COMPLETIONS OF NORMAL AFFINE SURFACES WITH A TRIVIAL MAKAR-LIMANOV INVARIANT

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Abstract. We study normal affine surfaces with non-trivial algebraic $\mathbb{C}_+$-actions in term of their completions. As a generalization of a result of Gizatullin [5], we prove that a normal affine surface has a trivial Makar-Limanov invariant if and only if it is completable by a zigzag.

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Introduction

For a connected normal affine surface $V = Spec\,(A)$ over $\mathbb{C}$, the Makar-Limanov invariant of V [9] is the subalgebra $ML\,(V) \subset A$ of all regular functions invariant under every algebraic $\mathbb{C}_+$-actions on V. Constant functions are certainly contained in $ML\,(V)$, and we say that the Makar-Limanov invariant of V is *trivial* (or that V is an *ML-surface*) if $ML\,(V) = \mathbb{C}$. In [1], Bandman and Makar-Limanov have discovered a link between ML-surfaces and *geometrically quasihomogeneous* surfaces studied by Gizatullin in [5], that issurfaces whose automorphism group has a Zariski open orbit with a finite complement. More precisely, they have established that, on a nonsingular ML-surface V, there exist at least two non-trivial algebraic $\mathbb{C}_+$-actions which generate a subgroup H of the automorphism group $Aut\,(V)$ of V such that the orbit $H.v$ of a general closed point $v \in V$ has finite complement. By Gizatullin [5] such a surface is rational and either isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$or can be obtained from a nonsingular projective surface $\bar{V}$ by deleting an ample divisor of a special form, called a *zigzag*. This is just a linear chain of nonsingular rational curves. More generally, in this paper we prove the following theorem:

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Theorem. *A normal affine surface V non isomorphic to $\mathbb{C}^*\times\mathbb{A}^1$ has a trivial Makar-Limanov invariant if and only if it is completable by a zigzag.*

1. Rulings and completions of normal surfaces

We use the following terminology :

- A surface is a connected, reduced, normal $\mathbb{C}$-scheme of finite type and of dimension 2.
- The intersection number of two divisors D_1 and D_2 on a surface V regular at the points of $D_1 \cap D_2$ is denoted by $(D_1 \cdot D_2)$. The self-intersection number of a divisor $D \subset V_{reg}$ is denoted by $\left(D^2\right) = (D \cdot D)$.
- For a morphism $f : W \to V$ between normal varieties and for a divisor D on V we denote by $q^{-1}(D)$ the set-theoretic preimage of D, whereas $q^*(D)$ denotes its preimage considered as a cycle.
- An $\mathbb{A}^1$-fibration (a $\mathbb{P}^1$-fibration) on a surface V is a surjective morphism $\rho : V \to Z$ on a nonsingular curve Z with general fibers isomorphic to the affine line $\mathbb{A}^1$ (to the projective line $\mathbb{P}^1$, respectively). The fibers of ρ which either are not isomorphic to $\mathbb{A}^1$ (respectively $\mathbb{P}^1$) or are not reduced are called *degenerate*.
- An SNC-divisor D on a surface is a divisor with normal crossing singularities whose irreducible components are nonsingular.
- For a normal affine surface V we call a *completion* of V an open embedding $i : V \hookrightarrow \bar{V}$ of V into a normal projective surface $\bar{V}$, nonsingular along $B = \bar{V} \setminus i(V)$ and such that B is an SNC-divisor. We say that the completion is minimal if B contains no (-1)-curve which meets at most two other components transversally in a single point.
- For an isolated singularity (V, P) of a normal surface, a minimal embedded resolution of p is a birational morphism $\pi : W \to V$ such that W is nonsingular, $W \setminus \pi^{-1}(P) \simeq V \setminus \{P\}$ and $\pi^{-1}(P)$ is an SNC-divisor which contains no (-1)-curve meeting at most two other components transversally in a single point.

Definition 1.1. A *zigzag* B on a normal projective surface $\bar{V}$ is a connected SNC-divisor with nonsingular rational curves as irreducible components and whose dual graph is a linear chain. If $Supp\,(B) = \bigcup_{i=1}^n B_i$, the irreducible components B_i, $1 \leq i \leq n$, of B can be ordered in such a way that

$$(B_i \cdot B_j) = \begin{cases} 1 & \text{if} \quad |i-j| = 1 \\ 0 & \text{if} \quad |i-j| > 1 \end{cases}.$$

A zigzag with such an ordering on the set of its components is called *oriented* and the sequence $\left(\left(B_1^2\right), \ldots, \left(B_n^2\right)\right)$ is called the *type* of B. For an oriented zigzag B, the components B_1 and B_n are called the boundaries of B. Given an irreducible component B_{i_0} of B we denote by $B_{i_0}^{\pm}$ the component $B_{i_0 \pm 1}$ provided it does exist. A zigzag B is called minimal if it contains no (-1)-curve.

Let $C \subset \bar{V}$ be an SNC-divisor. A zigzag B of C is a zigzag with support contained in C and such that no irreducible component of B corresponds to a ramification vertex of the dual graph of C. A zigzag B which is maximal for the inclusion of supports is called maximal. If C itself is not a zigzag, then we call a maximal zigzag B of C simple if only one boundary of B meets a ramification vertex of the dual graph of C. We call it double if this happens for both boundaries of B.

We say that a normal affine surface V is completable by a zigzag if there exists a completion $\bar{V}$ of V such that $B := \bar{V} \setminus V$ is a zigzag.

Properties of $\mathbb{P}^1$-fibrations on normal projective surfaces. We recall some properties of $\mathbb{P}^1$-fibrations on a normal projective surface. The following lemma is well known for a nonsingular surface $\bar{V}$ (see [12, Lemma 1.4.1, p.195]).

Lemma 1.2. *Let $\bar{q}:\bar{V}\to\bar{Z}$ be a $\mathbb{P}^1$-fibration. If $F:=\sum_{1=1}^{p}n_iC_i$ is a fiber of $\bar{q}$ with irreducible components C_i then the following hold.*

1. *The morphism $\bar{q}$ admits a section $S\subset\bar{V}$.*
2. *If F is irreducible and $P=F\cap S$ is a regular point of $\bar{V}$ then F is non-degenerate.*
   Now assume that F is degenerate, then :
3. *The support of F is connected.*
4. *If a singular point P of $\bar{V}$ is contained in a unique curve C_i then it is a cyclic quotient singularity. In this case the proper transform of C_i in a minimal embedded resolution $\pi:\bar{W}\to\bar{V}$ of P meets a terminal component of $\pi^{-1}(P)$.*
5. *If C_i does not contain any singular point of $\bar{V}$ then it is nonsingular ($C_i\simeq\mathbb{P}^1$) and $\left(C_i^2\right)<0$.*
6. *If C_i and C_j, $i\neq j$, are nonsingular and do not contain any singular point of $\bar{V}$ then $(C_i\cdot C_j)=0$ or 1.*
7. *For any three distinct indices i, j and l, either $C_i\cap C_j\cap C_l=\emptyset$ or $C_i\cap C_j\cap C_l$ is a singular point P of $\bar{V}$.*
8. *If F is contained in $\bar{V}\setminus Sing\left(\bar{V}\right)$ then at least one of the C_i, say C_1, is a (-1)-curve. If $\tau:\bar{V}\to\bar{V}_1$ denotes the contraction of C_1 then $\bar{q}$ factors as*
   $$\bar{q}:\bar{V}\xrightarrow{\tau}\bar{V}_1\xrightarrow{\bar{q}_1}\bar{Z},$$
   where $\bar{q}_1:\bar{V}_1\to\bar{Z}$ is a $\mathbb{P}^1$-fibration. Hence all but one irreducible component of F can be contracted succesively to obtain a non-degenerate fiber. Therefore F is an SNC-divisor whose dual graph $\Gamma(F)$ is a tree.
9. *If F is contained in $\bar{V}\setminus Sing\left(\bar{V}\right)$ and if one of the n_i, say n_1, is equal to 1, then there exists a (-1)-curve among the C_i, $2\leq i\leq p$.*

Proof. We let $\phi:\bar{W}\to\bar{V}$ be a minimal embedded resolution of singularities. We denote by $\tilde{q}$ the $\mathbb{P}^1$-fibration on $\bar{W}$ lifting $\bar{q}$ and by $\tilde{S}$ a section of $\tilde{q}$. Then $S:=\phi\left(\tilde{S}\right)$ is a section of $\bar{q}$, and so (1) follows. In the nonsingular case, (2) is a consequence of the existence of a section of $\bar{q}$ and $(3)-(9)$ follow from the genus formula.

In the normal case, (3) and $(5)-(9)$ follow at once from the nonsingular case and (4) can be proved in the same way as Lemma 1.4.4 in [12, p.196]. To show (2) we let $F=\bar{q}^{-1}(z_0)$, $z_0\in Z$, be an irreducible fiber of $\bar{q}$. Its total transform $\phi^{-1}(F)$ is the fiber $\tilde{F}=\tilde{q}^{-1}(z_0)$ of $\tilde{q}$. If $P\in F$ is a singular point of $\bar{V}$ then $\phi^{-1}(P)\subset\bar{W}$ contains no (-1)-curve which meets at most two other components transversally in a single point. Then assertions (7) and (8) on $\bar{W}$ imply that $\phi^{-1}(P)$ contains no (-1)-curve at all. It follows from (8) that the proper transform F' of F is the unique (-1)-curve in $\tilde{F}$. Thus F has to be a non-reduced fiber of $\bar{q}$ for otherwise F' has multiplicity one in $\tilde{F}$ which contradicts (9). Provided that $P_0=S\cap F$ is a regular point of $\bar{V}$, F does not contain any singular point of $\bar{V}$ and so is non-degenerate, which proves (2). □

Remark 1.3. Note that by (7) and (8), a (-1)-curve E contained in a degenerate fiber $F\subset\bar{V}_{reg}$ of $\bar{q}$ cannot be a ramification vertex of the dual graph of $F\cup S$.

We introduce the following :

Definition 1.4. Let $F \subset \bar{V}_{reg}$ be a degenerate fiber of a $\mathbb{P}^1$-fibration $\bar{q} : \bar{V} \to \bar{Z}$ over a nonsingular projective curve $\bar{Z}$ and S be a section of $\bar{q}$. A maximal zigzag D of F (see 1.1) is called terminal if either $D = F$ or D is a maximal simple zigzag of F which does meet S.

In the following lemma we precise the position of (-1)-curves in a degenerate fiber of a $\mathbb{P}^1$-fibration.

Lemma 1.5. *Let $\bar{q} : \bar{V} \to \bar{Z}$ be a $\mathbb{P}^1$-fibration on a normal projective surface $\bar{V}$ over a nonsingular projective curve $\bar{Z}$. Let S be a section of $\bar{q}$ and let $F \subset \bar{V}_{reg}$ be a degenerate fiber of $\bar{q}$. If $F \cup S$ is not a zigzag then the following assertions hold.*

1) At least one (-1)-curve E in F is contained in a maximal terminal zigzag of F .

2) If all such (-1)-curve are contained in the same maximal terminal zigzag D of F then every ramification vertex of the dual graph $\Gamma(F \cup S)$ of $F \cup S$ belongs to the shortest path in $\Gamma(F \cup S)$ which joins D and S.

Proof. Given a (-1)-curve E in F we let $\tau_E : \bar{V} \to \bar{V}_1$ be the contraction of E and we consider the factorization

$$\bar{q} : \bar{V} \stackrel{\tau_E}{\to} \bar{V}_1 \stackrel{\bar{q}_1}{\to} \bar{Z},$$

where $\bar{q}_1 : \bar{V}_1 \to \bar{Z}$ is a $\mathbb{P}^1$-fibration with a degenerate fiber $F_1 := \tau_E(F) \subset (\bar{V}_1)_{reg}$ and a section $S_1 = \tau_E(S)$. By our assumption the graph $\Gamma(F \cup S)$ has a ramification vertex so that $F \cup S$ has at least 4 irreducible components. By 1.3, E is a component of a maximal zigzag D of F.

We consider first the case that $F \cup S = E_1 \cup E_2 \cup E_S \cup S$ has 4 irreducible components, where E_S meets S . It is easily seen that E_S corresponds to a ramification vertex of $\Gamma(F \cup S)$. Then E_1 and E_2 are both maximal terminal zigzags of F and at least one of them is a (-1)-curve, which proves the first assertion in this case. The second assertion follows then at once since E_S is a unique ramification vertex of $\Gamma(F \cup S)$.

To show (1) we may assume that F is not a zigzag for otherwise our statement is evidently true. We also suppose that $F \cup S$ has $n > 4$ irreducible components, and we assume on the contrary that every (-1)-curve E in F is contained either in a maximal simple zigzag of F which meets S or in a maximal double zigzag of F. We denote this maximal zigzag by $D = D(E)$. By our assumption the contraction τ_E of E gives a one-to-one correspondance between the maximal simple zigzags of $F \cup S$ and the maximal simple zigzags of $F_1 \cup S_1$. Moreover none of the maximal terminal zigzags of F is affected by this contraction. Since F_1 has one less irreducible components that F we can conclude by induction that there is a (-1)-curve E_1 in F_1 which belongs to a maximal terminal zigzag of F_1. Then $\tau_E^{-1}(E_1)$ is a (-1)-curve contained in a maximal terminal zigzag of F, a contradiction. Thus assertion (1) is proved.

To prove (2) we may suppose that F is not a zigzag and that $F \cup S$ has $n > 4$ irreducible components. We let E be a (-1)-curve in D. If $D \neq E$ then the contraction τ_E of E yields a bijection between maximal terminal zigzags of F_1 and those of F. Since D is the only maximal terminal zigzag of F affected by the contraction of E it follows from (1) that $\tau_E(D)$ contains a (-1)-curve. In fact it contains all (-1)-curves as in (1), and so we are done by induction.

In case $D = E$ we let H be a ramification vertex of $\Gamma(F \cup S)$ such that E is a branch of $\Gamma(F \cup S)$ at H. Then H has valency 3 for otherwise $\tau_E(H)$ is a ramification vertex of $\Gamma(F_1 \cup S_1)$ and hence none of the maximal terminal zigzags of F_1 contains a (-1)-curve which contradicts (1). Thus if $F_1 \cup S_1$ is a zigzag then we are done. If $F_1 \cup S_1$ is not a zigzag then $\tau_E(H)$ is contained in a maximal zigzag D_1 of F_1. If either D_1 meets S_1 or D_1

is double then τ_E provides a bijective correspondance between the maximal terminal zigzags of F different from E and those of F_1. Since these maximal zigzags of F were not affected by the contraction of E it follows that none of the maximal terminal zigzags of F_1 contains a (-1)-curve which again contradicts (1). Therefore D_1 is a maximal terminal zigzag of F_1 and it contains a (-1)-curve E_1 by (1). Our induction hypothesis then implies that every ramification vertex of $\Gamma(F_1 \cup S_1)$ belongs to the shortest path from E_1 to S_1 in $\Gamma(F_1 \cup S_1)$. As H is the only ramification vertex of $\Gamma(F \cup S)$ which is eliminated by the contraction of E, we conclude that every such ramification vertex belongs to the shortest path from E to S in $\Gamma(F \cup S)$. This proves the second assertion. □

Properties of $\mathbb{A}^1$-fibrations on normal affine surfaces. Given a normal affine surface V together with an $\mathbb{A}^1$-fibration $q : V \to Z$ over a nonsingular affine curve Z, we let $\bar{V}$ be a minimal completion of V. Since V is affine the divisor $B := \bar{V} \setminus V$ is connected. The $\mathbb{A}^1$-fibration q on V induces a rational map $\bar{q} : \bar{V} \dashrightarrow \bar{Z}$, where $\bar{Z}$ denotes a nonsingular projective model of Z. The closures of the fibers of q in $\bar{V}$ define a pencil of nonsingular rational curves with at most one base point on B. If necessary, this base point and all infinitely near ones can be eliminated by a succession of blow-ups with centers outside of V. Thus we may suppose that $\bar{q}$ is a well-defined $\mathbb{P}^1$-fibration on $\bar{V}$.

1.6. In this way we arrive at a completion $\bar{V}$ of V with the following properties :

(1) $\bar{V}$ is a normal projective surface, nonsingular along $B := \bar{V} \setminus V$, with a $\mathbb{P}^1$-fibration $\bar{q} : \bar{V} \to \bar{Z}$ such that the following diagram commutes

$$\begin{array}{ccc} V & \hookrightarrow & \bar{V} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \bar{q}} \\ Z & \hookrightarrow & \bar{Z} \end{array}$$

(2) B is a connected SNC-divisor and can be written as $B = H \cup S \cup G$, where S is a section of $\bar{q}$, $H = \bigcup H_j$, where $H_j := \bar{q}^{-1}(z_j)$ with $z_j \in \bar{Z} \setminus Z$,and the connected components of G are trees of nonsingular rational curves.

(3) We can write $G = \bigcup_{i=}^{s} G_i$, where $\bar{q}(G_i) = z_i \in Z$ and where $z_1, \ldots, z_s \in Z$ are the points such that the fiber $q^{-1}(z_i) \subset V$ is degenerate. Thus $\bar{q}^{-1}(z_i) = G_i \cup \overline{q^{-1}(z_i)}$, $1 \leq i \leq s$, where $\overline{q^{-1}(z_i)}$ denotes the closure of $q^{-1}(z_i)$ in $\bar{V}$.

One can moreover assume that the boundary divisor B contains no (-1)-curve except maybe the section S. Since B contains no singular point of $\bar{V}$, it follows that every H_j is a nonsingular rational curve. In the sequel, such a completion will be called a *good completion of V with respect to q*.

For degenerate fibers of an $\mathbb{A}^1$-fibration on a normal affine surface V, there exists the following description.

Lemma 1.7. *(Miyanishi* [12, Lemmas 1.4.2 and 1.4.4, p.196]*) If $q : V \to Z = \mathbb{A}^1$ is an $\mathbb{A}^1$-fibration then the following assertions hold :*

(1) *Every irreducible component C of $q^{-1}(z)$ is a connected component of $q^{-1}(z)$ and is a rational curve with only one place at infinity. Hence C is isomorphic to $\mathbb{A}^1$ provided it is nonsingular.*

(2) *Every such component C contains at most one singular point of V.*

(3) *The surface V has at most cyclic quotient singularities.*
(4) *IfC contains a singular point P of V and if $\pi : W \to V$ is a minimal embedded resolution of P then the closure $\bar{C}'$ in W of the proper transform C' of C meets a terminal component of $\pi^{-1}(P)$.*

2. Completions of ML-surfaces

This section is devoted to the proof of the following theorem :

Theorem 2.1. *A normal affine surface V has a trivial Makar-Limanov invariant if and only if it is completable by a zigzag.*

To reformulate our statement we need the following lemma.

Lemma 2.2. *(See* e.g. [4]*). If V is a normal affine surface then the following assertions are equivalent :*

(1) *There exists an $\mathbb{A}^1$-fibration $q : V \to Z$ over a nonsingular affine curve Z.*
(2) *The surface V contains a principal Zariski open subset U which is a cylinder : $U \simeq C \times \mathbb{A}^1$.*
(3) *There exists a non-trivial algebraic $\mathbb{C}_+$-action on V.*

As a consequence we obtain :

Corollary 2.3. *For a normal affine surface V the following assertions are equivalent :*

(1) *The Makar-Limanov invariant of V is trivial.*
(2) *There exists at least two nontrivial algebraic $\mathbb{C}_+$-actions on V such that their general orbits do not coincide.*
(3) *There exists at least two $\mathbb{A}^1$-fibrations $q_1 : V \to Z_1$ and $q_2 : V \to Z_2$ over nonsingular affine curves Z_1 and Z_2, such that the general fibers of q_1 and q_2 do not coincide.*

Thus, Theorem 2.1 can be equivalently formulated as follows.

Theorem 2.4. *A normal affine surface is completable by a zigzag if and only if it admits two $\mathbb{A}^1$-fibrations whose general fibers do not coincide.*

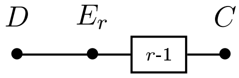
Normal affine surfaces completable by a zigzag.

This section is closely related to the work of Danilov and Gizatullin [5] and [6], where the case of nonsingular surfaces completable by a zigzag was treated. Let us mention first some usefull technical results about zigzags on normal projective surfaces. The following construction will be frequently used in the sequel.

Definition 2.5. Let $\bar{V}$ be a normal projective surface, and let C and D be two irreducible nonsingular curves on $\bar{V}$ which intersect transversally at a single nonsingular point of $\bar{V}$. By the *iterative modification* of $\bar{V}$ *with center* (C, D), *lenght* $r \in \mathbb{N}^*$ *and divisors* $E_1, \ldots E_r$, we mean the birational morphism $\sigma : \bar{W} \to \bar{V}$, where $\bar{W}$ is a normal projective surface, obtained by the following blow-up procedure :

- Step 1 is the blow-up $\sigma_1 : \bar{W}_1 \to \bar{V}$ of the intersection point of C and D with exceptional curve $E_1 \subset \bar{W}_1$.
- Step k for $2 \leq k \leq r$ is the blow-up $\sigma_k : \bar{W}_k \to \bar{W}_{k-1}$ of the intersection point of E_{k-1} and the proper transform of D in $\bar{W}_{k-1}$, with exceptional curve $E_k \subset \bar{W}_k$.

We let $\sigma := \sigma_r \circ \cdots \circ \sigma_1 : \bar{W} := \bar{W}_r \to \bar{V}$. If $C' \subset \bar{W}$ ($D' \subset \bar{W}$) denotes the proper transform of $C \subset \bar{V}$ (of $D \subset \bar{V}$ respectively) then $\left(C'^2\right) = \left(C^2\right) - 1$, $\left(D'^2\right) = \left(D^2\right) - r$, $\left(E_r^2\right) = -1$ and $\left(E_i^2\right) = -2$, $1 \leq i \leq r-1$. For the dual graph of the total transform of $C \cup D$ in $\bar{W}$ we use the following notation :



In 2.6-2.9 below we establish some usefull properties of affine surfaces completable by a zigzag.

Lemma 2.6. *Let $\bar{V}$ be a normal projective surface and $B \subset \bar{V}$ be a zigzag such that $\bar{V}$ is nonsingular along B and $V := \bar{V} \setminus B$ is affine. If B is irreducible then $\left(B^2\right) > 0$. If B is reducible then it contains an irreducible component C with $\left(C^2\right) \geq -1$.*

Proof. Since $V = \bar{V} \setminus B$ is affine, by a theorem of Goodman [7] there exists an ample divisor D on $\bar{V}$ such that $Supp\,(D) = B$. Thus the first assertion follows. Let further B be reducible : $B = \cup_{i=1}^{n} C_i$ with C_i irreducible, $n \geq 2$, and let $D = \sum_{i=1}^{n} m_i C_i$ with $m_i > 0$ for all $1 \leq i \leq n$. Since B is a zigzag we have $\left(C_i \cdot \sum_{j \neq i} C_j\right) \leq 2$. From

$$(D \cdot B) = \sum_{i=1}^{n} m_i \left(C_i \cdot B\right) = \sum_{i=1}^{n} m_i \left(\left(C_i^2\right) + \left(C_i \cdot \sum_{j \neq i} C_j\right)\right) > 0$$

we conclude that there exists i_0 with $\left(C_{i_0}^2\right) > -\left(C_{i_0} \cdot \sum_{j \neq i_0} C_j\right) \geq -2$, whence $\left(C_{i_0}^2\right) \geq -1$. □

Lemma 2.7. *Given a normal affine surface V completable by a zigzag, there exists a minimal completion $\bar{V}$ of V by an oriented zigzag B such that its left boundary C_1 has non-negative self-intersection.*

Proof. If B is irreducible then the assertion follows from lemma 2.6. Thus we may assume that $B = \cup_{i=1}^{n} C_i$ with $n \geq 2$. By lemma 2.6, $\left(C_{i_0}^2\right) \geq -1$ for some i_0, $1 \leq i_0 \leq n$. In fact $\left(C_{i_0}^2\right) \geq 0$ as B is minimal. If $i_0 = 1$ or $i_0 = n$ then, up to reversing the ordering, we are done. If not, we let i_0 be the minimal indice such that $\left(C_i^2\right) \geq 0$, and we denote $C\,(B) := C_{i_0}$ and $d\,(B) = d\,(C_1, C\,(B)) = i_0 - 1$. Thus $\left(C_i^2\right) \leq -2$ for every component C_i to the left of $C\,(B)$.

Since $C\,(B)$ is not a boundary of B, the successor $C\,(B)^+$ of $C\,(B)$ exists, and so we can perform the iterative modification $\sigma : \bar{W} \to \bar{V}$ of $\bar{V}$ with center $\left(C\,(B), C\,(B)^+\right)$, length $c+1$ and divisors $E_1, \ldots, E_c, E_{c+1}$ with $c := \left(C\,(B)^2\right)$. Then we get that $\left(C\,(B)'^2\right) = \left(E_{c+1}^2\right) = -1$. If $\tau : \bar{W} \to \bar{W}_1$ is the contraction of $C\,(B)'$ then $\left(\tau\left(C\,(B)'^-\right)^2\right) = \left(\left(C\,(B)^-\right)^2\right) + 1$ and $\left(\tau\left(E_{c+1}\right)^2\right) = 0$. By iterating this procedure, we obtain that $\left(\left(C\,(B)^-\right)^2\right) = -1$ and $\left(C\,(B)^2\right) = 0$. Contracting $C\,(B)^-$ and all (-1)-curves that arise successively to the left of $C\,(B)$ we arrive at a new completion $\bar{V}_1$ of V by a zigzag B_1 with $d\,(B_1) < d\,(B)$. Since under this procedure, no (-1)-curve has been created on the right of $C\,(B_1)$, it follows that $\bar{V}_1$ is a minimal completion of V. Now the proof can be completed by induction. □

Corollary 2.8. *A normal affine surface completable by a zigzag is rational.*

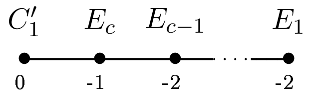
Proof. It is enough to show that there exists a completion $\bar{W}$ of V and a nonsingular rational curve $C \subset \bar{W}_{reg}$ with $\left(C^2\right) > 0$. Let $\bar{V}$, B and C_1 be as in lemma 2.7. If $\left(C_1^2\right) > 0$ then we are done. If not then B is reducible as it is the support of an ample divisor. By our assumptions $\left(C_1^2\right) = 0$ and $\left(C_2^2\right) \leq 0$. After blowing-up with center in $C_1 \setminus C_2$, the proper transform of C_1 becomes a (-1)-curve, and we contract it, obtaining a completion of V with $\left(C_2^2\right)$ increased by one. By iterating this procedure we get a completion $\bar{W}$ of V and a nonsingular rational curve $C \subset \bar{W}_{reg}$ with $\left(C^2\right) > 0$. □

Lemma 2.9. *If V is a normal affine surface completable by a zigzag then the following assertions hold.*

(1) *If V is completable by a zigzag of type $(0,0)$ then $V \simeq \mathbb{A}^2$.*
(2) *If V is completable by a zigzag of type $(0,0,0)$ then $V \simeq \mathbb{C}^* \times \mathbb{A}^1$.*
(3) *If $V \not\simeq \mathbb{C}^2$ and $V \not\simeq \mathbb{C}^* \times \mathbb{C}$ then there exists a completion $\bar{V}$ of V by an oriented zigzag of type $(0,0,k_1,\ldots k_m)$, where $k_i \leq -2$, $1 \leq i \leq m$.*

Proof. We let $\bar{W}$ be a minimal completion of V by an oriented zigzag $B = \cup_{i=1}^{n} C_i$ such that its left boundary C_1 is a curve with non-negative self-intersection.

(1) If $B = C_1$ then $c := \left(B^2\right) > 0$ because B is the support of an ample divisor. Let $D \subset \bar{W}$ be a nonsingular curve germ meeting C_1 transversally in a single point, and consider the iterative modification $\sigma : \bar{W}_1 \to \bar{W}$ of $\bar{W}$ with center (D, C_1), length c and divisors $E_1, \ldots E_c$ (see 2.5). Then the total transform B_1 of B is a zigzag whose left boundary is the proper transform $C_1'$of C_1. Moreover $\left(C_1'^2\right) = 0$, $\left(E_c^2\right) = -1$ and $\left(E_i^2\right) = -2$, $1 \leq i \leq c-1$. Thus B is now replaced by a zigzag with the following dual graph:



Let $\pi : \bar{W}_2 \to \bar{W}_1$ be the blow-up of a point $v \in C_1' \setminus E_c$ with exceptional component $E \subset \bar{W}_2$. Then the proper transform of C_1' in $\bar{W}_2$ is a (-1)-curve that can be contracted to obtain a completion $\bar{V}$ of V by a zigzag of type $(0,0,-2,\ldots,-2)$.

(2) If $B \neq C_1$ and $c = \left(C_1^2\right) > 0$ then by applying the same procedure as in (1) we obtain a new minimal completion $\bar{W}_1$ of V by a reducible zigzag such that $\left(C_1^2\right) = 0$. Performing, if necessary, elementary transformations we obtain a minimal completion by a zigzag with $\left(C_1^2\right) = \left(C_2^2\right) = 0$. We must distinguish then the following three cases :

- $B = C_1 \cup C_2$. Since $\bar{W}_1$ is rational the linear system $|C_1|$ defines a $\mathbb{P}^1$-fibration $\bar{q} : \bar{W}_1 \to \bar{Z} = \mathbb{P}^1$ whose restriction to V is an $\mathbb{A}^1$-fibration $q : V \to Z = \bar{Z} \setminus \{\bar{q}(C_1)\} \simeq \mathbb{A}^1$. Thus $\bar{W}_1$ is a good completion of V with respect to q. Moreover, every fiber $\bar{q}^{-1}(z)$, $z \in Z$, coincides with the closure of $q^{-1}(z)$ in $\bar{V}$, and, being connected, it is irreducible. Therefore, by virtue of lemma 1.2(2), $\bar{q}$ has no degenerate fiber, and hence $\bar{W}_1$ is nonsingular. From $\left(C_1^2\right) = \left(C_2^2\right) = 0$ we finally deduce $\bar{W}_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ so that $V = \bar{W}_1 \setminus (C_1 \cup C_2)$ is isomorphic to $\mathbb{A}^2$.
- If $B = C_1 \cup C_2 \cup C_3$ and $\left(C_3^2\right) = 0$ then the linear system $|C_1|$ defines a $\mathbb{P}^1$-fibration $\bar{q} : \bar{W}_1 \to \bar{Z} = \mathbb{P}^1$ whose restriction to V is an $\mathbb{A}^1$-fibration $q : V \to Z = \bar{Z} \setminus \{\bar{q}(C_1), \bar{q}(C_3)\} \simeq \mathbb{C}^*$. Thus $\bar{W}_1$ is a good completion of V with respect to q and we

can again conclude that $\bar{q}$ has no degenerate fiber. Hence $\bar{W}_1$ is a nonsingular surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Finally we have $V = \bar{W}_1 \setminus (C_1 \cup C_2 \cup C_3) \simeq \mathbb{C}^* \times \mathbb{A}^1$.

- It remains to consider the case $B = C_1 \cup C_2 \cup G$, where either $G = C_3$ with $\left(C_3^2\right) \neq 0$ or $G = \cup_{i=3}^n C_i$ with $n > 3$. The linear system $|C_1|$ defines a $\mathbb{P}^1$-fibration $\bar{q} : \bar{W}_1 \to \mathbb{P}^1$ having C_2 as a cross-section. Since G is connected and does not intersect C_1 it must be contained in a fiber F of $\bar{q}$. Moreover, F must be a singular fiber of $\bar{q}$ for otherwise, we would have $F = C_3$ and hence, $0 = \left(F^2\right) = \left(C_3^2\right) \neq 0$, a contradiction. In virtue of lemma 1.2, every C_i with $3 \leq i \leq n$ has negative self-intersection. Since the initial completion $\bar{W}$ has been assumed minimal and since our transformations do not affect the curves C_i for $3 \leq i \leq n$, we conclude that $\left(C_i^2\right) \leq -2$ for all $3 \leq i \leq n$.

□

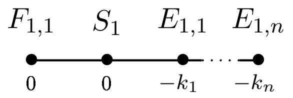
The next proposition proves one of the two implications of theorem 2.1.

Proposition 2.10. *If V is a normal affine surface non-isomorphic to $\mathbb{C}^* \times \mathbb{A}^1$ and completable by a zigzag then V has a trivial Makar-Limanov invariant.*

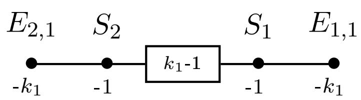
Proof. If V admits a completion $\bar{V}$ by a zigzag of type $(0,0)$ then, by case (1) of lemma 2.9, $V \simeq \mathbb{C}^2$ which has a trivial Makar-Limanov invariant. Therefore we may assume from now on that case (3) of lemma 2.9 holds, that is V has a completion $\bar{V}_1$ by a zigzag B_1 of type $(0,0,-k_1,\ldots,-k_n)$ with $k_i \geq 2$, $1 \leq i \leq n$. As in (1.6) we write

$$B_1 = F_{1,1} \cup S_1 \cup \left(\bigcup_{i=1}^{n} E_{1,i}\right),$$

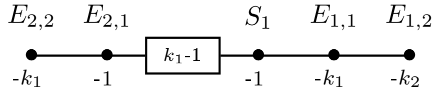
where $\left(F_{1,1}^2\right) = \left(S_1^2\right) = 0$ and $\left(E_{1,i}^2\right) = -k_i$, $1 \leq i \leq n$. The dual graph $\Gamma(B_1)$ is the following:



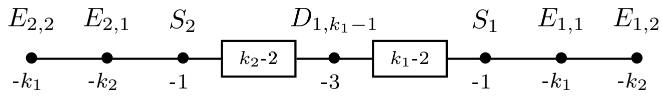
The linear system $|F_{1,1}|$ defines a $\mathbb{P}^1$-fibration $\bar{q}_1 : \bar{V}_1 \to \mathbb{P}^1$ with S_1 as a cross-section so that the restriction $q_1 : V \to \mathbb{A}^1$ of $\bar{q}_1$ to V is an $\mathbb{A}^1$-fibration. Thus it remains to find a second $\mathbb{A}^1$-fibration $q_2 : V \to \mathbb{A}^1$ such that the general fibers of q_1 and q_2 do not coincide. To do this we construct a completion $\bar{W}$ of V together with a birational morphism $\sigma_1 : \bar{W} \to \bar{V}_1$ which will also dominate a good completion $\bar{V}_2$ of V with respect to this $\mathbb{A}^1$-fibration q_2. It will be convenient in the sequel to denote the component $F_{1,1}$ of B by $E_{2,n}$.
If $n = 1$ then $\sigma_1 : \bar{W} \to \bar{V}_1$ is the iterative modification of $\bar{V}_1$ with center $(S_1, E_{2,1})$, length k_1 and divisors $D_1, \ldots, D_{k_1-1}, S_2$. For the total transform B of B_1 we obtain the following symmetrical dual graph:



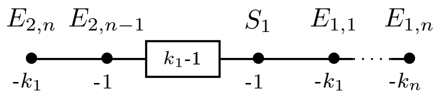
In case $n = 2$ we obtain $\sigma_1 : \bar{W} \to \bar{V}_1$ by the following procedure :
- Step 1 is the iterative modification $\pi_1 : \bar{W}_1 \to \bar{V}_1$ with center $(S_1, E_{2,2})$, length k_1 and divisors $D_{1,1}$, D_{1,k_1-1}, $E_{2,1}$. The dual graph of the total transform of B_1 is the following :



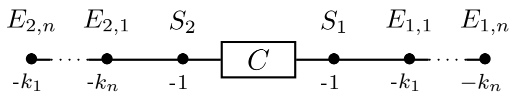
- Step 2 is the iterative modification $\pi_2 : \bar{W}_2 \to \bar{W}_1$ of $\bar{W}_1$ with center $\left(E_{2,1}^{+} = D_{1,k_1-1}, E_{2,1}\right)$, length $k_2 - 1$ and divisors $D_{2,1}, \ldots, D_{2,k_2-2}, S_2$ if $k_2 > 2$ or just S_2 if $k_2 = 2$.We then let $\bar{W} := \bar{W}_2$ and $\sigma_1 = \pi_1 \circ \pi_2 : \bar{W} \to \bar{V}_1$. The dual graph of the total transform $B = \sigma_1^{-1}(B_1)$ of B_1 has the following structure:



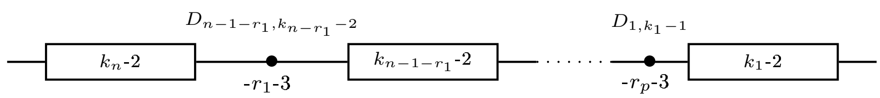
We observe that the same dual graph can be obtained from a zigzag of type $(0, 0, -k_2, -k_1)$ by reversing the ordering and the blow-up procedure.
In case $n \geq 3$, $\bar{W}$ is obtained from $\bar{V}_1$ by the following procedure :
-Step 1 is the iterative modification $\pi_1 : \bar{W}_1 \to \bar{V}_1$ with center $(S_1, E_{2,n})$, length k_1 and divisors $D_{1,1}, \ldots, D_{1,k_1-1}$, $E_{2,1}$. Then the dual graph of the total transform of B_1 is the following one:



-Step m, where $2 \leq m \leq n-1$, is the iterative modification $\pi_m : \bar{W}_m \to \bar{W}_{m-1}$ of $\bar{W}_{m-1}$ with center $\left(E_{2,n-m}^{+}, E_{2,n-m}\right)$, length $k_m - 1$ and divisors $D_{m,1}, \ldots, D_{m,k_m-2}, E_{2,n-m-1}$ if $k_m > 2$ or just $E_{2,n-m-1}$ if $k_m = 2$.
- Step n is the last step and consists of the iterative modification $\pi_n : \bar{W}_n \to \bar{W}_{n-1}$ of $\bar{W}_{n-1}$ with center $\left(E_{2,1}^{+}, E_{2,1}\right)$, length $k_n - 1$ and divisors $D_{n,1}, \ldots, D_{n,k_n-2}, S_2$ if $k_n > 2$ or just S_2 if $k_n = 2$.
Then we let $\bar{W} := \bar{W}_n$ and $\sigma_1 := \pi_1 \circ \cdots \circ \pi_n : \bar{W} \to \bar{V}_1$. For the total transform $B := \sigma_1^{-1}(B_1)$ of B_1 we obtain the following dual graph:



The dual graph of C looks like:



where $r_j \geq 0$ $(0 \leq j \leq p)$ depend on the number of (-2)-curves among the $E_{1,i}$, $1 \leq i \leq n$. Obviously, $V = \bar{W} \setminus B$. We observe as before that the same dual graph can be obtained from a zigzag of type $(0, 0, -k_n, \ldots, -k_1)$ by a symmetric blow-up procedure. Henceforth, the sub-zigzag

$$D := C \cup S_1 \cup \bigcup_{i=1}^{n-1} E_{1,i}$$

of B can be contracted to a nonsingular point. We denote this contraction by $\sigma_2 : \bar{W} \to \bar{V}_2$, and we let

$$B_2 = F_{2,1} \cup S_2 \cup \left(\bigcup_{i=1}^{n} E_{2,n-i+1} \right)$$

be the image of B by σ_2, where $F_{2,1} := E_{1,n}$. Then $V = \bar{V}_2 \setminus B_2$ where B_2 is a zigzag of type $(0, 0, -k_n, \ldots, -k_1)$.

The linear system $|F_{2,1}|$ defines then a $\mathbb{P}^1$-fibration $\bar{q}_2 : \bar{V}_2 \to \mathbb{P}^1$ whose restriction to V is a second $\mathbb{A}^1$-fibration $q_2 : V \to \mathbb{A}^1$. Moreover, since

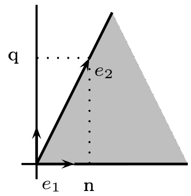
$$\sigma_2 \left(\sigma_1^* \left(F_{1,1} \right) \right) = \alpha S_2 + \sum_{i=1}^{n} \beta_i E_{2,i}$$

with $\alpha > 0$ and $\beta_i \geq 0$, $1 \leq i \leq n$, it follows that $\left(F_{2,1} \cdot \sigma_2 \left(\sigma_1^* \left(F_{1,1} \right) \right) \right) \geq 1$. Thus the general fibers of q_1 and q_2 do not coincide, whence V has a trivial Makar-Limanov invariant. □

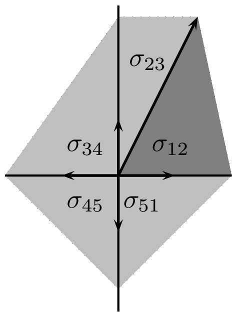
Finally we have the following proposition :

Proposition 2.11. *Every normal affine toric surface except for $\mathbb{C}^* \times \mathbb{C}^*$ and $\mathbb{C}^* \times \mathbb{A}^1$ has a trivial Makar-Limanov invariant. Consequently, every cyclic quotient singularity appears as a singular point of an ML-surface.*

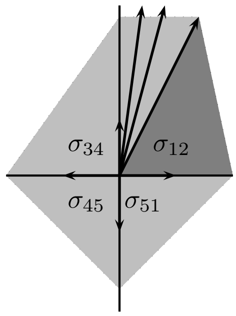
Proof. Recall that, given a 2-dimensional lattice N, an affine toric surface corresponds to a strictly convex rational polyhedral cone in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. If V is a normal affine toric surface non-isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}$, then there exists a basis of N such that V is given by the cone $\sigma_{12} = \langle e_1, e_2 \rangle$ with $e_1 = (1, 0)$ and $e_2 = (n, q)$ where n and q are coprime integers.



In order to construct a completion of V we need to include σ_{12} into a complete fan Δ in $N_{\mathbb{R}}$. This can be done *e.g.* in the following way.



Δ



$\tilde{\Delta}$

We let $\sigma_{ij} = \langle e_i, e_j \rangle$ with $e_3 = (0,1)$, $e_4 = (-1,0)$ and $e_5 = (0,-1)$.The only possibly singular cones (*i.e.* cones whose generators do not form a basis of N) in Δ are σ_{12} and σ_{23}. We can subdivide the cone σ_{23} if necessary to obtain a new fan $\tilde{\Delta}$ such that σ_{12} is the only possibly singular cone in $\tilde{\Delta}$.We denote by e_i for $6 \leq i \leq r$ the new generators introduced in this subdivision procedure. Then $\bar{V} := V\left(\tilde{\Delta}\right)$ is a completion of $V := V(\sigma_{12})$. We let $D_i = V(\tau_i)$ be the divisor on $\bar{V}$ corresponding to the cone $\tau_i = \langle e_i \rangle$ for $3 \leq i \leq r$. Then $B := \bar{V} \setminus V = D_3 \cup D_4 \cup \cdots \cup D_r$ is a zigzag, whence V has a trivial Makar-Limanov invariant by proposition 2.10. □

Completion of a normal affine surface with a trivial Makar-Limanov invariant. In this subsection we prove that, conversely, every ML-surface V is completable by a zigzag.

2.12. By corollary 2.3 there exist two $\mathbb{A}^1$-fibrations $q_1 : V \to Z_1 \simeq \mathbb{A}^1$ and $q_2 : V \to Z_2 \simeq \mathbb{A}^1$ whose general fibers do not coincide. We denote by $\bar{V}_1$ a good completion of V with respect to q_1, with a boundary divisor $B = H \cup S \cup G \subset \left(\bar{V}_1\right)_{reg}$ as in 1.6. Thus q_1 extends to a $\mathbb{P}^1$-fibration $\bar{q}_1 : \bar{V}_1 \to \bar{Z}_1 = \mathbb{P}^1$ so that $H = \bar{q}_1^{-1}(\infty) =: F_\infty$ is a non-degenerate fiber of $\bar{q}_1$ over the point $\infty := \bar{Z}_1 \setminus Z_1$, and $S \simeq \mathbb{P}^1$ is a section.

We let $\bar{q}_2 : \bar{V}_1 \dashrightarrow \bar{Z}_2 \simeq \mathbb{P}^1$ be the rational map which extends $q_2 : V \to Z_2$. We let $\bar{T}_2$ be the closure in $\bar{V}_1$ of a general fiber T_2 of q_2. The point $\bar{T}_2 \setminus T_2$ belongs to F_∞, for otherwise the restriction of q_1 to a general fiber of q_2 would be constant and the general fibers of these two $\mathbb{A}^1$-fibration would coincide, in contrary to our assumption. As G is disjoint from F_∞, the map $\bar{q}_2$ has no base point on G, and so $\bar{q}_2|_G$ must be locally constant. Moreover $\bar{q}_2|_{S\setminus\{P_0\}} = \infty$, for otherwise q_2 would be bounded whence constant along a general fiber of q_1. Since $S \cup G$ is connected, it follows that $\bar{q}_2|_{(S\cup G)\setminus\{P_0\}} = \infty$.

Lemma 2.13. *If $\bar{q}_2 : \bar{V}_1 \dashrightarrow \bar{Z}_2$ is a morphism then $G = \emptyset$, $B = F_\infty \cup S$ is a zigzag and $V \simeq \mathbb{A}^2$.*

Proof. If $\bar{q}_2 : \bar{V}_1 \to \bar{Z}_2$ is a morphism then it is a $\mathbb{P}^1$-fibration and its general fiber meets F_∞ at one point. It follows that F_∞ is a section of $\bar{q}_2$ and $S \cup G$ is contained in the fiber $\bar{q}_2^{-1}(\infty) \subset \left(\bar{V}_1\right)_{reg}$. Moreover $\bar{q}_2^{-1}(\infty) = S \cup G$ as $\bar{q}_2^{-1}(\infty) \subset \bar{V}_1 \setminus V$. Since $\bar{V}_1$ is a minimal completion of V it follows that $S \cup G$ contains no (-1)-curve whence is a non-degenerate fiber of $\bar{q}_2$ (see (5) of 1.2). Thus $\left(S^2\right) = 0$, $G = \emptyset$ and $\bar{q}_2^{-1}(\infty) = S$ so that the zigzag $B = F_\infty \cup S$ is of type $(0,0)$ and $V \simeq \mathbb{A}^2$ by lemma 2.9. □

2.14. If $\bar{q}_2$ is not a morphism then $\bar{q}_2$ defines a linear pencil with a unique base point $P \in F_\infty$. Suppose that $P = P_0 := S \cap F_\infty$. If we blow-up the point P_0 into an exceptional component E, the proper transform F'_∞ of F_∞ is a (-1)-curve. By contracting F'_∞, we obtain a new completion of V in which $\left(S^2\right)$ has decreased by one. By applying these transformations with center P_0 several times, we arrive at the situation that the linear pencil $\bar{q}_2 : \bar{V}_1 \dashrightarrow \bar{Z}_2 \simeq \mathbb{P}^1$ has no base point on the proper transform of S. So we may assume from the very beginning that $\bar{V}_1$ is a good completion of V with respect to q_1 such that $\bar{q}_2$ has a unique base point $P \in F_\infty \setminus S$. Note that this new completion $\bar{V}_1$ of V is not necessarily minimal, but anyhow the only possible (-1)-curve in the boundary B is a section S of $\bar{q}_1$. Observe also that, as $\bar{V}_1$ is obtained from a given good completion $\bar{V}$ of V with respect to $\bar{q}_1$ by means of elementary transformations with centers in F_∞, $\bar{V}_1 \setminus V$ is a zigzag if and only if $\bar{V} \setminus V$ is.

The following proposition proves the second implication of theorem 2.1.

Proposition 2.15. *If V is a ML-surface with an $\mathbb{A}^1$-fibration $q : V \to Z \simeq \mathbb{A}^1$ then, for any good completion $\bar{V}$ of V with respect to q as in 1.6, the divisor $B = \bar{V} \setminus V$ is a zigzag. Moreover the $\mathbb{A}^1$-fibration q has at most one degenerate fiber.*

Proof. If $\bar{q}_2 : \bar{V} \to \bar{Z}_2$ is a morphism then, by lemma 2.13, B is a zigzag and we are done. We now suppose that $\bar{q}_2$ is not a morphism. Due to 2.14 we can also suppose that the unique base point P of the linear pencil $\bar{q}_2$ belongs to $F_\infty \setminus S$. We let $\pi : \bar{W} \to \bar{V}_1$ be a minimal resolution of the base points of $\bar{q}_2$, and we denote by $\tilde{q}_2 : \bar{W} \to \bar{Z}_2$ the $\mathbb{P}^1$-fibration which lifts $\bar{q}_2$. The last (-1)-curve arising from this elimination procedure gives rise to a section S_2 of $\tilde{q}_2$, and it is a unique (-1)-curve in $\pi^{-1}(P)$. Since $\bar{q}_2|_{S \cup G} = \infty$, the proper transform of $S \cup G$ in $\bar{W}$ is contained in the fiber $\tilde{q}_2^{-1}(\infty)$. If $\tilde{T}_2$ is a general fiber of $\tilde{q}_2$ then the point $\tilde{T}_2 \setminus T_2$ belongs to $\pi^{-1}(P)$. It follows that the proper transform of F_∞ in $\bar{W}$ is disjoint from $\tilde{T}_2$ whence is contained in a fiber of $\tilde{q}_2$. Since $P \in F_\infty \setminus S$ the proper transform of $B = F_\infty \cup S \cup G$ is connected and so is contained in $\tilde{q}_2^{-1}(\infty) \subset \bar{W}_{reg}$. As $\tilde{q}_2^{-1}(\infty) \subset \bar{W} \setminus V$ is then degenerate, by (8) of lemma 1.2 it must contain a (-1)-curve. Since no such curve can be contained in $G \cup \left(\pi^{-1}(P) \cap \tilde{q}_2^{-1}(\infty)\right)$ it follows that the proper transform of S or F_∞ is a (-1)-curve. Since these two curves meet and are contained in a maximal simple zigzag of $\tilde{q}_2^{-1}(\infty)$ which intersects the section S_2, we deduce from lemma 1.5 that $\tilde{q}_2^{-1}(\infty) \cup S_2$ is a zigzag. Therefore G is connected and is a zigzag, whence q has a unique degenerate fiber. It follows that $B = F_\infty \cup S \cup G$ is a zigzag. □

More generally we have the following theorem.

Theorem 2.16. *If V is an ML-surface then the boundary divisor $C := \bar{V} \setminus V$ of any minimal completion $\bar{V}$ of V is a zigzag.*

The proof is done in 2.17-2.20 below. Remind that $\bar{V}$ is a minimal completion of V iff C is an SNC-divisor containing no (-1)-curve which meets at most two other irreducible components transversally in a single point (see 1.1). Since V is affine C is connected. The $\mathbb{A}^1$-fibration $q_1 : V \to \mathbb{A}^1$ extends to a rational map $\bar{q}_1 : \bar{V} \dashrightarrow \mathbb{P}^1$ with at most one base point P on C.

Lemma 2.17. *If $\bar{q}_1 : \bar{V} \to \mathbb{P}^1$ is a morphism then C is a zigzag.*

Proof. Since the closure $\bar{T}_1$ of a general fiber T_1 of q_1 intersects C in a single point it follows that there exists a unique irreducible component S of C which is a section of $\bar{q}_1$. If $C = S$ we are done.

If S is a terminal component of C then $C\setminus S$ is connected whence contained in a unique fiber F of $\bar{q}_1$. Moreover since $\bar{q}_1^{-1}(\infty) \subset C$ we get that $F = F_\infty = \bar{q}_1^{-1}(\infty)$ and $F_\infty = \overline{C \setminus S} \subset \bar{V}_{reg}$. By the minimality of C, it then follows from 1.3 that $\overline{C \setminus S}$ cannot contain a (-1)-curve. Hence the fiber F_∞ of $\bar{q}_1$ is nondegenerate, and so $C = S \cup F_\infty$ is a zigzag with two components.

If S is not a terminal component of C we denote by $G_1, \ldots, G_n$ the connected components of $\overline{C \setminus S}$. Then every G_i is contained in a fiber F_i of $\bar{q}_1$, whence with the same argument as above G_i cannot contain a (-1)-curve. Since one of the F_i, say F_n, is the fiber $F_\infty = \bar{q}_1^{-1}(\infty) \subset \bar{V}_{reg}$ it follows that $G_n = F_\infty \simeq \mathbb{P}^1$. Hence $\bar{V}$ is a minimal good completion of V with respect to q_1 (see 1.6). Thus according to proposition 2.15 $n = 2$ and $C = F_\infty \cup S \cup G_1$ is a zigzag. □

2.18. Therefore we may suppose in the sequel that that neither q_1 nor q_2 extends to a morphism on $\bar{V}$. We let $P \in C$ be the unique base point of the rational map $\bar{q}_1 : \bar{V} \dashrightarrow \bar{Z}_1$, and $\pi : \bar{W} \to \bar{V}$ be a minimal resolution of P. That is $\bar{q}_1$ lifts to a $\mathbb{P}^1$-fibration $\tilde{q}_1 : \bar{W} \to \bar{Z}_1$ and $\pi^{-1}(P)$ contains a unique (-1)-curve S which is a section of $\tilde{q}_1$. Since the closure $\tilde{T}_1$ in $\bar{W}$ of a general fiber of q_1 meets $\pi^{-1}(C)$ in a single point it follows that every connected component of the proper transform C' of C in $\bar{W}$ is contained in a fiber of $\tilde{q}_1$.

Lemma 2.19. *If P belongs to just one irreducible component D of C then C is a zigzag.*

Proof. In this case C' is connected whence contained in the fiber F_∞ of $\tilde{q}_1$. Thus $F_\infty \subset \bar{W}_{reg}$ does not contain (-1)-curves except maybe for the proper transform D' of D. Indeed, by the minimality of C and of the resolution of P, such a (-1)-curve in F_∞, different from D', must be a ramification point of C', which is excluded by 1.3. If the fiber F_∞ does not contain a (-1)-curve then it is nondegenerate, so $F_\infty = D'$ with $\left(D'^2\right) = 0$ and $C = D$ is a zigzag. We now suppose that $C \neq D$. Then F_∞ is degenerate and D' is a unique (-1)-curve in F_∞. Therefore D is a terminal component of C for otherwise D' is a ramification vertex of $\Gamma(F_\infty \cup S)$, which contradicts 1.3. If C is not a zigzag then $F_\infty \cup S$ is not a zigzag either since it contains C'. To eliminate this possibility we note that if $\pi^{-1}(P)$ is a zigzag then D' is contained in a maximal simple zigzag of F_∞ which meets S, and otherwise D' is contained in a maximal double zigzag of F_∞. But both these possibilities are excluded by lemma 1.5. Hence C is a zigzag. □

The following lemma completes the proof of theorem 2.16.

Lemma 2.20. *In the situation of 2.18, if P belongs to two irreducible components say D_1 and D_2 of C then C is a zigzag.*

Proof. In this case the proper transform C' of C has two connected components C'_1 and C'_2, where D'_i is a terminal component of C'_i, $i = 1, 2$. Therefore either C' is entirely contained in the fiber F_∞ of $\tilde{q}_1$, or there exists another fiber F_1 of $\tilde{q}_1$ such that say $C'_1 \subset F_1$ and $C'_2 \subset F_\infty$. The latter happens if and only if $D'_1 \cup \pi^{-1}(P) \cup D'_2$ is a zigzag. Indeed, otherwise at some step $k \geq 2$ of the resolution procedure, we must have blown-up a simple point $P_k \in \pi_{k-1}^{-1}(D_1 \cup D_2)$ into an exceptional component E_k. As E_k is terminal in the dual graph of $D'_1 \cup \pi_k^{-1}(P) \cup D'_2$ we then conclude that $\pi_k^{-1}(C) \setminus E_k$ is connected. Since all further blow-ups have their centers over E_k it follows that the proper transform of $\pi_{k-1}^{-1}(C)$ in $\bar{W}$ contains C' and is connected. This implies that C' is entirely contained in a fiber of $\tilde{q}_1$.

1) We first suppose that C' is contained in the fiber $F_\infty \subset \bar{W}_{reg}$ of $\tilde{q}_1$. For $i = 1, 2$ we consider the shortest paths joining D'_i to S in the tree $\Gamma(F_\infty \cup S)$, and we denote by D_0 the vertex where they meet. Since C' is not connected it follows that D_0 is contained in $\overline{F_\infty \setminus C'}$

and is a ramification vertex of $\Gamma(F_\infty \cup S)$. Moreover F_∞ is degenerate and the only possible (-1)-curves in F_∞ are D_1' and D_2'. So by (1) of lemma 1.5 at least one of the D_i', say D_1' is a (-1)-curve contained in a maximal terminal zigzag of F_∞. Clearly, this zigzag contains also C_1'. This implies that D_1 is not a ramification vertex of $\Gamma(C)$ for otherwise D_1' is a ramification vertex of $\Gamma(F_\infty \cup S)$ which contradicts 1.3.

If either C_2' is not a zigzag or D_2 is a ramification vertex of $\Gamma(C)$ then D_1' is a unique (-1)-curve contained in a maximal terminal zigzag of F_∞ and there exists a ramification vertex H' of $\Gamma(F_\infty \cup S)$ which is not contained in the shortest path joining D_1' to S in $\Gamma(F_\infty \cup S)$. Indeed in the first case C_2' is not a zigzag whence it contains such a ramification vertex H', and in the second case we can choose $H' = D_2'$. This contradicts (2) of lemma 1.5 hence, C_2' is a zigzag and D_2 is not a ramification vertex of $\Gamma(C)$. Thus $C = C_1 \cup C_2$ is a zigzag too.

2) We now suppose that C' is not entirely contained in a fiber of $\tilde{q}_1$. Thus $D_1' \cup \overline{\pi^{-1}(P) \cup D_2'}$ is a zigzag. Moreover there exist two connected components, say G_1 and G_2, of $\overline{\pi^{-1}(C) \setminus S}$ and two different fibers F_1 and $F_2 = F_\infty$ of $\tilde{q}_1$ such that $C_i' \subset G_i \subset F_i$ for $i = 1, 2$. Since $F_\infty \subset \bar{W}_{reg}$ we can deduce similarly as in lemma 2.19 that $F_\infty \cup S$ is a zigzag. This implies that C_2 is a zigzag and D_2 is not a ramification vertex of $\Gamma(C)$. We let $\tau_\infty : \bar{W} \to \bar{W}_1$ be the contraction of F_∞ to a nondegenerate fiber of a $\mathbb{P}^1$-fibration. That is, $\tau_\infty(F_\infty) \simeq \mathbb{P}^1$ is a nondegenerate fiber $\hat{F}_\infty = \hat{q}_1^{-1}(\infty)$ of the resulting $\mathbb{P}^1$-fibration $\hat{q}_1 : \bar{W}_1 \to \bar{Z}_1$. Since the components of F_1 are not affected by this contraction it follows that $\tau_\infty(D_1')$ is the only possible (-1)-curve in $\tau_\infty(G_1) \subset \hat{F}_1 = \tau_\infty(F_1)$. Moreover as $D_1' \cup \pi^{-1}(P) \cup D_2'$ is a zigzag, $\tau_\infty(D_1')$ is contained in the maximal simple zigzag of $\tau_\infty(G_1)$ which meets the section $\hat{S}$ of $\hat{q}_1$.

If $\tau_\infty(G_1)$ contains no (-1)-curve then $\bar{W}_1$ is a good completion of V with respect to q_1, and it follows from proposition 2.15 that $\tau_\infty\left(G_1 \cup \hat{S}\right)$ is a zigzag. Thus C_1 is a zigzag too and D_1 is not a ramification vertex of $\Gamma(C)$.

Otherwise $\tau_\infty(D_1')$ is a unique (-1)-curve of $\tau_\infty(G_1)$. Starting with $\tau_\infty(D_1')$ we can successively contract the (-1)-curves which arise in $\tau_\infty(G_1)$ to obtain a minimal good completion $\bar{W}_2$ of V with respect to q_1. Hence the image of $\tau_\infty\left(G_1 \cup \hat{S}\right)$ in $\bar{W}_2$ is a zigzag by proposition 2.15. Since $\tau_\infty(D_1')$ is contained in a maximal simple zigzag of $\tau_\infty(G_1)$ which meets $\hat{S}$ it follows that none of the possible ramification vertices of $\tau_\infty\left(G_1 \cup \hat{S}\right)$ has been eliminated by the above contractions. This means that $\tau_\infty\left(G_1 \cup \hat{S}\right)$ is also a zigzag. Thus C_1 is a zigzags and D_1 is not a ramification vertex of $\Gamma(C)$. Hence $C = C_1 \cup C_2$ is a zigzag too. □

We complete our discussion by a characterization of the affine plane. We need the following lemma.

Lemma 2.21. *(seealso* [1]*)Let V be an ML-surface and $q_i : V \to Z_i \simeq \mathbb{A}^1$, $i = 1, 2$, be two $\mathbb{A}^1$-fibrations whose general fibers do not coincide. Then $\phi_{12} := q_1 \times q_2 : V \to \mathbb{A}^2$ is a surjective, quasi-finite morphism.*

Proof. We let $\bar{V}$ be a good completion of V with respect to q_1 by a zigzag $B = G \cup S \cup F_\infty$ as in 2.12, and we denote by $\bar{q}_1 : \bar{V} \to \bar{Z}_1 \simeq \mathbb{P}^1$ the $\mathbb{P}^1$-fibration which extends q_1. If the $\mathbb{A}^1$-fibration $q_2 : V \to Z_2 \simeq \mathbb{A}^1$ extends to a $\mathbb{P}^1$-fibration $\bar{q}_2 : \bar{V} \to \bar{Z}_2 \simeq \mathbb{P}^1$ then $V \simeq \mathbb{A}^2$ by lemma 2.13 and q_1 and q_2 are coordinates on V which proves the assertion. So we may assume from now on that $\bar{q}_2 : \bar{V} \dashrightarrow \bar{Z}_2 \simeq \mathbb{P}^1$ is a linear pencil with a unique base point $P \in F_\infty \setminus S$

(see 2.14). Therefore $\bar{q}_2|_{S\cup G}=\infty$ and $\bar{T}_2\setminus T_2=P$ for the closure $\bar{T}_2$ of a general fiber T_2 of q_2.

To prove that ϕ_{12} is quasi-finite it is sufficient to show that none of the irreducible components of a fiber of q_2 is contained in a fiber of q_1. Suppose on the contrary that there exists an irreducible component C of a fiber F_1 of q_1 which is contained in a fiber F_2 of q_2. If F_1 were a nondegenerate fiber of q_1 then its closure $\bar{F}_1=\bar{C}$ in $\bar{V}$ would meet S in a single point P_1. Since $\bar{q}_2|_C$ is constant and finite and $\bar{q}_2\left(\bar{C}\cap S\right)=\infty$ it follows that P_1 would be a base point of $\bar{q}_2$ which is impossible. Thus by proposition 2.15 F_1 is a unique degenerate fiber of q_1 and hence, $\bar{C}$ meets G (see 2.12). Since $q_2|_C$ is constant and finite and $\bar{q}_2\left(G\cap\bar{C}\right)=\infty$ it follows that $Q=G\cap\bar{C}$ is a base point of $\bar{q}_2$ which is again impossible. Thus there is no such curve C on V and hence, ϕ_{12} is quasi-finite.

The normalisation of every irreducible component C of a fiber of q_2 is isomorphic to $\mathbb{A}^1$ by lemma 1.7. Hence the restriction of q_1 to C is nonconstant and surjective, and so $\phi:V\to\mathbb{A}^2$ is a surjection as required. □

Corollary 2.22. *A normal affine surface V is isomorphic to $\mathbb{A}^2$ if and only if it admits two $\mathbb{A}^1$-fibrations whose general fibers meet in a single point.*

Proof. We let $q_i:V\to Z_i\simeq\mathbb{A}^1$, $i=1,2$ be two $\mathbb{A}^1$-fibrations as above. The morphism $\phi:=q_1\times q_2:V\to\mathbb{A}^2$ is surjective and quasi-finite by lemma 2.21. Since the general fibers of q_1 and q_2 meet in a single point it follows that ϕ is birational. By the Zarisky Main Theorem (see *e.g.* [8]) there exists a factorization

$$\phi:V\overset{\phi'}{\to}X\overset{u}{\to}\mathbb{A}^2$$

where ϕ' is an open immersion and $u:X\to\mathbb{A}^2$ is finite and birational whence an isomorphism. Then $\phi'=\phi$ is an isomorphism too as ϕ is surjective. □

To conclude we provide a series of examples of nonsingular affine surfaces in $\mathbb{A}^3$ with easily computable completions, and we distinguish ML-surfaces among these.

Example 2.23. We consider the hypersurface $V:=V_{P,n}$ of $\mathbb{A}^3=Spec\mathbb{C}\left[x,y,z\right]$, with equation $x^nz=P\left(y\right)$, where $P=\prod_{i=1}^{r}\left(y-y_i\right)$ is a polynomial with r simple roots. Let us show if that V has a nontrivial Makar-Limanov invariant provided $n,r\geq 2$ (see [**?**] and [11] for a purely algebraic proof of this result). By theorem 2.16 it is sufficient to find a minimal completion $\bar{V}$ of V such that $B=\bar{V}\setminus V$ is not a zigzag. We proceed as follows. We consider the birational morphism

$$\begin{array}{ccc} V & \overset{\phi_0}{\to} & V_0:=\mathbb{A}^2\subset\bar{V}_0:=\mathbb{P}^1\times\mathbb{P}^1 \\ (x,y,z) & \mapsto & (x,y) \end{array}$$

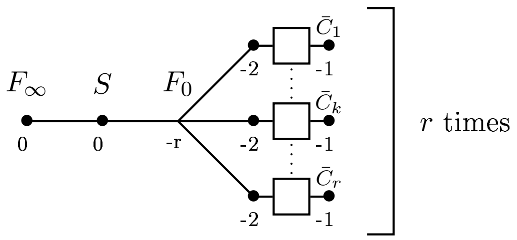
and we let $S=\mathbb{P}^1\times\{\infty\}\subset\bar{V}_0$, $F_\infty=\{\infty\}\times\mathbb{P}^1\subset\bar{V}_0$ and $F_0=\{0\}\times\mathbb{P}^1\subset\bar{V}_0$. We denote by $C_i\subset V$ the curve $x=0,y=y_i$ for $1\leq i\leq r$; these are the irreducible components of the degenerate fiber of the $\mathbb{A}^1$-fibration $pr_1\circ\phi_0$ on V . Then $\phi_0\left(C_i\right)=\left(0,y_i\right)\subset F_0$ is a point. We let $V_i=V\setminus\left(\bigcup_{j\neq i}C_i\right)\simeq\mathbb{A}^2$ with coordinates (x,u_i), where $u_i:=x^{-n}\left(y-y_i\right)=\prod_{j\neq i}\left(y-y_j\right)^{-1}z$. The restriction of ϕ_0 to V_i is given by

$$\begin{array}{ccc} V_i\simeq\mathbb{A}^2 & \overset{\phi_0|_{V_i}}{\to} & \mathbb{A}^2 \\ (x,u_i) & \mapsto & (x,x^nu_i+y_i) \end{array}.$$

We let $\pi_1 : \bar{V}_1 \to \bar{V}_0$ be the blow-up of $\bar{V}_0$ in the points $\phi_0(C_i)$ with exceptional divisors $E_{1,i}$ for $1 \le i \le r$. Clearly $\phi_0 : V \to V_0$ lifts to a morphism $\phi_1 : V \to V_1 \subset \bar{V}_1 \setminus (F'_\infty \cup S' \cup F'_0)$. Moreover $\phi_1(V_i) \subset V_{1,i} := \bar{V}_1 \setminus \left(F'_\infty \cup S' \cup F'_0 \cup \left(\bigcup_{j\neq i} E_{1,j}\right)\right) \simeq \mathbb{A}^2$, and ϕ_1 is given by

$$\begin{array}{ccc} V_i \simeq \mathbb{A}^2 & \stackrel{\phi_1|_{V_i}}{\to} & \mathbb{A}^2 \\ (x, u_i) & \mapsto & (x, x^{n-1}u_i + y_{1,i}) \end{array}$$

for some $y_{1,i} \in \mathbb{C}$. Iterating the construction, after n blow-ups as above we arrive at an open embedding $\phi_n : V \hookrightarrow \bar{V}$ of V in a nonsingular projective surface $\bar{V}$. We let $B = \bar{V} \setminus V$. If $\bar{C}_i$ denotes the closure of $\phi_n(C_i)$ in $\bar{V}$ then the dual graph of $B \cup \bar{C}_1 \cup \cdots \cup \bar{C}_r$ has the following structure:



where $\square$ stands for a linear chain of (-2)-curves of length $n-3$ (provided $n \ge 3$) .

Thus $\bar{V}$ is a minimal completion of V by an SNC-divisor B, which is a zigzag iff $r = 1$. Hence by proposition 2.10 and 2.15 V has a trivial Makar-Limanov invariant iff $n = 1$ or $n > 1$ and $r = 1$. The interested reader is referred to [**?**, **?**] for a more systematic study of these surfaces and to [2] for more explicit examples of surfaces with $\mathbb{C}_+$-actions.

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