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Introduction

Avant propos

Le groupe de tresses B_n, introduit par Artin en 1926 ([1]), joue un rôle remarquable dans plusieurs domaines des mathématiques, en particulier dans la théorie des nœuds.
Les Théorèmes d'Alexander et de Markov établissent une correspondance entre nœuds (et entrelacs) et tresses. Plus précisément deux entrelacs sont isotopes si et seulement si les tresses qui les représentent sont reliées par une suite de mouvements élémentaires dans la tour $\cup_{n\geq 2}B_n$.
Cela implique que la recherche des invariants d'entrelacs correspond à la construction des *traces de Markov* sur la tour de $\cup_{n\geq 2}\mathbb{C}[B_n]$, c'est-à-dire, des familles de fonctionnelles linéaires qui vérifient les conditions du théorème de Markov. Cette construction resta longtemps purement théorique, jusqu'aux années 80 et à la découverte du polynôme de Jones ([58]). La construction algébrique de ce polynôme est basée sur la définition (inductive) d'une trace de Markov sur les *algèbres de Hecke* ([23]), qui sont des quotients de dimension finie des algèbres des groupes de tresses.
Les travaux de Jones conduisent à la question naturelle suivante:

"Peut-on définir d'autres invariants d'entrelacs avec des constructions analogues au polynôme de Jones (et son extension, le polynôme HOMFLY-PT) ?"

Le premier résultat de cette thèse (dans l'ordre chronologique) est une réponse affirmative à cette question. Plus précisément dans le chapitre 5 nous nous intéressons aux algèbres de Hecke *cubiques*, qui sont d'autres quotients des algèbres des groupes de tresses. En suivant l'approche de Jones, nous construisons deux nouveaux invariants d'entrelacs dans $\mathbb{R}^3$. Ces invariants, différents des invariants HOMFLY-PT et de Kauffman, sont récursivement calculables et définis univoquement par deux relations skein (figure 1).

L'autre sujet de cette thèse est l'étude des tresses et, plus généralement, des tresses singulières sur les surfaces. Étant donnée une surface F on peut définir le groupe de tresses $B(n,F)$ avec une construction analogue à celle de B_n ([42]). Ces groupes sont une généralisation naturelle du groupe fondamental de la surface F et ils sont liés aux *Mapping class groups* et à la théorie des espaces de configurations ([17]). Un sous-groupe remarquable de $B(n,F)$ est le groupe de tresses pures $P(n,F)$, qui est le noyau de la projection de $B(n,F)$ dans le groupe symétrique à n éléments.
Nous exhibons de nouvelles présentations, simples, pour les groupes de tresses et de tresses pures sur les surfaces. Ces présentations sont des extensions de présentations usuelles de B_n et du groupe fondamental de la surface. Le nombre de générateurs et relations est inférieur aux autres présentations connues et, à notre connaissance, le cas d'une surface à bord (de genre $g\geq 1$) est nouveau dans la littérature.
L'intérêt pour les groupes de tresses est également motivé par la recherche d'invariants d'entrelacs sur les 3-variétés. En effet, il existe une généralisation du théorème de Markov pour les 3-variétés ([86]), qui relie les entrelacs de la variété M avec les tresses sur la surface F, où F est la surface associée à la décomposition à livre ouvert de M ([73]). Nous étendons aux

3-variétés la construction de Jones et nous obtenons un résultat partiel, en construisant une *trace de Markov* sur un certain quotient de l'algèbre de $B(n, F)$.
Les tresses singulières sont des tresses ayant un nombre fini de points doubles. Les tresses singulières à n brins sur le disque , avec la composition usuelle de chemins, forment le monoïde SB_n, appelé monoïde de tresses singulières à n brins sur le disque. Le monoïde $SB(n, F)$ de tresses singulières à n brins sur une surface F, a été introduit dans [48] afin de définir des invariants de type fini ([4]) pour les tresses sur les surfaces. Nous obtenons qu'il se plonge dans un groupe et que le problème du mot est résoluble dans $SB(n, F)$. Ces résultats découlent de la caractérisation des centralisateurs de ce monoïde, que nous obtenons en généralisant des techniques de Fenn, Rolfsen et Zhu pour SB_n.
Nous détaillons nos résultats dans les paragraphes suivants.

Tresses sur les surfaces

Dans le premier chapitre nous démontrons des nouvelles présentations pour les groupes de tresses $B(n, F)$.

Théorème 1. (Théorème 1.1.1)
Soit F une surface orientable de genre $g \geq 1$ et avec p composantes de bord. Le groupe $B(n, F)$ admet la présentation suivante.

- *Générateurs:* $\sigma_1, \ldots, \sigma_{n-1}, a_1, \ldots, a_g, b_1, \ldots, b_g, z_1, \ldots, z_{p-1}$.
- *Relations:*
  - *Relations de tresses, i.e.*

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \, ; \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \textit{pour } |i - j| \geq 2 \, . \end{aligned}$$

  - *Relations mixtes:*

$$\begin{array}{ll}
(R1) & a_r \sigma_i = \sigma_i a_r \quad (1 \leq r \leq g; \ i \neq 1) \, ; \\
 & b_r \sigma_i = \sigma_i b_r \quad (1 \leq r \leq g; \ i \neq 1) \, ; \\
(R2) & \sigma_1^{-1} a_r \sigma_1^{-1} a_r = a_r \sigma_1^{-1} a_r \sigma_1^{-1} \quad (1 \leq r \leq g) \, ; \\
 & \sigma_1^{-1} b_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} b_r \sigma_1^{-1} \quad (1 \leq r \leq g) \, ; \\
(R3) & \sigma_1^{-1} a_s \sigma_1 a_r = a_r \sigma_1^{-1} a_s \sigma_1 \quad (s < r) \, ; \\
 & \sigma_1^{-1} b_s \sigma_1 b_r = b_r \sigma_1^{-1} b_s \sigma_1 \quad (s < r) \, ; \\
 & \sigma_1^{-1} a_s \sigma_1 b_r = b_r \sigma_1^{-1} a_s \sigma_1 \quad (s < r) \, ; \\
 & \sigma_1^{-1} b_s \sigma_1 a_r = a_r \sigma_1^{-1} b_s \sigma_1 \quad (s < r) \, ; \\
(R4) & \sigma_1^{-1} a_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} a_r \sigma_1 \quad (1 \leq r \leq g) \, ; \\
(R5) & z_j \sigma_i = \sigma_i z_j \quad (i \neq n-1, j = 1, \ldots, p-1) \, ; \\
(R6) & \sigma_1^{-1} z_i \sigma_1 a_r = a_r \sigma_1^{-1} z_i \sigma_1 \quad (1 \leq r \leq g; \ i = 1, \ldots, p-1; \ n > 1) \, ; \\
 & \sigma_1^{-1} z_i \sigma_1 b_r = b_r \sigma_1^{-1} z_i \sigma_1 \quad (1 \leq r \leq g; \ i = 1, \ldots, p-1; \ n > 1) \, ; \\
(R7) & \sigma_1^{-1} z_j \sigma_1 z_l = z_l \sigma_1^{-1} z_j \sigma_1 \quad (j = 1, \ldots, p-1, \ j < l) \, ; \\
(R8) & \sigma_1^{-1} z_j \sigma_1^{-1} z_j = z_j \sigma_1^{-1} z_j \sigma_1^{-1} \quad (j = 1, \ldots, p-1) \, .
\end{array}$$

Les tresses géométriques correspondant aux générateurs sont les générateurs usuels du groupe de tresses d'Artin et de $\pi_1(F)$. Nous renvoyons au chapitre 1 et au Théorème 1.1.2 pour les présentations correspondant aux surfaces fermées et aux Théorèmes 1.5.2 and 1.5.3 pour le cas de surfaces non orientables. La preuve est inspirée d'une preuve de Morita pour la présentation d'Artin de B_n ([71]). Nous obtenons ensuite une présentation pour $P(n,F)$, dans le cas d'une surface orientable. Cette présentation est une extension de la présentation classique du groupe de tresses pures $P_n \subset B_n$ ([17]).

Théorème 2. (Théorème 1.6.1)
Soit F une surface orientable de genre $g \geq 1$ avec $p > 0$ composantes de bord. Le groupe $P(n,F)$ admet la présentation suivante:

- *Générateurs:*

$$\{A_{i,j} \mid 1 \leq i \leq 2g+p+n-2, 2g+p \leq j \leq 2g+p+n-1, i<j\}.$$

- *Relations:*

$$\begin{array}{ll}
(PR1) & A_{i,j}^{-1}A_{r,s}A_{i,j} = A_{r,s} \ \ si\ (i<j<r<s)\ ou\,(r+1<i<j<s), \\
& ou\,(i=r+1<j<s\ r<2g\ paire\ et\ r\geq 2g)\,; \\
(PR2) & A_{i,j}^{-1}A_{j,s}A_{i,j} = A_{i,s}A_{j,s}A_{i,s}^{-1} \ \ si\ (i<j<s)\,; \\
(PR3) & A_{i,j}^{-1}A_{i,s}A_{i,j} = A_{i,s}A_{j,s}A_{i,s}A_{j,s}^{-1}A_{i,s}^{-1} \ \ si\ (i<j<s)\,; \\
(PR4) & A_{i,j}^{-1}A_{r,s}A_{i,j} = A_{i,s}A_{j,s}A_{i,s}^{-1}A_{j,s}^{-1}A_{r,s}A_{j,s}A_{i,s}A_{j,s}^{-1}A_{i,s}^{-1} \\
& si\ (i+1<r<j<s)\ ou \\
& (i+1=r<j<s\ \ r<2g\ impaire\ \ et\ r>2g)\,; \\
(ER1) & A_{r+1,j}^{-1}A_{r,s}A_{r+1,j} = A_{r,s}A_{r+1,s}A_{j,s}A_{r+1,s}^{-1} \\
& si\ r<2g\ paire\,; \\
(ER2) & A_{r-1,j}^{-1}A_{r,s}A_{r-1,j} = A_{r-1,s}A_{j,s}A_{r-1,s}^{-1}A_{r,s}A_{j,s}A_{r-1,s}A_{j,s}^{-1}A_{r-1,s}^{-1} \\
& si\ \ r<2g\ impaire\ .
\end{array}$$

Le Théorème 1.6.2 fournit un résultat analogue pour les surfaces orientables fermées.
Le groupe de tresses pures P_n est un produit semi-direct de P_{n-1} et de F_n, le groupe libre de rang n, où l'action induite de P_{n-1} sur l'abelianisé de F_n est triviale. On dit alors que P_n est un produit *quasi-direct* de P_{n-1} et de F_n. Par conséquent, $\bigcap_{d=0}^{\infty} I(P_n)^d = \{0\}$ et $I(P_n)^d/I(P_n)^{d+1}$ est un $\mathbb{Z}$-module libre pour tout $d \geq 0$, où I^k est la puissance k-ième de l'idéal d'augmentation de $\mathbb{Z}P_n$. Ce résultat est fondamental dans la théorie de Vassilev pour les entrelacs dans $\mathbb{R}^3$ (voir [74]).
Le groupe $K_n(F)$, qui est la clôture normale de P_n dans $P(n,F)$, est étudié dans [48]. On démontre que $K_n(F)$ est un produit quasi-direct iteré de groupes libres de rang infini et on construit un invariant universel de type fini pour les tresses sur une surface fermée.
Nous introduisons le groupe $Y(n,F)$, défini comme la clôture normale dans $P(n,F)$ de $P(n,E)$, où E est la surface obtenue en enlevant les anses de F. Nous obtenons que $Y(n,F)$, qui contient

proprement $K(n,F)$, est un produit quasi-direct iteré de groupes libres (Proposition 1.6.3). Par conséquent, $\bigcap_{d=0}^{\infty} I(Y(n,F))^d = \{0\}$ et $I(Y(n,F))^d / I(Y(n,F))^{d+1}$ est un $\mathbb{Z}$-module libre pour tout $d \geq 0$. D'autre part, lorsque F est une surface de genre $g \geq 1$ à bord, $P(n,F)$ est un produit semi-direct iteré de groupes libres de rang fini, mais il n'est pas quasi-direct, à cause des relations (ER1) et (ER2) dans le Théorème 2 (voir section 1.6.3).
Nous remarquons aussi que la relation (R4) dans le Théorème 1 implique qu'il n'existe pas un invariant universel *multiplicatif* de type fini pour les tresses sur les surfaces de genre ≥ 1 ([7]).

Graphes et présentations de tresses

Dans le chapitre 2 nous poursuivons la recherche de présentations pour les groupes de tresses sur les surfaces. Sergiescu ([84]) a démontré que l'on peut associer à tout graphe à n sommets sur le plan (connexe, sans boucles ni intersections) une présentation pour le groupe de tresses B_n. Ce résultat a été ensuite généralisé pour des autres familles de graphes. Les présentations ainsi obtenues sont en général très redondantes mais elles permettent de relier les relations des tresses à la géométrie du graphe. En particulier, les présentations par graphes ont été utilisées dans le problème de conjugaison pour B_n ([18]) et dans le problème de plongement des monoides de tresses positives dans B_n ([53]).
Nous allons donc considérer le cas des graphes sur une surface F et des groupes de tresses correspondant. Nous démontrons que l'on peut associer à tout graphe à n sommets sur la sphère (connexe, sans boucles et intersections) une présentation pour le groupe de tresses sur la sphère $B(n,S^2)$ (Théorème 2.2.1) et nous déduisons quelques résultats sur les automorphismes de $B(n,S^2)$ (Corollaire 2.3.1). Nous démontrons aussi que $Out(B(n,S^2))$ est isomorphe à $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, pour $n \geq 4$ (Proposition 2.4.2). Les automorphismes ϕ_1, ϕ_2 de $B(n,S^2)$ définis par $\phi_1(\sigma_j) = \sigma_j^{-1}$ pour $j = 1, \ldots, n-1$ et $\phi_2(\sigma_j) = \sigma_j U$ pour $j = 1, \ldots, n-1$, où $U = (\sigma_1 \cdots \sigma_{n-1})^n$ est le générateur du centre de $B(n,S^2)$, sont des représentants pour les générateurs de $Out(B(n,S^2))$.

Tresses singulières sur les surfaces

Dans le chapitre 3, nous étudions le monoïde de tresses singulières, $SB(n,F)$. Les générateurs de $B(n,F)$, leur inverses, plus des générateurs singuliers $\tau_1, \ldots \tau_{n-1}$, qui correspondent à des tresses avec un point double, forment un ensemble de générateurs pour $SB(n,F)$. Nous démontrons que les tresses sur les surfaces satisfont une propriété analogue aux tresses singulières sur le disque ([40]).

Théorème 3. (Théorème 3.3.2)
Pour tout $x \in SB(n,F)$, les propriétés suivantes sont équivalentes:

1. $\sigma_j x = x \sigma_k$,
2. $\sigma_j^r x = x \sigma_k^r$, *pour quelques* $r \in \mathbb{Z} \setminus \{0\}$,
3. $\sigma_j^r x = x \sigma_k^r$, *pour tout* $r \in \mathbb{Z}$,
4. $\tau_j x = x \tau_k$,

5. $\tau_j^r x = x\tau_k^r$, *pour quelques* $r \in \mathbb{Z} \setminus \{0\}$.

L'idée de la preuve est de considérer les tresses à n brins sur la surface F comme des *mapping classes* de la surface $F \setminus \mathcal{P}$, où $\mathcal{P}$ est un ensemble de n points distincts. En particulier, dans les Théorèmes 3.3.1 and 3.3.2 on traduit les relations du Théorème 3 en termes d'action de tresses sur les classes d'isotopies d'arcs (un arc est un plongement de l'intervalle unitaire avec extremités dans $\mathcal{P}$). Comme application du Théorème 3 et d'une propriété de réduction pour les tresses singulières (Lemme 3.4.2), on déduit des preuves simples pour les résultats suivants:

Théorème 4. (Théorème 3.4.1)
Le monoïde $SB(n, F)$ se plonge dans un groupe.

Théorème 5. (Théorème 3.5.2)
Le problème du mot pour $SB(n, F)$ est résoluble.

Algèbres de Hecke sur les surfaces

Dans le chapitre 4 nous rappelons quelques définitions et constructions classiques (algèbres de Hecke, traces de Markov et construction algébrique du polynome d'HOMFLY-PT) qui nous seront utiles dans le chapitre 5, et nous introduisons les algèbres de Hecke sur la surface F comme le quotient

$$H_n(q, F) = \mathbb{C}[B(n, F)]/(\sigma_j^2 + (1-q)\sigma_j - q\,,\ j = 1, \ldots, n-1)\,,$$

où σ_j sont les générateurs usuels des groupes des tresses. Nous construison une trace de Markov pour le cas $q = 1$.

Théorème 6. (Théorème 4.1.1)
Soit $\widehat{\pi}$ l'ensemble des classes de conjugaison de $\pi_1(F)$ et $\widehat{\pi}^0 = \widehat{\pi} - \{1\}$. Soit $S(\mathbb{C}\widehat{\pi}^0)$ l'algèbre tensorielle symétrique de $\mathbb{C}\widehat{\pi}^0$. Pour tout $z \in \mathbb{C}$, il y a une (unique) famille $\mathcal{T}_n$ de fonctionnelles linéaires

$$\mathcal{T}_n : H_n(1, F) \to S(\mathbb{C}\widehat{\pi}^0)$$

telles que

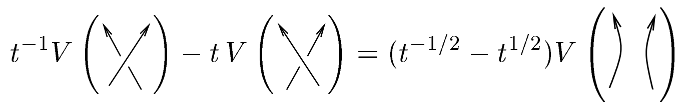
- $\mathcal{T}_n(xy) = \mathcal{T}_n(yx) \quad \forall x, y \in H_n(1, F)\,;$
- $\mathcal{T}_{n+1}(x\sigma_n) = z\mathcal{T}_n(x) \quad \forall x \in H_n(1, F)\,;$
- $\mathcal{T}_{n+1}(\sigma_n \cdots \sigma_1 A \sigma_1 \cdots \sigma_n x) = \widehat{A}\mathcal{T}_n(x) \quad \forall x \in H_n(1, F)\ \forall A \in B(1, F)\,;$
- $\mathcal{T}_n(1) = 1\,,$

où $\widehat{A}$ dénote la classe de conjugaison de $A \in B(1, F) \cong \pi_1(F)$.

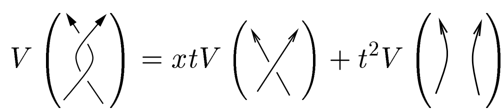
Nous pensons que ce résultat s'étend aux algèbres $H_n(q, F)$ (voir aussi [78]). Toutefois, les calculs sont bien plus compliqués et l'utilisation d'un ordinateur semble nécessaire. Nous remarquons que ces algébres ont été précédemment étudiées dans le cas particulier $F = S^1 \times I$ ([66], [77]). En suivant l'approche de Jones, une trace de Markov ainsi que l' invariant d'entrelacs correspondant ont été ainsi construits dans le cas du tore solide $F \times I$. Le module de skein pour le tore solide avait été précédemment calculé par Turaev ([87], [88]).

Invariants d'entrelacs satisfaisants une relation skein cubique

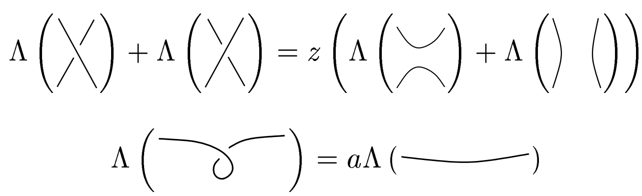
Dans le chapitre 5 on considère une autre généralisation des algèbres de Hecke et on définit deux nouveaux invariants polynomiaux qui sont calculables récursivement et qui sont différents de polynômes d'HOMFLY-PT et Kauffman. Nous rappelons que le polynôme de Jones vérifie la relation skein (d'écheveau) suivante :



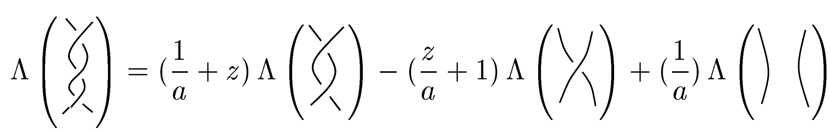
En autres termes, on considère trois entrelacs avec le même diagramme (même projection sur le plan) sauf au voisinage du croisement representé en figure. Etant donné un diagramme planaire d'un noeud, on peut changer certains croisements pour obtenir un nouveau diagramme qui représente le diagramme trivial. De cette manière on peut utiliser la relation skein ci-dessus pour un calcul récursif de V. En remplaçant le facteur $(t^{-1/2} - t^{1/2})$ par x on obtient l'invariant HOMFLY-PT. On peut remarquer que la relation qui définit le polynôme de HOMFLY-PT est *quadratique*. En effet, en rajoutant un croisement positif on obtient la relation skein suivante:



Le polynôme de Kauffman est l'autre extension connue du polynôme de Jones et il est défini par les relations skein suivantes sur les diagrammes non orientés.



Quelques manipulations élémentaires montrent que Λ vérifie une relation skein cubique:



On a récemment démontré que cette relation ne peut pas être complète, c'est-à-dire, elle n'est pas suffisante pour un calcul récursif de Λ ([31]). La recherche d'un système complet de relations skein dont une cubique, est particulièrement intéressante et difficile. En collaboration avec L.Funar ([8]) nous avons obtenu deux nouveaux invariants *cubiques.*

Théorème 7. (Théorème 5.1.1)
Ils existent deux invariants $I_{(\alpha,\beta)}$ et $I^{(z,\delta)}$ qui sont uniquement définis par les deux relations skein en figure 1 (et par leur valeur sur le noeud trivial qui est traditionellement 1). Ces invariants prennent valeurs dans

$$\frac{\mathbb{Z}[\alpha,\,\beta,\,(2\alpha-\beta^2)^{\pm\epsilon/2},\,(\alpha^2+2\beta)^{\pm\epsilon/2}]}{(H_{(\alpha,\beta)})},$$

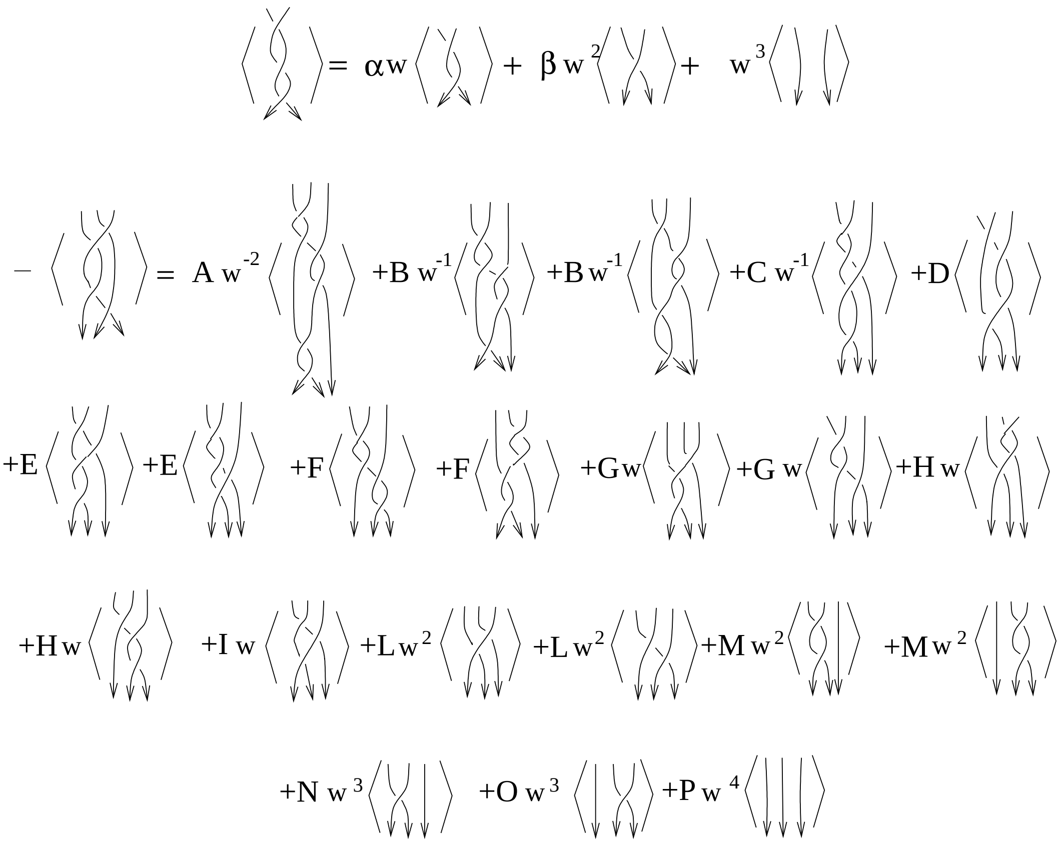


Figure 1: Les relations skein.

et respectivement

$$\frac{\mathbb{Z}[z^{\pm\epsilon/2}, \delta^{\pm\epsilon/2}]}{(P^{(z,\delta)})},$$

où $\epsilon - 1 \in \{0,1\}$ *est le nombre de composantes mod 2 et*

$$\begin{aligned} H_{(\alpha,\beta)} \quad := \quad & 8\alpha^6 - 8\alpha^5\beta^2 + 2\alpha^4\beta^4 + 36\alpha^4\beta - 34\alpha^3\beta^3 + 17\alpha^3 + 8\alpha^2\beta^5 + 32\alpha^2\beta^2 - \\ & -36\alpha\beta^4 + 38\alpha\beta + 8\beta^6 - 17\beta^3 + 8, \end{aligned}$$

et respectivement

$$P^{(z,\delta)} := z^{23} + z^{18}\delta - 2z^{16}\delta^2 - z^{14}\delta^3 - 2z^9\delta^4 + 2z^7\delta^5 + \delta^6 z^5 + \delta^7.$$

Ici on denote par (Q) *l'idéal engendré par l'élément* Q *dans l'algèbre respective.*

Les polynômes $A, B, C, .., P$ correspondant à $I_{(\alpha,\beta)}$ sont donnés ci-dessous. Pour obtenir les coefficients associés à $I^{(z,\delta)}$, il suffit de faire le changement de variable $w = (-z^4/(\delta z))^{1/2}$, $\alpha = -(z^7+\delta^2)/(z^4\delta)$ et $\beta = (\delta - z^2)/z^3$ dans le tableau 1.

$w = ((\alpha^2 + 2\beta)/(2\alpha - \beta^2))^{1/2}$	$A = (\beta^2 - \alpha)$
$B = (\alpha^2 - \alpha\beta^2 - \beta)$	$C = (\alpha^2 - \alpha\beta^2)$
$D = (1 + 2\alpha\beta + \alpha^2\beta^2 - \alpha^3)$	$E = (1 + \alpha\beta + \alpha^2\beta^2 - \alpha^3)$
$F = (1 + 2\alpha\beta - \beta^3)$	$G = (\alpha\beta^3 - 2\alpha - 2\alpha^2\beta)$
$H = (\alpha\beta^3 - 2\alpha - 2\alpha^2\beta + \beta^2)$	$I = (\alpha^4 - \alpha^3\beta^2 - 2\alpha^2\beta - 3\alpha)$
$L = (2\alpha^3\beta + 3\alpha^2 - \alpha^2\beta^3 - \alpha\beta^2)$	$M = (\beta^4 - 2\beta - 3\alpha\beta^2 + \alpha^2)$
$N = (1 + 4\alpha\beta + 3\alpha^2\beta^2 - \alpha^3 - \alpha\beta^4 - \beta^3)$	$O = (1 + 3\alpha\beta + 3\alpha^2\beta^2 - \alpha^3 - \alpha\beta^4)$
$P = (3\beta^2 - \beta^5 - 2\alpha - 3\alpha^2\beta + 4\alpha\beta^3)$	

Tableau 1

La preuve du Théorème est une extension de l'approche de Jones et elle est detaillée dans les sections 1, 2 et 3 du chapitre 5. La première des relations skeins ci-dessus provient de considérations sur les quotients cubiques des algèbres de groupes de tresses $\mathbb{C}[B_n]$. On définit l'algèbre de Hecke cubique par analogie avec les algèbres de Hecke classiques (voir [23]):

$$H(Q,n) = \mathbb{C}[B_n]/(Q(\sigma_j)\,;\; j = 1,\ldots,n-1)\,,$$

où $Q(\sigma_j) = \sigma_j^3 - \alpha\,\sigma_j^2 - \beta\,\sigma_j - 1,\ \alpha,\,\beta \in \mathbb{C}$.
Notre but est de construir des traces de Markov sur la tour d'algèbres de Hecke cubiques, qui définissent des invariants pour les entrelacs. La différence entre les algèbres de Hecke usuelles et celles cubiques est de la même nature que celle entre les groupes de Coxeter sphériques (et donc finis) et ceux hyperboliques (en général infinis). En effet, pour $Q(0) \neq 0$ on a (voir [28]):

- $\dim_{\mathbb{C}} H(Q,3) = 24$, et $H(Q,3)$ est isomorphe à l'algèbre du groupe tétraédral $< 2,3,3 >$ d'ordre 24 (i.e. $SL(2,\mathbf{Z}_3)$).
- $\dim_{\mathbb{C}} H(Q,4) = 648$, et $H(Q,4)$ est isomorphe à l'algèbre du groupe G_{25}, selon la classification de Shepard-Todd ([85]).
- $H(Q,5)$ est l'algèbre de Hecke cyclotomique du groupe G_{32}, qui est d'ordre 155520. Il est conjecturé que cette algèbre est libre de dimension finie, ce qui impliquerait, en utilisant le théorème de déformation de Tits, qu'elle est isomorphe à l'algèbre de G_{32}.
- $\dim_{\mathbb{C}} H(Q,n) = \infty$ pour $n \geq 6$.

En particulier la définition directe d'une trace sur $H(Q,n)$, $n \geq 6$ se heurte au problème de la dimension infinie.
Pour rester justement dans un contexte de dimension finie on introduit les quotients $K_n(\alpha,\,\beta)$, en rajoutant une relation de plus qui vit dans $H(Q,3)$. La forme exacte de cette relation est:
$\sigma_2\,\sigma_1^2\,\sigma_2 + A\,\sigma_1^2\,\sigma_2^2\,\sigma_1^2 + B\,\sigma_1\,\sigma_2^2\,\sigma_1^2 + B\,\sigma_1^2\,\sigma_2^2\,\sigma_1 + C\,\sigma_1^2\,\sigma_2\,\sigma_1^2 + D\,\sigma_1\,\sigma_2^2\,\sigma_1 + E\,\sigma_1\,\sigma_2\,\sigma_1^2 +$
$E\,\sigma_1^2\,\sigma_2\,\sigma_1 + F\,\sigma_2^2\,\sigma_1^2 + F\,\sigma_1^2\,\sigma_2^2 + G\,\sigma_2\,\sigma_1^2 + G\,\sigma_1^2\,\sigma_2 + H\,\sigma_2^2\,\sigma_1 + H\,\sigma_1\,\sigma_2^2 + I\,\sigma_1\,\sigma_2\,\sigma_1 +$
$L\,\sigma_2\,\sigma_1 + L\,\sigma_1\,\sigma_2 + M\,\sigma_1^2 + M\,\sigma_2^2 + N\,\sigma_1 + O\,\sigma_2 + P = 0$
où $A, B, \ldots, P$ sont les polynômes du tableau 1.

Remarque Les algèbres $K_n(\alpha,\,\beta)$ sont de dimension finie pour tout n.

On donne une explication intuitive du choix de cette relation. L'algèbre $H(Q,3)$ est semi-simple (pour Q générique) et se décompose comme $\mathbb{C}^3 \oplus M_2^{\oplus 3} \oplus M_3$, où M_m est l'algèbre des matrices $m \times m$. Si on quotiente par le facteur $\mathbb{C} \oplus M_2^{\oplus 2} \oplus M_3$ on obtient l'algèbre de Hecke usuelle $H_q(3)$. De même l'algèbre de Birman-Wenzl qui est liée au polynôme de Kauffman, s'obtient en passant au quotient par $\mathbb{C} \oplus M_2^2$. Dans notre cas, on prend le quotient par $\mathbb{C}^3$. Notre résultat principal est une conséquence immédiate du résultat technique ci-dessous:

Théorème 8. (Théorème 5.1.2) *Il y a exactement quatre valeurs de $(z, \widehat{z})$ pour lesquelles il existe une (unique) trace de Markov $\mathcal{T}$ sur la tour $K_*(\alpha, \beta)$ avec les paramètres $(z, \widehat{z})$, c'est-à-dire:*

1. $\mathcal{T}(xy) = \mathcal{T}(yx)$, *pour tout $x, y \in K_n(\alpha, \beta)$, et tout n.*
2. $\mathcal{T}(x\sigma_{n-1}) = z\mathcal{T}(x)$, *pour tout $x \in K_n(\alpha, \beta)$, et tout n.*
3. $\mathcal{T}(x\sigma_{n-1}^{-1}) = \widehat{z}\mathcal{T}(x)$, *pour tout $x \in K_n(\alpha, \beta)$, et tout n.*

Le premier couple $(z, \widehat{z})$ est

$$z = (2\alpha - \beta^2)/(\alpha\beta + 4),\ \widehat{z} = -(\alpha^2 + 2\beta)/(\alpha\beta + 4),$$

et la trace associée est $\mathcal{T}_{\alpha,\beta} : K_n(\alpha, \beta) \to \mathbb{Z}[\alpha, \beta, 1/(\alpha\beta + 4)]/(H_{(\alpha,\beta)})$.
Les trois autres solutions ne sont pas des fonctions rationnelles et c'est plus convenable de considérer α, β et $\widehat{z}$ comme fonctions de z, δ, où $\delta = z^2(\beta z + 1)$. Plus précisément on a une trace de Markov

$$\mathcal{T}^{(z,\delta)} : K_*(\alpha, \beta) \to \mathbb{Z}[z^{\pm 1}, \delta^{\pm 1}]/(P^{(z,\delta)}),$$

où $\beta = (\delta - z^2)/z^3,\ \alpha = -(z^7 + \delta^2)/(z^4\delta),\ \widehat{z} = -z^4/\delta$.

Idée de la preuve.- D'abord tout élément de $K_{n+1}(\alpha, \beta)$ peut être écrit comme combinaison linéaire d'éléments du type $a\sigma_n^\eta c$, où $a, b \in K_n(\alpha, \beta)$ et $\eta = \{0, 1, 2\}$. Ceci implique que une trace de Markov sur $K_n(\alpha, \beta)$ s'étend d'une manière unique à une trace de Markov sur $K_{n+1}(\alpha, \beta)$. La partie compliquée concerne donc l'existence d'une telle trace de Markov.

Notre méthode, fortement inspirée de [11], est une amélioration de celle utilisée dans [43]. On définit un graphe géant dont les sommets sont les éléments du semi-groupe abélien engendré par le groupe libre à $n - 1$ générateurs. Les arêtes correspondent aux éléments qui diffèrent par exactement une relation parmi les relations qui définissent $K_n(\alpha, \beta)$. One donne une orientation sur les arêtes, en choisissant un processus de réduction des mots, sauf pour les arêtes correspondant aux commutations: $a\sigma_i\sigma_j b \to a\sigma_j\sigma_i b$ ($\mid i - j \mid > 1$), qui restent non orientées.

On prouve que, par rapport à l'ordre partiel ainsi défini, ils existent des éléments minimaux (peut-être plusieurs) dans chaque composante connexe du graphe. Ensuite on considère la suite ascendante de graphes qui modélise les fonctionnelles sur la tour d'algèbres $K_*(\alpha, \beta)$ satisfaisant les conditions 2.) et 3.) ci-dessus.L'unicité des éléments minimaux pour la réunion de graphes est équivalente à un nombre fini d'obstructions.

Plus précisément on montre que toute fonctionnelle comme avant qui est bien définie sur $K_4(\alpha, \beta)$ admet une extension à tous les $K_n(\alpha, \beta)$, $n \geq 5$. Si l'on rajoute maintenant la condition de commutativité 1.) (pour en faire une trace de Markov) on montre à nouveau qu'on peut se ramener à la commutativité dans $K_4(\alpha, \beta)$.

En particulier ces obstructions sont en nombre fini, ce qui nous a permis de les traiter à l'aide d'un ordinateur. Les valeurs des paramètres se trouvent en utilisant la commutativité sur $K_3(\alpha, \beta)$ et ensuite les calculs explicits montrent que toutes les obstructions appartiennent à l'idéal engendré par le polynôme $H_{(\alpha,\beta)}$ (et respectivement $P^{(z,\delta)}$). $\square$

Maintenant, étant donnée une trace de Markov $\mathcal{T}$ on définit un invariant pour les entrelacs à l'aide de la formule standard:

$$I(x) = \left(\frac{1}{z\widehat{z}}\right)^{\frac{n-1}{2}} \left(\frac{\widehat{z}}{z}\right)^{\frac{e(x)}{2}} \mathcal{T}(x),$$

où $x \in B_n$ est une tresse dont la clôture est l'entrelacs L et $e(x)$ est la somme des exposants de x. On trouve ainsi les invariants $I_{(\alpha,\beta)}$ et $I^{(z,\delta)}$ du Théorème 7. Des calculs explicites montrent que:

- Ces invariants distinguent les nœuds avec au plus 10 croisements, ayant le même invariant HOMFLY-PT.
- $I_{(\alpha,\beta)} = I_{(-\beta,-\alpha)}$ pour les nœuds amphichirals et $I_{(\alpha,\beta)}$ détecte la chiralité de tous les nœuds avec au plus 10 croisements, dont la chiralité n'est pas detectée par les polynômes de Kauffman et HOMFLY-PT et le 2-cables de HOMFLY-PT.
- Tout comme HOMFLY-PT, Kauffman et leurs 2-cables, les invariants $I_{(\alpha,\beta)}$ et $I^{(z,\delta)}$ semblent ne pas distinguer les nœuds mutants.

Il est très difficile, à l'état actuel, de comprendre à quel point ces polynômes diffèrent des polynômes usuelles de Kauffman et HOMFLY-PT. En particulier, on se pose la question si les indéterminations engendrées par les polynômes H et P sont essentielles.

Conjecture. *Il y a une trace de Markov sur $H(Q,n)$ à valeurs dans une extension algébrique de $\mathbb{Z}[\alpha, \beta]$ qui relève la trace de Markov sous-jacente à $I_{(\alpha,\beta)}$.*

Remarquons que les polynômes H et P définissent des courbes algébriques planes non rationelles, donc on ne peut pas expliciter une variable. Dans l'Appendice du chapitre 5 on donne un tableau avec les valeurs des polynômes $I_{(\alpha,0)}(K)$ et $I_{(0,\beta)}(K)$ pour tous les nœuds avec au plus 8 croisements.

Contents

Chapter 1

Braids on surfaces

1.1 Presentations for surface braid groups

Let F be an orientable surface and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of n distinct points of F. A *geometric braid* on F based at $\mathcal{P}$ is an n-tuple $\Psi = (\psi_1, \ldots, \psi_n)$ of paths $\psi_i : [0,1] \to F$ such that

- $\psi_i(0) = P_i, \ i = 1, \ldots, n;$
- $\psi_i(1) \in \mathcal{P}, \ i = 1, \ldots, n;$
- $\psi_1(t), \ldots, \psi_n(t)$ are distinct points of F for all $t \in [0,1]$.

The usual product of paths defines a group structure on the set of braids up to homotopies among braids. This group, denoted $B(n,F)$, does not depend on the choice of $\mathcal{P}$ and it is called the braid group on n strings on F. On the other hand, let be $F_nF = F^n \setminus \Delta$, where Δ is the big diagonal, i.e. the n-tuples $x = (x_1, \ldots x_n)$ for which $x_i = x_j$ for some $i \neq j$. There is a natural action of Σ_n on F_nF by permuting coordinates. We call the orbit space $\hat{F}_nF = F_nF/\Sigma_n$ *configuration space*. Then the braid group $B(n,F)$ is isomorphic to $\pi_1(\hat{F}_nF)$. We recall that the pure braid group $P(n,F)$ on n strings on F is the kernel of the natural projection of $B(n,F)$ in the permutation group Σ_n. This group is isomorphic to $\pi_1(F_nF)$. The first aim of this chapter is to give (new) presentations for braid groups on orientable surfaces.

A p-punctured surface of genus $g \geq 1$ is the surface obtained by deleting p points on a closed surface of genus $g \geq 1$.

Theorem 1.1.1 *Let F be an orientable p-punctured surface of genus $g \geq 1$, with $p \geq 1$. The group $B(n,F)$ admits the following presentation (see also Section 1.2.2):*

- *Generators:* $\sigma_1, \ldots, \sigma_{n-1}, a_1, \ldots, a_g, b_1, \ldots, b_g, z_1, \ldots, z_{p-1}$.
- *Relations:*
  - *Braid relations, i.e.*

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}\,; \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \textit{for } |i-j| \geq 2\,. \end{aligned}$$

– *Mixed relations:*

$$
\begin{array}{ll}
(R1) & a_r\sigma_i = \sigma_i a_r \quad (1 \le r \le g;\ i \ne 1)\,; \\
 & b_r\sigma_i = \sigma_i b_r \quad (1 \le r \le g;\ i \ne 1)\,; \\
(R2) & \sigma_1^{-1}a_r\sigma_1^{-1}a_r = a_r\sigma_1^{-1}a_r\sigma_1^{-1} \quad (1 \le r \le g)\,; \\
 & \sigma_1^{-1}b_r\sigma_1^{-1}b_r = b_r\sigma_1^{-1}b_r\sigma_1^{-1} \quad (1 \le r \le g)\,; \\
(R3) & \sigma_1^{-1}a_s\sigma_1 a_r = a_r\sigma_1^{-1}a_s\sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1}b_s\sigma_1 b_r = b_r\sigma_1^{-1}b_s\sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1}a_s\sigma_1 b_r = b_r\sigma_1^{-1}a_s\sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1}b_s\sigma_1 a_r = a_r\sigma_1^{-1}b_s\sigma_1 \quad (s < r)\,; \\
(R4) & \sigma_1^{-1}a_r\sigma_1^{-1}b_r = b_r\sigma_1^{-1}a_r\sigma_1 \quad (1 \le r \le g)\,; \\
(R5) & z_j\sigma_i = \sigma_i z_j \quad (i \ne 1, j = 1, \ldots, p-1)\,; \\
(R6) & \sigma_1^{-1}z_i\sigma_1 a_r = a_r\sigma_1^{-1}z_i\sigma_1 \quad (1 \le r \le g;\ i = 1, \ldots, p-1;\ n > 1)\,; \\
 & \sigma_1^{-1}z_i\sigma_1 b_r = b_r\sigma_1^{-1}z_i\sigma_1 \quad (1 \le r \le g;\ i = 1, \ldots, p-1;\ n > 1)\,; \\
(R7) & \sigma_1^{-1}z_j\sigma_1 z_l = z_l\sigma_1^{-1}z_j\sigma_1 \quad (j = 1, \ldots, p-1,\ j < l)\,; \\
(R8) & \sigma_1^{-1}z_j\sigma_1^{-1}z_j = z_j\sigma_1^{-1}z_j\sigma_1^{-1} \quad (j = 1, \ldots, p-1)\,.
\end{array}
$$

Theorem 1.1.2 *Let F be a closed orientable surface of genus $g \ge 1$. The group $B(n, F)$ admits the following presentation:*

- *Generators:* $\sigma_1, \ldots, \sigma_{n-1}, a_1, \ldots, a_g, b_1, \ldots, b_g$.
- *Relations:*
  - *Braid relations as in Theorem 1.1.1.*
  - *Mixed relations:*

$$
\begin{array}{ll}
(R1) & a_r\sigma_i = \sigma_i a_r \quad (1 \le r \le g;\ i \ne 1)\,; \\
 & b_r\sigma_i = \sigma_i b_r \quad (1 \le r \le g;\ i \ne 1)\,; \\
(R2) & \sigma_1^{-1}a_r\sigma_1^{-1}a_r = a_r\sigma_1^{-1}a_r\sigma_1^{-1} \quad (1 \le r \le g)\,; \\
 & \sigma_1^{-1}b_r\sigma_1^{-1}b_r = b_r\sigma_1^{-1}b_r\sigma_1^{-1} \quad (1 \le r \le g)\,; \\
(R3) & \sigma_1^{-1}a_s\sigma_1 a_r = a_r\sigma_1^{-1}a_s\sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1}b_s\sigma_1 b_r = b_r\sigma_1^{-1}b_s\sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1}a_s\sigma_1 b_r = b_r\sigma_1^{-1}a_s\sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1}b_s\sigma_1 a_r = a_r\sigma_1^{-1}b_s\sigma_1 \quad (s < r)\,; \\
(R4) & \sigma_1^{-1}a_r\sigma_1^{-1}b_r = b_r\sigma_1^{-1}a_r\sigma_1 \quad (1 \le r \le g)\,; \\
(TR) & [a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}] = \sigma_1\sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2\sigma_1\,,
\end{array}
$$

where $[a, b] := aba^{-1}b^{-1}$.

We may assume that Theorem 1.1.1 provides also a presentation for $B(n, F)$, when F is an orientable surface with p boundary components. We recall that the first presentations of braid groups on closed surfaces were found by Scott ([82]), afterwards revised by Kulikov and Shimada ([63]). Recently González-Meneses reduced significantly the number of generators ([46]). Our presentation has the same number of generators than González-Meneses'one, but it uses the standard generators of the fundamental group of the surface and the number of relations is smaller. At our knowledge, the case of punctured surfaces is new in the literature. Our proof is inspired by Morita's combinatorial proof for the classical presentation of Artin's braid group ([71]). We will explain this approach while proving Theorem 1.1.1. After that we will show how to make this technique fit for obtaining Theorem 1.1.2. We remark that our argument is quite shorter than previous ones, since we do not need a presentation for surface pure braid groups. In Section 1.5 we give presentations for braid groups on non orientable surfaces.

1.2 Preliminaries

1.2.1 Fadell-Neuwirth fibrations

The main tool one uses is the Fadell-Neuwirth fibration, with its generalisation and corresponding exact sequences. As observed in [34], if F is a surface (closed or punctured, orientable or not), the map $\theta : F_nF \to F_{n-1}F$ defined by

$$\theta(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$$

is a fibration with fiber $F \setminus \{x_1, \dots, x_{n-1}\}$. The exact homotopy sequence of the fibration gives us the exact sequence

$$\cdots \pi_2(F_nF) \to \pi_2(F_{n-1}F) \to \pi_1(F \setminus \{x_1, \dots, x_{n-1}\})$$

$$\to P(n, F) \to P(n-1, F) \to 1.$$

Since a punctured surface (with at least one puncture) has the homotopy type of a one dimensional complex, we deduce

$$\pi_k(F_nF) \cong \pi_k(F_{n-1}F) \cong \cdots \cong \pi_k(F), \quad k \geq 3$$

and

$$\pi_2(F_nF) \subseteq \pi_2(F_{n-1}F) \subseteq \cdots \subseteq \pi_2(F)\,.$$

If F is an orientable surface and $F \neq S^2$, all higher homotopy groups are trivial. Thus, if F is an orientable surface different from the sphere we can conclude that there is an exact sequence

$$(PBS) \quad 1 \longrightarrow \pi_1(F \setminus \{x_1, \dots, x_{n-1}\}) \longrightarrow P(n, F) \xrightarrow{\theta} P(n-1, F) \to 1,$$

where θ is the map that "forgets" the last path pointed at x_n.

The problem of the existence of a section for (PBS) has been completely solved in [52]. It is possible to show that θ admits a section, when F has punctures. On the other hand, when F is a closed orientable surface of genus $g \geq 2$, (PBS) splits if and only if $n = 2$. An explicit section is shown in [17] in the case of the torus.

1.2.2 Geometric interpretations of generators and relations

Let F be an orientable surface. Let $\widetilde{B}(n,F)$ be the group with the presentation given in Theorem 1.1.1 or Theorem 1.1.2 respectively. The geometric interpretation for generators of $\widetilde{B}(n,F)$, when F is a closed surface of genus $g \geq 1$ is the same as in [46], except that we represent F as a polygon L of $4g$ sides with the standard identification of edges (see also Section 1.5.3). We can consider braids as paths on L, which we draw with the usual "over and under" information at the crossing points. Figure 1.1 presents the generators of $\widetilde{B}(n,F)$ realized as braids on L.

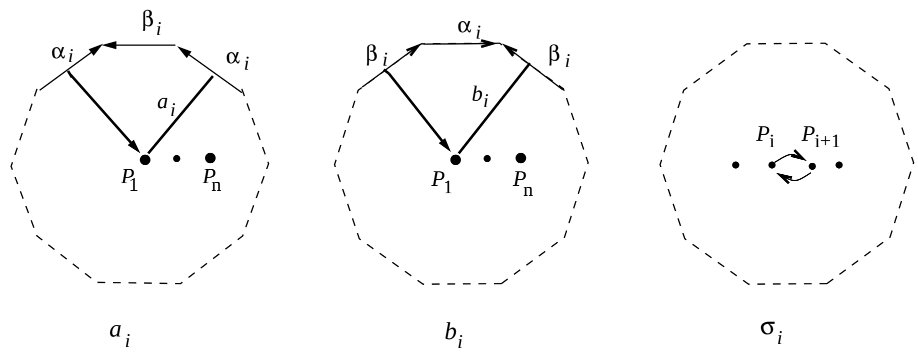


Figure 1.1: Generators as braids (for F an orientable closed surface).

Note that in the braid a_i (respectively b_i) the only non trivial string is the first one, which goes through the wall α_i (the wall β_i). Remark also that $\sigma_1 \ldots, \sigma_{n-1}$ are the classical braid generators on the disk.

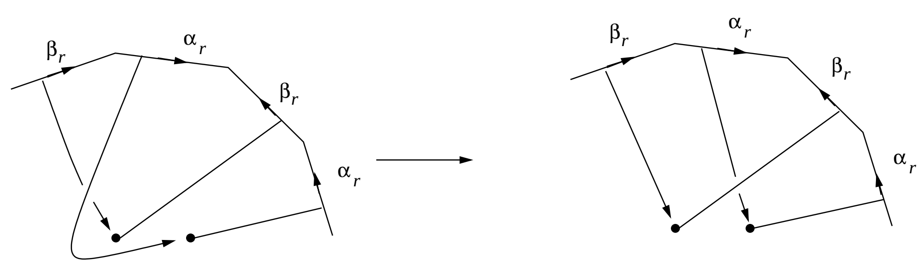


Figure 1.2: Geometric interpretation for relation (R4) in Theorem 1.1.1; homotopy between $\sigma_1^{-1} a_r \sigma_1^{-1} b_r$ (on the left) and $b_r \sigma_1^{-1} a_r \sigma_1$ (on the right).

It is easy to check that the relations above hold in $B(n,F)$. The non trivial strings of a_r and σ_i when $i \neq 1$, may be considered to be disjoint and then $(R1)$ holds in $B(n,F)$. On the other hand, $\sigma_1^{-1} a_r \sigma_1^{-1}$ is the braid whose the only non trivial string is the second one, which goes through the the wall α_r and disjoint from the corresponding non trivial string of a_r. Then $\sigma_1^{-1} a_r \sigma_1^{-1}$ and a_r commute. Similarly we have that $\sigma_1^{-1} b_r \sigma_1^{-1}$ and b_r commute and $(R2)$ is verified. The case of $(R3)$ is similar. Figure 1.2 presents a sketch of a homotopy between $\sigma_1^{-1} a_r \sigma_1^{-1} b_r$ and $b_r \sigma_1^{-1} a_r \sigma_1$. Thus, $(R4)$ holds in $B(n,F)$.

Let s_r (respectively t_r) be the first string of a_r (respectively b_r), for $r = 1, \ldots, 2g$, and consider all the paths $s_1, t_1, \ldots, s_g, t_g$. We cut L along them and we glue the pieces along the edges of L. We obtain a new fundamental domain (see Figure 1.3, for the case of a surface of genus 2), called L_1, with vertex P_1. On L_1 it is clear that $[a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}]$ is equivalent to

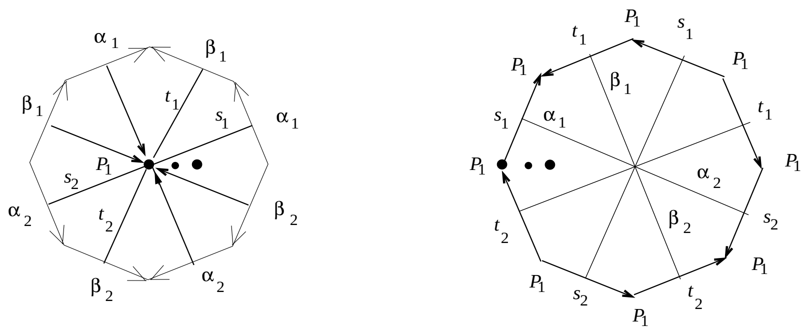


Figure 1.3: The fundamental domain L_1.

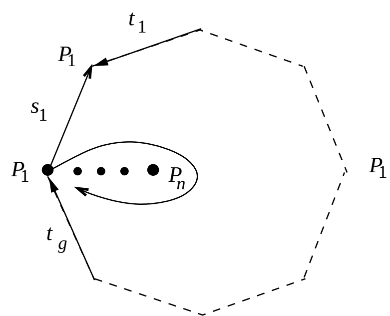


Figure 1.4: Braid $[a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}]$.

the braid of Figure 1.4, equivalent to the braid $\sigma_1\sigma_2 \dots \sigma_{n-1}^2 \dots \sigma_2\sigma_1$ and then (TR) is verified in $B(n, F)$.

There is an analogous geometric interpretation of generators of $\widetilde{B}(n, F)$, for F an orientable p-punctured surface. The definition of generators σ_i, a_j, b_j is the same as above. We only have to add generators z_i, where the only non trivial string is the first one, which is a loop around the i-th puncture (Figure 1.5). As above, relations can be easily checked on corresponding paths (Figure 1.6).

Remark that a loop of the first string around the p-th puncture can be represented by the geometric braid corresponding to the element

$$[a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}]\sigma_1^{-1} \cdots \sigma_{n-1}^{-1} \cdots \sigma_1^{-2} z_1 \cdots z_{p-1} .$$

Therefore, one has natural morphisms $\phi_n : \widetilde{B}(n, F) \to B(n, F)$. We prove that ϕ_n are actually isomorphisms.

1.3 Outline of the proof of Theorem 1.1.1

1.3.1 The inductive assertion

We outline the ideas of the proof for F a surface of genus g with one puncture. One applies an induction on the number n of strands. For $n = 1$, $\widetilde{B}(1, F) = \pi_1(F) = B(1, F)$, then ϕ_1 is an isomorphism.

Consider the subgroup $B^0(n, F) = \pi^{-1}(\Sigma_{n-1})$ and the map

$$\theta : B^0(n, F) \to B(n - 1, F)$$

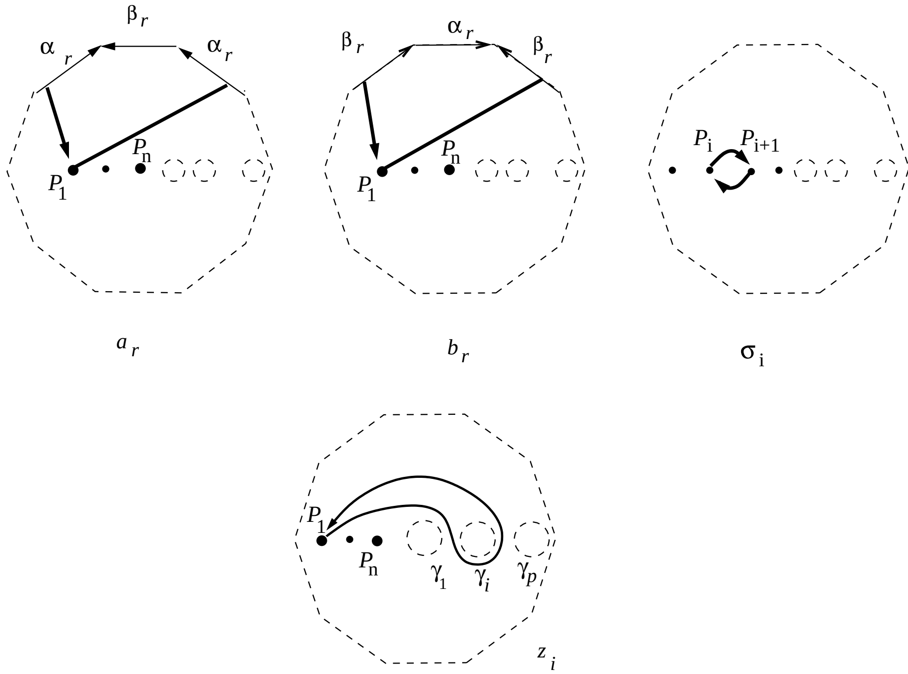


Figure 1.5: Generators as braids (for F an orientable surface with p punctures).

which "forgets" the last string. Now, let $\widetilde{B}^0(n,F)$ be the subgroup of $\widetilde{B}(n,F)$ generated by $a_1,\ldots,a_g,b_1,\ldots,b_g,\sigma_1,\ldots,\sigma_{n-2},\ \tau_1,\ldots,\tau_{n-1},\omega_1,\ldots,\omega_{2g}$, where

$$\begin{aligned}
\tau_j &= \sigma_{n-1}\cdots\sigma_{j+1}\sigma_j^2\sigma_{j+1}^{-1}\cdots\sigma_{n-1}^{-1} \qquad (\tau_{n-1}=\sigma_{n-1}^2)\,;\\
\omega_{2r-1} &= \sigma_{n-1}^{-1}\cdots\sigma_1^{-1}a_r\sigma_1\cdots\sigma_{n-1} \quad r=1,\ldots,g\,;\\
\omega_{2r} &= \sigma_{n-1}^{-1}\cdots\sigma_1^{-1}b_r\sigma_1\cdots\sigma_{n-1} \quad r=1,\ldots,g\,.
\end{aligned}$$

We construct the following diagram:

$$\begin{array}{ccc}
\widetilde{B}^0(n,F) & \xrightarrow{\tilde{\theta}} & \widetilde{B}(n-1,F)\\
\Big\downarrow{\scriptstyle \phi_{n|\widetilde{B}^0(n,F)}} & & \Big\downarrow{\scriptstyle \phi_{n-1}}\\
B^0(n,F) & \xrightarrow{\theta} & B(n-1,F)
\end{array}$$

The map $\tilde{\theta}$ is defined as $\phi_{n-1}^{-1}\theta\phi_{n|B^0(n,F)}$. It is well defined, since ϕ_{n-1} is an isomorphism by the inductive assumption, and it is onto. In fact, $\tilde{\theta}(a_i)=a_i,\tilde{\theta}(b_i)=b_i$ for $i=1,\ldots,g$ and $\tilde{\theta}(\sigma_j)=\sigma_j$ for $j=1,\ldots,n-2$.

1.3.2 The existence of a section

The morphism $\tilde{\theta}$ has got a natural section $s:\widetilde{B}(n-1,F)\to\widetilde{B}^0(n,F)$ defined as: $s(\sigma_j)=\sigma_j, s(a_i)=a_i, s(b_i)=b_i$ for $j=1,\ldots,n-2$ and $i=1,\ldots,2g$.

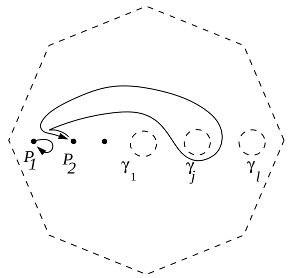


Figure 1.6: The braids $\sigma_1^{-1}z_j\sigma_1$ and $\sigma_1^{-1}z_j\sigma_1^{-1}$. The non trivial string of $\sigma_1^{-1}z_j\sigma_1$ can be considered disjoint from the non trivial string of z_l, for $j < l$. Similarly, the braid $\sigma_1^{-1}z_j\sigma_1^{-1}$ commutes with the braid z_j.

Remark 1.3.1 *Geometrically this section consists of adding a straight strand just to the left of the puncture.*

Given a group G and a subset $\mathcal{G}$ of elements of G we set $\langle\mathcal{G}\rangle$ for the subgroup of G generated by $\mathcal{G}$ and $\langle\langle\mathcal{G}\rangle\rangle$ for the subgroup of G normally generated by $\mathcal{G}$. From now on, given a, b two elements of a group G, we set $a^b = b^{-1}ab$ and ${}^b a = bab^{-1}$.

Lemma 1.3.1 *Let $\mathcal{G} = \{\tau_1, \ldots, \tau_{n-1}, \omega_1, \ldots, \omega_{2g}\}$. Then $Ker(\tilde{\theta}) = \langle\mathcal{G}\rangle$.*

Proof: We set $\beta = \tau_1 \cdots \tau_{n-1} = \sigma_{n-1} \cdots \sigma_2\sigma_1^2\sigma_2 \cdots \sigma_{n-1}$ and $\gamma = \beta^{-1}\tau_1\beta = \sigma_{n-1}^{-1} \cdots \sigma_2^{-1}$ $\sigma_1^2\sigma_2 \cdots \sigma_{n-1}$. By construction we have $\langle\mathcal{G}\rangle \subset Ker(\tilde{\theta})$. The existence of a section s implies that $Ker(\tilde{\theta}) = \langle\langle\mathcal{G}\rangle\rangle$. In fact, suppose that there is such $x \in Ker(\tilde{\theta})$ that $x \notin \langle\langle\mathcal{G}\rangle\rangle$. Thus, there is a word $x' \neq 1$ on generators $a_1, \ldots, a_g, b_1, \ldots, b_g, \sigma_1, \ldots, \sigma_{n-2}$, of $\widetilde{B}^0(n, F)$ such that $\tilde{\theta}(x') = 1$, because all other generators of $\widetilde{B}^0(n, F)$ are in $\langle\mathcal{G}\rangle$. This is false, since $x' = s(\tilde{\theta}(x'))$. To prove that $\langle\mathcal{G}\rangle$ is normal, we need to show that $h^c, {}^c h \in \langle\mathcal{G}\rangle$ for all generators c of $\widetilde{B}^0(n, F)$ and for all $h \in \mathcal{G}$.

i) Let c be the classical braid generator σ_j, $j = 1, \ldots, n-2$. It is clear that $\tau_i^{\sigma_j}$ and ${}^{\sigma_j}\tau_i$ $(i = 1, \ldots, n-1)$ belong to $\langle\tau_1, \ldots, \tau_{n-1}\rangle$, since it is already true in classical braid groups ([71], [83]). On the other hand, $\omega_i^{\sigma_j} = {}^{\sigma_j}\omega_i = \omega_i$ $(i = 1, \ldots, 2g)$.

ii) Let $c = a_r$ or $c = b_r$ $(r = 1, \ldots, g)$. Commutativity relations imply $\tau_j^c = {}^c\tau_j = \tau_j$ $(j = 2, \ldots n-1)$. Note that

$$\begin{aligned} {}^{a_r}\tau_1 &= {}^{\beta\omega_{2r-1}^{-1}}\gamma \quad \text{and} \quad \tau_1^{a_r} = {}^{\tau_1^{-1}\beta\omega_{2r-1}}\gamma \quad \text{for} \quad r = 1, \ldots, g\,; \\ {}^{b_r}\tau_1 &= {}^{\beta\omega_{2r}^{-1}}\gamma \quad \text{and} \quad \tau_1^{b_r} = {}^{\tau_1^{-1}\beta\omega_{2r}}\gamma \quad \text{for} \quad r = 1, \ldots, g\,. \end{aligned}$$

We show only the first equation (the other is similar). By iterated application of $[a_r, \sigma_1 a_r^{-1}\sigma_1] = 1$ we obtain:

$$\begin{aligned} {}^{a_r}\tau_1 &= \sigma_{n-1} \cdots \sigma_2 a_r\sigma_1 a_r^{-1} a_r\sigma_1 a_r^{-1}\sigma_1\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{n-1}^{-1} = \\ &= \sigma_{n-1} \cdots \sigma_2 a_r\sigma_1 a_r^{-1}\sigma_1 a_r^{-1}\sigma_1 a_r\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{n-1}^{-1} = \\ &= \sigma_{n-1} \cdots \sigma_2\sigma_1 a_r^{-1}\sigma_1\sigma_1 a_r\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{n-1}^{-1} = {}^{\beta\omega_{2r-1}^{-1}}\gamma\,. \end{aligned}$$

Set $a_{2,s} = \sigma_1^{-1} a_s \sigma_1$ for $s = 1, \ldots, g$ and respectively $b_{2,s} = \sigma_1^{-1} b_s \sigma_1$ for $s = 1, \ldots, g$. In the same way as above we find that:

$$\begin{array}{ll} (RC1) & (\sigma_1^2)^{a_r} = {}^{a_{2,r}}(\sigma_1^2) \quad (r = 1, \ldots, g)\,; \\ & (\sigma_1^2)^{b_r} = {}^{b_{2,r}}(\sigma_1^2) \quad (r = 1, \ldots, g)\,; \\ (RC2) & {}^{a_r}(\sigma_1^2) = (\sigma_1^2)^{a_{2,r}\sigma_1^{-2}} \quad (r = 1, \ldots, g)\,; \\ & {}^{b_r}(\sigma_1^2) = (\sigma_1^2)^{b_{2,r}\sigma_1^{-2}} \quad (r = 1, \ldots, g)\,. \end{array}$$

Now, remark that relations $(R3)$ and $(R4)$ imply the following relations:

$$\begin{array}{ll} (R3') & a_r \sigma_1 a_s \sigma_1^{-1} = \sigma_1 a_s \sigma_1^{-1} a_r \quad (r < s)\,; \\ & b_r \sigma_1 a_s \sigma_1^{-1} = \sigma_1 a_s \sigma_1^{-1} b_r \quad (r < s)\,; \\ & a_r \sigma_1 b_s \sigma_1^{-1} = \sigma_1 b_s \sigma_1^{-1} a_r \quad (r < s)\,; \\ & b_r \sigma_1 b_s \sigma_1^{-1} = \sigma_1 b_s \sigma_1^{-1} b_r \quad (r < s)\,; \\ (R4') & a_r \sigma_1^{-1} b_r \sigma_1^{-1} = \sigma_1 b_r \sigma_1^{-1} a_r \quad (1 \le r \le g)\,; \end{array}$$

Relations $(RC1), (RC2), (R3'), (R4')$ combined with relations $(R2), (R3), (R4)$ give:

$$\begin{array}{l} {}^{a_r}a_{2,s} = a_{2,s} \quad (s < r)\,; \\ {}^{b_r}a_{2,s} = a_{2,s} \quad (s < r)\,; \\ {}^{a_r}b_{2,s} = b_{2,s} \quad (s < r)\,; \\ {}^{b_r}b_{2,s} = b_{2,s} \quad (s < r)\,; \\ a_{2,r}^{a_r} = {}^{a_{2,r}\sigma_1^{-2}}a_{2,r} \quad (1 \le r \le g)\,; \\ b_{2,r}^{b_r} = {}^{b_{2,r}\sigma_1^{-2}}b_{2,r} \quad (1 \le r \le g)\,; \\ {}^{a_r}a_{2,r} = {}^{\sigma_1^2}a_{2,r} \quad (1 \le r \le g)\,; \\ {}^{b_r}b_{2,r} = {}^{\sigma_1^2}b_{2,r} \quad (1 \le r \le g)\,; \\ a_{2,s}^{a_r} = {}^{[a_{2,r},\sigma_1^{-2}]}(a_{2,s}) \quad (r < s)\,; \\ b_{2,s}^{a_r} = {}^{[a_{2,r},\sigma_1^{-2}]}(b_{2,s}) \quad (r < s)\,; \\ a_{2,s}^{b_r} = {}^{[b_{2,r},\sigma_1^{-2}]}(a_{2,s}) \quad (r < s)\,; \\ b_{2,s}^{b_r} = {}^{[b_{2,r},\sigma_1^{-2}]}(b_{2,s}) \quad (r < s)\,; \\ {}^{a_r}a_{2,s} = {}^{[\sigma_1^2,a_{2,r}^{-1}]}(a_{2,s}) \quad (s < r)\,; \\ {}^{b_r}a_{2,s} = {}^{[\sigma_1^2,b_{2,r}^{-1}]}(a_{2,s}) \quad (s < r)\,; \\ {}^{a_r}b_{2,s} = {}^{[\sigma_1^2,a_{2,r}^{-1}]}(b_{2,s}) \quad (s < r)\,; \\ {}^{b_r}b_{2,s} = {}^{[\sigma_1^2,b_{2,r}^{-1}]}(b_{2,s}) \quad (s < r)\,; \\ b_{2,r}^{a_r} = (a_{2,r}\sigma_1^{-2}a_{2,r}^{-1})b_{2,r}[\sigma_1^{-2}, a_{2,r}] \quad (1 \le r \le g)\,; \\ {}^{a_r}b_{2,r} = \sigma_1^2 b_{2,r}[a_{2,r}^{-1}, \sigma_1^2] \quad (1 \le r \le g)\,; \\ {}^{b_r}a_{2,r} = a_{2,r}\sigma_1^{-2} \quad (1 \le r \le g)\,; \\ a_{2,r}^{b_r} = a_{2,r}b_{2,r}\sigma_1^2 b_{2,r}^{-1} \quad (1 \le r \le g)\,. \end{array}$$

A consequence of these identities and relation $(R1)$ is that $\omega_i^{a_r}, {}^{a_r}\omega_i, \omega_i^{b_r}, {}^{b_r}\omega_i \in \langle \mathcal{G} \rangle$, for $i, r = 1, \ldots, g$. □

Lemma 1.3.2 *Set also* $\{\omega_1, \ldots, \omega_{2g}, \tau_1, \ldots \tau_{n-1}\}$ *in* $B^0(n, F)$ *for* $\{\phi_n(\omega_1), \ldots, \phi_n(\omega_{2g}), \phi_n(\tau_1), \ldots \phi_n(\tau_{n-1})\}$. *Then* $Ker(\theta)$ *is freely generated by* $\{\omega_1, \ldots, \omega_{2g}, \tau_1, \ldots, \tau_{n-1}\}$.

Proof: The diagram

$$\begin{array}{ccc} P(n,F) & \xrightarrow{\theta} & P(n-1,F) \\ \big\downarrow & & \big\downarrow \\ B^0(n,F) & \xrightarrow{\theta} & B(n-1,F) \end{array}$$

is commutative and the kernels of horizontal maps are the same. As stated in Section 1.2.1, $Ker(\theta) = \pi_1(F \setminus \{P_1, \ldots, P_{n-1}\}, P_n)$. If the fundamental domain is changed as in Figure 1.7 and the non trivial strings of ω_j, τ_i are considered as loops of the fundamental group of $F \setminus \{P_1, \ldots, P_{n-1}\}$ based on P_n, it is clear that $\pi_1(F \setminus \{P_1, \ldots, P_{n-1}\}, P_n) = \langle \omega_1, \ldots, \omega_{2g}, \tau_1, \ldots \tau_{n-1} | \ \emptyset \ \rangle$. □

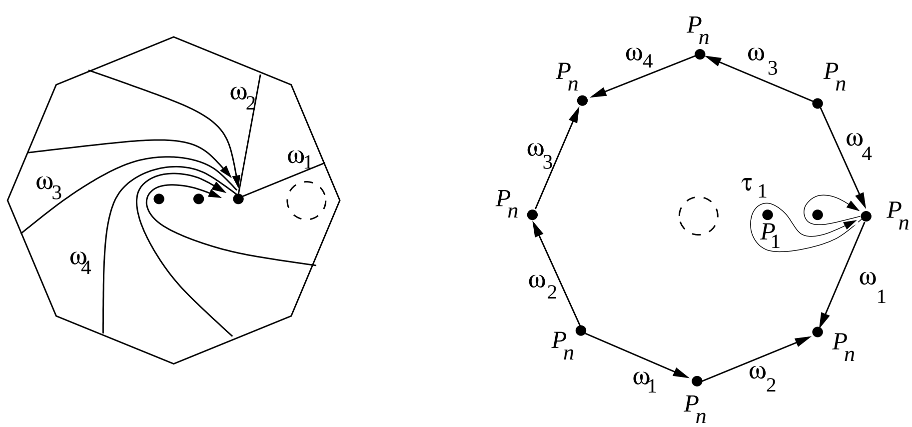


Figure 1.7: Interpretation of ω_j, τ_i as loops of the fundamental group.

Lemma 1.3.3 $\phi_{n|\widetilde{B}^0(n,F)}$ *is an isomorphism.*

Proof: From the previous Lemmas it follows that the map from $Ker(\tilde{\theta})$ to $Ker(\theta)$ is an isomorphism. The Five Lemma and the inductive assumption conclude the proof. □

1.3.3 End of the proof

In order to show that ϕ_n is an isomorphism, let us remark first that it is onto. In fact, from Lemma 1.3.3 it follows that the image of $\widetilde{B}(n, F)$ contains P_n and on the other hand $\widetilde{B}(n, F)$ surjects on Σ_n. Since the index of $B^0(n, F)$ in $B(n, F)$ is n, it is sufficient to show that

$[\widetilde{B}(n,F) : \widetilde{B}^0(n,F)] \leq n$. Consider the elements $\rho_j = \sigma_j \cdots \sigma_{n-1}$ (we set $\rho_n = 1$) in $\widetilde{B}(n,F)$. We claim that $\bigcup_i \rho_i \widetilde{B}^0(n,F) = \widetilde{B}(n,F)$. We only have to show that for any (positive or negative) generator g of $\widetilde{B}(n,F)$ and $i = 1, \ldots, n$ there exists $j = 1, \ldots, n$ and $x \in \widetilde{B}^0(n,F)$ such that

$$g\rho_i = \rho_j x \ .$$

If g is a classical braid, this result is well-known ([24]). Other cases come almost directly from the definition of ω_j. Thus every element of $\widetilde{B}(n,F)$ can be written in the form $\rho_i \widetilde{B}^0(n,F)$. Since $\rho_i^{-1}\rho_j \notin \widetilde{B}^0(n,F)$ for $i \neq j$ we are done. □

The previous proof holds also for $p > 1$. This time $\widetilde{B}^0(n,F)$ is the subgroup of $\widetilde{B}(n,F)$ generated by $a_1, \ldots, a_g, b_1, \ldots, b_g, \sigma_1, \ldots, \sigma_{n-2},\ \tau_1, \ldots, \tau_{n-1},\ \omega_1, \ldots, \omega_{2g},\ \zeta_1, \ldots, \zeta_{p-1}$ where τ_j, ω_r are defined as above and $\zeta_j = \sigma_{n-1}^{-1} \cdots\ \sigma_1^{-1}\ z_j \sigma_1 \cdots \sigma_{n-1}$. □

1.4 Proof of Theorem 1.1.2

1.4.1 About the section

The steps of the proof are the same. We set again $B^0(n,F) = \pi^{-1}(\Sigma_{n-1})$. Let $\widetilde{B}^0(n,F)$ be the subgroup of $\widetilde{B}(n,F)$ generated by $a_1, \ldots, a_g, b_1, \ldots, b_g$, $\sigma_1, \ldots, \sigma_{n-2}$, $\tau_1, \ldots, \tau_{n-1}$, $\omega_1, \ldots,$ ω_{2g}, where τ_j, ω_r are defined as above. Remark that $\tau_1 \in \langle \mathcal{G} \rangle$ since from (TR) relation, the following relation

$$\tau_1 = [\omega_1, \omega_2^{-1}] \cdots [\omega_{2g-1}, \omega_{2g}^{-1}] \tau_{n-1}^{-1} \cdots \tau_2^{-1} \ ,$$

holds in $\widetilde{B}^0(n,F)$. When F is a closed surface the corresponding $\tilde{\theta}$ has no section (see Section 1.2.1). Nevertheless, we are able to prove the analogous of Lemma 1.3.1 (see section 1.4.2).

Lemma 1.4.1 *Let F be a closed surface. Then $Ker(\tilde{\theta})$ is generated by $\{\omega_1, \ldots,\ \omega_{2g},\ \tau_2, \ldots,$ $\tau_{n-1}\}$.*

The following Lemma is analogous to Lemma 1.3.2.

Lemma 1.4.2 *Let F be a closed surface and set also $\{\omega_1, \ldots, \omega_{2g}, \tau_2, \ldots, \tau_{n-1}\}$ in $B^0(n,F)$ for $\{\phi_n(\omega_1), \ldots, \phi_n(\omega_{2g}), \phi_n(\tau_2), \ldots, \phi_n(\tau_{n-1})\}$. $Ker(\theta)$ is freely generated by $\{\omega_1, \ldots, \omega_{2g}, \tau_2, \ldots,$ $\tau_{n-1}\}$.*

Let $\rho_j = \sigma_j \cdots \sigma_{n-1}$ (where $\rho_n = 1$). We may conclude by checking that for any generator g of $\widetilde{B}(n,F)$ (or its inverse) and $i = 1, \ldots, n$ there exists $j = 1, \ldots, n$ and $x \in \widetilde{B}^0(n,F)$ such that

$$g\rho_i = \rho_j x \ ,$$

which is a sub-case of previous situation. □

1.4.2 Proof of Lemma 1.4.1

To conclude the proof of Theorem 1.1.2, we give the demonstration of Lemma 1.4.1. Let us begin with the following Lemma.

Lemma 1.4.3 *Let F be a closed surface and $\mathcal{G} = \{\tau_2, \ldots, \tau_{n-1}, \omega_1, \ldots, \omega_{2g}\}$. The subgroup $\langle \mathcal{G} \rangle$ is normal in $\widetilde{B}^0(n, F)$*

Proof: It suffices to consider relations in Lemma 1.3.1. Remark that from relations shown in Lemma 1.3.1, it follows also that the set

$$\{\gamma\tau_j\gamma^{-1} | j = 1, \ldots n-1, \gamma \text{ word on } \{\omega_1^{\pm 1}, \ldots, \omega_{2g}^{\pm 1}\}\},$$

is a system of generators for $\langle\langle \tau_1, \ldots, \tau_{n-1} \rangle\rangle \equiv \langle\langle \tau_{n-1} \rangle\rangle$. □

In order to prove Lemma 1.4.1, let us consider the following diagram

$$\begin{array}{ccccc}
Ker\tilde{\theta} & \overset{}{\underset{i}{\hookrightarrow}} & \widetilde{B}^0(n,F) & \overset{\tilde{\theta}}{\longrightarrow} & \widetilde{B}(n-1,F) \\
\downarrow t_n & & \downarrow q_n & \nearrow \tilde{\theta}' & \\
Ker\tilde{\theta}' & \underset{i'}{\hookrightarrow} & \widetilde{B}^0(n,F)/\langle\langle \tau_{n-1} \rangle\rangle & &
\end{array}$$

In this diagram q_n is the natural projection, $\tilde{\theta}'$ is defined by $\tilde{\theta}' \circ q_n = \tilde{\theta}$ and t_n is defined by $i' \circ t_n = q_n \circ i$. Since t_n is well defined and onto we deduce that $Ker(t_n) = \langle\langle \tau_{n-1} \rangle\rangle$. Now, $\tilde{\theta}'$ has a natural section $s : \widetilde{B}(n-1, F) \to \widetilde{B}^0(n, F)/\langle\langle \tau_{n-1} \rangle\rangle$ defined as $s(a_i) = [a_j]$, $s(b_i) = [b_j]$ and $s(\sigma_j) = [\sigma_j]$, where $[x]$ is a representative of $x \in \widetilde{B}^0(n, F)$ in $\widetilde{B}^0(n, F)/\langle\langle \tau_{n-1} \rangle\rangle$. Thus, using the same argument as in Lemma 1.3.1, we derive that $Ker(\tilde{\theta}') = \langle\langle \mathcal{K} \rangle\rangle$, where $\mathcal{K} = \{[\omega_1], \ldots, [\omega_{2g}], [\tau_2], \ldots [\tau_{n-1}]\}$. From Lemma 1.4.3 it follows that $\langle \mathcal{K} \rangle = \langle\langle \mathcal{K} \rangle\rangle$. Moreover, since $\tau_i \in \langle\langle \tau_{n-1} \rangle\rangle$ for $i = 1, \ldots, n-2$, $Ker(\tilde{\theta}') = \langle [\omega_1], \ldots, [\omega_{2g}] \rangle$.

From the exact sequence

$$1 \to \langle\langle \tau_{n-1} \rangle\rangle \to Ker(\tilde{\theta}) \to Ker(\tilde{\theta}') \to 1$$

it follows that the set $\{\omega_1, \ldots, \omega_{2g}\}$ and a system of generators for $\langle\langle \tau_{n-1} \rangle\rangle$ form a system of generators for $Ker(\tilde{\theta})$. From the remark in Lemma 1.4.3 it follows that $Ker(\tilde{\theta}) = \langle \tau_2, \ldots, \tau_{n-1}, \omega_1, \ldots, \omega_{2g} \rangle$. □

1.5 Other presentations and remarks

1.5.1 Braids on p-punctured spheres

We recall that the exact sequence

$$1 \longrightarrow \pi_1(F \setminus \{P_1, \ldots, P_{n-1}\}, P_n) \longrightarrow P(n, F) \overset{\theta}{\longrightarrow} P(n-1, F) \to 1$$

holds also when $F = S^2$ ([35]). Thus, previous arguments may be repeated in the case of the sphere, to obtain a new proof for the well-known presentation of braid groups on the sphere as quotients of classical braid groups. When F is a p-punctured sphere, our argument leads to the following result.

Theorem 1.5.1 *Let F be a p-punctured sphere. The group $B(n,F)$ admits the following presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, z_1, \dots, z_{p-1}$.
- *Relations:*
  - *Braid relations, i.e.*

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}\,; \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \quad \textit{for } |i-j| \geq 2\,. \end{aligned}$$

  - *Mixed relations:*

$$\begin{array}{ll} (R1) & z_j\sigma_i = \sigma_i z_j \quad (i \neq 1, j = 1, \dots, p-1)\,; \\ (R2) & \sigma_1^{-1} z_j \sigma_1 z_l = z_l \sigma_1^{-1} z_j \sigma_1 \quad (j = 1, \dots, p-1,\ j < l)\,; \\ (R3) & \sigma_1^{-1} z_j \sigma_1^{-1} z_j = z_j \sigma_1^{-1} z_j \sigma_1^{-1} \quad (j = 1, \dots, p-1)\,; \end{array}$$

We remark that this presentation coincides with the presentation shown in [66].

1.5.2 Braids on non-orientable surfaces

Previous techniques can be used in the case of non-orientable surfaces to prove the following Theorems.

Theorem 1.5.2 *Let F be a non-orientable p-punctured surface of genus $g \geq 1$, with $p \geq 1$. The group $B(n,F)$ admits the following presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_g, z_1, \dots, z_{p-1}$.
- *Relations:*
  - *Braid relations, i.e.*

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}\,; \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \quad \textit{for } |i-j| \geq 2\,. \end{aligned}$$

  - *Mixed relations:*

$$\begin{array}{ll} (R1) & a_r\sigma_i = \sigma_i a_r \quad (1 \leq r \leq g;\ i \neq 1)\,; \\ (R2) & \sigma_1^{-1} a_r \sigma_1^{-1} a_r = a_r \sigma_1^{-1} a_r \sigma_1 \quad (1 \leq r \leq g)\,; \\ (R3) & \sigma_1^{-1} a_s \sigma_1 a_r = a_r \sigma_1^{-1} a_s \sigma_1 \quad (s < r)\,; \\ (R4) & z_j\sigma_i = \sigma_i z_j \quad (i \neq 1, j = 1, \dots, p-1)\,; \\ (R5) & \sigma_1^{-1} z_i \sigma_1 a_r = a_r \sigma_1^{-1} z_i \sigma_1 \quad (1 \leq r \leq g;\ i = 1, \dots, p-1;\ n > 1)\,; \\ (R6) & \sigma_1^{-1} z_j \sigma_1 z_l = z_l \sigma_1^{-1} z_j \sigma_1 \quad (j = 1, \dots, p-1,\ j < l)\,; \\ (R7) & \sigma_1^{-1} z_j \sigma_1^{-1} z_j = z_j \sigma_1^{-1} z_j \sigma_1^{-1} \quad (j = 1, \dots, p-1)\,. \end{array}$$

Theorem 1.5.3 *Let F be a closed non-orientable surface of genus $g \geq 2$. The group $B(n,F)$ admits the following presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_g$.
- *Relations:*
  - *Braid relations as in Theorem 1.1.1.*
  - *Mixed relations:*

$$\begin{array}{ll} (R1) & a_r\sigma_i = \sigma_i a_r \quad (1 \leq r \leq g;\ i \neq 1)\,; \\ (R2) & \sigma_1^{-1}a_r\sigma_1^{-1}a_r = a_r\sigma_1^{-1}a_r\sigma_1 \quad (1 \leq r \leq g)\,; \\ (R3) & \sigma_1^{-1}a_s\sigma_1 a_r = a_r\sigma_1^{-1}a_s\sigma_1 \quad (s < r)\,; \\ (TR) & a_1^2 \cdots a_g^2 = \sigma_1\sigma_2\cdots\sigma_{n-1}^2\cdots\sigma_2\sigma_1\,. \end{array}$$

We give only a geometric interpretation for the generators. To represent a braid in F we consider the surface as a polygon of $2g$ sides as in Figure 1.8, and we make an additional cut: define the path e as in the left hand of the Figure 1.8 and cut the polygon along it. We get F represented as in the right hand side of the same figure, where we can also see how we choose the points $P_1, \dots, P_n$. We show generators in Figure 1.9. Generators σ_j and z_j are as above. For all $r = 1, \dots, g$, the braid a_r consists on the first string passing through the r-th wall, while the other strings are trivial paths. Relations can be easily verified drawing corresponding braids. The relation (TR) in Theorem 1.5.3 is shown in [46]. We remark that Theorem 1.5.3 provides also a presentation for braid groups on the projective plane (see also [90]).

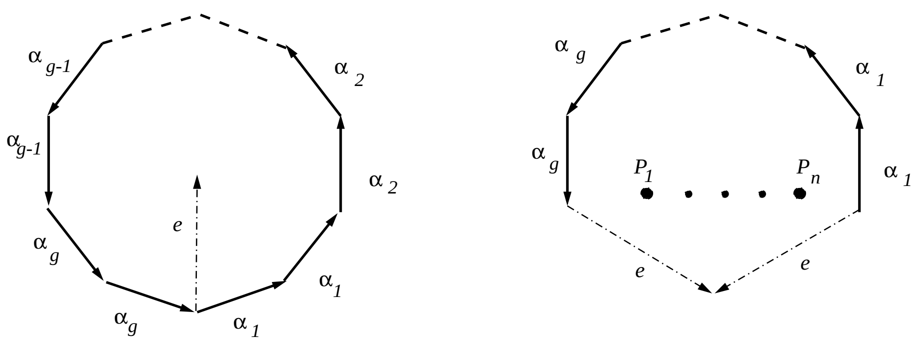


Figure 1.8: Representation of a non-orientable surface F.

1.5.3 González-Meneses' presentations

Let F be a closed orientable surface of genus $g \geq 1$. Using the same arguments outlined in previous Sections we may provide an other presentation for $B(n,F)$.

Theorem 1.5.4 *Let F be a closed orientable surface of genus $g \geq 1$. The group $B(n,F)$ admits the following presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, b_1, \dots, b_{2g}$.

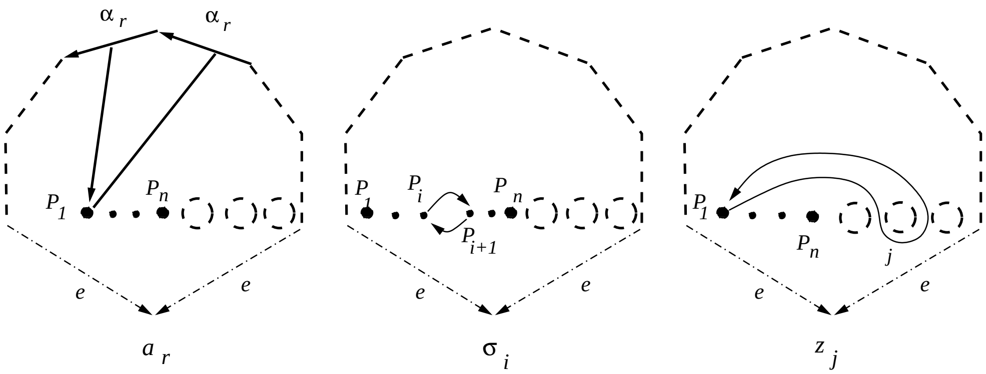


Figure 1.9: Generators as braids (for F a non-orientable surface).

- *Relations:*
  - *Braid relations as in Theorem 1.1.1.*
  - *Mixed relations:*

$$
\begin{array}{ll}
(R1) & b_r\sigma_i = \sigma_i b_r \quad (1 \le r \le 2g; i \ne 1)\,; \\
(R2) & b_s\sigma_1^{-1}b_r\sigma_1^{-1} = \sigma_1 b_r \sigma_1^{-1} b_s \quad (1 \le s < r \le 2g)\,; \\
(R3) & b_r\sigma_1^{-1}b_r\sigma_1^{-1} = \sigma_1^{-1} b_r \sigma_1^{-1} b_r \quad (1 \le r \le 2g)\,; \\
(TR) & b_1 b_2^{-1} \dots b_{2g-1} b_{2g}^{-1} b_1^{-1} b_2 \dots b_{2g-1}^{-1} b_{2g} = \sigma_1\sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2\sigma_1\,.
\end{array}
$$

A closed orientable surface F of genus $g \ge 1$ is represented as a polygon L of $4g$ sides, where opposite edges are identified. Figure 1.10 gives a geometric interpretation of generators. Relations can be easily verified on corresponding braids.

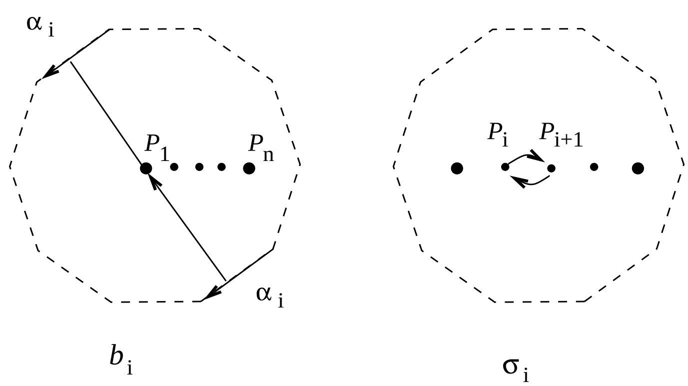


Figure 1.10: Generators as braids (for F an orientable closed surface).

The presentation in Theorem 1.5.4 is close to González-Meneses' presentation.

Theorem 1.5.5 ([46]) *Let F be a closed orientable surface of genus $g \ge 1$. The group $B(n, F)$ admits the following presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_{2g}$.

- *Relations:*

$$
\begin{array}{ll}
(1) & \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\,, \\
(2) & \sigma_i\sigma_j = \sigma_j\sigma_i \quad \textit{for}\ |i-j| \geq 2\,, \\
(3) & [a_r, A_{2,s}] = 1 \quad (1 \leq r,s \leq 2g;\ r \neq s)\,, \\
(4) & [a_r, \sigma_i] = 1 \quad (1 \leq r \leq 2g; i \neq 1)\,, \\
(5) & [a_1 \dots a_r, A_{2,r}] = \sigma_1^2 \quad (1 \leq r \leq 2g)\,, \\
(6) & a_1 \dots a_{2g} a_1^{-1} \dots a_{2g}^{-1} = \sigma_1\sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2\sigma_1\,,
\end{array}
$$

where $A_{2,r} = \sigma_1^{-1}(a_1 \dots a_{r-1}a_{r+1}^{-1} \dots a_{2g}^{-1})\sigma_1^{-1}$.

Remark that the geometric interpretation of b_j corresponds to the braid generator a_j when j is odd and respectively to a_j^{-1}, when j is even. Tedious computations show that relations in Theorem 1.5.4 (after replacing generators b_j's with a_j's) imply relations in Theorem 1.5.5. In the same way, Theorem 1.5.3 can be also verified directly, checking that the relations in Theorem 1.5.3 imply all relations of the González-Meneses' presentation for braid groups on non orientable closed surfaces in [46]. However, we remark that the presentation in Theorem 1.5.3 is simpler and with less relations than González-Meneses' one.

On the other hand, it seems difficult to give an algebraic proof of the equivalence between presentation in Theorem 1.1.2 and presentation in Theorem 1.5.5.

1.5.4 Applications

We conclude this Section with some remarks. Let F be a surface, possibly with boundary. Consider a connected subsurface $E \subset F$, such that every boundary component of E either is a boundary component of F or lies in the interior of F. We suppose also that E contains $\mathcal{P}$. It is known ([75]) that the natural map $\psi_n : B(n,E) \to B(n,F)$ induced by the inclusion $E \subseteq F$ is injective if and only if $\overline{F \setminus E}$ does not contain a disk D^2. We may provide an analogous characterisation about surjection.

Proposition 1.5.1 *Let F be a surface of genus $g \geq 1$ with $p \geq 0$ boundary components, and let E be a subsurface of F. The natural map $\psi_n : B(n,E) \to B(n,F)$ induced by the inclusion $E \subseteq F$ is surjective if and only if $\overline{F \setminus E}$ is a disjoint union of disks.*

Proof: When E is obtained from F removing k disks, the natural map $\psi_n : B(n,E) \to B(n,F)$ is onto and Theorems 1.1.1, 1.1.2, 1.5.2 and 1.5.3 give a description of $Ker(\psi_n)$. Remark that the natural morphism

$$\psi_1 : \pi_1(E, P_1) \to \pi_1(F, P_1)$$

is a surjection if and only if $\overline{F \setminus E}$ is a disjoint union of disks. Now consider a pure braid $p \in P(n,F)$ as a n-tuple of paths $(p_1, \dots, p_n)$ and let $\chi : P(n,F) \to \pi_1(F)^n$ be the map defined by $\chi(p) = (p_1, \dots, p_n)$. The following commutative diagram holds

$$
\begin{array}{ccc}
P(n,E) & \xrightarrow{\ \chi\ } & \pi_1(E)^n \\
\Big\downarrow{\scriptstyle (\psi_n)_{|P(n,E)}} & & \Big\downarrow{\scriptstyle \psi_1 \times \cdots \times \psi_1} \\
P(n,F) & \xrightarrow{\ \chi\ } & \pi_1(F)^n
\end{array}
$$

Since χ is surjective ([17]) we deduce that $(\psi_n)_{|P(n,E)}$ is not surjective on $P(n,F)$ when ψ_1 is not surjective. Thus, since $\psi_n^{-1}(P(n,F))$ belongs to $P(n,E)$, it follows that ψ_n is not surjective on $B(n,F)$ when ψ_1 is not surjective. □

Remark 1.5.1 *The fact that the natural map $\psi_n : B(n,E) \to B(n,F)$ is onto when E is obtained from F removing k disks can also be obtained from the remark that $B(n,E)$ is a subgroup of $B(n+k,F)$ and that the map ψ_n corresponds to the usual projection $B(n+k,F) \to B(n,F)$. The existence of a braid combing in $B(n+k,F)$ ([66]) implies the claim.*

Proposition 1.5.2 *Let F be a orientable surface of genus $g \geq 1$, possibly with boundary. Let $N_n(F)$ be the normal closure of B_n in $B(n,F)$. The quotient $B(n,F)/N_n(F)$ is isomorphic to $H_1(F)$, the first homology group of the surface F.*

Proof: It is sufficient to replace all σ_j with 1 in Theorems 1.1.1 and 1.1.2. □

1.6 Surface pure braid groups

Several presentations for surface braid groups are known, when F is a closed surface or a holed disk ([46], [52], [66], [82]). In Theorem 1.6.1 we provide a presentation for pure braid groups on orientable surfaces with boundary. This presentation is close to the standard presentation of the pure braid group P_n on the disk. We provide also the analogous presentation for pure braid groups on orientable closed surfaces.

1.6.1 Presentations for surface pure braid groups

Theorem 1.6.1 *Let F be an orientable surface of genus $g \geq 1$ with $p > 0$ boundary components. $P(n,F)$ admits the following presentation:*

- *Generators:*

$$\{A_{i,j} \mid 1 \leq i \leq 2g+p+n-2, 2g+p \leq j \leq 2g+p+n-1, i < j\}.$$

- *Relations:*

$$\begin{array}{ll}
(PR1) & A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{r,s} \ \ \textit{if } (i<j<r<s) \textit{ or} (r+1<i<j<s), \\
 & \textit{or} (i=r+1<j<s \textit{ for even } r<2g \textit{ or } r \geq 2g)\,; \\
(PR2) & A_{i,j}^{-1} A_{j,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1} \ \ \textit{if } (i<j<s)\,; \\
(PR3) & A_{i,j}^{-1} A_{i,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1} \ \ \textit{if } (i<j<s)\,; \\
(PR4) & A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1} A_{j,s}^{-1} A_{r,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1} \\
 & \textit{if } (i+1<r<j<s) \textit{ or} \\
 & (i+1=r<j<s \textit{ for odd } r<2g \textit{ or } r>2g)\,; \\
(ER1) & A_{r+1,j}^{-1} A_{r,s} A_{r+1,j} = A_{r,s} A_{r+1,s} A_{j,s} A_{r+1,s}^{-1} \\
 & \textit{if } r \textit{ even and } r<2g\,; \\
(ER2) & A_{r-1,j}^{-1} A_{r,s} A_{r-1,j} = A_{r-1,s} A_{j,s} A_{r-1,s}^{-1} A_{r,s} A_{j,s} A_{r-1,s} A_{j,s}^{-1} A_{r-1,s}^{-1} \\
 & \textit{if } r \textit{ odd and } r<2g\,.
\end{array}$$

Proof: The choice of the notation is motivated by the notation for standard generators of P_n from [15]. Let $\widetilde{P}(n-1,F)$ be the group defined by above presentation. We give in Figure 1.11 a picture of corresponding braids on the surface. Let $h = 2g+p-1$. In respect of the presentation for $B(n,F)$ given in Theorem 1.1.1, the elements $A_{i,j}$ are the following braids:

- $A_{i,j} = \sigma_{j-h}\cdots\sigma_{i+1-h}\sigma_{i-h}^2\sigma_{i+1-h}^{-1}\cdots\sigma_{j-h}^{-1}$, for $i \geq 2g+p$;
- $A_{i,j} = \sigma_j\cdots\sigma_1 z_{i-2g}^{-1}\sigma_1^{-1}\cdots\sigma_{j-h}^{-1}$, for $2g < i < 2g+p$;
- $A_{2i,j} = \sigma_j\cdots\sigma_1 a_{g-i+1}^{-1}\sigma_1^{-1}\cdots\sigma_{j-h}^{-1}$, for $1 \leq i \leq g$;
- $A_{2i-1,j} = \sigma_{j-h}\cdots\sigma_1 b_{g-i+1}^{-1}\sigma_1^{-1}\cdots\sigma_{j-h}^{-1}$, for $1 \leq i \leq g$.

The relations (PR1), ... , (PR4) correspond to the classical relations for P_n. The new relations arise when we consider two generators $A_{2i,j}$, $A_{2i-1,k}$, for $1 \leq i \leq g$ and $j \neq k$. They correspond to two loops based at two different points which go around the same handle. Relations (ER1) and (ER2) can be verified by explicit pictures or using relations in Theorem 1.1.1. The

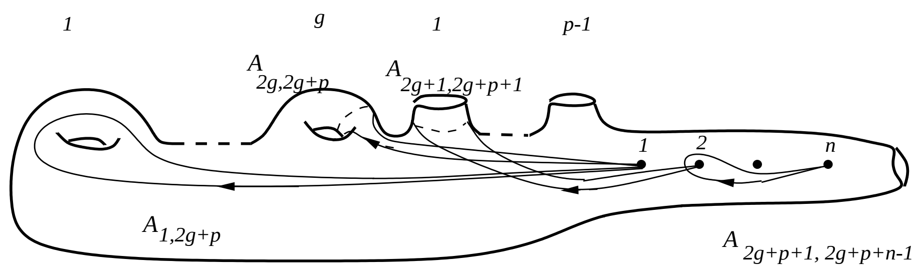


Figure 1.11: Geometric interpretation of $A_{i,j}$. We mark again with $A_{i,j}$ the only non trivial string of the braid $A_{i,j}$

technique to prove that $(PR1),\dots,(ER2)$ is a complete system of relations for $P(n,F)$ is well known ([46], [52], [66], [82]). As shown in [57], given an exact sequence

$$1 \to A \to B \to C \to 1\,,$$

and presentations $\langle G_A, R_A\rangle$ and $\langle G_C, R_C\rangle$, we can derive a presentation $\langle G_B, R_B\rangle$ for B, where G_B is the set of generators G_A and coset representatives of G_C. The relations R_B are given by the union of three sets. The first corresponds to relations R_A, and the second one to writing each relation in C in terms of corresponding coset representatives as an element of A. The last set corresponds to the fact that the action under conjugation of each coset representative of generators of C (and their inverses) on each generator of A is an element of A. We can apply this result on (PBS) sequence. The presentation is correct for $n = 1$. By induction, suppose that for $n-1$, $\widetilde{P}(n-1,F) \cong P(n-1,F)$. The set of elements $A_{i,2g+n+p-1}$ $(i = 1,\dots,2g+n+p-2)$ is a system of generators for $\pi_1(F \setminus \{P_1,\dots,P_{n-1}\},P_n)$. To show that $(PR1),\dots,(ER2)$ is a complete system of relations for $P(n,F)$ it suffices to prove that relations $R_{P(n,F)}$ are a consequence of relations $(PR1),\dots,(ER2)$. Since $\pi_1(F\setminus\{P_1,\dots,P_{n-1}\},P_n)$ is a free group on the given generators, we just have to check the second and the third set of relations. Consider as coset representative for the generator $A_{i,j}$ in $P(n-1,F)$ the generator $A_{i,j}$ in $P(n,F)$. Relations lift directly to relations in $P(n,F)$. The action of $A_{i,j}^{-1}$ on $\pi_1(F\setminus\{P_1,\dots,P_{n-1}\},P_n)$ may be deduced from that of $A_{i,j}$. In fact, relations (PR2) and (PR3) imply that

$$A_{i,j}A_{i,2g+n+p-1}A_{j,2g+n+p-1} = A_{i,2g+n+p-1}A_{j,2g+n+p-1}A_{i,j}\ ,$$

for all $i < j < 2g+n+p-1$, and from this relation and relations (PR2) we deduce that

$$A_{i,j}A_{i,2g+n+p-1}A_{i,j}^{-1} = A_{j,2g+n+p-1}^{-1}A_{i,2g+n+p-1}A_{j,2g+n+p-1}\,,$$

for all $i < j < 2g+n+p-1$). It follows that

$$A_{s,j}A_{i,2g+n+p-1}A_{s,j}^{-1} \in \langle A_{1,2g+n+p-1},\ldots,A_{2g+n+p-2,2g+n+p-1}\rangle\,,$$

for all $s < j < 2g+n+p-1$.

Thus we have proved that $\langle A_{1,2g+n+p-1},\ldots,A_{2g+n+p-2,2g+n+p-1}\rangle$ is a normal subgroup and that also the third set of relations of $R_{P(n,F)}$ is a consequence of $(PR1),\ldots,(ER2)$. □

In the same way we can prove the following Theorem.

Theorem 1.6.2 *Let F be an orientable closed surface of genus $g \geq 1$. $P(n,F)$ admits the following presentation:*

- *Generators:* $\{A_{i,j} \mid 1 \leq i \leq 2g+n-1, 2g+1 \leq j \leq 2g+n, i < j\}$.
- *Relations:*

$$\begin{array}{ll}
(PR1) & A_{i,j}^{-1}A_{r,s}A_{i,j} = A_{r,s} \ \textit{if}\ (i<j<r<s)\ \textit{or}\,(r+1<i<j<s),\\
 & \textit{or}\,(i=r+1<j<s\ \textit{for even}\ r<2g\ \textit{or}\ r>2g)\,;\\
(PR2) & A_{i,j}^{-1}A_{j,s}A_{i,j} = A_{i,s}A_{j,s}A_{i,s}^{-1} \ \textit{if}\ (i<j<s)\,;\\
(PR3) & A_{i,j}^{-1}A_{i,s}A_{i,j} = A_{i,s}A_{j,s}A_{i,s}A_{j,s}^{-1}A_{i,s}^{-1} \ \textit{if}\ (i<j<s)\,;\\
(PR4) & A_{i,j}^{-1}A_{r,s}A_{i,j} = A_{i,s}A_{j,s}A_{i,s}^{-1}A_{j,s}^{-1}A_{r,s}A_{j,s}A_{i,s}A_{j,s}^{-1}A_{i,s}^{-1}\\
 & \textit{if}\ (i+1<r<j<s)\ \textit{or}\\
 & (i+1=r<j<s\ \textit{for odd}\ \ r<2g\ \textit{or}\ r>2g)\,;\\
(ER1) & A_{r+1,j}^{-1}A_{r,s}A_{r+1,j} = A_{r,s}A_{r+1,s}A_{j,s}A_{r+1,s}^{-1}\\
 & \textit{if}\ r\ \textit{even and}\ r<2g\,;\\
(ER2) & A_{r-1,j}^{-1}A_{r,s}A_{r-1,j} = A_{r-1,s}A_{j,s}A_{r-1,s}^{-1}A_{r,s}A_{j,s}A_{r-1,s}A_{j,s}^{-1}A_{r-1,s}^{-1}\\
 & \textit{if}\ r\ \textit{odd and}\ r<2g\,;\\
(TR) & [A_{2g,2g+k}^{-1}, A_{2g-1,2g+k}]\cdots[A_{2,2g+k}^{-1}, A_{1,2g+k}] = \prod_{l=2g+1}^{2g+k-1} A_{l,2g+k}\times\\
 & \times \prod_{j=2g+k+1}^{2g+n} A_{2g+k,j}\ \ k=1,\ldots,n\,.
\end{array}$$

Remark 1.6.1 *Let E be a holed disk. Theorem 1.6.1 provides a presentation for $P(n,E)$ ([66]). Let us recall that $P(n,E)$ is a (proper) subgroup of P_{n+k}, where k is the number of holes in E.*

Remark 1.6.2 *We recall that P_n embeds in $P(n,F)$ ([75]) and thus P_n is isomorphic to the subgroup*

$$\langle A_{i,j}\,|\,2g+1 \leq i < j \leq 2g+n\rangle\,,$$

when F is a closed surface and P_n is isomorphic to

$$P_n = \langle A_{i,j} \,|\, 2g+p \leq i < j \leq 2g+p+n-1 \rangle \,,$$

when F is a surface with $p > 0$ boundary components. Consider the sub-surface E obtained removing g handles from F. The group $P(n,E)$ embeds in $P(n,F)$ ([75]) and it is isomorphic to the subgroup

$$\langle \{A_{i,j} \,|\, 2g+1 \leq i < j \leq 2g+n\} \cup \{A_{2k-1,l}, A_{2k,l}^{-1} A_{2k-1,l}^{-1} A_{2k,l} \,|\, 1 \leq k \leq g \,,\, 2g+1 \leq l \leq 2g+n\} \rangle \,,$$

when F is a closed surface and respectively to the subgroup

$$\langle \{A_{i,j} \,|\, 2g+1 \leq i \leq 2g+p+n-2,\, 2g+p \leq j \leq 2g+p+n-1,\, i < j\} \cup$$
$$\cup \{A_{2k-1,l},\, A_{2k,l}^{-1} A_{2k-1,l}^{-1} A_{2k,l} \,|\, 1 \leq k \leq g,\, 2g+p \leq l \leq 2g+p+n-1\} \rangle \,,$$

when F is a surface with $p > 0$ boundary components.

Remark 1.6.3 *When F is a surface with genus, from relation (ER1) we deduce that generators $A_{i,j}$ for $2g+p \leq i < j \leq 2g+n+p-1$, which generate a subgroup isomorphic to P_n, are redundant. Then Theorem 1.6.1 provides a (homogeneous) presentation for $P(n,F)$ with $(2g+p-1)n$ generators.*

1.6.2 Remarks on the normal closure of P_n in $P(n,F)$

As corollary of previous presentations we give an easy proof of a well-known fact on $K_n(F)$, the normal closure of classical pure braid group P_n in $P(n,F)$ ([45]).

Lemma 1.6.1 *Let $\chi : P(n,F) \to \pi_1(F)^n$ be the map defined by $\chi(p) = (p_1, \ldots, p_n)$. Let F be a closed orientable surface possibly with boundary. Let $K_n(F)$ be the normal closure of P_n in $P(n,F)$. Then*

$$Ker(\chi) = K_n(F) \,.$$

Proof: We outline the case of a surface F with boundary. The inclusion $K_n(F) \subseteq Ker(\chi)$ is obvious. The quotient group $\dfrac{P(n,F)}{K_n(F)}$ is isomorphic to the group $\widehat{P}(n,F)$ with generators $\{A_{i,j} \,|\, 1 \leq i \leq 2g+p-1,\, 2g+p \leq j \leq 2g+p+n-1\}$ and relations $\{[A_{i,j}, A_{k,l}] = 1, j \neq l\}$. The morphism χ induces an isomorphism between $\widehat{P}(n,F)$ and $\pi_1(F)^n$. □

Proposition 1.6.1 *Let F be an orientable surface possibly with boundary. When F is a torus*

$$[P(n,F), P(n,F)] = K_n(F) \,.$$

Otherwise the strict inclusion holds:

$$[P(n,F), P(n,F)] \supset K_n(F) \,.$$

Proof: The inclusion $K_n(F) \subset [P(n,F), P(n,F)]$ follows from relation (ER1). Suppose that $[P(n,F), P(n,F)] = K_n(F) = Ker(\chi)$ for $g > 1$. It follows that $\dfrac{P(n,F)}{Ker(\chi)}$ is abelian. This is false since $\pi_1(F)^n$ is not abelian for $g > 1$. Let $w \in [P(n,F), P(n,F)]$. The sum of exponents $A_{i,j}$ in w must be zero. The projection of $\chi(w)$ on any coordinate is the sub-word of w consisting of the generators associated to corresponding strand. Since the sum of exponents is zero, if F is a torus this projection is trivial and the claim follows. □

1.6.3 Almost-direct products

Let us recall the following definition:

Definition 1.6.1 *We say that a group G is residually a $\mathcal{P}$-group if for each $g \in G$, $g \neq 1$, there exists a normal subgroup N of G such that $g \notin N$ and G/N has the property $\mathcal{P}$.*

The *lower central series* $\{G_i\}_{i \geq 0}$ of a group G is the series of groups G_i defined by $G_1 = G$, $G_{i+1} = [G, G_i]$, where $[G, G_i]$ is the subgroup of G generated by all commutators $hkh^{-1}k^{-1}$, for $h \in G$ and $k \in G_i$. Set $G_{(i)} = G_i/G_{i+1}$. Magnus proved that free groups, as well as fundamental groups of orientable surfaces, are residually torsion free nilpotents (see [6] for an outline of the proof). We refer to [67] for more results on residual nilpotence. We recall just that if G is residually torsion free nilpotent then G is *biorderable* ([81]). The group G is called bi-orderable if there exists a total order $<$ on G such that for all g, h, k in G the relation $g < h$ implies that $kg < kh$ and $gk < hk$.
In this section we consider subgroups of surface braid groups having the following properties:

1. $\bigcap_{d=0}^{\infty} I(G)^d = \{0\}$;
2. $I(G)^d/I(G)^{d+1}$ is a free $\mathbb{Z}$-module for all $d \geq 0$, where I_G^k means the k-th power of the augmentation ideal of the group ring of the group G.

Free groups have properties 1) and 2) (see [41]). We stress that $\bigcap_{d=0}^{\infty} I(G)^d = \{0\}$ implies that G is residually nilpotent.

Definition 1.6.2 *Let A, C be two groups. If C acts on A and the induced action on the abelianization of A is trivial, we say that $A \rtimes C$ is an* almost-direct product *of A and C.*

Proposition 1.6.2 ([36]) *Let A, C be two groups. If C acts on A and the induced action on the abelianization of A is trivial, then*

$$I(A \rtimes C)^m = \sum_{k=0}^{m} I(A)^k \otimes I(C)^{m-k} \quad \textit{for all } m \geq 0$$

and

$$(A \rtimes C)_{(m)} = A_{(m)} \rtimes C_{(m)}\,.$$

The pure braid group P_n is an *almost-direct product* of free groups ([37]). In particular P_n inherits the properties of free groups that we described above. These properties have been used in [74] in order to construct an universal finite type invariant for braids.
The group $K_n(F)$, the normal closure of classical pure braid group P_n in $P(n, F)$, is an almost-direct product of (infinitely generated) free groups ([48]). Moreover, it has been constructed an universal finite type invariant for braids on surfaces, where the group $K_n(F)$ plays the rôle of P_n.
Let F be a surface with boundary components. Consider the sub-surface E obtained removing the handles of F. Let $Y_n(F)$ be the normal closure of $P(n, E)$ in $P(n, F)$. Using our presentation for surface pure braid groups, we prove that the group $Y_n(F)$, which contains properly $K_n(F)$, is an almost-direct product of free groups.

Proposition 1.6.3 *The group $Y_n(F)$ is an almost-direct product of free groups.*

Proof: We sketch a proof for F orientable surface with one boundary component. Let $\pi_1(F)^n$ be provided with presentation

$$\langle A_{j,2g+k}\ j=1,\ldots,2g,\ k=1,\ldots,n | [A_{j,2g+k}, A_{l,2g+q}] = 1 \text{ for all } j,l=1,\ldots,2g,\ 1\le k\ne q\le n\,\rangle\,,$$

where $A_{j,2g+k}$ are the loops defined in Theorem 1.6.1. Let $F_{g,n}$ be the group with presentation

$$\langle \widetilde{A_{j,2g+k}}\ j=1,\ldots,g,\ k=1,\ldots,n | [\widetilde{A_{j,2g+k}}, \widetilde{A_{l,2g+q}}] = 1 \text{ for all } j,l=1,\ldots,g,\ 1\le k\ne q\le n\,\rangle\,.$$

Let $\mu : \pi_1(F)^n \to F_{g,n}$ be the map defined by $\mu(A_{2i-1,2g+k}) = \widetilde{A_{i,2g+k}}$ and $\mu(A_{2i,2g+k}) = 1$. One can proceed as in Lemma 1.6.1 for showing that $Ker(\mu\circ\chi) = Y_n(F)$. Thus the following commutative diagram holds:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & F_{g,1} & \longrightarrow & F_{g,n} & \longrightarrow & F_{g,n-1} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow{\scriptstyle \mu\circ\chi} & & \uparrow{\scriptstyle \mu\circ\chi} & & \\
1 & \longrightarrow & Ker(\theta) & \longrightarrow & P(n,F) & \stackrel{\theta}{\longrightarrow} & P(n-1,F) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & G_n & \longrightarrow & Y_n(F) & \stackrel{\theta}{\longrightarrow} & Y_{n-1}(F) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

where $F_{g,1}$ is the free group on g generators and $G_n = Y_n(F)\cap Ker(\theta)$ is a free group.

Lemma 1.6.2 *The following set is a system of generators for G_n.*

$$\{\gamma A_{j,2g+n}\gamma^{-1} | 2g < j < 2g+n \ \ \textit{and}\ 1\le j<2g,\ j\ \ \textit{even}\ \}$$

where γ is a word on $\{A^{\pm 1}_{2k-1,2g+n} | 1\le k\le g\}$.

Proof: Consider the vertical sequence

$$1 \longrightarrow G_n \longrightarrow Ker(\theta) \longrightarrow F_{g,1} \longrightarrow 1\,.$$

Recall that $Ker(\theta) = \pi_1(F\setminus\{P_1,\ldots,P_{n-1}\}, P_n)$. A set of free generators for this group is given by $\{A_{j,2g+n} | 1\le j<2g+n\}$. The map $\mu\circ\chi$ sends $A_{j,2g+n}$ in 1 for $2g<j<2g+n$ and $1\le j<2g$, j even. On the other hand, $\mu\circ\chi(A_{2k-1,2g+n}) = \widetilde{A_{k,2g+1}}$ for $k=1,\ldots,g$. $\square$

Recall that the existence of a section for θ implies that $Y_{n-1}(F)$ acts by conjugation on G_n and thus on the abelianization $G_n/[G_n, G_n]$. The following Lemma concludes the proof. □

Lemma 1.6.3 *The action of $Y_{n-1}(F)$ by conjugation on $G_n/[G_n, G_n]$ is trivial.*

Proof: Let $t \in \{A_{j,k} | 2g < k < 2g+n, 2g < j < k \text{ and } 1 \leq j < 2g, \text{ j even}\}$ and $f \in \{A_{j,2g+n} | 2g < j < 2g+n \text{ and } 1 \leq j < 2g, \text{ j even}\}$. We need to verify that every t acts trivially on $G_n/[G_n, G_n]$. Presentation in Theorem 1.6.1 shows that

$$(A) \quad tft^{-1} \equiv f \pmod{[G_n, G_n]},$$

for every t and f. Now consider the action of t on $A_{2s,2g+n}$, for $s = 1, \ldots, g$. We refer once again to Theorem 1.6.1 for showing that for every $t \in \{A_{j,k} | 2g < k < 2g+n, 2g < j < k \text{ and } 1 \leq j < 2g, j \text{ even}\}$,

$$(B) \quad tA_{2s-1,2g+n}t^{-1} = hA_{2s-1,2g+n} \quad (1 \leq s \leq g),$$

where $h \in G_n$. Let γ be a word on $\{A^{\pm 1}_{2k-1,2g+n} | 1 \leq k < g\}$. From (A) and (B) it follows that, for every $t \in \{A_{j,k} | 2g < k < 2g+n, 2g < j < k \text{ and } 1 \leq j < 2g, j \text{ even}\}$, $t\gamma f\gamma^{-1}t^{-1} = t\gamma t^{-1}tft^{-1}t\gamma^{-1}t^{-1} = h\gamma tft^{-1}\gamma^{-1}h^{-1} \equiv h\gamma f\gamma^{-1}h^{-1} \equiv \gamma f\gamma^{-1}$, where h is an element of G_n. □

Remark 1.6.4 *We notice that classical techniques do not apply to the whole group $P(n, F)$. The main obstruction is that, even when the exact sequence (PBS) splits, the action of $P(n, F)$ on the abelianisation of $\pi_1(F \setminus \{x_1, \ldots, x_{n-1}\})$ is not trivial, because of relations (ER1) and (ER2). In particular, when F is a surface of genus $g \geq 1$, it is presently unknown whether the graded group associated to the lower central series of $P(n, F)$ is torsion free.*

Remark 1.6.5 *According to [51] and [54], the mapping class group of a pointed surface (see Definition 3.2.1) is residually finite. As $P(n, F)$ is a (normal) subgroup of the mapping class group of a pointed surface ([15]), it follows that $P(n, F)$ is residually finite.*

Chapter 2

Braid presentations via graphs

2.1 Introduction

To any planar, connected graph with n vertices, without loops or intersections, it can be associated a presentation for the braid group B_n (Sergiescu, [83] and [84]). To each edge e of the graph we associate the braid β_e which is a clockwise half-twist along e (see Figure 2.1).

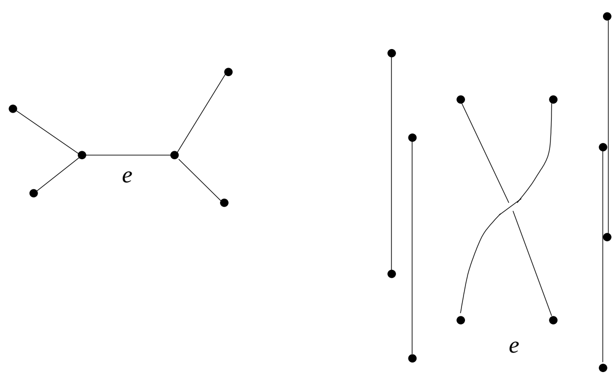


Figure 2.1: Edges and geometric braids.

Sergiescu provided a complete set of relations using this set of generators for B_n. Afterwards, Birman, Ko and Lee ([18]) extended this result to inner-complete graphs in order to give a new proof for the conjugation problem in B_n. Recently Han and Ko ([53]) showed that it is possible to associate braid group presentations to a more general family of graphs (*linearly spanned graphs*) containing above graphs. We recall also that these presentations turned out useful in other related contexts (see for instance [13] and [14]). In this chapter we provide an analogous result for sphere braids (Theorem 2.2.1) and we prove some results on automorphisms of $B(n, S^2)$. In particular, we prove that the outer automorphisms group of $B(n, S^2)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

2.2 Sphere braid groups presentations via graphs

2.2.1 Definitions and Statement of the Main Theorem

Definition 2.2.1 *Let Γ be a graph on an orientable surface F. The graph Γ is called* normal *if Γ is connected, finite and it has no loops or intersections.*

Let Γ be a normal graph on F. Let $S(\Gamma)$ be the set of vertices of Γ. We associate to the edges of Γ the corresponding geometric braids on F (Figure 2.1) and we define $B(\Gamma, F)$ the subgroup of $B(S(\Gamma), F)$ generated by these braids. In the following we will use the same notation for elements in the free group generated by the set $X_\Gamma = \{\sigma \mid \sigma \text{ edge of } \Gamma\}$ and corresponding braids in $B(\Gamma, F)$. It can be easily verified that $B(\Gamma, F) = B(S(\Gamma), F)$ if $F = D^2$ or S^2. Otherwise $B(\Gamma, F) \subseteq N_{S(\Gamma)}F$, where $N_{S(\Gamma)}F$ is the normal closure of $B_{S(\Gamma)}$ in $B(S(\Gamma), F)$. Proposition 1.5.2 shows that the inclusion $N_{S(\Gamma)}F \subset B(S(\Gamma), F)$ is proper.
From now on, Γ is a normal graph on S^2.
Suppose that Γ is not a tree. The set $S^2 \setminus \Gamma$ is the disjoint union of a finite number of open disks $D_1, \ldots, D_m$, $m > 1$. The boundary of D_j on S^2 is a subgraph $\Gamma(D_j)$ of Γ. We choose a point O in the interior of $\Gamma(D_j)$ and an edge σ of $\Gamma(D_j)$, with vertices v_1 and v_2. We suppose that the triangle $O\,v_1 v_2$ is anti clock-wise oriented. To the subgraph $\Gamma(D_j)$ we associate a polygon P_j with p_j edges as follows.
We suppose that the edges $e_1, \ldots, e_{p_j}$ and the vertices $x_1, \ldots, x_{p_j}$ of P are anti clock-wise oriented. The edge e_1 is labelled with the edge $\sigma(e_1) = \sigma$ of $\Gamma(D_j)$. The vertices x_1, x_2 of e_1 are labelled with the vertices $v(x_1) = v_1, v(x_2) = v_2$. Each edge e_{i+1} of P (we set $e_{p_j+1} = e_1$) is labelled with $\sigma(e_{i+1}) \subset \Gamma(D_j)$, the first edge on the left of $\sigma(e_i)$ adjacent to $v(x_{i+1})$. The vertex x_{i+2} is labelled with the other vertex adjacent to $\sigma(e_{i+1})$. If $v(x_{i+1})$ is a uni-valent vertex then $\sigma(e_{i+1}) = \sigma(e_i)$ and $v(x_{i+2}) = v(x_i)$.

Definition 2.2.2 *The sequence $\sigma(e_1) \ldots \sigma(e_{p_j})$ defined below is the (anti clock-wise oriented) pseudo-cycle associated to D_j.*

The pseudo-cycle $\sigma(e_1) \ldots \sigma(e_{p_j})$ is uniquely defined up to cyclic permutation.

Definition 2.2.3 *Let $\gamma = \sigma(e_1) \ldots \sigma(e_p)$ be a pseudo-cycle of Γ. If there exist a pair i, j, $1 \leq i, j \leq p$ such that $\sigma(e_i) = \sigma(e_j)$, we say that*

- *$\sigma(e_i)$ is the start edge of a reversing if $j \neq i - 1$ (we set $e_0 = e_p$).*
- *$\sigma(e_i)$ is the end edge of a reversing if $j \neq i + 1$ (we set $e_{p+1} = e_1$).*

We set $\sigma_1 \ldots \sigma_p$ for the pseudo-cycle $\sigma(e_1) \ldots \sigma(e_p)$.
Let Δ be a maximal tree of Γ. We start from x going through σ and at the vertex y we choose the first edge on the left. We iterate this process until meeting an uni-valent vertex, say z, where we go back through the edge corresponding to z and we start again the process. In this way we come back to x after we passed two times through each edge of Δ.

Definition 2.2.4 *Set $\delta_{x,\sigma}(\Delta)$ for the word in X_Γ corresponding to the above circuit (Figure 2.2).*

The element $\delta_{x,\sigma}(\Delta)$ corresponds to a braid where the only non trivial string corresponds to the vertex x. The projection of this string on the sphere is a simple (oriented) loop bounding a disk containing all other vertices.

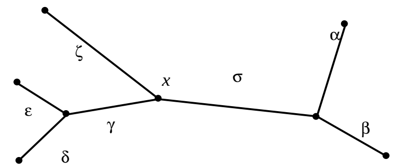


Figure 2.2: $\delta_{x,\sigma}(\Delta) = \sigma\alpha^2\beta^2\sigma\gamma\delta^2\epsilon^2\gamma\zeta^2$.

Theorem 2.2.1 *Let Γ be a normal graph with n vertices. The braid group $B(n, S^2)$ admits the presentation $\langle X_\Gamma \mid R_\Gamma \rangle$, where $X_\Gamma = \{\sigma \mid \sigma \text{ edge of } \Gamma\}$ and R_Γ is the set of following relations:*

- *Separate relations (SR): if $\sigma_i \cap \sigma_j = \emptyset$ then $\sigma_i\sigma_j = \sigma_j\sigma_i$;*
- *Adjacency relations (AR): if σ_i, σ_j have a common vertex, then $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$;*
- *Nodal relations (NR): if $\{\sigma_1, \sigma_2, \sigma_3\}$ have only one common vertex and they are clockwise ordered (Figure 2.3), then*

$$\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2\,;$$

- *Pseudo-cycle relations (PR): if $\sigma_1 \dots \sigma_m$ is a pseudo-cycle and σ_1 is not the start edge or σ_m the end edge of a reversing (Definition 2.2.3 and Figure 2.4), then*

$$\sigma_1\sigma_2 \cdots \sigma_{m-1} = \sigma_2\sigma_3 \cdots \sigma_m\,.$$

- *Tree relations (TR): $\delta_{x,\sigma}(\Delta) = 1$, for every maximal tree $\Delta \subseteq \Gamma$, every vertex $x \in \Delta$ and every edge $\sigma \in \Delta$ adjacent to x.*

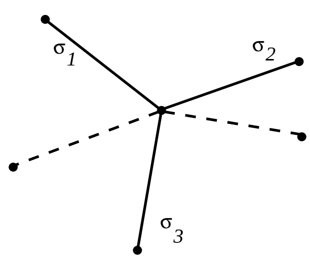


Figure 2.3: Nodal relation.

Remark 2.2.1 *The pseudo-cycle $\sigma_3^2\sigma_4\sigma_1\sigma_2$ in Figure 2.4 (respectively the pseudo-cycle $\sigma_4\sigma_1\sigma_2\sigma_3^2$) does not verify the hypothesis of the definition of the relation (PR). Spherical braids corresponding to the words $\sigma_3^2\sigma_4\sigma_1$ and $\sigma_3\sigma_4\sigma_1\sigma_2$ (respectively the braids corresponding to the words $\sigma_4\sigma_1\sigma_2\sigma_3$ and $\sigma_1\sigma_2\sigma_3^2$) do not represent the same element in $B(n, S^2)$.*

Definition 2.2.5 *Let Γ be a normal graph and let $\sigma_1 \dots \sigma_p$ be a pseudo-cycle of Γ. We set $R_{\sigma_1 \dots \sigma_p}$ for the set of (PR) relations satisfied, up to cyclic permutation, by the pseudo-cycle $\sigma_1 \dots \sigma_p$.*

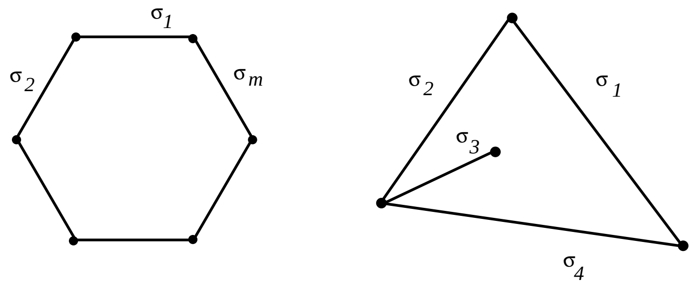


Figure 2.4: Pseudo-cycle relation; on the left $\sigma_1\sigma_2\cdots\sigma_{m-1} = \sigma_2\cdots\sigma_m = \cdots = \sigma_m\cdots\sigma_{m-2}$. On the right $\sigma_1\sigma_2\sigma_3^2 = \sigma_2\sigma_3^2\sigma_4 = \sigma_3^2\sigma_4\sigma_1$ and $\sigma_3\sigma_4\sigma_1\sigma_2 = \sigma_4\sigma_1\sigma_2\sigma_3$.

Remark 2.2.2 *The definition of pseudo-cycle extends naturally to a tree* Γ*. In particular* $\delta_{x,\sigma_1}(\Gamma)$ *is the word* $\sigma_1\cdots\sigma_p$ *in the free group generated by the set* X_Γ*, where the sequence* $\sigma_1\ldots\sigma_p$ *is the pseudo-cycle associated to* $S^2\setminus\Gamma$*. The set of (TR) relations of* Γ *implies the set of relations* $R_{\sigma_1\ldots\sigma_p}$*.*

Remark 2.2.3 *Let* $\gamma\subseteq\Gamma$ *be a star. For any clock-wise ordered subset* $\{\sigma_{i_1},\ldots,\sigma_{i_j}\,|\,j\geq 2\}$ *of edges of* γ *the following relation holds in the group* $\langle X_\Gamma\,|\,R_\Gamma\rangle$*:*

$$\sigma_{i_1}\ldots\sigma_{i_j}\sigma_{i_1} = \sigma_{i_j}\sigma_{i_1}\ldots\sigma_{i_j}\,.$$

2.2.2 Geometric interpretation of relations

The natural map $\phi_\Gamma : \langle X_\Gamma\,|\,R_\Gamma\rangle \to B(\Gamma, S^2)$ is an homomorphism. It is geometrically evident that the relations (AR) and (SR) hold in $B(\Gamma, S^2)$. Let Γ contain a triangle σ_1, σ_2, τ as in Figure 2.9. Corresponding spherical braids verify the relation $\tau = \sigma_1\sigma_2\sigma_1^{-1}$ and thus $\tau\sigma_1 = \sigma_1\sigma_2$ in $B(\Gamma, S^2)$. The relation $\sigma_1\sigma_2 = \sigma_2\tau$ follows from the braid relation $\sigma_1\sigma_2\sigma_1^{-1} = \sigma_2^{-1}\sigma_1\sigma_2$. Let $\sigma_1, \sigma_2, \sigma_3$ be arranged as in Figure 2.5. We add three edges τ_1, τ_2, τ_3. The nodal relation follows from pseudo-cycle relations on triangles $\tau_1\sigma_2\sigma_3$, $\tau_2\sigma_1\sigma_3$ and $\tau_3\sigma_1\sigma_2$. In fact, $\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\tau_3\sigma_3\sigma_1 = \sigma_2\sigma_3\tau_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2$. All other pseudo-cycle relations follow from induction on the length of the cycle.

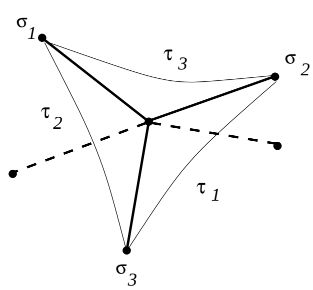


Figure 2.5: Nodal relation holds in $B(\Gamma, S^2)$.

Let $x, \sigma, \Delta \subset \Gamma$. The word $\delta_{x,\sigma}(\Delta)$ corresponds to the geometric braid in $B(\Gamma, S^2)$, where the only non vertical string is the string associated to the vertex x going around (with clock-wise orientation) all other strings (Figure 2.6). This braid is isotopic to the trivial braid and then $\delta_{x,\sigma}(\Delta) = 1$ in $B(\Gamma, S^2)$.

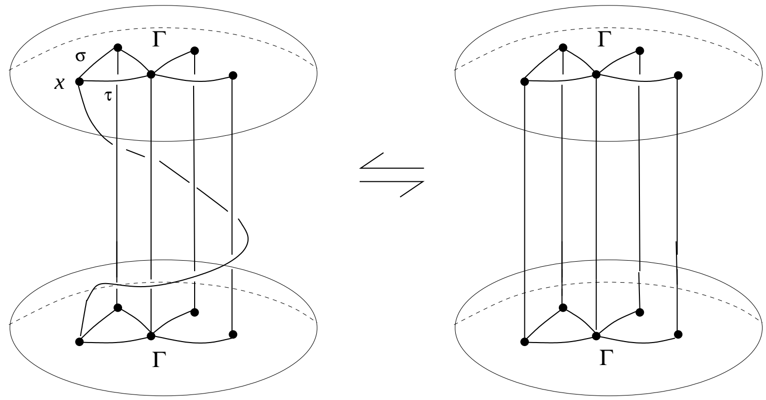


Figure 2.6: The braid $\delta_{x,\sigma}(\Delta)$ associated to the tree $\Delta = \Gamma \setminus \tau$.

2.3 Proof of Theorem 2.2.1

2.3.1 Preliminaries

The steps of the proof are similar to [84]. We need some preliminary Lemmas.

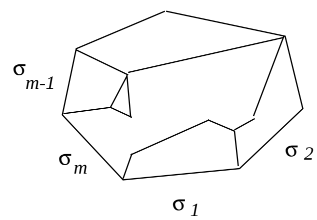


Figure 2.7: Projection of the graph Γ on the face bounded by $\sigma_1, \sigma_2, \ldots, \sigma_m$.

Lemma 2.3.1 *Let Γ be a normal graph on the sphere and let $\sigma_1 \ldots \sigma_m$ be a pseudo-cycle on Γ. Let $\langle X_\Gamma \mid R_\Gamma \rangle$ be the group defined in Theorem 2.2.1.*
The groups $\langle X_\Gamma \mid R_\Gamma \rangle$ and $\langle X_\Gamma \mid R_\Gamma \setminus \{R_{\sigma_1 \ldots \sigma_m}\} \rangle$ are isomorphic.

Proof: We can represent the graph Γ on the plane, projecting Γ on the face bounded by the pseudo-cycle $\sigma_1 \ldots \sigma_m$ (Figure 2.7). We need to show that the relation

$$\sigma_m \cdots \sigma_2 = \sigma_{m-1} \cdots \sigma_1$$

holds in $\langle X_\Gamma \mid R_\Gamma \setminus \{R_{\sigma_1 \ldots \sigma_m}\} \rangle$.

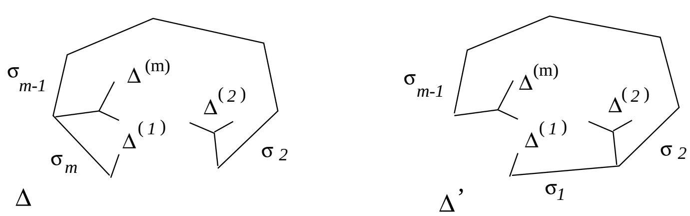


Figure 2.8: The maximal trees Δ and Δ'.

Consider two maximal trees Δ, Δ' in Γ such that the tree Δ contains $\sigma_2, \ldots, \sigma_m$, the tree Δ' contains $\sigma_1, \ldots, \sigma_{m-1}$, and $\Delta \cup \{\sigma_1\} = \Delta' \cup \{\sigma_m\}$. The graph $\Delta \setminus \{\sigma_2, \ldots, \sigma_m\} = \Delta' \setminus \{\sigma_1, \ldots, \sigma_{m-1}\}$ is a set of subtrees $\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(m)}$ of Γ (Figure 2.8). The (TR) relations on Δ and Δ' yield the following relation:

$$(A) \quad \sigma_m \cdots \sigma_2\, \alpha_2\, \sigma_2\, \alpha_3 \cdots \sigma_m\, \alpha_1 = \sigma_{m-1} \cdots \sigma_1\, \alpha_1\, \sigma_1\, \alpha_2 \cdots \sigma_{m-1}\, \alpha_m \ ,$$

where $\alpha_1, \ldots, \alpha_m$ are sub-words associated to corresponding sub-trees $\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(m)}$. Let $\alpha_1 = \zeta_1 \cdots \zeta_{n-1}\zeta_n\zeta_{n+1} \cdots \zeta_q$, where $\zeta_{n-1} = \zeta_n$ et $\zeta_i \neq \zeta_j$, for $i, j > n+1$. We apply the (PR) relation on the pseudo-cycle $\zeta_n\zeta_{n+1} \cdots \zeta_q\sigma_1\alpha_2 \cdots \sigma_m\zeta_1 \cdots \zeta_{n-2}\zeta_{n-1}$,

$$\zeta_n\zeta_{n+1} \cdots \zeta_q\sigma_1\alpha_2 \cdots \sigma_m\zeta_1 \cdots \zeta_{n-2} = \zeta_{n+1} \cdots \zeta_q\sigma_1\alpha_2 \cdots \sigma_m\zeta_1 \cdots \zeta_{n-2}\zeta_{n-1}\zeta_n \ ,$$

and we derive

$$\zeta_q^{-1} \cdots \zeta_{n+1}^{-1}\zeta_n\zeta_{n+1} \cdots \zeta_q\sigma_1\alpha_2 \cdots \sigma_m\zeta_1 \cdots \zeta_{n-2} = \sigma_1\alpha_2 \cdots \sigma_m\zeta_1 \cdots \zeta_{n-2}\zeta_{n-1} \, .$$

We premultiply by σ_1^{-1} and we apply (AR) relations in order to obtain

$$\zeta_n\zeta_{n+1} \cdots \zeta_q\sigma_1\zeta_q^{-1} \cdots \zeta_{n+1}^{-1}\zeta_n^{-1}\alpha_2 \cdots \sigma_m\zeta_1 \cdots \zeta_{n-2} = \alpha_2 \cdots \sigma_m\zeta_1 \cdots \zeta_{n-2}\zeta_{n-1} \, .$$

It follows that

$$\alpha_1\sigma_1\alpha_1^{-1} = \zeta_1 \cdots \zeta_{n-1}\alpha_2 \cdots \sigma_m\zeta_1 \cdots \zeta_{n-2}\zeta_{n-1}\zeta_{n-2}^{-1} \cdots \zeta_1^{-1}\sigma_m^{-1} \cdots \alpha_2^{-1}\zeta_{n-1}^{-1} \cdots \zeta_1^{-1} \, .$$

From iterated applications of (AR) and (NR) relations it follows that

$$\sigma_m\zeta_1 \cdots \zeta_{n-2}\zeta_{n-1}\zeta_{n-2}^{-1} \cdots \zeta_1^{-1}\sigma_m^{-1} = \zeta_{n-1}^{-1}\zeta_{n-2}^{-1} \cdots \zeta_1^{-1}\sigma_m\zeta_1 \cdots \zeta_{n-2}\zeta_{n-1} \, .$$

Then

$$\alpha_1\sigma_1\alpha_1^{-1} = \alpha_2\sigma_2 \cdots \alpha_m\sigma_m\alpha_m^{-1} \cdots \sigma_2^{-1}\alpha_2^{-1}$$

holds in $\langle X_\Gamma \,|\, R_\Gamma \setminus \{R_{\sigma_1 \ldots \sigma_m}\}\rangle$. The braids α_1 and $\alpha_m^{-1} \cdots \sigma_2^{-1}\alpha_2^{-1}$ commute, and thus

$$\alpha_2\sigma_2 \cdots \alpha_m\sigma_m\alpha_1 = \alpha_1\sigma_1\alpha_2 \cdots \alpha_m \, .$$

From equation (A), it follows that

$$\sigma_m \cdots \sigma_2 = \sigma_{m-1} \cdots \sigma_1 \, .$$

□

The Following Lemmas establish that for any graph Γ' obtained removing or adding “triangles” to Γ the groups $\langle X_{\Gamma'} \,|\, R_{\Gamma'}\rangle$ and $\langle X_\Gamma \,|\, R_\Gamma\rangle$ defined in Theorem 2.2.1 are isomorphic.

Lemma 2.3.2 *Let σ_1, σ_2 be two adjacent edges of Γ, which are not contained in any pseudo-cycle. Let $\Gamma' = \Gamma \cup \tau$ be the graph obtained adding an edge τ to Γ to form an anti clock-wise triangle $\tau\sigma_1\sigma_2$ (Figure 2.9). If $B(\Gamma, S^2) = \langle X_\Gamma \,|\, R_\Gamma\rangle$ then $B(\Gamma', S^2) = \langle X_{\Gamma'} \,|\, R_{\Gamma'}\rangle$.*

Proof: By Tietze's transformation we obtain that $\langle X_{\Gamma'} \,|\, R_\Gamma,\, \tau = \sigma_1\sigma_2\sigma_1^{-1}\rangle$ is a presentation for $B(\Gamma, S^2) = B(\Gamma', S^2)$. Since $\{R_\Gamma,\, \tau = \sigma_1\sigma_2\sigma_1^{-1}\} \subset R_{\Gamma'}$, $\langle X_{\Gamma'} \,|\, R_{\Gamma'}\rangle$ is a presentation for $B(\Gamma', S^2)$. □

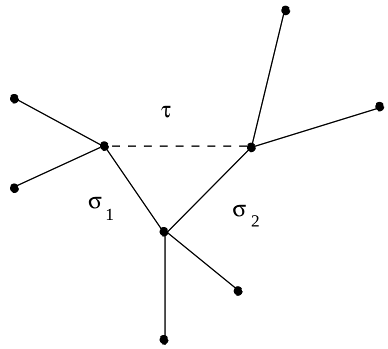


Figure 2.9: Adding or removing a triangle.

Lemma 2.3.3 *Let τ be an edge of Γ' bounding only one pseudo-cycle, which is an anti clockwise triangle $\tau\sigma_1\sigma_2$ (Figure 2.9). Let $\Gamma = \Gamma' \setminus \tau$. If $\langle X_{\Gamma'} \,|\, R_{\Gamma'} \rangle$ is a presentation for $B(\Gamma', S^2)$ then $\langle X_\Gamma \,|\, R_\Gamma \rangle$ is a presentation for $B(\Gamma, S^2)$.*

Proof: We need to show that the set of relations $R_{\Gamma'}$ are verified in the group $\langle X_{\Gamma'} \,|\, R_\Gamma, \tau = \sigma_1\sigma_2\sigma_1^{-1} \rangle$. Sergiescu showed (Lemma 1.3 in [84]) that the relations (SR), (AR) and (NR) for the graph Γ' are a consequence of the relations (SR), (AR) and (NR) for the graph Γ and the relation $\tau = \sigma_1\sigma_2\sigma_1^{-1}$. The edge τ belongs to the pseudo-cycle $\tau\sigma_1\sigma_2$. The corresponding pseudo-cycle relations derive from $\tau = \sigma_1\sigma_2\sigma_1^{-1}$ and $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. We prove that the relation $\delta_{x,\sigma}(\Delta) = 1$, for any δ, x, σ, is a consequence of the set of relations $\{R_\Gamma,\, \tau = \sigma_1\sigma_2\sigma_1^{-1}\}$. If $\tau \notin \Delta$ the claim follows. Suppose that $\tau \in \Delta$. We have two cases (we refer to the Figure 2.10):

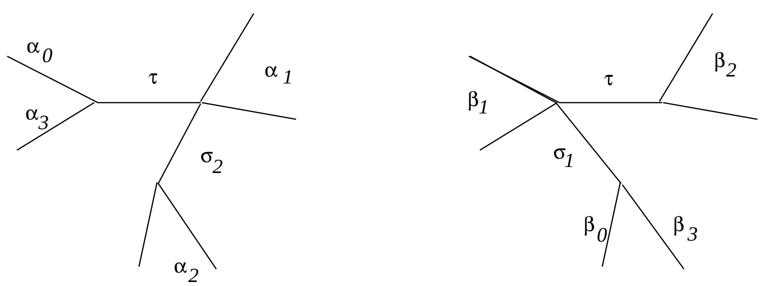


Figure 2.10: Tree relations for $\Gamma' = \Gamma \cup \tau$ are generated by the set $\{R_\Gamma \cup \tau = \sigma_1\sigma_2\sigma_1^{-1}\}$.

1. $\delta_{x,\sigma}(\Delta) = \beta_0\sigma_1\beta_1\tau\beta_2\tau\sigma_1\beta_3$, where β_i are the sub-words obtained following the rest of Δ;

2. $\delta_{x,\sigma}(\Delta) = \alpha_0\tau\alpha_1\sigma_2\alpha_2\sigma_2\tau\alpha_3$, where α_i are the sub-words obtained following the rest of Δ.

Replace $\tau = \sigma_1\sigma_2\sigma_1^{-1}$. In the first case, from (SR) it follows that $\sigma_1\beta_2 = \beta_2\sigma_1$ and then

$$\beta_0\sigma_1\beta_1\sigma_1\sigma_2\sigma_1^{-1}\beta_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_1\beta_3 = \beta_0\sigma_1\beta_1\sigma_1\sigma_2\sigma_1^{-1}\beta_2\sigma_1\sigma_2\beta_3 = \beta_0\sigma_1\beta_1\sigma_1\sigma_2\beta_2\sigma_2\beta_3$$

One deduces that the relation $\delta_{x,\sigma}(\Delta) = 1$ holds in $\langle X_{\Gamma'} \,|\, R_\Gamma, \tau = \sigma_1\sigma_2\sigma_1^{-1} \rangle$. In the second case, let write $\alpha_2 = \sigma_3\zeta_3\sigma_3 \ldots \sigma_p\zeta_p\sigma_p$, where σ_k, for $k = 1, \ldots, p$, corresponds to an edge of Γ

adjacent to σ_1 and σ_2 and ζ_k, for $k = 1, \ldots, p$, corresponds to a tree disjoint from σ_1 and σ_2. The elements σ_k $(k = 1, \ldots, p)$ and $\sigma_1\sigma_2\sigma_1^{-1}$ commute:

$$\sigma_k\sigma_1\sigma_2\sigma_1^{-1} = \sigma_2^{-1}\sigma_1\sigma_2\sigma_k = \sigma_1\sigma_2\sigma_1^{-1}\sigma_k\,.$$

One derives that $[\alpha_2, \sigma_1\sigma_2\sigma_1^{-1}] = 1$. From $\sigma_1\alpha_1 = \alpha_1\sigma_1$, it follows that

$$\begin{aligned}
\alpha_0\sigma_1\sigma_2\sigma_1^{-1}\alpha_1\sigma_2\alpha_2\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\alpha_3 &= \alpha_0\sigma_1\sigma_2\alpha_1\sigma_1^{-1}\sigma_2\alpha_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_1\alpha_3 = \\
\alpha_0\sigma_1\sigma_2\alpha_1\sigma_1^{-1}\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\alpha_2\sigma_1\alpha_3 &= \alpha_0\sigma_1\sigma_2\alpha_1\sigma_2\alpha_2\sigma_1\alpha_3\,.
\end{aligned}$$

Therefore $\delta_{x,\sigma}(\Delta) = 1$ holds in $\langle X_{\Gamma'} \mid R_\Gamma,\, \tau = \sigma_1\sigma_2\sigma_1^{-1}\rangle$. We remark that τ belongs also to the pseudo-cycle P bounding the other connected component of $S^2 \setminus \Gamma$. According to Lemma 2.3.1 we can suppose that the set of pseudo-cycle relations R_P are redundant and thus $\langle X_{\Gamma'} \mid R_{\Gamma'}\rangle$ is isomorphic to $\langle X_{\Gamma'} \mid R_\Gamma,\, \tau = \sigma_1\sigma_2\sigma_1^{-1}\rangle$. □

2.3.2 Inductive steps

Definition 2.3.1 *A* node *is a vertex of valence greater than two. We call the valence of* Γ*,* $v(\Gamma)$*, the sum of valences of all nodes of* Γ*.*

In order to prove Theorem 2.2.1, we proceed by induction on the number of connected components of $S^2 \setminus \Gamma$.
i) Let Γ be a tree. We recall that the braid group on the sphere is a quotient of the braid group on the disk.

Theorem 2.3.1 ([35]) *The group* $B(n, S^2)$ *admits the following presentation*

- *Generators:* $\sigma_1, \ldots, \sigma_{n-1}$.
- *Relations:*

$$\begin{aligned}
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}\,; \\
\sigma_i\sigma_j &= \sigma_j\sigma_i \quad \textit{for } |i-j| \geq 2\,; \\
\sigma_1\sigma_2\cdots\sigma_{n-1}^2\cdots\sigma_2\sigma_1 &= 1\,.
\end{aligned}$$

Then, it follows that Theorem 2.2.1 holds when Γ is a straight line and $v(\Gamma) = 0$. Suppose that Theorem 2.2.1 holds for all trees with valence less than $q > 0$ and let Γ be a tree such that $v(\Gamma) = q$. Let v_0 be an uni-valent vertex of Γ. As in Definition 2.2.4, we run on the tree Γ starting from v_0 and choosing to turn on the right at each node. Let v_1 be the vertex preceding the first node. Let v_2 be the first uni-valent vertex after v_0 (see Figure 2.11).
We replace the edge τ between v_1 and the first node with an edge τ_1 joining v_1 to the vertex v_2 (see Figure 2.12). The graph Γ_1 so obtained is such that $v(\Gamma_1) < v(\Gamma)$, and then Theorem 2.2.1 holds for Γ_1. From Lemma 2.3.2 it follows that Theorem 2.2.1 is verified for the graph Γ_2 obtained adding an edge τ_2 between v_1 and the other vertex adjacent to v_2 (see Figure 2.13). From Lemma 2.3.3 one deduces that Theorem 2.2.1 holds for the graph $\Gamma_3 = \Gamma_2 \setminus \{\tau_1\}$. Iterating the process we derive that the result holds for the initial tree Γ.
ii) Suppose that Theorem 2.2.1 holds when the number of connected components of $S^2 \setminus \Gamma$ is less than $p > 1$. Let Γ be a normal graph such that $S^2 \setminus \Gamma$ has p connected components. We remove an edge σ_1 which bounds two pseudo-cycles $\sigma_1, \sigma_2 \ldots, \sigma_n$ and $\sigma_1, \tau_2, \ldots, \tau_m$ of Γ. We encounter two cases.

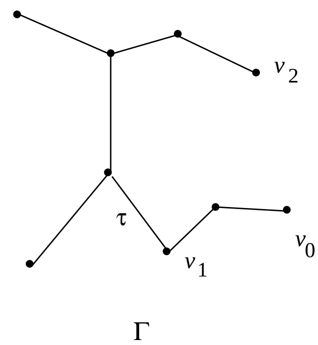


Figure 2.11: We suppose Γ embedded in the sphere.

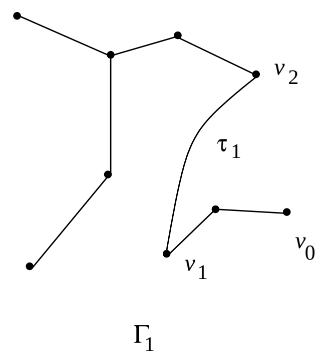


Figure 2.12: Replacing the edge τ withτ_1.

1. σ_n is not the end edge of a reversing. From induction hypothesis and Tietze's transformation, we deduce that $\langle X_\Gamma \mid R_{\Gamma\setminus\{\sigma_1\}}, \sigma_1 \cdots \sigma_{n-1} = \sigma_2 \ldots \sigma_n\rangle$ is a presentation for $B(\Gamma, S^2)$. According to Lemma 2.3.1 we can suppose that pseudocycle relations for the cycle $\sigma_2, \ldots, \sigma_n, \tau_m, \ldots, \tau_2$ are redundant. Therefore R_Γ contains the set $\{R_{\Gamma\setminus\{\sigma_1\}}, \sigma_1 \cdots \sigma_{n-1} = \sigma_2 \ldots \sigma_n\}$ and we conclude that $\langle X_\Gamma \mid R_\Gamma\rangle$ is a presentation for $B(\Gamma, S^2)$.

2. If σ_n is the end edge of a reversing, there exists $l < n$ such that $\sigma_l = \sigma_{l+1}$, and $\sigma_i \neq \sigma_j$ for $l + 1 \leq i < j \leq n$. It follows that the relation

$$\sigma_{l+1}\sigma_{l+2} \cdots \sigma_n\sigma_1\sigma_2 \ldots \sigma_{l-1} = \sigma_{l+2} \cdots \sigma_n\sigma_1\sigma_2 \ldots \sigma_{l-1}\sigma_l$$

holds in $\langle X_\Gamma \mid R_\Gamma\rangle$. Multiplying by $\sigma_1^{-1}\sigma_n^{-1} \cdots \sigma_{l+2}^{-1}$ and applying (AR) relations we obtain

$$\sigma_2 \ldots \sigma_{l-1}\sigma_l = \sigma_1^{-1}\sigma_n^{-1} \cdots \sigma_{l+2}^{-1}\sigma_{l+1}\sigma_{l+2} \cdots \sigma_n\sigma_1\sigma_2 \ldots \sigma_{l-1} =$$

$$= \sigma_{l+1}\sigma_{l+2} \cdots \sigma_n\sigma_1\sigma_n^{-1} \cdots \sigma_{l+2}^{-1}\sigma_{l+1}^{-1}\sigma_2 \ldots \sigma_{l-1} ,$$

what yields

$$\sigma_1 = \sigma_n^{-1} \cdots \sigma_{l+1}^{-1}\sigma_2 \cdots \sigma_{l-1}\sigma_l\sigma_{l-1}^{-1} \cdots \sigma_2^{-1}\sigma_{l+1} \cdots \sigma_n .$$

Thus the above argument concludes the proof.

□

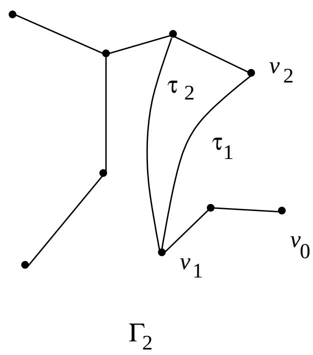


Figure 2.13: Adding and removing triangulations.

Remark 2.3.1 *In order to show Theorem 2.2.1, one can also consider a normal graph* Γ *on the disk and add to Sergiescu's relations for* $B(\Gamma, D^2)$ *([84]) all (TR) relations for* Γ*. Let* $\langle X_\Gamma \mid Z_\Gamma \rangle$ *be the group so obtained. Lemmas 2.3.2 and 2.3.3 as well as the inductive steps hold true and then* $\langle X_\Gamma \mid Z_\Gamma \rangle$ *is isomorphic to* $B(n, S^2)$*. Anyway, our approach allows to consider* Γ *as embedded on* S^2 *and to prove in an algebraic way the redundance of the pseudo-cycle relations.*

2.3.3 Automorphisms and isometries

The presentation of Theorem 2.2.1 is redundant however useful because we can read the relations on the graph Γ.

Definition 2.3.2 *Let* F *be a sub-set of* $\mathbb{R}^2$ *(*S^2*). The symmetry group of* F*,* $\Sigma(F)$*, is the set of congruent transformations of* $\mathbb{R}^2$ *(*S^2*) that leave* F *invariant. We denote* $\Sigma(F)^+$ *the sub-group of* $\Sigma(F)$ *generated by rotations. The symmetry group* $\Sigma(F)$ *is discrete if the set* $\{\phi(P) | \phi \in \Sigma(F)\}$ *is discrete for any point* $P \in F$*.*

Corollary 2.3.1 *Every finite group* H *of* $O(3)$ *is isomorphic to a subgroup of* $Aut(B(n, S^2))$*, for some* n*.*

Proof: Let Γ be a normal graph on the sphere such that $\Sigma(\Gamma) = H$. Since relations of $B(\Gamma, S^2)$ hold by rotations we associate to every rotation $\rho \in H$ the corresponding automorphism $\bar{\rho}$ of $B(\Gamma, S^2)$. To every reflection $\chi \in H$ we associate the morphism $\bar{\chi}$ that moves the generator σ of $B(\Gamma, S^2)$ in the braid $\chi(\sigma)^{-1}$. This map is an automorphism of $B(\Gamma, S^2)$. The subgroup K of $Aut(B(\Gamma, S^2))$ generated by the set $\{\bar{g} \mid g$ generator of $H\}$ is isomorphic to H. $\square$

We recall that B_∞ is the inductive limit of the sequence $B_1 \subset B_2 \subset \ldots$. Sergiescu showed that it is possible to associate to every infinite graph on the plane, locally finite and without loops or intersections, a presentation for B_∞ ([84]). The following Corollary is the analogous of Corollary 2.3.1

Corollary 2.3.2 *Let* F *be a subset of* $\mathbb{R}^2$*. Let* $\Sigma(F)$ *be discrete. Then* $\Sigma(F)$ *is isomorphic to a subgroup of* $Aut(B_\infty)$*.*

Proof: For every subset $F \subset \mathbb{R}^2$ such that $\Sigma(F)$ is discrete, there exists an infinite graph Γ locally finite such that $\Sigma(F) = \Sigma(\Gamma)$ (see for instance [27]). $\square$

2.4 The outer automorphisms group of $B(n, S^2)$

Let G be a group. Let $Aut(G)$ be the automorphism group of G, $Inn(G)$ the inner automorphism group of G and $Out(G) = Aut(G)/Inn(G)$ the outer automorphism group of G. We conclude the Chapter with the computation of the outer automorphisms group of sphere braid groups. We recall that $Out(B_n) = \mathbb{Z}_2$ and the morphism $\phi : B_n \to B_n$ defined by $\phi(\sigma_j) = \sigma_j^{-1}$ for $j = 1, \ldots, n-1$ is the generator of $Out(B_n)$ ([33]).

Definition 2.4.1 *For an element $x \in B(n, S^2)$, written in standard generators of Theorem 2.3.1 as $x = \prod_i \sigma_{j_i}^{\lambda_i}$ we denote by $e(x) = \sum_i \lambda_i \mod 2(n-1)$, the exponent sum of x.*

It follows from the presentation in Theorem 2.3.1 that the exponent sum is well-defined (it is independent on the word chosen in the generators). Let $\bar{\phi}$ in $Out(B(n, S^2))$ and let $\phi \in Aut(B(n, S^2))$ a co-representative of $\bar{\phi}$. The exponent sum is invariant up to inner automorphisms, and thus we can set $e(\bar{\phi}(x))$ for $e(\phi(x))$, for $x \in B(n, S^2)$.
Let $ZB(n, S^2)$ be the center of $B(n, S^2)$. Let $\mathcal{M}(n, S^2)$ be the *mapping class group* of the n-punctured sphere (see next chapter).

Proposition 2.4.1 *The group $Out(\dfrac{B(n, S^2)}{ZB(n, S^2)})$ is isomorphic to $\mathbb{Z}_2$, for $n \geq 4$.*

Proof: The quotient $\dfrac{B(n, S^2)}{ZB(n, S^2)}$ is isomorphic to $\mathcal{M}(n, S^2)$ ([15]) and the group $Out(\mathcal{M}(n, S^2))$ is isomorphic to $\mathbb{Z}_2$ for $n \geq 4$ ([56]). □

Proposition 2.4.2 *The group $Out(B(n, S^2))$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, for $n \geq 4$.*

Proof: The subgroup $ZB(n, S^2)$ is isomorphic to $\mathbb{Z}_2$ and it is generated by the element $U = (\sigma_1 \cdots \sigma_{n-1})^n$ ([15]). Let id be the identity map in $Aut(B(n, S^2))$ and let ϕ_1 be the map defined by $\phi_1(\sigma_j) = \sigma_j^{-1}$ for $j = 1, \ldots, n-1$. Since the relations in Theorem 2.3.1 are symmetric, $\phi_1 \in Aut(B(n, S^2))$.

Let ϕ_2 be the map defined by $\phi_2(\sigma_j) = \sigma_j U$ for $j = 1, \ldots, n-1$. By the definition of ϕ_2 and the fact that U has order two and it generates $ZB(n, S^2)$, we derive that $\phi_2(U) = (U)^{n(n-1)+1} = U$. It follows that $\phi_2 \circ \phi_2 = id$, and thus $\phi_2 \in Aut(B(n, S^2))$. In the same way, one can verify that the map ϕ_3, defined by $\phi_3(\sigma_j) = \sigma_j^{-1} U$ for $j = 1, \ldots, n-1$, is an automorphism of $B(n, S^2)$.
Any automorphism $\phi \in Aut(B(n, S^2))$ induces an automorphism ϕ' of $Aut(B(n, S^2)/ZB(n, S^2))$. Moreover, if $\phi \in Inn(B(n, S^2))$, then $\phi \in Inn(B(n, S^2)/ZB(n, S^2))$. Therefore, we obtain a map $\psi : Out(B(n, S^2)) \to Out(B(n, S^2)/ZB(n, S^2))$. Let $\bar{\phi_1}, \bar{\phi_2}, \bar{\phi_3}$ be the images of ϕ_1, ϕ_2, ϕ_3 in $Out(B(n, S^2))$. Since $e(\bar{\phi_1}(\sigma_j)) = -1 \mod 2(n-1)$, for all $j = 1, \ldots, n-1$, it follows that $\bar{\phi_1}$ is a non trivial element of $Out(B(n, S^2))$ for $n \geq 3$.
Since U has order two and it generates $ZB(n, S^2)$, we deduce that all automorphisms of $B(n, S^2)$ leave invariant U. If an outer automorphism $\widetilde{\phi}$ belongs to $Ker(\psi)$, then, up to inner isomorphism, for $j = 1, \ldots, n-1$, $\widetilde{\phi}(\sigma_j) = \sigma_j$ or $\widetilde{\phi}(\sigma_j) = \sigma_j U$. The relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ imply that either $\widetilde{\phi}(\sigma_j) = \sigma_j$ for all $j = 1, \ldots, n-1$, or $\widetilde{\phi}(\sigma_j) = \sigma_j U$ for all $j = 1, \ldots, n-1$. Thus $Ker(\psi)$ is generated by $\bar{\phi_2}$.
We prove that, for $n \geq 4$, ϕ_2 is not an inner automorphism of $B(n, S^2)$, and thus $Ker(\psi)$ is isomorphic to $\mathbb{Z}_2$. Since $Out(B(n, S^2))/Ker(\psi)$ is trivial or isomorphic to $\mathbb{Z}_2$ (Proposition 2.4.1), it follows that $Out(B(n, S^2))$ has order two or four.

Consider the exact sequence

$$1 \longrightarrow P(n,S^2) \longrightarrow B(n,S^2) \xrightarrow{\rho} \Sigma_n \longrightarrow 1\,.$$

Remark that $Z\Sigma_n = \{1\}$ and $ZP(n,S^2) = ZB(n,S^2)$. On the other hand, the automorphism ϕ_2 leaves invariant the group $P(n,S^2)$. Moreover, the restriction of ϕ_2 on $P(n,S^2)$ is the identity map (if $x \in P(n,S^2)$ then $e(x)$ is an even number). Let $\hat{\phi}_2 : \Sigma_n \to \Sigma_n$ be the automorphism induced by ϕ_2. This automorphism is the identity on Σ_n. The automorphism ϕ_2 is not the identity on $B(n,S^2)$, since $U \neq 1$.
Suppose that ϕ_2 is an inner automorphism. Let $\alpha \in B(n,S^2)$ such that $\phi_2(\beta) = \alpha\beta\alpha^{-1}$ for all $\beta \in B(n,S^2)$. Since $\hat{\phi}_2 = id$, then $\rho(\alpha) \in Z\Sigma_n$ and thus $\rho(\alpha) = 1$. It follows that α belongs to $ZP(n,S^2)$. Since $ZP(n,S^2) = ZB(n,S^2)$, we deduce that ϕ_2 is the identity on $B(n,S^2)$, which is false.
When $n = 2k, k > 1$, the automorphism ϕ of $B(n,S^2)/ZB(n,S^2)$ defined by $\phi(\sigma_j) = \sigma_j^{-1}$ for $j = 1,\ldots,n-1$ can be choosen as representative of the generator of $Out(B(n,S^2)/ZB(n,S^2))$. Thus ψ is onto and $Out(B(n,S^2))$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.
When $n = 2k+1$, we derive that $e(\bar{\phi_2}(\sigma_j)) = 2k+1$ mod $4k$ and $e(\bar{\phi_3}(\sigma_j)) = 2k-1$ mod $4k$, and then, for $k > 1$, the elements $\bar{\phi_1}, \bar{\phi_2}, \bar{\phi_3}$ are distinct non trivial elements of $Out(B(n,S^2))$. It follows that $Out(B(n,S^2))$ is generated by $\bar{\phi_1}, \bar{\phi_2}, \bar{\phi_3}$ and it is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

□

Remark 2.4.1 *The group $B(3,S^2)$ has order 12 and it is isomorphic to the group T, the semi-direct product of $\mathbb{Z}_3$ by $\mathbb{Z}_4$ ([35]). It can be verified that $Aut(B(3,S^2)) = B(3,S^2)$ and $Inn(B(3,S^2)) = \mathbb{Z}_6$ and then $Out(B(3,S^2)) = \mathbb{Z}_2$. In this case, $e(\bar{\phi_1}(\sigma_j)) = e(\bar{\phi_2}(\sigma_j)) = 3$ mod 4, and it is simple to verify that $\phi_1 \equiv \phi_2$. We remark also that $Out(B(2,S^2))$ is trivial.*

Let F be an orientable surface different from the sphere, and let $\mathcal{M}(n,F)$ be the mapping class group of the surface F with n-punctures (see Definition 3.2.1).

Remark 2.4.2 *The characterisation of $Out(\mathcal{M}(n,F))$ is completely solved (see [56] and [70]). The braid group $B(n,F)$ is a subgroup of $\mathcal{M}(n,F)$ ([15]), but no result is known about $Out(B(n,F))$, except when F is the annulus ([30]).*

Chapter 3

Singular braids

3.1 Definitions and results

Singular braids have been introduced in [3] and [16] as an extension of classical braids on n strings. String of singular braids are allowed to intersect in a finite number of double points (*singular crossing*). The isotopy classes of these singular braids, with the analogous multiplication, form the monoid of singular braids on n strings on D^2, denoted by SB_n. The generators of the monoid are the usual generators σ_j of B_n and their inverses, plus new monoid generators $\tau_1, \ldots, \tau_{n-1}$, where τ_j corresponds to the singular braid with a crossing point involving the j-th string and the $(j+1)$-th one. Baez and Birman showed also a complete system of relations for this monoid. To the usual braid relations (and the invertibility of σ_j) we need to add following relations:

- $\tau_i\,\sigma_j\,\sigma_i = \sigma_j\,\sigma_i\,\tau_j \quad \text{for } |i-j| = 1\,;$
- $\tau_i\,\tau_j = \tau_j\,\tau_i \quad \text{for } |i-j| \geq 2\,;$
- $\tau_i\,\sigma_j = \sigma_j\,\tau_i \quad \text{for } |i-j| \geq 2\,;$
- $\tau_i\,\sigma_i = \sigma_i\,\tau_i \quad \text{for } i = 1, \ldots, n-1\,.$

Singular braids are related to finite type invariants for knots. Several properties of singular braids have been studied ([5], [26], [32], [39], [40] and [44]). In particular it has been shown that SB_n embeds in a group, that the word problem for SB_n is solvable and that it exists a Markov Theorem for singular braids. An interesting conjecture for singular braids concerns the embedding of the singular braid monoid SB_n in the group ring of the braid group B_n ([16], [40], [92]). On the other hand one can extend the surface braid group $B(n,F)$ to the singular braid monoid on n strings on F, $SB(n,F)$. This monoid has been introduced in [48], in order to define finite type invariants for surface braids. A system of generators for $SB(n,F)$ is given by the generators of $B(n,F)$, their inverses and the singular generators $\tau_1, \ldots, \tau_{n-1}$, which correspond to the singular braid generators of SB_n. González-Meneses ([49]) provided a presentation for $SB(n,F)$, when F is an orientable closed surface. In the last section we give a presentation for the monoid $SB(n,F)$, when F is an orientable surface, possibly with boundary. Fenn, Rolfsen and Zhu ([40]) gave a characterisation of those elements of the braid group B_n which commute with usual generators σ_j. They proved an analogous result for SB_n. In particular, it was established that the sub-monoids of those elements commuting with σ_j on one side or with singular generator τ_j in the other side are the same. We extend these results to surface braid groups $B(n,F)$ (for an orientable surface F) and corresponding singular braid monoids $SB(n,F)$:

Theorem 3.1.1 *For all $x \in SB(n,F)$, the following properties are equivalent:*

1. $\sigma_j x = x\sigma_k$,
2. $\sigma_j^r x = x\sigma_k^r$, *for some* $r \in \mathbb{Z} \setminus \{0\}$,
3. $\sigma_j^r x = x\sigma_k^r$, *for all* $r \in \mathbb{Z}$,
4. $\tau_j x = x\tau_k$,
5. $\tau_j^r x = x\tau_k^r$, *for some* $r \in \mathbb{N} \setminus \{0\}$.

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of n distinct points of F. An arc is an embedding $a : [0,1] \to F$ such that $a(0), a(1) \in \mathcal{P}$ and $a(x) \notin \mathcal{P}$ for all $x \in (0,1)$. As in [40] the main idea is to consider braids as mapping classes of the surface $F \setminus \mathcal{P}$ (Section 3.2.1) and to study the action of braids on isotopy classes of arcs (Sections 3.2.2 and 3.2.3). In particular, in Theorem 3.3.1 and 3.3.2 we identify all solutions x of $\sigma_j x = x\sigma_k$ by a natural criterion involving braids as geometrical objects having what Fenn, Rolfsen and Zhu called (j,k)*-bands*. As application of Theorem 3.1.1 and of a reduction property of singular braids (Lemma 3.4.2), we obtained simple proofs for the following statements.

Theorem A (Theorem 3.4.1) *$SB(n,F)$ embeds in a group.*

Theorem B (Theorem 3.5.2) *The word problem for $SB(n,F)$ is solvable.*

Remark 3.1.1 *Let M be a monoid, with presentation $\langle G \mid R \rangle$. The monoid M embeds in a group if and only if it embeds in the group defined by the presentation $\langle G \mid R \rangle$.*

3.2 Preliminaries

3.2.1 Mapping class groups

We recall some definitions about mapping class groups. The main references for this Section are [15], [64] and [76]. From now on, let F be a compact, connected, oriented surface and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of n distinct points in the interior of F.

Definition 3.2.1 *We denote $\mathcal{H}(F,\mathcal{P})$ the group of orientation-preserving homeomorphisms $h : F \to F$, such that $h(\mathcal{P}) = \mathcal{P}$. The punctured mapping class group of F relatively to $\mathcal{P}$ is defined to be the group of isotopy classes of elements of $\mathcal{H}(F,\mathcal{P})$. The punctured mapping class group does not depend on the choice of $\mathcal{P}$. We denote this group $\mathcal{M}(n,F)$.*

In the following we consider a simple closed curve in $F \setminus \mathcal{P}$ as an embedding $c : S^1 \to F \setminus \mathcal{P}$ which does not intersect the boundary of F. By abuse of notation, we use the symbol c to denote the image of c. The simple closed curve c is essential if it does not bound a disc in $F \setminus \mathcal{P}$. The simple closed curve c is generic, if it does not bound a disc in F containing 0 or 1 point of $\mathcal{P}$.
Two simple closed curves c, c' are isotopic if there exists a continuous family $h_t \in \mathcal{H}(F,\mathcal{P})$, $t \in [0,1]$ such that h_0 is the identity and $h_1(c) = c'$. We denote $c \simeq c'$.

Definition 3.2.2 *Let $c : S^1 \to F \backslash \mathcal{P}$ be a simple closed curve. Choose an embedding $A : [0,1] \times S^1 \to F \setminus \mathcal{P}$ such that $A(1/2, z) = c(z)$ for all $z \in S^1$, and we consider the homeomorphism $T \in \mathcal{H}(F, \mathcal{P})$ defined by*

$$(T \circ A)(t, z) = A(t, e^{2i\pi t} z), t \in [0,1], z \in S^1,$$

and T is the identity on the exterior of the image of A. The Dehn twist along c is defined to be the element $\gamma \in \mathcal{M}(n, F)$ which represents T (figure 3.1).

We recall that the definition of γ does not depend on the choice of A and that two isotopic generic simple closed curves define the same Dehn twist.

Definition 3.2.3 *An arc is an embedding $a : [0,1] \to F$ such that $a(0), a(1) \in \mathcal{P}$ and $a(x) \notin \mathcal{P}$ for all $x \in (0,1)$. A (j,k)-arc is an arc such that $a(0) = P_j$ and $a(1) = P_k$.*

By abuse of notation, we use the symbol a to denote the image of a. Note that two (j,k)-arcs are isotopic if and only if they can be connected by a continuous family of (j,k)-arcs. As above the isotopy of the arcs a and b is denoted by $a \simeq b$.

Definition 3.2.4 *Let $a : [0,1] \to F$ be an arc. Choose an embedding $A : D^2 \to F$ such that:*

- $a(t) = A(t - 1/2)$ *for all* $t \in [0,1]$,
- $A(D^2) \cap \mathcal{P} = \{a(0), a(1)\}$,

and we consider the homeomorphism $T \in \mathcal{H}(F, \mathcal{P})$ defined by

$$(T \circ A)(z) = A(t, e^{2i\pi|z|} z), z \in D^2,$$

and T is the identity on the exterior of the image of A. The braid twist along a is defined to be the element $\alpha \in \mathcal{M}(n, F)$ which represents T (figure 3.1).

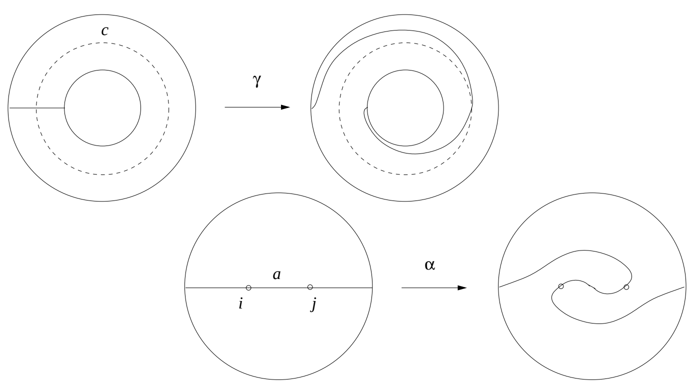


Figure 3.1: The Dehn twist γ and the braid twist α.

Note that the definition of α does not depend on the choice of A and that two isotopic arcs define the same braid twist.
The Isotopy Extension Theorem defines a map from $B(n, F)$ to $\mathcal{M}(n, F)$. It is well-known that $B(n, F)$ embeds in $\mathcal{M}(n, F)$ when $g > 1$. Thus we are allowed to consider braids as elements of $\mathcal{M}(n, F)$.

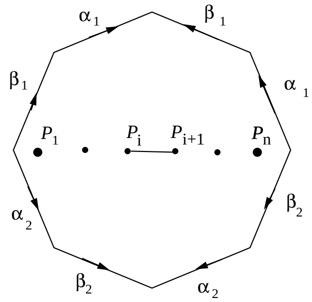


Figure 3.2: The fundamental domain of an orientable closed surface of genus 2.

We represent F and we choose the points $P_1, \ldots, P_n$ as in Chapter 1. We fix a segment $[i, i+1]$ with end points P_i and P_{i+1} for $i = 1, \ldots, n-1$ as in Figure 3.2.
Let σ_i, a_j, b_j, z_l be the braid generators defined in Chapter 1. The braid generator σ_i corresponds to the braid twist defined by the segment $[i, i+1]$. On the other hand, let ψ_j be the non trivial string of a_j (respectively b_j). We consider two generic simple closed curves c_0, c_1 on the fundamental domain of F as in Figure 3.3 and we choose an embedding $A : [0,1] \times S^1 \to F \setminus \mathcal{P}$ of the annulus such that

- $A(3/4, S^1) = c_1$,
- $A(1/4, S^1) = c_0$,
- $A(1/2, S^1) = \psi_j$.

The braid a_j (respectively b_j) corresponds to the homeomorphism $\gamma_0 \gamma_1^{-1}$, where γ_i is the Dehn twist along c_i (Figure 3.3). This homeomorphism is the identity on the exterior of A. Similarly, to the generator z_k we associate an element of the punctured mapping class group defined by two generic simple closed curves c_0', c_1' around the k-th boundary component.

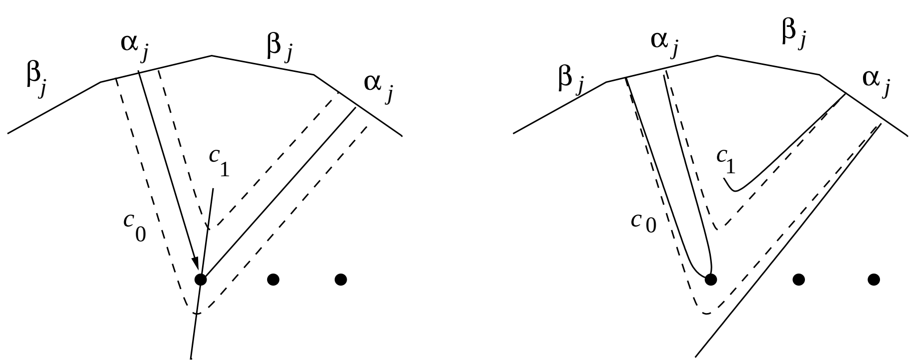


Figure 3.3: The homeomorphism associated with a generator a_j.

3.2.2 Braids and arcs

We adapt some definitions and propositions introduced in [40]. We can represent braids as a collection $\Psi = (\psi_1, \ldots, \psi_n)$ of n disjoint strings in $F \times I$ such that ψ_i runs monotically in $t \in I$ from the point in $P_i \times 0$ to some point $P_k \times 1 \in \mathcal{P} \times 1$. As above an isotopy is a deformation through braids with fixed ends and two braids are considered equivalent if they are isotopic. Similarly to the classical case, we notice that two equivalent braids are related by an horizontal isotopy ([86]). The product of braids corresponds to composing of mapping classes of F. One can have a braid β acts on F (up to isotopy in $\mathcal{H}(F, \mathcal{P})$) on the left or on the right respectively as follows, $*\beta : F \to F$ corresponds to a mapping class $F \times 0 \to F \times 1$ and $\beta* : F \to F$ corresponds to a mapping class $F \times 1 \to F \times 0$. In particular braids act on the right and on the left on the set of arcs on F up to isotopy in $\mathcal{H}(F, \mathcal{P})$.

Definition 3.2.5 *A* ribbon *is an embedding*

$$R : [0,1] \times [0,1] \to F \times [0,1]\,,$$

such that $R(s,t) \in F \times t$. Let β be a braid and let A be a $(j-k)$-arc in $F \times 0$. Then the isotopy corresponding to β moves A through a ribbon which is **proper** *for β, meaning that*

- *$R(0,t)$ and $R(1,t)$ trace out two strings of the braid, while the rest of the ribbon is disjoint from β;*
- *$R(s,0) = A$ and $R(s,1) = A * \beta$.*

Proposition 3.2.1 ([40]) *Let $\beta \in B(n,F)$ and let A and B be arcs. Then $A * \beta = B$ if and only if there is a proper ribbon for β connecting $A \subset F \times 0$ to $B \subset F \times 1$.*

Definition 3.2.6 *We say that $\beta \in B(n,F)$ has a (j,k)-band if there exists a ribbon proper for β and connecting $[j, j+1] \times 0$ to $[k, k+1] \times 1$.*

Proposition 3.2.2 ([40]) *Let $\beta \in B(n,F)$. The braid β has a (j,k)-band if and only if $[j, j+1] * \beta = [k, k+1]$. If β has a (j,k)-band then $\sigma_j \beta = \beta \sigma_k$.*

Proposition 3.2.3 *Let $\beta \in B(n,F)$. If $\sigma_j^r \beta = \beta \sigma_k^r$ for some integer r, then $\{j, j+1\} * \beta = \{k, k+1\}$.*

Proof: The case r odd is trivial, since it suffices to consider the associated permutation. Thus, let r be even. Then, $\beta \sigma_k^r \beta^{-1} \in P(n,F)$. Let $\chi_{j,j+1} : P(n,F) \to P(2,F)$ be the map that forgets all strands except the j-th one and the $(j+1)$-th one. Suppose that $\{j\} * \beta \notin \{k, k+1\}$. It follows

$$1 = \chi_{j,j+1}(\beta \sigma_k^r \beta^{-1}) = \chi_{j,j+1}(\sigma_j^r) = \sigma_1^r.$$

This is false, since P_n embeds in $P(n,F)$ ([75]) and $P_2 = \langle \sigma_1^2 \rangle = \mathbb{Z}$. □

3.2.3 Isotopy invariants

Let a and b be two simple closed curves in F. Following [38] and [76], the index of intersection of a and b is

$$I(a,\, b) = \ \inf\{|a' \cap b'|\,;\, a' \simeq a,\, b' \simeq b\}\,.$$

Adapting above definition to arcs, the index of intersection of the arcs a and b is

$$I(a,\, b) = \ \inf\{|int(a') \cap int(b')|\,;\, a' \simeq a,\, b' \simeq b\}\,.$$

We note that $I(a,\, b) = \inf\{|a' \cap b|\,;\, a' \simeq a\}$ and that if $a \simeq b$ then $I(a,\, b) = 0$.
Let a be an arc and let α the braid twist associated to a. The square of α is a Dehn twist. We set $\widehat{a}$ for a generic simple closed curve such that $T_{\widehat{a}} = \alpha^2$ where $T_{\widehat{a}}$ is the Dehn twist along $\widehat{a}$. Each generic simple closed curves $\widehat{a}$ bounds a disk $D(\widehat{a})$ in F containing an arc a' isotopic to a.

Proposition 3.2.4 ([76]) *Let a and b be two simple closed curves and n any integer Then*

$$I(T_a^n(b), b) = |n| I(a, b)^2 \,,$$

where T_a^n is the n-th power of the Dehn twist along a.

Lemma 3.2.1 *If A is a (l,k)-arc, with $\{k,l\} \cap \{j, j+1\} = \emptyset$ such that $A * \sigma_j = A$, then $I(A, [j, j+1]) = 0$, i.e. A and $[j, j+1]$ are disjoint up to isotopy.*

Proof: The hypothesis $A * \sigma_j = A$ implies that $\widehat{A} * \sigma_j = \widehat{A}$ and therefore $T_{\widehat{[j,j+1]}}(\widehat{A}) = \widehat{A}$. It follows that $I(T_{\widehat{[j,j+1]}}(\widehat{A}), \widehat{A}) = 0$. From the formula in Proposition 3.2.4 we deduce $I(\widehat{A}, \widehat{[j, j+1]}) = 0$, which implies $I(A, [j, j+1]) = 0$. □

Lemma 3.2.2 *If A is a $(j, j+1)$-arc, such that $A * \sigma_j^r = A$ for some r, then $A \simeq [j, j+1]$.*

Proof: Using the same argument as in Lemma 3.2.1 we can suppose that $I(\widehat{A},\, \widehat{[j, j+1]}) = 0$. Let $\widehat{A}' \simeq \widehat{A}$ such that $|\widehat{A}' \cap \widehat{[j, j+1]}| = \emptyset$. The disks $D(\widehat{A}')$, $D(\widehat{[j, j+1]})$ cannot be disjoint (they contains P_j). Thus, either $D(\widehat{A}') \subseteq D(\widehat{[j, j+1]})$ or $D(\widehat{[j, j+1]}) \subseteq D(\widehat{A}')$. We deduce that $\widehat{A}' \simeq \widehat{[j, j+1]}$ and therefore $[j, j+1]$ and A are isotopic. □

The following corollary is the analogous for braid twists of the well-known result on Dehn twists (see e.g. Theorem 7.5 in [55]).

Corollary 3.2.1 *Let a be an (i, j)-arc in F and let T_a be the braid twist along a. Let b be a (k, l)-arc, where $\{i, j\} \cap \{k, l\} = \emptyset$. The braid twist T_a and T_b commute if and only if $I(a, b) = 0$.*

Proof: It is evident that if a and b are disjoint (up to isotopy) T_a and T_b commute. Recall that as in the case of Dehn twist $T_a T_b T_a^{-1} = T_{T_a(b)}$. Commutation hypothesis implies that $T_{T_a(b)} = T_b$. Now, consider braids associated to $T_{T_a(b)}$ and T_b. Thus, with the previous notation $b * T_{T_a(b)} = b * T_b = b$ and Lemma 3.2.2 implies $T_a(b)$ isotopic to b, i.e. $b * T_a = b$. From Lemma 3.2.1 we deduce that a and b are disjoint (up to isotopy). □

3.3 Statements of Main Theorems

3.3.1 Centralisers of $B(n, F)$

Let us state the first Theorem on centralisers of $B(n, F)$.

Theorem 3.3.1 *For each $\beta \in B(n, F)$, the following properties are equivalent:*

1. *$\sigma_j \beta = \beta \sigma_k$,*

2. *$\sigma_j^r \beta = \beta \sigma_k^r$, for any integer r,*

3. *$\sigma_j^r \beta = \beta \sigma_k^r$, for some nonzero integer r,*

4. *β has a $(j,\ k)$-band,*

5. *$[j, j+1] * \beta = [k, k+1]$.*

Proof: From Proposition 3.2.2 it follows that (5) $\Rightarrow$ (4) $\Rightarrow$ (1), and it is obvious that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). It remains to show that (3) $\Rightarrow$ (5). Suppose that for some r, $\sigma_j^r \beta = \beta \sigma_k^r$. By Proposition 3.2.3 this equation is possible only if $\{j, j+1\} * \beta = \{k, k+1\}$. From the remark that $\beta^{-1} \sigma_j^r \beta = \sigma_k^r$ and that σ_k^r has a proper (k, k)-band, we conclude that there is a proper ribbon R for $\beta^{-1} \sigma_j^r \beta$ from $[k, k+1] \times 0$ to $[k, k+1] \times 1$. Define $A = \beta * [k, k+1] = [k, k+1] * \beta^{-1}$. We may assume (up to isotopy) that $R(\cdot, 1/3) = A \times 1/3$ and $R(\cdot, 2/3) = A \times (2/3)$. Then there is a ribbon for σ_j^r connecting A to A. From Proposition 3.2.1 we deduce that $A * \sigma_j^r = A$. By Lemma 3.2.2, $A = [j, j+1]$ and the claim is proved. $\square$

Remark 3.3.1 *Definitions 3.2.5 and 3.2.6 can be extended naturally to mapping class groups. The Theorem 3.3.1 holds true for any $\beta \in \mathcal{M}(n, F)$.*

3.3.2 Singular ribbons

Theorem 3.3.1 can be extended to $SB(n, F)$. Two singular braids x_1 and x_2 are equivalent if there exists an isotopy H_t of $F \times [0,1]$ such that $H_0 = id_{F \times [0,1]}$ and $H_1(x_1) = x_2$. Let B be a ball of radius ϵ centered at the singularity p. Denote $s_t = H_t(x_1)$, $p_t = H_t(p)$ and $B_t = H_t(B)$. We can suppose without loss of generality, that B_t is the ball of radius ϵ centered at p_t and that $B_t \cap s_t$ is as in Figure 3.4.

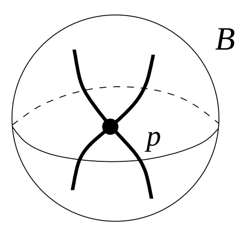


Figure 3.4: The neighbourhood of the singularity p.

We stress that equivalent singular braids do not need to be related by a level preserving isotopy H in $F \times [0,1]$.

Definition 3.3.1 *A* singular ribbon *is a map $R : I \times I \to F \times I$ such that R embeds $I \times t$ into $F \times t$, except for finitely many points t, for which the image is a single point in $F \times t$. One also assumes, at these singular points, that there is a tangent plane in $F \times t$ for the singular ribbon.*

As for braids, we say that a singular ribbon is *proper* for a singular braid if it sends $\{0, 1\} \times I$ along two of its strings and the image is disjoint from the other strings of the singular braid. An isotopy of a singular braid can be extended to an isotopy of any of its proper singular ribbons. In contrast to the ordinary situation, it is not always possible to find a singular ribbon proper for a given singular braid and a given arc $A = R(I, 0)$. Anyway, we have the following.

Proposition 3.3.1 ([40]) *If a singular ribbon R is proper for the singular braid x and $R(I, 0)$ and $R(I, 1)$ are isotopic as arcs to $[j, j+1] \times 0$ and $[k, k+1] \times 1$ respectively, then $\sigma_j x = x\sigma_k$ in $SB(n, F)$.*

Definition 3.3.2 *A singular braid has a (j, k)-band if it has a proper (singular) ribbon connecting $[j, j+1] \times 0$ to $[k, k+1] \times 1$.*

Remark 3.3.2 *For a singular braid x, having a (j, k)-band is a sufficient condition for satisfying $\sigma_j x = x\sigma_k$.*

Lemma 3.3.1 *Let $\beta \in B(n, F)$ and $y \in SB(n, F)$, such that both $\beta\sigma_i y$ and βy have (singular) (j, k)-bands. Then $\beta\tau_i y$ also has a (singular) (j, k)-band.*

Proof: The proof is an immediate extension of Lemma 6.4 in [40]. Let $A = [j, j+1] * \beta$. Since βy has a (j, k)-band, we have a proper ribbon R such that $R(I, 0)$ and $R(I, 1)$ are isotopic as arcs to $[j, j+1] \times 0$ and $[k, k+1] \times 1$. After an isotopy we may suppose R as the composition of two ribbons R_1 and R_2 for β and y, such that $R_1(I, 1) = R_2(I, 0) = A$. The hypothesis that $\beta\sigma_i y$ has a (j, k)-band implies that $A * \sigma_i = A$. Considerations on associated permutation show that either $\{j, j+1\} * \beta = \{i, i+1\}$ (case 1) or $\{j, j+1\} * \beta \cap \{i, i+1\} = \emptyset$ (case 2). In the first case, A is an $(i, i+1)$-arc. Since $A * \sigma_i = A$, Lemma 3.2.2 implies that $A = [i, i+1]$ and then β has a (j, i)-band. On the other hand R_2 is a proper band connecting $[i, i+1] \times 0$ to $[k, k+1] \times 1$ and then y has a (i, k)-band. Combining these bands with the obvious singular (i, i)-band for τ_i provides a (j, k)-band for $\beta\tau_i y$.

If $\{j, j+1\} * \beta$ and $\{i, i+1\}$ are disjoint sets, $A = R_1(I, 1) = R_2(I, 0)$ is disjoint from $[i, i+1]$ (Lemma 3.2.1). Thus we may insert τ_i between β and y so that the singular strands are disjoint from the band, and we conclude that $\beta\tau_i y$ has a (j, k)-band.

□

Similarly to singular braids, singular surface braids have the *cancellation property.*

Proposition 3.3.2 ([49]) *Left and right cancellation hold in $SB(n, F)$, that is for all $x, y, z \in SB(n, F)$ the equation $xy = xz$ (respectively $yx = zx$) implies $y = z$.*

3.3.3 Centralisers on $SB(n, F)$

The following Theorem is an extension of Theorem 3.3.1 to $SB(n, F)$.

Theorem 3.3.2 *For each $x \in SB(n, F)$, the following properties are equivalent:*

1. $\sigma_j x = x\sigma_k$,
2. $\sigma_j^r x = x\sigma_k^r$, *for some nonzero integer r,*
3. $\sigma_j^r x = x\sigma_k^r$, *for any r,*
4. $\tau_j x = x\tau_k$,

5. *$\tau_j^r x = x\tau_k^r$, for some positive integer r,*

6. *x has a (possibly singular) (j, k)-band.*

Proof: It is clear that (6) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2) and (6) $\Rightarrow$ (4) $\Rightarrow$ (5).

- (4) $\Rightarrow$ (6). Since the order of singularities on a string is an isotopy invariant, we deduce that $\{j, j+1\} * x = \{k, k+1\}$ and that the j-th and $(j+1)$-th strings are disjoint from the other strings. Let $p_0, \ldots, p_m$ be the ordered set of singularities of $\tau_j x$ on the $j-$th string. Let $\tau_j x = \tau_{l_0}\beta_1\tau_{l_1}\cdots\beta_m\tau_{l_m}\beta_{m+1}$, where τ_{l_q} is the singular generator corresponding to p_q and $\beta_{q+1} \in SB(n, F)$. On the other hand we write $x\tau_k = \beta_1\tau_{l_1}\beta_2\cdots\beta_m\tau_{l_m}\beta_{m+1}\tau_{l_{m+1}}$. The isotopy "increases" of one the index of all singular generators. The trivial singular braid near τ_{l_q} provides a (l_q, l_{q+1})-band for β_{q+1}. We combine these bands with the obvious singular (l_q, l_q)-bands for τ_{l_q} in order to obtain a singular (j, k)-band for x. The case (5) $\Rightarrow$ (6) is analogous.

- (2) $\Rightarrow$ (6). We outline the proof that is the same as in [40]. We proceed by induction on the number of singular generators in x. Assume $x = \beta\tau_i y$, where β is a surface braid. The hypothesis implies that $\sigma_j^{2r} x = x\sigma_k^{2r}$ and then $\beta^{-1}\sigma_j^{2r}\beta\tau_i y = \tau_i y\sigma_k^{2r}$. Since $\beta^{-1}\sigma_j^{2r}\beta$ is a pure surface braid, the τ_i in $\tau_i y\sigma_k^{2r}$ corresponds under some homomorphism, to the τ_i in $\beta^{-1}\sigma_j^{2r}\beta\tau_i y$. Hence the image, under that homeomorphism, of the trivial singular band near the first τ_i provides a band for $\beta^{-1}\sigma_j^{2r}\beta$. Therefore, τ_i commutes with $\beta^{-1}\sigma_j^{2r}\beta$. It follows that $\tau_i\beta^{-1}\sigma_j^{2r}\beta y = \tau_i y\sigma_k^{2r}$. By Proposition 3.3.2, we have $\beta^{-1}\sigma_j^{2r}\beta y = y\sigma_k^{2r}$, i.e. $\sigma_j^{2r}\beta y = \beta y\sigma_k^{2r}$. We deduce that βy has a (j, k)-band and from the existence of a (i, i)-band for $\beta^{-1}\sigma_j^{2r}\beta$ we deduce $\beta^{-1}\sigma_j^{2r}\beta\sigma_i y = \sigma_i y\sigma_k^{2r}$. It follows that also $\beta\sigma_i y$ has a (j, k)-band. Lemma 3.3.1 concludes the proof.

□

3.4 The monoid $SB(n, F)$ embeds in a group

3.4.1 Extended singular braids

From now on given a set E we denote by $(E)^*$ the free monoid generated by E. Let $G_{B(n,F)}$ be the set of generators of $B(n, F)$ and $G_{B(n,F)}^{-1}$ the set of their inverses. Let T be the set of singular generators τ_j. We can associate to $SB(n, F)$ the group defined as follows:

- *Generators:* $G_{B(n,F)}, G_{B(n,F)}^{-1}, T$ and the additional set T^{-1} of singular generators $\overline{\tau_j}$, for $j = 1, \ldots, n-1$:

- *Isotopy relations (IR):* the relations for $SB(n, F)$ (see Section 3.6) and the additional relations obtained by substituting $\overline{\tau_j}$ for τ_j .

- *Birth-death relations (BDR):* $\tau_j\overline{\tau_j} = \overline{\tau_j}\tau_j = 1$.

We call this group the singular braid group on F, $SG(n, F)$. Let ϕ be the natural homomorphism from $SB(n, F)$ to $SG(n, F)$. The aim of this Section is to show that ϕ is injective.

There is a geometric interpretation for $SG(n, F)$ analogous to the one proposed by Fenn, Keyman and Rourke for the group SG_n associated to SB_n ([39]). Now, we have two types of singular points, that we label respectively with a black point, corresponding to τ_j and with an open blob, corresponding to $\overline{\tau_j}$. We call these singular braids having two type of singular crossing, *extended singular braids.*

Given two words α, β in $(G^{\pm 1}_{B(n,F)} \cup T^{\pm 1})^*$, we write $\alpha \underset{I}{\sim} \beta$ if they are equivalent under relations of type IR. In other words, there exists an isotopy from α to β. We can suppose that the isotopy preserves balls of radius ϵ centered at the labeled singularities. Also we can give a geometric interpretation of birth-death relations, that correspond to two extended singular braids that coincide except an open set V where we allow the birth or the death of a pair of singular crossings of different types. We can assume that V is disjoint from the other strings.

Lemma 3.4.1 *Two words in $(G^{\pm 1}_{B(n,F)} \cup T)^*$ which represent the same element in $SG(n,F)$ have the same number of singular generators.*

Proof: From the presentation of $SG(n,F)$ it follows that there is a group homomorphism from $SG(n,F)$ to the integers which maps $G^{\pm 1}_{B(n,F)}$ in 0, τ_j in 1 and $\overline{\tau_j}$ in -1. □

Lemma 3.4.2 *Consider two words $A = \alpha_1 \tau_{j_1} \alpha_2 \ldots \alpha_m \tau_{j_m} \alpha_{m+1}$ and $B = \alpha'_1 \tau_{i_1} \alpha'_2 \ldots \alpha'_m \tau_{i_m} \alpha'_{m+1}$, where $\alpha_i, \alpha'_i \in (G^{\pm 1}_{B(n,F)})^*$ and $\tau_{i_l}, \tau_{j_l} \in (T)^*$. If $A = B$ in $SG(n,F)$, then there exists $k \in \{1, \ldots, m\}$ such that*

$$\alpha_1 \sigma_{j_1} \alpha_2 \ldots \alpha_m \tau_{j_m} \alpha_{m+1} = \alpha'_1 \tau_{i_1} \alpha'_2 \ldots \alpha'_k \sigma_{i_k} \alpha'_{k+1} \ldots \alpha'_m \tau_{i_m} \alpha'_{m+1}\,,$$

$$\alpha_1 \sigma^{-1}_{j_1} \alpha_2 \ldots \alpha_m \tau_{j_m} \alpha_{m+1} = \alpha'_1 \tau_{i_1} \alpha'_2 \ldots \alpha'_k \sigma^{-1}_{i_k} \alpha'_{k+1} \ldots \alpha'_m \tau_{i_m} \alpha'_{m+1}$$

hold in $SG(n,F)$, where τ_{j_1} in A is replaced by σ_{j_1} or $\sigma^{-1}_{j_1}$ and τ_{i_k} in B is replaced by σ_{i_k} or $\sigma^{-1}_{i_k}$ respectively.

Proof: If the words A and B represent the same element of $SG(n,F)$, then there is a finite sequence of isotopies and birth-deaths relations relaying A to B. Let p_0 be the singularity point corresponding to τ_{j_1} in A. We encounter two cases.

1. Suppose that p_0 during the sequence does not match a "death" and thus τ_{j_1} is sent in some τ_{i_k} in B. The isotopies and birth-death relations can move the singularity p_0 but we can suppose that they do not modify the interior of a ball $B(p_0)$ of radius ϵ centered at p_0. Let us consider the singular braid A_1 obtained by modifying A only inside $B(p_0)$, where we substitute the singularity p_0 with the positive crossing σ_{j_1}. It follows that A_1 is equivalent in $SG(n,F)$ to the singular braid B_1, which corresponds to the singular braid B except that τ_{i_k} is replaced by σ_{i_k}.

2. Suppose that p_0 matches a "death" at the step d_1 of the sequence. It follows that at a step $b_1 < d_1$, there is a birth $\tau_{s(b_1)}\overline{\tau_{s(b_1)}}$ (or $\overline{\tau_{s(b_1)}}\tau_{s(b_1)}$) on the j-th and the $j+1$-th string. Restart from the step b_1 and set p_1 and p'_1 the opposite singularities corresponding to $\tau_{s(b_1)}$ and $\overline{\tau_{s(b_1)}}$. We iterate the process following p_1 and so on. Since the sequence is finite there exists a birth b_q such that the corresponding singularity p_q is sent by the sequence in a singular point of B, corresponding to some singular generator τ_{i_k}. As above consider the singular braid A_1 obtained by replacing τ_{j_1} with the positive crossing σ_{j_1} and the singular braid B_1 obtained by replacing τ_{i_k} with σ_{i_k}. Define $B(p_i)$ and $B(p'_i)$

the balls of radius ϵ centered at p_i and p_i'. During the sequence of isotopies and "birth-deaths" from A to B we can suppose that except in steps of type d_l or b_l the balls $B(p_i)$ and $B(p_i')$ remain balls of radius ϵ centered at the corresponding singularities. Replace A with A_1. At each step b_i we modify singular braids only inside a small open set $V \supset B(p_i) \cup B(p_i')$ replacing p_i with a positive crossing $\sigma_{s(b_i)}$ and p_i' with a negative crossing $\sigma_{s(b_i)}^{-1}$. This corresponds to substitute the birth b_i with an horizontal isotopy. It follows that, at each step d_i (recall that $b_i < d_i$) the subword $\tau_{s(d_i)}\overline{\tau_{s(d_i)}}$ (or $\overline{\tau_{s(d_i)}}\tau_{s(d_i)}$) is replaced by $\sigma_{s(d_i)}\sigma_{s(d_i)}^{-1}$ (or $\sigma_{s(d_i)}^{-1}\sigma_{s(d_i)}$). Thus we substitute $\sigma_{s(d_i)}\sigma_{s(d_i)}^{-1}$ (or $\sigma_{s(d_i)}^{-1}\sigma_{s(d_i)}$) with the empty word, which corresponds to an horizontal isotopy. This procedure gives a sequence of isotopy and birth-death relations from A_1 to B_1.

□

3.4.2 Singular braids embed in extended singular braids

Theorem 3.4.1 *The monoid $SB(n,F)$ embeds in a group.*

Proof: Consider two words A, B in $(G^{\pm 1}_{B(n,F)} \cup T)^*$. Let $A = B$ in $SG(n,F)$. We want to show that $A = B$ in $SB(n,F)$. We proceed by induction on the number m of singular generators in A and B. We set $SB(n,F) = \coprod_m SB(n,F)^m$, where $SB(n,F)^m$ means the set of singular braids with m singularities. If $m = 0$ the statement is true, because $B(n,F) = SB(n,F)^0$ embeds in $SG(n,F)$. In fact, there is a *retraction* morphism $r : SG(n,F) \to B(n,F)$, which sends each braid generator to itself and singular generators τ_i to identity. The composition $\phi_{|B(n,F)} \circ r$ is the identity on $B(n,F)$. Now, suppose that the statement is true for $m-1$ singular generators, and set $A = \alpha_1\tau_{j_1}\alpha_2 \dots \alpha_m\tau_{j_m}\alpha_{m+1}$ and $B = \alpha_1'\tau_{i_1}\alpha_2' \dots \alpha_m'\tau_{i_m}\alpha_{m+1}'$, where α_i, α_i' are words in $(G^{\pm 1}_{B(n,F)})^*$. We can suppose α_1 to be the empty word. Lemma 3.4.2 implies that the equalities

$$\sigma_{j_1}\alpha_2 \dots \alpha_m\tau_{j_m}\alpha_{m+1} = \alpha_1'\tau_{i_1}\alpha_2' \dots \alpha_k'\sigma_{i_k}\alpha_{k+1}' \dots \alpha_m'\tau_{i_m}\alpha_{m+1}'$$

$$\sigma_{j_1}^{-1}\alpha_2 \dots \alpha_m\tau_{j_m}\alpha_{m+1} = \alpha_1'\tau_{i_1}\alpha_2' \dots \alpha_k'\sigma_{i_k}^{-1}\alpha_{k+1}' \dots \alpha_m'\tau_{i_m}\alpha_{m+1}' \,,$$

hold in $SG(n,F)$ for some $k \in \{1, \dots, m\}$. Thus we derive

$$\sigma_{j_1}^2\alpha_1'\tau_{i_1}\alpha_2' \dots \alpha_k' = \alpha_1'\tau_{i_1}\alpha_2' \dots \alpha_k'\sigma_{i_k}^2 \text{ in } SG(n,F)$$

and by induction on the number of singular generators this relation holds true also in $SB(n,F)$. From Theorem 3.3.2 it follows:

$$\tau_{j_1}\alpha_1'\tau_{i_1}\alpha_2' \dots \alpha_k' = \alpha_1'\tau_{i_1}\alpha_2' \dots \alpha_k'\tau_{i_k} \text{ in } SB(n,F). \tag{3.1}$$

and thus

$$\alpha_2\tau_{j_2}\alpha_3 \dots \alpha_m\tau_{j_m}\alpha_{m+1} = \alpha_1'\tau_{i_1}\alpha_2' \dots \alpha_k'\alpha_{k+1}' \dots \alpha_m'\tau_{i_m}\alpha_{m+1}' \text{ in } SG(n,F)$$

which is also true in $SB(n,F)$, by induction. We deduce that

$$\tau_{j_1}\alpha_2\tau_{j_2} \dots \alpha_m\tau_{j_m}\alpha_{m+1} = \tau_{j_1}\alpha_1'\tau_{i_1}\alpha_2' \dots \alpha_k'\alpha_{k+1}' \dots \alpha_m'\tau_{i_m}\alpha_{m+1}'$$

holds in $SB(n,F)$, and (3.1) implies

$$\tau_{j_1}\alpha_2\tau_{j_2}\dots\alpha_m\tau_{j_m}\alpha_{m+1} = \alpha_1^{'}\tau_{i_1}\alpha_2^{'}\dots\alpha_k^{'}\tau_{i_k}\alpha_{k+1}^{'}\dots\alpha_m^{'}\tau_{i_m}\alpha_{m+1}^{'}$$

in $SB(n,F)$. □

Remark 3.4.1 *Recently González-Meneses showed us another proof of Theorem 3.4.1, when F is a closed orientable surface ([50]).*

3.5 The word problem is solvable

Theorem 3.5.1 *The word problem for $B(n,F)$ is solvable.*

Proof: It suffices to prove the word problem for the pure braid group $P(n,F)$. Let F be a surface of genus g with $p>0$ boundary components and let $\{A_{i,j} \mid 1 \leq i \leq 2g+p+n-2, 2g+p \leq j \leq 2g+p+n-1, i<j\}$ be the set of generators of $P(n,F)$ defined in Theorem 1.6.1. The algorithm is similar to the classical braid combing. When F has boundary, the (PBS) exact sequence splits (Section 1.2.1) and then

$$P(n,F) \cong \pi_1(F\setminus\{x_1,\dots,x_{n-1}\},x_n) \rtimes \pi_1(F\setminus\{x_1,\dots,x_{n-2}\},x_{n-1}) \rtimes \cdots \rtimes \pi_1(F,x_1)\,,$$

where the fundamental group $\pi_1(F\setminus\{x_1,\dots,x_{j-1}\},x_j)$ is freely generated by the set $\{A_{i,j}\,|\,i<j\}$. We use relations in Theorem 1.6.1 to move all letters in $\{A_{i,n}^{\pm1}\,|\,i<n\}$ on the right hand side to obtain a word $X_{n-1}\beta_n$ equivalent to β. Let β_n' the reduced word obtained removing all the subwords of the form xx^{-1} or $x^{-1}x$, in β_n. The algorithm will end in $n-1$ steps and we obtain a word $\beta''=\beta_1'\cdots\beta_n'$ equivalent to β, where β_j' $(j=1,\dots,n)$ is a reduced word on $\{A_{i,j}\,|\,i<j\}$. Since $\{A_{i,j}\,|\,i<j\}$ is a free system of generators for the free group $\pi_1(F\setminus\{x_1,\dots,x_{j-1}\},x_j)$, the word β'' is unique. The case of closed surfaces is similar ([46]). □

Lemma 3.5.1 *Consider two words A,B in $(G^{\pm1}_{B(n,F)}\cup T)^*$. Let $A=\alpha_1\tau_{j_1}\alpha_2\dots\alpha_m\tau_{j_m}\alpha_{m+1}$ and $B=\alpha_1^{'}\tau_{i_1}\alpha_2^{'}\dots\alpha_m^{'}\tau_{i_m}\alpha_{m+1}^{'}$, where $\alpha_i,\alpha_i'\in(G^{\pm1}_{B(n,F)})^*$ and $\tau_{i_l},\tau_{j_l}\in(T)^*$, for $1\leq i,l\leq m+1$. A and B represent the same element in $SB(n,F)$ if and only if there exists $k\in\{1,\dots,m\}$ such that*

$$\alpha_1\sigma_{j_1}\alpha_2\dots\alpha_m\tau_{j_m}\alpha_{m+1} = \alpha_1^{'}\tau_{i_1}\alpha_2^{'}\dots\alpha_k^{'}\sigma_{i_k}\alpha_{k+1}^{'}\dots\alpha_m^{'}\tau_{i_m}\alpha_{m+1}^{'}\,,$$

$$\alpha_1\sigma_{j_1}^{-1}\alpha_2\dots\alpha_m\tau_{j_m}\alpha_{m+1} = \alpha_1^{'}\tau_{i_1}\alpha_2^{'}\dots\alpha_k^{'}\sigma_{i_k}^{-1}\alpha_{k+1}^{'}\dots\alpha_m^{'}\tau_{i_m}\alpha_{m+1}^{'}\,,$$

hold in $SB(n,F)$, where τ_{j_1} in A is replaced by σ_{j_1} or $\sigma_{j_1}^{-1}$ and τ_{i_k} in B is replaced by σ_{i_k} or $\sigma_{i_k}^{-1}$ respectively.

Proof: The "if" part follows from Lemma 3.4.2. Conversely, we suppose α_1 to be the empty word and we proceed as in Theorem 3.4.1. The equalities

$$\sigma_{j_1}\alpha_2\dots\alpha_m\tau_{j_m}\alpha_{m+1} = \alpha_1^{'}\tau_{i_1}\alpha_2^{'}\dots\alpha_k^{'}\sigma_{i_k}\alpha_{k+1}^{'}\dots\alpha_m^{'}\tau_{i_m}\alpha_{m+1}^{'}$$

$$\sigma_{j_1}^{-1}\alpha_2\dots\alpha_m\tau_{j_m}\alpha_{m+1} = \alpha_1^{'}\tau_{i_1}\alpha_2^{'}\dots\alpha_k^{'}\sigma_{i_k}^{-1}\alpha_{k+1}^{'}\dots\alpha_m^{'}\tau_{i_m}\alpha_{m+1}^{'}\,,$$

in $SB(n,F)$ imply that

$$\sigma_{j_1}^2\alpha_1'\tau_{i_1}\alpha_2'\dots\alpha_k' = \alpha_1'\tau_{i_1}\alpha_2'\dots\alpha_k'\sigma_{i_k}^2$$

holds in $SG(n,F)$ and, by Theorem 3.4.1, also in $SB(n,F)$. From Theorem 3.3.2 we derive that $\alpha_1'\tau_{i_1}\alpha_2'\dots\alpha_k'$ has a (j_1,i_k)-band, and thus

$$\alpha_2\dots\alpha_m\tau_{j_m}\alpha_{m+1} = \alpha_1'\tau_{i_1}\alpha_2'\dots\alpha_k'\alpha_{k+1}'\dots\alpha_m'\tau_{i_m}\alpha_{m+1}' \,.$$

Then we conclude as in Theorem 3.4.1. □

Theorem 3.5.2 *The word problem for $SB(n,F)$ is solvable.*

Proof: Let A and B be two words in $(G_{B(n,F)}^{\pm 1}\cup T)^*$. We proceed by induction on the number m of singularities in A. For $m=0$ we have $SB(n,F)^0 = B(n,F)$ and for $B(n,F)$ the word problem is solvable (Theorem 3.5.1). Let $A=\alpha_1\tau_{j_1}\alpha_2\dots\alpha_m\tau_{j_m}\alpha_{m+1}$ and $B = \alpha_1'\tau_{i_1}\alpha_2'\dots\alpha_m'\tau_{i_m}\alpha_{m+1}'$. Let $\chi_j^{(1)}=\sigma_j$ and $\chi_j^{(2)}=\sigma_j^{-1}$. Set the words

$$A_1^{(s)} = \alpha_1\chi_{j_1}^{(s)}\alpha_2\dots\alpha_m\,\tau_{j_m}\alpha_{m+1}\,,\ B_r^{(s)} = \alpha_1'\tau_{i_1}\alpha_2'\dots\alpha_r'\chi_{i_r}^{(s)}\alpha_{r+1}'\dots\alpha_m'\tau_{i_m}\alpha_{m+1}'\,,$$

for $s=1,2$ and $r=1,\dots,m$. Then all the $A_1^{(s)}$ and $B_r^{(s)}$ have $m-1$ singular generators. By the induction hypothesis there exists an algorithm deciding whether $A_1^{(s)}=B_r^{(s)}$ in $SB(n,F)$ or not. From Lemma 3.5.1, $A=B$ in $SB(n,F)$ if and only if there exists $r=1,\dots,m$ such that $A_1^{(1)}=B_r^{(1)}$ and $A_1^{(2)}=B_r^{(2)}$. □

3.6 Monoid presentations

In following Theorems we provide presentations for singular braid monoids on orientable surfaces. Relations can easily verified on corresponding braids. To prove that it is a complete system of relations one can repeat arguments in [49].

Theorem 3.6.1 *Let F be an orientable surface with $p>0$ boundary components. The monoid $SB(n,F)$ admits the following presentation:*

- *Generators:* $\sigma_1^{\pm1},\dots,\sigma_{n-1}^{\pm1},a_1^{\pm1},\dots,a_g^{\pm1},b_1^{\pm1},\dots,b_g^{\pm1},z_1^{\pm1},\dots,z_{p-1}^{\pm1},\tau_1,\dots,\tau_{n-1}$.
- *Relations:*
  - *Group relations:*

$$\begin{aligned}\sigma_i^{-1}\sigma_i = \sigma_i\sigma_i^{-1} = 1 \quad (1\le i\le n-1)\,;\\ a_r^{-1}a_r = a_ra_r^{-1} = b_r^{-1}b_r = b_rb_r^{-1} = 1 \quad (1\le r\le g)\,;\\ z_j^{-1}z_j = z_jz_j^{-1} = 1 \quad (1\le j\le p-1)\,.\end{aligned}$$

  - *Braid relations, i.e:*

$$\begin{aligned}(R1)\quad & \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\,;\\ (R2)\quad & \sigma_i\sigma_j = \sigma_j\sigma_i \quad \textit{for } |i-j|\ge 2\,.\end{aligned}$$

– *Mixed relations:*

$$\begin{array}{ll}
(R3) & a_r\sigma_i = \sigma_i a_r \quad (1 \leq r \leq g;\ i \neq 1)\,; \\
 & b_r\sigma_i = \sigma_i b_r \quad (1 \leq r \leq g;\ i \neq 1)\,; \\
(R4) & \sigma_1^{-1}a_r\sigma_1^{-1}a_r = a_r\sigma_1^{-1}a_r\sigma_1^{-1} \quad (1 \leq r \leq g)\,; \\
 & \sigma_1^{-1}b_r\sigma_1^{-1}b_r = b_r\sigma_1^{-1}b_r\sigma_1^{-1} \quad (1 \leq r \leq g)\,; \\
(R5) & \sigma_1^{-1}a_s\sigma_1 a_r = a_r\sigma_1^{-1}a_s\sigma_1 \quad (1 \leq s < r \leq g)\,; \\
 & \sigma_1^{-1}b_s\sigma_1 b_r = b_r\sigma_1^{-1}b_s\sigma_1 \quad (1 \leq s < r \leq g)\,; \\
 & \sigma_1^{-1}a_s\sigma_1 b_r = b_r\sigma_1^{-1}a_s\sigma_1 \quad (1 \leq s < r \leq g)\,; \\
 & \sigma_1^{-1}b_s\sigma_1 a_r = a_r\sigma_1^{-1}b_s\sigma_1 \quad (1 \leq s < r \leq g)\,; \\
(R6) & \sigma_1^{-1}a_r\sigma_1^{-1}b_r = b_r\sigma_1^{-1}a_r\sigma_1 \quad (1 \leq r \leq g)\,; \\
(R7) & z_j\sigma_i = \sigma_i z_j \quad (i \neq 1, 1 \leq j \leq p-1)\,; \\
(R8) & \sigma_1^{-1}z_i\sigma_1 a_r = a_r\sigma_1^{-1}z_i\sigma_1 \quad (1 \leq r \leq g;\ 1 \leq i \leq p-1;\ n > 1)\,; \\
 & \sigma_1^{-1}z_i\sigma_1 b_r = b_r\sigma_1^{-1}z_i\sigma_1 \quad (1 \leq r \leq g;\ 1 \leq i \leq p-1;\ n > 1)\,; \\
(R9) & \sigma_1^{-1}z_j\sigma_1 z_l = z_l\sigma_1^{-1}z_j\sigma_1 \quad (1 \leq j < l \leq p-1)\,; \\
(R10) & \sigma_1^{-1}z_j\sigma_1^{-1}z_j = z_j\sigma_1^{-1}z_j\sigma_1^{-1} \quad (1 \leq j \leq p-1)\,.
\end{array}$$

– *Singular relations:*

$$\begin{array}{ll}
(R11) & \tau_i\,\sigma_j\,\sigma_i = \sigma_j\,\sigma_i\,\tau_j \quad \textit{for } |i-j| = 1\,; \\
(R12) & \tau_i\,\tau_j = \tau_j\,\tau_i \quad \textit{for } |i-j| \geq 2\,; \\
(R13) & \tau_i\,\sigma_j = \sigma_j\,\tau_i \quad \textit{for } |i-j| \geq 2\,; \\
(R14) & \tau_i\,\sigma_i = \sigma_i\,\tau_i \quad (1 \leq i \leq n-1)\,; \\
(R15) & (a_{i,r}a_{i+1,r})\tau_i = \tau_i(a_{i,r}a_{i+1,r}) \quad (1 \leq i \leq n-1; 1 \leq r \leq g)\,; \\
 & (b_{i,r}b_{i+1,r})\tau_i = \tau_i(b_{i,r}b_{i+1,r}) \quad (1 \leq i \leq n-1; 1 \leq r \leq g)\,; \\
(R16) & \tau_i a_{j,r} = a_{j,r}\tau_i \quad (j \neq i, i+1; 1 \leq r \leq g)\,; \\
 & \tau_i b_{j,r} = b_{j,r}\tau_i \quad (j \neq i, i+1; 1 \leq r \leq g)\,; \\
(R17) & (z_{i,r}z_{i+1,r})\tau_i = \tau_i(z_{i,r}z_{i+1,r}) \quad (1 \leq i \leq n-1; 1 \leq r \leq p-1)\,; \\
(R18) & \tau_i z_{j,r} = z_{j,r}\tau_i \quad (j \neq i, i+1; 1 \leq r \leq p-1)\,,
\end{array}$$

where $a_{i,r} = \sigma_{i-1}^{-1}\cdots\sigma_1^{-1}a_r\sigma_1^{-1}\cdots\sigma_{i-1}^{-1}$, $b_{i,r} = \sigma_{i-1}^{-1}\cdots\sigma_1^{-1}b_r\sigma_1^{-1}\cdots\sigma_{i-1}^{-1}$ *and* $z_{i,r} = \sigma_{i-1}^{-1}\cdots\sigma_1^{-1}$ $z_r\sigma_1\cdots\sigma_{i-1}$.

Theorem 3.6.2 *Let F be a closed orientable surface of genus $g \geq 1$. The monoid $SB(n,F)$ admits the following presentation:*

- *Generators:* $\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}, a_1^{\pm 1}, \ldots, a_g^{\pm 1}, b_1^{\pm 1}, \ldots, b_g^{\pm 1}, \tau_1, \ldots, \tau_{n-1}$.
- *Relations:*
  – *Group relations:*

$$\sigma_i^{-1}\sigma_i = \sigma_i\sigma_i^{-1} = 1 \quad (1 \leq i \leq n-1)\,;$$
$$a_r^{-1}a_r = a_r a_r^{-1} = b_r^{-1}b_r = b_r b_r^{-1} = 1 \quad (1 \leq r \leq g)\,.$$

– *Braid relations, i.e.*

$$(R1)\quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\,;$$
$$(R2)\quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \textit{for}\ |i-j|\geq 2\,.$$

– *Mixed relations:*

$$\begin{array}{ll}
(R3) & a_r\sigma_i = \sigma_i a_r \quad (1\leq r\leq g;\ i\neq 1)\,;\\
 & b_r\sigma_i = \sigma_i b_r \quad (1\leq r\leq g;\ i\neq 1)\,;\\
(R4) & \sigma_1^{-1}a_r\sigma_1^{-1}a_r = a_r\sigma_1^{-1}a_r\sigma_1^{-1} \quad (1\leq r\leq g)\,;\\
 & \sigma_1^{-1}b_r\sigma_1^{-1}b_r = b_r\sigma_1^{-1}b_r\sigma_1^{-1} \quad (1\leq r\leq g)\,;\\
(R5) & \sigma_1^{-1}a_s\sigma_1 a_r = a_r\sigma_1^{-1}a_s\sigma_1 \quad (1\leq s< r\leq g)\,;\\
 & \sigma_1^{-1}b_s\sigma_1 b_r = b_r\sigma_1^{-1}b_s\sigma_1 \quad (1\leq s< r\leq g)\,;\\
 & \sigma_1^{-1}a_s\sigma_1 b_r = b_r\sigma_1^{-1}a_s\sigma_1 \quad (1\leq s< r\leq g)\,;\\
 & \sigma_1^{-1}b_s\sigma_1 a_r = a_r\sigma_1^{-1}b_s\sigma_1 \quad (1\leq s< r\leq g)\,;\\
(R6) & \sigma_1^{-1}a_r\sigma_1^{-1}b_r = b_r\sigma_1^{-1}a_r\sigma_1 \quad (1\leq r\leq g)\,;\\
(TR) & [a_1,b_1^{-1}]\cdots[a_g,b_g^{-1}] = \sigma_1\sigma_2\cdots\sigma_{n-1}^2\cdots\sigma_2\,.\sigma_1\,.
\end{array}$$

– *Singular relations:*

$$\begin{array}{ll}
(R7) & \tau_i\,\sigma_j\,\sigma_i = \sigma_j\,\sigma_i\,\tau_j \quad \textit{for}\ |i-j|=1\,;\\
(R8) & \tau_i\,\tau_j = \tau_j\,\tau_i \quad \textit{for}\ |i-j|\geq 2\,;\\
(R9) & \tau_i\,\sigma_j = \sigma_j\,\tau_i \quad \textit{for}\ |i-j|\geq 2\,;\\
(R10) & \tau_i\,\sigma_i = \sigma_i\,\tau_i \quad (1\leq i\leq n-1)\,;\\
(R11) & (a_{i,r}a_{i+1,r})\tau_i = \tau_i(a_{i,r}a_{i+1,r}) \quad (1\leq i\leq n-1; 1\leq r\leq g)\,;\\
 & (b_{i,r}b_{i+1,r})\tau_i = \tau_i(b_{i,r}b_{i+1,r}) \quad (1\leq i\leq n-1; 1\leq r\leq g)\,;\\
(R12) & \tau_i a_{j,r} = a_{j,r}\tau_i \quad (j\neq i,i+1; 1\leq r\leq g)\,;\\
 & \tau_i b_{j,r} = b_{j,r}\tau_i \quad (j\neq i,i+1; 1\leq r\leq g)\,,
\end{array}$$

where $[a,b] := aba^{-1}b^{-1}$ *and* $a_{i,r}$, $b_{i,r}$ *are defined as in previous Theorem.*

Chapter 4

Generalized Hecke Algebras

4.1 Introduction

Any braid x in B_n yields an oriented link $\widehat{x}$ by closing up the strands of the braid as in figure 4.1. The up to down orientation of braid strands induces an orientation for the closure. Alexander was the first to observe that any oriented link can be identified with the *closure* of a braid ([2]). On the other hand, Markov provided moves relating two braids with the same closure.

Proposition 4.1.1 (Markov) *Two closed braids $\widehat{x}, \widehat{y}$ are equivalent links if and only if one can relate the braids x, y in $\cup_{n\geq 2} B_n$ by a sequence of the following elementary moves:*

- Conjugation*: $z \in B_n$ is replaced by $czc^{-1} \in B_n$, for some $c \in B_n$.*
- Stabilization *(or its inverse, namely a* destabilization*): $z \in B_n$ is replaced by $z\sigma_n^{\pm 1} \in B_{n+1}$.*

To obtain invariants of links, we may proceed constructing suitable functionals on $\mathbb{C}[B_n]$, called *Markov Traces*. This process is well-known for Hecke algebras, which are finite dimensional quotients of $\mathbb{C}[B_n]$.

Definition 4.1.1 *The Hecke algebra $H_n(q)$ (of type $\mathcal{A}$) is the quotient*

$$H_n(q) = \mathbb{C}[B_n]/(\sigma_j^2 + (1-q)\sigma_j - q\,,\ j = 1, \ldots, n-1)\,.$$

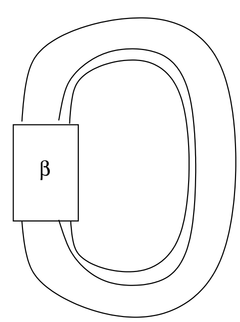


Figure 4.1: The closure of the braid β.

Famous construction of Jones polynomial using Hecke algebras posed the problem of similar constructions on other quotients of $\mathbb{C}[B_n]$. We would extend Jones' approach to the case of links in 3-manifolds. The first task is to define and verify the existence of Markov traces on $\mathbb{C}[B(n,F)]$, or suitable quotients. We propose to consider the following quotient of $\mathbb{C}[B(n,F)]$.

Definition 4.1.2 *Let F be a surface with at least one boundary component. The surface Hecke algebra $H_n(q,F)$ is the quotient*

$$H_n(q,F) = \mathbb{C}[B(n,F)]/(\sigma_j^2 + (1-q)\sigma_j - q\,,\ j = 1,\ldots,n-1)\,,$$

where σ_j is the classic generator of B_n.

The Hecke algebra $H_1(q,F)$ is the group algebra of $B(1,F) \cong \pi_1(F)$ We remark that the natural embedding $B_n \to B(n,F)$ induces an embedding of the usual Hecke algebra $H_n(q)$ into $H_n(q,F)$. On the other hand, when F has boundary, $B(n,F)$ embeds naturally in $B(n+1,F)$, and thus we can define the tower $\cup_{n\geq 1} H_n(q,F)$.
We succeeded finding a Markov trace on $\cup_{n\geq 1} H_n(q,F)$ for the specialization $q = 1$. Let $\widehat{\pi}$ be the set of conjugation classes of $\pi_1(F)$ and $\widehat{\pi}^0 = \widehat{\pi} - \{1\}$. Let $S(\mathbb{C}\widehat{\pi}^0)$ be the symmetric algebra of the vector space $\mathbb{C}\widehat{\pi}^0$.

Theorem 4.1.1 *For any $z \in \mathbb{C}$, there exists an unique family $\mathcal{T}_n$ of linear functionals*

$$\mathcal{T}_n : H_n(1,F) \to S(\mathbb{C}\widehat{\pi}^0)$$

such that

1. $\mathcal{T}_n(xy) = \mathcal{T}_n(yx) \quad \forall x,y \in H_n(1,F)\,;$
2. $\mathcal{T}_{n+1}(x\sigma_n) = z\mathcal{T}_n(x) \quad \forall x \in H_n(1,F)\,;$
3. $\mathcal{T}_{n+1}(\sigma_n\cdots\sigma_1 A \sigma_1\cdots\sigma_n x) = \widehat{A}\mathcal{T}_n(x) \quad \forall x \in H_n(1,F)\ \forall A \in B(1,F)\,;$
4. $\mathcal{T}_n(1) = 1\,,$

where $\widehat{A}$ denotes the conjugation class of the element $A \in B(1,F) \cong \pi_1(F)$.

We notice that when M is a handlebody, i.e. $M = F \times I$, where F is a holed disk, it has been proved that the 3*rd skein module* of links in M is isomorphic to $S(\mathbb{C}\widehat{\pi}^0)$ ([78]). A Markov trace and corresponding link invariant have been constructed in the case of the solid torus ([65] and [77]).
Quantum invariants for links in arbitrary 3-manifolds were defined by Reshetikhin and Turaev in [80]. In the case when the manifold is S^3 one obtains the colored Jones polynomial at roots of unity. In general is presently unknown whether these invariants come from a polynomial evaluated at roots of unity.

4.2 Preliminaries

4.2.1 Markov traces

Let us recall some classic results. The natural embedding $B_n \to B_{n+1}$ induces the injection $\mathbb{C}[B_n] \to \mathbb{C}[B_{n+1}]$. Let $\mathcal{T}_n : \mathbb{C}[B_n] \to \mathbb{C}$ be a family of linear functionals fulfilling following conditions.

- $\mathcal{T}_n(xy) = \mathcal{T}_n(yx), \quad \forall x, y \in B_n;$
- $\mathcal{T}_{n+1}(x\sigma_n) = z\mathcal{T}_n(x) \quad \forall x \in B_n;$
- $\mathcal{T}_{n+1}(x\sigma_n^{-1}) = \widehat{z}\mathcal{T}_n(x) \quad \forall x \in B_n,$

for some $z, \widehat{z} \in \mathbb{C}^*$. Such a family is called a *Markov trace*, and usually one drops the subscript n from $\mathcal{T}$. For an element $x \in B_n$ written in terms of the standard generators as $x = \prod_i \sigma_i^{\lambda_i}$ we denote by $e(x)$ the exponent sum of x. Since relations in B_n are homogeneous, the exponent sum is well-defined.

Corollary 4.2.1 *Let $\mathcal{T}$ be a Markov trace. The function F associating to the closed braid $\hat{x}$ (for $x \in B_n$) the value*

$$F(x) = z^{-\frac{e(x)+n}{2}}\widehat{z}^{\frac{e(x)-n}{2}}\mathcal{T}(x)\,,$$

is a link invariant.

In order to find Markov traces we focus on Hecke algebras $H_n(q)$. We set again σ_j for the image of σ_j in $H_n(q)$. V. Jones and A. Ocneanu showed that:

Proposition 4.2.1 *For any $z \in \mathbb{C}$ there exists an unique Markov trace $\mathcal{T}$ on the Hecke algebras $H_n(q)$ verifying:*

- $\mathcal{T}_n(xy) = \mathcal{T}_n(yx), \quad \forall x, y \in H_n(q);$
- $\mathcal{T}_{n+1}(x\sigma_n) = z\mathcal{T}_n(x) \quad \forall x \in H_n(q);$
- $\mathcal{T}_n(1) = 1.$

The main idea in the proof is that every element in $H_n(q)$ can be written in terms of a linear combination of words $x\sigma_n y$ with $x, y \in H_n(q)$ and words from $H_n(q)$. Moreover, every element σ of $H_n(q)$ can be written in terms of the standard basis of $H_n(q)$ ([58]):

$$(\sigma_{i_1}\cdots\sigma_{i_1-r_1})(\sigma_{i_2}\cdots\sigma_{i_2-r_2})\cdots(\sigma_{i_p}\cdots\sigma_{i_p-r_p})\,,$$

where $1 \leq i_1 < i_2 < \cdots < i_p \leq n-1$ and $r_j \in \{0, 1, \ldots, i_j - 1\}$. The natural inclusion of B_n into B_{n+1} induces the inclusion on corresponding Hecke algebras. The above inductive basis allows to construct a functional $\mathcal{T}$ on $\mathcal{H} := \cup_{n=1}^{\infty} H_n(q)$ with values in $\mathbb{C}$. One proves easily that $\mathcal{T}$ is a trace. We remark that the definition of a Markov trace concerns also the behaviour of the trace with respect to the other stabilization. The condition $\mathcal{T}(a\sigma_n) = z\mathcal{T}(a)$ and the quadratic relation $\sigma_j^2 = (q-1)\sigma_j + q$ imply the relation $\mathcal{T}(a\sigma_n^{-1}) = \widehat{z}\mathcal{T}(a)$, where $\widehat{z} = \frac{z-q+1}{q}$.

4.2.2 Algebraic construction of HOMFLY-PT polynomial

A Markov trace gives rise to a link invariant and we derive the following corollary.

Corollary 4.2.2 *Set $\lambda = \frac{1-q+z}{qz}$. Consider the link L presented as the closure of the braid $x \in B_n$. Then*

$$X_L(q, \lambda) = (-\frac{(1-\lambda q)}{\sqrt{\lambda}(1-q)})^{n-1}(\sqrt{\lambda})^{e(x)}\mathcal{T}(x)\,,$$

is a link invariant.

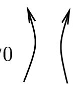
This is called HOMFLY-PT polynomial of L and it is common to change variables as follows:

$$t = \sqrt{\lambda}\sqrt{q}\,,\ x = \sqrt{q} - \frac{1}{\sqrt{q}}\,,$$

and to denote it

$$P_L(t,x) = X_L(q,\lambda)\,.$$

Let L_+, L_-, L_0 be oriented link diagrams that are identical, except in one crossing as in the figure below.

L_+  , L_-  , L_0  .

Thus we can express HOMFLY-PT polynomial with the usual *skein relation*.

Theorem 4.2.1 *$P_L(t,x)$ is the unique Laurent polynomial in t and x verifying the skein relation*

$t^{-1}P$ () $- t\,P$ () $= xP$ ()

and taking the value 1 *for the unknot.*

The Jones polynomial follows from the specialisation $P_L(\sqrt{-1}/t, \sqrt{-1}(\sqrt{t} - (1/\sqrt{t})))$.

4.3 Proof of Theorem 4.1.1

First, we recall that the condition "F with boundary" implies that $B(n,F)$ embeds naturally in $B(n+1,F)$. Therefore, there is no ambiguity to consider an element of $B(n,F)$ as an element of $B(m,F)$, for $m > n$.

Let us denote $\widehat{B}(n,F)$ the quotient $\dfrac{B(n,F)}{\langle\langle \sigma_j^2,\ j = 1,\ldots,n-1\rangle\rangle}$. The algebra $H_n(1,F)$ can be considered as the group algebra of $\pi_1(F)^n \rtimes \Sigma_n$, where Σ_n acts by permutation of the coordinates in $\pi_1(F)^n$.

Proposition 4.3.1 *The group $\widehat{B}(n,F)$ is isomorphic to the semi-direct product $\pi_1(F)^n \rtimes \Sigma_n$.*

Proof: Let $\chi : P(n,F) \to \pi_1(F)^n$ be the map which forgets about the braiding and keeps only the fundamental group information of each strand (see also Lemma 1.6.1). The following diagram holds:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P(n,F) & \longrightarrow & B(n,F) & \xrightarrow{\pi} & \Sigma_n & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\chi} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & \pi_1(F)^n & \longrightarrow & \widehat{B}(n,F) & \xrightarrow{\widehat{\pi}} & \Sigma_n & \longrightarrow & 1\,, \end{array}$$

where $\pi : B(n,F) \to \Sigma_n$ is the canonic projection of $B(n,F)$ onto the symmetric group Σ_n, which induces a surjection $\widehat{\pi}$ of $\widehat{B}(n,F)$ on Σ_n. Set s_j for the transposition $(j, j+1)$ in Σ_n.

The morphism $\widehat{\pi}$ is provided with the section $s : \Sigma_n \to \widehat{B}(n, F)$ which sends each generator s_j in the equivalence class of σ_j on $\widehat{B}(n, F)$. Thus the lower exact sequence splits and the claim follows. $\square$

Remark 4.3.1 *We have proved that the map χ induces an isomorphism $\bar{\chi} : \widehat{B}(n, F) \to \pi_1(F)^n \rtimes \Sigma_n$. The inverse $\phi : \pi_1(F)^n \rtimes \Sigma_n \to \widehat{B}(n, F)$ is defined as follows:*

- $\phi(p_1, 1, \cdots, 1) = p_1$, *where* $p_1 \in \pi_1(F) \cong B(1, F)$
- $\phi(1, \ldots, 1, p_i, 1, \ldots, 1) = \sigma_{i-1} \cdots \sigma_1 p_1 \sigma_1 \cdots \sigma_{i-1}$, *where* $p_i \in \pi_1(F) \cong B(1, F)$, *for* $i = 2, \ldots, n-1$.
- $\phi(1, \ldots, 1, \sigma) = s(\sigma)$, *where* $\sigma \in \Sigma_n$.

We can suppose that $s(\sigma)$ is written in the normal *form :*

$$(s_{i_1} \cdots s_{i_1 - r_1})(s_{i_2} \cdots s_{i_2 - r_2}) \cdots (s_{i_p} \cdots s_{i_p - r_p}),$$

where $1 \leq i_1 < i_2 < \cdots < i_p \leq n-1$ and $r_j \in \{0, 1, \ldots, i_j - 1\}$. The word $\phi \circ \bar{\chi}(\beta)$ is the normal form for the element $\beta \in \widehat{B}(n, F)$.

We set again σ_j for the image of σ_j in $H_n(1, F)$. The algebra $H_1(1, F)$ is the group algebra of $B(1, F)$, the group freely generated by $\{a_1, \ldots, a_g, b_1, \ldots, b_g, z_1, \ldots, z_{p-1}\}$.

Theorem 4.3.1 *Every element $x \in H_{n+1}(1, F)$ can be written as linear combination of words, each of one of the following types:*

1. w_{n-1};
2. $w_{n-1} \sigma_n v_{n-1}$;
3. $w_{n-1} \sigma_n \cdots \sigma_1 A \sigma_1 \cdots \sigma_n$,

where w_{n-1}, v_{n-1} are some words in $H_n(1, F)$ and A some word in $H_1(1, F)$.

Proof: Since $H_n(1, F)$ is the group algebra of $\widehat{B}(n, F)$, it suffices to prove that each word x in $\widehat{B}(n+1, F)$ can be written as a word of one of following types:

1. w_{n-1};
2. $w_{n-1} \sigma_n v_{n-1}$;
3. $w_{n-1} \sigma_n \cdots \sigma_1 A \sigma_1 \cdots \sigma_n$,

where w_{n-1}, v_{n-1} are some elements of $\widehat{B}(n, F)$ and A an element of $B(1, F)$. We proceed by induction on n. The claim is true for $n = 1$. Braid relations in Theorem 1.1.1 and the relation $\sigma_j^2 = 1$ imply $A \sigma_1 B \sigma_1 = \sigma_1 B \sigma_1 A$, for all A, B in $B(1, F)$. It yields that any element of $\widehat{B}(2, F)$ can be rewritten as a word $A \sigma_1^{\eta_1} B \sigma_1^{\eta_2}$, for A, B in $B(1, F)$ and $\eta_1, \eta_2 = \{0, 1\}$. Let $x \in \widehat{B}(n+1, F)$. There exists a word z equivalent to x, where σ_n appears at most twice. Let $\sigma_n x' \sigma_n x'' \sigma_n$, $x', x'' \in \widehat{B}(n, F)$ be a sub-word of x. We distinguish four cases

1. $x' = y\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}, x'' = y'\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}$, for some $y, y' \in \widehat{B}(n-1, F)$, $A, A' \in B(1, F)$.

$$\begin{aligned}&\sigma_n y\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n y'\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}\sigma_n = \\ &= y\underline{\sigma_n\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n} y'\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}\sigma_n =^* \\ &= yy'\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}\sigma_n\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n^2 = w\sigma_n v\,,\end{aligned}$$

where $w,\, v \in \widehat{B}(n, F)$.

2. $x' = y\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}, x'' = y'\sigma_{n-1}y''$, for some $y, y', y'' \in \widehat{B}(n-1, F)$, $A \in B(1, F)$.

$$\begin{aligned}&\sigma_n y\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n y'\sigma_{n-1}y''\sigma_n = \\ &= y\underline{\sigma_n\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n} y'\sigma_{n-1}y''\sigma_n =^* \\ &= yy'\sigma_{n-1}y''\sigma_n\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n^2 = w\sigma_n v\,,\end{aligned}$$

where $w,\, v \in \widehat{B}(n, F)$.

3. $x' = y\sigma_{n-1}y', x'' = y''\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}$, for some $y, y', y'' \in \widehat{B}(n-1, F)$, $A \in B(1, F)$.

$$\begin{aligned}&\sigma_n y\sigma_{n-1}y'\underline{\sigma_n\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n} =^* \\ &= \sigma_n\sigma_n\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n y\sigma_{n-1}\sigma_n\sigma_{n-1}y' = w\sigma_n v\,,\end{aligned}$$

where $w,\, v \in \widehat{B}(n, F)$.

4. $x' = y\sigma_{n-1}y', x'' = u\sigma_{n-1}u'$ for some $y, y', u, u' \in \widehat{B}(n-1, F)$.

$$\underline{\sigma_n y\sigma_{n-1}y'\sigma_n} u\sigma_{n-1}u'\sigma_n = y\sigma_{n-1}\sigma_n\sigma_{n-1}y'u\sigma_{n-1}\sigma_n u';$$

   (a) if $y'u = a\sigma_{n-2}b$ for some $a, b \in \widehat{B}(n-2, F)$,

$$\begin{aligned}&y\sigma_{n-1}\sigma_n\sigma_{n-1}y'u\sigma_{n-1}\sigma_n u' = y\sigma_{n-1}a\sigma_n\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}\sigma_n bu' = \\ &= y\sigma_{n-1}a\sigma_{n-2}\sigma_{n-1}\sigma_n\sigma_{n-1}\sigma_{n-2}bu' = w\sigma_n v\,,\end{aligned}$$

   where $w,\, v \in \widehat{B}(n, F)$.

   (b) if $y'u = a\sigma_{n-2}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-2}$ for some $a \in \widehat{B}(n-2, F)$, $A \in B(1, F)$,

$$\begin{aligned}&y\sigma_{n-1}\sigma_n\sigma_{n-1}y'u\sigma_{n-1}\sigma_n u' = \\ &= y\sigma_{n-1}a\underline{\sigma_n\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-2}\sigma_{n-1}\sigma_n} u' =^* \\ &w\sigma_n\cdots\sigma_1 A\sigma_1\cdots\sigma_n\,,\end{aligned}$$

   where $w \in \widehat{B}(n, F), A \in B(1, F)$.

Equivalences with $=^*$ are justified because $[g, \sigma_k\cdots\sigma_1 A\sigma_1\cdots\sigma_k] = 1$ for any $g \in \widehat{B}(k, F)$, $A \in B(1, F)$. Let $x = w\sigma_n u\sigma_n v$, $w, u, v \in \widehat{B}(n, F)$. We proceed as in 4.

1. if $u = a\sigma_{n-1}b$ for some $a, b \in \widehat{B}(n-1, F)$,

$$wa\sigma_n\sigma_{n-1}\sigma_n bv = w'\sigma_n v'\,,$$

where $w',\, v' \in \widehat{B}(n, F)$.

2. if $u = a\sigma_{n-2}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-2}$ for some $a \in \widehat{B}(n-1,F),\ A \in B(1,F)$,

$$wa\sigma_n\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n u = w'\sigma_n\cdots\sigma_1 A\sigma_1\cdots\sigma_n\,,$$

where $w' \in \widehat{B}(n,F)$.

3. if $u = a$ for some $a \in \widehat{B}(n-1,F)$,

$$wa\sigma_n\sigma_n v = w'\,,$$

where $w' \in \widehat{B}(n-1,F)$.

□

Let $A \in B(1,F)$. We set $A^{(i)} = \sigma_{i-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{i-1}$, for $i = 1,\ldots,n$, $(A^{(1)} = A)$.

Theorem 4.3.2 *Every element of $H_{n+1}(1,F)$ can be written uniquely as a linear combination of words each of one of the following types:*

1. w_{n-1};
2. $w_{n-1}\sigma_n\cdots\sigma_j$ *for* $j = 1,\ldots,n-1$;
3. $w_{n-1}\sigma_n\cdots\sigma_i A^{(i)}$ *for* $i = 1,\ldots,n$,

where w_{n-1} is a word in $H_n(1,F)$ and A a word in $H_1(1,F)$.

Proof: First, we prove that each word $x \in \widehat{B}(n+1,F)$ can be written as a word of one of following types:

1. w_{n-1};
2. $w_{n-1}\sigma_n\cdots\sigma_j$ for $j = 1,\ldots,n-1$;
3. $w_{n-1}\sigma_n\cdots\sigma_i A^{(i)}$ for $i = 1,\ldots,n$,

where w_{n-1} is an element of $\widehat{B}(n,F)$ and A an element of $B(1,F)$. From previous Theorem it suffices to show the claim for words of the type $w_{n-1}\sigma_n v_{n-1}$, where w_{n-1}, v_{n-1} are in $\widehat{B}(n,F)$ and A in $B(1,F)$.
We reason by induction. We encounter three possibilities:

1. if $v_{n-1} \in \widehat{B}(n-1,F)$ then $x = w_{n-1}v_{n-1}\sigma_n = w'_{n-1}\sigma_n$, where $w'_{n-1} \in \widehat{B}(n,F)$;

2. if $v_{n-1} = u_{n-2}\sigma_{n-1}\cdots\sigma_j$, where $u_{n-2} \in \widehat{B}(n-1,F)$, then

$$x = w_{n-1}u_{n-2}\sigma_n\sigma_{n-1}\cdots\sigma_j = w'_{n-1}\sigma_n\cdots\sigma_j\,,$$

where $w'_{n-1} \in \widehat{B}(n,F)$;

3. if $v_{n-1} = u_{n-2}\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}$, where $u_{n-2} \in \widehat{B}(n-1,F)$ and $A \in B(1,F)$, then $x = w_{n-1}u_{n-2}\sigma_n\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1} = w'_{n-1}\sigma_n A^{(n)}$, where $w'_{n-1} \in \widehat{B}(n,F)$.

Actually the word $w(x)$ can be rewritten as a word $\widehat{w}(x) = A_1^{(1)} \cdots A_n^{(n)}\sigma$, where $A_1, \ldots, A_n \in B(1,F)$ and σ is an element of $\widehat{B}(n,F)$ in the generators $\{\sigma_1, \ldots \sigma_n\}$. We can suppose σ written in the normal form (see Remark 4.3.1). We reason by induction on n. We have to check only the case $w(x) = w_{n-1}\sigma_n \cdots \sigma_i A^{(i)}$, where $w_{n-1} \in \widehat{B}(n,F)$. Suppose that $w_{n-1} = A_1^{(1)} \cdots A_{n-1}^{(n)}\sigma$, where $A_1, \ldots, A_{n-1}$ belong to $\widehat{B}(1,F)$ and σ is an element of Σ_n, written in its normal form. Thus,

$$w(x) = A_1^{(1)} \cdots A_{n-1}^{(n)}\sigma\sigma_n \cdots \sigma_i A^{(i)} = A_1^{(1)} \cdots A_{n-1}^{(n)}\sigma\underline{\sigma_n \cdots \sigma_i A^{(i)}\sigma_i \cdots \sigma_n}\sigma_n \cdots \sigma_i =$$

$$= A_1^{(1)} \cdots A_{n-1}^{(n-1)}A^{(n)}\sigma\,\sigma_n \cdots \sigma_i = A_1^{(1)} \cdots A_{n-1}^{(n-1)}A_n^{(n)}\sigma'\,,$$

where $A_n^{(n)} = A^{(n)}$ and $\sigma' = \sigma\sigma_n \cdots \sigma_i$ is an element of Σ_n written in its normal form.
The word $\widehat{w}(x)$ is the normal form $\phi \circ \bar{\chi}(x)$ of x (Remark 4.3.1). □

We can construct inductively the trace $\mathcal{T}$ on $\cup_{n\geq 1}H_n(1,F)$ using Theorem 4.3.2. Let x be an arbitrary word of $\widehat{B}(n+1,F)$. By Theorem 4.3.2, the element x is equivalent to a word in $\widehat{B}(n,F)$ or a word $w\sigma_n v$, with $w, v \in \widehat{B}(n,F)$ or a word $u\sigma_n \cdots \sigma_1 A\sigma_1 \cdots \sigma_n$, with $u \in \widehat{B}(n,F)$ and $A \in B(1,F)$. Assume that the $\mathcal{T}$ is defined on $H_n(1,F)$. Define now $\mathcal{T}(x) = z\mathcal{T}(wv)$ if $x = w\sigma_n v$ and $\mathcal{T}(x) = \widehat{A}\mathcal{T}(u)$ if $x = u\sigma_n \cdots \sigma_1 A\sigma_1 \cdots \sigma_n$. The map $\mathcal{T}$ is well defined and it extends by linearity to $H_{n+1}(1,F)$.
In order to prove the existence of $\mathcal{T}$, it remains to prove that $\mathcal{T}(xy) = \mathcal{T}(yx)$, for all $x, y \in \cup_{n\geq 1}H_n(1,F)$. Before continuing with proof, we note that the uniqueness of $\mathcal{T}$ follows immediately since for any $x \in H_n(1,F)$, $\mathcal{T}(x)$ can be computed inductively using rules 1), 2), 3), and 4) of Theorem 4.1.1.
We proceed with checking that $\mathcal{T}(xy) = \mathcal{T}(yx)$, for all $x, y \in \cup_{n\geq 1}H_n(1,F)$. We suppose that the assumption holds for all $x, y \in \widehat{B}(n,F)$ and we prove that $\mathcal{T}(xy) = \mathcal{T}(yx)$, for all $x, y \in \widehat{B}(n+1,F)$. We can suppose that $y = w\sigma_n v$, with $w, v \in \widehat{B}(n,F)$ or $y = u\sigma_n \cdots \sigma_1 A\sigma_1 \cdots \sigma_n$, with $u \in \widehat{B}(n,F)$ and $A \in B(1,F)$.

When $x \in \widehat{B}(n,F)$ the claim follows by the definition of $\mathcal{T}$.
For instance, let $y = u\sigma_n \cdots \sigma_1 A\sigma_1 \cdots \sigma_n$, with $u \in \widehat{B}(n,F)$ and $A \in B(1,F)$. Then,

$$\mathcal{T}(xy) = \widehat{A}\mathcal{T}(xu) = \widehat{A}\mathcal{T}(ux) = \mathcal{T}(yx)\,.$$

When $x = a\sigma_n \cdots \sigma_1 A'\sigma_1 \cdots \sigma_n$, with $a \in \widehat{B}(n,F)$ and $A \in B(1,F)$ we encounter two possibilities:

1. If $y = u\sigma_n \cdots \sigma_1 A\sigma_1 \cdots \sigma_n$, with $u \in \widehat{B}(n,F)$ and $A \in B(1,F)$, then

$$\mathcal{T}(xy) = \mathcal{T}(au\sigma_n \cdots \sigma_1 A'A\sigma_1 \cdots \sigma_n) = \widehat{A'A}\mathcal{T}(au) =$$

$$= \widehat{AA'}\mathcal{T}(ua) = \mathcal{T}(ua\sigma_n \cdots \sigma_1 AA'\sigma_1 \cdots \sigma_n) = \mathcal{T}(yx)\,.$$

2. If $y = w\sigma_n v$, with $w, v \in \widehat{B}(n,F)$, then

$$\mathcal{T}(xy) = \mathcal{T}(aw\sigma_n \cdots \sigma_1 A\sigma_1 \cdots \sigma_{n-1}v) = z\mathcal{T}(aw\sigma_{n-1} \cdots \sigma_1 A\sigma_1 \cdots \sigma_{n-1}v) =$$

$$= z\mathcal{T}(w\sigma_{n-1} \cdots \sigma_1 A\sigma_1 \cdots \sigma_{n-1}va) = \mathcal{T}(w\sigma_{n-1} \cdots \sigma_1 A\sigma_1 \cdots \sigma_n va) = \mathcal{T}(yx)\,.$$

Finally suppose that $x = a\sigma_n b$. Without loss of generality, we can suppose a and b to be the empty word.

Let $y = u\sigma_n \cdots \sigma_1 A \sigma_1 \cdots \sigma_n$, for some $u \in \widehat{B}(n, F)$, $A \in B(1, F)$.

1. Suppose $u = u'\sigma_{n-1} \cdots \sigma_1 A' \sigma_1 \cdots \sigma_{n-1}$, for some $y' \in \widehat{B}(n-1, F)$ and $A' \in B(1, F)$.

$$\begin{aligned}\mathcal{T}(\sigma_n y) = \mathcal{T}(u'\underline{\sigma_n\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}\sigma_n}\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_n) =\\ = \mathcal{T}(u'\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}\sigma_n\sigma_n) =\\ = z\mathcal{T}(u'\sigma_{n-1}\cdots\sigma_1 AA'\sigma_1\cdots\sigma_{n-1}) = z\widehat{AA'}\mathcal{T}(u')\,.\end{aligned}$$

$$\begin{aligned}\mathcal{T}(y\sigma_n) = \mathcal{T}(u'\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}\sigma_n\cdots\sigma_1 A\sigma_1\cdots\sigma_n\sigma_n) =\\ = z\mathcal{T}(u'\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}) =\\ = z\mathcal{T}(u'\sigma_{n-1}\cdots\sigma_1 A'A\sigma_1\cdots\sigma_{n-1}) = z\widehat{A'A}\mathcal{T}(u') = z\widehat{AA'}\mathcal{T}(u')\,.\end{aligned}$$

2. Suppose $u = u'\sigma_{n-1}u''$, for some $u', u'' \in \widehat{B}(n-1, F)$.

$$\begin{aligned}\mathcal{T}(y\sigma_n) = \mathcal{T}(u'\sigma_{n-1}u''\sigma_n\cdots\sigma_1 A\sigma_1\cdots\sigma_n\sigma_n) =\\ = z\mathcal{T}(u'\sigma_{n-1}u''\underline{\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}}) =\\ = z\mathcal{T}(u'\sigma_{n-2}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}u'') =\\ = z^2\mathcal{T}(u'\sigma_{n-2}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-2}u'')\,;\end{aligned}$$

$$\begin{aligned}\mathcal{T}(\sigma_n y) = \mathcal{T}(u'\sigma_n\sigma_{n-1}u''\underline{\sigma_n\cdots\sigma_1 A\sigma_1\cdots\sigma_n}) =\\ = z\mathcal{T}(u'\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_{n-1}u'') =\\ = z^2\mathcal{T}(u'\sigma_{n-2}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-2}u'')\,.\end{aligned}$$

Let $y = w\sigma_n v$, for some $w, v \in \widehat{B}(n, F)$.

1. Suppose $w = w'\sigma_{n-1}w''$, $v = v'\sigma_{n-1}v''$ for some $w', w'', v', v'' \in \widehat{B}(n-1, F)$.

$$\begin{aligned}\mathcal{T}(y\sigma_n) = \mathcal{T}(w'\sigma_{n-1}w''\sigma_n v'\sigma_{n-1}v''\sigma_n) =\\ = \mathcal{T}(w'\sigma_{n-1}w''v'\sigma_{n-1}\sigma_n\sigma_{n-1}v'') = z^2\mathcal{T}(w'w''v'v'') = \mathcal{T}(\sigma_n y)\,.\end{aligned}$$

2. Suppose $w = w'\sigma_{n-1}w''$, $v = v'\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}$, for some w', w'', $v' \in \widehat{B}(n-1, F)$ and $A \in B(1, F)$.

$$\mathcal{T}(y\sigma_n) = \mathcal{T}(w'\sigma_{n-1}w''\sigma_n v'\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n) = z\widehat{A}\mathcal{T}(w'w''v')\,;$$

$$\begin{aligned}\mathcal{T}(\sigma_n y) = \mathcal{T}(w'\sigma_n\sigma_{n-1}\sigma_n w''v'\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}) =\\ = z\mathcal{T}(w'w''v'\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}) = z\widehat{A}\mathcal{T}(w'w''v')\,.\end{aligned}$$

3. Suppose $w = w'\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}$, $v = v'\sigma_{n-1}v''$, for some v', v'', $w' \in \widehat{B}(n-1, F)$ and $A \in B(1, F)$. As in previous point

$$\mathcal{T}(y\sigma_n) = \mathcal{T}(w'\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n v'\sigma_{n-1}v''\sigma_n) = z\widehat{A}\mathcal{T}(w'v'v'')\,;$$

$$\mathcal{T}(\sigma_n y) = \mathcal{T}(w'\sigma_n\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n v'\sigma_{n-1}v'') = z\widehat{A}\mathcal{T}(w'v'v'')\,.$$

4. Suppose $w = w'\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}$, $v = v'\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}$, for some $v, w \in \widehat{B}(n-1, F)$, $A, A' \in B(1, F)$.

$$\begin{aligned}\mathcal{T}(y\sigma_n) &= \mathcal{T}('w\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n v'\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}\sigma_n)\\ &= \mathcal{T}(w'v'\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}\sigma_n) = \\ &= \widehat{A'}\widehat{A}\,\mathcal{T}(w'v')\,;\end{aligned}$$

$$\begin{aligned}\mathcal{T}(\sigma_n y) &= \mathcal{T}(\sigma_n w'\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n v'\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1})\\ &= \mathcal{T}(w'v'\sigma_n\sigma_{n-1}\cdots\sigma_1 A\sigma_1\cdots\sigma_{n-1}\sigma_n\sigma_{n-1}\cdots\sigma_1 A'\sigma_1\cdots\sigma_{n-1}) = \\ &= \widehat{A}\widehat{A'}\,\mathcal{T}(w'v')\,.\end{aligned}$$

□

Chapter 5

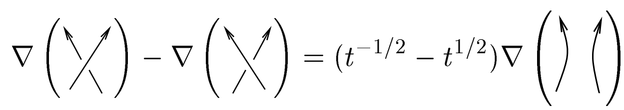
Cubic Hecke algebras and new invariants for links

This chapter is a joint work with L.Funar ([8]).

5.1 Introduction

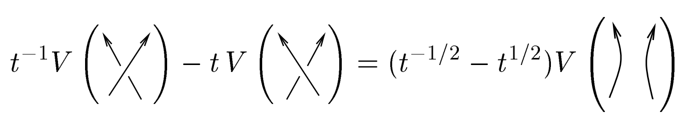
5.1.1 A short history

John Conway showed that the Alexander polynomial of a knot, when suitably normalized, satisfies the following skein relation:

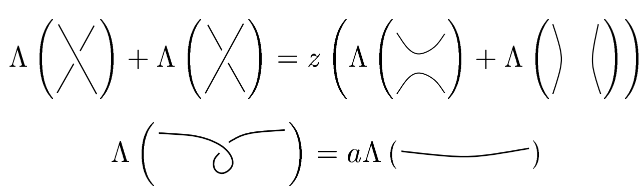


Given a knot diagram one can always change some of the crossings such that the modified diagram represents the unknot. Therefore one can use the skein relation for a recursive computation of ∇, although this algorithm is rather time consuming (exponential).

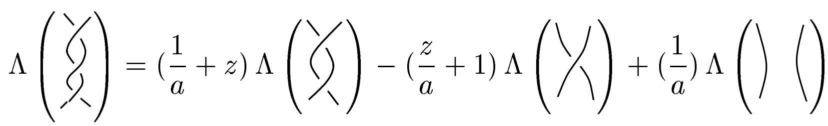
In the mid eighties Jones discovered another invariant verifying a different but quite similar skein relation, namely:



which was further generalized to HOMFLY-PT invariant (Chapter 4) by replacing the factor $(t^{1/2} - t^{-1/2})$ with a new variable x. The latter one was shown to specialize to both Alexander and Jones polynomials. The Kauffman polynomial is another extension of Jones polynomial which satisfies a skein relation in the realm of unoriented diagrams. Specifically the formulas



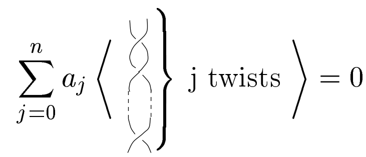
define a regular isotopy invariant of links, which can be renormalized (by using the writhe of the oriented diagram) in order to become a link invariant. Remark that some elementary manipulations show that Λ verifies a cubical skein relation:



It has been recently proved ([31], see also problem 1.59 [59]) that this relation alone is not sufficient for a recursive computation of Λ (whenever this is possible the skein relations are called complete).

These invariants were generalized to quantum invariants associated to Lie (super Lie, etc) algebras and their representations. Turaev ([89]) identified the HOMFLY-PT and Kauffman polynomials with the invariants obtained from the series A_n and B_n, C_n, D_n respectively. Kuperberg ([61]) defined the G_2 quantum invariant of knots by means of skein relations making use of trivalent graphs diagrams and exploited further these ideas for spiders of rank 2 Lie algebras. The skein relations satisfied by the quantum invariants coming from simple Lie algebras were approached also via weight systems and the Kontsevich integral in ([68, 69]) for the classical series and in ([9, 10]) for the case of g_2.

Notice that any link invariant coming from some R-matrix R verifies a skein relation of the type



which can be derived from the polynomial equation satisfied by the matrix R.

Let us mention that the skein relations are somewhat related to the representation theory of the Hopf algebra associated to R. In particular there are no other invariants whose skein relations are completely known and one expects that the invariants obtained from other super Lie algebras or by cabling the previous ones satisfy skein relations of degree at least 4 (as the G_2 invariant does).

This makes the search for an explicit set of complete skein relations, in which at least one relation is cubical, particularly difficult and interesting. This problem was first considered in [43] and solved in a particular case. In this chapter we complete the previous results by constructing a deformation of the previously considered quotients (of the *cubic Hecke algebras*, see Section 5.1.3) and of the Markov traces supported by these algebras. In particular the link invariants obtained this way will be recursively computable and different from HOMFLY-PT, Kauffman and the 2-cabling of HOMFLY-PT.

5.1.2 The main result

In this chapter we will define two link invariants by means of (a complete set of) skein relations. More precisely we will prove the following Theorem (see Section 5.5):

Theorem 5.1.1 (Main Theorem) *There exist two link invariants $I_{(\alpha,\beta)}$ and $I^{(z,\delta)}$ which are (uniquely) determined by the two skein relations shown in Figure 5.1 and their value for the unknot (which, traditionally, is 1). These invariants take values in*

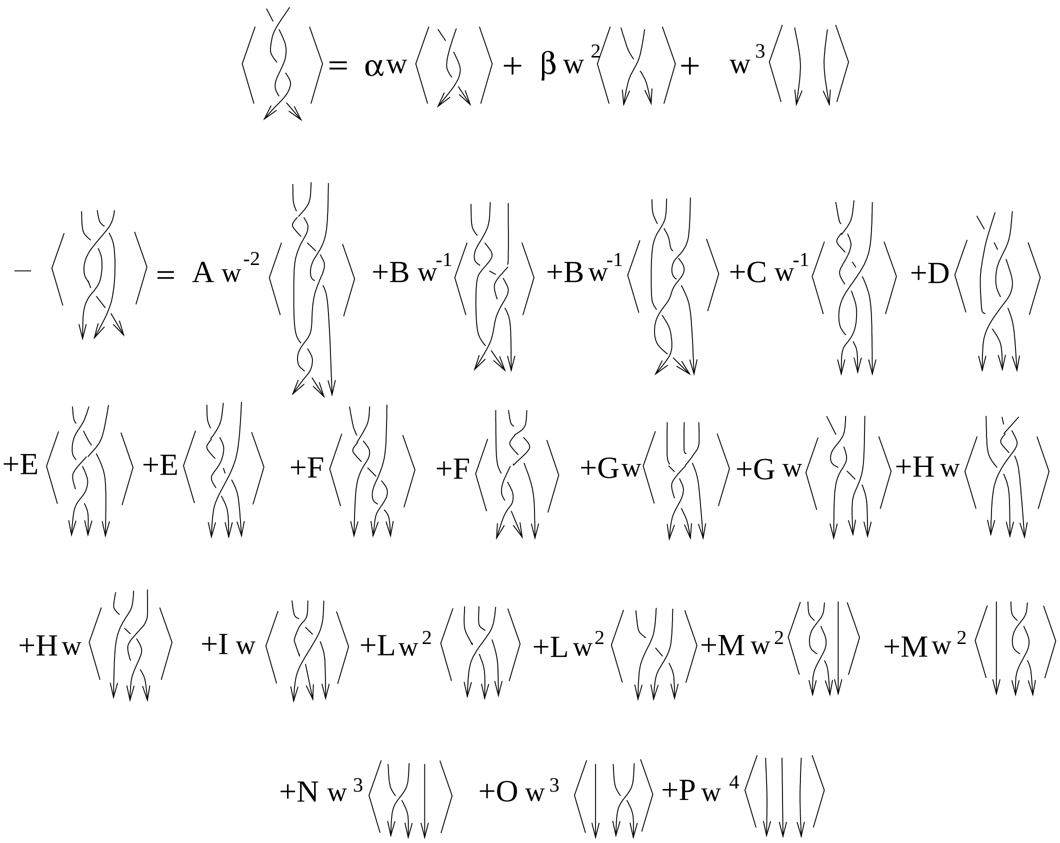


Figure 5.1: The skein relations

$$\frac{\mathbb{Z}[\alpha,\,\beta,\,(2\alpha-\beta^2)^{\pm\epsilon/2},\,(\alpha^2+2\beta)^{\pm\epsilon/2}]}{(H_{(\alpha,\beta)})},$$

and respectively

$$\frac{\mathbb{Z}[z^{\pm\epsilon/2},\delta^{\pm\epsilon/2}]}{(P^{(z,\delta)})},$$

where $\epsilon - 1 \in \{0,1\}$ *is the number of components mod 2 and*

$$\begin{aligned} H_{(\alpha,\beta)} \;&:=\; 8\alpha^6 - 8\alpha^5\beta^2 + 2\alpha^4\beta^4 + 36\alpha^4\beta - 34\alpha^3\beta^3 + 17\alpha^3 + 8\alpha^2\beta^5 + 32\alpha^2\beta^2 - \\ &\quad -36\alpha\beta^4 + 38\alpha\beta + 8\beta^6 - 17\beta^3 + 8, \end{aligned}$$

and respectively

$$P^{(z,\delta)} := z^{23} + z^{18}\delta - 2z^{16}\delta^2 - z^{14}\delta^3 - 2z^9\delta^4 + 2z^7\delta^5 + \delta^6 z^5 + \delta^7.$$

Here (Q) *denotes the ideal generated by the element* Q *in the algebra under consideration.*

The polynomials $A, B, C..., P$ corresponding to $I_{(\alpha,\beta)}$ are given in the table below. In order to obtain those corresponding to $I^{(z,\delta)}$ it suffices to set $w = (-z^4/(\delta z))^{1/2}$ and replace $\alpha = -(z^7+\delta^2)/(z^4\delta)$ and $\beta = (\delta - z^2)/z^3$ in the other entries of table 1.

$w = ((\alpha^2 + 2\beta)/(2\alpha - \beta^2))^{1/2}$	$A = (\beta^2 - \alpha)$
$B = (\alpha^2 - \alpha\beta^2 - \beta)$	$C = (\alpha^2 - \alpha\beta^2)$
$D = (1 + 2\alpha\beta + \alpha^2\beta^2 - \alpha^3)$	$E = (1 + \alpha\beta + \alpha^2\beta^2 - \alpha^3)$
$F = (1 + 2\alpha\beta - \beta^3)$	$G = (\alpha\beta^3 - 2\alpha - 2\alpha^2\beta)$
$H = (\alpha\beta^3 - 2\alpha - 2\alpha^2\beta + \beta^2)$	$I = (\alpha^4 - \alpha^3\beta^2 - 2\alpha^2\beta - 3\alpha)$
$L = (2\alpha^3\beta + 3\alpha^2 - \alpha^2\beta^3 - \alpha\beta^2)$	$M = (\beta^4 - 2\beta - 3\alpha\beta^2 + \alpha^2)$
$N = (1 + 4\alpha\beta + 3\alpha^2\beta^2 - \alpha^3 - \alpha\beta^4 - \beta^3)$	$O = (1 + 3\alpha\beta + 3\alpha^2\beta^2 - \alpha^3 - \alpha\beta^4)$
$P = (3\beta^2 - \beta^5 - 2\alpha - 3\alpha^2\beta + 4\alpha\beta^3)$	

Table 1

5.1.3 Cubic Hecke algebras

Definition 5.1.1 *The cubic Hecke algebra $H(Q,n)$ is the quotient*

$$H(Q,n) = \mathbb{C}[B_n]/(Q(\sigma_j)\,;\; j = 1, \dots, n-1)\,,$$

where $Q(\sigma_j) = \sigma_j^3 - \alpha\,\sigma_j^2 - \beta\,\sigma_j - 1,\ \alpha,\,\beta \in \mathbb{C}$.

Our aim is to construct Markov traces on the tower of cubic Hecke algebras since Markov traces define link invariants (Section 4.2.1). The cubic Hecke algebras are particular cases of the (generic) cyclotomic Hecke algebras introduced by Broué and Malle (see [20]) and studied in [21, 22] in connection with braid group representations. Let us stress that, for $Q(0) \neq 0$, the following results are known (see also [28, 20, 21, 22] and [29] p.148-149):

- $\dim_{\mathbb{C}} H(Q,3) = 24$, and $H(Q,3)$ is isomorphic to the group algebra of the binary tetrahedral group $< 2,3,3 >$ of order 24 (equivalently, $SL(2,\mathbb{Z}_3)$).
- $\dim_{\mathbb{C}} H(Q,4) = 648$, and $H(Q,4)$ is the group algebra of G_{25} in the Shepard-Todd classification (see [85]).
- $\dim_{\mathbb{C}} H(Q,5)$ is the cyclotomic Hecke algebra of group G_{32}, whose order is 155520. It is conjectured that this algebra is free of finite dimension which would imply (by using the Tits deformation theorem) that it is isomorphic to the group algebra of G_{32}.
- $\dim_{\mathbb{C}} H(Q,n) = \infty$ for $n \geq 6$.

Thus a direct definition of the trace on $H(Q,n)$ for $n \geq 6$ is highly a nontrivial matter, in particular it would involve the explicit solution of the conjugacy problem in these algebras which seems out of reach.

In order to deal with finite dimensional algebras one introduces smaller quotients $K_n(\alpha,\,\beta)$ by adding one more relation living in $H(Q,3)$. The exact form of this relation is

$$\sigma_2\,\sigma_1^2\,\sigma_2 + A\,\sigma_1^2\,\sigma_2^2\,\sigma_1^2 + B\,\sigma_1\,\sigma_2^2\,\sigma_1^2 + B\,\sigma_1^2\,\sigma_2^2\,\sigma_1 + C\,\sigma_1^2\,\sigma_2\,\sigma_1^2 + D\,\sigma_1\,\sigma_2^2\,\sigma_1 + E\,\sigma_1\,\sigma_2\,\sigma_1^2 + E\,\sigma_1^2\,\sigma_2\,\sigma_1 + F\,\sigma_2^2\,\sigma_1^2 + F\,\sigma_1^2\,\sigma_2^2 + G\,\sigma_2\,\sigma_1^2 + G\,\sigma_1^2\,\sigma_2 + H\,\sigma_2^2\,\sigma_1 + H\,\sigma_1\,\sigma_2^2 + I\,\sigma_1\,\sigma_2\,\sigma_1 + L\,\sigma_2\,\sigma_1 + L\,\sigma_1\,\sigma_2 + M\,\sigma_1^2 + M\,\sigma_2^2 + N\,\sigma_1 + O\,\sigma_2 + P = 0$$

where $A, B, \dots, P$ are the polynomials from table 1.

Remark 5.1.1 *The algebras $K_n(\alpha, \beta)$ are finite dimensional for any n.*

Let us explain the heuristics behind that choice for the additional relation. The algebra $H(Q,3)$ is semisimple (for generic Q) and decomposes as $\mathbb{C}^3 \oplus M_2^{\oplus 3} \oplus M_3$, where M_m is the algebra of $m \times m$ matrices. As explained in Section 5.2 the usual quadratic Hecke algebra $H_q(3)$ arises when the factor $\mathbb{C} \oplus M_2^{\oplus 2} \oplus M_3$ is killed. It is known that Jones and HOMFLY-PT polynomials can be derived by the unique Markov trace on the tower $H_q(n)$. In a similar way the Birman-Wenzl algebra, which yields the Kauffman polynomial ([43]) is obtained when we quotient by $\mathbb{C} \oplus M_2^2$. In our situation the extra relation kills exactly the factor $\mathbb{C}^3$.

The geometric interpretation of these relations is now obvious: the first skein relation in Figure 1 is the cubical relation corresponding to the quotients $H(Q,n)$ and the second skein relation defines the algebras $K_n(\alpha, \beta)$.

Our main theorem is a consequence of the more technical result below (see Sections 5.2,5.3,5.4).

Theorem 5.1.2 *For exactly four values of the $(z, \widehat{z})$ there exists an unique Markov traces $\mathcal{T}$ on $K_n(\alpha, \beta)$ with parameters $(z, \widehat{z})$ i.e. verifying:*

1. $\mathcal{T}(xy) = \mathcal{T}(yx)$,
2. $\mathcal{T}(x\sigma_{n-1}) = z\mathcal{T}(x)$,
3. $\mathcal{T}(x\sigma_{n-1}^{-1}) = \widehat{z}\mathcal{T}(x)$.

The first couple $(z, \widehat{z})$ is

$$z = (2\alpha - \beta^2)/(\alpha\beta + 4),\ \widehat{z} = -(\alpha^2 + 2\beta)/(\alpha\beta + 4),$$

and the corresponding trace is $\mathcal{T}_{\alpha,\beta} : K_n(\alpha, \beta) \to \mathbb{Z}[\alpha, \beta, 1/(\alpha\beta + 4)]/(H_{(\alpha,\beta)})$.
The other three solutions are not rational functions on the parameters and we prefer to give α, β and $\widehat{z}$ as functions of z, δ $(\delta = z^2(\beta z + 1))$. More precisely we set

$$\mathcal{T}^{(z,\delta)} : K_*(\alpha, \beta) \to \mathbb{Z}[z^{\pm 1}, \delta^{\pm 1}]/(P^{(z,\delta)}),$$

where

$$\beta = (\delta - z^2)/z^3,\ \alpha = -(z^7 + \delta^2)/(z^4\delta)\ \widehat{z} = -z^4/\delta.$$

5.1.4 Outline of the proof

As explained in Chapter 4, one can construct inductively a basis for the (quadratic) Hecke algebra $H_{n+1}(q)$. We will prove by recurrence on n (Lemma 5.3.1) that there is a surjection of $(K_n(\alpha, \beta), K_n(\alpha, \beta))$-bimodules

$$K_n(\alpha, \beta) \oplus K_n(\alpha, \beta) \otimes_{K_{n-1}(\alpha,\beta)} K_n(\alpha, \beta) \oplus K_n(\alpha, \beta) \otimes_{K_{n-1}(\alpha,\beta)} K_n(\alpha, \beta) \longrightarrow K_{n+1}(\alpha, \beta)$$

given by $x \oplus y \otimes z \oplus u \otimes v \to x + y\sigma_n z + u\sigma_n^2 v$.
Since there is a system of generators for $K_{n+1}(\alpha, \beta)$ constructed out of one for $K_n(\alpha, \beta)$, the extension of a Markov trace on $K_n(\alpha, \beta)$, if ever exists, it must be unique.
However the previous morphism is not injective and the most difficult step is to prove that the canonical extension is a well-defined linear functional and it satisfies the trace commutativity.

Let $A = \mathbb{Z}[\alpha, \beta, z, \widehat{z}]/H$, where H is an ideal of $\mathbb{Z}[\alpha, \beta, z, \widehat{z}]$. Our approach is to consider a *rewriting system* (we refer to [25] for a survey on the subject) for the tower of the ring algebra $A[F_n^*]$ where F_n^* is the free monoid in the generators $\{\sigma_1, \ldots, \sigma_{n-1}\}$. The method of proof is greatly inspired from [11].

First, one defines a graph which vertices are the elements of $A[F_n^*]$ and which edges correspond to elements differing by exactly one relation (from the set of relations defining $K_n(\alpha, \beta)$). One defines a reduction process for elements in $A[F_n^*]$ introducing the following orientations on some edges. The arrows show the orientation, if exactly one monomial is changed using one of the following rules

$$
\begin{aligned}
&a\sigma_j^3 b \to \alpha\, a\sigma_j^2 b + \beta\, a\sigma_j b + ab,\\
&a\sigma_{j+1}\sigma_j\sigma_{j+1} b \to a\sigma_j\sigma_{j+1}\sigma_j b,\\
&a\sigma_{j+1}\sigma_j^2\sigma_{j+1} b \to aS_j b,\\
&a\sigma_{j+1}\sigma_j^2\sigma_{j+1}^2 b \to aC_j b,\\
&a\sigma_{j+1}^2\sigma_j^2\sigma_{j+1} b \to aD_j b,
\end{aligned}
$$

where S_j, C_j and D_j are of the form $\sum_i P_i\, \sigma_j^{a_i}\sigma_{j+1}^{b_i}\sigma_j^{c_i}$, P_i are polynomials in α, β and $a_i, b_i, c_i \in \{0, 1, 2\}$. An element z is irreducible (or minimal) if there is not an element u such that $z \to u$. Several edges remain unoriented. They correspond to a change in a monomial of type

$$a\sigma_i\sigma_j b \to a\sigma_j\sigma_i b \text{ whenever } | i - j | > 1.$$

The reason for introducing the extra relations (which obviously hold in $H(Q, n)$) is to insure the existence of descending paths to some minimal points even if closed oriented loops may be found in the graph. Our aim is to show that this rewriting system is confluent, i.e. every connected component has at most one minimal element, up to unoriented equivalence (see Section 5.3.1). The existence is proved in Lemma 5.3.4 and we check the uniqueness by means of so-called Pentagon Lemma. Anyway, to show that the rewriting system is confluent, we shall enlarge our graph to a tower of graphs modeling not one algebra $K_n(\alpha, \beta)$ but the functionals on the whole tower $\cup_{n=2}^{\infty} K_n(\alpha, \beta)$ satisfying a recurrence condition which permits to reduce further the minimal elements. We will find that the Pentagon Lemma applies except a finite number of configurations that lie in $K_4(\alpha, \beta)$. Here the Colored Pentagon Lemma (see Section 5.3.2) can be applied and the problem is reduced to some algebraic computations. When we wish to check the commutativity condition for the functional to be actually a Markov trace, a constraint appears in $K_4(\alpha, \beta)$ on the variables $\alpha, \beta, z, \widehat{z}$. Then there are only two types of obstructions to the existence of a Markov traces:

- CPC obstructions (Colored Pentagon Condition), defining an ideal H in $\mathbb{Z}[\alpha, \beta, z, \widehat{z}]$.
- commutativity obstructions, implying a constraint on $\alpha, \beta, z, \widehat{z}$ (Section 5.4).

These finitely many obstructions have been checked by using the computer and all of them lie in the principal ideal generated by $H_{(\alpha,\beta)}$ (respectively $P^{(z,\delta)}$).

5.1.5 Properties of the invariants

In the next Section we will compute these obstructions and derive the existence of the two traces $\mathcal{T}_{(\alpha,\beta)}$ and $\mathcal{T}^{(z,\delta)}$. As explained in Chapter 4, given a Markov trace $\mathcal{T}$, we get a link invariant by setting:

$$I(\widehat{x}) = \left(\frac{1}{z\widehat{z}}\right)^{\frac{n-1}{2}} \left(\frac{\widehat{z}}{z}\right)^{\frac{e(x)}{2}} \mathcal{T}(x),$$

where $x \in \sigma_n$ is a braid representative of the link L and $e(x)$ is the exponent sum of x. Therefore we find two invariants $I_{(\alpha,\beta)}$ and $I^{(z,\delta)}$. We find that:

- they distinguish all knots with number crossing at most 10 that have the same HOMFLY-PT polynomial (and then they are independent from HOMFLY-PT). However, like HOMFLY-PT and Kauffman polynomials, they seem to not distinguish among mutants knots (in particular they don't separate Kinoshita-Terasaka and Conway knots).

- $I_{(\alpha,\beta)} = I_{(-\beta,-\alpha)}$ for amphicheiral knots, and $I_{(\alpha,\beta)}$ detects the chirality of all the knots with number crossing at most 10, where HOMFLY-PT and Kauffman polynomials fail.

- The invariant $I_{(\alpha,\beta)}$ is independent from the 2-cabling of HOMFLY-PT.

- $I_{(\alpha,\beta)}$ and $I^{(z,\delta)}$ have a *cubical* behaviour.

Let us explain briefly what we meant by *cubical behaviour.*

Definition 5.1.2 *A Laurent polynomial* $\sum_{j\in\mathbb{Z}} c_j a^j$ *is a* (n,k)*-polynomial (for* $n,k \in \mathbb{N}$*) if* $c_j = 0$ *for* $j \neq hn+k$*, for all* $h \in \mathbb{Z}$*.*

Remark 5.1.2

- *The HOMFLY-PT polynomial can be written as* $\sum_{k\in z} R_k(l)m^k$ *and respectively as* $\sum_{k\in z} S_k(m)l^k$*, where* $R_k(l)$ *and* $S_k(m)$ *are* $(2,k)$*-Laurent polynomials with* $R_{2k+1}(l) = S_{2k+1}(m) = 0$*.*

- *The Kauffman polynomial can be written as* $\sum_{k\in z} U_k(l)m^k$ *(respectively as* $\sum_{k\in z} T_k(m)l^k$*), where* $U_k(l)$ *and* $T_k(m)$ *are* $(2,k+1)$*-Laurent polynomials.*

In this respect the HOMFLY-PT and Kauffman polynomials have a quadratic behaviour.

Proposition 5.1.1 $I_{(\alpha,\beta)}$ *and* $I^{(z,\delta)}$ *have a cubical behaviour, i.e. for each link* L *there exists some* $l \in \{0,1,2\}$ *so that*

$$I_{(\alpha,\beta)}(L) = \frac{\sum_{k\in\mathbb{N}} P_k(\beta)\alpha^k}{\sum_{k\in\mathbb{N}} Q_k(\beta)\alpha^k} = \frac{\sum_{k\in\mathbb{N}} M_k(\alpha)\beta^k}{\sum_{k\in\mathbb{N}} N_k(\alpha)\beta^k},$$

where P_k, Q_k, M_k, N_k *are* $(3,k+l)$*-polynomials, and*

$$I^{(z,\delta)}(L) = \sum_{k\in\mathbb{Z}} H_k(\delta)z^k = \sum_{k\in\mathbb{Z}} G_k(z)\delta^k,$$

where H_k, G_k *are* $(3,k)$*-Laurent polynomials.*

5.2 Markov traces on $K_n(\alpha,\,\beta)$

5.2.1 A base for the cubic Hecke algebra $H(Q,3)$

Usually cubic Hecke algebras are defined as the quotients

$$H(Q,n) = \mathbb{C}[B_n]/(Q(\sigma_j)\,;\; j = 1,\dots,n-1)$$

of the group algebra of the braid group by the ideal generated by $Q(\sigma_j)$, cubic polynomial with parameters α and β, i.e. $Q(\sigma_j) = \sigma_j^3 - \alpha\,\sigma_j^2 - \beta\,\sigma_j - \gamma,\ \alpha,\,\beta,\,\gamma \in \mathbb{C}$. One may consider $\gamma = 1$ in the cubic polynomial since $H(Q,\infty)$ and $H(\gamma^{-1}Q,\infty)$ (the towers of cubic Hecke algebras) are isomorphic. In [43] it was shown that:

Proposition 5.2.1 *For all cubic polynomials Q with $Q(0) \neq 0$ one has $dim_{\mathbb{C}} H(Q,3) = 24$. A convenient base of the vector space $H(Q,3)$ is*
$e_1 = 1,\, e_2 = \sigma_1,\, e_3 = \sigma_1^2,\, e_4 = \sigma_2,\, e_5 = \sigma_2^2,\, e_6 = \sigma_1\sigma_2,\, e_7 = \sigma_2\sigma_1,\, e_8 = \sigma_1^2\sigma_2,\, e_9 = \sigma_2\sigma_1^2,\, e_{10} = \sigma_1\sigma_2^2,\, e_{11} = \sigma_2^2\sigma_1,\, e_{12} = \sigma_1^2\sigma_2^2,\, e_{13} = \sigma_2^2\sigma_1^2,\, e_{14} = \sigma_1\sigma_2\sigma_1,\, e_{15} = \sigma_1^2\sigma_2\sigma_1,\, e_{16} = \sigma_1\sigma_2\sigma_1^2,\, e_{17} = \sigma_1\sigma_2^2\sigma_1^2,\, e_{18} = \sigma_1^2\sigma_2\sigma_1^2,\, e_{19} = \sigma_1^2\sigma_2^2\sigma_1,\, e_{20} = \sigma_1\sigma_2^2\sigma_1,\, e_{21} = \sigma_1^2\sigma_2^2\sigma_1^2,\, e_{22} = \sigma_2\sigma_1^2\sigma_2,\; e_{23} = \sigma_2\sigma_1^2\sigma_2\sigma_1 = \sigma_1\sigma_2\sigma_1^2\sigma_2,\, e_{24} = \sigma_2\sigma_1^2\sigma_2\sigma_1^2 = \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1 = \sigma_1^2\sigma_2\sigma_1^2\sigma_2.$

We refer also to [43] for the following identities:

$$\sigma_{j+1}\sigma_j^2\sigma_{j+1}\sigma_j = \sigma_j\sigma_{j+1}\sigma_j^2\sigma_{j+1},$$
$$\sigma_{j+1}^2\sigma_j^2\sigma_{j+1} = \sigma_j\sigma_{j+1}^2\sigma_j^2 + \alpha(\sigma_{j+1}\sigma_j^2\sigma_{j+1} - \sigma_j\sigma_{j+1}^2\sigma_j) + \beta(\sigma_j^2\sigma_{j+1} - \sigma_j\sigma_{j+1}^2),$$
$$\sigma_{j+1}\sigma_j^2\sigma_{j+1}^2 = \sigma_j^2\sigma_{j+1}^2\sigma_j + \alpha(\sigma_{j+1}\sigma_j^2\sigma_{j+1} - \sigma_j\sigma_{j+1}^2\sigma_j) + \beta(\sigma_{j+1}\sigma_j^2 - \sigma_{j+1}^2\sigma_j).$$

5.2.2 The homogeneous quotient of rank 3

The quotient $P(\infty)$ of $H(Q,\infty)$ is homogeneous if any identity $F(\sigma_i, \sigma_{i+1}, ..., \sigma_j) = 0$, which holds in $P(\infty)$ remains valid under the translation of indices i.e. also $F(\sigma_{i+k}, \sigma_{i+k+1}, ..., \sigma_{j+k}) = 0$, for $k \in \mathbb{Z}, k \geq 1 - i$. One considers the Markov traces supported by the quotients $K_n(\alpha,\,\beta) = H(Q,n)/I_n$, where I_n is the (two-sided) ideal generated by:

$$\begin{aligned}&\sigma_j\sigma_{j-1}^2\sigma_j + (\beta^2-\alpha)\sigma_{j-1}^2\sigma_j^2\sigma_{j-1}^2 + (\alpha^2-\alpha\beta^2-\beta)\sigma_{j-1}\sigma_j^2\sigma_{j-1}^2 + (\alpha^2-\alpha\beta^2-\beta)\sigma_{j-1}^2\sigma_j^2\,\sigma_{j-1} + (\alpha^2 - \\ &\alpha\beta^2)\sigma_{j-1}^2\sigma_j\sigma_{j-1}^2 + (1+2\alpha\beta+\alpha^2\beta^2-\alpha^3)\sigma_{j-1}\sigma_j^2\sigma_{j-1} + (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)\sigma_{j-1}\sigma_j\sigma_{j-1}^2 + (1+\alpha\beta+ \\ &\alpha^2\beta^2-\alpha^3)\sigma_{j-1}^2\sigma_j\sigma_{j-1} + (1+2\alpha\beta-\beta^3)\sigma_j^2\sigma_{j-1}^2 + (1+2\alpha\beta-\beta^3)\sigma_{j-1}^2\sigma_j^2 + (\alpha\beta^3-2\alpha-2\alpha^2\beta)\sigma_j\sigma_{j-1}^2 + \\ &(\alpha\beta^3-2\alpha-2\alpha^2\beta)\sigma_{j-1}^2\sigma_j + (\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)\sigma_j^2\sigma_{j-1} + (\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)\sigma_{j-1}\sigma_j^2 + \\ &(\alpha^4-\alpha^3\beta^2-2\alpha^2\beta-3\alpha)\sigma_{j-1}\sigma_j\sigma_{j-1} + (2\alpha^3\beta+3\alpha^2-\alpha^2\beta^3-\alpha\beta^2)\sigma_j\sigma_{j-1} + (2\alpha^3\beta+3\alpha^2 - \\ &\alpha^2\beta^3-\alpha\beta^2)\sigma_{j-1}\sigma_j + (\beta^4-2\beta-3\alpha\beta^2+\alpha^2)\sigma_{j-1}^2 + (\beta^4-2\beta-3\alpha\beta^2+\alpha^2)\sigma_j^2 + (1+4\alpha\beta+ \\ &3\alpha^2\beta^2-\alpha^3-\alpha\beta^4-\beta^3)\sigma_{j-1} + (1+3\alpha\beta+3\alpha^2\beta^2-\alpha^3-\alpha\beta^4)\sigma_j + 3\beta^2-\beta^5-2\alpha-3\alpha^2\beta+4\alpha\beta^3,\end{aligned}$$

where $j = 1,\dots,n-1$. Then $K_\infty(\alpha,\,\beta)$ is a homogeneous quotient of $H(Q,\infty)$.

Remark 5.2.1 *$H(Q,3)$ is a semisimple algebra which decomposes generically as $\mathbb{C}^3 \oplus M_2^{\oplus 3} \oplus M_3$, where M_n is the algebra of $n\times n$ matrices. The morphism into $\mathbb{C}^3$ is obtained via the abelianization map and that into M_2 is part of the projection onto the quadratic Hecke algebra defined by a divisor of Q (which is $\mathbb{C}^2 \oplus M_2$). One identifies then $K_3(\alpha,\,\beta) \cong M_2^{\oplus 3} \oplus M_3$.*

In fact it suffices to show that the ideal I_3 is a vector space of dimension 3. Let R be the span of R_0, R_1, R_2, where

$R_0 := \sigma_2\sigma_1^2\sigma_2 + (\beta^2-\alpha)\,\sigma_1^2\sigma_2^2\sigma_1^2 + (\alpha^2-\alpha\beta^2-\beta)\sigma_1\sigma_2^2\sigma_1^2 + (\alpha^2-\alpha\beta^2-\beta)\sigma_1^2\sigma_2^2\sigma_1 + (\alpha^2-\alpha\beta^2)\sigma_1^2\sigma_2\sigma_1^2 + (1+2\alpha\beta+\alpha^2\beta^2-\alpha^3)\sigma_1\sigma_2^2\sigma_1 + (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)\sigma_1\sigma_2\sigma_1^2 + (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)\sigma_1^2\sigma_2\sigma_1 + (1+2\alpha\beta-\beta^3)\sigma_2^2\sigma_1^2 + (1+2\alpha\beta-\beta^3)\sigma_1^2\sigma_2^2 + (\alpha\beta^3-2\alpha-2\alpha^2\beta)\sigma_2\sigma_1^2 + (\alpha\beta^3-2\alpha-2\alpha^2\beta)\sigma_1^2\sigma_2 + (\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)\sigma_2^2\sigma_1 + (\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)\sigma_1\sigma_2^2 + (\alpha^4-\alpha^3\beta^2-2\alpha^2\beta-3\alpha)\sigma_1\sigma_2\sigma_1 + (2\alpha^3\beta+3\alpha^2-\alpha^2\beta^3-\alpha\beta^2)\sigma_2\sigma_1 + (2\alpha^3\beta+3\alpha^2-\alpha^2\beta^3-\alpha\beta^2)\sigma_1\sigma_2 + (\beta^4-2\beta-3\alpha\beta^2+\alpha^2)\sigma_1^2 + (\beta^4-2\beta-3\alpha\beta^2+\alpha^2)\sigma_2^2 + (1+4\alpha\beta+3\alpha^2\beta^2-\alpha^3-\alpha\beta^4-\beta^3)\sigma_1 + (1+3\alpha\beta+3\alpha^2\beta^2-\alpha^3-\alpha\beta^4)\sigma_2 + 3\beta^2-\beta^5-2\alpha-3\alpha^2\beta+4\alpha\beta^3,$

$R_1 := \sigma_1 R_0 = \sigma_1\sigma_2\sigma_1^2\sigma_2 - \beta\sigma_1^2\sigma_2^2\sigma_1^2 + (1+\alpha\beta)\sigma_1\sigma_2^2\sigma_1^2 + (1+\alpha\beta)\sigma_1^2\sigma_2^2\sigma_1 + (1+\alpha\beta)\sigma_1^2\sigma_2\sigma_1^2 (-\alpha^2\beta-2\alpha)\sigma_1\sigma_2^2\sigma_1 + (-\alpha^2\beta-2\alpha)\sigma_1\sigma_2\sigma_1^2 + (-\alpha^2\beta-2\alpha)\sigma_1^2\sigma_2\sigma_1 + (\beta^2-\alpha)\sigma_2^2\sigma_1^2 + (\beta^2-\alpha)\sigma_1^2\sigma_2^2 + (\alpha^2-\alpha\beta^2)\sigma_2\sigma_1^2 + (\alpha^2-\alpha\beta^2)\sigma_1^2\sigma_2 + (\alpha^2-\alpha\beta^2-\beta)\sigma_2^2\sigma_1 + (\alpha^2-\alpha\beta^2-\beta)\sigma_1\sigma_2^2 + (\alpha^3\beta+\beta+3\alpha^2)\sigma_1\sigma_2\sigma_1 + (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)\sigma_2\sigma_1 + (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)\sigma_1\sigma_2 + (1+2\alpha\beta-\beta^3)\sigma_1^2 + (1+2\alpha\beta-\beta^3)\sigma_2^2 + (\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)\sigma_1 + (\alpha\beta^3-2\alpha-2\alpha^2\beta)\sigma_2 + \beta^4-2\beta-3\alpha\beta^2+\alpha^2,$

$R_2 := \sigma_1 R_1 = \sigma_1^2\sigma_2\sigma_1^2\sigma_2 + \sigma_1^2\sigma_2^2\sigma_1^2 - \alpha\sigma_1\sigma_2^2\sigma_1^2 - \alpha\sigma_1^2\sigma_2^2\sigma_1 - \alpha\sigma_1^2\sigma_2\sigma_1^2 + \alpha^2\sigma_1\sigma_2^2\sigma_1 + (\alpha^2+\beta)\sigma_1\sigma_2\sigma_1^2 + (\alpha^2+\beta)\sigma_1^2\sigma_2\sigma_1 + (-\beta)\sigma_2^2\sigma_1^2 + (-\beta)\sigma_1^2\sigma_2^2 + (1+\alpha\beta)\sigma_2\sigma_1^2 + (1+\alpha\beta)\sigma_1^2\sigma_2 + (1+\alpha\beta)\sigma_2^2\sigma_1 + (1+\alpha\beta)\sigma_1\sigma_2^2 + (-\alpha^3\beta-\alpha\beta+1)\sigma_1\sigma_2\sigma_1 + (-\alpha^2\beta-2\alpha)\sigma_2\sigma_1 + (-\alpha^2\beta-2\alpha)\sigma_1\sigma_2 + (\beta^2-\alpha)\sigma_1^2 + (\beta^2-\alpha)\sigma_2^2 + (-\alpha\beta^2+\alpha^2-\beta)\sigma_1 + (-\alpha\beta^2+\alpha^2)\sigma_2 + 1 + 2\alpha\beta - \beta^3.$

Lemma 5.2.1 *As vector spaces $R \cong I_3$ in $H(Q,3)$.*

Proof: Remark that

$$\sigma_1 R_0 = R_0\sigma_1 = R_1,\ \sigma_1 R_1 = R_1\sigma_1 = R_2,\ \sigma_1 R_2 = R_2\sigma_1 = R_0 + \beta R_1 + \alpha R_2.$$

and after some messy computations (computer aided) we obtain that

$$\sigma_2 R_0 = R_0\sigma_2 = R_1,\ \sigma_2 R_1 = R_1\sigma_2 = R_2,\ \sigma_2 R_2 = R_2\sigma_2 = R_0 + \beta R_1 + \alpha R_2.$$

From these relations we find that $xR_0y \in R$ for all $x, y \in H(Q,3)$, hence $I_3 \subset R$. The other inclusion is trivial. □

5.2.3 Uniqueness of Markov trace on $K_n(\alpha,\,\beta)$

From now on we will work with the group ring $\mathbb{Z}[\alpha,\beta][B_\infty]$ instead of $\mathbb{C}[B_\infty]$.

Definition 5.2.1 *Let $z, \widehat{z} \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$, R be a $\mathbb{Z}[\alpha,\beta,z,\widehat{z}]$ −module and H an ideal of R.*
i) $\mathcal{T}$ is an admissible functional on $K_\infty(\alpha,\,\beta)$ (taking values in R/H) if the following conditions are fulfilled:

$$\mathcal{T}(x\sigma_n y) = z\mathcal{T}(xy) \quad \text{for any } x, y \in K_n(\alpha,\,\beta),$$

$$\mathcal{T}(x\sigma_n^{-1} y) = \widehat{z}\mathcal{T}(xy) \quad \text{for any } x, y \in K_n(\alpha,\,\beta).$$

ii) An admissible functional $\mathcal{T}$ is a Markov trace if

$$\mathcal{T}(ab) = \mathcal{T}(ba) \ \text{ for any } a, b \in K_n(\alpha,\,\beta).$$

Remark 5.2.2 *Markov traces on the quadratic Hecke algebras (see [58]) have the following multiplicative property:* $\mathcal{T}(x\sigma_n) = \mathcal{T}(x)\mathcal{T}(\sigma_n)$, *for* $x \in H(Q,n)$, *which implies that:* $\mathcal{T}(xy) = \mathcal{T}(x)\mathcal{T}(y)$, *for* $x \in H(Q,n), y \in < 1, \sigma_n, \sigma_{n+1}, ..., \sigma_{n+k} >$.

However we cannot expect that this property will extend to higher level algebras and Markov traces on them.

Definition 5.2.2 *The Markov trace* $\mathcal{T}$ *is multiplicative if* $\mathcal{T}(x\sigma_n^k) = \mathcal{T}(x)\mathcal{T}(\sigma_n^k)$ *holds when* $x \in H(Q,n), k \in \mathbb{Z}$.

Remark 5.2.3 *In the case of cubic Hecke algebras the Markov traces are multiplicative. In fact using the identity* $\sigma_n^2 = \alpha\sigma_n + \beta + \sigma_n^{-1}$ *we derive then the multiplicativity for* $k = 2$, *and by recurrence for all* k. *In particular if* $\mathcal{T}$ *is a Markov trace it follows that* $\mathcal{T}(a\sigma_n^2 b) = t\mathcal{T}(ab)$ $a, b \in B_n$, *where* $t = \alpha z + \beta + \widehat{z}$.

One can state now the unique extension property of Markov traces.

Proposition 5.2.2 *For fixed* $(z,t) \in (\mathbb{C}^*)^2$ *there exists at most one Markov trace on* $K_n(\alpha, \beta)$ *with parameters* (z,t).

Proof: Define recursively the modules L_n by

$L_2 = H(Q,2)$,

$L_3 = \mathbb{C} < \sigma_1^i \sigma_2^j \sigma_1^k;\ i,j,k \in \{0,1,2\} >$,

$L_{n+1} = \mathbb{C} < a\sigma_n^{\varepsilon} b \,|\, a, b \in$ basis of $L_n,\ \varepsilon \in \{1,2\} > \oplus L_n$.

We need the following result

Lemma 5.2.2 *Under the natural projection* π *on* $K_n(\alpha, \beta)$, L_n *surjects onto* $K_n(\alpha, \beta)$.

Proof: For $n = 2$ it is clear. For $n = 3$ we know that $\sigma_2\sigma_1^2\sigma_2,\ \sigma_1\sigma_2\sigma_1^2\sigma_2,\ \sigma_1^2\sigma_2\sigma_1^2\sigma_2 \in \pi(L_3)$.

Consider now $w \in K_{n+1}(\alpha, \beta)$ represented by a word in the σ_i's having only positive exponents. We assume that the degree of the word in the variable σ_n is minimal among all linear combinations of words (with positive exponents) representing w.

If the degree is less or equal to 1 there is nothing to prove.

If the degree is 2 then either $w = u\sigma_n^2 v$, $u, v \in K_n(\alpha, \beta)$ so using the induction hypothesis we are done, or else $w = u\sigma_n z\sigma_n v$, where $u, z, v \in K_n(\alpha, \beta)$. Therefore $z = x\sigma_{n-1}^{\varepsilon} y$ where $x, y \in K_{n-1}(\alpha, \beta)$ by the induction and $\varepsilon \in \{0,1,2\}$. If $\varepsilon = 0$ then w can be reduced to $uz\sigma_n^2 v$. If $\varepsilon = 1$ then $w = u\sigma_n x\sigma_{n-1} y\sigma_n v = ux\sigma_{n-1}\sigma_n\sigma_{n-1} yv$ hence the degree of w can be lowered by 1, which contradicts our assumption. If $\varepsilon = 2$ then $w = ux\sigma_n\sigma_{n-1}^2\sigma_n yv$. One derives

$\sigma_n\sigma_{n-1}^2\sigma_n \in \mathbb{C} < \sigma_{n-1}^i \sigma_n^j \sigma_{n-1}^k,\ i,j,k \in \{0,1,2\} >$,

hence we reduced the problem to the case when w is a word of type $u'\sigma_n^2 v'$.

If the degree of w is at least 3 we will contradict the minimality. In fact w contains either a subword $w' = \sigma_n^a u\sigma_n^b$, $u \in K_n(\alpha, \beta)$ and $a + b \geq 3$, or else a subword $w'' = \sigma_n u\sigma_n v\sigma_n$, $u, v \in K_n(\alpha, \beta)$.

In the first case using the induction we can write $u = x\sigma_{n-1}^{\varepsilon} y$, $x, y \in K_{n-2}(\alpha, \beta)$.

If $\varepsilon = 0$ then $w' = \sigma_n^{a+b} xy = \alpha\sigma_n^{a+b-1} xy + \beta\sigma_n^{a+b-2} xy + \sigma_n^{a+b-3} xy$, hence the degree of w can be lowered by 1.

If $\varepsilon = 1$ then $w' = \sigma_n^{a-1} x\sigma_n\sigma_{n-1}\sigma_n y\sigma_n^{b-1} = \sigma_n^{a-1} x\sigma_{n-1}\sigma_n\sigma_{n-1} y\sigma_n^{b-1}$, and again its degree can be reduced by one unit.

If $\varepsilon = 2$ then a or b equals 2. Set $a = 2$. We can write
$w' = x\sigma_n^2\sigma_{n-1}^2\sigma_n y\sigma_n^{b-1} = x\sigma_{n-1}\sigma_n^2\sigma_{n-1}^2 y\sigma_n^{b-1} + \alpha(\sigma_n\sigma_{n-1}^2\sigma_n - \sigma_{n-1}\sigma_n^2\sigma_{n-1})y\sigma_n^{b-1} +$
$+\beta(\sigma_{n-1}^2\sigma_n - \sigma_{n-1}\sigma_n^2)y\sigma_n^{b-1}$.
still contradicting the minimality of the degree of w.

In the second case we can write also $u = x\sigma_{n-1}^{\varepsilon}y$, $v = r\sigma_{n-1}^{\delta}s$ with $x, y, r, s \in K_{n-1}(\alpha,\, \beta)$.

If ε or δ equals 1 then, after some obvious commutation the word w'' contains the subword $\sigma_n\sigma_{n-1}\sigma_n$ which can be replaced by $\sigma_{n-1}\sigma_n\sigma_{n-1}$ hence lowering its degree.

If $\varepsilon = \delta = 2$ then $w'' = x\sigma_n\sigma_{n-1}^2\sigma_n yr\sigma_{n-1}^2\sigma_n s$. We use the homogeneity to replace $\sigma_n\sigma_{n-1}^2\sigma_n$ by a sum of elements of type $\sigma_{n-1}^i\sigma_n^j\sigma_{n-1}^k$. Each term of the expression of w'' which comes from a factor having $j < 2$ has the degree less than it had before. The remaining terms are $x\sigma_{n-1}^i\sigma_n^2\sigma_{n-1}^k yr\sigma_{n-1}^2\sigma_n s$, so they contains a subword $\sigma_n^2 u\sigma_n$ whose degree we already know that it can be reduced as above. This proves our claim. □

Now the Markov traces $\mathcal{T}$ on $H(Q, \infty)$ are multiplicative hence $\mathcal{T}(x\sigma_n^{\varepsilon}y) = \mathcal{T}(\sigma_n^{\varepsilon})\mathcal{T}(yx)$ holds, and $K_n(\alpha,\, \beta)$ it is an algebra hence $yx \in K_n(\alpha,\, \beta)$. Therefore the extension of $\mathcal{T}$, by recursion, from $K_n(\alpha,\, \beta)$ to $K_{n+1}(\alpha,\, \beta)$ if ever exists it is unique. This ends the proof of our proposition. □

5.3 CPC Obstructions

5.3.1 The pentagonal condition

The following Lemma is also a consequence of the previous one:

Lemma 5.3.1 *There is a surjection of* $(K_n(\alpha,\, \beta), K_n(\alpha,\, \beta))$*-bimodules*
$K_n(\alpha,\, \beta) \oplus K_n(\alpha,\, \beta) \otimes_{K_{n-1}(\alpha,\, \beta)} K_n(\alpha,\, \beta) \oplus K_n(\alpha,\, \beta) \otimes_{K_{n-1}(\alpha,\, \beta)} K_n(\alpha,\, \beta) \longrightarrow K_{n+1}(\alpha,\, \beta)$
given by
$x \oplus y \otimes z \oplus u \otimes v \to x + y\sigma_n z + u\sigma_n^2 v.$

In particular admissible functionals are unique up to the choice of $\mathcal{T}(1) \in R$. Look now at the algebra $K_*(\alpha,\, \beta)$. We wish to use the following transformations on the words (one way):
(C0)(j+1) $a\sigma_{j+1}^3 b \to aE_{j+1}b$,
(C1)(j) $a\sigma_{j+1}\sigma_j\sigma_{j+1}b \to a\sigma_j\sigma_{j+1}\sigma_j b$,
(C2)(j) $a\sigma_{j+1}\sigma_j^2\sigma_{j+1}b \to aS_j b$,
(C12)(j) $a\sigma_{j+1}\sigma_j^2\sigma_{j+1}^2 b \to aC_j b$,
(C21)(j) $a\sigma_{j+1}^2\sigma_j^2\sigma_{j+1}b \to aD_j b$,
where $E_{j+1} = \alpha\sigma_{j+1}^2 + \beta\sigma_{j+1} + 1$, $S_j = \sigma_{j+1}\sigma_j^2\sigma_{j+1} - R_{(0,j)}$, $C_j = \sigma_j^2\sigma_{j+1}^2\sigma_j + \alpha(\sigma_{j+1}\sigma_j^2\sigma_{j+1} - \sigma_j\sigma_{j+1}^2\sigma_j) + \beta(\sigma_{j+1}\sigma_j^2 - \sigma_{j+1}^2\sigma_j)$ and $D_j = \sigma_j\sigma_{j+1}^2\sigma_j^2 + \alpha(\sigma_{j+1}^2\sigma_j^2\sigma_{j+1}^2 - \sigma_j\sigma_{j+1}^2\sigma_j) + \beta(\sigma_j^2\sigma_{j+1} - \sigma_j\sigma_{j+1}^2)$, $j = 0, \ldots, n-2$. Our aim is to reduce the degree of σ_{n-1} as much as possible in $K_n(\alpha,\, \beta)$. According to the previous Lemma every word in $K_n(\alpha,\, \beta)$ is equivalent to a sum of words of type $\sum_i x_i\sigma_{n-1}^{\varepsilon_i}y_i$. Unfortunately we are forced to use also the transformations
$\sigma_i\sigma_j \leftrightarrow \sigma_j\sigma_i$ for $\mid i - j \mid > 1$,
which have to be used in both directions.
Assume this is the reduction process we want to carry out. We eventually obtain a sum $\sum_i x_i\sigma_{n-1}^{\varepsilon_i}y_i$ with $x_i, y_i \in K_{n-1}(\alpha,\, \beta)$. Of course this *normal form* for the word we started with is not unique since we may perform again permutations of its letters in each term. But if any two such normal forms are equivalent under permutations of its letters (i.e. of the letters

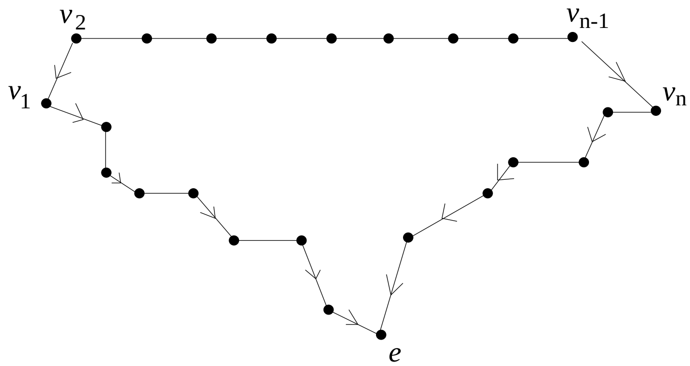


Figure 5.2: The pentagon condition

$\sigma_i\sigma_j$ with $\mid i-j \mid > 1$) then we will get an almost canonical description of the basis of $K_n(\alpha, \beta)$. Indeed the last assumption is equivalent to say that the surjection of the previous Lemma is an isomorphism. Unfortunately this is not the case. One can however obtain the obstructions to the uniqueness of this almost canonical form as follows.

We return now to the module of admissible functionals. The last group of relations enables us to make a further reduction, namely

$a\sigma_{n-1}b \to z\, ab$,

$a\sigma_{n-1}b \to t\, ab$.

This way we can reduce a word to a linear combination of words lying in $K_{n-1}(\alpha, \beta)$. Assume that we are using a recurrence on n. This means that each element of $K_{n-1}(\alpha,\beta)$ can be uniquely reduced to an element of R (the value of the functional on the element). Thus it suffices to check the obstructions directly on the values in order to obtain that the functional is well-defined. One formalizes this at follows. Let Γ be a semi-oriented graph. This means that some of its edges are oriented and the remaining ones are unoriented. A path $v_1v_2...v_n$ is a semi-oriented path if either $v_j \to v_{j+1}$ or else v_jv_{j+1} is unoriented, for all j. If all edges of the chain are unoriented we say that its endpoints are unoriented equivalents.

Definition 5.3.1 *$v_1v_2...v_n$ is an open pentagon configuration in Γ (o.p.c.) if $v_2 \to v_1$, $v_2v_3...v_{n-1}$ is an unoriented path and $v_{n-1} \to v_n$.*

We state first the *pentagon condition* for semi-oriented graphs:

Definition 5.3.2 *Γ verifies the pentagon condition (PC) if for any open pentagon configuration $v_1v_2...v_n$ there exist semi-oriented paths $v_1x_1x_2...x_me$ and $v_ny_1y_2...y_pe$ having the same endpoint (see Figure 5.2).*

Set now $x \leq y$ if there exists a semi-oriented path from y to x in Γ. Of course $\leq$ is not always a partial order relation. A necessary and sufficient condition is that no closed semi-oriented loops exist in Γ. One says that x is minimal if $y \leq x$ implies that y is unoriented equivalent to x.

Lemma 5.3.2 *Suppose that the (PC) holds. If a connected component C of the graph Γ has a minimal element m_C then it is unique up to unoriented equivalence.*

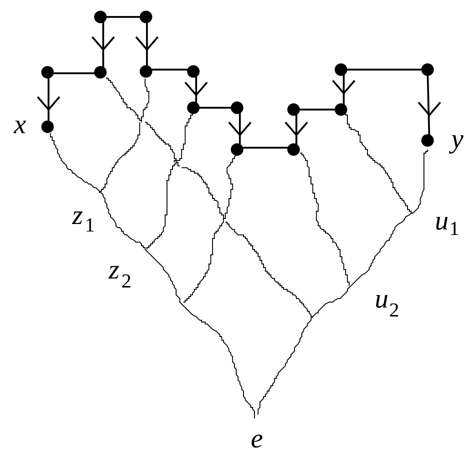


Figure 5.3: Proof of Pentagon Lemma

Proof: Consider two minimal elements x and y which lie in C. Then there exists some path $xx_0x_1...x_ny$ joining them. Since x is minimal the closest oriented edge (if ever exists) is ingoing, and the same is true for y. If this path is not unoriented again from minimality there are at least two oriented edges. Therefore open pentagon configurations (i.e. those configurations where (PC) applies) exist. We apply then (PC) iteratively whenever such configurations exist or has appeared. When this process stops we find two semi-oriented $xz_1z_2...z_pe$ and $yu_1u_2...u_se$ having the same endpoint e. So $e \leq x$ and $e \leq y$. Again from minimality these paths must be unoriented so x and y are unoriented equivalent (see Figure 5.3). □

Remark 5.3.1 *A priori one cannot say too much about the existence of such minimal elements. If $\leq$ had been a partial order with descending chain condition then the existence of minimal elements would be standard. However in the present case even if $\leq$ is not a partial order the existence of minimal elements can be established.*

5.3.2 The colored pentagon condition: the definition of Γ_n

Suppose now we have a sequence of disjoint graphs Γ_n. In every Γ_n there exists a distinguished subset of vertices V_n^0 which are minimal elements in their connected components. Suppose that each connected component admits at least one minimal element. Each such vertex from V_n^0 has exactly one outgoing edge going to a vertex of Γ_{n-1}. We color these new edges in red. Set Γ_n^* for the union of all Γ_j, $j \leq n$ and with the red edges added in each rank j.

Definition 5.3.3 Γ_n^* *is* coherent *if any connected component of* Γ_n *has an unique minimal element (with respect to* Γ_n^**) in* Γ_0 *up to unoriented equivalence.*

We state now the *colored* version of the Pentagon Lemma for this type of graphs.

Definition 5.3.4 *We say that* Γ_n *verifies the colored pentagon condition (CPC) if for any open pentagon configuration* $v_1v_2...v_n$ *in* Γ_n *there exist bicoloured semi-oriented paths (in* Γ_n^**) from* v_1 *and* v_n *having the same endpoint. In addition if* xy *is an unoriented edge in* Γ_n *with* $x, y \in V_n^0$ *then there exist semi-oriented paths in* Γ_n^* *starting from* x *and* y *with red edges and having the same endpoint (see the Figure 5.4).*

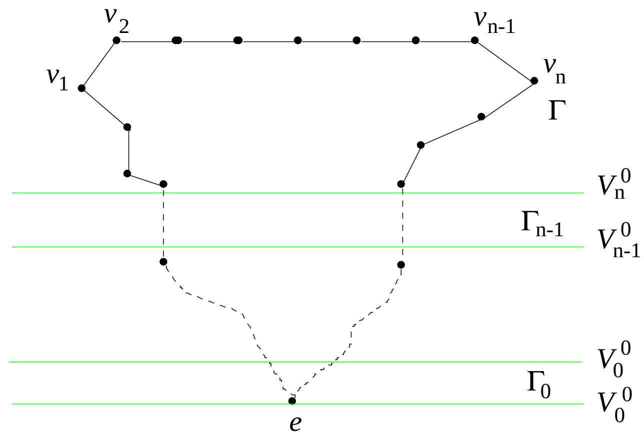


Figure 5.4: The colored Pentagon Condition

Lemma 5.3.3 *Suppose that Γ^*_{n-1} is coherent and the (CPC) condition is fulfilled. Then Γ^*_n is coherent.*

Proof: The proof is similar to that of Pentagon Lemma. □

Now we are ready define our graph Γ_n. Its vertices are the elements of the ring algebra $\mathbb{Z}[\alpha,\beta,z,\widehat{z}]F^*_n$, where F^*_n is the free monoid in the n letters $\{\sigma_1,\sigma_2,...,\sigma_n\}$. The vertices of Γ_0 will be the elements of $\mathbb{Z}[\alpha,\beta,z,\widehat{z}]$. Two vertices $v=\sum_i \alpha_i x_i$ and $w=\sum_i \beta_i y_i$, $\alpha_i,\beta_i \in \mathbb{Z}[\alpha,\beta,z,\widehat{z}]$ and x_i, $y_i \in F^*_n$, are related by an oriented edge if exactly one monomial of v is changed following one of the rules
(C0)(j) $a\sigma_j^3 b \to aE_j b$,
(C1)(j) $a\sigma_{j+1}\sigma_j\sigma_{j+1}b \to a\sigma_j\sigma_{j+1}\sigma_j b$,
(C2)(j) $a\sigma_{j+1}\sigma_j^2\sigma_{j+1}b \to aS_j b$,
(C12)(j) $a\sigma_{j+1}\sigma_j^2\sigma_{j+1}^2 b \to aC_j b$,
(C21)(j) $a\sigma_{j+1}^2\sigma_j^2\sigma_{j+1}b \to aD_j b$,
where E_j,S_j,C_j,D_j as above. An unoriented edge between v and w corresponds to a change in a monomial of v of type
(P_{ij}) $a\sigma_i\sigma_j b \to a\sigma_j\sigma_i b$ whenever $\mid i-j \mid > 1$.
Remark that the use of (C12) and (C21) is somewhat ambiguous since we may always use (C2) for a subword. Their role is to break in some sense the closed oriented loops in Γ_n. Consider now the following sets of words in the σ_i's:
$W_0=\{1\}$,
$W_{n+1}=W_n\cup W_n\sigma_{n+1}Z_n\cup W_n\sigma_{n+1}^2 Z_n$.
$Z_n=\{\sigma_n^{i_0}\sigma_{n-1}^{i_1}...\sigma_{n-p}^{i_p}; i_1,i_2,...,i_p\in\{1,2\}, p=0,n-1\}$.
Let V^0_n be the set of vertices corresponding to elements of the $\mathbb{Z}[\alpha,\beta,z,\widehat{z}]$-module generated by W_n.

Lemma 5.3.4 *Each connected component of Γ_n has a minimal element in V^0_n, not necessarily unique.*

Proof: We prove our claim by induction on n. For $n=0$ it is obvious. Let now w be a word in the σ_i's having only positive exponents. If its degree in σ_n is zero or one we apply the induction hypothesis and we are done. If the degree is 2 and it contains the subword σ_n^2 we are able to apply the induction hypothesis. One can also suppose that no exponents greater than 2 occur by using (C0) several times. If the degree is 2 then $w = x\sigma_n y\sigma_n z$ with $x, y, z \in F_{n-1}^*$. The induction applied to y implies that $w \geq x\sigma_n a\sigma_{n-1}^{\varepsilon} bz$ with $a, b \in F_{n-1}^*$. Then several transforms of type (P_{nj}) and (Cε) will do the job. Consider now that the degree is strictly greater than 2. So we have a subword of type

$\sigma_n^{\alpha} x \sigma_n^{\beta}$ with $3 \leq \alpha + \beta \leq 4$

or else one of the type

$\sigma_n x \sigma_n y \sigma_n$.

The second case reduce to the first one as above. Next say that $x \geq a\sigma_{n-1}^{\varepsilon} b$, $a, b \in F_{n-2}^*$. Several applications of (P_{nj}) leads us to consider the word $\sigma_n^{\alpha}\sigma_{n-1}^{\varepsilon}\sigma_n^{\beta}$. If $\varepsilon = 1$ we apply two times (C1) and we are done. Otherwise we shall apply (C$\alpha\beta$) and then (C1) if $\alpha \neq \beta$ or both (C12) and (C21) and then (C1) if $\alpha = \beta = 2$. This proves that every vertex descends to V_n^0. But these vertices have not outgoing edges as can be easily seen. When we use the unoriented edges some new vertices have to be added. But it is easy to see that these also does not have outgoing edges. Since any vertex has a semi-oriented path ending in V_n^0 we are done. □

Remark 5.3.2 *The moves (C12) and (C21) are really necessary for the conclusion of the previous Lemma to remain valid. For instance look at the case* $\alpha = \beta = 0$. *From* $\sigma_{j+1}\sigma_j^2\sigma_{j+1}^2$ *only (C2) can be applied and in the linear combination we obtain the factor* $\sigma_{j+1}^2\sigma_j^2\sigma_{j+1}$. *If we continue, then at each stage we shall find one of these two monomials. When all possible reductions are performed at the second stage we recover the word* $\sigma_{j+1}\sigma_j^2\sigma_{j+1}^2$ *so we have a closed oriented loop in the graph. This connected component has no minimal element unless we enlarge the graph by using the extra transformations (C12) and (C21). For general* α, β *a similar argument holds, and it can be checked by a computer program. If one does not use (C12) or (C21) then the reduction process for* $\sigma_{j+1}\sigma_j^2\sigma_{j+1}^2$ *yields at the sixth stage a sum of words generating an oriented loop.*

5.3.3 The bicoloured graph $\Gamma_n^*(H)$: the sub-module H

We are able now to define the bicoloured graph $\Gamma_n^*(H)$. The red edges are defined as follows. Each minimal vertex $v = \sum_{i,k} \alpha_{(i,k)} x_{(i,k)} \sigma_n^k y_{(i,k)}$, where $k = 0, 1, 2$, is joined by a red edge to $w = \sum_{i,k} \alpha_{(i,k)} u_k x_{(i,k)} y_{(i,k)}$, which is a vertex of Γ_{n-1}, where we set $u_0 = 1, u_1 = z, u_2 = t$. Finally $\Gamma_0(H)$ is the graph having the vertices corresponding to the module R and two vertices are connected by an unoriented edge iff the corresponding elements lie in the same coset of R/H, H being a certain submodule of R. The submodule H is necessary because going on different descending paths we might obtain different elements. Then, we have to find whether there exists H so that $\Gamma_n^*(H)$ is coherent.

We will test the conditions of coherence of each $\Gamma_n^*(H)$ by recurrence on n. Notice that for $n = 1, 2$ there are no conditions on H. Our strategy is to make use of the Colored Pentagon Lemma in the following way. For those configurations that we cannot prove the (PC) holds directly we shall check that the (CPC) (which is weaker since it concerns all the tower $\Gamma_n^*(H)$) is verified.

Consider an open pentagon configuration (abbreviated o.p.c.) $[w_0, w_1, ..., w_n]$. This means that $w_1 \to w_0$, $w_1, ..., w_{n-1}$ are unoriented equivalent and $w_{n-1} \to w_n$. We say that this o.p.c. is irreducible if none of the vertices $w_1, w_2, ..., w_{n-1}$ has an outgoing edge.

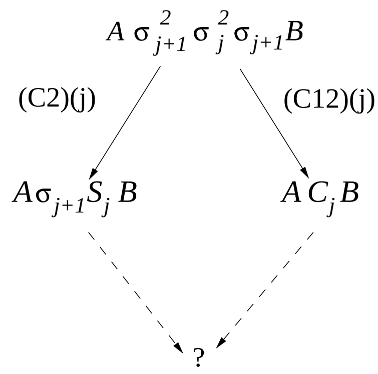


Figure 5.5: The o.p.c. for $A\sigma_{j+1}^2\sigma_j^2\sigma_{j+1}B$

Lemma 5.3.5 *i) In order to verify (PC) it suffices to restrict to irreducible configurations.*

ii) It suffices to verify (PC) only for monomials from F_n^*.

iii) Suppose $w_j' = Aw_jB$, for $j = 0, \ldots, n$ (so A, B are not touched by any transform) in the o.p.c..

If (PC) holds for $[w_0, w_1, ..., w_n]$ it also holds for $[w_0', w_1', ..., w_n']$.

iv) Suppose that (PC) holds for $[w_0, w_1, ..., w_n]$ and for $[y_0, y_1, ..., y_m]$. Then for all A, B, C the (PC) is valid also for

$[Aw_0By_1C, Aw_1By_1C, ..., Aw_{n-1}By_1C, Aw_{n-1}By_2C, Aw_{n-1}By_3C, ...$
$.., Aw_{n-1}By_{m-1}C, Aw_{n-1}By_mC]$.

In fact when we fix the endpoints of the o.p.c. we can mix the unoriented edges of each subjacent o.p.c. in any order we want. Let $(i_k, j_k) \in \{0, 1, ..., n\} \times \{0, 1, ..., m\}, k = 1, p$ such that $i_0 = 0 < i_1 \leq i_2 \leq ... \leq i_p, j_p = m > j_{p-1} \geq ... \geq 0$, and $i_{k+1} - i_k + j_{k+1} - j_k = 1$ for all k. Then the o.p.c. $[Aw_{i_0}By_{j_0}C, Aw_{i_1}By_{j_1}C, ..., Aw_{i_p}By_{j_p}C]$ fulfills the (PC).

Proof: i) We may always decompose a configuration into irreducible ones and iterate the construction.

ii) The reduction transforms on different monomials commute with each other so we are done.

iii) Obvious.

iv) The reductions of x_{n-1} and y_1 commute again with each other. □

Thus the top line of a o.p.c. corresponds to a word w_1 and a sequence of permutations of its letters giving in order $w_2, w_3, ..., w_{n-1}$. We may suppose that $w = w_1$ has no proper subwords w_1' which fulfill the following two conditions:

i) Set $w = Aw'B$. Then each of the considered permutations acts only on the letters of A, of B or w'. Thus the transform w" of w' is equivalent to w'.

ii) The reduction transforms performed at w_1 and w_2 acts actually on w' and w".

Now we can study the (PC) for irreducible configurations as in Lemma 5.3.5.
The first step is to check if the (PC) condition holds when the top line is trivial ($n = 2$) and there are two or more outgoing edges. For instance, see Figure 5.5. A and B are subwords not touched by reductions and on the subword $\sigma_{j+1}^2\sigma_j^2\sigma_{j+1}$ one can apply (Cij) or (C2). C_j and S_j are as in Section 5.3.1.

Lemma 5.3.6 *If the top line is trivial then the (PC) holds.*

Proof: By Lemma 5.3.5 we have a finite number of cases to test. These are the words of the form abc, where ab and bc are subwords belong to the set $\{\sigma_{j+1}^3, \sigma_{j+1}\sigma_j\sigma_{j+1}, \ \sigma_{j+1}\sigma_j^2\sigma_{j+1}, \ \sigma_{j+1}^2\sigma_j^2\sigma_{j+1}, \ \sigma_{j+1}\sigma_j^2\sigma_{j+1}^2\}$, $j = 1, \ldots, n-2$. The number of cases to study can be easily reduced, since

- If b is the identity, the (PC) trivially holds.
- By homogeneity of the reductions (C ε)(j) it suffices to consider $j = 1$.
- For a word $w = w_1, \ldots, w_l$ its symmetric is the word $w^* = w_l, \ldots, w_1$. If the (PC) holds for abc, (PC) holds also for the symmetric word $(abc)^*$ (this result follows from the form of reductions).
- Several cases, as $\sigma_{j+1}^3\sigma_j\sigma_{j+1}$, can be easily tested at hand.

The non trivial cases appear when a (Cij)-move (and then a (C2)-move) can be applied. Actually, we have to check only $\sigma_{j+1}^2\sigma_j^2\sigma_{j+1}$, since $\sigma_{j+1}\sigma_j^2\sigma_{j+1}^2$ is its symmetric and the cases $\sigma_{j+1}^{\varepsilon_1}\sigma_j^2\sigma_{j+1}^{\varepsilon_2}$ ($\varepsilon_i = 2, 3$) are consequences of these ones. Then, we start from the situation depicted in Figure 5.5 (A, B are empty words). If we apply (Cij) whether is possible on $\sigma_{j+1}S_j$, after a long and messy computation we find the same minimal element associated to C_j. □

Remark 5.3.3 *Using a computer program one can get the oriented graph associated to the word $\sigma_{j+1}^2\sigma_j^2\sigma_{j+1}$ (Figure 5.6). The vertices are of the type $\sum c_j w_j$, c_j polynomials in α, β and w_j words in σ_j, σ_{j+1}. An oriented edge between an outgoing vertex a and an ingoing vertex b indicates that the reduction procedure applied to a yields b. When there are no subwords $\sigma_{j+1}^2\sigma_j^2\sigma_{j+1}$ or $\sigma_{j+1}\sigma_j^2\sigma_{j+1}^2$ the edges are spotted. As we already noticed in Remark 5.3.2, if we apply six times the procedure without (Cij) we find a loop.*

Let us study the case when the top line is non trivial. By Lemma 5.3.6 we can suppose that w_1 and w_{n-1} have each one exact one outgoing edge. In particular, when a (Cij)-move can be applied, we choose always the edge (Cij) in the reduction process.
Now the top line is determined by the sequence of transpositions of the letters of w. Let l be the length of w. Otherwise this is the same to giving a permutation $\sigma \in S_l$ with a prescribed decomposition into transpositions. Set T_j for the transposition which interchanges the letters on the positions j and $j+1$. Notice that for a fixed w not all σ are suitable. In fact only a subset of the group of permutations, which we call permitted, may work. Say $P(w)$ is the set of permitted permutations. If $e_w : \{1, 2, ..., l\} \longrightarrow \{1, 2, ..., n-1\}$ is the evaluation map

$e_w(j)$ =index of the letter lying in position j on w

then $T_j\sigma$ is permitted (where $\sigma \in P(w)$) iff

$\mid e_{\sigma(w)}(j) - e_{\sigma(w)}(j+1) \mid > 1$.

Say that two permitted permutations σ and σ' are equivalent if for the o.p.c. corresponding to σ and σ' the (PC) is valid or not for both in same time.

Lemma 5.3.7 *i) Suppose that $\sigma_1 T_j T_i \sigma_2 \in P(w)$, $\mid i - j \mid > 1$. Then $\sigma_1 T_i T_j \sigma_2 \in P(w)$ and these two permutations are equivalent.*

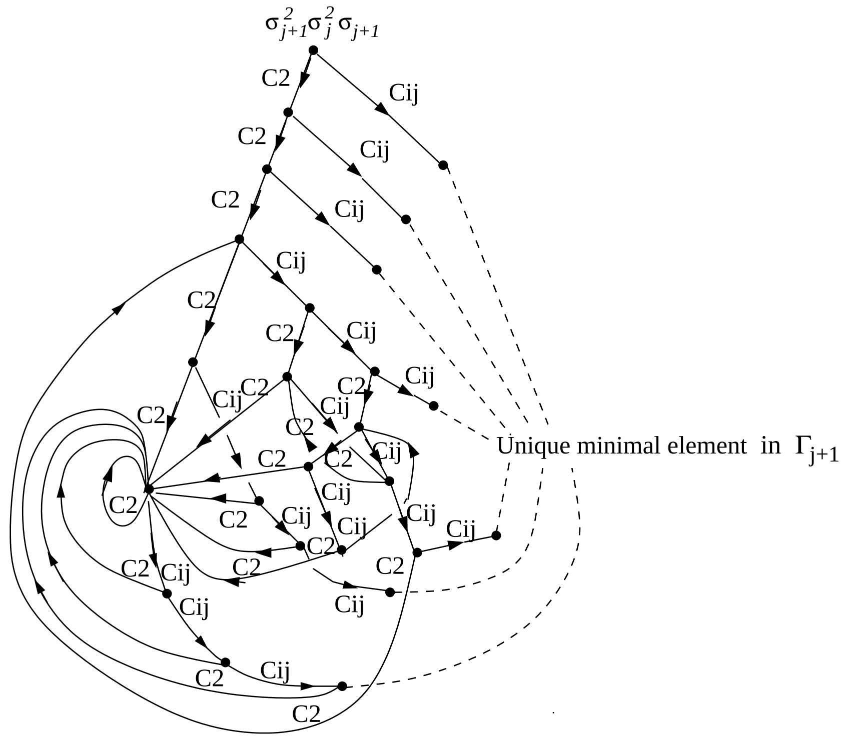


Figure 5.6: The graph underlying to $\sigma_{j+1}^2\sigma_j^2\sigma_{j+1}$

ii) Suppose that $\sigma_1 T_{i+1} T_i T_{i+1} \sigma_2 \in P(w)$. *Then* $\sigma_1 T_i T_{i+1} T_i \sigma_2 \in P(w)$ *and these two permutations are equivalent. The converse is still true.*

iii) If $\sigma_1 T_i^2 \sigma_2 \in P(w)$ *then* $\sigma_1\sigma_2$ *is permitted and equivalent to the previous one.*

Proof: The existence in the first case is equivalent to $\mid e_{\sigma_2(w)}(j) - e_{\sigma_2(w)}(j+1) \mid > 1$ and $\mid e_{\sigma(w)}(i) - e_{\sigma(w)}(i+1) \mid > 1$, so it is symmetric. In the second case also it is equivalent to $\mid e_{\sigma_2(w)}(j+\varepsilon_1) - e_{\sigma_2(w)}(j+\varepsilon_2) \mid > 1$ for all $\varepsilon_j \in \{0,1,2\}$, so it is again symmetric. The equivalence is trivial. □

One uses a graphical representation for the decomposition of σ into transpositions similar to the braid pictures (see Figure 5.7), where we specify on the top and bottom lines of the diagram the values of the evaluation maps.
This figure encodes all information about the o.p.c. because the two words w and $\sigma(w)$ have unique reduction. For the moment one draws only those trajectories of the six (to ten) elements which enter in the two blocks which reduces. Suppose for instance that the two reduction moves are two (C0). So $w = xiiiy$ and $\sigma(w) = x'jjjy'$. Say that $i = j$. The trajectories of the i's may be disjointed since the transposition acting on the couple ii is trivial in fact. So the possible trajectories fit into 4 cases which may be seen in Figure 5.8.a,b,c,d.

Suppose now we have two trajectories of i and $j \neq i$ which intersects. First of way we derive that $\mid i - j \mid > 1$. Orient all the arcs from the top to the bottom.

Lemma 5.3.8 *i) Suppose that the arcs labeled* i *and* j *have algebraic intersection number 0. Then we can replace the diagram by an equivalent one where the arcs are disjoint.*

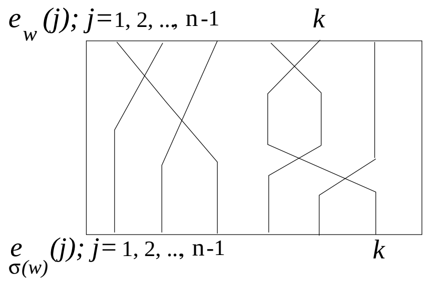


Figure 5.7: The complete diagram associated to an o.p.c.

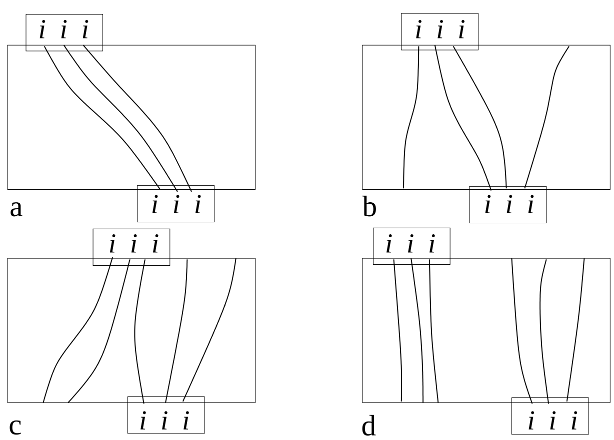


Figure 5.8: The essential trajectories for (C0)(i)-(C0)(i)

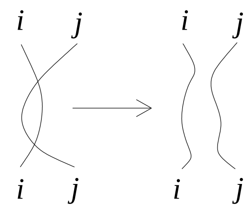


Figure 5.9: Disjointing trajectories

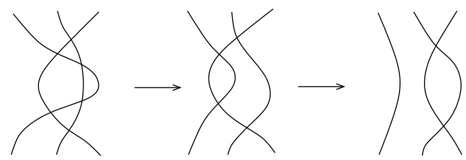


Figure 5.10: Non minimal biangle procedure

ii) Suppose that the arcs labeled i and j have algebraic intersection number 1. Then we can replace the diagram by an equivalent one where the arcs have exactly one intersection point.

Proof: We consider the diagram is that from Figure 5.9.

We can assume that the biangle in the middle is minimal, hence it does not contain any other biangle. In fact we can apply repeatedly the disjointedness procedure only for minimal biangles. Such biangle have two walls: one coming from i and the other from j. From minimality no other arc cross twice the same wall (see Figure 5.10).

Let consider the region L and R such that: the set of arcs labeled by something not commuting with j is contained in L, and those labeled by some k not commuting with i are contained in R. Then the situation is that from Figure 5.11.

Thus all arcs which cross the biangle are labeled by some k which commutes with both i and j. The same commutation transforms may be performed whenever we make the arcs i and j disjoint. □

A similar reasoning permits to say that the diagrams from Figure 5.12 are equivalent.

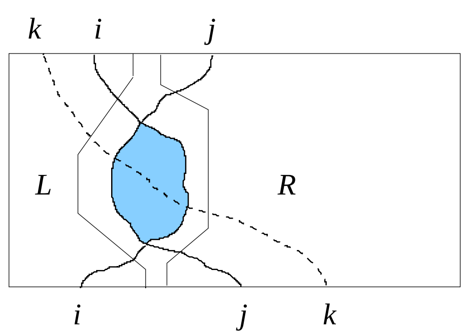


Figure 5.11: The regions R and L

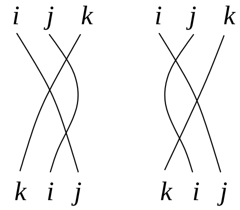


Figure 5.12: Equivalent diagrams

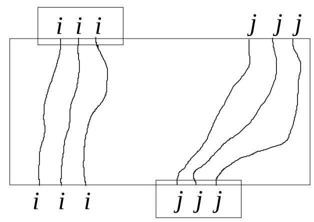


Figure 5.13: The diagram for (C0)(i)-(C0)(j) when $\mid i - j \mid > 1$

When the triangle in the middle is not touched by any arc then it is a simple consequence of lemma 5.3.8 ii). If it is minimal, any arc which cross it is labeled by something which commutes with j.
Remark now the similitude of Figures 5.9 and 5.12 with the Reidemester's moves on link diagrams. So we can actually isotopy our arcs leaving the endpoints fixed and keeping the tangent (in a C^1-approximation of arcs) away from the horizontal.
Now we can continue our discussion on the trajectories of i's and j's. If $\mid i - j \mid = 1$ the trajectories are disjoint so there are as in Figure 5.13.

If i and j commute then there are essentially sixteen diagrams (up to isotopy) which can be seen in Figure 5.14.

In order to represent graphically the possible diagrams for the (C1), (C2), (C12), (C21) moves we shall figure the trajectories of a couple of neighbor points having the same label as a single thicker trajectory. This may be done since every arc crossing the dashed region (see Figure 5.15) between the trajectories of the the two i's has a label commuting with i. In addition the trajectories of i and $i+1$ are disjoint.

Suppose we are in the case (C1)(i)-(C0)(j). For $j \neq i-1, i, i+1, i+2$ the twelve diagrams from above appear appropriately labeled. For $j = i-1, i, i+2$ some diagrams are not realized because the arcs labeled by $i-1$ and i does not intersect, so several cases have to be left. For $j = i+1$ another diagram have to be considered, that from Figure 5.16.

The same situation we encounter when we describe the possible trajectories for the couple of reduction transforms (C2)-(C0), (C12)-(C0), (C21)-(C0). A simple analysis shows that in the remaining cases the only new diagrams are those from Figure 5.17.

The other ones are obtained from the previous twelve using the suitable labeling, and taking into account the constraints of disjointedness imposed by the labels. We say now that a

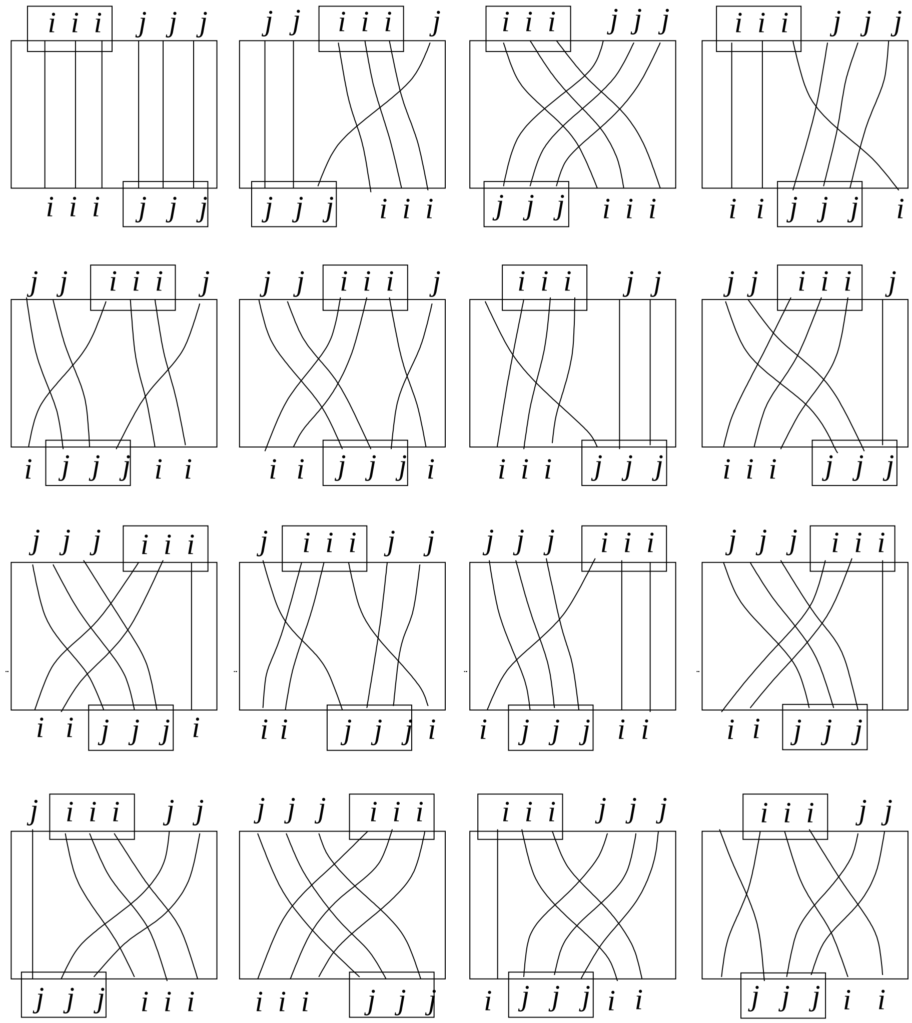


Figure 5.14: The 16 diagrams for (C0)(i) -(C0)(j) in the commuting case

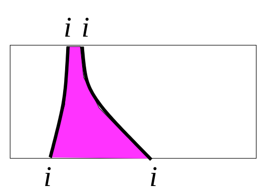


Figure 5.15: The graphical representation of the dashed region

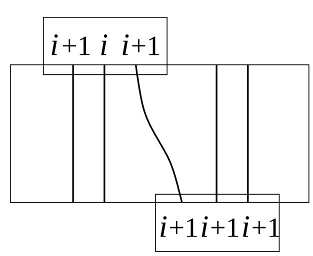


Figure 5.16: The new diagram for (C1)(i)-(C0)(i+1)

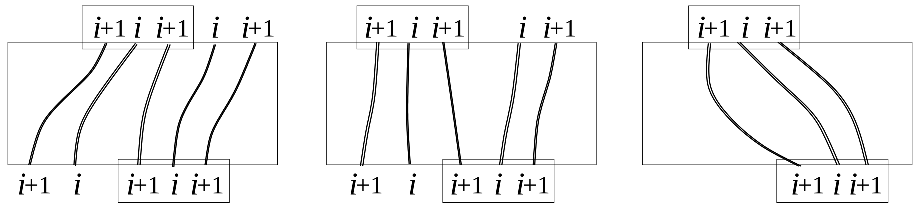


Figure 5.17: The new diagrams for (Cx)(i)-(Cy)(i) $x, y \neq 0$

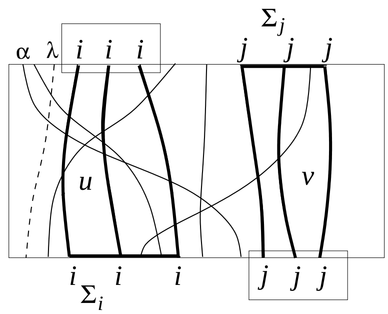


Figure 5.18: The whole figure of a non-interactive diagram without crossings

diagram is *interactive* if there is some marked arc relating the top and bottom blocks where the reduction transforms act. Our task will be to eliminate the non-interactive diagrams where the (PC) trivially holds.

Lemma 5.3.9 *The usual (PC) is valid in Γ_n for non-interactive diagrams.*

Proof: We consider first the case where no crossings of the essential arcs exist. The typical case is that from Figure 5.13. We draw now all trajectories as in Figure 5.18. We have the dashed regions U and V which are bounded by the i's arcs and respectively j's arcs.

Everything crossing the regions U and V commutes with i and j respectively. We claim first that U and V are tangent to the end lines from left and right respectively. If not there exists some arc labeled λ lying to the left of U. Assume that this arc is the first from the left having this property. In particular λ commutes with every label α which stands to the left of λ. Thus we may perform these commutation transforms at any moment, to get λ on the first position. Since λ does not cross U we may leave it on the the first position replacing the o.p.c. by an equivalent one. Thus the new configuration corresponds to a word which is not minimal with respect to the reduction procedure (see Lemma 5.3.5 and the subsequent comments).
Let now Σ_i be the convex hull of the three points labeled i coming from essential arcs and lying on the bottom line. Similarly set Σ_j for the convex hull of the j's on the top line. Every arc which arrive on Σ_i must cross U hence is labeled by some k commuting with i. We can move these endpoints using the commutation rules from the left or the right according to the following principle: if the start point of the arc labeled k is in the left of the block of i''s on the top line, then we move to the left. Otherwise we move to the right. The only problem which we can have is in the following case: the start point of some k is in the left of the arc labeled l, both arrive on Σ_i, but this time the endpoint of l is in the left of k. A topological argument shows that these two arcs cross each other. Therefore k and l are commuting and

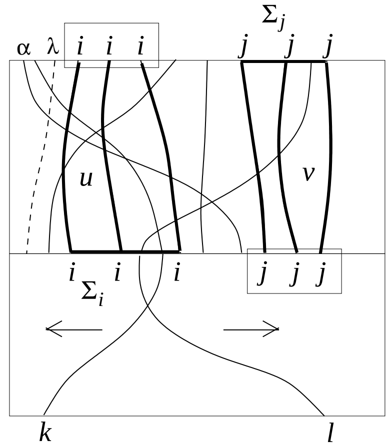


Figure 5.19: The simplification of a non-interactive diagram without crossings of essential arcs

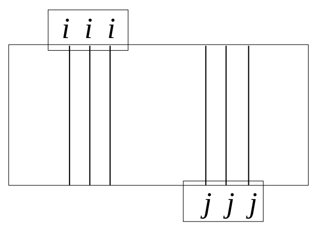


Figure 5.20: The standard non-interactive diagram

we can perform our transforms as it was said (see Figure 5.19).

Finally we recover a diagram which this time has crossings but is equivalent to the standard one of Figure 5.20.

Without loss of generality we can set $\alpha = \beta = 0$ in the reduction transforms in order to simplify the notation. Suppose now that the reduction transforms $AiiiB \to AB$ and $CjjjD \to CD$ are also performed. We may use the simplification transforms (commutations which are still valid even if the i or the j are collapsed) for above for each word: to AB in the part of j's and to CD in the part of i's. Due to the particular form of the standard diagram we shall get (see the Figure 5.18) the words $UjjjV$ and $U'iiiV'$ respectively, with $UV = U'V'$. So again the use of a reduction transform will get the same word. Thus the (PC) is satisfied for these configurations. It is almost the same reasoning for the other non-interactive diagrams without crossings.
It remains the case when crossings of essential arcs appear. But the commutation transforms may be also be performed in such way that the starting points of j's on the top line will be all on the same part with respect to the iii block. In other words we make Σ_j and the block iii disjoint. The same is true for the bottom line. The worst case is again when iii is in the left of Σ_j on the top line and down the situation is reversed. But again i and j commutes with everything starts or arrives on the convex hulls of $iii \cup \Sigma_j$ and $jjj \cup \Sigma_i$. So we can rearrange them to obtain the same order in the top and bottom lines. This ends the proof of the Lemma.
□

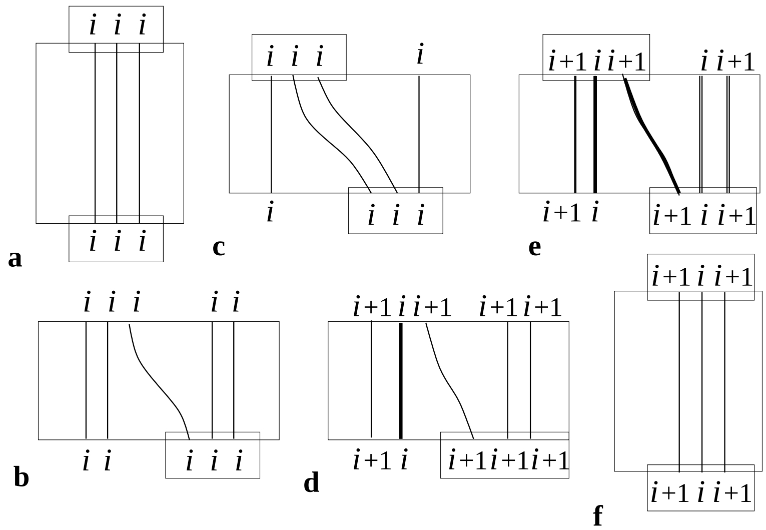


Figure 5.21: The normal forms of interactive configurations

So it remains to look at the interactive configurations. The same reasoning permits us to restrict to the normal forms drawn in Figure 5.21.a-f.
Some of the trajectories may be thick trajectories. The cases **a**,**b**,**c**,**d** and **f** are trivially verified because only the consistency of relations defining $K_3(\alpha, \beta)$ is involved.
Let us check a subcase of **d**, corresponding to (Cε)-C(0). The monomial has the form $w = \sigma_{i+1}\sigma_i^{\varepsilon}\sigma_{i+1}x\sigma_{i+1}^2$ which is unoriented equivalent to $w' = \sigma_{i+1}\sigma_i^{\varepsilon}x\sigma_{i+1}^3$. Here x commutes with σ_{i+1} and so we may suppose it lies in F_{i-1}^*. Therefore $x \to x_0\sigma_{i-1}^{j_1}\sigma_{i-2}^{j_2}...\sigma_{i-p}^{j_p}$, with $x_0 \in F_{i-2}^*$. So again we can restrict to the case $x_0 = 1$. Consider the case $\varepsilon = 2$ (the others are trivial). Set $q = \sigma_{i-2}^{j_2}...\sigma_{i-p}^{j_p}$. We have the following situation

$$\begin{array}{ccc} & w \;\text{———}\; w' & \\ \swarrow & & \searrow \\ S_j\sigma_{i-1}^{j_1}\sigma_{i+1}^2 q & & \sigma_{i+1}\sigma_i^2 E_j\sigma_{i-1}^{j_1}q \end{array}$$

where S_j, E_j as above. From Lemmas 3.5 and 3.6 it follows that (PC) holds for

$$\begin{array}{ccc} & \sigma_{i+1}\sigma_i^2\sigma_{i+1}^3\sigma_{i-1}^{j_1}q & \\ \swarrow & & \searrow \\ S_j\sigma_{i+1}^2\sigma_{i-1}^{j_1}q & & \sigma_{i+1}\sigma_i^2 E_j\sigma_{i-1}^{j_1}q \end{array}$$

Since $S_j\sigma_{i-1}^{j_1}\sigma_{i+1}^2 q$ is unoriented equivalent to $S_j\sigma_{i+1}^2\sigma_{i-1}^{j_1}q$ we have done. All other cases but **e** are similar.

In the case **e** the situation is different. Using the commutation rules, as above we must preserve the term $\sigma_{i-1}^{j_1}$. So we must check the configurations

$w = x\sigma_{i+1}^{\alpha}\sigma_i^{\varepsilon}\sigma_{i+1}^{\beta}\sigma_{i-1}^{\mu}\sigma_i^{\delta}\sigma_{i+1}^{\gamma}\sigma_{i-2}^{j_2}...\sigma_{i-p}^{j_p}$,

where $x \in F_{i-1}^*$. At this point one cannot prove that the (PC) holds. In fact it does not hold since the surjection of Lemma 5.3.1 has a nontrivial kernel in rank $n = 3$. Fortunately we proved that the configurations that don't verify (CPC) come from a finite number of obstructions. Therefore one can define H as the ideal containing all these obstructions, and see whether it is nontrivial.

Lemma 5.3.10 *It suffices to consider $x = 1, p = 1$.*

Proof: One observes that any admissible functional $\mathcal{T}$ on $K_\infty(\alpha, \beta)$ satisfies:
$\mathcal{T}(xuv) = \mathcal{T}(u)\mathcal{T}(xv)$ for $x, v \in H(Q, m)$ and $u \in < 1, \sigma_m, \sigma_{m+1}, ..., \sigma_{m+k} >$.
In fact for $k = 0$ this is the multiplicativity of $\mathcal{T}$. For $k > 0$ then in the reduction process one replaces u by $\alpha\sigma_m^\varepsilon$ where $\mathcal{T}(u) = \alpha\mathcal{T}(\sigma_m^\varepsilon)$. When reducing again one derives $\mathcal{T}(xuv) = \alpha\mathcal{T}(\sigma_m^\varepsilon)\mathcal{T}(xv)$.
Further the (CPC) is equivalent to the existence of an admissible functional. □

We have therefore to check the o.p.c. corresponding to following couples

$$\sigma_3^\xi\,\sigma_2^\epsilon\,\sigma_1^\nu\,\sigma_3^\mu\,\sigma_2^\delta\,\sigma_3^\gamma \text{ and } \sigma_3^\xi\,\sigma_2^\epsilon\,\sigma_3^\mu\,\sigma_1^\nu\,\sigma_2^\delta\,\sigma_3^\gamma \quad \xi, \epsilon, \mu, \nu, \delta, \gamma = 1 \text{ or } 2$$

Then the only possible obstructions to the existence of Markov trace come out from these couples. In Section 5.5 we study these obstructions and we find the ideal H in R containing them.

5.4 The computation of obstructions

5.4.1 Commutativity obstructions

We are now concerned with the commutativity constraints:

$$\mathcal{T}(ab) = \mathcal{T}(ba) \text{ for all } a,\, b.$$

At the first stage (i.e. $K_3(\alpha,\, \beta)$) we obtain the identities

$$\mathcal{T}(\sigma_2\,\sigma_1^2\,\sigma_2) = \mathcal{T}(\sigma_1^2\,\sigma_2^2),\ \mathcal{T}(\sigma_1\,\sigma_2\,\sigma_1^2\,\sigma_2) = \mathcal{T}(\sigma_2\,\sigma_1\,\sigma_2\,\sigma_1^2).$$

Thus the following equations should be satisfied:

$$\mathcal{T}(R_0) = \mathcal{T}(R_1) = 0.$$

i.e.
$(-\beta^3+3\alpha\beta+4)t^2+(3\alpha^2-7\alpha\beta^2-6\beta+2\beta^4)t+ (3\beta^2-\beta^5-2\alpha-3\alpha^2\beta+4\alpha\beta^3)+ (2\alpha\beta^3+\beta^2-6\alpha^2\beta-10\alpha)zt+(-3\alpha^3+7\alpha^2\beta^2+9\alpha\beta+4-\beta^3-2\alpha\beta^4)z+ (3\alpha^3\beta+7\alpha^2-\alpha^2\beta^3-\alpha\beta^2+2\beta)z^2 = (\beta^2-2\alpha)t^2+(4+5\alpha\beta-2\beta^3)t+ (\beta^4-2\beta-3\alpha\beta^2+\alpha^2)+(2\beta+5\alpha^2-2\alpha\beta^2)zt+(\beta^2+2\alpha\beta^3-5\alpha^2\beta-6\alpha)z+(4+\alpha^2\beta^2+\alpha\beta-2\alpha^3)z^2 = 0$
These yield the following values for the parameters:

- either
$$z = \frac{-\beta^2+2\alpha}{\alpha\beta+4}, \quad t = \frac{\alpha^2+2\beta}{\alpha\beta+4},$$
- or else
$$t = \frac{2\alpha z - 2z^2 + \beta}{2+\beta z}, \quad \text{where } z \text{ verifies } (\alpha\beta+1)z^3 + (\alpha+\beta^2)z^2 + 2\beta z + 1 = 0.$$

One checks then the commutativity constraints by induction on n. It suffices to consider $b \in \{\sigma_1, ..., \sigma_n\}$ and a lying in a system of generators of $K_{n+1}(\alpha, \beta)$, let us say W_n (Section 3.2). For $b = \sigma_i$, $i < n$ it is obvious. It remains to check whenever $\mathcal{T}(a\sigma_n) = \mathcal{T}(\sigma_n a)$. We have three cases

i) $a \in K_n(\alpha, \beta)$.
ii) $a = x\sigma_n y$, $x, y \in K_n(\alpha, \beta)$.
iii) $a = x\sigma_n^2 y$, $x, y \in K_n(\alpha, \beta)$.

which will be discussed in combination with the six subcases

1) $x \in K_{n-1}(\alpha, \beta)$, and $y \in K_{n-1}(\alpha, \beta)$.
2) $x \in K_{n-1}(\alpha, \beta)$, and $y = u\sigma_{n-1}v$, $u, v \in K_{n-1}(\alpha, \beta)$.
3) $x \in K_{n-1}(\alpha, \beta)$, and $y = u\sigma_{n-1}^2 v$, $u, v \in K_{n-1}(\alpha, \beta)$.
4) $x = r\sigma_{n-1}s$, $r, s \in K_{n-1}(\alpha, \beta)$, $y = u\sigma_{n-1}v$, $u, v \in K_{n-1}(\alpha, \beta)$.
5) $x = r\sigma_{n-1}s$, $r, s \in K_{n-1}(\alpha, \beta)$, $y = u\sigma_{n-1}^2 v$, $u, v \in K_{n-1}(\alpha, \beta)$.
6) $x = r\sigma_{n-1}^2 s$, $r, s \in K_{n-1}(\alpha, \beta)$, $y = u\sigma_{n-1}^2 v$, $u, v \in K_{n-1}(\alpha, \beta)$.

Now (*,i), (1,ii) and (1,iii) are trivial.

(2,ii) $\mathcal{T}(\sigma_n x\sigma_n u\sigma_{n-1}v) = tz\mathcal{T}(xuv) = \mathcal{T}(x\sigma_n u\sigma_{n-1}v\sigma_n)$.
(2,iii) $\mathcal{T}(\sigma_n x\sigma_n^2 u\sigma_{n-1}v) = (\alpha t + \beta z + 1)\mathcal{T}(xu\sigma_{n-1}v) = (\alpha t + \beta z + 1)z\mathcal{T}(xuv)$
$= \mathcal{T}(xu\sigma_{n-1}\sigma_n\sigma_{n-1}^2 v) = \mathcal{T}(x\sigma_n^2 u\sigma_{n-1}v\sigma_n)$.
(3,ii) $\mathcal{T}(\sigma_n x\sigma_n u\sigma_{n-1}^2 v) = t^2\mathcal{T}(xuv) = \mathcal{T}(\sigma_n^2\sigma_{n-1}^2)\mathcal{T}(xuv) = \mathcal{T}(\sigma_n\sigma_{n-1}^2\sigma_n)\mathcal{T}(xuv) =$
$= \mathcal{T}(xu\sigma_n\sigma_{n-1}^2\sigma_n v) = \mathcal{T}(x\sigma_n u\sigma_{n-1}^2 v\sigma_n)$.
(3,iii) $\mathcal{T}(\sigma_n x\sigma_n^2 u\sigma_{n-1}^2 v) = (\alpha t + \beta z + 1)\mathcal{T}(xu\sigma_{n-1}^2 v) = (\alpha t + \beta z + 1)t\mathcal{T}(xuv)$
$= \mathcal{T}(xuv)\mathcal{T}(\sigma_n^2\sigma_{n-1}^2\sigma_n) = \mathcal{T}(x\sigma_n^2 u\sigma_{n-1}^2 v\sigma_n)$.

For the other cases, we need also to know the form of su. Set $su = p\sigma_{n-2}^{\varepsilon}q$ with $p, q \in K_{n-2}(\alpha, \beta)$ where $\varepsilon = 0$, 1 or 2. We can show by a direct computation that the equalities hold also for $(4, ii), (4, iii), (6, ii)$, and $(6, iii)$. Using Maple we have found that in the cases $(5, ii)$ and $(5, iii)$ for $su = p\sigma_{n-2}^2 q$ there are only two new equations, which are not consequences of the identities $\mathcal{T}(R_0) = \mathcal{T}(R_1) = 0$). Specifically we have three obstructions in each case, namely the polynomial coefficients of $\mathcal{T}(rp\sigma_{n-2}^2 qv)$, $\mathcal{T}(rp\sigma_{n-2}qv)$ and $\mathcal{T}(rpqv)$.

- from $(5, ii)$ we have
  - the coefficient of $\mathcal{T}(rp\sigma_{n-2}^2 qv)$ yields the equation $L := 3\alpha\beta^4 + 5\alpha^2\beta^5 - 2\alpha\beta + 2\alpha^4\beta - 7\alpha^3\beta^3 - 7\alpha^2\beta^2 - \alpha\beta^7 + \alpha^3 + (13\alpha^3\beta^2 - 10\alpha^2\beta^4 + 13\alpha^2\beta - 6\alpha\beta^3 - 2\alpha^4 + 3\alpha + 2\alpha\beta^6)t + (-6\alpha^3\beta - \alpha\beta^5 - 6\alpha^2 + 3\alpha\beta^2 + 5\alpha^2\beta^3)t^2 + (-16\alpha^4\beta^2 - 5\alpha\beta^2 - 2\alpha^2 + 3\alpha^5 + 2\alpha\beta^5 - 13\alpha^3\beta + 11\alpha^3\beta^4 - 2\alpha^2\beta^6)z + (-2\alpha\beta^4 + 15\alpha^4\beta + 2\alpha^2\beta^5 - 11\alpha^3\beta^3 + 15\alpha^3 + 6\alpha\beta)zt + (-3\alpha - \alpha^3\beta^5 + 6\alpha^4\beta^3 - 3\alpha^3\beta^2 + 2\alpha^2\beta^4 - 9\alpha^5\beta - 9\alpha^2\beta - 10\alpha^4)z^2 = 0$,
  - the coefficient of $\mathcal{T}(rp\sigma_{n-2}qv)$ vanishes is equivalent to $M := \alpha - \alpha^4 + 6\alpha^2\beta - 2\alpha^5\beta - 2\alpha\beta^3 + 7\alpha^4\beta^3 + 11\alpha^3\beta^2 + \alpha\beta^6 - 7\alpha^2\beta^4 - 5\alpha^3\beta^5 + \alpha^2\beta^7 + (-21\alpha^3\beta - 2\alpha^2\beta^6 + 2\alpha\beta^2 + 14\alpha^2\beta^3 - 13\alpha^4\beta^2 - 7\alpha^2 + 10\alpha^3\beta^4 - 2\alpha\beta^5 + 2\alpha^5)t + (-7\alpha^2\beta^2 + 6\alpha^4\beta + 10\alpha^3 + \alpha\beta^4 + \alpha^2\beta^5 - 5\alpha^3\beta^3)t^2 + (-3\alpha^6 + 2\alpha^3\beta^6 + 5\alpha\beta + 11\alpha^2\beta^2 + 16\alpha^5\beta^2 + 8\alpha^3 + 25\alpha^4\beta - 11\alpha^4\beta^4 - 4\alpha\beta^4 - 10\alpha^3\beta^3)z + (11\alpha^4\beta^3 - 14\alpha^2\beta + 10\alpha^3\beta^2 - \alpha + 4\alpha\beta^3 - 15\alpha^5\beta - 27\alpha^4 - 2\alpha^3\beta^5)zt + (4\alpha\beta^2 - 4\alpha^2\beta^3 + \alpha^4\beta^5 + 19\alpha^5 - \alpha^3\beta^4 + 4\alpha^2 - 3\alpha^4\beta^2 + 21\alpha^3\beta - 6\alpha^5\beta^3 + 9\alpha^6\beta)z^2 = 0$,

– the coefficient of $\mathcal{T}(rpqv)$ from which one derives $N := 12\alpha^2\beta^3 + \alpha\,\beta^8 - 6\alpha^2\beta^6 - 2\alpha^2 + 3\alpha\beta^2 + 11\alpha^3\beta^4 - 4\beta^5\alpha - 6\alpha^4\beta^2 - 7\alpha^3\beta + (-21\alpha^3\beta^3 + 7\alpha\beta^4 + 5\alpha^3 + 10\alpha^4\beta - 2\alpha\beta^7 - 2\alpha\beta - 17\alpha^2\beta^2 + 12\alpha^2\beta^5)t + (-4\alpha^4 + 10\alpha^3\beta^2 - 3\alpha + \alpha\beta^6 + 5\alpha^2\beta - 6\alpha^2\beta^4 - 3\alpha\beta^3)t^2 + (3\alpha + 3\alpha\beta^3 + 2\alpha^2\beta^7 + 16\alpha^3\beta^2 - 2\alpha\beta^6 - 7\alpha^4 - 13\alpha^5\beta + 5\alpha^2\beta - 13\alpha^3\beta^5 + 25\alpha^4\beta^3)z + (\alpha^2 - 12\alpha^3\beta + 10\alpha^5 + 13\alpha^3\ \beta^4 - \alpha^2\beta^3 - 2\alpha^2\ \beta^6 + 2\alpha\beta^5 - 24\alpha^4\beta^2 - 5\alpha\beta^2)zt + (5\alpha^3 + 4\alpha^3\beta^3 + 14\alpha^5\beta^2 + 8\alpha^4\beta + 7\alpha^2\beta^2 + \alpha^3\beta^6 + 5\alpha\beta - 2\alpha^2\beta^5 - 6\alpha^6 - 7\alpha^4\beta^4)z^2 = 0.$

- from $(5, iii)$ one obtains the obstructions:
  – the coefficient of $\mathcal{T}(rp\sigma_{n-2}^2 qv)$ yields $-\alpha L = 0$,
  – the coefficient of $\mathcal{T}(rp\sigma_{n-2} qv)$ yields $-\alpha M = 0$,
  – the coefficient of $\mathcal{T}(rpqv)$ yields $-\alpha N = 0$.

5.4.2 The CPC obstructions for n=4

As pointed out in Section 5.3 the coherence of $\Gamma_n^*(H)$ depends on the following couples:

$$\sigma_3^\xi\,\sigma_2^\epsilon\,\sigma_1^\nu\,\sigma_3^\mu\,\sigma_2^\delta\,\sigma_3^\gamma \text{ et } \sigma_3^\xi\,\sigma_2^\epsilon\,\sigma_3^\mu\,\sigma_1^\nu\,\sigma_2^\delta\,\sigma_3^\gamma \quad \xi,\epsilon,\mu,\nu,\delta,\gamma = 1 \text{ or } 2$$

Recall that for a word $w = w_1, \dots, w_l$ its symmetric is the word $w^* = w_l, \dots, w_1$. Since $\mathcal{T}(w) = \mathcal{T}(w^*)$ holds one can reduce ourselves to the study of 24 couples. The couples that we must check are the following:

- $(1.i)$: $\sigma_3\,\sigma_2\,P_i\,\sigma_2^2\,\sigma_3$ and $\sigma_3\,\sigma_2\,P_i'\,\sigma_2^2\,\sigma_3$,
- $(2.i)$: $\sigma_3\,\sigma_2\,P_i\,\sigma_2\,\sigma_3^2$ and $\sigma_3\,\sigma_2\,P_i'\,\sigma_2\,\sigma_3^2$,
- $(3.i)$: $\sigma_3\,\sigma_2^2\,P_i\,\sigma_2\,\sigma_3^2$ and $\sigma_3\,\sigma_2^2\,P_i'\,\sigma_2\,\sigma_3^2$,
- $(4.i)$: $\sigma_3^2\,\sigma_2^2\,P_i\,\sigma_2^2\,\sigma_3$ and $\sigma_3^2\,\sigma_2^2\,P_i'\,\sigma_2^2\,\sigma_3$,
- $(5.i)$: $\sigma_3^2\,\sigma_2\,P_i\,\sigma_2^2\,\sigma_3^2$ and $\sigma_3^2\,\sigma_2\,P_i'\,\sigma_2^2\,\sigma_3^2$,
- $(6.i)$: $\sigma_3^2\,\sigma_2^2\,P_i\,\sigma_2\,\sigma_3$ and $\sigma_3^2\,\sigma_2^2\,P_i'\,\sigma_2\,\sigma_3$,

where $P_1 = \sigma_1\,\sigma_3$, $P_2 = \sigma_1^2\,\sigma_3$, $P_3 = \sigma_1\,\sigma_3^2$, $P_4 = \sigma_1^2\,\sigma_3^2, P_1' = \sigma_3\,\sigma_1$, $P_2' = \sigma_3\,\sigma_1^2$, $P_3' = \sigma_3^2\,\sigma_1$, $P_4' = \sigma_3^2\,\sigma_1^2$.
From now on we denote the corresponding couples by the respective label (i, j). For general α, β the computation is very long and and we needed a computer program. For more information about the code, see Remark 5.6.2.

One finds 15 different obstructions from these CPC obstructions, and the following identities among the obstructions: $(5.2) = -\alpha(3.2)$, $(6.2) = \alpha(1.2)$, $(1.4) = -\alpha(1.2)$. Thus, we must consider the couples $(1,2)$, $(2,4)$, $(3,2)$, $(3,3)$, $(3,4)$, $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$, $(5,3)$, $(5,4)$, $(6,4)$.
The exact form of the obstructions will be given in the next Section.

5.5 The existence of Markov traces

5.5.1 Statements

Theorem 5.5.1 *There exists an unique Markov trace*

$$\mathcal{T}_{(\alpha,\,\beta)} : K_*(\alpha,\,\beta) \to \frac{\mathbb{Z}[\alpha,\,\beta,\,(4+\alpha\beta)^{-1}]}{(H_{(\alpha,\,\beta)})}$$

with parameters $z = (2\alpha - \beta^2)/(\alpha\beta + 4)$ *and* $\widehat{z} = -(\alpha^2 + 2\beta)/(\alpha\beta + 4)$*, where* $H_{(\alpha\,,\beta)} := 8\alpha^6 - 8\alpha^5\beta^2 + 2\alpha^4\beta^4 + 36\alpha^4\beta - 34\alpha^3\beta^3 + 17\alpha^3 + 8\alpha^2\beta^5 + 32\alpha^2\beta^2 - 36\alpha\beta^4 + 38\alpha\beta + 8\beta^6 - 17\beta^3 + 8.$

It is convenient now to put $\delta = z^2(\beta z + 1)$, so that the obstructions below in the second case become Laurent polynomials in z and δ.

Theorem 5.5.2 *Set* $\alpha = -(z^7 + \delta^2)/(z^4\delta)$*,* $\beta = (\delta - z^2)/z^3$ *and* $\widehat{z} = -z^2/(\beta z + 1) = -z^4/\delta$*. There exists an unique Markov trace with parameters* $(z, \widehat{z})$

$$\mathcal{T}^{(z,\,\delta)} : K_{(\alpha,\,\beta)} \to \frac{\mathbb{Z}[z^{\pm 1}, \delta^{\pm 1}]}{(P^{(z,\,\delta)})}$$

where $P^{(z,\,\delta)} = z^{23} + z^{18}\delta - 2z^{16}\delta^2 - z^{14}\delta^3 - 2z^9\delta^4 + 2z^7\delta^5 + \delta^6 z^5 + \delta^7$*.*

5.5.2 Proof of Theorem 5.5.1

The parameters z, t have to satisfy the condition

$$\mathcal{T}(R_0) = \mathcal{T}(R_1) = 0$$

Consider first the simple solutions $z = (2\alpha - \beta^2)/(\alpha\beta + 4)$ and $t = (\alpha^2 + 2\beta)/(\alpha\beta + 4)$. We set $\mathcal{T}_{(\alpha,\,\beta)}$ for the admissible functional associated to these values of the parameters. Notice that in this case $\widehat{z} = -t$. Set $u := 1/(\alpha\beta + 4)$, $z_0 := 2\alpha - \beta^2$ and $t_0 := \alpha^2 + 2\beta =: -\widehat{z}_0$.

The commutativity obstructions

The equations encountered above for $(5, ii)$ amount to

- $u^2\beta \mathrm{H}_{(\alpha,\,\beta)} = 0$,
- $-u^2(\alpha\beta + 2)\mathrm{H}_{(\alpha,\,\beta)} = 0$,
- $u^2(\alpha - \beta^2)\mathrm{H}_{(\alpha,\,\beta)} = 0$.

CPC obstructions

- (1.2): $-u^3\alpha(\alpha - \beta^2)\mathrm{H}_{(\alpha\,,\beta)}W$,
- (2.4): $u^3(\alpha - \beta^2)(\alpha^2 + \beta)\mathrm{H}_{(\alpha\,,\beta)}W$,
- (3.2): $u^3(-\alpha^2\beta^2 + 2 + \alpha\beta + \alpha^3)\mathrm{H}_{(\alpha,\,\beta)}W$,
- (3.3): $u^3(\alpha\beta + 2)\mathrm{H}_{(\alpha,\beta)}W$,

- (3.4): $u^3\alpha\beta(\alpha-\beta^2)\mathrm{H}_{(\alpha,\beta)}W$,
- (4.1): $-u^3(\alpha-\beta^2)(\alpha^2+\beta)\mathrm{H}_{(\alpha,\beta)}W$,
- (4.2): $u^3\alpha(\alpha^3+2+2\alpha\beta-\alpha^2\beta^2-\beta^3)\mathrm{H}_{(\alpha,\beta)}W$,
- (4.3): $u^3\alpha(\alpha^3-\alpha^2\beta^2-2-\beta^3)\mathrm{H}_{(\alpha,\beta)}W$,
- (4.4): trivial,
- (5.3): $-u^3(\beta^2+2\alpha+2\alpha^2\beta)\mathrm{H}_{(\alpha,\beta)}W$,
- (5.4): $u^3\alpha(-\alpha^3\beta^2-\beta^2-\alpha^2\beta+\alpha^4)\mathrm{H}_{(\alpha,\beta)}W$,
- (6.4): $-u^3\alpha(\beta+2\alpha^2)(\alpha-\beta^2)\mathrm{H}_{(\alpha,\beta)}W$,

where $W=(\alpha+2-\beta)(\alpha^2-2\alpha+4+\alpha\beta+2\beta+\beta^2)=\alpha^3+8-\beta^3+6\alpha\beta$.

5.5.3 Proof of Theorem 5.5.2

There are three more solutions of $\mathcal{T}(R_0)=\mathcal{T}(R_1)=0$, given by $t=\frac{2\alpha z-2z^2+\beta}{2+\beta z}$, where z verifies $(\alpha\beta+1)z^3+(\alpha+\beta^2)z^2+2\beta z+1=0$. In this case the obstructions are better expressed as rational functions on z and β.

The commutativity obstructions

- $-ZB_1/(z^7(z\beta+1)^4)=0$,
- $-ZB_2/(z^9(z\beta+1)^5)=0$,
- $ZB_3/(z^7(z\beta+1)^5)=0$.

The CPC obstructions

- (1.2): $-ZB_4B_5B_6/(z^{13}(z\beta+1)^8)$,
- (2.4): $-ZB_4B_6B_7/(z^{15}(z\beta+1)^9)$,
- (3.2): $ZB_4B_8/(z^{15}(z\beta+1)^9)$,
- (3.3): $-ZB_4B_9/(z^{11}(z\beta+1)^7)$,
- (3.4): $ZB_4B_5B_6\beta/(z^{13}(z\beta+1)^8)$,
- (4.1): $ZB_4B_6B_7/(z^{15}(z\beta+1)^9)$,
- (4.2): $ZB_4B_5B_{10}/(z^{17}(z\beta+1)^{10})$,
- (4.3): $ZB_4B_5B_{11}/(z^{17}(z\beta+1)^{10})$,
- (4.4): trivial,
- (5.3): $-ZB_4B_{12}/(z^{13}(z\beta+1)^8)$,
- (5.4): $-ZB_4B_5B_{13}/(z^{19}(z\beta+1)^{11})$,

- (6.4): $-ZB_4B_5B_6B_{14}/(z^{17}(z\beta+1)^{10})$,

where $Z, B_1, \ldots, B_{14}$ are the following polynomials in z, β:

- $Z = 1+7z\beta+21z^2\beta^2+z^3+35z^3\beta^3+35z^4\beta^4+21z^5\beta^5+7z^6\beta^6+z^7\beta^7+z^9\beta^6+8z^8\beta^5+23z^7\beta^4+32z^6\beta^3+23z^5\beta^2+8z^4\beta-2z^6+z^9-z^9\beta^3-5z^8\beta^2-6z^7\beta$,
- $B_1 = 3z^3+z^4\beta+1+z\beta$,
- $B_2 = 5z^3+10z^4\beta+6z^5\beta^2+z^6\beta^3+4z^6+2z^7\beta+1+3z\beta+3z^2\beta^2+z^3\beta^3$,
- $B_3 = \beta+2z\beta^2+4z^3\beta+5z^4\beta^2+z^5\beta^3+z^2\beta^3-2z^5$,
- $B_4 = (z\beta+z^2\beta+1+z-z^2)(z\beta+1+2z^3)(z^4\beta^2-z^3\beta^2+z^2\beta^2+1+2z\beta-z-2z^2\beta+2z^2+3z^3\beta+z^3+z^4\beta+z^4)$,
- $B_5 = 1+z^3+z^2\beta^2+2z\beta$,
- $B_6 = z^3\beta^3+1+2z\beta+2z^2\beta^2+z^3$,
- $B_7 = 1+4z\beta+6z^2\beta^2+2z^3+4z^3\beta^3+z^4\beta^4+z^6\beta^3+4z^5\beta^2+5z^4\beta+z^6)$,
- $B_8 = z^2\beta^3+\beta+2z\beta^2-2z^2-z^3\beta$,
- $B_9 = 1+6z\beta+16z^2\beta^2+3z^3+25z^3\beta^3+25z^4\beta^4+16z^5\beta^5+6z^6\beta^6+z^7\beta^7+3z^8\beta^5+13z^7\beta^4+24z^6\beta^3+24z^5\beta^2+13z^4\beta+z^7\beta+z^6+z^9$,
- $B_{10} = 1+6z\beta+16z^2\beta^2+3z^3+25z^3\beta^3+25z^4\beta^4+16z^5\beta^5+6z^6\beta^6+z^7\beta^7+z^9\beta^6+7z^8\beta^5+20z^7\beta^4+31z^6\beta^3+28z^5\beta^2+14z^4\beta+z^6+z^9+z^9\beta^3+2z^8\beta^2+2z^7\beta$,
- $B_{11} = 6z\beta+16z^2\beta^2+3z^3+10z^8\beta^2+5z^8\beta^5+z^7\beta^7+z^9\beta^6+12z^7\beta+12z^7\beta^4+19z^6\beta^3+20z^5\beta^2+12z^4\beta+6z^6\beta^6+3z^9\beta^3+5z^6+z^9+1+25z^3\beta^3+25z^4\beta^4+16z^5\beta^5$,
- $B_{12} = 2\beta+4z^5\beta^3-2z^5+2z^4\beta^5+8z\beta^2+12z^2\beta^3-2z^2+8z^3\beta^4+3z^4\beta^2-2z^3\beta+z^6\beta^4$,
- $B_{13} = 1+8z\beta+29z^2\beta^2+63z^3\beta^3+80z^6\beta^3+29z^7\beta^7+13z^9\beta^6+17z^9\beta^3+91z^4\beta^4+57z^5\beta^2+23z^4\beta+4z^3+6z^6+4z^9+91z^5\beta^5+63z^6\beta^6+39z^8\beta^5+70z^7\beta^4+30z^8\beta^2+22z^7\beta+z^{12}+z^9\beta^9-z^{12}\beta^6+z^{10}\beta^4+2z^{10}\beta^7+8z^8\beta^8-3z^{11}\beta^5+3z^{11}\beta^2+7z^{10}\beta$,
- $B_{14} = 2+8z\beta+12z^2\beta^2+4z^3+8z^3\beta^3+2z^4\beta^4+z^6\beta^3+6z^5\beta^2+9z^4\beta+2z^6$.

Notice that $Z(z,\beta) = P^{(z,\delta)}(z,\delta)$.

5.5.4 Corollaries

Corollary 5.5.1 • *There exists an unique Markov trace*

$$\mathcal{T} : K_*(0,\, 2\lambda) \to \frac{\mathbb{Z}[\lambda]}{(8\lambda^6 - 17\lambda^3 + 1)},$$

with parameters $z = -\lambda^2$, $t = \lambda$ and $\widehat{z} = -\lambda$,

- *respectively*
$$\mathcal{T} : K_*(-2\lambda,\, 0) \to \frac{\mathbb{Z}[\lambda]}{(8\lambda^6 - 17\lambda^3 + 1)},$$
with parameters $z = -\lambda$*,* $t = \lambda^2$ *and* $\widehat{z} = -\lambda^2$*.*

We have a similar situation for the other three solutions. In fact for $\alpha = 0$, we derive $z = -(t-\beta)^2$, where t satisfies $(t^3 - 4\beta t^2 + 5\beta^2 t + 1 - 2\beta^3) = 0$. In particular $\widehat{z}^3 - \beta\widehat{z}^2 + 1 = 0$ because $\widehat{z} = t - \beta$.

Corollary 5.5.2
- *There exists an unique Markov trace*
$$\mathcal{T} : K_*(0,\, \frac{\lambda^3+1}{\lambda^2}) \to \frac{\mathbb{Z}[\lambda^{\pm 1}]}{(\lambda^9 - 2\lambda^6 + \lambda^3 + 1)},$$
with parameters $z = -\lambda^2$*,* $\widehat{z} = \lambda$ *and* $t = \dfrac{2\lambda^3+1}{\lambda^2}$*,*

- *and respectively*
$$\mathcal{T} : K_*(-\frac{\lambda^3+1}{\lambda^2},\, 0) \to \frac{\mathbb{Z}[\lambda^{\pm 1}]}{(\lambda^9 - 2\lambda^6 + \lambda^3 + 1)},$$
with parameters $z = \lambda$*,* $\widehat{z} = -\lambda^2$ *and* $t = -\dfrac{2\lambda^3+1}{\lambda^2}$*.*

5.6 The invariants

5.6.1 The definition of $I_{(\alpha,\beta)}$

As in Section 5.2 we set $z = (2\alpha - \beta^2)/(\alpha\beta + 4)$, $t = (\alpha^2 + 2\beta)/(\alpha\beta + 4)$, $u := 1/(\alpha\beta + 4)$, $z_0 := 2\alpha - \beta^2$ and $t_0 := \alpha^2 + 2\beta =: -\widehat{z}_0$ (notice that in this case $\widehat{z} = -t$).

Definition 5.6.1 *Let us set for an oriented link* L
$$I_{(\alpha,\beta)}(L) = \left(\frac{1}{z\widehat{z}}\right)^{\frac{n-1}{2}} \left(\frac{\widehat{z}}{z}\right)^{\frac{e(x)}{2}} \mathcal{T}_{(\alpha,\beta)}(x) \in \frac{\mathbb{Z}[\alpha,\, \beta,\, z_0^{\pm\epsilon/2}, \widehat{z}_0^{\pm\epsilon/2}]}{(H_{(\alpha,\beta)})},$$
where $x \in B_n$ *is any braid whose closure is isotopic to* L*. Here* $\epsilon - 1$ *is the number of components mod 2.*

Lemma 5.6.1 $I_{(\alpha,\beta)}$ *is well-defined.*

Proof: Since $\sigma_j^{-1} = \sigma_j^2 - \alpha\sigma_j - \beta$, we can suppose that x is a positive braid. All the elements in $\Gamma_0(H)$ associated to x are polynomials in the variables z, t of degree at most $n-1$. The substitutions $z = uz_0$ and $t = ut_0$ imply that, if $\mathcal{T}_{(\alpha,\beta)}(x)$ and $\mathcal{T}_{(\alpha,\beta)}(x)'$ are representatives of the trace of x, then $\mathcal{T}_{(\alpha,\beta)}(x)' - \mathcal{T}_{(\alpha,\beta)}(x) = u^{n-1}G(\alpha,\beta)\mathrm{H}_{(\alpha,\beta)}$, where $G(\alpha,\beta)$ is a polynomial in α, β. It follows
$$I_{(\alpha,\beta)}(L) = \left(\frac{1}{z_0\widehat{z}_0}\right)^{\frac{n-1}{2}} \left(\frac{\widehat{z}_0}{z_0}\right)^{\frac{e(x)}{2}} \widetilde{\mathcal{T}}_{(\alpha,\beta)}(x),$$
where
$$\widetilde{\mathcal{T}}_{(\alpha,\beta)}(x) := u^{-n+1}\mathcal{T}_{(\alpha,\beta)}(x) \in \frac{\mathbb{Z}[\alpha,\, \beta]}{(\mathrm{H}_{(\alpha,\beta)})}.$$
□

5.6.2 The cubical behaviour

Proposition 5.6.1 *For any link K there exists some $l \in \{0, 1, 2\}$ such that*

$$I_{(\alpha,\beta)}(K) = \frac{\sum_{k\in\mathbb{N}} P_k(\beta)\alpha^k}{\sum_{k\in\mathbb{N}} Q_k(\beta)\alpha^k} = \frac{\sum_{k\in\mathbb{N}} M_k(\alpha)\beta^k}{\sum_{k\in\mathbb{N}} N_k(\alpha)\beta^k}$$

where P_k, Q_k, M_k, N_k are $(3, k+l)$-polynomials.

Proof: We will show that M_k, N_k are $(3, k+l)$-polynomials. The other case is analogous. Suppose first that $x \in B_n^+$, where B_n^+ is the set of positive braids and n is such that $x \notin B_{n-1}^+$. Then $e(x) = |x|$ where $|x|$ means the length of x. In the process computing the value of the trace on the word x we make two types of reductions: either one uses the relations in some $K_n(\alpha, \beta)$, or else one replaces $a\sigma_l b$ by zab (respectively $a\sigma_l^2 b$ by tab), where a, b are subwords, and this way one lowers the rank n. Using the relations the word y is replaced by $\sum_s(\sum_{k\in\mathbb{N}} D_{k,s}(\alpha)\beta^k)y_s$ where the y_s are a finite number of words in B_n and the coefficients $D_{k,s}(\alpha)$ are $(3, k+e(x)-l_s)$-polynomials where $l_s = |y_s|$. In the second case the word w is replaced by $zw' + tw''$ where $|w'| = |w| - 1$ and $|w''| = |w| - 2$. When we introduce the z and t as functions on α and β one finds that

$$\mathcal{T}_{(\alpha,\beta)}(x) = \sum_{k\in\mathbb{N}} u^{s_k} V_k(\alpha)\beta^k,$$

where $0 \le s_k \le n-1$ and $V_k(\alpha)$ are $(3, k+e(x))$-polynomials. In particular

$$\widetilde{\mathcal{T}}_{(\alpha,\beta)}(x) = \sum_{k\in\mathbb{N}} u^{s_k-n+1} V_k(\alpha)\beta^k.$$

Now $u^{s_k-n+1} = \sum_{k\in\mathbb{N}} Y_k(\alpha)\beta^k$ where $Y_k(\alpha)$ are $(3, k)$-polynomials. Thus it follows

$$\widetilde{\mathcal{T}}_{(\alpha,\beta)}(x) = \sum_{k\in\mathbb{N}} L_k(\alpha)\beta^k,$$

where $L_k(\alpha)$ are $(3, k+e(x))$-polynomials.

The same is true for non necessarily positive $x \in B_n$, by getting rid of the negative exponents (using the cubic relation).

Taking into account the normalization factor in front of the trace we obtain the claim. □

Corollary 5.6.1 $I_{(\alpha,0)}(K) = \sum_{i\in\mathbb{N}} a_{3i}\alpha^{3i}$ *and, respectively,* $I_{(0,\beta)}(K) = \sum_{i\in\mathbb{Z}} b_{3i}\beta^{3i}$*, where* $a_{3i}, b_{3i} \in \mathbb{Z}[\frac{1}{2}]$.

5.6.3 Chirality and other properties of $I_{(\alpha,\beta)}$

Lemma 5.6.2 *Set $x^* \in B_n$ for the word one obtains from x when each σ_j^ϵ is replaced by $\sigma_j^{-\epsilon}$. Then $\mathcal{T}_{(\alpha,\beta)}(x) = \mathcal{T}_{(-\beta,-\alpha)}(x^*)$ holds true. Consequently for amphicheiral K, $I_{(\alpha,\beta)}(K) = I_{(-\beta,-\alpha)}(K)$ is fulfilled.*

Proof: Let $Q(\sigma_j)^*$ (respectively R_0^*) denotes the image of $Q(\sigma_j)$ (respectively R_0) after the substitutions $\alpha \to -\beta$, $\beta \to -\alpha$ and $\sigma_l \to \sigma_l^{-1}$ for $l = 1, \ldots, n-1$. It is easy to check that $Q(\sigma_j)^* = \sigma_j^{-3}Q(\sigma_j) = 0$. Using a computer we verified that $R_0^* = R_1 = 0$. Since $\mathrm{H}_{(\alpha,\beta)} = \mathrm{H}_{(-\beta,-\alpha)}$ we are done. □

The following properties have been checked with a computer program:

- $I_{(\alpha,\beta)}$ is independent from HOMFLY-PT and in particular it distinguishes knots that have the same HOMFLY-PT polynomial. The knots 5.1 and 10.132 have the same HOMFLY-PT polynomial but different $I_{(\alpha,0)}$ and $I_{(0,\beta)}$ invariants. This holds true for the other three couples of prime knots with number crossing ≤ 10 that HOMFLY-PT fails to distinguish, i.e. (8.8, 10.129), (8.16, 10.156), (10.25, 10.56).

- $I_{(\alpha,\beta)}$ detects the chirality of those knots with crossing number at most 10, where HOMFLY-PT fails i.e. the knots 9.42, 10.48, 10.71, 10.91, 10.104 and 10.125).

- The Kauffman polynomial does not detect the chirality of 9.42 and 10.71 (see [79]). Therefore $I_{(\alpha,\beta)}$ is independent from the Kauffman polynomial.

- The 2-cabling of HOMFLY-PT does not detect the chirality of 10.71 (this result was kindly communicated by H. Morton). Therefore $I_{(\alpha,\beta)}$ is independent from the 2-cabling of HOMFLY-PT.

- $I_{(\alpha,\beta)}$ does not distinguish a well-known pair of mutant knots, the Conway knot (C) and the Kinoshita-Terasaka knot (KT).

5.6.4 The definition of $I^{(z,\delta)}$

Definition 5.6.2 *Let us set for an oriented link L*

$$I^{(z,\delta)}(L) = \left(\frac{1}{z\widehat{z}}\right)^{\frac{n-1}{2}} \left(\frac{\widehat{z}}{z}\right)^{\frac{e(x)}{2}} \mathcal{T}^{(z,\delta)}(x) \in \frac{\mathbb{Z}[z^{\pm\epsilon/2}, \delta^{\pm\epsilon/2}]}{(P^{(z,\delta)})},$$

where $x \in B_n$ is any braid whose closure is isotopic to L and α, β, t, $\widehat{z}$ as in Theorem 5.5.2. Here $\epsilon - 1$ is the number of components mod 2, $\epsilon \in \{1,2\}$.

Remark 5.6.1 *This invariant doesn't detect the amphicheirality of knots. Also $I^{(z,\delta)}$ does not distinguish the Conway knot and the Kinoshita-Terasaka knot.*

Proposition 5.6.2

$$I^{(z,\delta)}(K) = \sum_{k\in\mathbb{Z}} H_k(\delta) z^k = \sum_{k\in\mathbb{Z}} G_k(z)\delta^k,$$

where H_k, G_k are $(3,k)$-Laurent polynomials.

Proof: The proof is analogous to the proof of Proposition 5.6.1. □

Remark 5.6.2 *For evaluating obstructions and traces of braids we used a Delphi code. The input is a word (or a linear combinations) and we restricted to words representing 5-braids. One transforms first the word to a sum of positive words, by using the cubic relations. Furthermore the transformations C_i and Cij are used in order to reduce the shape of the word as much as possible. When it gets stalked one allows permutations of the letters. The final result is the value of the trace on the braid element. The program is available on* www-fourier.ujf-grenoble.fr/~bellinge.html.

5.6.5 Comments

As explained in Section 5.1.3, there are three essentially distinct link invariants which come from Markov traces on the cubic Hecke algebras. For each quadratic factor P_i of the cubic polynomial Q one has a Markov trace which factors through $H(P_i, n)$, yielding a reparameterized HOMFLY-PT invariant. The two others are the Kauffman polynomial and $I_{(\alpha,\beta)}$ (or $I^{(z,\delta)}$). It would be very interesting to find whether there exists some relation among them. First of way one expects there exists a lift of the invariant we described to a genuine two-parameter invariant.

Conjecture 5.6.1 *There exists a Markov trace on $H(Q, n)$ taking values in an algebraic extension of $\mathbb{Z}[\alpha, \beta]$ lifting the Markov trace underlying $I_{(\alpha,\beta)}$. In other words the non-determinacy $H_{(\alpha,\beta)}$ in $I_{(\alpha,\beta)}$ can be removed.*

Notice that the polynomials H and P define irreducible planar algebraic curves which are non-rational. In particular one cannot express explicitly the invariants as one variable polynomial. How far are these invariants from the usual Kauffman and HOMFLY-PT polynomials is hard to determine in the present state. One might expect they give rise to some nice weight systems for particular values of the parameters, which should be compared with those coming from Lie algebras.

5.7 Appendix

The values of the polynomials for $I_{(\alpha, 0)}(K)$ and $I_{(0, \beta)}(K)$ are displayed below for all knots with no more than 8 crossings. The second column is a braid representative for the knot. A bold entry in the table is the coefficient of α^0 (respectively β^0). The other entries are the non zero coefficients of α^{3k} and β^{3k} respectively, for $k \in \mathbb{Z}$. For example,

$$I_\alpha(6.2) = [-5 - \frac{19}{4}\alpha^3 - \frac{1}{2}\alpha^6]; \quad I_\beta(6.2) = [-16\beta^{-3} + 19 - 2\beta^3].$$

The entry “A" in the last column means that the knot is amphicheiral.

3.1	σ_1^3	**-1** − 1/4	−8 **2**	
4.1	$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$	8 **10** 1	−8 **10** − 1	*A*
5.1	σ_1^5	**0** 7/8 1/8	−24 **4**	
5.2	$\sigma_1^2\sigma_2^2\sigma_1^{-1}\sigma_2$	**2** 17/8 1/4	−8 **2**	
6.1	$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1}\sigma_3\sigma_2$	−8 **-16** − 10 − 1	**1**	
6.2	$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^3$	**-5** − 19/4 − 1/2	−16 **19** − 2	
6.3	$\sigma_1^{-1}\sigma_2^2\sigma_1^{-2}\sigma_2$	**-3** − 1/2	**-3** 1/2	*A*
7.1	σ_1^7	**0** − 5/8 − 9/16 − 1/16	−56 **8**	
7.2	$\sigma_1^{-1}\sigma_3^3\sigma_2\sigma_1^2\sigma_3^{-1}\sigma_2$	**-3** − 11/2 − 21/8 − 1/4	−64 − 64 **-6**	
7.3	$\sigma_1^2\sigma_2\sigma_1^{-1}\sigma_2^4$	**-1** − 7/4 − 19/16 − 1/8	−64 48 **-4**	
7.4	$\sigma_1^2\sigma_2\sigma_3^2\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2$	**0** − 17/8 − 9/4 − 1/4	−64 + 128 **-78** 8	
7.5	$\sigma_1^4\sigma_2\sigma_1^{-1}\sigma_2^2$	**0** − 9/8 − 9/8 − 1/8	−24 **4**	
7.6	$\sigma_1\sigma_2^{-1}\sigma_1^{-2}\sigma_3\sigma_2^3\sigma_3$	**-4** − 37/8 − 1/2	−24 **20** − 2	
7.7	$\sigma_1\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2$	−8 **-20** − 21/2 − 1	**-19** 37/2 − 2	
8.1	$\sigma_1^{-1}\sigma_2\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_4^2\sigma_3\sigma_2\sigma_4^{-1}$	16 **43** 37 12 1	−64 144 **-88** 9	
8.2	$\sigma_1^{-1}\sigma_2^5\sigma_1^{-1}\sigma_2$	**4** 59/8 23/8 1/4	−24 **36** − 4	
8.3	$\sigma_1^{-2}\sigma_2^{-1}\sigma_1\sigma_4^2\sigma_3\sigma_4^{-1}\sigma_2^{-1}\sigma_3$	−8 **-8** − 1	8 **-8** 1	*A*
8.4	$\sigma_1^3\sigma_3\sigma_2^{-1}\sigma_3^{-2}\sigma_1\sigma_2^{-1}$	8 **8** 3/4	8 **-24** 19 − 2	
8.5	$\sigma_1^3\sigma_2^{-1}\sigma_1^3\sigma_2^{-1}$	**1** 3 19/8 1/4	−24 **36** − 4	
8.6	$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_2^3\sigma_3^2$	**5** 21/2 21/4 1/2	**1**	
8.7	$\sigma_1^4\sigma_2^{-2}\sigma_1\sigma_2^{-1}$	**3** 9/4 1/4	16 **-25** 3	
8.8	$\sigma_1^{-1}\sigma_2\sigma_1^2\sigma_3^{-1}\sigma_2^2\sigma_3^{-2}$	**3** 17/4 1/2	16 **-21** 5/2	
8.9	$\sigma_1^{-1}\sigma_2\sigma_1^{-3}\sigma_2^3$	**-7** − 9 − 1	**-7** 9 − 1	*A*
8.10	$\sigma_1^{-1}\sigma_2^2\sigma_1^{-2}\sigma_2^3$	**1** 2 1/4	8 **-8** 1	
8.11	$\sigma_1^{-1}\sigma_2^2\sigma_3^{-1}\sigma_2\sigma_3^2\sigma_1^{-1}\sigma_2$	8 **21** 147/8 6 1/2	−64 136 **-79** 8	
8.12	$\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_3\sigma_4^{-1}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$	24 **44** 21 2	−24 **44** − 21 2	*A*
8.13	$\sigma_1^2\sigma_2\sigma_3^{-1}\sigma_2\sigma_1^{-1}\sigma_3^{-2}\sigma_2$	8 **12** 21/4 − 1/2	8 **-28** 39/2 − 2	
8.14	$\sigma_1^2\sigma_2^2\sigma_1^{-1}\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_2$	**6** 85/8 21/4 1/2	−8 **18** − 2	
8.15	$\sigma_1^2\sigma_2^{-1}\sigma_1\sigma_3^2\sigma_2^2\sigma_3$	**0** − 17/8 − 9/4 − 1/4	64 − 32 **4**	
8.16	$\sigma_1^2\sigma_2^{-1}\sigma_1^2\sigma_2^{-1}\sigma_1\sigma_2^{-1}$	**-3** 3/2 1/4	**-7** 1	
8.17	$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^2\sigma_1^{-2}\sigma_2$	**-11** − 19/2 − 1	**-11** 19/2 − 1	*A*
8.18	$\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$	−8 **-16** − 10 − 1	8 **-16** 10 − 1	*A*
8.19	$\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^2\sigma_1$	**0** 3/8 1/16	64 − 64 **1**	
8.20	$\sigma_1^3\sigma_2\sigma_1^{-3}\sigma_2$	**5** 9/2 1/2	−8 **0**	
8.21	$\sigma_1\sigma_2^{-2}\sigma_1^2\sigma_2^3$	**1** − 1 − 1/8	8 **0**	

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