{L}$ with $\mathbf{C}_6$; $\mathbf{D}$ is therefore a rational curve. Let me denote by $u : \mathbb{P}^1 \to \mathbf{X}$ the simple map having $\mathbf{D}$ its image; for $\lambda \in \mathbb{C}$, define $u_\lambda : \mathbb{P}^1 \to \mathbf{X}$ by $u_\lambda(\zeta) := u(\lambda\zeta)$. If $\lambda \neq 1$ I get a map

$$\overline{u} := (u, u_\lambda) : \mathbb{P}^1 \to \mathbf{X} \times \mathbf{X}$$

which is not contained in the diagonal.

The symmetric product $\mathbf{X}^{(2)}$ defined by

$$\mathbf{X}^{(2)} := \mathbf{X} \times \mathbf{X} / \sim \qquad (x, x') \sim (x', x)$$

is singular along the diagonal. A resolution of singularities is given by $\mathbf{X}^{[2]}$, the Hilbert scheme of finite algebraic schemes $(\mathbf{Y}, \mathcal{O}_\mathbf{Y})$ with $\mathbf{Y}_{red} \subset \mathbf{X}$ and *lenght* $\mathcal{O}_\mathbf{Y} = 2$. For $\mathbf{Y}_{red}$ consisting of two distinct points $(\mathbf{Y}, \mathcal{O}_\mathbf{Y})$ is nothing but a pair of (reduced) points in $\mathbf{X}$. The difference comes from points $x \in \mathbf{X}$ whose structure sheaf is non-reduced; such points are defined locally by the ideal $(a\mathrm{x} + b\mathrm{y}, \mathfrak{m}^2)$, $[a\!:\!b] \in \mathbb{P}^1$, where $\mathfrak{m} \subset \mathbb{C}[\mathrm{x}, \mathrm{y}]$ is the maximal ideal of the origin. There is a birational morphism $\mathbf{X}^{[2]} \dashrightarrow \mathbf{X}^{(2)}$. Beauville proves in [Be] that for $\mathbf{X}$ a $K3$ surface, $\mathbf{X}^{[2]}$ is a hyperkähler 4-fold.

Any hyperkähler manifold $\mathbf{V}$ posses a 2-sphere $\mathcal{S}_\mathbf{V}$ of (integrable) complex structures which are all compatible with a fixed metric (of Kähler) type. It is a consequence of the results proved in [V] that there is a dense subset $S_0 \subset \mathcal{S}_\mathbf{V}$ such that for all $I \in \mathcal{S}_\mathbf{V}$ the manifold $(\mathbf{V}, I)$ is *of general type*; it has the property that the closed analytic subvarieties are complex even-dimensional; in particular they do not carry any complete curves.

The map $\overline{u}$ induces a (simple) rational map

$$\overline{u} : \mathbb{P}^1 \dashrightarrow \mathbf{X}^{[2]}$$

which extends to a morphism by the valuative criterion of properness.

Let $A \in H_2(\mathbf{X}^{[2]}, \mathbb{Z})$ denote the homology class represented by the image of $\overline{u}$. Abstract index computation shows that the expected (real) dimension of simple parameterized (J-holomorphic) maps representing the class A is $2r$, where

$$r = 4(1 - 0) + c_1(\mathbf{X}^{[2]}) \cdot A = 4$$

Quotienting out the $Sl(2, \mathbb{C})$-action on $\mathbb{P}^1$, I deduce that the expected dimension of non-parameterized simple J-holomorphic curves representing the homology class A equals $2(4 - 3) = 2$.

Let me consider again the universal moduli space $\mathcal{M}_0^l(\mathbf{X}^{[2]}, A)$ together with the projection

$$\Pi : \mathcal{M}_0^l(\mathbf{X}^{[2]}, A) \longrightarrow \mathcal{J}^l$$

I say that for an integrable complex structures J induced by the hyperkähler structure of $\mathbf{X}^{[2]}$ either the fibre $\Pi^{-1}(J)$ is empty or J is not a regular value for Π. I should notice that the construction above shows that the image of Π intersects $\mathcal{S}_{\mathbf{X}^{[2]}}$.

Assume the contrary: let $J \in \mathcal{S}_{\mathbf{X}^{[2]}}$ which is regular for Π and there are simple J-holomorphic rational maps representing the homology class A. Then the image of Π will contain an open neighborhood of J. As I remarked earlier, the set S_0 of complex structures I on $\mathbf{X}^{[2]}$ for which $(\mathbf{X}^{[2]}, I)$ is of general type is dense in $\mathcal{S}_{\mathbf{X}^{[2]}}$. So I shall find such an I contained in the image of Π which means that there are complete (rational) curves on $(\mathbf{X}^{[2]}, I)$. This contradicts the choice of I. $\diamond$

This example is not really satisfactory for proving the remark 3.6.5 since the integrable complex structures induced by the hyperkähler structure of a manifold represent a very small part of the set of all almost complex structures. It remains an interesting problem to give an explicit example where Π is nowhere submersive.

3.7 The four-dimensional case

The question if the map Π above has non-empty smooth fibres is a deep and difficult question which is related to the existence of a non-vanishing Gromov-Witten invariant. Fortunately in dimension four this "emptyset problem" is solved by the following simple criterion which can be found in [HLS]:

Proposition 3.7.1 *Suppose $(\mathbf{X}, \omega, J)$ is a four-dimensional symplectic manifold with an ω-compatible almost complex structure. Consider $A \in H_2(\mathbf{X}, \mathbb{Z})$ with the property*

$$c_1(\mathbf{X}) \cdot A \geq 1$$

Then the condition **(surj)** *is satisfied for any immersed pseudo-holomorphic curve $u : (\mathbf{C}, j) \to (\mathbf{X}, J)$.*

Proof Because the map u is an immersion, there is a complex structure preserving splitting into the tangent and normal bundles:

$$u^* T\mathbf{X} = \mathcal{T} \oplus \mathcal{N}$$

with respect to the metric $u^* g_J$. According to this splitting we may decompose our operator into its tangent and normal part $\mathcal{D} = \mathcal{D}_T \oplus \mathcal{D}_N$

$$\mathcal{D}_T : (\Omega^0(\mathbf{C}, \mathcal{T}) \oplus T_j N_g) \oplus \Omega^0(\mathbf{C}, \mathcal{N}) \longrightarrow \Omega^{0,1}_{jJ}(\mathbf{C}, \mathcal{T})$$

$$\mathcal{D}_T((\xi_T, y), \xi_N) = D_u \xi_T + J(u) \cdot \mathrm{d}u \cdot \mathcal{I}_y + \mathrm{proj}_{\Omega^{0,1}_{jJ}(\mathbf{C}, \mathcal{T})} D_u \xi_N =: P(\xi_T, y) + \mathrm{proj}_{\Omega^{0,1}_{jJ}(\mathbf{C}, \mathcal{T})} D_u \xi_N$$

and

$$\mathcal{D}_N : \Omega^0(\mathbf{C}, \mathcal{N}) \longrightarrow \Omega^{0,1}_{jJ}(\mathbf{C}, \mathcal{N})$$

$$\mathcal{D}_N(\xi) = \mathrm{proj}_{\Omega^{0,1}_{jJ}(\mathbf{C}, \mathcal{N})} D_u \xi$$

It is shown in [HLS] thm 1' page 6 that the operator $\mathcal{D}_N$ is surjective.

Let us focus now on the tangent directions: in order to prove the surjectivity of $\mathcal{D}$ it is enough to prove that P is so. Let me notice that when restricted to the tangent bundle $\mathcal{T}$, D_u is nothing but the usual $\overline{\partial}$-operator because the complex structure of the tangent bundle is integrable; this implies that $P = \overline{\partial} + P_1$. I have to treat separately the three cases:

(i) $g = 0$: $Coker(\overline{\partial}) = H^1(\mathbb{P}^1, \mathcal{T}) \cong H^0(\mathbb{P}^1, K^{\otimes 2}) = 0$; $\overline{\partial}$ is surjective and P is also.

For higher genus curves $\overline{\partial}$ always has cokernel.

(ii) $g = 1$: $Coker(\overline{\partial}) = H^1(\mathbb{E}, \mathcal{O}) = H^0(\mathbb{E}, \mathcal{O}) = \mathbb{C}$;

(iii) $g \geq 2$: $Ker(\overline{\partial}) = H^0(\mathbf{C}, \mathcal{T}) = 0$. The index of $\overline{\partial}$ can be computed using Riemann-Roch and gives

$$index(\overline{\partial}) = (1-g) + 2(1-g) = 3(1-g)$$

on the other hand $index(\overline{\partial}) = h^0(\mathcal{T}) - h^1(\mathcal{T}) = -h^1(\mathcal{T})$. We deduce that $Coker(\overline{\partial}) = H^1(\mathbf{C}, \mathcal{T})$ is of complex dimension $3(g-1)$.

In both cases the operator P_1 will save use: according to the $\overline{\partial}$-Hodge decomposition

$$\Omega^{0,1}(\mathcal{T}) = \mathcal{H}^{0,1}(\mathcal{T}) \oplus Im(\overline{\partial})$$

We can see that the $\overline{\partial}$-operator leaves "uncovered" precisely the $\overline{\partial}$-harmonic forms. But the deformation space $N_{g,1} \to N_g$ is complete at $j \in N_g$ and therefore

$$\mathcal{H}^{0,1}(\mathcal{T}) \cong H^{0,1}(\mathcal{T}) \cong \{\mathcal{I}_y \mid y \in T_j N_g\}$$

This proves that P is not only epimorphism, but is in fact an isomorphism. □

3.8 Orientation of the space of J-holomorphic curves

If $\mathbf{V}$ is a smooth manifold, then it is orientable if and only if the real rank-one bundle $\bigwedge^{max} T\mathbf{V}$ or (equivalently) $\bigwedge^{max} T^*\mathbf{V}$ is trivial. An orientation is given by the choice of a global non-vanishing section. If o and o' are two such sections they define the same orientation if and only if o'/o is a strictly positive function on $\mathbf{V}$.

In order to make clear the method used for orienting the space of pseudo-holomorphic maps I have to give some elementary facts concerning index bundles. Let me start with a technical lemma:

Lemma 3.8.1 *Let $L : \mathcal{V} \to \mathcal{W}$ be a linear Fredholm operator and suppose $\psi : \mathbb{C}^N \to \mathcal{W}$ is a linear map such that $L \oplus \psi : \mathcal{V} \oplus \mathbb{C}^N \to \mathcal{W}$ is surjective. Then there is the following exact sequence*

$$0 \longrightarrow Ker\, L \longrightarrow Ker\,(L \oplus \psi) \longrightarrow \mathbb{C}^N \longrightarrow Coker\, L \longrightarrow 0$$

Proof First one has to give a description of the cokernel of L.

$$\begin{aligned} Coker\, L &= \frac{\mathcal{W}}{Im\, L} = \frac{Im\, L + Im\, \psi}{Im\, L} \cong \frac{Im\, \psi}{Im\, L \cap Im\, \psi} = \frac{\psi(\mathbb{C}^N / Ker\, \psi)}{\psi(pr_{\mathbb{C}^N}(Ker(L \oplus \psi))/Ker\, \psi)} \\ &= \frac{\mathbb{C}^N / Ker\, \psi}{pr_{\mathbb{C}^N}(Ker(L \oplus \psi))/Ker\, \psi} \cong \frac{\mathbb{C}^N}{pr_{\mathbb{C}^N}\, Ker(L \oplus \psi)} \end{aligned}$$

Using this isomorphism it becomes clear the exactness of the sequence. □

What's the use of this lemma? For a linear Fredholm operator $L : \mathcal{V} \to \mathcal{W}$, where $\mathcal{V}, \mathcal{W} \to \mathbf{T}$ are Banach bundles over a manifold $\mathbf{T}$, the *index bundle* of L is defined to be

$$index\, L := [Ker\, L] \ominus [Coker\, L]$$

which should be viewed as an element in $KO(\mathbf{T})$. The problem with this definition is to make sense only if L has a kernel of constant rank along $\mathbf{T}$ (let me remind that $\dim Ker L - \dim Coker L = constant$). The important thing is that *the determinant of the index bundle* is well-defined. Indeed, let $t \in \mathbf{T}$ be a point; the operator $L_t : \mathcal{V}_t \to \mathcal{W}_t$ has a finite dimensional cokernel and therefore we may find a neighborhood $U_t \hookrightarrow \mathbf{T}$ of t with the property that there is a morphism $\psi : \underline{\mathbb{C}}^N \to \mathcal{W}$ such that $L \oplus \psi : \mathcal{V} \oplus \underline{\mathbb{C}}^N \to \mathcal{W}$ is surjective over U_t. Here $\underline{\mathbb{C}}^N \to U_t$ denotes the trivial bundle.

Remark I should point out that I do not assume L nor ψ to be complex-linear and that $\mathbb{C}^N$ stands for $\mathbb{R}^{2N}$; anyway, if *à priori* one finds an $\mathbb{R}^{odd}$ he may take the direct sum with $\mathbb{R}$ and gets a $\mathbb{R}^{even}$. ⋄

Lemma 3.8.1 above tells us that

$$\bigwedge^{max} Ker\, L \otimes \left(\bigwedge^{max} Coker\, L \right)^{-1} \cong \bigwedge^{max} Ker(L \oplus \psi)$$

so the determinant of the index bundle is well defined locally. A simple argument shows that this construction do not depend on the choice of ψ and therefore this bundle is globally defined.

In our case there is a commutative diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\mathcal{D}} & \mathcal{E} \\ & \searrow \quad \swarrow & \\ & N_g(\mathbf{X}, A, J) & \end{array}$$

where $\mathcal{V}_{u,j} = T_{(u,j)}\chi_A = \Omega^0(u^*T\mathbf{X}) \oplus T_j N_g$ and $\mathcal{E}_{(u,j)} = \Omega^{0,1}(u^*T\mathbf{X})$. Both bundles have natural complex structures which is given by J. On a vector $(\xi, y) \in \mathcal{V}_{(u,j)}$ the operator $\mathcal{D}$ acts as follows

$$\mathcal{D}_{(u,j)}(\xi, y) = D_u\xi + J(u) \cdot \mathrm{d}u \cdot \mathcal{I}_y = \overline{\partial}_u \xi + \frac{1}{4} N_J(\partial u, \xi) + J \cdot \mathrm{d}u \cdot \mathcal{I}_y$$

where $\overline{\partial}_u$ denotes the $\overline{\partial}$-operator in the complex vector bundle $u^*T\mathbf{X} \to \mathbf{C}_j$ and N_J is the Nijenhuis tensor of J. It is important to notice that $\mathcal{D}$ is not complex linear. The tangent space of $N_g(\mathbf{X}, A, J)$ at a point (u, j) is the kernel of $\mathcal{D}_{(u,j)}$ and therefore it has a canonical orientation if the top wedge-product of $Ker\, \mathcal{D}$ has a canonical trivialization. Because J was chosen generic, $\mathcal{D}$ is surjective and therefore

$$\bigwedge^{max} Ker\, \mathcal{D} = \det index\, \mathcal{D}.$$

The homotopy $[0,1] \ni t \mapsto \mathcal{D}^t$, where $\mathcal{D}^t : \mathcal{V} \to \mathcal{W}$ is defined at a point $(u,j) \in N_g(\mathbf{X}, A, J)$ by

$$\mathcal{D}^t_{(u,j)}(\xi, y) := \overline{\partial}_u \xi + \frac{t}{4} N_J(\partial u, \xi) + t \cdot J(u) \cdot \mathrm{d}u \cdot \mathcal{I}_y$$

All these operators are Fredholm being compact perturbations of $\overline{\partial}$. This way we deform $\mathcal{D}$ which is not complex linear into $\overline{\partial}$ which is, so its kernel and cokernel over each point (u,j) are J-complex spaces and therefore carry a natural orientation. The problem is again that the dimension of the kernel of $\overline{\partial}$ may jump, but we can handle this problem with the previous trick: in this case we have the operator $\overline{\mathcal{D}} = (\mathcal{D}^t)_{t \in [0,1]}$

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\overline{\mathcal{D}}} & \mathcal{E} \\ & \searrow \quad \swarrow & \\ & N_g(\mathbf{X}, A, J) \times [0,1] & \end{array}$$

Consider a point $(u,j) \in N_g(\mathbf{X}, A, J)$. $\mathcal{D} = \mathcal{D}^1$ being surjective, $\mathcal{D}^t$ will be also for t close to 1. Because $[0,1]$ is a compact parameter space, we can find a neighborhood $U_{(u,j)} \hookrightarrow N_g(\mathbf{X}, A, J)$ of (u,j) and a morphism $\overline{\psi} = (\psi^t)_{t \in [0,1]} : \mathbb{C}^N \to \mathcal{E}$ with the properties

- $\mathcal{D}^t \oplus \psi^t : \mathcal{V} \oplus \mathbb{C}^N \to \mathcal{E}$ are surjective for all $t \in [0,1]$;
- ψ^t is the zero morphism for t close to 1 ;
- ψ^t are all complex linear.

Let me explain why this is possible: the first condition can be clearly satisfied because the cokernels of all $\mathcal{D}^t$'s are finite dimensional; the second condition can be satisfied because $\mathcal{D}^1$ is already surjective. If we have a morphism $\overline{\varphi} = (\varphi^t)_{t \in [0,1]} : \mathbb{R}^N \to \mathcal{E}$ a morphism satisfying the first two properties then we may define $\overline{\psi} = (\psi^t)_{t \in [0,1]} : \mathbb{C}^N = \mathbb{R}^N \otimes_{\mathbb{R}} \mathbb{C} \to \mathcal{E}$ its complexification

$$\psi^t(v \otimes 1) = \varphi(v) \quad \text{and} \quad \psi^t(v \otimes i) = J\varphi(v)$$

This procedure preserve the first two conditions but now $\overline{\psi}$ satisfies also the third condition.

Now we are almost done: the bundle $Ker(\overline{\mathcal{D}} \oplus \overline{\psi})$ is well defined over $U_{(u,j)} \times [0,1]$ and the restriction to $U_{(u,j)} \times \{0\}$ is complex, with complex structure given by J and $\mathbb{C}^N$; hence it carries a natural orientation. The top wedge-product

$$\bigwedge^{max} Ker(\overline{\mathcal{D}} \oplus \overline{\psi}) \longrightarrow U_{(u,j)} \times [0,1]$$

is a real line bundle whose restriction to the 0-end is trivial and therefore it is globally trivial. The lemma above insures that the construction do not depend on the choice of $\overline{\phi}$.

Remark 3.8.2 This proof tells a little bit more: it says, in the transverse situation, that if we take (locally) the direct sum $\mathcal{V} \oplus \mathbb{C}^N$, we can deform the tangent space of $N_g(\mathbf{X}, A, J) \times \mathbb{C}^N$ at $(u,j) \times 0$ (which is split inside $\mathcal{V} \oplus \mathbb{C}^N$) into a J-complex subspace. $\diamond$

3.9 Almost complex structure on the space of J-curves

In this section I want to prove that if the Nijenhuis tensor of the almost complex structure J on $\mathbf{X}$ is suitably small then the moduli space $N_g(\mathbf{X}, A, J)$ of pseudo-holomorphic curves admits an almost complex structure.

The "Zariski tangent space" to $N_g(\mathbf{X},A,J)$ at (u,j) is given by

$$T_{(u,j)}N_g(\mathbf{X},A,J) = \{(\xi,y) \in \Omega^0(u^*T\mathbf{X}) \oplus T_jN_g \mid \mathcal{D}_{(u,j)}(\xi,y) = D_u\xi + J(u)\cdot \mathrm{d}u \cdot \mathcal{I}_y = 0\}$$

Let me introduce on the space $\Omega^0(u^*T\mathbf{X}) \oplus T_jN_g$ the scalar product

$$\langle (\xi,y),(\xi',y')\rangle := \int_{\mathbf{C}_j} (u^\star g)_x(\xi_x,\xi'_x)\mathrm{d}\gamma_j(x) + g_{\mathrm{WP}}(y,y')$$

In the formula above g_{WP} represents the Weil-Petersson metric on $\overline{N}_g$; γ_j is the metric on $\mathbf{C}_j$ induced by the projective embedding $\phi : \overline{N}_{g,1} \to \mathbb{P}^N$; $u^\star g$ is the metric induced by g on the vector bundle $u^*T\mathbf{X} \to \mathbf{C}_j$. In the case when $A = 0 \in H_2(\mathbf{X},\mathbb{Z})$, so $N_g(\mathbf{X},A,J) = \mathbf{X}\times N_g$, the above scalar product at a point $u : \mathbf{C}_j \mapsto x \in \mathbf{X}$ is given by the formula

$$\langle (\xi,y),(\xi',y')\rangle = g_x(\xi,\xi')\cdot vol_{\gamma_j}(\mathbf{C}_j) + g_{\mathrm{WP}}(y,y')$$

which is almost the product metric on $\mathbf{X}\times N_g$. I shall denote by $W^2(u^*T\mathbf{X}) \oplus T_jN_g$ the completion with respect to the norm given by the scalar product.

On $\Omega^{0,1}_{jJ}(u^*T\mathbf{X})$ I introduce the scalar product

$$\langle \alpha,\beta\rangle : \;= \int_{\mathbf{C}_j} (u^\star g)_x(\alpha_x,\beta_x)\mathrm{d}\gamma_j(x)$$

and its completion with respect to this norm will be denoted by $W^2(\Omega^{0,1}_{jJ}\otimes u^*T\mathbf{X})$. For these norms the operator

$$\mathcal{D}_{(u,j)} : W^2(u^*T\mathbf{X}) \oplus T_jN_g \longrightarrow W^2(\Omega^{0,1}_{jJ}\otimes u^*T\mathbf{X})$$

is unbounded and Fredholm whose kernel will be the "Zariski tangent space" at (u,j) to $N_g(\mathbf{X},A,J)$; by elliptic regularity, the elements in its kernel are smooth.

Let me notice that both $\Omega^0(u^*T\mathbf{X}) \oplus T_jN_g$ and $\Omega^{0,1}_{jJ}(u^*T\mathbf{X})$ are complex vector spaces and the scalar products defined on them are compatible with the complex structure. If $\mathbf{J}$ denotes the complex structure on $\Omega^0(u^*T\mathbf{X}) \oplus T_jN_g$, then the formula

$$\begin{aligned}\Theta(v,v') &:= \langle \mathbf{J}v,v'\rangle \qquad\qquad \forall\ v=(\xi,y),\ v'=(\xi',y') \in W^2(u^*T\mathbf{X})\oplus T_jN_g \\ &= \int_{\mathbf{C}_j} (u^\star\omega)_x(\xi_x,\xi'_x)\mathrm{d}\gamma_j(x) + \omega_{\mathrm{WP}}(y,y')\end{aligned}$$

defines a skew-symmetric two-form on it. In a completely analogous way, I define the metric $\overline{G}$ on $W^2(u^T\mathbf{X}) \oplus T_jN_g$. I want to see when Θ is non-degenerate on $Ker\,\mathcal{D}_{(u,j)}$. This is equivalent saying that for all $v \in Ker\,\mathcal{D}_{(u,j)}$ there is a $w \in Ker\,\mathcal{D}_{(u,j)}$ such that $\Theta(v,w) \neq 0$.

There are orthogonal decompositions into closed subspaces

$$W^2(u^*T\mathbf{X}) \oplus T_jN_g = Ker\,\mathcal{D}_{(u,j)} \oplus (Ker\,\mathcal{D}_{(u,j)})^\perp$$
$$W^2(\Theta^{0,1}_{jJ}\otimes u^*T\mathbf{X}) = Im\,\mathcal{D}_{(u,j)} \oplus (Im\,\mathcal{D}_{(u,j)})^\perp$$

For a vector $v = (\xi,y) \in Ker\,\mathcal{D}_{(u,j)}$ let

$$\mathbf{J}v = (\mathbf{J}v)_0 + (\mathbf{J}v)^\perp$$

be the orthogonal decomposition of the vector $\mathbf{J}v = (J\xi, J_{N_g}y)$ (here J_{N_g} stands for the complex structure on N_g). I want to compute

$$\begin{aligned}\Theta(v, (\mathbf{J}v)_0) &= \Theta\left(v, \mathbf{J}v - p_\perp(\mathbf{J}v)\right) = \|v\|^2 - \Theta\left(v, p_\perp(\mathbf{J}v)\right) = \|v\|^2 - \langle \mathbf{J}v, p_\perp(\mathbf{J}v)\rangle \\ &\geq \|v\|^2 - \|\mathbf{J}v\| \cdot \|p_\perp(\mathbf{J}v)\| \geq \|v\|^2 - \|p_\perp\| \cdot \|\mathbf{J}v\| \cdot \|\mathbf{J}v\| = 0\end{aligned}$$

I have proved that

$$\Theta(v, (\mathbf{J}v)_0) \geq 0 \qquad \forall v \in Ker\, \mathcal{D}_{(u,j)}$$

For Θ to be non-degenerate it is sufficient to have strict inequality above. It can be seen that $\Omega(v, (\mathbf{J}v)_0) = 0$ if and only if equality holds in the Cauchy-Schwartz inequality. This happens precisely when

$$p_\perp(\mathbf{J}v) = l \cdot \mathbf{J}v$$

with $l \in \mathbb{R}_+$ a constant i.e. if and only if $\mathbf{J}v \in \left(Ker\, \mathcal{D}_{(u,j)}\right)^\perp$, $l = 1$.

Remark 3.9.1 The computations above show that this "bad case" occurs if and only if

$$\mathbf{J} Ker\, \mathcal{D}_{(u,j)} \cap \left(Ker\, \mathcal{D}_{(u,j)}\right)^\perp \neq 0$$

I should notice that such a situation is quite an unnatural one since the dimension of the first vector space equals the codimension of the second one. The "generic" intersection of two such vector spaces is transverse. It might be possible that varying $J \in \mathcal{J}(\mathbf{X}, \omega)$ and using a different metrics on $\mathbf{X}$ (for instance $f^2 g_J$, with $f : \mathbf{X} \to \mathbb{R}$ a nowhere vanishing function) in order to define the scalar product, such a transversality can be achieved. But it is difficult to decide when such "generic" J's and f's really exist. $\diamond$

Definition 3.9.2 *In the spirit of the previous remark, I shall call the almost complex structure J on $\mathbf{X}$ ω-*good *if it is both generic (in the usual sense) and*

$$\mathbf{J} Ker\, \mathcal{D}_{(u,j)} \cap \left(Ker\, \mathcal{D}_{(u,j)}\right)^\perp = 0$$

for all $(u, j) \in N_g(\mathbf{X}, A, J)$.

The analysis done before shows that if J is good, then the space $N_g(\mathbf{X}, A, J)$ carries an almost complex structure $\mathbb{J}$ which is obtained from the polar decomposition of Θ with respect to $\overline{G}$; there is an induced metric

$$G(v, w) := \Theta(v, \mathbb{J}w)$$

on $N_g(\mathbf{X}, A, J)$. According to proposition 1.2.8, $\mathbb{J}$ it is obtained as follows: the automorphism A of the tangent space with the property that

$$\Theta(v, w) = \langle Av, w\rangle \qquad \forall v, w \in T_{(u,j)} N_g(\mathbf{X}, A, J)$$

has the concrete description

$$\Theta(v,w) = \langle \mathbf{J}v, w\rangle = \langle (\mathbf{J}v)_0 + (\mathbf{J}v)^{\perp}, w\rangle = \langle (\mathbf{J}v)_0, w\rangle$$

so that

$$Av = (\mathbf{J}v)_0 = \mathrm{proj}_{Ker\,\mathcal{D}_{(u,j)}}\mathbf{J}v$$

Then $\mathbb{J} = A\cdot|A|^{-1}$, where the absolute value $|A|$ of A is the positive operator given by $|A| := \sqrt{A^*A}$. The inequality

$$\Theta(v,\mathbb{J}v) = \langle Av, \mathbb{J}v\rangle = -\langle \mathbb{J}Av, v\rangle = \langle |A|v, v\rangle > 0 \qquad \forall v \in TN_g(\mathbf{X}, A, J)$$

shows that Θ is non-degenerate on any $\mathbb{J}$-invariant subspace of $TN_g(\mathbf{X}, A, J)$.

In what follows I want to study some properties of the natural map $\varpi : N_g(\mathbf{X}, A, J) \to N_g$ which associates to a map (u, j) the conformal structure j of the domain of definition of u. This map ϖ is $\mathcal{C}^\infty$ in the case when J is $\omega -$ *good.*

Definition 3.9.3 *For a $\mathcal{C}^\infty$ map $h : \mathbf{M} \to \mathbf{N}$ between smooth manifolds I shall call a fibre $h^{-1}(n)$, $n \in \mathbf{N}$, h-*general *if*

$$\mathrm{rank}\,\mathrm{d}h_m = \max_{m'\in\mathbf{M}} \mathrm{rank}\,\mathrm{d}h_{m'} \qquad \forall m \in h^{-1}(n)$$

The definition is correct since the rank of the differential $\mathrm{d}h_{m'}$ is bounded by the dimension of $\mathbf{N}$. It is easy to see that any smooth map $h : \mathbf{M} \to \mathbf{N}$ has *non-empty* and *smooth* h-general fibres; indeed, let $m_0 \in \mathbf{M}$ be a point where $\mathrm{d}h$ has maximal rank r. Then it will have (the same) maximal rank in an open neighborhood $\mathscr{V}_0$ of m_0 inside $\mathbf{M}$. The restriction of h to $\mathscr{V}_0$ has constant rank and therefore its image $h(\mathscr{V}_0)$ is a smooth, locally closed submanifold of $\mathbf{N}$. The inverse image $\mathscr{V} := h^{-1}(h(\mathscr{V}_0))$ is an open subset of $\mathbf{M}$ and the restriction

$$h : \mathscr{V} \longrightarrow h(\mathscr{V}_0)$$

is a smooth map between smooth manifolds. Moreover, by very construction, h is surjective at the point $m_0 \in \mathscr{V}$. Standard Sard theorem insures that the regular values of h form a dense open set in $h(\mathscr{V}_0)$. I should notice that this definition differs from that of regular fibres of a map.

If I define the "universal curve" over $N_g(\mathbf{X}, A, J)$ to be

$$N_{g,1}(\mathbf{X}, A, J) := N_g(\mathbf{X}, A, J) \times_{N_g} N_{g,1},$$

I get the commutative diagram

$$\begin{array}{ccccc} \mathbf{F}_j \times \mathbf{C}_j & \stackrel{i_j}{\hookrightarrow} & N_{g,1}(\mathbf{X}, A, J) & \stackrel{\varpi_1}{\longrightarrow} & N_{g,1} \\ \downarrow & & \mathrm{p}\downarrow & & \downarrow \\ \mathbf{F}_j & \stackrel{\iota_j}{\hookrightarrow} & N_g(\mathbf{X}, A, J) & \stackrel{\varpi}{\longrightarrow} & N_g \end{array}$$

where $\mathbf{F}_j$ denotes a ϖ-general fibre; it is nothing else but the space of simple J-holomorphic curves representing the 2-homology class $A \in H_2(\mathbf{X})$ and having *fixed* domain of definition $\mathbf{C}_j$.

Definition 3.9.4 *Using the evaluation map*

$$ev : N_{g,1}(\mathbf{X}, A, J) \longrightarrow \mathbf{X}$$
$$ev((u, j), s) := u(s) \qquad \forall s \in \mathbf{C}_j$$

I can define the following cohomology class $[\Omega]$ *on* $N_g(\mathbf{X}, A, J)$

$$\langle [\Omega], [B] \rangle := \langle ev^*\omega \cup \varpi_1^*\gamma, \mathrm{p}^{-1}(B) \rangle$$

for any 2-cycle $[B] \in H_2(N_g(\mathbf{X}, A, J))$ *(here* B *is a representative of its class). The cohomology class* $[\Omega]$ *is represented by the two form* Ω *defined to be the integral over the fibres of* $N_{g,1}(\mathbf{X}, A, J) \to N_g(\mathbf{X}, A, J)$ *of the four form* $ev^*\omega \wedge \varpi_1^*\gamma$*.*

Let me remind that ω is the symplectic form on $\mathbf{X}$ and γ is the restriction of the standard Kähler form of $\mathbb{P}^N$ to $\overline{N}_{g,1}$ *via* an embedding $\overline{N}_{g,1} \to \mathbb{P}^N$. When restricted to the fibres $\mathbf{F}_j$, $[\Omega]$ has a very nice description as a DeRham 2-form:

Lemma 3.9.5 The restriction of $\iota_j^*[\Omega]$ to any $\mathbf{F}_j$ is represented by:

$$(\iota_j^*\Omega)_u(\xi_1, \xi_2) = \int_{\mathbf{C}_j} (u^\star\omega)_s(\xi_{1,s}, \xi_{2,s}) \mathrm{d}\gamma_j(s) = (\iota_j^*\Theta)_u(\xi_1, \xi_2) \qquad \forall \xi_1, \xi_2 \in T_u\mathbf{F}_j$$

where $u^\star\omega$ denotes the skew-symmetric form on $u^*T\mathbf{X}$ induced by ω.

Proof For $B \in H_2(\mathbf{F}_j)$ a 2-cycle,

$$\langle \iota_j^*[\Omega], [B] \rangle = \langle ev^*\omega \cup \varpi_1^*\gamma, [B \times \mathbf{C}_j] \rangle = \langle ev_j^*\omega \cup \gamma_j, [B \times \mathbf{C}_j] \rangle = \langle ev_j^*\omega \cup \gamma_j/[\mathbf{C}_j], [B] \rangle$$

where γ_j denotes the restriction of γ to the fibre $\mathbf{C}_j$. This equality shows that $\iota_j^*\Omega$ can be represented by the fibre integral of the four-form $ev_j^*\omega \cup \gamma_j$ on $\mathbf{F}_j \times \mathbf{C}_j \to \mathbf{F}_j$. At any point $(u, s) \in \mathbf{F}_j \times \mathbf{C}_j$, one may see that, as a differential form

$$(ev_j^*\omega \wedge \gamma_j)_{(u,s)} = (ev_{j,s}^*\omega)_u \wedge (\gamma_j)_s$$

because the derivatives of ev_j in the $\mathbf{C}_j$-direction are annihilated taking the exterior product with γ_j (for fixed $s \in \mathbf{C}_j$, I denoted $ev_{j,s} : \mathbf{F}_j \to \mathbf{X}$ the map $u \mapsto u(s),\ u \in \mathbf{F}_j$). If $\xi_1, \xi_2 \in T_u\mathbf{F}_j$ are two vectors,

$$(ev_{j,s}^*\omega)_u(\xi_1, \xi_2) = \omega\left((\mathrm{d}\, ev_{j,s})_u\xi_1, (\mathrm{d}\, ev_{j,s})_u\xi_2\right) = \omega\left(\xi_{1,u(s)}, \xi_{2,u(s)}\right) = (u^\star\omega)_s(\xi_{1,s}, \xi_{2,s}) \qquad \square$$

For any $v = (\xi, y) \in T_{(u,j)}N_g(\mathbf{X}, A, J)$

$$\mathcal{D}_{(u,j)}(\mathbf{J}v) = -\frac{1}{2} J N_J(\cdot, \xi)$$

This equality says that if the Nijenhuis tensor has a small norm, $\mathbf{J}v$ is "almost" in the kernel of $\mathcal{D}_{(u,j)}$ and therefore the almost complex structure J might be ω-good. Let denote by

$$W^{1,2}(u^*T\mathbf{X}) \hookrightarrow W^2(u^*T\mathbf{X})$$

the completion of $\Omega^0(u^*T\mathbf{X})$ with respect to o $W^{1,2}$-norm. Then the restriction

$$\mathcal{D}_{(u,j)} : W^{1,2}(u^*T\mathbf{X}) \oplus T_jN_g \longrightarrow W^2(\Omega^{0,1}_{jJ} \otimes u^*T\mathbf{X})$$

is a continuous Fredholm operator. It has a continuous pseudo right-inverse

$$\mathcal{R}_{(u,j)} : W^2(\Omega^{0,1}_{jJ} \otimes u^*T\mathbf{X}) \longrightarrow W^{1,2}(u^*T\mathbf{X}) \oplus T_jN_g$$

having the property that

$$\mathcal{R}_{(u,j)}\mathcal{D}_{(u,j)}(v) = \mathrm{proj}_{(Ker\,\mathcal{D}_{(u,j)})^{\perp}} v \qquad \forall v \in W^{1,2}(u^*T\mathbf{X}) \oplus T_jN_g$$
$$\mathcal{R}_{(u,j)}(w) = 0 \qquad \forall w \in \left(\mathrm{Im}\,\mathcal{D}_{(u,j)}\right)^{\perp}$$

Assume now that there is $v \in Ker\,\mathcal{D}_{(u,j)}$ such that $\mathbf{J}v \in \left(Ker\,\mathcal{D}_{(u,j)}\right)^{\perp}$. Then

- $\mathcal{D}_{(u,j)}v = 0 \quad \Rightarrow \quad \mathcal{D}_{(u,j)}(\mathbf{J}v) = -1/2JN_J(\cdot,\xi)$
- $\mathbf{J}v = \mathcal{R}_{(u,j)}\mathcal{D}_{(u,j)}(\mathbf{J}v) = -1/2\mathcal{R}_{(u,j)}(JN_J(\cdot,\xi))$.

Taking the norms in the previous equality I obtain

$$|(\xi,y)| \le \frac{1}{2}\|\mathcal{R}_{(u,j)}\| \cdot \|N_J\| \cdot |\xi|$$

where $\|N_J\| := \sup_{x\in\mathbf{X}} \|N_J\|_x$. Because $|(\xi,y)| \ge |\xi|$, I deduce that a necessary condition for this "bad situation" to occur is: $1 \le 1/2\|\mathcal{R}_{(u,j)}\| \cdot \|N_J\|$. This proves the:

Lemma 3.9.6 • If

$$\|N_J\| < 2 \cdot \frac{1}{\|\mathcal{R}_{(u,j)}\|}$$

then the two-form Θ is non-degenerate on the tangent space of $N_g(\mathbf{X},A,J)$ at (u,j).

• If

$$\|N_J\| < \inf_{(u,j)\in N_g(\mathbf{X},A,J)} 2 \cdot \frac{1}{\|\mathcal{R}_{(u,j)}\|} \qquad (\spadesuit)$$

then Θ is a non-degenerate 2-form. In other words, if J is generic and its Nijenhuis Tensor is small enough, then J is ω-good.

For integrable J the operator $\mathcal{D}_{(u,j)}$ is complex linear and therefore Θ is obviously non-degenerate on its kernel. The meaning of the lemma above is that if one makes a $\mathcal{C}^1$-small perturbation from an integrable almost complex structure, then Θ remains non-degenerate.

Remark 3.9.7 (i) First of all it is not clear that the infimum above should be a positive number, because the space $N_g(\mathbf{X},A,J)$ is non-compact. The scalar products on each $T_{(u,j)}N_g(\mathbf{X},A,J)$

depend on (the norm of) the derivative of the map u. If these derivatives are bounded, then $\inf_{(u,j)} 1/\|\mathcal{R}_{(u,j)}\| > 0$. If the differentials of the maps $(u,j) \in N_g(\mathbf{X}, A, J)$ are unbounded, then J-holomorphic 2-spheres appear in the limit. Consequently we may say that either $\mathbf{X}$ carries J-holomorphic 2-spheres (which is a good news) or our infimum is strictly positive and the inequality (♠) makes sense. It seems a difficult problem to give an estimate for this infimum.

(ii) In the integrable case the previous procedure gives us the standard (integrable) almost complex structure on $N_g(\mathbf{X}, A, J)$.

(iii) **(A shortcoming)** Assuming that the Nijenhuis tensor N_J has a very small sup-norm, we may find in a neighborhood of J integrable almost complex structures. Indeed, let us consider the functional

$$\mathcal{J}_\tau \ni K \overset{\varphi}{\mapsto} \sup_{x \in \mathbf{X}} \|N_K\|^2(x) \in \mathbb{R}$$

$\mathcal{J}_\tau$ is an open set inside the space of all almost complex structures on $\mathbf{X}$; φ is a non-constant function and if, for a certain J, $\varphi(J)$ is (very) small then it might exist in its neighborhood an almost complex structure $J' \in \mathcal{J}_\tau$ with the property that $\varphi(J') = 0$. But this means that J' is integrable and therefore $(\mathbf{X}, J')$ is a complex manifold. The conclusion is that the previous construction seems to be applicable only to $\mathcal{C}^1$-small deformations of Kähler manifolds. ⋄

Another interesting question is the following: there is a natural map

$$\varpi : N_g(\mathbf{X}, A, J) \longrightarrow N_g$$

which associates to a map (u,j) the conformal structure j of its domain. The problem is to see how far it falls short this map being pseudo-holomorphic. The differential at a point (u,j) of ϖ is given by

$$\mathrm{d}\varpi_{(u,j)}(\xi, y) = y.$$

Let us notice that in the case when $(\mathbf{X}, J)$ is a complex manifold, this differential is complex linear; in the case when the Nijenhuis tensor is small one should expect that ϖ won't be anymore a holomorphic map, but close being so.

It turns out that this is indeed the case. We saw already that

$$\Theta(\cdot,\cdot) = \langle A\cdot, \cdot\rangle \quad \text{with} \quad A(\cdot) = \mathrm{proj}_{Ker\,\mathcal{D}_{(u,j)}} \mathbf{J}(\cdot).$$

For $v = (\xi, y)$ we have the inequality

$$|Av - \mathbf{J}v| = |(\mathbf{J}v)^\perp| = \frac{1}{2}|\mathcal{R}_{(u,j)}\left(JN_J(\cdot,\xi)\right)| \le \frac{1}{2}\|\mathcal{R}_{(u,j)}\| \cdot \|N_J\| \cdot |\xi| \le \frac{1}{2}\|\mathcal{R}_{(u,j)}\| \cdot \|N_J\| \cdot |v|$$

which implies the

Lemma 3.9.8

$$\|A - \mathbf{J}\|_{(u,j)} \le \frac{1}{2}\|\mathcal{R}_{(u,j)}\| \cdot \|N_J\|$$

In the notations of proposition 1.2.8, we have that the absolute value of J is $|J| = \sqrt{J^*J} = \mathrm{Id}$. Because the polar decomposition is continuous for the $\mathcal{C}^0$-topology, we obtain that the absolute value of A is $|A| = \sqrt{A^*A} = \mathrm{Id} + O(\|N_J\|)$ and consequently the almost complex structure

$$\mathbb{J}_{(u,j)} = \mathbf{J} + O(\|N_J\|)$$

I want now to see whether or not the form Ω is non-degenerate on the relative tangent bundle $T_{rel} := Ker\, \mathrm{d}\varpi$. At a point $(u, j) \in N_g(\mathbf{X}, A, J)$, $T_{rel} = Ker\, D_u \hookrightarrow Ker \mathcal{D}_{(u,j)}$ and using the metric G there is a direct sum decomposition $Ker\, \mathcal{D}_{(u,j)} = Ker\, D_u \oplus (Ker\, D_u)^{\perp_G}$. For $v \in Ker\, D_u$, let me compute

$$\begin{aligned}\Omega(v, \mathrm{proj}_{Ker\, D_u} \mathbb{J}v) &= G(\mathbb{J}v, \mathrm{proj}_{Ker\, D_u} \mathbb{J}v) = G(\mathrm{proj}_{Ker\, D_u} \mathbb{J}v, \mathrm{proj}_{Ker\, D_u} \mathbb{J}v) \\ &= |\mathrm{proj}_{Ker\, D_u} \mathbb{J}v|^2_G \geq 0\end{aligned}$$

Recall that $v \in Ker\, D_u$, that $\|\mathbb{J} - \mathbf{J}\| = O(\|N_J\|$ and that the failure of D_u to be J-linear is given by an amount of $\|N_J\|$. Therefore, if N_J has some very small norm, it is reasonable to expect that $\mathrm{proj}_{Ker\, D_u} \mathbb{J}v \neq 0$ for all $v \in Ker\, D_u$. In this case Ω is non-degenerate on the relative tangent bundle of $N_g(\mathbf{X}, A, J) \to N_g$.

The conclusions of this section can be summarized in the following

Theorem 3.9.9 *Let $(\mathbf{X}, \omega, J)$ be a symplectic manifold and $A \in N_2(\mathbf{X}, \mathbb{Z})$ a 2-homology class.*

(i) If J is ω-good, then the space $N_g(\mathbf{X}, A, J)$ admits an almost complex structure.

(ii) If the sup-norm on $\mathbf{X}$ of the Nijenhuis tensor N_J of the almost complex structure J is suitably small, then either there are J-holomorphic bubbles on $\mathbf{X}$ or J is ω-good. In the second case, the almost complex structure $\mathbb{J}$ has the property that

$$\|\mathbb{J} - \mathbf{J}\| = O(\|N_J\|)$$

where $\mathbf{J}$ denotes, for a J-holomorphic map $u : \mathbf{C}_j \to \mathbf{X}$, the complex structure induced on $\Omega^0(u^*T\mathbf{X}) \oplus T_j N_g$ by J and J_{N_g}.

(iii) Under the same assumptions as in (ii), the canonical projection

$$\varpi : N_g(\mathbf{X}, A, J) \longrightarrow N_g$$

fails to be $\mathbb{J}J_{N_g}$-holomorphic by an amount of $O(\|N_J\|)$ and the form Ω is non-degenerate on the relative tangent bundle of $\varpi : N_g(\mathbf{X}, A, J) \to N_g$. Consequently, every ϖ-general fibre $(\mathbf{F}_j, \iota_j^*\Omega)$ is a symplectic manifold. The metric G and the symplectic form $\iota_j^*\Omega$ define an almost complex structure $\mathbf{I}_j$ with the property that $\|\mathbf{I}_j - \mathbf{J}\| = O(\|N_J\|)$.

Note The first thing to say about this result is that it is very imprecise. If the first conclusion, i.e. that the moduli space of J-curves is almost complex, is likely to be true in many situations, the conclusions (ii) and (iii) use the assumption that $\|N_J\|$ is (very) small, but it is not given a clear estimate for the acceptable bound. In the proposition above $\|\mathbb{J} - \mathbf{J}\|$ is computed with respect to the norm induced by the scalar product $\langle \cdot, \cdot \rangle$. But we may consider the norm

$$|(\xi, y)|_{sup} := \sup_{x \in \mathbf{C}} |\xi_x|_{u^* g_J} + |y|_{g_{\mathrm{WP}}}$$

on $T_u N_g(\mathbf{X}, A, J) \oplus \mathbf{J} T_u N_g(\mathbf{X}, A, J)$, which is a finite dimensional vector space; consequently it is equivalent with the other one and therefore $|\mathbb{J} - \mathbf{J}|_{sup}$ is small if $\|N_J\|$ is small. This remark will be useful in the applications. $\diamond$

In the case when there is an almost complex structure on $N_g(\mathbf{X}, A, J)$, it is a trivial fact that this space has a canonical orientation. There remains however a question: what kind of object is $N_g(\mathbf{X}, A, J)$? It is certainly a smooth manifold in the case when J is chosen to be generic, but in general it is not smooth. However, as it was proved by B.Siebert, it is a Hausdorff space in the Gromov topology and admits a natural compactification. It the case when $\mathbf{X}$ is a projective variety, $N_g(\mathbf{X}, A, J)$ is a quasiprojective variety. In the next section I shall briefly present how our moduli spaces of curves can naturally be compactified.

3.10 Gromov compactness

An important feature of the space $N_g(\mathbf{X}, A, J)$ is that it can be compactified; moreover, in some well-defined situations, J-holomorphic bubbles appear in the limit. This section is devoted to give a brief account about the Gromov compactification of the space of pseudo-holomorphic curves, the main references being the papers [PW], [RT1], [Si]. Let me start with the following

Definition 3.10.1 (stable J-map) $(\mathbf{C}, \mathbf{x}, u)$ *is a J-holomorphic stable map if*

- $\mathbf{C}$ *is a connected, reduced, complete marked algebraic curve whose singularities are at most double points;*
- *for any irreducible component $\mathbf{C}_\nu$ of $\mathbf{C}$ the restriction $u_{|\mathbf{C}_\nu}$ is J-holomorphic;*
- *any smooth genus 0 (resp. 1) which is contracted must have at least three (resp. one) special point i.e. marked or singular.*

The reason for introducing this definition is given by the following theorem whose proof is given by prop.3.1 of [RT1] or thm.6.2 of [PW]):

Theorem 3.10.2 *Consider a sequence $(u_k)_k$ of stable J-holomorphic maps $u_k : (\mathbf{C}_{j_k}, \mathbf{x}) \to \mathbf{X}$, $j_k \in N_g$, with bounded energy and suppose that $j_k \stackrel{k\to\infty}{\longrightarrow} j_\infty \in \overline{N}_g$. Then there is a subsequence of $(u_k)_k$ which converges to a stable J-curve $(\mathbf{C}, \mathbf{x}, u)$.*

In the theorem above convergence means:

(i) the image curves converge to the image of the limit map in the Hausdorff topology. Moreover, the energy of the limit map is the same as the energy of the convergent subsequence;

(ii) on compacts subsets $\mathcal{K} \subset (\mathbf{C} - \mathbf{C}_{sing})$ the convergence is in $\mathcal{C}^\infty(\mathcal{K})$.

If $\mathbf{C}_\infty$ denotes the fibre of $\overline{N}_{g,1}$ over j_∞, then the curve $\mathbf{C}$ is obtained from $\mathbf{C}_\infty$:

(i) either joining some trees of $\mathbb{P}^1$'s at some points of $\mathbf{C}_\infty$

(ii) or first joining chains of $\mathbb{P}^1$'s at some double points of $\mathbf{C}_\infty$ (separating this way the components) and after adding the trees of $\mathbb{P}^1$'s on this new curve.

$\mathbf{C}_\infty$ is called *the principal component* of $\mathbf{C}$ and the attached $\mathbb{P}^1$'s form *the bubble component*. Whenever a bubble is attached at a marked point, it will "leave" the principal component and "land" on a bubble. This happens in such a way that the evaluation map corresponding to this marked point is continuous. Let us remark also that the arithmetic genus of $\mathbf{C}$ and $\mathbf{C}_{principal}$ coincide.

A natural question is when this bubbling appears. Let me remind that using a fixed a projective embedding $\phi : \overline{N}_{g,1} \to \mathbb{P}^N$ we obtain on each fibre $\mathbf{C}_j$, $j \in \overline{N}_{g,1}$ a metric γ_j. On $\mathbf{X}$ we have the metric $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$ so for any map $u_j : \mathbf{C}_j \to \mathbf{X}$ we may consider its sup-norm

$$|u_j| := sup_{p\in\mathbf{C}_j} |\mathrm{d}u(p)|_{\gamma_j, g_J}$$

When we consider the converging subsequence $u_k : \mathbf{C}_{j_k} \to \mathbf{X}$ given by the theorem two things may happen with the sequence of the norms $(|u_k|)_k$: it is either bounded or unbounded. It is proved in the given references that bubbling appears precisely when this sequence is unbounded; there will be (isolated) points where the differential of these maps grows bigger and bigger. A (repeated) rescaling procedure will give the bubbles.

In order to compactify spaces like $N_g(\mathbf{X}, A, J)$, one has to add, as biundary component, isomorphism classes of stable J-maps. This construction is explained, for instance, in [LiTi] and [Si].

Theorem 3.10.3 *If $(\mathbf{X}, \omega, J)$ is a symplectic manifold with an ω-tame almost complex structure J, then the space*

$$\overline{M}^{hol}_{g,k}(\mathbf{X}, A, J) := \{(\mathbf{C}, \mathbf{x}, u) \text{ stable } J-\text{holomorphic} \mid u_*[\mathbf{C}] = A,\ g(\mathbf{C}) = g,\ |\mathbf{x}| = k\}/\text{iso}$$

is compact and Hausdorff and it carries a "virtual fundamental class"

$$[\![\overline{M}^{hol}_{g,k}(\mathbf{X}, A, J)]\!] \in H_{2((3-n)(g-1)+k+c_1(\mathbf{X})\cdot A)}(\overline{M}^{hol}_{g,k}(\mathbf{X}, A, J))$$

Moreover there is a continuous evaluation map

$$ev : \overline{M}^{hol}_{g,k}(\mathbf{X}, A, J) \longrightarrow \mathbf{X}^k$$

given by the marked points.

For proving this theorem B.Siebert uses $\overline{M}_{g,k}$ as parameter space for curves of genus g and k marked points, but the theorem is still true if one considers a finite covering of it as long as no bubbles appear in the limit maps. Working with global a cover of $\overline{M}_{g,k}$ may be useful in the differential setting because it carries a universal curve whose fibres give representatives for the points in the base space. In particular I may take $\overline{N}_g$ as parameter space (see section 3.5 for its definition) and I get the

Corollary 3.10.4 *If $(\mathbf{X}, \omega, J)$ is a symplectic manifold with an ω-tame almost complex structure, then the space $N_g(\mathbf{X}, A, J)$ has a compactification $\overline{N}_g(\mathbf{X}, A, J)$ obtained "adding" the isomorphism classes of stable J-maps representing the homology class $A \in H_2(\mathbf{X}, \mathbb{Z})$ and whose principal components are fibres of $\overline{N}_{g,1} \to \overline{N}_g$.*

Let me point out that B.Siebert's construction do not use any genericity assumption on J. If the almost complex structure is generic, one has the additional information that the boundary $\overline{N}_g(\mathbf{X}, A, J) - N_g(\mathbf{X}, A, J)$ is of real codimension at least two (cf. [RT2]) and therefore $\overline{N}_g(\mathbf{X}, A, J)$ carries a natural orientation class which is obtained extending that of $N_g(\mathbf{X}, A, J)$.

3.11 Applications

As I said in the first section of this second part, my guiding idea was to investigate if it is possible to produce rational curves on symplectic varieties. Mori's theory shows that if the canonical bundle of a projective variety is not nef, then there are rational curves on it i.e. non-constant morphisms $\mathbb{P}^1 \to \mathbf{X}$. They are obtained using the "bend-and-break" procedure which is unfortunately strongly algebraic. The example 3.3.3 suggests that letting the conformal structure of our curves to vary we may expect degeneracy of the complex structure, in other words we may expect to find curves with strictly smaller (geometric) genus. Such a technique would provide us a recursive method to get curves of smaller and smaller genus.

(I) The projective case

In this case, the good choice of the parameter space for the curves of genus g is $\overline{M}_g$ because of its functorial properties. Now $(\mathbf{X}, J)$ denotes a smooth, complex projective variety ; hyperplane sections give rise to smooth holomorphic curves of some big genus g on it and representing a certain homology class $A \in H_2(\mathbf{X}, \mathbb{Z})$. There are two possibilities for $M_g(\mathbf{X}, A)$: bubbling either appears or it doesn't in the Gromov compactification $\overline{M}_g(\mathbf{X}, A, J)$. If bubbles occur, I am done because I get the rational curve on $\mathbf{X}$. So let me assume that there is no bubbling phenomenon. In this case $\overline{M}_g(\mathbf{X}, A, J)$ has a projective scheme structure and there is a morphism $\varpi : \overline{M}_g(\mathbf{X}, A, J)_{red} \to \overline{M}_g$; here $\overline{M}_g(\mathbf{X}, A, J)_{red}$ denotes the reduced scheme underlying $\overline{M}_g(\mathbf{X}, A, J)$. The image of ϖ will be a complete subvariety of $\overline{M}_g$ and the aim is to be able to decide whether or not it intersects the boundary divisor. According to S.Diaz's result, this will be the case if the dimension of the image is at least $g-1$.

Difficulties appear in the case when $\overline{M}_g(\mathbf{X}, A, J)$ has no reduced points because the only one thing which can be computed (at least theoretically) is the dimension of the Zariski tangent space of $\overline{M}_g(\mathbf{X}, A, J)$. If this scheme has no reduced points, case which occur as it was shown by D.Mumford in his paper [Mu2], the dimension of the Zariski tangent space is strictly bigger than the dimension of the reduced space at every point.

In the case when $\overline{M}_g(\mathbf{X}, A, J)$ has reduced points, these will form a Zariski open set. For computing $\dim Im\, \varpi$, let me consider a smooth and reduced point $(u, j) \in M_g(\mathbf{X}, A, J)$ which represents a morphism $u : \mathbf{C} \to \mathbf{X}$. Corresponding to the exact sequence:

$$0 \longrightarrow T\mathbf{C} \longrightarrow u^*T\mathbf{X} \longrightarrow N_{\mathbf{C}|\mathbf{X}} \longrightarrow 0$$

I get

$$0 \to H^0(u^*T\mathbf{X}) \to H^0(N_{\mathbf{C}|\mathbf{X}}) \overset{\theta}{\to} H^1(T\mathbf{C}) \to H^1(u^*T\mathbf{X}) \to H^1(N_{\mathbf{C}|\mathbf{X}}) \to 0$$

Here I assumed that $g \geq 2$, because the genus of the curves we find *à priori* on a projective variety is high. The dimension of the image of ϖ is just the dimension of the image of the map $\theta : H^0(N_{\mathbf{C}|\mathbf{X}}) \to H^1(T\mathbf{C})$ i.e.

$$\dim Im\, \varpi = h^1(N_{\mathbf{C}|\mathbf{X}}) - h^1(u^*T\mathbf{X}) + 3(g-1)$$

and using S.Diaz's theorem we get the following

Criterion 3.11.1 *If*

$$2(g-1) \geq h^1(u^*T\mathbf{X}) - h^1(N_{\mathbf{C}|\mathbf{X}}) \qquad (\clubsuit)$$

then the compactification $\overline{M}_g(\mathbf{X}, A, J)$ contains curves of strictly smaller geometric genus.

Such an approach is much too weak compared with Mori's technique in the case when the canonical bundle of a projective manifold is not nef. The approach might be interesting for projective varieties whose canonical bundle is trivial (Calabi-Yau varieties) which are supposed to carry low genus curves, even rational ones.

The simplest case should be that of $K3$ surfaces, but difficult questions I don't know to answer appear already. It is known that the projective $K3$ surfaces are parameterized by a countable family $\bigcup_{g\geq 2} \mathcal{F}_g$ with the property that:

(i) the $K3$ surfaces in $\mathcal{F}_2$ are obtained as double covers of $\mathbb{P}^2$ along a smooth sextic;

(ii) if $g \geq 3$ and $\mathbf{X} \in \mathcal{F}_g$ then there is a very ample line bundle $L \to \mathbf{X}$ with the property that $L^2 = 2(g-1)$ and which defines a projective embedding $\mathbf{X} \hookrightarrow \mathbb{P}^g$. The transverse hyperplane sections of $\mathbf{X}$ are smooth curves of genus g.

The generic elements in these families have the property that $Pic(\mathbf{X}) = \mathbb{Z}L$; they are called $K3$ surfaces of general type. It is known from Mumford that any $K3$ surface carries rational curves ; a detailed proof of the fact that $K3$ surfaces of general type carry rational curves in any (positive) degree can be found in [Ch]. If I want to apply the previous techniques to reprove this result I am running into the following problem I don't know to answer:

I am starting with a $K3$ surface of general type of genus $g \geq 3$. The transverse hyperplane sections in $\mathbf{X}$ are smooth curves of genus g ; on the other hand, because $\mathbf{X}$ is of general type, any curve on $\mathbf{X}$ (with arithmetic genus g) is obtained as a hyperplane section. If $A \in H_2(\mathbf{X}, \mathbb{Z})$ denotes the class of the hyperplane section, the space $\overline{M}_g(\mathbf{X}, A)$ is g-dimensional. Now I am looking at the projection $\varpi : \overline{M}_g(\mathbf{X}, A) \to \overline{M}_g$ and I would like to know the dimension of its image, so I would need to estimate the dimension of the fibres. It is already a difficult task, but S.Mukai proves in [Mk] that for $g \geq 11$ and odd, if H and H' are two hyperplanes which intersect $\mathbf{X}$ transversally along $\mathbf{C}_H$ and $\mathbf{C}_{H'}$ respectively, these two curves are not isomorphic. This is a very strong result and shows that ϖ is in fact birational on the image. So, $\operatorname{Im}\varpi$ is a g-dimensional subvariety of $\overline{M}_g$ and therefore it intersects the boundary divisor $\Delta_g := \overline{M}_g - M_g$ along a $(g-1)$-dimensional subvariety. It can quite easily be shown that there is a Zariski open subset in $\operatorname{Im}\varpi \cap \Delta_g$ which represents curves whose singularity is an ordinary double point. These curves have geometric genus equal $(g-1)$ and therefore the space $\overline{M}_{g-1}(\mathbf{X}, A)$ is $(g-1)$-dimensional. I would need to know again the dimension of the general fibres of $\varpi : \overline{M}_{g-1}(\mathbf{X}, A) \to \overline{M}_{g-1}$; in fact, in order to be able to continue the inductive procedure until I find rational curves on $\mathbf{X}$, I would need again that ϖ is generically a finite map. This leads to the question I don't know the answer:

Question 3.11.2 *Suppose that* $\mathbf{D}$, $\mathbf{C}$ *are two projective curves and* $\mathbf{X}$ *is a* $K3$ *surface of general type. Is it possible to have a morphism*

$$f : \mathbf{D} \times \mathbf{C} \longrightarrow \mathbf{X}$$

with the following property: there is a Zariski open subset $\mathbf{D}^0 \subset \mathbf{D}$ *such that the maps* $f_d : \mathbf{C} \to \mathbf{X}$, $d \in \mathbf{D}^0$, *defined by* $f_d(p) := f(d,p)$ *for all* $p \in \mathbf{C}$ *are birational on the image?*

In order to have affirmative answer to this question it would be sufficient that for a morphism $u : \mathbf{C} \to \mathbf{X}$ which is birational on the image $H^0(\mathbf{C}, u^*T_{\mathbf{X}}) = 0$ (one may actually assume that u is, differential geometrically, an immersion).

(II) The Kähler case

Problems appear from the very beginning: in to order to see if the complex structure of some curves degenerate I must have some curve to start with; in fact, on a general Kähler manifold there are no curves at all and therefore all Gromov-Witten invariants must vanish, as they do not depend on the almost complex structure used to define them.

Donaldson's result says that there are almost complex structures J on $\mathbf{X}$ $\mathcal{C}^0$-close to the integrable one which are compatible with the Kähler form and having the property that exist J-holomorphic curves on $\mathbf{X}$; the problem is that in general the Nijenhuis tensor of such J's is big and I can not construct anymore the almost complex structure on the space of curves.

Actually it is hopeless to find any low "arithmetic genus" J-holomorphic curves on such a bad Kähler manifolds.

Therefore I shall assume that our Kähler manifold $(\mathbf{X}, \omega, J_0)$ carries holomorphic curves. I am again in the same situation as before: if bubbling doesn't occur, the compactification $\overline{N}_g(\mathbf{X}, A, J_0)$ carries a complex structure and the projection ϖ is holomorphic. Its image is a complex submanifold of $\overline{N}_g$ and therefore an algebraic subvariety: consequently I can check (theoretically) the inequality (♣) to see whether or not the complex structure of the curves degenerate.

(III) The symplectic case

Again, I would like to find out what is happening if J-holomorphic bubbles do not appear in the limit, to have criteria insuring that in the compactification $\overline{N}_g(\mathbf{X}, A, J)$ there are curves with degenerated conformal structure. This is equivalent asking whether the image of the continuous map

$$\varpi : \overline{N}_g(\mathbf{X}, A, J) \longrightarrow \overline{N}_g$$

intersects the boundary divisor of $\overline{N}_g$. In the complex analytic case I had an upper bound of the dimension of complete *algebraic* subvarieties of N_g; in the symplectic case ϖ is only a continuous map, so what I need is an upper bound for the degree of non-vanishing (rational) homology groups of N_g. Such a result exists indeed (see [Ha] and [HL], page 16):

Theorem (J.Harer) *M_g has the homotopy type of a complex of dimension $4g-5$ and moreover $H_{4g-5}(M_g, \mathbb{Q}) \neq 0$. In particular, $H_k(M_g, \mathbb{Q}) = 0$ for $k \geq 4g-4$.*

As it is pointed out in [HL] *loc. cit.*, the theorem holds also for finite Galois covers of M_g, in particular for the space N_g I am working with. Compared with the upper bound on the dimension of complete algebraic subvarieties of M_g, this result is much weaker and the computations done for the standard projective plane show that one can not expect ϖ to have such a high dimensional image, at least not in the projective case.

A second problem which arises is what kind of object is $\varpi(\overline{N}_g(\mathbf{X}, A, J)) \hookrightarrow \overline{N}_g$ itself: in order to apply a result like Harer's one, it should carry some rational homology class. In this case I get the following obvious:

Criterion 3.11.3 *If* $\operatorname{Im} \varpi \subset \overline{N}_g$ *carry a non-zero homology class in $H_\delta(\overline{N}_g)$ with $\delta \geq 4(g-1)$, then* $\operatorname{Im} \varpi$ *intersects the boundary divisor of $\overline{N}_g$.*

The existence of a homology class supported on the image of a map is not at all obvious. I shall study this problem in the case when J is generic and its Nijenhuis tensor is small. Certainly, such conditions on the almost complex structure are very restrictive; examples of manifolds which do admit such J's are projective (or Kähler) varieties with non-vanishing Gromov-Witten invariants. If pseudo-holomorphic bubbles do not appear in the limit, $\overline{N}_g(\mathbf{X}, A, J)$ is a smooth, compact manifold and moreover, according to theorem 3.9.9, it has a complex structure $\mathbb{J}$ on its tangent space such that the differential of the projection ϖ fails to be complex linear by an amount of $O(\|N_J\|)$. The genericity assumption on J implies that ϖ is a smooth map between smooth compact manifolds and the theorem 3.9.9(iii) says that every ϖ-general fibre is symplectic with symplectic two form $\iota^*\Omega$ and compatible almost complex structure $\mathbf{I}$.

Proposition 3.11.4 *If the almost complex structure J is generic and its Nijenhuis tensor is small enough, then* $\operatorname{Im} \varpi$ *carries a non-zero homology class of dimension*

$$2\delta := 6(g-1) - \dim Coker\, D_u$$

where $(u, j) \in \overline{N}_g(\mathbf{X}, A, J)$ *is any point where* $\mathrm{d}\varpi$ *has maximal rank.*

Proof Using the form Ω on $\overline{N}_g(\mathbf{X}, A, J)$ introduced in definition 3.9.4 it is easy to define now a homology class of expected dimension supported on the image of ϖ

$$[\mathrm{Im}\,\varpi] := \varpi_*([\overline{N}_g(\mathbf{X}, A, J)] \cap \Omega^r) \in H_{2\delta}(\mathrm{Im}\,\varpi)$$

where $2r :=$ dimension of a ϖ-general fibre $=\dim Ker\, D_u$. Taking the cap-product with Ω^r the fibres of ϖ are "killed" and therefore the dimension of the homology class $[\mathrm{Im}\,\varpi]$ is

$$2\delta = \dim N_g(\mathbf{X}, A, J) - 2r = 6(g-1) + 2n(1-g) + c_1(A) - \dim Ker\, D_u = 6(g-1) - \dim Coker\, D_u$$

It remains to prove that this homology class is non-zero: let $\overline{\omega}_{\mathrm{WP}}$ be the Kähler form of $\overline{N}_g$. It is a positive $(1,1)$-form with respect to the complex structure $J_{\overline{N}_g}$. For $k \geq 0$, the restriction of $\overline{\omega}^k_{\mathrm{WP}}$ to any $J_{\overline{N}_g}$-invariant $2k$-plane in $T_j\overline{N}_g,\ j \in \overline{N}_g$ is its (positive) volume form. I want to evaluate

$$\langle \overline{\omega}^\delta_{\mathrm{WP}}, [\mathrm{Im}\,\varpi] \rangle = \langle \varpi^* \overline{\omega}^\delta_{\mathrm{WP}}, [\mathbf{X}] \cap \Omega^r \rangle = \langle \varpi^* \overline{\omega}^\delta_{\mathrm{WP}} \cup \Omega^r, [\mathbf{X}] \rangle$$

I claim that this integral is strictly positive: indeed, at each point $(u, j) \in \overline{N}_g(\mathbf{X}, A, J)$ there is the orthogonal decomposition

$$T_{(u,j)}\overline{N}_g(\mathbf{X}, A, J) = Ker\, \mathrm{d}\varpi_{(u,j)} \oplus (Ker\, \mathrm{d}\varpi_{(u,j)})^{\perp_G}$$

where I consider the orthogonal with respect to the metric G introduced in section 3.9 which is compatible with the almost complex structure $\mathbf{I}$. Both of the direct summands are $\mathbb{J}$-invariants; furthermore, because $\mathrm{d}\varpi$ fails to be complex linear by an amount of $O(\|N_J\|)$, I deduce that if N_J is small enough, $\varpi^*\omega_{\mathrm{WP}}$ is non-degenerate on $(Ker\, \mathrm{d}\varpi_{(u,j)})^{\perp_G}$. Because $\varpi^*\omega_{\mathrm{WP}}$ vanishes on $Ker\, \mathrm{d}\varpi$ and Ω is non-degenerate on $Ker\, \mathrm{d}\varpi$, the evaluation is positive. □

An immediate consequence of this proposition is the following

Corollary 3.11.5 *Under the same assumptions on* J *as before, if*

$$\dim Coker\, D_u \leq 4(g-1)$$

then in the compactification $\overline{N}_g(\mathbf{X}, A, J)$ *must appear* J*-holomorphic curves whose domain have geometric genus strictly smaller than* g*.*

Proof Assume the contrary: then $\langle \omega^\delta_{\mathrm{WP}}, [Im\,\varpi] \rangle > 0$ with $\delta \geq g-1$. On the other hand, by the result of E.Looienga in [L2], we know that $[\omega^{d-1}_{\mathrm{WP}}] = 0$ in $H^{2(d-1)}(M_g)$. The contradiction proves the corollary. □

The problem is that *à priori* it is rather difficult to explicitly compute the dimension of the cokernel of D_u. The way out is the Riemann-Roch formula

$$\dim_{\mathbb{R}} Ker\ D_u - \dim_{\mathbb{R}} Coker\ D_u = 2n(1-g) + 2c_1(A)$$

and the following criterion, analogous to proposition 3.2.1:

Proposition 3.11.6 *Let $(\mathbf{X}^{2n}, \omega, J)$ be a symplectic manifold whose almost complex structure is ω-good and has small Nijenhuis tensor. Assume that in the compactification $\overline{N}_g(\mathbf{X}, A, J)$ do not appear bubbles in the limit. If $\mathbf{F} = \varpi^{-1}(j)$ is any ϖ-general fibre, then*

$$\dim_{\mathbb{R}} Ker\ D_u \leq 2n \qquad \forall u \in \mathbf{F}$$

Proof It goes along the very same line as that of proposition 3.2.1: I assume that the dimension of the kernel is at least $2(n+1)$ and no bubbles appear in the compactification. Let me consider the map

$$ev : \mathbf{F} \times \mathbf{C}_j \longrightarrow \mathbf{X}$$

and, for $q_0 \in \mathbf{C}_j$, take $ev_{q_0} : \mathbf{F} \to \mathbf{X}$ be the evaluation at the point q_0. It is a smooth map between smooth manifolds, so it has general fibres (see definition 3.9.3) whose dimension is at least $2(n+1) - 2n = 2$; let $\mathbf{F}_{q_0 \to x_0}$, $x_0 \in \mathbf{X}$ be one of them. The hypothesis on J to be ω-good tells us, cf. theorem 3.9.9, that $(\mathbf{F}, \Omega, \mathbf{I})$ is a symplectic manifold. The tangent space

$$T_u \mathbf{F}_{q_0 \to x_0} = \{\xi \in T_u \mathbf{F} \mid \xi(q_0) = 0\}$$

can be seen to be $\mathbf{I}$-invariant (because for $u \in \mathbf{F}$, $\mathbf{I}_u \in \Gamma(\mathbf{C}, End(u^*T\mathbf{X}))$) and therefore $(\mathbf{F}_{q_0 \to x_0}, \Omega, \mathbf{I})$ is a symplectic submanifold of $(\mathbf{F}, \Omega, \mathbf{I})$. Donaldson's result says that there are almost $\mathbf{I}$-holomorphic curves on it whose tangent space are as close being $\mathbf{I}$-invariant as wanted ; let $\mathbf{D} \hookrightarrow \mathbf{F}_{q_0 \to x_0}$ be a curve such that $\langle \Omega, \mathbf{D} \rangle > 0$. The evaluation map

$$ev : \mathbf{D} \times \mathbf{C}_j \to \mathbf{X}$$

has the property that $ev(u, q_0) = x_0$ for all points $u \in \mathbf{D}$ and consequently

$$\langle ev_{q_0}^* \omega, [\mathbf{D} \times \{q_0\}] \rangle = 0.$$

On the other hand, because $(ev^*\omega \wedge \gamma_j)_{(u,j)} = (ev_q^*\omega)_u \wedge (\gamma_j)_q$

$$\int_{\mathbf{C}_j} \left(\int_{\mathbf{D}} ev_q^* \omega \right) d\gamma_j(q) = \int_{\mathbf{D} \times \mathbf{C}_j} ev^* \omega \wedge \gamma_j = \int_{\mathbf{D}} (ev^* \omega \wedge \gamma_j)/[\mathbf{C}_j] = \int_{\mathbf{D}} \Omega > 0$$

which implies the existence of points $q \in \mathbf{C}_j$ having the property that

$$\langle ev_q^* \omega, [\mathbf{D} \times \{q\}] \rangle > 0$$

As $[\mathbf{D} \times \{q_0\}]$ and $[\mathbf{D} \times \{q\}]$ are homotopic, I get a contradiction which proves that the kernel of D_u is at most $2n$-dimensional. □

Assembling the previous two results we get:

Proposition 3.11.7 *If J is generic with small Nijenhuis tensor and*

$$c_1(A) \geq (n-2)g + 2$$

then in the compactification $\overline{N}_g(\mathbf{X}, A, J)$ there are either J-holomorphic bubbles or curves with strictly smaller geometric genus.

Remark 3.11.8 For proving these results I used both the genericity of J and the smallness of $\|N_j\|$. It might be possible to avoid the genericity assumption on J using the techniques introduced by B.Siebert in his paper [Si] and working in the total space of a finite rank bundle over the space of curves. The very restrictive condition in my approach is that on the Nijenhuis tensor which makes sure that on $\overline{N}_g(\mathbf{X}, A, J)$ there are some good structures (almost complex structure, symplectic structure on the fibres of ϖ). It might be possible that such nice structures exist for a much wider class of symplectic manifolds but the big problem is how could they be "detected". $\diamond$

Part IV

GIT on $\overline{M}_{g,k}(\mathbf{X}, A)$

4.1 The setup

In this part I want to study the following problem: assume that a connected reductive linear algebraic group G acts on an irreducible projective algebraic variety $\mathbf{X}$ *via* a linearization on the very ample line bundle $\mathcal{O}_{\mathbf{X}}(1) \to \mathbf{X}$. Then there is a canonically induced G-action on the moduli space $\overline{M}_{g,k}(\mathbf{X}, A)$ and one is tempted to make its geometric invariant theoretic (GIT) quotient. On the other hand, the homology class $A \in H_2(\mathbf{X}, \mathbb{Z})$, if it is "nice" enough in a sense that will be made precise, induces a homology class $\widehat{A} \in H_2(\widehat{\mathbf{X}}, \mathbb{Z})$, where $\widehat{\mathbf{X}}$ denotes the GIT quotient of $\mathbf{X}$ by G, so one may talk about the space $\overline{M}_{g,k}(\widehat{\mathbf{X}}, \widehat{A})$. The natural question which raises is whether there is any link between these two objects. The motivation for this study is to answer (at least partially) the question whether the Gromov-Witten invariants of $\mathbf{X}$ and $\widehat{\mathbf{X}}$ are related or not. The main result I obtain in this direction is theorem 4.8.3 which states that it exists a relation between the *rational* Gromov-Witten invariants of $\mathbf{X}$ and $\widehat{\mathbf{X}}$, under the strong assumption that they are both flag varieties.

The structure of the study is as follows: in sections 4.2-4.4, I briefly recall the definition and the construction of the moduli space of stable maps $\overline{M}_{g,k}(\mathbf{X}, A)$. The reason for doing it is that I need in my computations the description of the projective embedding of $\overline{M}_{g,k}(\mathbf{X}, A)$. Section 4.5 presents some basic notions of geometric invariant theory (GIT) I need. My study starts actually in section 4.6, where I give an algebraic criterion (proposition 4.6.1) for deciding when a stable map $[(\mathbf{C}, \mathbf{x}, u)]$ defines a G-semi-stable point in the moduli space $\overline{M}_{g,k}(\mathbf{X}, A)$. The rest of the section is devoted to find necessary conditions on one hand and sufficient conditions on the other hand for this criterion to be satisfied. An important result is theorem 4.6.7 which asserts that if the image of a stable map $[(\mathbf{C}, \mathbf{x}, u)]$ is contained in the G-semi-stable locus of the projective variety $\mathbf{X}$, then it defines a G-semi-stable point in $\overline{M}_{g,k}(\mathbf{X}, A)$. In section 4.7 I test the semi-stability criterion 4.6.7 and I study the relation between the moduli spaces $\overline{M}_{g,k}(\mathbf{X}, A)$ and $\overline{M}_{g,k}(\widehat{\mathbf{X}}, \widehat{A})$ in a concrete example. The conclusions are presented in section 4.8 where I state and prove theorem 4.8.3 which relates the rational GW-invariants of $\mathbf{X}$ and $\widehat{\mathbf{X}}$ in the case when both these varieties are flag manifolds. Finally, in section 4.8 I present a symplectic approach to the problem. The motivation comes from the aim to get a better understanding of the algebraic conditions in criterion 4.6.1. Using the differential version of the Atiyah-Singer index theorem in the relative case, proved by J.M.Bismut, H.Gillet, D.Freed and Ch.Soulé, I explicitely compute the moment map for the $\mathbb{C}^*$-action on a certain space $M_{\mathbf{C},k}(\mathbf{X}, A)$ defined in this section, induced by the $\mathbb{C}^*$-section on $\mathbf{X}$. As a consequence I obtain a geometric, transparent proof of theorem 4.6.7.

4.2 Reminders about the Hilbert schemes

All the results in this section are due to A. Grothendieck and can be found in the article [Gk]. The starting data are a complex projective variety $\mathcal{O}_{\mathbf{X}}(1) \to \mathbf{X}$ and a polynomial P with rational coefficients. For any complex scheme S denote:

$$\underline{Quot}^P_{\mathbf{X}}(S) := \left\{ \begin{array}{c} \text{coherent quotients } \mathcal{Q} \text{ of } \mathcal{O}_{S\times\mathbf{X}}, \text{ flat}/S \\ \text{such that } \forall s \in S,\ \chi(\mathcal{Q}_s(\nu)) = P(\nu)\ \forall \nu \geq 0 \end{array} \right\} \Big/ \ \mathtt{iso}$$

Two quotients $\mathcal{Q}$ and $\mathcal{Q}'$ of $\mathcal{O}_{S\times\mathbf{X}}$ are equivalent if there is an isomorphism $\mathtt{g} : \mathcal{Q} \to \mathcal{Q}'$ making the diagram

$$\begin{array}{ccccc} \mathcal{O}_{S\times\mathbf{X}} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ \| \,\mathrm{id} & & \downarrow \mathrm{g} & & \\ \mathcal{O}_{S\times\mathbf{X}} & \longrightarrow & \mathcal{Q}' & \longrightarrow & 0 \end{array}$$

commutative. In [Gk] a general existence theorem of Quot-schemes has been stated but I shall state below only the case I am interested in:

Theorem 4.2.1 (A.Grothendieck) *The functor:*

$$\underline{Quot}^P_{\mathbf{X}} : \underline{\texttt{Schemes}} \longrightarrow \underline{\texttt{Sets}}$$

$$S \mapsto \underline{Quot}^P_{\mathbf{X}}(S)$$

is represented by a projective scheme $\boldsymbol{\mathcal{H}} := \mathcal{H}ilb^P_{\mathbf{X}}$, the Hilbert scheme of closed subschemes of $\mathbf{X}$ whose Hilbert polynomial equals P.

This means that there is a coherent sheaf $\mathscr{Q} \to \boldsymbol{\mathcal{H}} \times \mathbf{X}$ flat over $\boldsymbol{\mathcal{H}}$ and a surjective homomorphism $\rho : \mathcal{O}_{\boldsymbol{\mathcal{H}}\times\mathbf{X}} \to \mathscr{Q}$ satisfying the following universality property: for any scheme S and any quotient $\mathcal{O}_{S\times\mathbf{X}} \to \mathcal{Q} \to 0$, flat over S, there is a unique map $f : S \to \boldsymbol{\mathcal{H}}$ such that $\mathcal{Q}$ is equivalent with $(f, \mathrm{id}_{\mathbf{X}})^* \mathscr{Q}$.

Notice that $\mathscr{Q}$ being a coherent quotient of $\mathcal{O}_{\boldsymbol{\mathcal{H}}\times\mathbf{X}}$ defines a closed subscheme $\boldsymbol{\mathcal{Z}}$ of $\boldsymbol{\mathcal{H}} \times \mathbf{X}$ and the natural projection $\boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{H}}$ is flat and has the property that for any geometric point $h \in \boldsymbol{\mathcal{H}}$ the fibre $\boldsymbol{\mathcal{Z}}_h$ is a closed subscheme of $\mathbf{X}$ whose Hilbert polynomial is P.

In order to prove that $\boldsymbol{\mathcal{H}}$ is projective, Grothendieck uses the following:

Result 4.2.2 *There is an integer $l > 0$ such that for all coherent quotients*

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbf{X}} \longrightarrow \mathcal{Q} \longrightarrow 0$$

with Hilbert polynomial P the following three conditions hold:

(a) $H^i(\mathbf{X}, \mathcal{Q}(n)) = 0 \quad \forall n \geq l \quad \forall i > 0$;

(b) $H^i(\mathbf{X}, \mathcal{I}(n)) = 0 \quad \forall n \geq l \quad \forall i > 0$;

(c) $H^0(\mathbf{X}, \mathcal{I}(l+k)) = H^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(k)) \otimes H^0(\mathbf{X}, \mathcal{I}(l)) \quad \forall k \geq 0.$

The condition (a) implies that $\dim H^0(\mathbf{X}, \mathcal{Q}(n)) = P(n)$ for all $n \geq l$ and the condition (b) says that there is an exact sequence

$$0 \longrightarrow H^0(\mathbf{X}, \mathcal{I}(n)) \longrightarrow H^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(n)) \longrightarrow H^0(\mathbf{X}, \mathcal{Q}(n)) \longrightarrow 0$$

for $n \geq l$. The projective embedding of $\boldsymbol{\mathcal{H}}$ is obtained as follows:

$$\boldsymbol{\mathcal{H}} \longrightarrow Gr_{P(l)}\left(H^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(l))^{\vee}\right)$$

$$h \mapsto H^0(\mathbf{X}, \mathscr{Q}_h(l))^{\vee}$$

Lemma 4.2.3 *The map above is injective.*

Proof If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two quotients of $\mathcal{O}_{\mathbf{X}}$ such that $H^0(\mathbf{X}, \mathcal{Q}_1(l))^\vee = H^0(\mathbf{X}, \mathcal{Q}_2(l))^\vee$ then from the equalities

$$H^0(\mathbf{X}, \mathcal{I}_{1,2}(l)) = \bigcap_{\beta \in H^0(\mathbf{X}, \mathcal{Q}_{1,2}(l))^\vee} Ker\, \beta$$

it follows that $H^0(\mathbf{X}, \mathcal{I}_1(l)) = H^0(\mathbf{X}, \mathcal{I}_2(l)) =: V$. The condition (c) together with Serre's theorem guarantees that for some $k \gg 0$, the homomorphism

$$V \otimes \mathcal{O}_{\mathbf{X}}(k) \longrightarrow \mathcal{I}_{1,2}(l+k)$$

is surjective and therefore $\mathcal{I}_1 = \mathcal{I}_2$ because $\mathcal{I}_{1,2} = \mathrm{Im}\,((V \otimes \mathcal{O}_{\mathbf{X}}(k)) \otimes \mathcal{O}_{\mathbf{X}}(-l-k) \longrightarrow \mathcal{O}_{\mathbf{X}})$. □

I want to describe also the ample line bundle on the Grassmannian

$$\mathfrak{Gr} := Gr_{P(l)}\left(H^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(l))^\vee\right)$$

Let $\mathcal{E} \subset H^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(l))^\vee$ be a $P(l)$-dimensional linear subspace. Then

$$K_{\mathcal{E}} := \bigcap_{\beta \in \mathcal{E}} Ker\, \beta \subset H^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(l))$$

is a $P(l)$-codimensional linear subspace. The fibre of the universal rank $P(l)$ quotient bundle $\mathfrak{Q} \to \mathfrak{Gr}$ is:

$$\mathfrak{Q}_{\mathcal{E}} = \frac{H^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(l))}{K_{\mathcal{E}}}$$

The ample line bundle is $\det \mathfrak{Q} \to \mathfrak{Gr}$.

4.3 The variety $\overline{M}_{g,k}(\mathbf{X}, A)$

The reference for this section is the article [FP]. Let me recall that $\mathbf{X}$ is an irreducible, reduced complex projective variety and $A \in H_2(\mathbf{X}, \mathbb{Z})$ is a fixed homology class.

I shall give the definition of $\overline{M}_{g,k}(\mathbf{X}, A)$ in the algebraic setting. Let start with:

Definition 4.3.1 *For a complex algebraic scheme S, a family of k-pointed, genus g stable maps consists of the data $(\mathcal{C}, \mathbf{x}, \pi, S, u)$ where:*

(i) $\pi : \mathcal{C} \to S$ is a flat morphism and $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_k) : S \to \mathcal{C}$ is a family of k sections ;

(ii) $u : \mathcal{C} \to \mathbf{X}$ is a morphism such that for all geometric points $s \in S$ its restriction $u_s : (\mathcal{C}_s, \mathbf{x}_s) \to \mathbf{X}$ is a stable map (see definition 3.10.1).

Two such families $(\mathcal{C}_1,, \mathbf{x}_{1,2}, \pi_{1,2}, S, u_{1,2})$ are isomorphic if there is an isomorphism $\tau : \mathcal{C}_2 \to \mathcal{C}_1$ such that: $u_2 = u_1 \circ \tau$ and $\mathbf{x}_1 = \tau \circ \mathbf{x}_2$.

Define the contravariant functor:

$$\overline{\mathcal{M}}_{g,k}(\mathbf{X}, A) : \underline{\texttt{Schemes}} \longrightarrow \underline{\texttt{Sets}}$$

$$S \mapsto \left\{ \begin{array}{c} \text{families of } k\text{-pointed, genus } g \text{ stable maps}/S \\ \text{such that } u_{s*}[\mathcal{C}_s] = A \quad \forall s \in S \end{array} \right\} \Big/ \texttt{iso}$$

It is proved in the given reference that:

Theorem 4.3.2 *There exists a projective, coarse moduli space $\overline{M}_{g,k}(\mathbf{X}, A)$ coarsely representing the functor $\overline{\mathcal{M}}_{g,k}(\mathbf{X}, A)$.*

This means that the geometric points of $\overline{M}_{g,k}(\mathbf{X}, A)$ parameterize the *equivalence classes* of genus g, k pointed stable maps representing the homology class $A \in H_2(\mathbf{X}, \mathbb{Z})$.

The very ample line bundle $\mathcal{O}_{\mathbf{X}}(1) \to \mathbf{X}$ defines a projective embedding:

$$\begin{array}{ccc} \mathcal{O}_{\mathbf{X}}(1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^r}(1) \\ \downarrow & & \downarrow \\ \mathbf{X} & \longrightarrow & \mathbb{P}^r \end{array}$$

The image of A under the canonical map $H_2(\mathbf{X}, \mathbb{Z}) \to H_2(\mathbb{P}^r, \mathbb{Z})$ will be $d[L]$, where $[L]$ represents the class of a line in the projective space. Therefore $\overline{M}_{g,k}(\mathbf{X}, A)$ is a closed subscheme of $\overline{M}_{g,k}(\mathbb{P}^r, d)$. What I want to present below in more detail is the description of its ample line bundle since I need this later on.

Lemma 4.3.3 *A map $u : (\mathbf{C}, \mathbf{x}) \to \mathbb{P}^r$ is stable if and only if*

$$L_{\mathbf{C}} := \omega_{\mathbf{C}}(x_1 + \ldots + x_k) \otimes u^*\mathcal{O}_{\mathbb{P}^r}(3)$$

is ample.

Proof $L_{\mathbf{C}}$ is ample if and only if its restiction to each irreducible component of $\mathbf{C}$ is ample i.e. it has positive degree. If u_α denotes the restriction of u to the $\mathbf{C}_\alpha$ component of $\mathbf{C}$, then

$$L_{\mathbf{C}|\mathbf{C}_\alpha} = \omega_{\mathbf{C}_\alpha}(\text{contact points} + \text{marked points}) \otimes u_\alpha^*\mathcal{O}_{\mathbb{P}^r}(3).$$

It is immediate that the degree of the righthandside is positive if and only if the stability condition is satisfied. $\square$

Lemma 4.3.4 *There is a number $f = f(g, k, r, d) > 0$ such that for all genus g, k pointed stable map $(\mathbf{C}, \mathbf{x}, u)$ representing the class $d \cdot [L] \in H_2(\mathbb{P}^r, \mathbb{Z})$ the line bundle $L_{\mathbf{C}}^{\otimes f}$ is very ample and $H^1(\mathbf{C}, L_{\mathbf{C}}^{\otimes f}) = 0$.*

Proof the result can be deduced from the following facts:

- if $L_\alpha \to \mathbf{C}_\alpha$ is a line bundle whose degree is $\deg L_\alpha \geq 2g(\mathbf{C}_\alpha) + 1$, then it is very ample;
- there are only finitely many types of graphs associated to the stable maps as in the lemma.

Consider $f = f(g, k, r, d)$ such that the resticton $L_\alpha^{\otimes f} := L_{\mathbf{C}|\mathbf{C}_\alpha}^{\otimes f}$ to the component $\mathbf{C}_\alpha$ of $\mathbf{C}$ has $\deg L_\alpha^f \geq 2g(\mathbf{C}) + 1$. Notice that in this case $H^1(\mathbf{C}, L_{\mathbf{C}}^{\otimes f}) = 0$ already. The line bundle $L_{\mathbf{C}}^{\otimes f} \to \mathbf{C}$ is very ample if and only if its global sections separate the points and the directions. For the moment only the restictions $L_\alpha^{\otimes f} \to \mathbf{C}_\alpha$ are known to be very ample, so points and directions can be separated for each irreducible component. There are still two problems: one needs to glue the sections over the $\mathbf{C}_\alpha$'s into global sections over $\mathbf{C}$ and furthermore these sections should separate the points lying on different irreducible components. These things can be done if one can impose prescribed values for the (partial) sections over the $\mathbf{C}_\alpha$'s at the contact points with the other irreducible components. However this requires a sufficiently high degree for the restrictions $L_\alpha \to \mathbf{C}_\alpha$. One can obtain such a high degree raising the value of f. That one may chose a f which depends only on (g, k, r, d) comes from the fact that only finitely many types of graphes associated to such stable maps occur. $\square$

I shall fix once for all such a f. Then the Riemann-Roch formula says that

$$h^0(\mathbf{C}, L_{\mathbf{C}}^{\otimes f}) = f(2g-2+k+3d) - g + 1 =: N$$

which can be seen to be independent of $(\mathbf{C}, \mathbf{x}, u)$. The same is true for the degree $\deg L_{\mathbf{C}}^{\otimes f} = f(2g-2+k+3d) =: e$. The linear system associated to $L_{\mathbf{C}}^{\otimes f}$ together with the map u defines an embedding

$$u_{\mathbf{Y}} : \mathbf{C} \longrightarrow \mathbb{P}\left(H^0(\mathbf{C}, L_{\mathbf{C}}^{\otimes f})^\vee\right) \times \mathbb{P}^r \cong \mathbb{P}^{N-1} \times \mathbb{P}^r =: \mathbf{Y}$$

whose image has bidegree (e, d). Notice that this embedding records both the curve $\mathbf{C}$ and the morphism u ; more precisely, $\mathbf{C}$ is determined up to a choice of an isomorphism between $\mathbb{P}\left(H^0(\mathbf{C}, L_{\mathbf{C}}^{\otimes f})^\vee\right)$ and $\mathbb{P}^{N-1}$ i.e. up to a $PGl(N)$-action.

The restriction to $\mathbf{C}$ of the very ample line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{P}^{N-1}}(1) \boxtimes \mathcal{O}_{\mathbb{P}^r}(1) \to \mathbf{Y}$ is:

$$\mathcal{L}_{|\mathbf{C}} = \omega_{\mathbf{C}}(x_1 + \ldots + x_k)^{\otimes f} \otimes u^* \mathcal{O}_{\mathbb{P}^r}(3f+1)$$

The Hilbert polynomial of $\mathbf{C}$, viewed as a subvariety of $\mathbf{Y}$, is

$$\chi(\mathcal{O}_{\mathbf{C}}(n)) = n\langle c_1(\mathcal{L}), [\mathbf{C}]\rangle + \chi(\mathbf{C}) = n(e+d) + 1 - g = P(n)$$

which is again independent of the stable curve $(\mathbf{C}, \mathbf{x}, u)$. If $\boldsymbol{\mathcal{H}} := \mathcal{H}ilb_{\mathbf{Y}}^P$ denotes the Hilbert scheme of subschemes of $\mathbf{Y}$ whose Hilbert polynomial is P, then to each stable map $(\mathbf{C}, \mathbf{x}, u)$ one can associate a point in $\boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$ as follows

$$(\mathbf{C}, \mathbf{x}, u) \mapsto (u_{\mathbf{Y}*}\mathbf{C}, u_{\mathbf{Y}}(x_1), \ldots, u_{\mathbf{Y}}(x_k))$$

The natural $PGl(N)$ action on $\mathbb{P}^{N-1}$ induces an action on $\boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$ and it can be seen that two maps $(\mathbf{C}_{1,2}, \mathbf{x}_{1,2}, u_{1,2})$ are isomorphic if and only if they are in the same $PGl(N)$ orbit. The stability condition can be translated into the fact that the stabilizer of any stable map under this action is finite.

Recall that $\boldsymbol{\mathcal{H}}$ being a *fine* moduli space for the corresponding functor, there is the following diagram

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \boldsymbol{\mathcal{H}} \times \mathbf{Y} \\ \text{flat} \downarrow & & \\ \boldsymbol{\mathcal{H}} & & \end{array}$$

such that for all geometric points $h \in \boldsymbol{\mathcal{H}}$, $\mathcal{C}_h$ represents the point h. Notice further that because $\deg P = 1$, the fibres $\mathcal{C}_h$ are all 1-dimensional subschemes of $\mathbf{Y}$ but they are not necessarily connected or reduced. Let $\mathscr{I} \subset \boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$ be the closed incidence subscheme defined by

$$\mathscr{I} := \{(h, x_1, \ldots, x_k) |\ x_1, \ldots, x_k \in \mathcal{C}_h\}$$

The k marked points give rise to k sections:

$$\mathbf{x}_1, \ldots, \mathbf{x}_k : \mathscr{I} \longrightarrow \mathcal{C}_{\mathscr{I}} := \mathscr{I} \times_{\boldsymbol{\mathcal{H}}} \mathcal{C}.$$

Using the projections:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \boldsymbol{\mathcal{H}} \times \mathbf{Y} \\ & p_{\mathbb{P}^r} \swarrow & \searrow p_{\mathbf{Y}} \\ & \mathbb{P}^r & \mathbf{Y} \end{array}$$

one can construct two line bundles on $\mathcal{C}_{\mathscr{I}}$:

$$L_1 := p_{\mathbf{Y}}^* \mathcal{L} \quad \text{and} \quad L_2 := \omega_{\mathcal{C}_{\mathscr{I}}/\mathscr{I}}(\mathbf{x}_1 + \ldots + \mathbf{x}_k)^{\otimes f} \otimes p_{\mathbb{P}^r}^* \mathcal{O}_{\mathbb{P}^r}(3f + 1).$$

There is an open subscheme $\mathscr{U} \subset \mathscr{I}$ defined by the properties:

(i) $\mathcal{C}_h$ is connected, reduced with at most ordinary double points (i.e. $\mathcal{C}_h$ is quasi-stable);

(ii) the projection $\mathcal{C}_h \to \mathbb{P}^{N-1}$ is an embedding;

(iii) the markings lie in the non-singular locus of $\mathcal{C}_h$;

(iv) the degrees of $L_{1|\mathcal{C}_h}$ and of $L_{2|\mathcal{C}_h}$ agree on each irreducible component of $\mathcal{C}_h$.

The theorem of the cube of Mumford says that there is a "maximal" closed subscheme $\mathcal{S}$ of $\mathscr{U}$ over which these two line bundles agree; $\mathcal{S}$ corresponds to the locus of stable maps.

One of the main results in [FP] is the following:

Theorem 4.3.5

$$\overline{M}_{g,k}(\mathbb{P}^r, d) = \mathcal{S}/PGl(N)$$

is a separated and proper scheme, projective over $\mathbb{C}$.

The problem with these moduli spaces is that they can be very singular and even non-reduced at every point. There is however a class of varieties, the *flag varieties*, for which the space of *rational* stable maps is quite nice. A flag variety is, by definition, the quotient space G/P, where G is a connected linear algebraic group and P is a parabolic subgroup of it. The result I am interested in is:

Theorem 4.3.6 *If* $\mathbf{X}$ *is a flag variety, the moduli spaces* $\overline{M}_{0,k}(\mathbf{X}, A)$ *are normal and their singularities are orbifold-like. Moreover, all these varieties are irreducible.*

Proof The proof for first part of the the theorem is given in the same paper [FP]. The irreducibility is established in the article [Th].

4.4 The projective embedding of $\overline{M}_{g,k}(\mathbb{P}^r, d)$

First I shall give the projective embedding of $\boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$. Theorem 4.2.1 says that $\boldsymbol{\mathcal{H}}$ is projective with an embedding given by

$$\boldsymbol{\mathcal{H}} \longrightarrow Gr_{P(l)}\left(H^0(\mathbf{Y}, \mathcal{L}^l)^{\vee}\right)$$

$$\mathbf{C} \mapsto H^0(\mathbf{Y}, \mathcal{O}_{\mathbf{C}} \otimes \mathcal{L}^l)^{\vee}$$

for some (big) $l > 0$. Notice also that

$$\underbrace{\mathcal{L}\boxtimes\cdots\boxtimes\mathcal{L}}_{k\ \text{times}}\longrightarrow \mathbf{Y}^k$$

is very ample. With these in mind, the projective embedding of $\boldsymbol{\mathcal{H}}\times\mathbf{Y}^k$ is constructed as follows: given a point $(\mathbf{C}, x_1, \ldots, x_k) \in \boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$ there is a surjection

$$\mathscr{W} := H^0(\mathbf{Y}, \mathcal{L}^l)^{(0)} \oplus \bigoplus_{j=1}^{k} H^0(\mathbf{Y}, \mathcal{L}^l)^{(j)} \longrightarrow H^0(\mathbf{Y}, \mathcal{O}_{\mathbf{C}} \otimes \mathcal{L}^l) \oplus \bigoplus_{j=1}^{k} \mathcal{L}^l_{x_j} \longrightarrow 0$$

and

$$\dim H^0(\mathbf{Y}, \mathcal{O}_{\mathbf{C}} \otimes \mathcal{L}^l) \oplus \bigoplus_{j=1}^{k} \mathcal{L}^l_{x_j} = P(l) + k =: q + k$$

Let me emphasize that I have used upper indices for the same vector space $H^0(\mathbf{Y}, \mathcal{L}^l)$ because they will play different roles in what follows.

The projective embedding of $\boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$ in a projective space is given by:

$$\begin{array}{ccccc} \boldsymbol{\mathcal{H}} \times \mathbf{Y}^k \longrightarrow & & Gr_{q+k}(\mathscr{W}^\vee) & \longrightarrow & \mathbb{P}\left(\bigwedge^{q+k} \mathscr{W}^\vee\right) \\ (\mathbf{C}, \mathbf{x}) & \mapsto & \left(H^0(\mathbf{Y}, \mathcal{O}_{\mathbf{C}} \otimes \mathcal{L}^l) \oplus \bigoplus_{j=1}^{k} \mathcal{L}^l_{x_j}\right)^\vee & \mapsto & \det\left(H^0(\mathbf{Y}, \mathcal{O}_{\mathbf{C}} \otimes \mathcal{L}^l) \oplus \bigoplus_{j=1}^{k} \mathcal{L}^l_{x_j}\right)^\vee \end{array}$$

The ample line bundle over $\boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$ is $\det \mathfrak{Q}_k := \det\left(\mathfrak{Q} \boxplus (\mathcal{L}^l)^{\boxplus k}\right)$, where $\mathfrak{Q}$ is the universal quotient bundle described in the section 4.2. In this case the fibre is

$$\mathfrak{Q}_{k_{(\mathbf{C},\mathbf{x})}} = \left(\mathfrak{Q} \boxplus (\mathcal{L}^l)^{\boxplus k}\right)_{(\mathbf{C},\mathbf{x})} = H^0(\mathbf{Y}, \mathcal{O}_{\mathbf{C}} \otimes \mathcal{L}^l) \oplus \bigoplus_{j=1}^{k} \mathcal{L}^l_{x_j}$$

The bundle $\mathfrak{Q}_k$ and consequently $\det \mathfrak{Q}_k$ is invariant under the $PGl(N)$-action. For this reason it descends to a line bundle over $\overline{M}_{g,k}(\mathbf{X}, A) = \mathcal{S}/PGl(N)$ where $\mathcal{S} \subset \boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$ (see theorem 4.3.5). Using the Nakai-Moishezon criterion, it is proved in [FP] that this quotient line bundle remains ample.

4.5 Notions of geometric invariant theory

In this section I want to recall some very basic definitions and results in GIT; all of them can be found in the reference [MFK]. The goal of the invariant theory is to study the existence of quotients $\mathbf{X}/G$, where $\mathbf{X}$ is an algebraic scheme and G is a group acting on it. Mumford's results say that there are "nice" quotients in the case when the group is linearly reductive.

Definition 4.5.1 (theorem) *Let G be a complex, linear algebraic group. It is called* linearly reductive *if equivalently:*

(i) any finite dimensional representation of G is completely reducible;

(ii) its unipotent radical is trivial: this is defined to be the largest connected normal unipotent subgroup of G;

(iii) G is the complexification of its maximal compact subgroup.

Definition 4.5.2 ([MFK], page 30) *Let $\mathbf{X}, \mathcal{L}, G$ be a prescheme/$\mathbb{C}$, an invertible sheaf over $\mathbf{X}$ and a linear algebraic group acting with an action σ on $\mathbf{X}$ respectively. A G-linearization of $\mathcal{L}$ is by definition an isomorphism φ such that:*

$$\begin{array}{ccccc} & \sigma^*\mathcal{L} \xrightarrow{\varphi} p_{\mathbf{X}}^*\mathcal{L} & & \mathcal{L} \\ & \searrow \quad \swarrow & & \downarrow \\ & G \times \mathbf{X} & \xrightarrow{\sigma} & \mathbf{X} \\ & \downarrow p_{\mathbf{X}} & & \\ \mathcal{L} \longrightarrow & \mathbf{X} & & \end{array}$$

and satisfying the usual condition for a group action.

In more down-to-earth terms, a G-linearization of $\mathcal{L}$ is a commutative diagram

$$\begin{array}{ccc} G \times \mathcal{L} & \xrightarrow{\varphi} & \mathcal{L} \\ \downarrow & & \downarrow \\ G \times \mathbf{X} & \xrightarrow{\sigma} & \mathbf{X} \end{array}$$

such that for all $g \in G$ and $x \in \mathbf{X}$,

$$\varphi_g : \mathcal{L}_x \longrightarrow \mathcal{L}_{\sigma(g,x)}$$

is an isomorphism. The definition above simply says that the linear maps φ_g (satisfying the condition $\varphi_{gg'} = \varphi_g \varphi_{g'}$) fit together in an algebraic fashion as $g \in G$ varies.

Example 4.5.3 An important example when such a situation occurs is the following: assume that a G is a complex linear algebraic group acting linearly on $\mathbb{C}^{r+1}$ and $\mathbf{X}$ is a quasi-projective variety inside $\mathbb{P}^r$. The action of G descends to an action on $\mathbb{P}^r$; assume that $\mathbf{X}$ is invariant under it. Because the fibre $\mathcal{O}_{\mathbb{P}^r}(1)_{[z]}$ is equal to $\mathrm{Hom}\,(\mathbb{C}z, \mathbb{C})$, G has a linearization in $\mathcal{O}_{\mathbb{P}^r}(1)$ defined by

$$\langle g \cdot \beta, v \rangle := \langle \beta, g^{-1} \cdot v \rangle \quad \forall \beta \in \mathcal{O}_{\mathbb{P}^r}(1)_{[z]} \;\; \forall v \in \mathbb{C}z.$$

This in turn induces a linearization of $\mathcal{L} := \imath_{\mathbf{X}}^* \mathcal{O}_{\mathbb{P}^r}(1) \to \mathbf{X}$.

A slightly more general situation is that of a complex linear algebraic group acting on the projective variety $\mathbf{X}$ by means of a linearization through an ample line bundle $L \to \mathbf{X}$. The ampleness of L means that there is $n > 0$ such that the linear system associated to L^n gives an embedding $\mathbf{X} \hookrightarrow \mathbb{P}\left(H^0(\mathbf{X}, L^n)^{\vee}\right)$. The linearized action of G on $L \to \mathbf{X}$ induces an action on the complex vector space $H^0(\mathbf{X}, L^n)^{\vee}$. Now we are back again in the previous situation. $\diamond$

There is a general definition for the semi-stable points for the linearly reductive group actions on schemes but for my purposes it will be enough to restrict myself to the case presented in the example 4.5.3.

Definition 4.5.4 (Proposition) *Assume that G is a connected linearly reductive complex algebraic group acting on $\mathbb{C}^{r+1}$. Assume further that a projective variety $\mathbf{X} \subset \mathbb{P}^r$ is left invariant by the induced G-action on $\mathbb{P}^r$. A point $x \in \mathbf{X}$ is called* semi- stable *for this action if equivalently:*

(i) for a representative $x' \in \mathbb{C}^{r+1}$ of x, $0 \notin \overline{Gx'}^{\mathbb{C}^{r+1}}$;

(ii) there is $n > 0$ and a G-invariant section $s \in H^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(n))$ with $s(x) \neq 0$.
The (open) subset of $\mathbf{X}$ of semi-stable points is denoted $\mathbf{X}^{ss}(\mathcal{O}_{\mathbf{X}}(1))$.

If the orbit $Gx' \subset \mathbb{C}^{r+1}$ is closed or, equivalently, the action of G on $\mathbf{X}_s := \{y | s(y) \neq 0\}$ is closed, the point $x \in \mathbf{X}$ is called stable. *The set of stable points is denoted $\mathbf{X}^s(\mathcal{O}_{\mathbf{X}}(1))$; the set of stable points with finite stabilizer is denoted by $\mathbf{X}^s_{(0)}(\mathcal{O}_{\mathbf{X}}(1))$.*

The reason for introducing these concepts relies in the following problem: given a group action on a variety, when does a "quotient" exist? There are two important notions of quotients:

Definition 4.5.5 (categorical quotient) *Given an action σ of the algebraic group G on the algebraic variety $\mathbf{X}$, a pair $(\widehat{\mathbf{X}}, \phi)$ is called* a categorical quotient *of $\mathbf{X}$ by G if:*

(i) the diagram:

$$\begin{array}{ccc} G \times \mathbf{X} & \xrightarrow{\sigma} & \mathbf{X} \\ \downarrow p_{\mathbf{X}} & & \downarrow \phi \\ \mathbf{X} & \xrightarrow{\phi} & \widehat{\mathbf{X}} \end{array}$$

is commutative.

(ii) given any algebraic variety $\mathbf{Z}$ acted on by G and a G-morphism $\psi : \mathbf{X} \longrightarrow \mathbf{Z}$, there is a unique map $\widehat{\psi}$ making the diagram

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\psi} & \mathbf{Z} \\ \phi \downarrow & \nearrow \widehat{\psi} & \\ \widehat{\mathbf{X}} & & \end{array}$$

commutative.

Definition 4.5.6 (geometric quotient) *Given an action σ of the algebraic group G on the algebraic variety $\mathbf{X}$, a pair $(\widehat{\mathbf{X}}, \phi)$ is called* a geometric quotient *of $\mathbf{X}$ by G if:*

(i) $\phi \circ \sigma = \phi \circ p_{\mathbf{X}}$ as in definition 4.5.5

(ii) ϕ is surjective and the geometric fibres of ϕ are precisely the orbits of the geometric points of $\mathbf{X}$

(iii) ϕ is submersive i.e. $U \subset \widehat{\mathbf{X}}$ is open if and only if $\phi^{-1}(U) \subset \mathbf{X}$ is open

(iv) $\Gamma(U, \mathcal{O}_{\widehat{\mathbf{X}}}) = \Gamma(\phi^{-1}(U), \mathcal{O}_{\mathbf{X}})^G$.

A very useful tool for detecting the semi-stable points corresponding to the action of a linearly reductive group, in a given situation, is the following criterion which reduces this problem to that of finding the semi-stable points corresponding to $\mathbb{C}^*$ actions.

Theorem 4.5.7 (Hilbert-Mumford criterion) *In the situation of the proposition 4.5.4, $x \in \mathbf{X}$ is G-semi-stable if and only if it is $\mathbb{C}^*$-semi-stable for all one parameter subgroups (1-PS) $\lambda : \mathbb{C}^* \to G$. The same statement holds for the stable points.*

The use of this result consists in the following: if λ is a representation of $\mathbb{C}^*$ in a vector space $\mathscr{V}$, then there is a (finite) direct sum decomposition

$$\mathscr{V} = \bigoplus_{m \in \mathbb{Z}} \mathscr{V}_m$$

where $\mathscr{V}_m = \{v \in \mathscr{V} | \ \lambda(t)v = t^m v \ \forall t \in \mathbb{C}^*\}$. Indeed, $\mathbb{C}^*$ being commutative, all the matrices $\lambda(t), t \in \mathbb{C}^*$ can be diagonalized simultaneously in the same basis. The eigenvalues are characters of $\mathbb{C}^*$ which are precisely $t \mapsto t^m$. A point $x' \in \mathscr{V}$, viewed as an affine space, has coordinates $x' = (x'_m)_{m \in \mathbb{Z}}$ in the above basis. It is clear that $0 \in \mathscr{V}$ is not in the closure of the $\mathbb{C}^*$-orbit of x' if and only if there are non-zero x'_m 's for both positive and negative values of m. Let introduce the consecrated notation:

$$\mu(x', \lambda) := -\min\{m | x'_m \neq 0\}$$

This remark together with the fact that if λ is a representation of $\mathbb{C}^*$ into $\mathscr{V}$, then λ^{-1} given by $\lambda^{-1}(t) := \lambda(t^{-1})$ still defines a representation, implies the:

Corollary 4.5.8 *If a linearly reductive group G acts on a vector space $\mathscr{V}$, a point $x' \in \mathscr{V}$ is G-semi-stable if and only if for any 1-PS $\lambda : \mathbb{C}^* \to G$:*

$$\mu(x', \lambda) \geq 0$$

The next result justifies the large number of definitions introduced so far:

Theorem 4.5.9 ([MFK], thm. 1.10 page 38) *(A) Let $\mathbf{X}$ be a complex algebraic variety and G a linearly reductive group acting on $\mathbf{X}$ via a G-linearized invertible sheaf $L \to \mathbf{X}$. Then a categorical quotient $(\widehat{\mathbf{X}}, \phi)$ of $\mathbf{X}^{ss}(L)$ by G exists . Moreover:*

(i) $\phi : \mathbf{X}^{ss}(L) \to \widehat{\mathbf{X}}$ is affine and submersive

(ii) there is an ample invertible sheaf $M \to \widehat{\mathbf{X}}$ such that $\phi^* M = L^{n_0}$ for some $n_0 > 0$ (so $\widehat{\mathbf{X}}$ is quasi-projective)

(iii) there is an open set $\widehat{\mathbf{U}} \subset \widehat{\mathbf{X}}$ such that $\mathbf{X}^s(L) = \phi^{-1}(\widehat{\mathbf{U}})$ and $(\widehat{\mathbf{U}}, \phi_{|\mathbf{X}^s(L)})$ is a geometric quotient of $\mathbf{X}^s(L)$ by G.

(B) If moreover $\mathbf{X}$ is proper over $\mathbb{C}$ (e.g. it is projective) and $L \to \mathbf{X}$ is ample then $\widehat{\mathbf{X}}$ is projective over $\mathbb{C}$. In fact:

$$\widehat{\mathbf{X}} = \mathtt{Proj}\left(\sum_n H^0(\mathbf{X}, L^n)^G\right)$$

Remark 4.5.10 *In case (B) of the previous theorem, because $M \to \widehat{\mathbf{X}}$ is ample, for large $m > 0$*

$$H^0(\widehat{\mathbf{X}}, M^m) = H^0(\mathbf{X}, L^{mn_0})^G \qquad \diamond$$

In part (A) of the theorem the general concept of semi-stability is used which was not defined in 4.5.4. This shouldn't be a problem because, anyway, $L \to \mathbf{X}^{ss}(L)$ is ample and in the rest of the paper I shall be always in the situation (B).

The consecrated notation for $\widehat{\mathbf{X}}$ is $\mathbf{X}//G$ and it is called *the geometric invariant quotient* of $\mathbf{X}$ for the action of G.

Later on I shall use the following easy:

Lemma 4.5.11 *Assume that on the complex projective variety* $\mathbf{X}$ *with an ample line bundle* $L \to \mathbf{X}$ *a linearly reductive group* G *acts via a linearization of* L*. Then the map*

$$\phi : \mathbf{X}^{ss}(L) \longrightarrow \widehat{\mathbf{X}}$$

doesn't contract any complete curve.

Proof Assume the contrary, that there is a complete, irreducible curve $\mathbf{C} \subset \mathbf{X}^{ss}(L)$ which is contracted by ϕ. The theorem 4.5.9 says that there is a categorical quotient $\phi : \mathbf{X}^{ss}(L) \to \widehat{\mathbf{X}}$ and an ample line bundle $M \to \widehat{\mathbf{X}}$ such that $\phi^* M = L^{\otimes n_0}_{|\mathbf{X}^{ss}(L)}$. Because $\mathbf{C}$ is assumed to be contracted, the evaluation $\deg_{\mathbf{C}} L = 0$. But this is not possible since L is ample. □

In the section 4.8 I shall make use of the following result due to F.Kirwan which describes the cohomology ring of the GIT quotient of a complex projective variety $\mathbf{X}$:

Proposition 4.5.12 (F.Kirwan) *If the action of* G *on* $\mathcal{O}_{\mathbf{X}}(1) \to \mathbf{X}$ *is such that* $\mathbf{X}^{ss} = \mathbf{X}^{s}_{(0)}$, *then:*

(i) *the restriction map* $H^*_G(\mathbf{X};\mathbb{Q}) \to H^*_G(\mathbf{X}^{ss};\mathbb{Q})$ *is surjective;*

(ii) *the natural projection* $\mathbf{X}^{ss} \times_G \mathrm{EG} \to \widehat{\mathbf{X}}$ *defines an isomorphism* $H^*(\widehat{\mathbf{X}};\mathbb{Q}) \to H^*_G(\mathbf{X}^{ss};\mathbb{Q})$.

Here $\mathrm{EG} \to \mathrm{BG}$ *denotes the universal* G*-bundle over the classifying* G*-space.*

Proof See [Ki], §14. □

4.6 Computation of semi-stable points on $\overline{M}_{g,k}(\mathbf{X}, A)$

In this section I shall finally start to study the problem presented in the section 4.1. The first question to answer is which are the semi-stable points of $\overline{M}_{g,k}(\mathbf{X}, A)$ for the induced group action.

Let me recall the setup: G is a connected, linearly reductive, complex algebraic group acting on the complex, irreducible, reduced, projective variety $\mathbf{X}$ by means of a linearization of the very ample line bundle $\mathcal{O}_{\mathbf{X}}(1) \to \mathbf{X}$. Then the G-action extends to one on $\mathcal{O}_{\mathbb{P}^r}(1) \to \mathbb{P}^r := \mathbb{P}\left(H^0(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(1))^{\vee}\right)$ and $\mathbf{X}$ is preserved under this action. Fix a homology class $A \in H_2(\mathbf{X}, \mathbb{Z})$; inside $\mathbb{P}^r$ it will represent d times the class of the line in the projective space. I explained in the section 4.3 that $\overline{M}_{g,k}(\mathbf{X}, A)$ is a closed subscheme of $\overline{M}_{g,k}(\mathbb{P}^r, d)$.

Recall from 4.3.5 that $\overline{M}_{g,k}(\mathbb{P}^r, d)$ is the *quotient* $\mathcal{S}/PGl(N)$, where $\mathcal{S}$ is some subscheme of $\boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$. As before, $\mathbf{Y} := \mathbb{P}^{N-1} \times \mathbb{P}^r$ and $\boldsymbol{\mathcal{H}} := \mathcal{H}ilb^{P}_{\mathbf{Y}}$ is the Hilbert scheme of closed subschemes of $\mathbf{Y}$ having fixed Hilbert polynomial P (of degree 1).

There is a natural action of G on $\mathbf{Y} = \mathbb{P}^{N-1} \times \mathbb{P}^r$ which is trivial on the first factor and is the given one on the second; it is important to notice that this G-action and the $PGl(N)$-action commute. This action induces a G-action on $\mathcal{H}$ also: if $\mathbf{C}$ is a closed subscheme of $\mathbf{Y}$, then an element $g \in G$ acts on it by: $(g, \mathbf{C}) \mapsto g_* \mathbf{C}$, where in this case g is viewed as a self map of $\mathbf{Y}$; because g is linear, it will preserve the Hilbert polynomial. Again one should notice that the G-action and the $PGl(N)$-action commute.

Let me come back to the stable maps: $\mathcal{S}$ is the subscheme of $\mathcal{H} \times \mathbf{Y}^k$ corresponding to stable maps. There is an induced action:

$$G \times \mathcal{S} \longrightarrow \mathcal{S}$$
$$(g, (\mathbf{C}, \mathbf{x}, u)) \mapsto (\mathbf{C}, \mathbf{x}, g \circ u)$$

This map is correctly defined because a stable map is transformed into a stable map. Moreover, as G is connected $(g \circ u)_*[\mathbf{C}]$ represents the same homology class as $u_*[\mathbf{C}]$ (inside both $\mathbf{X}$ and $\mathbb{P}^r$). Because the G- and $PGl(N)$-actions on $\mathcal{S}$ commute, there is an induced G-action on $\mathcal{S}/PGl(N) = \overline{M}_{g,k}(\mathbb{P}^r, d)$ which leaves $\overline{M}_{g,k}(\mathbf{X}, A)$ invariant. Moreover, this action leaves invariant the fibres of the map: $\varpi : \overline{M}_{g,k}(\mathbf{X}, A) \to \overline{M}_{g,k}$.

The fact that the two actions of G and $PGl(N)$ commute is important also because it implies that the G-semi-stable points of $\overline{M}_{g,k}(\mathbb{P}^r, d)$ are the images of the G-semi-stable points of $\mathcal{S}$. This means that I shall be able to work with (honest) maps instead of equivalence classes of maps.

My next task is to make the linearized G-action on $\det \mathfrak{Q}_k \to \mathcal{H} \times \mathbf{Y}^k$ explicit. I start recalling how the projective embedding of $\mathcal{H} \times \mathbf{Y}^k$ is obtained.

$$\mathcal{H} \times \mathbf{Y}^k \longrightarrow \mathbb{P}\left(\bigwedge^{q+k} \mathscr{W}^\vee\right)$$
$$(\mathbf{C}, \mathbf{x}) \mapsto \det\left(\left(H^0(\mathbf{Y}, \mathcal{O}_\mathbf{C} \otimes \mathcal{L}^l) \oplus \bigoplus_{j=1}^{k} \mathcal{L}^l_{x_j}\right)^\vee\right)$$

where $\mathscr{W} = H^0(\mathbf{Y}, \mathcal{L}^l)^{(0)} \oplus \bigoplus_{j=1}^k H^0(\mathbf{Y}, \mathcal{L}^l)^{(j)}$.

The G-action is as follows:

(i) G acts on

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}^{N-1}}(1) \boxtimes \mathcal{O}_{\mathbb{P}^r}(1) \longrightarrow \mathbf{Y} = \mathbb{P}^{N-1} \times \mathbb{P}^r$$

trivially on the first factor and in the given way on the second one.

(ii) this induces a natural action

$$G \times H^0(\mathbf{Y}, \mathcal{L}^l) \longrightarrow H^0(\mathbf{Y}, \mathcal{L}^l)$$
$$(g, S) \mapsto gS$$

where $(gS)_y = gS_{g^{-1}y}$ for all $y \in \mathbf{Y}$ and $S \in H^0(\mathbf{Y}, \mathcal{L}^l)$. In fact:

$$H^0(\mathbf{Y}, \mathcal{L}^l) = H^0(\mathbb{P}^{N-1}, \mathcal{O}_{\mathbb{P}^{N-1}}(1)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$$

and G acts only on the second factor.

(iii) the dual action on $H^0(\mathbf{Y},\mathcal{L}^l)^\vee$ is defined

$$(g,\Sigma)\mapsto g\Sigma \quad \text{where} \quad \langle g\Sigma, S\rangle := \langle \Sigma, g^{-1}S\rangle \quad \forall S\in H^0(\mathbf{Y},\mathcal{L}^l)$$

The action on $\bigwedge^{q+k}\mathscr{W}^\vee$ is the obviously induced one.

In order to find the G-semi-stable points I shall use the Hilbert-Mumford criterion 4.5.7. For $\lambda:\mathbb{C}^*\to G$ a 1-PS, there is a (finite) direct sum decomposition corresponding to the characters of $\mathbb{C}^*$:

$$H^0(\mathbf{Y},\mathcal{L}^l)=\bigoplus_{m\in\mathbb{Z}} H^0(\mathbf{Y},\mathcal{L}^l)_m$$

$$\lambda(t)S=t^mS \quad \forall t\in\mathbb{C}^* \quad \forall S\in H^0(\mathbf{Y},\mathcal{L}^l)_m$$

I want to know when a point:

$$\left[\bigwedge^{q+k}\left(H^0(\mathbf{Y},\mathcal{L}^l)^{(0)}\oplus\bigoplus_{j=1}^k \mathcal{L}^l_{x_j}\right)^\vee\right]\in\mathbb{P}\left(\bigwedge^{q+k}\mathscr{W}^\vee\right)$$

is λ-semi-stable, so I have to study the $\mathbb{C}^*$-orbit of a representative of this point in $\bigwedge^{q+k}\mathscr{W}^\vee$.

Let $\sigma_1,\ldots,\sigma_q$ be a basis of $\mathrm{Hom}_{\mathbb{C}}\left(H^0(\mathbf{C},\mathcal{L}^l)^{(0)},\mathbb{C}\right)$ and $\tau_1,\ldots,\tau_k$ generators of $\mathrm{Hom}\,(\mathcal{L}^l_{x_j},\mathbb{C})$, $j=1,\ldots,k$. Notice that the choice of the τ_j's is equivalent with the choice of representatives $x'_j\in\mathbb{C}^N\times\mathbb{C}^{r+1}$ of $x_j=(x_{j,1},x_{j,2})\in\mathbf{Y}$ because, by definition, $\mathcal{L}_{x_j}=\mathcal{O}_{\mathbb{P}^{N-1}}(1)_{x_{j,1}}\otimes\mathcal{O}_{\mathbb{P}^r}(1)_{x_{j,2}}$. Using the epimorphism

$$\mathscr{W}=H^0(\mathbf{Y},\mathcal{L}^l)^{(0)}\oplus\bigoplus_{j=1}^k H^0(\mathbf{Y},\mathcal{L}^l)^{(j)}\xrightarrow{\imath_{\mathbf{C}}} H^0(\mathbf{C},\mathcal{L}^l)\oplus\bigoplus_{j=1}^k\mathcal{L}^l_{x_j}\longrightarrow 0$$

$\sigma_1,\ldots,\sigma_q,\tau_1,\ldots,\tau_k$ can be extended to linear functionals on $\mathscr{W}$

$$\mathscr{S}_1,\ldots,\mathscr{S}_q: H^0(\mathbf{Y},\mathcal{L}^l)^{(0)}\longrightarrow\mathbb{C}$$

$$\langle\mathscr{S}_j,S\rangle:=\langle\sigma_j,\imath_{\mathbf{C}}S\rangle \quad j=1,\ldots,q \quad \forall S\in H^0(\mathbf{Y},\mathcal{L}^l)^{(0)}$$

and

$$\mathscr{T}_1,\ldots,\mathscr{T}_k: H^0(\mathbf{Y},\mathcal{L}^l)\longrightarrow\mathbb{C}$$

$$\langle\mathscr{T}_j,S\rangle:=\langle\tau_j,S(x_j)\rangle \quad j=1,\ldots,k \quad \forall S\in H^0(\mathbf{Y},\mathcal{L}^l)^{(j)}$$

The $\mathscr{T}_j$'s represent just the evaluations of the homogeneous polynomial S at the points x'_j representing x_j.

Remark I should emphasize that the linear functionals $\mathscr{S}_j$ act only on $H^0(\mathbf{Y},\mathcal{L}^l)^{(0)}$ and evaluate identically zero on the other copies $H^0(\mathbf{Y},\mathcal{L}^l)^{(j)}$, $j\neq 0$. Similar remark is valid for the $\mathscr{T}_j$'s: they evaluate identically zero on $H^0(\mathbf{Y},\mathcal{L}^l)^{(j')}$, $j'\neq j$. $\diamond$

The semistability condition is now

$$0 \notin \overline{\mathbb{C}^* \cdot \mathscr{S}_1 \wedge \ldots \wedge \mathscr{S}_q \wedge \mathscr{T}_1 \wedge \ldots \wedge \mathscr{T}_k}^{\wedge^{q+k} \mathscr{W}^\vee} \qquad \textbf{(ss)}$$

The condition **(ss)** is equivalent to the existence of $S_1, \ldots, S_{q+k} \in \mathscr{W}$ such that

$$0 \notin \overline{\{\langle \lambda(t) \cdot (\mathscr{S}_1 \wedge \ldots \wedge \mathscr{S}_q \wedge \mathscr{T}_1 \wedge \ldots \wedge \mathscr{T}_k), S_1 \wedge \ldots \wedge S_{q+k} \rangle \mid t \in \mathbb{C}^*\}}^{\mathbb{C}} \qquad \textbf{(ss}'\textbf{)}$$

Each vector S_j is the sum of $1+k$ vectors corresponding to the direct sum decomposition of $\mathscr{W}$. Moreover, as I pointed out in the remark above, each of the $\mathscr{S}_j$'s and $\mathscr{T}_j$'s evaluate non-zero only on vectors in a certain component of $\mathscr{W}$. Therefore the condition **(ss')** is equivalent to the existence of

$$S_1, \ldots, S_q \in H^0(\mathbf{Y}, \mathcal{L}^l)^{(0)} \quad \text{and} \quad S_{q+j} \in H^0(\mathbf{Y}, \mathcal{L}^l)^{(j)} \;\; j = 1, \ldots, k$$

such that **(ss')** holds. Because the $\mathbb{C}^*$ action of the 1-PS λ of G induces the decomposition $H^0(\mathbf{Y}, \mathcal{L}^l) = \oplus_{m \in \mathbb{Z}} H^0(\mathbf{Y}, \mathcal{L}^l)_m$, I can further assume that

$$S_j \in H^0(\mathbf{Y}, \mathcal{L}^l)^{(0)}_{m_j} \quad \text{for} \quad j = 1, \ldots, q$$

and

$$S_{q+j} \in H^0(\mathbf{Y}, \mathcal{L}^l)^{(j)}_{m_{q+j}} \quad \text{for} \quad j = 1, \ldots, k.$$

I am now in position to compute

$$\begin{aligned}
& \langle \lambda(t) \cdot (\mathscr{S}_1 \wedge \ldots \wedge \mathscr{S}_q \wedge \mathscr{T}_1 \wedge \ldots \wedge \mathscr{T}_k), S_1 \wedge \ldots S_q \wedge S_{q+1} \wedge \ldots \wedge S_{q+k} \rangle \\
= & \langle \mathscr{S}_1 \wedge \ldots \wedge \mathscr{S}_q \wedge \mathscr{T}_1 \wedge \ldots \wedge \mathscr{T}_k, \lambda(t^{-1}) S_1 \wedge \ldots \wedge \lambda(t^{-1}) S_q \wedge \lambda(t^{-1}) S_{q+1} \wedge \ldots \wedge \lambda(t^{-1}) S_{q+k} \rangle \\
= & \langle \mathscr{S}_1 \wedge \ldots \wedge \mathscr{S}_q \wedge \mathscr{T}_1 \wedge \ldots \wedge \mathscr{T}_k, t^{-m_1} S_1 \wedge \ldots \wedge t^{-m_q} S_q \wedge t^{-m_{q+1}} S_{q+1} \wedge \ldots \wedge t^{-m_{q+k}} S_{q+k} \rangle \\
= & \; t^{-\sum_{j=1}^{q+k} m_j} \cdot \langle \mathscr{S}_1 \wedge \ldots \wedge \mathscr{S}_q, S_1 \wedge \ldots \wedge S_q \rangle \cdot \textstyle\prod_{j=1}^{k} \langle \mathscr{T}_j, S_{q+j} \rangle \\
= & \; t^{-\sum_{j=1}^{q+k} m_j} \cdot \langle \sigma_1 \wedge \ldots \wedge \sigma_q, \imath_{\mathbf{C}}^* S_1 \wedge \ldots \wedge \imath_{\mathbf{C}}^* S_q \rangle \cdot \textstyle\prod_{j=1}^{k} S_{q+j}(x'_j)
\end{aligned}$$

This computation shows that the semi-stability property **(ss')** can be translated into the following:

Proposition 4.6.1 *The point $[(\mathbf{C}, \mathbf{x}, u)] \in \overline{M}_{g,k}(\mathbf{X}, A)$ is G-semi-stable if and only if for any 1-PS $\lambda : \mathbb{C}^* \to G$ there are sections*

$$S_1, \ldots, S_q, S_{q+1}, \ldots, S_{q+k} \in H^0(\mathbf{Y}, \mathcal{L}^l)$$

satisfying the properties:

(i) $\lambda(t) S_j = t^{m_j} S_j$ *for* $j = 1, \ldots, q+k$ *with* $\displaystyle\sum_{j=1}^{q+k} m_j = 0$ *;*

(ii) $\{\imath_{\mathbf{C}}^* S_1, \ldots, \imath_{\mathbf{C}}^* S_q\}$ *is a basis for* $H^0(\mathbf{C}, \mathcal{L}^l)$ *;*

(iii) $S_{q+j}(x_j) \neq 0$ *for* $j = 1, \ldots, k$.

For deducing some geometrical properties satisfied by the G-semi-stable maps, I shall use the following consequence of this proposition:

Corollary 4.6.2 *If the point* $[(\mathbf{C},\mathbf{x},u)] \in \overline{M}_{g,k}(\mathbf{X},A)$ *is* G*-semi-stable, then for all* $1-PS$ $\lambda:\mathbb{C}^* \to G$ *there are sections*

$$S_j \in H^0(\mathbf{Y},\mathcal{L}^l)_{m_j} \qquad j=1,\dots,q$$

satisfying the properties:

(i) $\{\imath_{\mathbf{C}}^* S_1,\dots,\imath_{\mathbf{C}}^* S_q\}$ *is a basis for* $H^0(\mathbf{C},\mathcal{L}^l)$ *;*

(ii) either all the integers $\{m_j\}_{j=1,\dots,q}$ *are zero or among them there are strictly positive and strictly negative ones.*

Proof For a fixed $1-PS$ of G, there are two possibilities in the previous proposition: either all the m_j's vanish for $j=1,\dots,q$ and we are done, or it is not so. Assume that all $m_j \geq 0$ for $j=1,\dots,q$. Because the sum $\sum_{j=1}^{j=q+k} m_j = 0$, it follows that there must exist a $m_{q+h}<0$. We know that $S_{m_{q+h}}(x_{q+h}) \neq 0$ and therefore the restriction $\imath_{\mathbf{C}}^* S_{m_{q+h}} \neq 0$. Because $\{\imath_{\mathbf{C}}^* S_1,\dots,\imath_{\mathbf{C}}^* S_q\}$ is a basis of $H^0(\mathbf{C},\mathcal{L}^l)$, one can write $\imath_{\mathbf{C}}^* S_{m_{q+h}}$ as a non-zero linear combination of these vectors. Now all it remains to be done is to replace a section from the original set which appears in this linear combination with this new vector. □

Now I can proceed to deduce some geometrical properties satisfied by the G-semi-stable points $[(\mathbf{C},\mathbf{x},u)]$. I start with the following easy lemma:

Lemma 4.6.3

$$\mathbf{Y}^{ss}(\mathcal{L}) = \mathbb{P}^{N-1} \times (\mathbb{P}^r)^{ss}(\mathcal{O}_{\mathbb{P}^r}(1))$$

Proof $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{N-1}}(1) \boxtimes \mathcal{O}_{\mathbb{P}^r}(1) \to \mathbf{Y}$ is very ample and the associated linear system gives the Segre embedding:

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & \mathcal{O}_{\mathbb{P}^{M-1}}(1) \\ \downarrow & & \downarrow \\ \mathbf{Y} & \xrightarrow{\text{Segre}} & \mathbb{P}^{M-1} = \mathbb{P}^{N(r+1)-1} \end{array}$$

The linearized G action on $\mathcal{L}$ induces an action:

$$G \times \mathbb{C}^M \longrightarrow \mathbb{C}^M \quad \text{where} \quad \mathbb{C}^M = \mathbb{C}^N \otimes \mathbb{C}^{r+1}$$
$$g\cdot(x' \otimes y') = x' \otimes gy' \quad \forall x' \in \mathbb{C}^N \quad \text{and} \quad \forall y' \in \mathbb{C}^{r+1}$$

For $(x,y) \in \mathbf{Y}$ a semi-stable point, I chose a representative $(x',y') \in \mathbb{C}^N \times \mathbb{C}^{r+1}$ of it. In coordinates on $\mathbb{C}^N$, $x' = (x'_0,\dots,x'_{N-1})$. The image of the point (x',y') in $\mathbb{C}^M$ is $(x'_0\cdot y',\dots,x'_{N-1}\cdot y')$; it is semi-stable if and only if:

$$0 \notin \overline{G\cdot(x'_0\cdot y',\dots,x'_{N-1}\cdot y')}^{\mathbb{C}^M}$$

Because not all of $x'_0,\dots,x'_{N-1}$ are zero, this last condition is equivalent to: $0 \notin \overline{G\cdot y'}^{\mathbb{C}^{r+1}}$ i.e. $y' \in \mathbb{P}^r$ is semi-stable. □

The next proposition gives a geometrical restriction which must satisfied by the G-semi-stable maps in $\overline{M}_{g,k}(\mathbf{X}, A)$.

Proposition 4.6.4 *If $(\mathbf{C}, \mathbf{x}, u) \in \overline{M}_{g,k}(\mathbf{X}, A)$ is a stable map which is G-semi-stable, then for each 1-PS $\lambda : \mathbb{C}^* \to G$ there exist an irreducible component $\mathbf{C}_\delta$ of $\mathbf{C}$ such that the image of the map*

$$u_{|\mathbf{C}_\delta} : \mathbf{C}_\delta \longrightarrow \mathbf{X}$$

in <u>*not*</u> *contained in the λ-unstable locus of $\mathbf{X}$.*

Proof The line bundle $\mathcal{L}^l \to \mathbf{Y}$ is again very ample and its associated linear system gives an embedding

$$\begin{array}{ccc} \mathcal{L}^l & \longrightarrow & \mathcal{O}_{\mathbb{P}^{R-1}}(1) \\ \downarrow & & \downarrow \\ \mathbf{Y} & \xrightarrow{|\mathcal{L}^l|} & \mathbb{P}^{R-1} = \mathbb{P}\left(H^0(\mathbf{Y}, \mathcal{L}^l)^\vee\right) \end{array}$$

The G-action on $\mathcal{L}^l \to \mathbf{Y}$ induces a linearised G-action on $\mathcal{O}_{\mathbb{P}^{R-1}}(1) \to \mathbb{P}^{R-1}$. For a stable map $(\mathbf{C}, \mathbf{x}, u)$ which is G-semi-stable and $\lambda : \mathbb{C}^* \to G$ a 1-PS of G, corollary 4.6.2 ensures the existence of sections $S_j \in H^0(\mathbf{Y}, \mathcal{L}^l)_{m_j}$ whose restictions to $\mathbf{C}$ give a basis of $H^0(\mathbf{C}, \mathcal{L}^l)$; in particular, they are linearly independent. Because one has the direct sum decomposition

$$H^0(\mathbf{Y}, \mathcal{L}^l) = \bigoplus_{m \in \mathbb{Z}} H^0(\mathbf{Y}, \mathcal{L}^l)_m$$

these sections can be completed with sections

$$S'_{q+1} \in H^0(\mathbf{Y}, \mathcal{L}^l)_{m'_{q+1}}, \ldots, S'_R \in H^0(\mathbf{Y}, \mathcal{L}^l)_{m'_R}$$

to a basis of $H^0(\mathbf{Y}, \mathcal{L}^l)$. This basis defines coordinates on $\mathbb{C}^R \cong H^0(\mathbf{Y}, \mathcal{L}^l)^\vee$ in which the λ-action is diagonal.

Claim There exists an irreducible component $\mathbf{C}_\delta$ of $\mathbf{C}$ having the property that among $\{S_1, \ldots, S_q\}$ there are two sections $S_j \in H^0(\mathbf{Y}, \mathcal{L}^l)_{m_j}$ and $S_{j'} \in H^0(\mathbf{Y}, \mathcal{L}^l)_{m_{j'}}$ such that $m_{j'} \leq 0$ and $m_j \geq 0$ and their restriction to $\mathbf{C}_\delta$ is non-zero.

I know already that there are two sections S_α and S_τ such that $m_\alpha \leq 0$ and $m_\tau \geq 0$. Let $\mathbf{C}_\alpha$ and $\mathbf{C}_\tau$ respectively two irreducible components of $\mathbf{C}$ on which these two sections do not vanish. Because $\mathbf{C}$ is connected, there is a chain of irreducible components $\mathbf{C}_\alpha, \mathbf{C}_\beta, \ldots, \mathbf{C}_\tau$ connecting these two components. Since $\{S_1, \ldots, S_q\}$ is a basis of $H^0(\mathbf{C}, \mathcal{L}^l)$ and $\mathcal{L}^l \to \mathbf{C}$ is very ample (see lemma 4.3.4), it follows that there are sections $S_{\alpha\beta}, S_{\beta\gamma}, \ldots, S_{\sigma\tau}$ with the property that: $S_{\alpha\beta}$ does not vanish at a (certain) point in $\mathbf{C}_\alpha \cap \mathbf{C}_\beta$, $S_{\beta\gamma}$ does not vanish at a (certain) point in $\mathbf{C}_\beta \cap \mathbf{C}_\gamma, \ldots, S_{\sigma\tau}$ does not vanish at a (certain) point in $\mathbf{C}_\sigma \cap \mathbf{C}_\tau$. Notice that $S_{\alpha\beta}$ does not vanish on both $\mathbf{C}_\alpha$ and $\mathbf{C}_\beta$; similar remarks hold true for the other sections. Let $m_{\alpha\beta}, m_{\beta\gamma}, \ldots$ denote the weights of the sections $S_{\alpha\beta}, S_{\beta\gamma}, \ldots$ respectively. If $m_{\alpha\beta} \geq 0$, then the component $\mathbf{C}_\alpha$ satisfies the requirement of the claim. If it is not the case, I look at the chain $\mathbf{C}_\beta, \ldots, \mathbf{C}_\tau$ whose length is one less than the length of $\mathbf{C}_\alpha, \ldots, \mathbf{C}_\tau$. Because at the end $\mathbf{C}_\tau$ the weight m_τ

is positive, an induction argument on the lenght of the connecting chain shows that there must exist an irreducible component $\mathbf{C}_\delta$ of the chain $\mathbf{C}_\alpha, \ldots, \mathbf{C}_\tau$ obeying property of the claim.

Let me look now at the image of a point $p \in \mathbf{C}_\delta$ inside $\mathbb{P}^{R-1}$: a representative $p' \in \mathbb{C}^r$ of it will have non-zero coordinates with both positive and negative weights (for the λ action). According to the definition, $\mu(p', \lambda) \geq 0$, so p is λ-semi-stable. Lemma 4.6.3 says that in this case $u(p)$ is in the λ-semi-stable locus of $\mathbf{X}$. □

This proposition has the shortcoming that it says nothing about the position of the image of u *vis à vis* the G-semi-stable points. A question which raises naturally is:

Question 4.6.5 Is it true that if $(\mathbf{C}, \mathbf{x}, u) \in \overline{M}_{g,k}(\mathbf{X}, A)$ is G-semi-stable then the image of some

$$u_{|\mathbf{C}_\alpha} : \mathbf{C}_\alpha \longrightarrow \mathbf{X}$$

is not contained in the G-unstable locus of $\mathbf{X}$?

The answer for this question is yes if we are dealing with torus actions and the curve $\mathbf{C}$ is irreducible:

Corollary 4.6.6 *Suppose that a torus T acts on $\mathbf{X}$. If $(\mathbf{C}, \mathbf{x}, u) \in \overline{M}_{g,k}(\mathbf{X}, A)$ is a T-semi-stable point and $\mathbf{C}$ is irreducible, the image of u is* <u>*not*</u> *contained in the T-unstable locus of $\mathbf{X}$.*

Proof By the Hilbert-Mumford criterion 4.5.7

$$\mathbf{X}_T^{ss} = \bigcap_{\lambda \ 1-PS \,\mathrm{of}\, T} \mathbf{X}_\lambda^{ss}$$

Since $\mathbf{C}$ is assumed to be irreducible, the proposition 4.6.4 implies that the image of u intersects the λ-unstable locus of $\mathbf{X}$ in finitely many points; denote by $\mathbf{C}^0(\lambda)$ the Zariski open subset of $\mathbf{C}$ consisting of points which are mapped by u into the λ-semi-stable locus of $\mathbf{X}$. Because there are countably many $1 - PS$'s in a torus,

$$\mathbf{C} \neq \bigcup_{\lambda \ 1-PS \,\mathrm{of}\, T} (\mathbf{C} - \mathbf{C}^0(\lambda)) \qquad \square$$

In what follows I want to prove a weakened converse of the proposition 4.6.4. It may be useful in the case when the unstable locus $\mathbf{X}^{\mathrm{unstable}}(\mathcal{O}_{\mathbf{X}}(1))$ has large codimension in $\mathbf{X}$. In this case it is reasonable to think that "many" curves in $\mathbf{X}$ won't meet it at all.

Theorem 4.6.7 *Let $(\mathbf{C}, \mathbf{x}, u) \in \overline{M}_{g,k}(\mathbf{X}, A)$ be a stable map satisfying the assumption:*

$$\mathrm{Im}\,(u : \mathbf{C} \to \mathbf{X}) \subset \mathbf{X}^{ss}(\mathcal{O}_{\mathbf{X}}(1))$$

Then the point $(\mathbf{C}, \mathbf{x}, u) \in \overline{M}_{g,k}(\mathbf{X}, A)$ is G-semi-stable.

Proof Theorem 4.5.9 says that the GIT quotient $\widehat{\mathbf{X}} = \mathbf{X}//G$ is projective because $\mathcal{O}_{\mathbf{X}}(1) \to \mathbf{X}$ is very ample; in fact it is a projective subvariety of $\widehat{\mathbb{P}^r} = \mathbb{P}^r//G$. This last geometric quotient can be described as

$$\widehat{\mathbb{P}^r} = \texttt{Proj}\left(\sum_n H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))^G\right)$$

Let's denote by $\phi : (\mathbb{P}^r)^{ss} \to \widehat{\mathbb{P}^r}$ the quotient map. There is an invertible sheaf $M \to \widehat{\mathbb{P}^r}$ having the property that

$$\phi^* M = \mathcal{O}_{\mathbb{P}^r}(m_0)_{|(\mathbb{P}^r)^{ss}} \quad \text{for some} \quad m_0 > 0$$

and also, by remark 4.5.10

$$H^0(\widehat{\mathbb{P}^r}, M^n) \overset{\phi^*}{\cong} H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(nm_0))^G$$

for large values of n.

The assumption that the image of the stable map is contained in the semi-stable locus of $\mathbf{X}$ implies that there is a commutative diagram

$$\begin{array}{ccc} (\mathbf{C}, \mathbf{x}) & \overset{u}{\longrightarrow} & \mathbf{X}^{ss} \\ & \widehat{u} \searrow & \downarrow \phi \\ & & \widehat{\mathbf{X}} \end{array}$$

Remark 4.6.8 The map $\widehat{u} : (\mathbf{C}, \mathbf{x}) \to \widehat{\mathbf{X}}$ is still stable. Indeed, problems would appear only if $\widehat{u}$ contracted some $\mathbb{P}^1$-components, without enough special points on them, which are not contracted by u (u is stable map). Lemma 4.5.11 says that this can not happen. $\diamond$

In virtue of the lemma 4.6.3, the GIT quotient of $\mathbf{Y}$ by G is $\widehat{\mathbf{Y}} = \mathbb{P}^{N-1} \times \widehat{\mathbb{P}^r}$ and the quotient map $\psi : \mathbf{Y}^{ss}(\mathcal{L}) \longrightarrow \widehat{\mathbf{Y}}$ is just $\psi = (\mathrm{id}_{\mathbb{P}^{N-1}}, \phi)$. Let me define the line bundle

$$\mathcal{M} := \mathcal{O}_{\mathbb{P}^{N-1}}(m_0) \boxtimes M \longrightarrow \widehat{\mathbf{Y}}.$$

It has the property that

$$\psi^* \mathcal{M} = \mathcal{O}_{\mathbb{P}^{N-1}}(m_0) \boxtimes \phi^* M = \left(\mathcal{O}_{\mathbb{P}^{N-1}}(m_0) \boxtimes \mathcal{O}_{\mathbb{P}^r}(m_0)\right)_{|\mathbf{Y}^{ss}} = \mathcal{L}^{m_0}{}_{|\mathbf{Y}^{ss}}$$

and it can be easily checked that

$$H^0(\widehat{\mathbf{Y}}, \mathcal{M}^n) \overset{\psi^*}{\cong} H^0(\mathbf{Y}, \mathcal{L}^{nm_0})^G$$

for large n. There is again a commutative diagram

$$\begin{array}{ccc} (\mathbf{C}, \mathbf{x}) & \overset{u_{\mathbf{Y}}}{\longrightarrow} & \mathbf{Y}^{ss} = \mathbb{P}^{N-1} \times (\mathbb{P}^r)^{ss} \\ & \widehat{u}_{\mathbf{Y}} \searrow & \downarrow \psi = (\mathrm{id}_{\mathbb{P}^{N-1}}, \phi) \\ & & \widehat{\mathbf{Y}} = \mathbb{P}^{N-1} \times \widehat{\mathbb{P}^r} \end{array}$$

Because $u_{\mathbf{Y}}$ is an embedding, $\widehat{u}_{\mathbf{Y}}$ is also. The 1-dimensional subvariety $\widehat{\mathbf{Z}} := \widehat{u}_{\mathbf{Y}*}\mathbf{C}$ of $\widehat{\mathbf{Y}}$ has Hilbert polynomial:

$$\begin{aligned}\widehat{P}(n) &= h^0(\mathbf{C}, \widehat{u}_{\mathbf{Y}}^*\mathcal{M}^n) - h^1(\mathbf{C}, \widehat{u}_{\mathbf{Y}}^*\mathcal{M}^n)\\ &= h^0(\mathbf{C}, u_{\mathbf{Y}}^*\mathcal{L}^{nm_0}) - h^1(\mathbf{C}, u_{\mathbf{Y}}^*\mathcal{L}^{nm_0}) = P(nm_0)\end{aligned}$$

where P is the Hilbert polynomial of $u_{\mathbf{Y}*}\mathbf{C} \subset \mathbf{Y}$. It is independent of $(\mathbf{C}, \mathbf{x}, u) \in \overline{M}_{g,k}(\mathbf{X}, A)$ satisfying the hypothesis of the proposition.

The result 4.2.2 used in the proof of the theorem 4.2.1 concerning the existence of Hilbert schemes ensures the existence of an integer $k > 0$ such that for all $n \geq k$, $\mathcal{M}^n$ is generated by its global sections and for any closed subscheme $\widehat{\mathbf{Z}}$ of $\widehat{\mathbf{Y}}$ whose Hilbert polynomial is $\widehat{P}$ there is an epimorphism

$$H^0(\widehat{\mathbf{Y}}, \mathcal{M}^n) \longrightarrow H^0(\widehat{\mathbf{Y}}, \mathcal{O}_{\widehat{\mathbf{Z}}} \otimes \mathcal{M}) \longrightarrow 0.$$

I have to recall that for obtaining the projective embedding of $\overline{M}_{g,k}(\mathbb{P}^r, d)$ I had to chose a high enough power $\mathcal{L}^l \to \mathbf{Y}$ in order to fulfill the conditions required in the theorem 4.2.1. Because $\psi^*\mathcal{M} = \mathcal{L}^{m_0}{}_{|\mathbf{Y}^{ss}}$, I can chose, *from the very beginning*, l large enough such that $\psi^*\mathcal{M}^n = \mathcal{L}^l{}_{|\mathbf{Y}^{ss}}$ with $n \geq k$ ($l = nm_0$).

The following three relations

$$\begin{cases} H^0(\mathbf{C}, u_{\mathbf{Y}}^*\mathcal{L}^l) = H^0(\mathbf{C}, \widehat{u}_{\mathbf{Y}}^*\mathcal{M}^n) \\ H^0(\widehat{\mathbf{Y}}, \mathcal{M}^n) \longrightarrow H^0(\mathbf{C}, \widehat{u}_{\mathbf{Y}}^*\mathcal{M}^n) \longrightarrow 0 \\ H^0(\widehat{\mathbf{Y}}, \mathcal{M}^n) \overset{\psi^*}{\cong} H^0(\mathbf{Y}, \mathcal{L}^l)^G \end{cases}$$

prove that there are G-invariant sections $S_1, \ldots, S_q \in H^0(\mathbf{Y}, \mathcal{L}^l)^G$ such that the restrictions $\{\imath_{\mathbf{C}}^* S_1, \ldots, \imath_{\mathbf{C}}^* S_q\}$ form a basis of $H^0(\mathbf{C}, u_{\mathbf{Y}}^*\mathcal{L}^l)$. The problem with the marked points is easy: the hypothesis says that $u(x_1), \ldots, u(x_k) \in \mathbf{X}^{ss}(\mathcal{O}_{\mathbf{X}}(1))$ and therefore I can consider their images $\widehat{u}_{\mathbf{Y}}(x_1), \ldots, \widehat{u}_{\mathbf{Y}}(x_k) \in \widehat{\mathbf{Y}}$. The number n was chosen large enough to ensure that $\mathcal{M}^n$ is globally generated by its sections. Consequently, there exist $\widehat{S}_{q+1}, \ldots, \widehat{S}_{q+k} \in H^0(\widehat{\mathbf{Y}}, \mathcal{M}^n)$ such that $\widehat{S}_{q+j}(x_j) \neq 0$, $j = 1, \ldots, k$. Using again that $H^0(\widehat{\mathbf{Y}}, \mathcal{M}^n) \overset{\psi^*}{\cong} H^0(\mathbf{Y}, \mathcal{L}^l)^G$, I find the G-invariant sections $S_{q+1}, \ldots, S_{q+k} \in H^0(\mathbf{Y}, \mathcal{L}^l)^G$ with the property that $S_{q+j}(x_j) \neq 0$ for $j = 1, \ldots, k$.

The $q + k$ sections $S_1, \ldots, S_q, S_{q+1}, \ldots, S_{q+k}$ now obviously satisfy the conditions of the proposition 4.6.1. □

Corollary 4.6.9 *If the stable map* $[(\mathbf{C}, \mathbf{x}, u)]$ *has the property that* $\operatorname{Im} u \subset \mathbf{X}^s_{(0)}$, *then* $[(\mathbf{C}, \mathbf{x}, u)] \in \overline{M}_{g,k}(\mathbf{X}, A)^s_{(0)}$.

Proof In view of the previous proposition, the only thing to be proved is that, under the assumption that the image of u is contained in the locus of stable points with finite stabilizer of $\mathbf{X}$, the stabilizer in G of $[\mathbf{C}, \mathbf{x}, u]$ is still finite. Let me consider a representative $u : (\mathbf{C}, \mathbf{x}) \to \mathbf{X}$ of the point $[(\mathbf{C}, \mathbf{x}, u)]$ and define $H := \operatorname{Stab}_G[(\mathbf{C}, \mathbf{x}, u)]$. I assume that $\dim H > 0$. By definition, for any $h \in H$, there is an automorphism $\gamma_h \in \operatorname{Aut}(\mathbf{C}, \mathbf{x})$ having the property that $hu = u\gamma_h$. In particular, for all $h \in H$, $\operatorname{Im} hu = \operatorname{Im} u$. Let $p \in \mathbf{C}$ be an arbitrary point: by assumption $u(p)$ has finite stabilizer in G and therefore $\dim H \cdot u(p) = \dim H > 0$. But $H \cdot u(p) \subset \operatorname{Im} u$ which is one dimensional. I deduce that $\dim H = 1$. Let me look at the connected component of the identity H° of H: it is a connected 1-dimensional group and therefore isomorphic either to the multiplicative group $\mathbf{G}_m$ or to the additive group $\mathbf{G}_a$. In both cases $\lim_{t\to\infty} t\cdot u(p) \in \operatorname{Im} u$ will be fixed. This contradicts the assumption that $\operatorname{Im} u \subset \mathbf{X}^s_{(0)}$. □

Remark 4.6.10 (i) The conclusion of the lemma above is optimal: in the example treated in the section 4.7, the line $\{z_0 = z_2 = 0\}$ intersects the unstable locus only in two points but its stabilizer is $\mathbb{C}^*$.

(ii) I would like to point out that the conclusions in theorem 4.6.7 are much stronger than those required in the proposition 4.6.1 and this suggests that what I have proved is not the optimal result. I would like to present some problems one should be aware of in the attempt of extending this proposition:

Let $(\mathbf{C}, \mathbf{x}, u)$ be a stable map such that the image under u of any of its irreducible components $\mathbf{C}_\alpha$ is not contained in the unstable locus of $\mathbf{X}$. I shall restrict myself to the case when $\mathbf{C}$ is irreducible. It has finitely many points $\{y_1, \dots, y_\nu\}$ which are mapped into $\mathbf{X}^{\text{unstable}}$. Composing with the projection, I get an induced rational map

$$\widehat{u} : (\mathbf{C}, \mathbf{x}) \dashrightarrow \widehat{\mathbf{X}}.$$

It might happen that some of the marked points are among the y's, but it is not a problem to extend the map at a marked point because it lies on the smooth locus of $\mathbf{C}$; $\mathbf{X}$ being projective, $\widehat{u}$ extends at a smooth point of $\mathbf{C}$ by the valuative criterion of properness. One of the bad situations is when a point y is precisely a double point of $\mathbf{C}$: in this case one can extend the map above only from the normalization of $\mathbf{C}$ (at that point). The price to pay is that the arithmetic genus decreases strictly.

Another bad situation is the following: suppose for simplicity that we are dealing with a $\mathbb{C}^*$ action and $x \in \mathbf{X}$ is a stable point with finite stabilizer. Denote by A the integral (spherical) 2-homology class in $\mathbf{X}$ represented by $\overline{\mathbb{C}^* x}^{\mathbf{X}}$ and let's look at $\overline{M}_{0,0}(\mathbf{X}, A)$. It will contain the map: $u : \mathbb{P}^1 \to \mathbf{X}$, $u(t) = t \cdot x$ (the map, which is defined *à priori* only on $\mathbb{C}^*$ extends at 0 and ∞ by the valuative criterion of properness for $\mathbf{X}$). Because $\mathbf{X}^{ss} \to \widehat{\mathbf{X}}$ doesn't contract complete curves (see lemma 4.5.11), $u(0)$ or $u(\infty)$ must be unstable. Here comes the problem: the map $\widehat{u} : \mathbb{P}^1 \to \widehat{\mathbf{X}}$ is constant, so it won't be stable anymore (in the sense of definition 3.10.1).

For these reasons I don't know how to deal with more general situations than that presented in theorem 4.6.7. ⋄

4.7 An example

In this section, for an explicit example, I shall compute the spaces $\overline{M}_{g,k}(\mathbf{X}, A)$ and its GIT quotient on one hand, and $\overline{M}_{g,k}(\mathbf{X}//G, \widehat{A})$ on the other hand in order to see what happens in concrete situations. Unfortunately, for being able to make computations, I chose the example as simple as possible: the variety $\mathbf{X}$ will be the complex projective 3-space $\mathbb{P}^3$ and the group acting on it is $\mathbb{C}^*$. The linearized action is given by:

$$\mathbb{C}^* \times \mathbb{C}^4 \longrightarrow \mathbb{C}^4 \qquad t \times (z_0, z_1, z_2, z_3) \mapsto (tz_0, tz_1, t^{-1}z_2, t^{-1}z_3)$$

The unstable locus of $\mathbb{C}^3$ for this action is the union of the two disjoint lines

$$L' := \{z_0 = z_1 = 0\} \quad \text{and} \quad L'' := \{z_2 = z_3 = 0\}$$

and it can be seen that on $(\mathbb{P}^3)^{ss}$ the $\mathbb{C}^*$ action is free. The invariant quotient is $\mathbb{P}^1 \times \mathbb{P}^1$ and the corresponding quotient map is

$$\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$
$$[z_0 : z_1 : z_2 : z_3] \mapsto [z_0 z_2 : z_0 z_3] \times [z_1 z_2 : z_1 z_3]$$

I shall study the space $\overline{M}_{0,0}(\mathbb{P}^3, 1)$ i.e. the moduli space of stable rational, unmarked maps into $\mathbb{P}^3$ which represent the class of a line. In order to make some interesting computations I should have allowed marked points but in that case I can't carry over the computations anymore. It is easy to see what $\overline{M}_{0,0}(\mathbb{P}^3, 1)$ is just the space of lines in $\mathbb{P}^3$. Any such a line is determined by two linear equations. Let me notice further that the line $L := \{z_0 + z_2 = z_1 + z_3 = 0\}$ is completely contained in the stable locus of $\mathbb{P}^3$; according to theorem 4.6.7, it must define a semi-stable point of $\overline{M}_{0,0}(\mathbb{P}^3, 1)$. We will see that this is the case. Indeed, a projective embedding of it is given by

$$\overline{M}_{0,0}(\mathbb{P}^3, 1) \longrightarrow \mathbb{P}\left(\bigwedge^2 H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))\right) \cong \mathbb{P}^5$$
$$\mathbf{C} := \{H_1 = H_2 = 0\} \mapsto [H_1 \wedge H_2]$$

Via this embedding one can identify

$$\overline{M}_{0,0}(\mathbb{P}^3, 1) = \{Z_{01}Z_{23} - Z_{02}Z_{13} + Z_{03}Z_{12} = 0\}$$

where $Z_{ij} := z_i \wedge z_j$ are the coordinates on $\mathbb{P}^5$. The induced linearized $\mathbb{C}^*$ action is given by

$$\mathbb{C}^* \times \mathbb{C}^6 \longrightarrow \mathbb{C}^6$$
$$t \times (Z_{01}, Z_{02}, Z_{03}, Z_{12}, Z_{13}, Z_{23}) \mapsto (t^{-2}Z_{01}, Z_{02}, Z_{03}, Z_{12}, Z_{13}, t^2 Z_{23})$$

The line L is mapped into the point $(z_0 + z_2) \wedge (z_1 + z_3) = Z_{01} + Z_{03} - Z_{12} + Z_{23}$ which can be seen to be semi-stable.

Another thing to notice is that the only two unstable points in $\overline{M}_{0,0}(\mathbb{P}^3, 1)$ are $\{Z_{02} = Z_{03} = Z_{12} = Z_{13} = Z_{23} = 0\}$ and $\{Z_{01} = Z_{02} = Z_{03} = Z_{12} = Z_{13} = 0\}$ which correspond to the lines $L' = \{z_0 = z_1 = 0\}$ and $L'' = \{z_2 = z_3 = 0\}$.

The ring of invariants of $\mathbb{C}[Z_{i,j}]$ is generated by $Z_{02}, Z_{03}, Z_{12}, Z_{13}, Z_{01}Z_{23}$. Consequently, a projective embedding of the GIT quotient of $\mathbb{P}^5$ is given by the rational map

$$v : \mathbb{P}^5 \dashrightarrow \mathbb{P}^{10} \qquad \textbf{(Ver)}$$
$$[Z_{01}, Z_{02}, Z_{03}, Z_{12}, Z_{13}, Z_{23}] \longmapsto$$
$$[Z_{02}^2 : Z_{03}^2 : Z_{12}^2 : Z_{13}^2 : Z_{02}Z_{03} : Z_{02}Z_{12} : Z_{02}Z_{13} : Z_{03}Z_{12} : Z_{03}Z_{13} : Z_{12}Z_{13} : Z_{01}Z_{23}]$$

Now I have to study the other space, that of rational maps to $\mathbb{P}^3//\mathbb{C}^* = \mathbb{P}^1 \times \mathbb{P}^1$. The homology class induced on $\mathbb{P}^1 \times \mathbb{P}^1$ by the line $L \subset (\mathbb{P}^3)^{ss}$ is that of bidegree $(1, 1)$, so I have to clarify what the moduli space $\overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 1))$ is. Because $\mathbb{P}^1 \times \mathbb{P}^1$ is convex, it is a smooth variety of dimension 3 which contains the group $PSl(2)$ *via*

$$PSl(2) \longrightarrow \overline{M}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 1)) \qquad \tau \mapsto u_\tau := (\mathrm{id}_{\mathbb{P}^1}, \tau)$$

The problem is that a sequence of stable maps $(u_\tau)_\tau$ may degenerate to a stable map having reducible domain which consists of a principal component $\mathbb{P}^1$ and some bubble components. Because the image of such a *stable map* must represent the homology class $(1,1)$ of $\mathbb{P}^1\times\mathbb{P}^1$, it is quite easy to see that only one bubble component may appear. So, in the boundary of $\overline{M}_{0,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,1))$ I shall find the maps

$$\begin{aligned} u_\infty = (u_{1,\infty},u_{2,\infty}) : \mathbb{P}^1_{\text{principal}} \sqcup_0 \mathbb{P}^1_{\text{bubble}} &\longrightarrow \mathbb{P}^1\times\mathbb{P}^1 \\ u_{1,\infty|\mathbb{P}^1_{\text{principal}}} = \mathrm{id}_{\mathbb{P}^1} \qquad & u_{1,\infty|\mathbb{P}^1_{\text{bubble}}} = 0\in\mathbb{P}^1 \\ u_{2,\infty|\mathbb{P}^1_{\text{principal}}} = \mathtt{p}\in\mathbb{P}^1 \quad & u_{2,\infty|\mathbb{P}^1_{\text{bubble}}} \in PSl(2) \text{ such that } 0\mapsto \mathtt{p} \end{aligned}$$

For fixed $0\in\mathbb{P}^1_{\text{principal}}$ and $\mathtt{p}\in\mathbb{P}^1$, the maps above are equivalent, so they define the same point in $\overline{M}_{0,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,1))$. Consequently, the *equivalence classes* of such limit maps will be parameterized by the choices of 0 and $\mathtt{p}$ which both vary in a projective line. This means that the boundary divisor of $\overline{M}_{0,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,1))$ will be isomorphic with $\mathbb{P}^1\times\mathbb{P}^1$. There is actually a more concrete description of this space: the group $Sl(2)$ can be viewed and compactified inside $\mathbb{P}^4$ *via*

$$Sl(2)\subset\overline{Sl(2)} := \{a_0a_3 - a_1a_2 = a_4^2\}$$

The boundary divisor of $\overline{Sl(2)}$ has the equations $\{a_0a_3-a_1a_2=a_4=0\}$ i.e. it is a quadric in $\mathbb{P}^3$ which is isomorphic with the product of two projective lines. There is a $\mathbb{Z}_2$ action on $\mathbb{P}^4$ given by $[a_0:a_1:a_2:a_3:a_4]\sim[-a_0:-a_1:-a_2:-a_3:a_4]$ which is trivial on the hyperplane $\{a_4=0\}$. One should notice also that $PSl(2)=Sl(2)/\sim$. The compactification $\overline{PSl(2)}$ is by definition $\overline{Sl(2)}/\sim$ and its boundary divisor is $\mathbb{P}^1\times\mathbb{P}^1$. According to the previous discussion,

$$\overline{M}_{0,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,1)) = \overline{PSl(2)}$$

Now the relation between $\overline{M}_{0,0}(\mathbb{P}^3,1)//\mathbb{C}^*$ and $\overline{M}_{0,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,1))$ becomes quite transparent: remember that the equation of the space of lines in $\mathbb{P}^3$ is $\{Z_{01}Z_{23}=Z_{02}Z_{13}-Z_{03}Z_{12}\}$. Writing $Z_{01}Z_{23}=\big(\sqrt{Z_{01}Z_{23}}\big)^2$, one recovers the equation of $\overline{Sl(2)}$; the $\mathbb{Z}_2$ action is present also, since there are two possible choices for the square root unless $Z_{01}Z_{23}=0$. The two spaces are identified *via* the rational map **(Ver)**.

4.8 Some consequences

The context in which I place myself is again that of a complex, connected, linearly reductive group G which acts on the complex projective variety $\mathcal{O}_{\mathbf{X}}(1)\to\mathbf{X}$; let $\widehat{\mathbf{X}}$ denote the GIT quotient of $\mathbf{X}$. An element $A\in\mathrm{Im}\,(H_2(\mathbf{X}^{ss};\mathbb{Z})\to H_2(\mathbf{X},\mathbb{Z}))$ defines a homology class $\widehat{A}\in H_2(\widehat{\mathbf{X}};\mathbb{Z})$ *via* the quotient map $\phi:\mathbf{X}^{ss}\to\mathbf{X}$. From the theorem 4.6.7 I deduce that there is a rational map

$$\overline{M}_{g,k}(\mathbf{X},A) \dashrightarrow \overline{M}_{g,k}(\widehat{\mathbf{X}},\widehat{A})$$

whose domain of definition contains the open set of stable maps $[\mathbf{C},\mathbf{x},u]$ with the property that $\mathrm{Im}(u:\mathbf{C}\to\mathbf{X})\subset\mathbf{X}^{ss}$. The universality property of the categorical quotient ensures that the above rational map induces the commutative diagram

$$\widehat{\overline{M}_{g,k}(\mathbf{X},A)} := \overline{M}_{g,k}(\mathbf{X},A)//G \quad \overset{Q_k}{-\!-\!\to} \quad \overline{M}_{g,k}(\widehat{\mathbf{X}},\widehat{A})$$
$$\widehat{\mathtt{ev}}_k^{\mathbf{X}} \searrow \qquad \swarrow \mathtt{ev}_k^{\mathbf{X}//G} \qquad\qquad (\mathbf{CD})$$
$$(\mathbf{X}//G)^k$$

Under the assumption that the action of the group G on $\mathbf{X}$ is such that $\mathbf{X}^{ss} = \mathbf{X}^s_{(0)}$, I want to study some finiteness properties of the rational map Q_k; the result I want to prove is:

Proposition 4.8.1 *Assume that $\mathbf{X}^{ss} = \mathbf{X}^s_{(0)}$ and that there exist stable maps with irreducible domain of definition in $\overline{M}_{g,k}(\mathbf{X},A)$ whose image is contained in the semistable locus of $\mathbf{X}$. Then*

$$Q_k : \widehat{\overline{M}_{g,k}(\mathbf{X},A)} -\!-\!\to \overline{\operatorname{Im} Q_k}$$

is a generically one-to-one rational morphism.

Proof Let $u_1, u_2 : (\mathbf{C},\mathbf{x}) \to \mathbf{X}$ be two stable maps having the property that $\mathbf{C}$ is irreducible, their images are contained in $\mathbf{X}^{ss}$ and that $\widehat{u}_1 = \widehat{u}_2 \in \overline{M}_{g,k}(\widehat{\mathbf{X}},\widehat{A})$. I know from the theorem 4.6.7 that in this case $(\mathbf{C},\mathbf{x},u_{1,2})$ are semi-stable points in $\overline{M}_{g,k}(\mathbf{X},A)$. Let me define the map

$$v : \mathbf{C} \to \mathbf{X} \times \mathbf{X}$$
$$v(p) := (u_1(p), u_2(p)) \quad \forall\, p \in \mathbf{C}$$

If σ denotes the action of G on $\mathbf{X}$, I can consider its graph

$$\begin{array}{ccc} \Gamma_\sigma & \longrightarrow & G \times \mathbf{X} \times \mathbf{X} \\ \swarrow \quad \searrow & & \\ G \times \mathbf{X} \overset{\sigma}{\to} \mathbf{X} & & \end{array}$$

which comes with another two projections

$$\begin{array}{ccc} \Gamma_\sigma & \overset{\mathtt{pr}}{\longrightarrow} & \mathbf{X} \times \mathbf{X} \\ \mathtt{pr}_G \downarrow & & \\ G & & \end{array}$$

For $(x, g_0 x) \in (\mathbf{X}^{ss} \times \mathbf{X}^{ss}) \cap \operatorname{Im} \mathtt{pr}$,

$$\mathtt{pr}^{-1}(x, g_0 x) = \{(g,x,gx) |\ gx = g_0 x\} = \big\{(g,x,gx) |\ g_0^{-1} g x = x\big\} = g_0 \cdot \mathrm{Stab}_G x.$$

The assumption that $\mathbf{X}^{ss} = \mathbf{X}^s_{(0)}$ implies that this fibre is finite. Construct the following commuting diagram

$$\begin{array}{ccc} \Gamma_\sigma \times_{\mathbf{X}\times\mathbf{X}} \mathbf{C} =: \Gamma_{\sigma,\mathbf{C}} & \longrightarrow & \mathbf{C} \\ \downarrow & & \downarrow v \\ \Gamma_\sigma & \overset{\mathtt{pr}}{\longrightarrow} & \mathbf{X} \times \mathbf{X} \end{array}$$

Because $\operatorname{Im} u_1$, $\operatorname{Im} u_2 \subset \mathbf{X}^{ss}$ and $\widehat{u}_1 = \widehat{u}_2$, it follows that $\operatorname{Im} v \subset (\mathbf{X}^{ss} \times \mathbf{X}^{ss}) \cap \operatorname{Im} \mathtt{pr}$. I already remarked that the fibre of $\mathtt{pr}$ over a point $(x, g_0 x) \in (\mathbf{X}^{ss} \times \mathbf{X}^{ss}) \cap \operatorname{Im} \mathtt{pr}$ is finite. As a consequence, the map $\Gamma_{\sigma,\mathbf{C}} \to \mathbf{C}$ is surjective and with finite fibres, so $\Gamma_{\sigma,\mathbf{C}}$ is a complete curve. The image of the composed map

$$\begin{array}{ccc} \Gamma_{\sigma,\mathbf{C}} & \longrightarrow & \Gamma_\sigma \\ & \searrow & \downarrow \mathtt{pr}_G \\ & & G \end{array}$$

must be a finite number of points because G is affine and $\Gamma_{\sigma,\mathbf{C}}$ is complete; I denote by

$$\{g_1, \ldots, g_t\} := \operatorname{Im}(\Gamma_{\sigma,\mathbf{C}} \to G)\,.$$

The following description of $\Gamma_{\sigma,\mathbf{C}}$

$$\begin{aligned} \Gamma_{\sigma,\mathbf{C}} &= \{(g, x, gx, p) \in \Gamma_\sigma \times \mathbf{C} \mid (x, gx) = v(p) = (u_1(p), u_2(p))\} \\ &= \{(g, p) \in G \times \mathbf{C} \mid u_2(p) = gu_1(p)\} \end{aligned}$$

shows that $\mathbf{C}$ can be written

$$\mathbf{C} = \bigcup_{l=1}^{t} \mathbf{C}_l := \bigcup_{l=1}^{t} \{p \in \mathbf{C} \mid u_2(p) = g_l u_1(p)\}\,.$$

Each $\mathbf{C}_l$ is a Zariski closed subset of $\mathbf{C}$. Because $\mathbf{C}$ is assumed to be irreducible, I deduce that there exist a g such that $u_2 = gu_1$. But this means that $[\mathbf{C}, \mathbf{x}, u_1] = [\mathbf{C}, \mathbf{x}, u_2] \in \widehat{\overline{M}_{g,k}(\mathbf{X}, A)}$. $\square$

Remark 4.8.2 The reasoning of this proposition shows that for arbitrary $\mathbf{C}$, under the assumption that $\mathbf{X}^{ss} = \mathbf{X}^s_{(0)}$, the map Q_k is generically a finite to one map on its image. On the other hand, from the example treated in the section 4.7, one may see that this result is the best one can expect: the line $\{z_0 = z_2 = 0\}$ intersectes the unstable locus (transversally) in two points, but its stabilizer is the whole $\mathbb{C}^*$. $\diamond$

I want now to show how the previous results help, in some cases, to compute the GW-invariants of the GIT-quotient $\mathbf{X}//G$. I shall assume that all the G-semi-stable points of $\mathbf{X}$ have finite stabilizer in G i.e. $\mathbf{X}^{ss} = \mathbf{X}^s_{(0)}$. In this case, if $\phi : \mathbf{X}^{ss} \to \widehat{\mathbf{X}}$ denotes the quotient map, there is an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{T}_{\mathfrak{g}} \longrightarrow T\mathbf{X}^{ss} \longrightarrow \phi^* T\widehat{\mathbf{X}} \longrightarrow 0,$$

where $\mathcal{T}_{\mathfrak{g}}$ denotes the subvector bundle of $T\mathbf{X}^{ss}$ defined by the infinitesimal action of G; the fact that $\mathcal{T}_{\mathfrak{g}}$ is indeed a vector bundle is insured by the assumption that G acts with finite stabilizers on $\mathbf{X}^{ss}$. Moreover, this hypothesis implies also that $\mathcal{T}_{\mathfrak{g}}$ is trivial.

Consider now a 2-homology class $A \in H_2(\mathbf{X}^{ss}, \mathbb{Z})$ and denote, as usual, by $\widehat{A} \in H_2(\widehat{\mathbf{X}}, \mathbb{Z})$ the push-forward $\phi_* A$. The exact sequence above implies that

$$\langle c_1(\widehat{\mathbf{X}}), \widehat{A} \rangle = \langle \phi^* c_1(\widehat{\mathbf{X}}), A \rangle = \langle \jmath^*_{\mathbf{X}^{ss}} c_1(\mathbf{X}), A \rangle$$

I should recall that in order to compute the GW-invariants one uses some special cycles on the corresponding spaces of stable maps, called *virtual fundamental classes* (see the theorem 3.10.3). I want to compare now the dimensions of these cycles for $\overline{M}_{g,k}(\mathbf{X}, A)$ and $\overline{M}_{g,k}(\widehat{\mathbf{X}}, \widehat{A})$. If $n := \dim \mathbf{X}$ and $\gamma := \dim G$, then

$$\begin{aligned} \text{virt.dim.}\, \overline{M}_{g,k}(\mathbf{X}, A) &= (n - 3)(1 - g) + c_1(\mathbf{X}) \cdot A + k \\ \text{virt.dim.}\, \overline{M}_{g,k}(\widehat{\mathbf{X}}, \widehat{A}) &= (n - \gamma - 3)(1 - g) + c_1(\widehat{\mathbf{X}}) \cdot \widehat{A} + k. \end{aligned}$$

Their difference is

$$\text{virt.dim.}\,\overline{M}_{g,k}(\mathbf{X},A) - \text{virt.dim.}\,\overline{M}_{g,k}(\widehat{\mathbf{X}},\widehat{A}) = \gamma(1-g) = \gamma - \gamma g \leq \gamma.$$

In the previous inequality, equality holds if and only if $g=0$. This means that one may expect some "nice" relations only between the rational GW-invariants of $\mathbf{X}$ and $\widehat{\mathbf{X}}$. Because of technical difficulties, I shall study only the case when $\mathbf{X}$ and its GIT quotient $\widehat{\mathbf{X}}$ are both flag varieties. In such a case, the commutative diagram **(CD)** and the proposition 4.8.1 above imply the following:

Theorem 4.8.3 *Let G be a complex, connected, linearly reductive algebraic group acting on a irreducible flag variety $\mathbf{X}$ via a linearization of $\mathcal{O}_{\mathbf{X}}(1) \to \mathbf{X}$ in such a way that $\mathbf{X}^{ss} = \mathbf{X}^{s}_{(0)}$ and the* GIT *quotient $\widehat{\mathbf{X}} := \mathbf{X}//G$ is still a flag variety. Assume that $A \in H_2(\mathbf{X}^{ss},\mathbb{Z})$ is a 2-homology class with the property that there exist stable maps $(\mathbb{P}^1,\mathbf{x},u)$ such that $u_*[\mathbb{P}^1] = A$ and $u(\mathbb{P}^1) \subset \mathbf{X}^{ss}$. Let $\mathbb{A}/G := [\widehat{\overline{M}_{0,k}(\mathbf{X},A)}]$ and $\widehat{\mathbb{A}} := [\overline{M}_{0,k}(\widehat{\mathbf{X}},\widehat{A})]$ denote the corresponding $\mathbb{Q}$-fundamental classes, both of them having real dimension equal $2D$. Then there is the following commutative diagram*

$$\begin{array}{ccccccc}
 & & & & H^{2D}\left(\widehat{\overline{M}_{0,k}(\mathbf{X},A)}\right) & & \\
 & & & \nearrow & \uparrow & \searrow^{(\mathbb{A}/G)\cap\cdot} & \\
H^*_G(\mathbf{X})^{\otimes k} & \twoheadrightarrow & H^*(\widehat{\mathbf{X}})^{\otimes k} & & & & \mathbb{Q} \\
 & & & \searrow & & \nearrow_{\widehat{\mathbb{A}}\cap\cdot} & \\
 & & & & H^{2D}\left(\overline{M}_{0,k}(\widehat{\mathbf{X}},A)\right) & &
\end{array}$$

where the coefficients of all the (co)homology rings are rational.

Proof The assumptions of the theorem ensure that the spaces $\overline{M}_{0,k}(\mathbf{X},A)$ and $\overline{M}_{0,k}(\widehat{\mathbf{X}},\widehat{A})$ are normal, irreducible, with orbifold-like singularities (cf. 4.3.6). The GIT quotient $\widehat{\overline{M}_{0,k}(\mathbf{X},A)}$ is still irreducible and normal; by Zariski's main theorem and the proposition 4.8.1 I deduce that the map $Q_k : \widehat{\overline{M}_{0,k}(\mathbf{X},A)} - \to \overline{M}_{0,k}(\widehat{\mathbf{X}},\widehat{A})$ is birational.

The first arrow is surjective by Kirwan's result 4.5.12. For proving the commutativity of the first cell, I use the commutative diagram **(CD)**. For $\alpha \in H^{2D}(\widehat{\mathbf{X}}^k)$, the pull-back $\left(\mathtt{ev}_k^{\mathbf{X}//G}\right)^*\alpha \in H^{2D}\left(\overline{M}_{0,k}(\widehat{\mathbf{X}},\widehat{A})\right)$ is a rational multiple of the volume form $Vol_{\overline{M}_{0,k}(\widehat{\mathbf{X}},\widehat{A})}$ (which evaluates one on $\widehat{\mathbb{A}}$). Similarly, the pull-back $\left(\widehat{\mathtt{ev}}_k^{\mathbf{X}}\right)^*\alpha$ is a rational multiple of the volume form $Vol_{\widehat{\overline{M}_{0,k}(\mathbf{X},A)}}$ (which evaluates one on $\mathbb{A}/G$). Because the map Q_k is birational, these multiples must be the same. The second cell is commutative because $(Q_k)_*(\mathbb{A}/G) = \widehat{\mathbb{A}}$ and $(Q_k)^*Vol_{\overline{M}_{0,k}(\widehat{\mathbf{X}},\widehat{A})} = Vol_{\widehat{\overline{M}_{0,k}(\mathbf{X},A)}}$.
□

Remark 4.8.4 (Question) In the theorem above, the assumption that the GIT quotient $\widehat{\mathbf{X}}$ is still a flag variety is strong; *à fortiori*, the quotient $\widehat{\mathbf{X}}$ must be smooth. An instance of such a well-behaved (linearized) action is that studied in the previous section 4.7.

In view of the proposition 4.8.1, the rational map $\widehat{\overline{M}_{0,k}(\mathbf{X},A)} - \to \overline{M}_{0,k}(\widehat{\mathbf{X}},\widehat{A})$ is generically one-to-one on its image under the assumption that the class A can be represented by a stable map $(\mathbb{P}^1,\mathbf{x},u)$ such that $u(\mathbb{P}^1) \subset \mathbf{X}^{ss}$. Therefore the closure of the image of the map is a cycle of the same dimension as the virtual dimension of $\overline{M}_{0,k}(\widehat{\mathbf{X}},\widehat{A})$. The question is the following:

is this cycle precisely the virtual cycle of $\overline{M}_{0,k}(\widehat{\mathbf{X}}, \widehat{A})$? If the answer is yes, then the theorem above holds in this more general case when $\widehat{\mathbf{X}}$ is not necessarily a flag variety anymore. $\diamond$

4.9 The symplectic perspective of the problem

In the sequel I would like to recall some facts which show the strong link between the symplectic Marsden-Weinstein quotient and the GIT quotient. Here is the context in which these two concepts agree: let $\mathbf{X}$ be a smooth, complex projective variety and G a connected, linearly reductive, complex algebraic group acting on it *via* a linearization of the very ample line bundle $\mathcal{O}_{\mathbf{X}}(1) \to \mathbf{X}$. In this case, using the linear system associated to $\mathcal{O}_{\mathbf{X}}(1)$, we may assume that $G \subset Gl(r+1)$ and $\mathbf{X} \subset \mathbb{P}^r$ is a subvariety which is invariant under the induced action. The definition-theorem 4.5.1 says that $G = K^{\mathbb{C}}$, where K is the maximal compact subgroup of G. By an appropriate choice of coordinates on $\mathbb{C}^{r+1}$, it may be assumed that $K \subset U(r+1)$, so it will preserve the Fubini-Study metric on $\mathbb{P}^r$ and the induced Kähler form on $\mathbf{X}$. Corresponding to the K-action there is a moment map $m : \mathbf{X} \to \mathfrak{k}^*$ with values in the dual of the Lie algebra of K. An explicit formula for m is given in [MFK], page 146. The following theorem is a consequence of the work of G.Kempf, L.Ness (see [KN]) and V.Guillemin, S.Sternberg (see [GS]):

Theorem 4.9.1 ([MFK], page 148) *Assume that* $\mathbf{X}^{ss} = \mathbf{X}^s_{(0)}$. *Then*

(i) A point $x \in \mathbf{X}$ *is G-semi-stable if and only if* $\overline{Gx} \cap m^{-1}(0) \neq \emptyset$ *;*

(ii) The inclusion $m^{-1}(0)/K \longrightarrow X//G$ *is a homeomorphism.*

In the statement of the theorem, $\mathbf{X}^s_{(0)}$ denotes the stable points in $\mathbf{X}$ with finite stabilizer. Requirering that $\mathbf{X}^{ss} = \mathbf{X}^s_{(0)}$ is equivalent to the fact that the action of K on $m^{-1}(0)$ has finite stabilizers (see [GS], page 526). In this case, the equality

$$\langle \mathrm{d}m_x(w), a\rangle = \omega(V_a, w) \quad \forall x \in m^{-1}(0) \quad \forall w \in T_x\mathbf{X} \quad \forall a \in \mathfrak{k}$$

proves that 0 is a regular value for m. Here and in what follows, if $a \in \mathfrak{k}$, V_a denotes the vector field induced by it on $\mathbf{X}$.

It is natural to ask which is the symplectic counterpart of the GIT quotient of $\overline{M}_{g,k}(\mathbf{X}, A)$ studied before. A second reason for such a study is the hope that this way it will be easier to visualize the meaning of the proposition 4.6.1.

I explained already that if G acts on $\mathbf{X}$, it does so on $\overline{M}_{g,k}(\mathbf{X}, A)$ also and, moreover, it preserves the map $\varpi : \overline{M}_{g,k}(\mathbf{X}, A) \longrightarrow \overline{M}_{g,k}$. Here I have to assume that $2g - 2 + k > 0$ in order to ensure the existence of the space $\overline{M}_{g,k}$. Let $[(\mathbf{C}^{st}, \mathbf{y})] \in \overline{M}_{g,k}$ be a geometric point and let $\mathbf{F}_{[(\mathbf{C}^{st},\mathbf{y})]}$ be the corresponding geometric fibre of ϖ ; it is a projective scheme acted on by G *via* the induced linearization from $\det \mathfrak{Q}_k \to \overline{M}_{g,k}(\mathbf{X}, A)$.

Remark 4.9.2 It is easy to see that a point $[(\mathbf{C}, \mathbf{x}, u)] \in \mathbf{F}_{[\mathbf{C}^{st},\mathbf{y}]}$ is G-semi-stable if and only if $[(\mathbf{C}, \mathbf{x}, u)] \in \overline{M}_{g,k}(\mathbf{X}, A)$ is G-semi-stable.

This remark justifies the following construction: for a quasi-stable curve $\mathbf{C}$ of genus g, define

$$M_{\mathbf{C},k}(\mathbf{X}, A) := \left\{ u : (\mathbf{C}, \mathbf{x}) \to \mathbf{X} \;\middle|\; \begin{array}{l} (\mathbf{C}, \mathbf{x}, u) \text{ is a stable} \\ \text{map, } |\mathbf{x}| = k \;\, u_*[\mathbf{C}] = A \end{array} \right\}$$

Lemma 4.9.3 *$M_{\mathbf{C},k}(\mathbf{X},A)$ carries a natural quasi-projective scheme structure.*

Proof As in section 4.3, I can assume that $\mathbf{X}=\mathbb{P}^r$. Lemma 4.3.3 says that a map $(\mathbf{C},\mathbf{x},u)$ is stable if and only if $L_{(\mathbf{C},\mathbf{x},u)}=\omega_{\mathbf{C}}(x_1+\ldots+x_k)\otimes u^*\mathcal{O}_{\mathbb{P}^r}(3)\to\mathbf{C}$ is ample and, according to the lemma 4.3.4, there is an integer $f=f(g,k,r,d)>0$ with the property that $L^f_{(\mathbf{C},\mathbf{x},u)}\to\mathbf{C}$ is very ample; I denoted $N=\dim H^0(\mathbf{C},L^f_{\mathbf{C}})$. In this way, any stable map $(\mathbf{C},\mathbf{x},u)$ gave rise to a canonical embedding $\mathbf{C}\to\mathbb{P}\left(H^0\left(\mathbf{C},L^f_{(\mathbf{C},\mathbf{x},u)}\right)^\vee\right)$ into a space *isomorphic* to $\mathbb{P}^{n-1}$; the ambiguity in the choice of this isomorphism is given by elements in $PGl(N)$. In order to define the space $M_{\mathbf{C},k}(\mathbf{X},A)$ I shall use a *fixed* but otherwise arbitrary stable map $(\mathbf{C},\mathbf{x}_0,u_0)$. Denote by $\mathcal{O}_{\mathbf{C}}(1)\to\mathbf{C}$ the very ample line bundle defined by the *fixed embedding*

$$\mathbf{C}\xrightarrow{j_0}\mathbb{P}\left(H^0\left(\mathbf{C},L^f_{(\mathbf{C},\mathbf{x},u)}\right)^\vee\right)=\mathbb{P}^{N-1}$$

and let $e:=\deg_{\mathbf{C}}\mathcal{O}_{\mathbf{C}}(1)$. If $(\mathbf{C},\mathbf{x},u)$ is a stable map, the Hilbert polynomial of its graph $\Gamma_u\subset\mathbf{C}\times\mathbb{P}^r$ is:

$$P(n)=\chi\left(\mathcal{O}_{\Gamma_u}\otimes(\mathcal{O}_{\mathbf{C}}(n)\boxtimes\mathcal{O}_{\mathbb{P}^r}(n)\right)=n(e+d)+\chi(\Gamma_u)=n(d+e)+(1-g)$$

This reasoning proves that each stable map $(\mathbf{C},\mathbf{x},u)$ defines a point $(\Gamma_u,\mathbf{x})\in\mathcal{H}ilb^P_{\mathbf{C}\times\mathbb{P}^r}\times(\mathbf{C}\times\mathbb{P}^r)^k$. Using the embedding $j_0:\mathbf{C}\to\mathbb{P}^{N-1}$, the graph Γ_u can be view as a subvariety of $\mathbb{P}^{N-1}\times\mathbb{P}^r=\mathbf{Y}$ and its Hilbert polynomial with respect to the very ample line bundle $\mathcal{L}=\mathcal{O}_{\mathbb{P}^{N-1}}(1)\boxtimes\mathcal{O}_{\mathbb{P}^r}(1)\to\mathbf{Y}$ is also P.

Clearly, the same is true for any closed subscheme $\mathbf{Z}$ of $\mathbf{C}\times\mathbb{P}^r$: the Hilbert polynomial inside $\mathbf{C}\times\mathbb{P}^r$ with respect to $\mathcal{O}_{\mathbf{C}}(1)\boxtimes\mathcal{O}_{\mathbb{P}^r}(1)$ is the same as the Hilbert polynomial of its image $j_{0*}\mathbf{Z}$ inside $\mathbf{Y}$ with respect to $\mathcal{L}$. Indeed, $\chi(\mathcal{O}_{\mathbf{Z}}\otimes j_0^*\mathcal{L}^n)=\chi(\mathcal{O}_{j_{0*}\mathbf{Z}}\otimes\mathcal{L}^n)$. This way I get a *closed immersion*

$$\boldsymbol{\mathcal{H}}_{\mathbf{C}}:=\mathcal{H}ilb^P_{\mathbf{C}\times\mathbb{P}^r}\longrightarrow\mathcal{H}ilb^P_{\mathbf{Y}}=\boldsymbol{\mathcal{H}}.$$

The ample line of $\boldsymbol{\mathcal{H}}_{\mathbf{C}}$ is given by the restriction of $\det\mathfrak{Q}\to\boldsymbol{\mathcal{H}}$ (its definition is given in the end of the section 4.2). Recall from the theorem 4.3.5 that there is a subscheme $\mathcal{S}$ of $\boldsymbol{\mathcal{H}}\times\mathbf{Y}^k$ corresponding to the locus of genus g, k-marked stable maps to $\mathbb{P}^r$ which represent d times the generator of $H_2(\mathbb{P}^r,\mathbb{Z})$. Let me construct the following commuting diagram:

$$\begin{array}{ccc}\overline{M}_{\mathbf{C},k}(\mathbb{P}^r,d):=\boldsymbol{\mathcal{H}}_{\mathbf{C}}\times(\mathbf{C}\times\mathbb{P}^r)^k\times_{\boldsymbol{\mathcal{H}}\times\mathbf{Y}^k}\mathcal{S} & \longrightarrow & \mathcal{S}\\ \downarrow & & \downarrow\\ \boldsymbol{\mathcal{H}}_{\mathbf{C}}\times(\mathbf{C}\times\mathbb{P}^r)^k & \longrightarrow & \boldsymbol{\mathcal{H}}\times\mathbf{Y}^k\end{array}$$

The map $\overline{M}_{\mathbf{C},k}(\mathbb{P}^r,d)\to\boldsymbol{\mathcal{H}}_{\mathbf{C}}\times(\mathbf{C}\times\mathbb{P}^r)^k$ is an immersion and shows that $\overline{M}_{\mathbf{C},k}(\mathbb{P}^r,d)$ is a quasi-projective scheme. Its ample line bundle is determined by the restriction of $\det\mathfrak{Q}_k\to\boldsymbol{\mathcal{H}}\times\mathbf{Y}^k$ (for the definition of $\mathfrak{Q}_k$, see 4.4). One can see that $M_{\mathbf{C},k}(\mathbb{P}^r,d)$ is a subscheme of $\overline{M}_{\mathbf{C},k}(\mathbb{P}^r,d)$ *via* $(\mathbf{C},\mathbf{x},u)\mapsto(\Gamma_u,\mathbf{x})$ and therefore it is quasi-projective. □

Note Despite the notation, $\overline{M}_{\mathbf{C},k}(\mathbf{X},A)$ is not necessarily complete: indeed, just consider the case when $\mathbf{C}=\mathbb{P}^1$ and $k=0$. However, it is always compact in the Gromov topology (see 3.10), being the Gromov compactification of $M_{\mathbf{C},k}(\mathbf{X},A)$. It is projective if $(\mathbf{C},\mathbf{x}_0)$ is a *stable curve* in the sense of Deligne-Mumford (remember that I used a fixed stable map $(\mathbf{C}, \mathbf{x}_0, u_0)$ for the embedding $j_0 : \mathbf{C} \to \mathbb{P}^{N-1}$). Indeed, in this case the intersection of the $PGl(N)$-orbit of a stable map in $\mathcal{S} \to \boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$ with the image of $\boldsymbol{\mathcal{H}}_{\mathbf{C}} \times (\mathbf{C} \times \mathbb{P}^r)^k \to \boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$ consists of finitely many points. Consequently, the map $\overline{M}_{\mathbf{C},k}(\mathbb{P}^r, d) \to \overline{M}_{g,k}(\mathbb{P}^r, d)$ is finite. Because $\overline{M}_{g,k}(\mathbb{P}^r, d)$ is projective, the conclusion follows.

Remark 4.9.4 Notice also that because the ample line bundle on $\overline{M}_{\mathbf{C},k}(\mathbf{X}, A)$ is given by the restriction of $\mathfrak{Q}_k$, the G-semi-stable points are precisely those which lie above G-semi-stable points in $\overline{M}_{g,k}(\mathbf{X}, A)$ under the natural map $\overline{M}_{\mathbf{C},k}(\mathbf{X}, A) \to \overline{M}_{g,k}(\mathbf{X}, A)$. $\diamond$

The reason for introducing the space $M_{\mathbf{C},k}(\mathbf{X}, A)$ is that this way I can work with maps instead of equivalence classes of maps. For the symplectic point of view I shall consider $M_{\mathbf{C},k}(\mathbf{X}, A)$ with the reduced scheme structure i.e. as a quasi-projective variety. The Zariski tangent space of $M_{\mathbf{C},k}(\mathbf{X}, A)$ at a point $(\mathbf{C}, \mathbf{x}, u)$ is

$$T_{(\mathbf{C},\mathbf{x},u)} M_{\mathbf{C},k}(\mathbf{X}, A) = \left\{ (\xi, v_1, \dots, v_k) \;\middle|\; \begin{array}{l} \xi \in H^0(\mathbf{C}, u^* T\mathbf{X}) \\ v_j \in T_{x_j}\mathbf{C},\ j = 1, \dots, k \end{array} \right\}$$

In what follows, I shall compute the Kähler form on $M_{\mathbf{C},k}(\mathbf{X}, A)$ induced by its projective embedding. The description of the projective embedding of $\boldsymbol{\mathcal{H}} \times \mathbf{Y}^k$ given in section 4.4 and lemma 4.9.3 implies that the ample line bundle over $M_{\mathbf{C},k}(\mathbf{X}, A)$ is $\det(p_* \overline{E}^* \mathcal{L}^l) \otimes \overline{\mathtt{ev}}^* (\mathcal{L}^l)^{\boxtimes k}$, where

$$\begin{array}{ccc} \overline{E}^* \mathcal{L}^l & & \mathcal{L}^l = \mathcal{O}_{\mathbb{P}^{N-1}}(l) \boxtimes \mathcal{O}_{\mathbb{P}^r}(l) \\ \downarrow & & \downarrow \\ M_{\mathbf{C},k}(\mathbf{X}, A) \times \mathbf{C} & \xrightarrow{\overline{E} = (j_0, E)} & \mathbb{P}^{N-1} \times \mathbb{P}^r \\ p \downarrow & & \\ M_{\mathbf{C},k}(\mathbf{X}, A) & & \end{array}$$

and

$$\begin{array}{rcl} M_{\mathbf{C},k}(\mathbf{X}, A) & \xrightarrow{\overline{\mathtt{ev}} = (\overline{\mathtt{ev}}_1, \dots, \overline{\mathtt{ev}}_k)} & \left(\mathbb{P}^{N-1} \times \mathbb{P}^r\right)^k \qquad \overline{\mathtt{ev}}_{\mathtt{j}} = (j_0, \mathtt{ev}_{\mathtt{j}}) \\ (\mathbf{C}, \mathbf{x}, u) & \longmapsto & \left((j_0(x_1), u(x_1)), \dots, (j_0(x_k), u(x_k))\right). \end{array}$$

The Kähler form on $M_{\mathbf{C},k}(\mathbf{X}, A)$ induced by its projective embedding is $-1/2\pi i \times$ [curvature of $\det((p_* \overline{E}^* \mathcal{L}^l) \otimes \overline{\mathtt{ev}}^* (\mathcal{L}^l)^{\boxtimes k})$]. For computing this curvature, I need a Hermitian metric on $\mathcal{L}$ and a Kähler metric on the fibres of $M_{\mathbf{C},k}(\mathbf{X}, A) \times \mathbf{C} \xrightarrow{p} M_{\mathbf{C},k}(\mathbf{X}, A)$ i.e. on $\mathbf{C}$. The fibres of p will be all isometric, the Kähler form on them being

$$\gamma_{\mathbf{C}} := \frac{1}{e} j_0^* \omega_{\mathbb{P}^{N-1}} \qquad e := \deg_{\mathbf{C}} \mathcal{O}_{\mathbb{P}^{N-1}}(1).$$

This choice reflects the fact that for defining the space $M_{\mathbf{C},k}(\mathbf{X}, A)$ I have required the maps $(\mathbf{C}, \mathbf{x}, u)$ to have a *fixed* domain of definition. On $\mathcal{O}_{\mathbb{P}^{N-1}}(1)$ and $\mathcal{O}_{\mathbb{P}^r}(1)$ I consider the Hermitian metrics whose curvatures are $-2\pi i \omega_{\mathbb{P}^{N-1}}$ and $-2\pi i \omega_{\mathbb{P}^r}$ respectively, with $\omega_{\mathbb{P}^{N-1}}$ and ω_{Bp^r} the corresponding Fubini-Study forms. There is an induced Hermitian metric on $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{N-1}}(1) \boxtimes \mathcal{O}_{\mathbb{P}^r}(1)$ and *à fortiori* on $\overline{E}^* \mathcal{L}^l$.

It is easy to see which is the expression of the curvature of $\overline{\mathtt{ev}}^*(\mathcal{L}^l)^{\boxtimes k}$ at a point $(\mathbf{C},\mathbf{x},u)\in M_{\mathbf{C},k}(\mathbf{X},A)$:

$$\Omega_1 := -\frac{1}{2\pi i}(R^{\mathtt{ev}^*(\mathcal{L}^l)^{\boxtimes k}})_{(\mathbf{C},\mathbf{x},u)} = le\sum_{j=1}^{k}(\gamma_{\mathbf{C}})_{x_j} + l\sum_{j=1}^{k}\left(\mathtt{ev}_j^*\omega_{\mathbb{P}^r}\right)_{(\mathbf{C},\mathbf{x},u)}.$$

For computing the curvature of $\det(p_*\overline{E}^*\mathcal{L}^l)$ the first thing I notice is that this line bundle is actually the determinant of the derived direct image of $\overline{E}^*\mathcal{L}^l$, since by lemma 4.3.4 $p_*^1\overline{E}^*\mathcal{L}^l=0$. Consequently, I may apply the differential form of the Atiyah-Singer index theorem for families which is proved in a series of papers [BF1,2], [BGS1] by J.M.Bismut, D.Freed, H.Gillet, Ch.Soulé. According to theorem 0.1, [BGS1] page 51, if $\mathbf{C}$ is *a smooth curve*, the curvature

$$\Omega_2 := -\frac{1}{2\pi i}R^{\det p_*\overline{E}^*\mathcal{L}^l} = \int_{\mathbf{C}} Td\left(-\frac{1}{2\pi i}R^{T_{\mathbf{C}}}\right)\cdot exp\left(-\frac{1}{2\pi i}R^{\overline{E}^*\mathcal{L}^l}\right).$$

Here $R^{\det p_*\overline{E}^*\mathcal{L}^l}$ denotes the curvature of the reletive tangent bundle of the projection p (i.e. of $T_{\mathbf{C}}$) corresponding to the Kähler metric $\gamma_{\mathbf{C}}$ on $\mathbf{C}$ and $R^{\overline{E}^*\mathcal{L}^l}$ is the curvature of the line bundle $\overline{E}^*\mathcal{L}^l$ with respect to the Hermitian metric induced by that on $\mathcal{L}^l$. Let me denote $\gamma := (-1/2\pi i)R^{\det p_*\overline{E}^*\mathcal{L}^l}$; it is a real form of type $(1,1)$ on $\mathbf{C}$ and therefore $\gamma = f\gamma_{\mathbf{C}}$ with $f:\mathbf{C}\to\mathbb{R}$ a smooth function having the property that $\int_{\mathbf{C}} f\gamma_{\mathbf{C}} = 2(1-g)$. On the other hand,

$$-\frac{1}{2\pi i}R^{\overline{E}^*\mathcal{L}^l} = l\overline{E}^*\left(-\frac{1}{2\pi i}R^{\mathcal{L}}\right) = l\overline{E}^*(\omega_{\mathbb{P}^{N-1}}+\omega_{\mathbb{P}^r}) = l(e\gamma_{\mathbf{C}}+E^*\omega_{\mathbb{P}^r})$$

The form Ω_2 is the term of degree two in

$$\int_{\mathbf{C}}\left(1+\frac{1}{2}\gamma\right)\left(1+l(e\gamma_{\mathbf{C}}+E^*\omega_{\mathbb{P}^r})+\frac{l^2}{2}(e\gamma_{\mathbf{C}}+E^*\omega_{\mathbb{P}^r})^2\right).$$

I deduce that

$$\begin{aligned}\Omega_2 &= \frac{l^2}{2}\int_{\mathbf{C}}(e\gamma_{\mathbf{C}}+E^*\omega_{\mathbb{P}^r})^2 + \frac{l}{2}\int_{\mathbf{C}}(e\gamma_{\mathbf{C}}+E^*\omega_{\mathbb{P}^r})\wedge\gamma \\ &= \frac{l^2}{2}\int_{\mathbf{C}}(E^*\omega_{\mathbb{P}^r})^2 + 2eE^*\omega_{\mathbb{P}^r}\wedge\gamma_{\mathbf{C}} + \frac{l}{2}\int_{\mathbf{C}}E^*\omega_{\mathbb{P}^r}\wedge\gamma \\ &= \frac{l^2}{2}\int_{\mathbf{C}}(E^*\omega_{\mathbb{P}^r})^2 + l^2e\int_{\mathbf{C}}E^*\omega_{\mathbb{P}^r}\wedge\gamma_{\mathbf{C}} + \int_{\mathbf{C}}f\cdot E^*\omega_{\mathbb{P}^r}\wedge\gamma_{\mathbf{C}}.\end{aligned}$$

The computations above prove the

Proposition 4.9.5 *If* $\mathbf{C}$ *is a smooth curve, the curvature of* $\det(p_*\overline{E}^*\mathcal{L}^l)\otimes\overline{\mathtt{ev}}^*(\mathcal{L}^l)^{\boxtimes k}$ *divided by* $-2\pi i$ *is* $\Omega=\Omega_1+\Omega_2$. *It represents the Kähler form on* $M_{\mathbf{C},k}(\mathbf{X},A)$ *induced by the projective embedding described in lemma 4.9.3.*

Let me come back to my original set-up: a complex, connected, linear algebraic group G acts on a smooth, irreducible complex projective variety $\mathbf{X}$ via a linearization through a very ample line bundle $\mathcal{O}_{\mathbf{X}}(1)\to\mathbf{X}$. In this case, using the linear system associated to $\mathcal{O}_{\mathbf{X}}(1)$, I may assume that G acts on $\mathbb{C}^{r+1}$ and, by an appropriate choice of coordinates, I am allowed to assume that the maximal compact subgroup K of G is included in $U(r+1)$. There is an induced action of G (so, *à fortiori*, of K) on $M_{\mathbf{C},k}(\mathbf{X}, A)$ defined by: $g \times (\mathbf{C}, \mathbf{x}, u) \mapsto (\mathbf{C}, \mathbf{x}, gu)$. Because the K-action preserves the Fubiny-Study form $\omega_{\mathbb{P}^r}$ and the maps $\overline{E}$ and $\overline{\mathtt{ev}}$ are obviously both K-invariant, it follows that Ω is K-invariant also.

I shall restrict myself to the case of $\mathbb{C}^*$-actions; in a certain sense, this is allowed by the Hilbert-Mumford criterion 4.5.7. I want to find a moment map for the induced S^1-action on $M_{\mathbf{C},k}(\mathbf{X}, A)$. If such a moment map exists, it is uniquely defined, up to a scalar constant, by the Kähler form Ω.

The S^1-action on $\mathbf{X}$ gives rise to a vector field V on $\mathbf{X}$ having the property that $\mathcal{L}_V \omega = 0$ and $\mathcal{L}_V J = 0$ because S^1 acts on $\mathbf{X}$ by isometries. The vector field $W := JV$ does not preserve ω in general but still preserves the complex structure J on $\mathbf{X}$: $\mathcal{L}_W J = 0$. In fact the vector field W corresponds to the action of $\mathbb{R}_+^* \hookrightarrow \mathbb{C}^*$. There is a S^1-invariant moment map $m : \mathbf{X} \longrightarrow \mathbb{R}$ which has, by definition, the property that

$$\mathrm{d}m_x(w) = \omega(V, w) \qquad \forall\, w \in T_x\mathbf{X}.$$

Consequently $\mathrm{d}m_x(W_x) = \omega_x(V_x, W_x) = \|V_x\|^2$, so m is an increasing function along the flow lines of W. Let $(\varphi_r)_{r\in\mathbb{R}}$ be the 1-parameter group of diffeomorphisms corresponding to W. Because W preserves the complex structure of $\mathbf{X}$, the φ_r's are in fact holomorphic automorphisms of $\mathbf{X}$ given by the formula

$$\varphi_r(x) = \mathbf{e}^r \cdot x \qquad \forall\, x \in \mathbf{X}$$

where the dot corresponds to the action of $\mathbb{R}_+^* \hookrightarrow \mathbb{C}^*$ on $\mathbf{X}$. An immediate consequence is that

$$\mathbb{R} \ni r \mapsto m(\varphi_r(x)) \in \mathbb{R}$$

are increasing functions for all $x \in \mathbf{X}$ and therefore, given a point $x \in \mathbf{X}$ either:

(i) x is a critical point for m and $\varphi_r(x) = x$ for all $r \in \mathbb{R}$ or

(ii) x is not critical and $\lim\limits_{r\to\infty} \varphi_r(x)$, $\lim\limits_{r\to-\infty} \varphi_r(x)$ are (distinct) critical points for m.

Assume that $\mathbf{X}^{ss} = \mathbf{X}^s_{(0)}$; we saw already that in this case S^1 acts with finite stabilizers on $m^{-1}(0)$ and consequently 0 is a regular value for m. According to the theorem 4.9.1, a point $x \in \mathbf{X}$ is semi-stable if and only if $\overline{\mathbb{C}^* x} \cap m^{-1}(0) \neq \emptyset$. In fact, because the points of $m^{-1}(0)$ have finite stabilizers, the last condition is equivalent to $\mathbb{R}_+^* x \cap m^{-1}(0) \neq \emptyset$. Indeed, I can restrict myself to the $\mathbb{R}_+^*$-action because $m^{-1}(0)$ is S^1-invariant. Secondly, if the point $\overline{\mathbb{R}_+^* x} \cap m^{-1}(0)$ would be reached only in the limit $r = 0$ or $r = +\infty$, then it would be critical a point for m which lies on $m^{-1}(0)$. But 0 is a regular value for m, a contradiction.

The S^1-action on $\mathbf{X}$ induces a (holomorphic) S^1-action on $M_{\mathbf{C},k}(\mathbf{X}, A)$; let denote by $\mathcal{V}$ the vector field on $M_{\mathbf{C},k}(\mathbf{X}, A)$ determined by it. At a point $(\mathbf{C}, \mathbf{x}, u)$

$$\mathcal{V}_{(\mathbf{C},\mathbf{x},u)} = (u^*V, 0, \ldots, 0) \in H^0(\mathbf{C}, u^*T\mathbf{X}) \times T_{x_1}\mathbf{C} \times \ldots T_{x_k}\mathbf{C}$$

Proposition 4.9.6 *The function*

$$M : M_{\mathbf{C},k}(\mathbf{X}, A) \longrightarrow \mathbb{R}$$

$$M(\mathbf{C},\mathbf{x},u) := l^2\int_{\mathbf{C}}(m\circ E)E^*\omega_{\mathbb{P}^r} + l^2 e\int_{\mathbf{C}}(m\circ E)\gamma_{\mathbf{C}} + \frac{l}{2}\int_{\mathbf{C}} f(m\circ E)\gamma_{\mathbf{C}} + l\sum_{j=1}^{k} m\circ \mathtt{ev}_j$$

is a moment map for this action.

Proof Because $E, \mathtt{ev}_j, m, \omega_{\mathbb{P}^r}$ are all S^1-invariant, it follows that M is also. In order to prove that M is a moment map, all it remains to to prove is that the differential of M is the same as the contraction of the Kähler form Ω on $M_{\mathbf{C},k}(\mathbf{X},A)$ with the vector field $\mathcal{V}$. It what follows, the symbol "$\lrcorner$" will denote always a contraction of a differential form by a vector field.

The contraction $\mathcal{V}\lrcorner(\gamma_{\mathbf{C}})_{x_j} = 0$ because the $T_{x_1}\mathbf{C}\times\ldots\times T_{x_k}\mathbf{C}$-component of $\mathcal{V}$ is zero.

- $$\begin{aligned}\mathcal{V}\lrcorner(\mathtt{ev}_j^*\omega_{\mathbb{P}^r}) &= \mathtt{ev}_j^*\left((\mathtt{ev}_{j*}\mathcal{V})\lrcorner\,\omega_{\mathbb{P}^r}\right) = \mathtt{ev}_j^*(V_{\mathtt{ev}_j(\cdot)}\lrcorner\,\omega_{\mathbb{P}^r})\\ &= \mathtt{ev}_j^*(\mathrm{d}m_{\mathtt{ev}_j(\cdot)}) = \mathrm{d}(m\circ\mathtt{ev}_j)\end{aligned}$$
- $$\mathcal{V}\lrcorner\int_{\mathbf{C}} E^*\omega_{\mathbb{P}^r}\wedge\gamma_{\mathbf{C}} = \int_{\mathbf{C}}\mathcal{V}\lrcorner(E^*\omega_{\mathbb{P}^r}\wedge\gamma_{\mathbf{C}}) = \int_{\mathbf{C}}(\mathcal{V}\lrcorner E^*\omega_{\mathbb{P}^r})\wedge\gamma_{\mathbf{C}}$$

At a point $p\in\mathbf{C}$,

$$\begin{aligned}\mathcal{V}_p\lrcorner(E^*\omega_{\mathbb{P}^r})_p &= E^*\left(E_*\mathcal{V}_p\lrcorner\,\omega_{\mathbb{P}^r,u(p)}\right) = E^*\left(V_{u(p)}\lrcorner\,\omega_{\mathbb{P}^r,u(p)}\right) = E^*\left(V_{u(p)}\lrcorner\,\omega_{\mathbb{P}^r,u(p)}\right)\\ &= E^*(\mathrm{d}m_{u(p)}) = \mathrm{d}(m\circ E)_p.\end{aligned}$$

I deduce that

- $$\mathcal{V}\lrcorner\int_{\mathbf{C}} E^*\omega_{\mathbb{P}^r}\wedge\gamma_{\mathbf{C}} = \int_{\mathbf{C}}\mathrm{d}(m\circ E)\wedge\gamma_{\mathbf{C}} = \int_{\mathbf{C}}\mathrm{d}\left((m\circ E)\gamma_{\mathbf{C}}\right) = \mathrm{d}\int_{\mathbf{C}}(m\circ E)\gamma_{\mathbf{C}}$$
- $$\begin{aligned}\mathcal{V}\lrcorner\int_{\mathbf{C}} f\cdot E^*\omega_{\mathbb{P}^r}\wedge\gamma_{\mathbf{C}} &= \int_{\mathbf{C}}\mathcal{V}\lrcorner(f\cdot E^*\omega_{\mathbb{P}^r}\wedge\gamma_{\mathbf{C}}) = \int_{\mathbf{C}} f\cdot(\mathcal{V}\lrcorner E^*\omega_{\mathbb{P}^r})\wedge\gamma_{\mathbf{C}}\\ &= \int_{\mathbf{C}} f\cdot\mathrm{d}(m\circ E)\wedge\gamma_{\mathbf{C}} \overset{(\star)}{=} \int_{\mathbf{C}}\mathrm{d}\left(f(m\circ E)\gamma_{\mathbf{C}}\right) = \mathrm{d}\int_{\mathbf{C}} f(m\circ E)\gamma_{\mathbf{C}}\end{aligned}$$

For writing the equality $(\star)$, I have used that $\mathrm{d}f\wedge\gamma_{\mathbf{C}} = \mathrm{d}_{\mathbf{C}}f\wedge\gamma_{\mathbf{C}} = 0$.

- $$\begin{aligned}\mathcal{V}\lrcorner\int_{\mathbf{C}}(E^*\omega_{\mathbb{P}^r})^2 &= 2\int_{\mathbf{C}}(\mathcal{V}\lrcorner E^*\omega_{\mathbb{P}^r})\wedge E^*\omega_{\mathbb{P}^r} = 2\int_{\mathbf{C}}\mathrm{d}(m\circ E)E^*\omega_{\mathbb{P}^r}\\ &= 2\int_{\mathbf{C}}\mathrm{d}\left((m\circ E)E^*\omega_{\mathbb{P}^r}\right) = \mathrm{d}\left(2\int_{\mathbf{C}}(m\circ E)E^*\omega_{\mathbb{P}^r}\right)\end{aligned}$$

These equalities alltogether show that M is indeed a moment map. □

Lemma 4.9.7 *S^1 acts with finite stabilizers on $M^{-1}(0)$.*

Proof I claim that for $(\mathbf{C},\mathbf{x},u)\in M^{-1}(0)$ there is a point $p\in\mathbf{C}$ with the property that $m(u(p))=0$ i.e. $u(p)\in m^{-1}(0)$. Suppose that it is not so; in this case, either $\mathrm{Im}\,u\subset\{m<0\}$ or $\mathrm{Im}\,u\subset\{m>0\}$. Let assume that I am in the first case. At a point $(\mathbf{C},\mathbf{x},u)\in M_{\mathbf{C},k}(\mathbf{X},A)$,

$$M(\mathbf{C},\mathbf{x},u) = l^2\int_{\mathbf{C}}(m\circ u)u^*\omega_{\mathbb{P}^r} + l\int_{\mathbf{C}}\left(le+\frac{1}{2}f\right)(m\circ u)\gamma_{\mathbf{C}} + l\sum_{j=1}^{k} m(u(x_j)).$$

I recall that the smooth real-valued function f defined on $\mathbf{C}$ is the "quotient" $R^{T_{\mathbf{C}}}/\gamma_{\mathbf{C}}$, where $R^{T_{\mathbf{C}}}$ denotes the curvature of the tangent bundle of $\mathbf{C}$ with respect to the Kähler form $\gamma_{\mathbf{C}}$. This last form was defined in tems of a *fixed* projective embedding of $\mathbf{C}$; in particular, it does not depend on l. Since for obtaining the projective embedding of the Hilbert scheme I have to take large positive integral values for l, I may assume that l is large enough for $le+\frac{1}{2}f$ to be a strictly positive function on $\mathbf{C}$. Let me notice also that because u is a holomorphic map and $\omega_{\mathbb{P}^r}$ is a positive $(1,1)$-form, the $(1,1)$-form $u^*\omega_{\mathbb{P}^r}$ on $\mathbf{C}$ is still positive. It becomes now clear that if $\operatorname{Im} u \subset \{m < 0\}$, $M(\mathbf{C},\mathbf{x},u)$ will be negative also. This contradicts the choice of $(\mathbf{C},\mathbf{x},u)$ in the zero locus of M.

The lemma follows now because, by assumption, S^1 acts with finite stabilizers on $m^{-1}(0)$. □

According to the theorem 4.9.1, a point $(\mathbf{C},\mathbf{x},u)$ is $\mathbb{C}^*$-semi-stable if and only if there is $r \in \mathbb{R}$ such that

$$\mathbf{e}^r \cdot (\mathbf{C},\mathbf{x},u) = (\mathbf{C},\mathbf{x},\varphi_r \circ u) \in M^{-1}(0).$$

Using these symplectic techniques, I recover easily the

Theorem 4.9.8 (4.6.7) *If* $\mathbf{C}$ *is a smooth curve and* $(\mathbf{C},\mathbf{x},u)$ *is a stable map having the property that* $\operatorname{Im} u \subset \mathbf{X}^{ss}(\mathcal{O}_{\mathbf{X}}(1))$, *then the point* $[(\mathbf{C},\mathbf{x},u)]$ *defines a* G*-semi-stable point in* $M_{g,k}(\mathbf{X},A)$.

Proof Remark 4.9.4 says that it is enough proving that the map $(\mathbf{C},\mathbf{x},u) \in M_{\mathbf{C},k}(\mathbf{X},A)$ is G-semi-stable; according to the Hilbert-Mumford criterion, is sufficient to prove this for all 1-PS $\lambda : \mathbb{C}^* \to G$. For a fixed 1-PS λ of G, lemma 4.9.7 asserts that $M_{\mathbf{C},k}(\mathbf{X},A)^{ss(\lambda)} = M_{\mathbf{C},k}(\mathbf{X},A)^{s(\lambda)}_{(0)}$. For proving that $(\mathbf{C},\mathbf{x},u)$ is λ-semi-stable, I have to show that its $\mathbb{C}^*$-orbit meets the zero-level set of the moment map M on $M_{\mathbf{C},k}(\mathbf{X},A)$.

By hypothesis, $\operatorname{Im} u \subset \mathbf{X}^{ss} \subset \mathbf{X}^{ss(\lambda)}$. Let me assume, for instance, that $M(\mathbf{C},\mathbf{x},u) < 0$. The assumption ensures that under the $\mathbb{R}^*_+$-action all the points $u(p)$ meet the $m^{-1}(0)$-level. Consequently $\varphi_r(u(p)) > 0$, $\forall p \in \mathbf{C}$ for $r \gg 0$. For such a large r, the translated map $(\mathbf{C},\mathbf{x},\varphi_r \circ u)$ will have the property that $M(\mathbf{C},\mathbf{x},\varphi_r \circ u) > 0$. A continuity argument proves that there is (a unique) r_0 such that $(\mathbf{C},\mathbf{x},\varphi_{r_0} \circ u) \in M^{-1}(0)$. □

Remark 4.9.9 There is one more problem I would have liked to treat: given a map $(\mathbf{C},\mathbf{x},u) \in \overline{M}_{g,k}(\mathbf{X},A)$ which is the induced stable map in the Marsden-Weinstein quotient $m^{-1}(0)$? Unfortunately, here I am running again into questions I don't know to answer. Consider on $\mathbf{X}$ the vector field $\overline{W} := -mW$; it is the gradient of the function $\mathbf{X} \ni x \mapsto -m(x)^2$. Its associated 1-parameter group ψ_r will have the tendency to collapse, for $r \to +\infty$, all the semi-stable points of $\mathbf{X}$ to the zero level set of m, so one may talk about a map $\psi_\infty = \lim\limits_{r\to+\infty} \psi_r$. My guess is that, for a stable map $(\mathbf{C},\mathbf{x},u)$ in $\mathbf{X}$, the induced stable map in $m^{-1}(0)$ is $(\mathbf{C},\mathbf{x},\widehat{\psi}_\infty \circ u)$. Here $\widehat{\psi}_\infty$ denotes the composition of ψ_∞ with the quotient map $m^{-1}(0) \to m^{-1}(0)/S^1$. The essential problem is that I don't see why $\widehat{\psi}_\infty : \mathbf{X}^{ss} \to m^{-1}(0)/S^1$ should be holomorphic.

I would like to mention also that this method might be extended to the symplectic case, where no algebraic methods are available anymore. In the symplectic case it is useful to work with some global covers of $\overline{M}_{g,k}(\mathbf{X},A)$, because in this case one can define *globally* and not just fibrewise an analogue of the moment map M. Such an map would be a rel-$\overline{M}_{g,k}$-moment map. ◇

References

[Be] A.Beauville: *Variétés Kähleriennes dont la 1ère classe de de Chern est nulle*, J.Differential Geom. **18** (1983), 755-782

[Br] M.Brion: *Cohomologie équivariente des points semi-stables*, J.Reine Angew Math. **421** (1991), 125-140

[BF1] J.M.Bismut, D.S.Freed: *The analysis of elliptic families: metrics and connections on determinant bundles*, Commun. Math. Phys. **106** (1986), 159-176

[BF2] J.M.Bismut, D.S.Freed: *The analysis of elliptic families: Dirac operators, Êta invariants and the holonomy theorem*, Commun. Math. Phys. **107** (1986), 103-163

[BGS1,2,3] J.M.Bismut, H.Gillet, Ch.Soulé: *Analytic torsion and holomorphic determinant bundles*, Commun. Math. Phys. **115** (1988), 49-78, 79-126, 301-351

[Ch] Xi Chen: *Rational curves on $K3$ surfaces*, alg-geom/9804075

[CKM] H.Clemens, J.Kollár, S.Mori: *Higher dimensional complex geometry*, Astérisque 166 (1988)

[Di] S.Diaz: *A bound on the dimension of complete subvarieties of M_g*, Duke Math. Journal **51** No.2, 405-408

[Do] S.K.Donaldson: *Symplectic submanifolds and almost complex geometry*, Preprint Univ. of Oxford, 1995

[EE] C.J.Earle, J.Eells : *A Fibre bundle description of Teichmüller theory*, J.Differential Geom. **3** (1969), 19-43

[F] W.Fulton: *Intersection Theory*, Springer-Verlag Berlin Heidelberg New York Tokyo 1984

[FP] W.Fulton, R.Pandharipande: *Notes on stable maps and quantum cohomology*, alg-geom/ 9608011

[FO] K.Fukaya, K.Ono: *Arnold conjecture and Gromov-Witten invariants*, Warwick preprint **29** (1996)

[Go] R.Gompf: *A new construction of symplectic manifolds*, Ann. of Math. **142** (1995), 527-595

[GH] Ph.Griffiths, J.Harris: *Principles of Algebraic Geometry*, John Wiley & Sons, New York etc, 1978

[Gr] M.Gromov: *Pseudo-holomorphic curves in symplectic manifolds*, Invent.Math. **82** (1985), 307-347

[Gk] A.Grothendieck: *Techniques de construction et théorèmes d'existence en géométrie algébrique* IV*: les schémas de Hilbert*, Séminaire Bourbaki 1960/1961, No.221

[GS] V.Guillemin, S.Sternberg: *Geometric quantization and multiplicities of group representations*, Invent.Math. **67** (1982), 515-538

[Ha] J.Harer: *The virtual cohomological dimension of the mapping class group of an orientable* *surface*, Invent.Math. **84** (1986), 157-176

[Hu] B.Hunt: *Complex manifold geography in dimension* 2 *and* 3, J.Differential Geom. **30** (1989), 51-153

[HL] R.Hain, E.Looijenga: *Mapping class groups and moduli spaces of curves*, alg-geom/ 9607004

[HLS] H.Hofer, V.Lizan, J-C.Sikorav: *On genericity for holomorphic curves in* 4*-dimensional almost-complex manifolds*, preprint 1994

[J] J.Jost : *Two-dimensional Geometric Variational Problems*, J.Wiley&Son Chichester New York Brisbane Toronto, 1990

[Ki] F.C.Kirwan: *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Princeton Univ. Press, 1984

[KN] G.Kempf, L.Ness: *The length of vectors in representation spaces*, Lect. Notes in Math. **732**, 1979

[L1] E.Looijenga: *Smooth Deligne-Mumford compactifications by means of Prym level structures*, J.Alg.Geom. **3** (1994) 283-293

[L2] E.Looijenga: *On the tautological ring of* M_g, Invent.Math. **121** (1995), 411-419

[LM] H.B.Lawson, M.L.Michelsohn: *Spin Geometry*, Princeton University Press, Princeton New Jersey, 1989

[LiTi] J.Li, G.Tian: *Virtual moduli cycles and* GW*-invariants*, alg-geom/9602007

[McD1] D.McDuff: *Examples of simply connected non-Kählerian manifolds*, J.Differential Geom. **20** (1984), 267-277

[McD2] D.McDuff: *The local behavior of holomorphic curves in almost complex* 4*-manifolds*, J.Differential Geom. **34** (1991), 143-164

[McDS1] D.McDuff, D.Salamon: *Introduction to Symplectic Topology*, Clarendon Press Oxford, 1995

[McDS2] D.McDuff, D.Salamon: J*-holomorphic Curves and Quantum Cohomology*, Univ. Lect. Series of AMS, 1994

[Mk] S.Mukai: *Duality of polarized* $K3$ *surfaces*, Proc. of Algebraic Geometry Euroconference, Warwick 1996

[Mu1] D.Mumford: *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry edited by M.Artin and J.Tate

[Mu2] D.Mumford: *Further pathologies in algebraic geometry*, Amer.J.Math. **84** (1962), 642-648

[MFK] D.Mumford, J.Fogarty, F.Kirwan: *Geometric Invariant Theory*, Springer Verlag Berlin Heidelberg New York NY London, 1994

[OV] Ch.Okonek, A.Van de Ven: *Cubic forms and complex* 3*-folds*, L'Enseignement Mathématique **41** (1995), 297-333

[PW] T.H.Parker, J.G.Wolfson: *Pseudo-holomorphic maps and bubble trees*, J.Func.Anal. **3** (1993), 63-98

[RT1] Y.Ruan, G.Tian: *A mathematical theory of quantum cohomology*, J.Differential Geom. **42** (1995), 259-367

[RT2] Y.Ruan, G.Tian: *Higher genus symplectic invariants and sigma models coupled with gravity*, alg-geom/9601005

[Si] B.Siebert: *Gromov-Witten invariants for general symplectic manifolds*, dg-ga/9608005

[T1] C.H.Taubes: *More constraints on symplectic forms from Seiberg-Witten invariants*, Math. Res.Lett.2 **No.1** (1995), 9-13

[T2] C.H.Taubes: *The Seiberg-Witten and Gromov invariants*, Math.Res.Lett.2 **No.2** (1995), 221-238

[Th] J.F.Thomsen: *Irreducibility of* $\overline{M}_{0,n}(G/P,\beta)$, alg-geom/9707005

[V] M.Verbitsky: *Hyperkähler embeddings and holomorphic symplectic geometry*, GAFA **No.1** (1995), 92-104

[W1] S.A.Wolpert: *On the homology of the moduli space of stable curves*, Ann. of Math. **118** (1983, 491-523

[W2] S.A.Wolpert: *Chern forms and the Riemann tensor for the moduli space of curves*, Invent.Math. **85** (1986), 119-145