The Piecewise-Projective Representation of Thompson's Group T

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Abstract.

Using lambda coordinates from Teichmüller theory, we study the action of Thompson's group T on a relative Teichmüller space, which is defined in terms of piecewise projective homeomorphisms.

1 Introduction

Let T be the group of piecewise-modular homeomorphisms of $\mathbb{RP}^1$. It is well known that T is one of the Thompson groups, the first examples of finitely presented infinite simple groups ([McK-Tho], [Can-Flo-Par]). Thompson's groups have been studied in different fields such as group theory, homotopy theory, logic and dynamical systems [Ghy-Ser], [Bri].

In [Gre2], Peter Greenberg provides a geometrical study of the group T, using piecewise-projective geometry, called *CPP* geometry. More precisely, the author introduces the infinite-dimensional space of $PSL_2\mathbb{R}$-classes of $\mathcal{C}^1$-piecewise projective diffeomorphisms of $\mathbb{RP}^1$, with rational breaks, denoted $\mathcal{G}_r$ (T_G in [Gre2]). His goal is to study the action of the group T by precomposition on $\mathcal{G}_r$. There, the main result is a sketch of proof of the contractibility of $\mathcal{G}_r$. He further points out the analogy between this action and the action of the Mapping-Class group on the corresponding Teichmüller space.

This article forms an essential backdrop to [Gre2]. A key ingredient is an injection of *CPP* geometry into Penner's universal decorated Teichmüller space [Pen2], which provides a system of affine coordinates, called lambda lengths. The rest of the paper is organized as follows. In section 2, we define Thompson's group T, *CPP* geometry and give an example of a CPP-homeomorphism, which is central in Penner's theory, and is called there a log-lambda deformation. In section 3, we recall some material from the theory of decorated tesselations and state some results of Penner [Pen2]. Then, we inject *CPP* geometry into the theory of decorated tesselations. This is done in proposition 3.4 and corollary 3.5. Section 4 discusses two different systems of coordinates over $\mathcal{G}_r$. Coordinates of the first type are called lambda coordinates [Pen2]. They are provided by the injection discussed above and lead to the principal theorem of this paper: $\mathcal{G}_r$ is topologically a direct limit of Euclidean vector spaces (theorem 4.4). As a direct consequence, we retrieve the contractibility of $\mathcal{G}_r$ (corollary 4.6). Coordinates of the second type, called shift coordinates, are intrinsic to the space $\mathcal{G}_r$ and provide another interesting analytic point of view of $\mathcal{G}_r$ (theorem 4.9). In section 5, we recall the action of Thompson's group T on $\mathcal{G}_r$ and translate it in terms of coordinate changes in both systems of coordinates. The lambda coordinates' behavior under T provides a nice tool to show, via Ptolemy's formula [Pen1], that the action is discrete (proposition 5.6). Then, we follow more or less [Gre2] to describe the stabilizers, although we improve the argument in avoiding the use of a theorem from [Ghy-Ser] which says that elements of T have rational rotation numbers (lemma 5.8 and theorem 5.9). This discussion leads to the construction of a classifying space for T (corollary 5.14). We also points out the analogy with the Teichmüller space of a surface. Indeed, we show that T acts isometrically for a certain metric associated to shift-coordinates (proposition 5.10). Compare with a theorem of Royden which says that the Mapping-Class group of a surface coincides with the group of isometry for the Teichmüller space (with the Teichmüller metric) of the surface [Roy]. More recently, Gardiner and Harvey proved an analogue in the universal case [Gar-Har]. Still more recently, two other works tend to see Thompson's group as the Mapping-Class group of an infinite surface ([Kap-Ser],[DeF-Gar-Har]). Finally, the appendix deals with horocyclic lengths associated with bracelets [Gre1], and gives successively a complete proof of a conjecture given in the same article (lemma 5.4) and the proof of theorem 4.9.

2 Thompson's Group and Piecewise-Projective Geometry

2.1 Thompson's Group T

The tangent function $tan : z \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \mapsto tan(\pi z) \in \mathbb{R} \cup \{\infty\} = \widehat{\mathbb{R}}$ induces a homeomorphism between the abstract version of the circle $\mathbb{S}^1$ and the one-point compactification of the real line $\widehat{\mathbb{R}}$. Hence, we shall denote the circle by $\mathbb{S}^1$ or $\widehat{\mathbb{R}}$ depending on the model we are referring to.
The *projective* group $PSL_2\mathbb{R} = SL_2(\mathbb{R})/\{\pm I_2\}$ acts on the hyperbolic plane by *projective transformations* $z \in \mathbb{H}^2 \mapsto \frac{az+b}{cz+d} \in \mathbb{H}^2$ with $ad - bc = 1$, and fixes the boundary $\partial\mathbb{H}^2 = \widehat{\mathbb{R}}$ set-wise. The group of orientation-preserving homeomorphisms (resp. $\mathcal{C}^1$-diffeomorphisms) of the circle will be denoted $Homeo^+(\widehat{\mathbb{R}})$ (resp. $Diff_1^+(\widehat{\mathbb{R}})$), or $Homeo^+(\mathbb{S}^1)$ (resp. $Diff_1^+(\mathbb{S}^1)$).
The *modular* subgroup $PSL_2\mathbb{Z} \subset PSL_2\mathbb{R}$ acts transitively on the subset of rational numbers $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. Elements of $PSL_2\mathbb{Z}$ are called *modular transformations*. If x is rational, P_x is the subgroup of $PSL_2\mathbb{Z}$ of elements fixing x. Notice that the conjugation isomorphism $\gamma \in P_x \mapsto g\gamma g^{-1} \in P_\infty = \mathbb{Z}$, for $g \cdot x = \infty$ does not depend on the choice of g. In that sense, we say that the isomorphism $P_x \cong P_\infty$ is canonical. This assertion comes from the fact that modular transformations of hyperbolic type do not have rational fixed points.

Definition 2.1. (Thompson's group T) Let $T = PPSL_2\mathbb{Z}$ be the subgroup of $Homeo^+(\widehat{\mathbb{R}})$ of elements f for which there exists a finite sequence $x_1 < x_2 < \cdots < x_n < x_1$ with $n \geq 3$ and $x_i \in \widehat{\mathbb{Q}}$ such that $f_i = f_{|[x_i;x_{i+1}]}$ is the restriction of a modular transformation for each i.

The $PSL_2\mathbb{Z}$-invariance of the subset $\widehat{\mathbb{Q}}$ implies that T is stable under composition and inverse. The arithmetic nature of definition 2.1 turns the group T into a subgroup of $Diff_1^+(\widehat{\mathbb{R}})$ (proposition 1.8. [Mar]).

2.2 CPP Geometry

CPP geometry has been introduced and studied by P.Greenberg in [Gre1], [Gre2].

Definition 2.2. (CPP-homeomorphism) Let CPP be the subgroup of $Homeo^+(\widehat{\mathbb{R}})$ of elements f for which there exists a finite sequence $x_1 < x_2 < \cdots < x_n < x_1$ with $n \geq 3$ such that $f_i = f_{|[x_i;x_{i+1}]}$ is the restriction of a projective transformation for each i. For $x \in \widehat{\mathbb{R}}$, denote by $f_x^l \in PSL_2\mathbb{R}$ (resp. $f_x^r \in PSL_2\mathbb{R}$) the left (resp. right) germ of $f \in CPP$. The projective transformation $D_xf = (f_x^r)^{-1} \circ f_x^l$ is called the *shift* of f at x. If the shift is not the identity, x is a *breakpoint* of f. The set of breakpoints of f will be denoted $bk(f)$.

Elements of CPP will be simply called CPP-homeomorphisms . Since elements of T are of class $\mathcal{C}^1$, CPP contains T as a subgroup. Notice that such homeomorphism of the circle extends to a homeomorphism between two regions of $\mathbb{H}^2$ whose complementary sets are finite-sided ideal polygons. Compare with quasisymmetric homeomorphisms which are extended to quasiconformal homeomorphisms of $\mathbb{H}^2$. It seems somewhat difficult to extend naturally a CPP-homeomorphism on the whole half-plane without using quasiconformal extensions. However, the $\mathcal{C}^1$-condition allows f to act on horocycles. Let $\mathcal{H}$ be the set of horocycles of $\mathbb{H}^2$, provided with the Hausdorff topology on compact-subset of $\overline{\mathbb{D}^2}$, after conjugation by the Cayley transform. If $x \in \widehat{\mathbb{R}}$, a horocycle based on x is usually denoted h_x.

Proposition 2.3. *The following action:*

$$\begin{array}{ccccc} CPP & \times & \mathcal{H} & \longrightarrow & \mathcal{H} \\ (f & , & h_x) & \longmapsto & f \cdot h_x = f_x^l(h_x) = f_x^r(h_x) \end{array}$$

is well defined.

Proof: Let f be CPP-homeomorphism and $x \in \widehat{\mathbb{R}}$. Since it is of class $\mathcal{C}^1$, the shift transformation $D_x f$ is a parabolic transformation fixing x. Hence, any horocycle h_x based on x is invariant under $D_x f$, and it follows the equality $f_x^l(h_x) = f_x^r(h_x)$. This completes the proof. □

It is an important geometric data on CPP-homeomorphisms which will be used constantly in the sequel.

We shall now define the principal topological space studied in this article and previously in [Gre2]. Topologically, it will be a direct limit space. To see this, recall that $\widehat{\mathbb{Q}}$ is filtrated as follows. Let $S_1 = \{0; \infty; -1\}$. Suppose $S_n = \{x_1^n; \cdots; x_{3\cdot 2^{n-1}}^n\}$ is constructed in such a way that:

$$0 = x_1^n < \cdots < x_i^n < x_i^{n+1} < \cdots < x_{3\cdot 2^{n-1}}^n = -\frac{1}{n}$$

Let $i \in < 1; 3 \cdot 2^{n-1} >$. Since $x_i^n = \frac{p}{q}$ and $x_{i+1}^n = \frac{p'}{q'}$ are rational numbers, we can form the Farey sum of these two elements, namely $x_{2i}^{n+1} = \frac{p+p'}{q+q'}$ (if $i = 3 \cdot 2^{n-1}$ set $x_{n+1}^n = x_1^n$). Moreover, we define $x_{2i+1}^{n+1} = x_{i+1}^n$ for all $i \in < 1; 3 \cdot 2^{n-1} - 1 >$. We have defined in this way a set

$$S_{n+1} = \left\{ 0 = x_1^{n+1}; \cdots; x_{3\cdot 2^n}^{n+1} = -\frac{1}{n+1} \right\}$$

Then, $\widehat{\mathbb{Q}}$ is the direct limit of the sequence $(S_n)_{n\geq 1}$ under the inclusions $S_n \subset S_{n+1}$ for $n \geq 1$.

Definition 2.4. (The Greenberg space $\mathcal{G}_r$) Let $\widehat{\mathcal{G}_r}$ denote the space of CPP-homeomorphisms with rational breakpoints. The space $\mathcal{G}_r$ is defined as the subspace of $\widehat{\mathcal{G}_r}$ consisting of elements which fix simultaneously the points 0, -1 and ∞. These elements are said to be *normalized*. In the same manner, we define $\widehat{\mathcal{G}_{rn}}$ to be the space of CPP-homeomorphisms with breakpoints in S_n and $\mathcal{G}_{rn}$ its subspace of normalized elements, or equivalently, $\mathcal{G}_{rn} = \widehat{\mathcal{G}_{rn}} \cap \mathcal{G}_r$.

Since $PSL_2\mathbb{R}$ acts triply-transitively on CPP by post-composition, and does not change the set of breakpoints, $\mathcal{G}_r$ (resp. $\mathcal{G}_{rn}$) is homeomorphic to $PSL_2\mathbb{R} \backslash \widehat{\mathcal{G}_r}$ (resp. $PSL_2\mathbb{R} \backslash \widehat{\mathcal{G}_{rn}}$). We give CPP, $\mathcal{G}_{rn}$ and $\widehat{\mathcal{G}_{rn}}$ the $\mathcal{C}^1$ topology, with its associated metric:

$$d_1(f; g) = \sup_{x \in \mathbb{S}^1} \sup\{d_{\mathbb{S}^1}(f(x); g(x)); |f'(x) - g'(x)|\}$$

The space $\mathcal{G}_r$ (resp. $\widehat{\mathcal{G}_r}$) is given the direct-limit topology, under inclusions $\mathcal{G}_{rn} \subset \mathcal{G}_{rn+1}$ (resp. $\widehat{\mathcal{G}_{rn}} \subset \widehat{\mathcal{G}_{rn+1}}$).

Example 1. Generalized hyperbolic elements: For each increasing sequence $x = (x_i)_{i\in<1;2n>} \in (\widehat{\mathbb{R}})^{2n}$ and each $t > 0$, let us denote h_x^t the unique map which restricts to the hyperbolic transformation with fixed points x_{2i}, x_{2i+1} and with derivative $\frac{1}{t}$ (resp. t) at x_{2i} (resp. x_{2i+1}) on the interval $]x_{2i}; x_{2i+1}[$ and to the hyperbolic transformation with fixed points x_{2i+1}, x_{2i+2} and with derivative t (resp. $\frac{1}{t}$) at x_{2i+1} (resp. x_{2i+2}) on the interval $]x_{2i+1}; x_{2i+2}[$. By construction, h_x^t is a CPP homeomorphism with breakpoint set $bk(f) = \{x_1; \cdots; x_{2n}\}$. It can be thought of as a generalization of hyperbolic transformations in $PSL_2\mathbb{R}$.

The case $n = 2$ appears in a series of papers on decorated Teichmüller spaces ([Pen1],[Pen2]). In these papers, the previous transformations (with $n = 2$) are called *log-lambda deformations.* Let Q be an ideal quadrilateral with vertices $x = \{x_1, \cdots, x_4\}$, $t > 0$ and $e = (x_1; x_3)$ one of its two diagonals. Then h_t^x is called the *(Q,e)-log-lambda deformation* with length t. The next section is a brief overview of the Penner theory of Teichmüller spaces introduced in ([Pen1],[Pen2]).

3 Embedding CPP Geometry into Penner's Decorated Theory

3.1 Decorated Tesselations

We briefly recall the definition of the beautiful objects that are decorated tesselations, due to Penner.

Definition 3.1. (Tesselation) A *tesselation* τ is a countable and locally finite ideal triangulation of $\mathbb{H}^2$, with a given distinguished oriented edge, called the *doe.*

Notice that τ is also understood as a collection of geodesics. We let $\tau^{(0)}$ denote the vertices of the triangulation. It is a countable and dense subset of $\widehat{\mathbb{R}}$. There is a standard tesselation τ_*, called *Farey tesselation*, with underlying triangulation $PSL_2\mathbb{Z} \cdot T_0$, where T_0 is the ideal triangle with vertices $0, \infty$ and -1, and with *doe* the geodesic running from 0 to ∞. It is well known that $\tau_*^{(0)} = \widehat{\mathbb{Q}}$. Let $Tess'$ denote the set of all tesselations, and $Tess$ the $PSL_2\mathbb{R}$-normalized subset consisting of tesselations with *doe* e_0 and with triangle to the left of e_0 equal to T_0.

Definition 3.2. (Decorated tesselation) A *decorated tesselation* $\tilde{\tau} = (\tau, h)$ is a tesselation τ, with a countable and locally finite collection of horocycles $h = (h_x)_{x\in\tau^{(0)}} \in \mathcal{H}^{\tau^{(0)}}$, based on the set of vertices of τ.

There is also a standard decorated tesselation $\tilde{\tau}_*$, with underlying tesselation τ_*, and whose collection of horocycles corresponds to the set of Ford circles, based on $\widehat{\mathbb{Q}}$ (see [Rad] for a definition of Ford circles). We let $(h_x^F)_{x\in\widehat{\mathbb{Q}}}$ denote the collection of Ford circles. $\tilde{\tau}_*$ is called the *Farey-Ford decorated tesselation* and verify the property that any two horocycles intersect if and only if their base-points are the endpoints of a geodesic of τ_*. Moreover, the intersection is tangent. We call it the *tangent property.* Let $\widetilde{Tess}'$ denote the set of all decorated tesselations and $\widetilde{Tess}$ the subset of elements with underlying tesselation in $Tess$.

Let $\tau' \in \widetilde{Tess}$. It is combinatorically rigid in the sense that its *doe* provides two natural bijections:

$$\chi_\tau : \tau_* \to \tau \quad \text{and} \quad \chi_\tau^{(0)} : \widehat{\mathbb{Q}} \to \tau^{(0)}$$

χ_τ is called the characteristic map associated to τ and is uniquely determined by the fact that it maps *doe* to *doe* and triangles to triangles. $\chi_\tau^{(0)}$ is defined by extending χ_τ to the vertices of the respective tesselations. The one-to-one map $\tau \in Tess' \to \chi_\tau^{(0)} \in \widehat{\mathbb{R}}^{\widehat{\mathbb{Q}}}$ provides $tess'$ with the topology induced by the product topology on $\widehat{\mathbb{R}}^{\widehat{\mathbb{Q}}}$. The same argument allows to embed $\widetilde{Tess}'$ in the product $\mathcal{H}^{\widehat{\mathbb{Q}}}$, given the product topology.

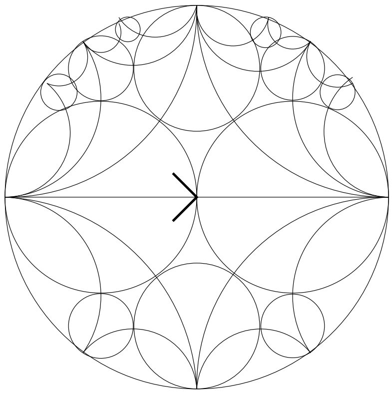


Figure 1: Farey-Ford decorated tesselation.

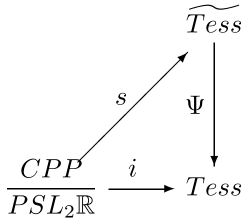
$Tess$ (resp. $\widetilde{Tess}$) is called the *universal Teichmüller space* (resp. *universal decorated Teichmüller space*) in the sense that it contains each of the classical Teichmüller spaces of surfaces with punctures. We close this section with a theorem due to Penner. By density of the sets $\tau_*^{(0)}$ and $\tau^{(0)}$ in $\widehat{\mathbb{R}}$, the map $\chi_\tau^{(0)}$ extends to a homeomorphism of the circle for any $\tau \in Tess'$. Denote this extension simply f_τ.

Theorem 3.3. *(Penner [Pen2]) The map $\tau \in Tess' \mapsto f_\tau \in Homeo^+(\widehat{\mathbb{R}})$ is a $PSL_2\mathbb{R}$-equivariant homeomorphism, with $Homeo^+(\widehat{\mathbb{R}})$ given the compact-open topology.*

3.2 Injecting $\mathcal{G}_r$ into $\widetilde{Tess}$

Since $CPP \subset Homeo^+(\widehat{\mathbb{R}})$, theorem 3.3 gives a one-to-one $PSL_2\mathbb{R}$-equivariant map $i : CPP \to Tess'$. Roughly speaking, this map consists of "pushing" the Farey tesselation by first moving its vertices. Moreover, proposition 2.3 shows that CPP-homeomorphisms push not only the vertices, but the horocycles as well. Let $\Psi : \widetilde{Tess} \to Tess$ be the forgetful map, thought of as a $\mathbb{R}^{\widehat{\mathbb{Q}}}$-fibration.

Proposition 3.4. *There exists a continuous one-to-one map*
$s : PSL_2\mathbb{R}\backslash CPP \to \widetilde{Tess}$ such that the diagram:



commutes.

Proof: Let $g \in PSL_2\mathbb{R}\backslash CPP$ and define the decorated tesselation $g \cdot \tilde{\tau}_*$ as follows:

Let $e = (x; y) \in \tau_*$ and set $g(e) = (g(x); g(y))$. Denote $g \cdot \tau_*$ the collection $(g(e))_{e\in\tau_*}$ with *doe* $g(e_0)$. In other words, $g \cdot \tau_*$ is $i(g)$.

Now, take any $z = g(x)$ in $(g\cdot\tau_*)^{(0)} = g(\widehat{\mathbb{Q}})$ and denote $h'_z = g\cdot h_x^F$ as defined in proposition 2.3, where h_x^F is the Ford circle based on $x \in \widehat{\mathbb{Q}}$. It is easy to see that the collection $h' = (h'_z)_{z\in(g\cdot\tau_*)^{(0)}}$ is locally finite in $\mathbb{H}^2$, hence it is a decoration of the tesselation $g\cdot\tau_*$. Let $g\cdot\tilde{\tau}_*$ denote the resulting decorated tesselation.The map s is defined by $s(g) = g \cdot \tilde{\tau}_*$.

It is clear that $\Psi \circ s = i$. It follows that s is one-to-one. To prove that s is continuous, recall from lemma A.2 in [Mal-Pen] that the horocycle $f \cdot h_x^F$ has Euclidean diameter $|f'(x)| d_x$ in $\mathbb{H}^2$, where d_x is the Euclidean diameter of h_x^F, if $x \neq \infty$. If $x = \infty$, $f \cdot h_\infty^F$ is the horizontal line located at height $\frac{1}{|f'(\infty)|}$. Thus, for any $x \in \widehat{\mathbb{Q}}$, the map $s_x : f \in PSL_2\mathbb{R} \backslash CPP \mapsto f \cdot h_x^F \in \mathcal{H}$ is continuous with the metric d_1 introduced in 2.2. Since $\widetilde{Tess} \subset \mathcal{H}^{\widehat{\mathbb{Q}}}$ is given the product topology, the map s is also continuous. □

Corollary 3.5. *The space $\mathcal{G}_r$ is continuously injected in $\widetilde{Tess}$.*

4 Global Coordinates on $\mathcal{G}_r$

4.1 Lambda Coordinates

Let I denote any subset of the circle. We say that I is decorated by h if $h = (h_x)_{x \in I}$ is a collection of horocycles, with one horocycle based on each point of the set I. If $(x, y) \in I^2$, and e is the geodesic with endpoints x and y, we define the quantity

$$d_h(h_x; h_y) = (-1)^{\frac{|h_x \cap h_y|}{2}} d_{\mathbb{H}^2}(h_x \cap e; h_y \cap e)$$

If h_x and h_y are tangent, this quantity is null.

Definition 4.1. (Lambda-length) Let $I \subset \widehat{\mathbb{R}}$ decorated by h and $e = (x; y)$ a geodesic with vertices in I. The quantity

$$\Lambda(h; e) = \sqrt{2 \exp\left(\frac{-d_h(h_x; h_y)}{2}\right)}$$

is called the *lambda-length* of h along e.

If $\tilde{\tau}$ is a decorated tesselation, the preceding definition applies to $I = \tau^{(0)}$ and $e \in \tau$. In that case, we rewrite $\Lambda(h; e) = \Lambda(\tilde{\tau}; e)$. Recall the characteristic map χ_τ associated with τ and defined in 3.1.

Theorem 4.2. *(Penner [Pen2]) The map:*

$$\begin{array}{rrcll} E: & \widetilde{Tess}: & \longrightarrow & {\mathbb{R}_+^*}^{\tau_*} & \\ & \tilde{\tau} & \longmapsto & E_{\tilde{\tau}}: & \tau_* \longrightarrow \mathbb{R}_+^* \\ & & & & e \longmapsto \Lambda(\tilde{\tau}; \chi_\tau(e)) \end{array}$$

is an embedding of $\widetilde{Tess}$ into ${\mathbb{R}_+^*}^{\tau_*}$ with the product topology.

This provides $\widetilde{Tess}$ with global affine coordinates. These coordinates are called *lambda coordinates* and are carried by Farey geodesics. Note that the tangent property of $\tilde{\tau}_*$ discussed in 3.1 implies that each coordinate of $\tilde{\tau}_*$ is $\sqrt{2}$.

We recall now some properties of log-lambda deformations in terms of lambda lengths. Let $\tilde{\tau}$ be a decorated tesselation and e one of its edges. Let Q denote the unique ideal quadrilateral in $\mathbb{H}^2 \setminus \{\tau \cup e\}$ (any other component of $\mathbb{H}^2 \setminus \{\tau \cup e\}$ is a triangle). Finally, let $\theta_{t,\tau,e}$ denote the (Q, e)-log lambda deformation with length t as defined in example 1.

Proposition 4.3. *1. $(\theta_{t,\tau,e})_{t \in \mathbb{R}_+^*}$ is a 1-parameter subgroup of CPP isomorphic to $\mathbb{R}_+^*$. If $\tau = \tau_*$, then this subgroup is contained in $\mathcal{G}_r$.*

2. Let $(e, e') \in \tilde{\tau}^2$, and set $\Lambda_t(e') = \Lambda(\theta_{t,\tau_*,e} \cdot \tilde{\tau}; \theta_{t,\tau_*,e}(e'))$. Then we have the following equalities:

$$1.\, \Lambda_t(e) = t\Lambda(\tilde{\tau}, e) \quad 2.\, \Lambda_t(e') = \Lambda(\tilde{\tau}, e') \quad \forall e' \in \tau \setminus \{e\}$$

Proof: 1. By definition of log-lambda deformations.
2. See proof of Lemma 3.2 in [Pen2]. □

We are now ready to prove the main result of this paper. Let us first introduce some notations. The filtration $\widehat{\mathbb{Q}} = \cup_{n\geq 1} S_n$, introduced in 2.2, defines a filtration $\tau_* = \cup_{n\geq 2}(\tau_*)_n$ by

$$(\tau_*)_n = \{e = (x;y) \in \tau_* \,|\, (x,y) \in (S_{n-1})^2\} \quad \forall n \geq 2$$

If $\Lambda \in {\mathbb{R}_+^*}^{\tau_*}$, let $Supp(\Lambda) = \{e \in \tau_* \,|\, \Lambda(e) \neq \sqrt{2}\} \subset \tau_*$ denotes the support of Λ. Let us introduce the following subspaces of ${\mathbb{R}_+^*}^{\tau_*}$:

$$(\mathbb{R}_+^*)_{\sqrt{2}}^{\tau_*} = \{\Lambda \in {\mathbb{R}_+^*}^{\tau_*} \,|\, card(Supp(\Lambda)) < \infty\}$$

$$(\mathbb{R}_+^*)_{\sqrt{2}}^{(\tau_*)_n} = \{\Lambda \in {\mathbb{R}_+^*}^{\tau_*} \,|\, Supp(\Lambda) \subset (\tau_*)_n\}$$

so that $(\mathbb{R}_+^*)_{\sqrt{2}}^{(\tau_*)_n}$ is homeomorphic to the Euclidean space $\mathbb{R}^{3(2^{n-1}-1)}$. Since $(\mathbb{R}_+^*)_{\sqrt{2}}^{\tau_*}$ is also the direct-limit of the sequence of spaces $\left((\mathbb{R}_+^*)_{\sqrt{2}}^{(\tau_*)_n}\right)_{n\geq 1}$ under the inclusions $(\mathbb{R}_+^*)_{\sqrt{2}}^{(\tau_*)_n} \subset (\mathbb{R}_+^*)_{\sqrt{2}}^{(\tau_*)_{n+1}}$, it is given the direct-limit topology. Finally let $\Lambda : \mathcal{G}_r \to {\mathbb{R}_+^*}^{\tau_*}$ be the composition $E \circ s$ (proposition 3.4 and theorem 4.2) and Λ_n its restriction to $\mathcal{G}_{rn}$.

Theorem 4.4. *The map* $\Lambda : \mathcal{G}_r \to (\mathbb{R}_+^*)_{\sqrt{2}}^{\tau_*}$ *is a homeomorphism.*

Proof: To prove that Λ maps $\mathcal{G}_r$ onto $(\mathbb{R}_+^*)_{\sqrt{2}}^{\tau_*}$, we use the corresponding filtrations and restrict to $\mathcal{G}_{rn}$. This is done in the following lemma:

Lemma 4.5. $\Lambda_n(\mathcal{G}_{rn}) = (\mathbb{R}_+^*)_{\sqrt{2}}^{(\tau_*)_n}$

Proof: We proceed by inclusions.
• $\subset$: If $e = (x;y) \in \tau_* \setminus (\tau_*)_n$, the ideal points x and y are included in a segment $[x_i; x_{i+1}]$ whose extremities are two consecutive points in S_n. By definition of $\mathcal{G}_{rn}$, any element of $\mathcal{G}_{rn}$ restricts to a projective transformation h_i on this segment. Thus, we have

$$d_h(f \cdot h_x^F; f \cdot h_y^F) = d_h(h_i(h_x^F); h_i(h_y^F)) = d_h(h_x^F; h_y^F)$$

This implies that $\Lambda(f)(e) = \sqrt{2}$. In other words, we have the direct inclusion.
• $\supset$: For the reverse inclusion, we define a natural order on τ_* and proceed by induction on this order. Let

$$e_1 = (0;\infty) < e_2 = (\infty;-1) < e_3 = (-1;0)$$

define the beginning of the order. Now suppose the order has been defined on $(\tau_*)_n$,. The set of edges $(\tau_*)_{n+1} \setminus (\tau_*)_n$ forms an ideal $3 \cdot 2^{n-1}$-gon with set of vertices $S_n = \{x_1^n; \cdots; x_{3\cdot 2^{n-1}}^n\}$ already ordered in 2.2. Then, we can write $(\tau_*)_{n+1} \setminus (\tau_*)_n = \{a_1; \cdots a_{3\cdot 2^{n-1}-1}\}$ with $a_i = (x_i^n; x_{i+1}^n)$. It has a natural order given by the indexes. Finally, by requiring that a_1 is greater than each element of $(\tau_*)_n$, we have defined a natural order on $(\tau_*)_{n+1}$.

Now, let $(t_1\sqrt{2}, \cdots, t_{3(2^{n-1}-1)}\sqrt{2}, \sqrt{2}, \cdots)$ be an element of $(\mathbb{R}_+^*)_{\sqrt{2}}^{(\tau_*)_n}$. We construct, inductively on the order (denoted k) of $(\tau_*)_n$, a sequence $(f_i)_{i\in<1;k>} \in CPP^k$ such that: $g_k = f_k \circ \cdots f_1 \in \mathcal{G}_{rn}$ and $\Lambda(g_k) = (t_1\sqrt{2}, \cdots, t_k\sqrt{2}, \sqrt{2}, \cdots)$.
• <u>Rank 1:</u> let $g_1 = f_1$ be the normalization of the log-lambda deformation θ_{t,τ_*,e_1}. According to proposition 4.3, we have $\Lambda(g_1)(e_1) = t_1\sqrt{2}$ and $\Lambda(g_1)(e) = \sqrt{2}$ if $e \neq e_1$.
• <u>Rank $k+1$ assuming rank k:</u> Assume that we have defined $f_1, \ldots, f_k$. Let f_{k+1} be the normalization of the log-lambda deformation $\theta_{t_{k+1},g_k(\tau_*),g_k(e_{k+1})}$ and set $g_{k+1} = f_{k+1} \circ g_k$. By assumption, we have $bk(g_k) \subset S_n$. Since the vertices of the quadrilateral associated with $e \in \tau_*$ belong to S_n, we have $bk(f_{k+1}) \subset g_k(S_n)$. Thus, with the inclusions $bk(g_{k+1}) \subset bk(g_k) \cup g_k^{-1}(bk(f_{k+1})) \subset S_n$, it is true that $g_{k+1} \in \mathcal{G}_{rn}$. Now by definition, we have:

$$\Lambda(g_{k+1})(e) = \Lambda(f_{k+1} \circ g_k)(e) = \Lambda((f_{k+1} \circ g_k)(\tilde{\tau}_*); (f_{k+1} \circ g_k)(e))$$

By proposition 4.3, the last term equals:

$$\Lambda(g_k \cdot \tilde{\tau}_*; g_k(e)) \text{ if } g_k(e) \neq g_k(e_{k+1})$$

and

$$t_{k+1}\Lambda(g_k \cdot \tilde{\tau}_*; g_k(e)) \text{ if } g_k(e) = g_k(e_{k+1})$$

This shows that

$$\Lambda(g_{k+1}) = (t_1\sqrt{2}, \cdots, t_k\sqrt{2}, t_{k+1}\sqrt{2}, \sqrt{2}, \cdots)$$

and finishes the proof of the lemma. □

Return to the proof of the theorem. Surjectivity of Λ follows from lemma 4.5 and the fact that $(\mathbb{R}_+^*)^{\tau_*}_{\sqrt{2}}$ is the direct-limit of the sequence $((\mathbb{R}_+^*)^{(\tau_*)_n}_{\sqrt{2}})_{n\geq 2}$. Since E and s are injective, the same holds for Λ, which is therefore bijective. Since $\mathcal{G}_r$ and $(\mathbb{R}_+^*)^{\tau_*}_{\sqrt{2}}$ have the direct-limit topology, it remains to be shown that Λ_n is a homeomorphism to complete the proof. Continuity follows from the fact that each component map $\Lambda_n(e) : f \in \mathcal{G}_{rn} \mapsto \Lambda_n(f)(e) \in \mathbb{R}_+^*$ is continuous. To prove that Λ_n^{-1} is continuous, assume that a sequence of maps $(f_p)_{p\geq 0}$ of $\mathcal{G}_{rn}$ and $f \in \mathcal{G}_{rn}$ are given such that $\Lambda_n(f_p)$ tends to $\Lambda_n(f)$ when p tends to ∞. Since $\Lambda(f_p)(e) = \sqrt{2} = \Lambda(f)(e)$ for all $e \notin (\tau_*)_n$, $\Lambda(f_p)$ tends to $\Lambda(f)$ for the product topology. Since E is open (theorem 4.2), $s(f_p)$ tends to $s(f)$. Moreover, the forgetful map $\Psi : \widetilde{Tess} \to Tess$ is continuous, which implies, via theorem 3.3, that $(f_p)_{p\geq 0}$ converges uniformly with limit f. Let us restrict to a fixed interval of the circle whose extremities are two consecutive points of S_n. We apply proposition 4.5.4. of [Bea] on this interval to conclude that $(f'_p)_{p\geq 0}$ converges uniformly with limit f' on this interval. Since there is a finite number of such intervals which cover the circle, we can conclude that $(f_p)_{p\geq 0}$ tends to f with the metric d_1. This proves that Λ_n^{-1} is continuous and ends the proof of the theorem. □

An immediate consequence of the theorem is:

Corollary 4.6. *$\mathcal{G}_r$ is a contractible space.*

We end this part with an interesting link between the $\mathcal{C}^1$-topology defined as the restriction of the topology corresponding to the metric d_1 and the lambda coordinates metric superior defined on $\mathcal{G}_r$ as follows:

$$d^\Lambda(f; g) = \sup_{e\in\tau_*} |\Lambda(f)(e) - \Lambda(g)(e)|$$

which is well defined since the range of Λ is $(\mathbb{R}_+^*)^{\tau_*}_{\sqrt{2}}$.

Proposition 4.7. *The identity map $id : (\mathcal{G}_r, d_1) \to (\mathcal{G}_r, d^\Lambda)$ is continuous.*

Proof: Fix $f \in \mathcal{G}_r$ and $\epsilon > 0$ once and for all. Let $\tau_*^0 = \{e = (x; y) \in \tau_* \,|\, (x, y) \in [0; 1]^2\}$. To show continuity at f, it suffices, up to $PSL_2\mathbb{R}$, to show that there exists $\alpha > 0$ such that:

$$d_1(f; g) < \alpha \Longrightarrow \sup_{e\in\tau_*^0} \left|\frac{\Lambda(f)(e) - \Lambda(g)(e)}{\sqrt{2}}\right| < \epsilon \qquad \forall g \in \mathcal{G}_r \tag{1}$$

From lemmas A.1. and A.2. in [Mal-Pen], we deduce easily the following formula for any $g \in CPP$:

$$\frac{\Lambda(g(h); g(e))}{\Lambda(h; e)} = \frac{|g(p) - g(q)|}{|p - q|}\sqrt{\frac{1}{g'(p)g'(q)}} \tag{2}$$

where $e = (p; q)$ lies in τ_*^0. Assume, up to symmetry, that $p < q$. Since f and g are $\mathcal{C}^1$, we can find two numbers $x^f_{p,q}$ and $x^g_{p,q}$ inside the segment $[p; q]$ such that $f'\left(x^f_{p,q}\right) = \frac{f(q)-f(p)}{q-p}$ and

$g'\left(x^g_{p,q}\right) = \frac{g(q)-g(p)}{q-p}$. Applied to both f and g, and combined with a triangular inequality, equation 2 then gives:

$$\left|\frac{\Lambda(f)(e)-\Lambda(g)(e)}{\sqrt{2}}\right| \leq \left|\frac{f'\left(x^f_{p,q}\right)}{\sqrt{f'(p)f'(q)}} - \frac{f'\left(x^g_{p,q}\right)}{\sqrt{f'(p)f'(q)}}\right| + \left|\frac{f'\left(x^g_{p,q}\right)}{\sqrt{f'(p)f'(q)}} - \frac{g'\left(x^g_{p,q}\right)}{\sqrt{g'(p)g'(q)}}\right| \quad (3)$$

Since f' is uniformly continuous over $[0;1]$ and $|x^g_{p,q} - x^f_{p,q}| < q-p$, we can find a finite subset $A_f \subset \tau^0_*$ such that:

$$\forall e \in \tau^0_* \setminus A_f \qquad |f'\left(x^g_{p,q}\right) - f'\left(x^f_{p,q}\right)| < \frac{\epsilon}{2\inf_{x\in[0;1]} f'(x)}$$

(f' does not vanish). It follows easily that:

$$\left|\frac{f'\left(x^f_{p,q}\right)}{\sqrt{f'(p)f'(q)}} - \frac{f'\left(x^g_{p,q}\right)}{\sqrt{f'(p)f'(q)}}\right| \leq \frac{\epsilon}{2} \qquad \forall e \in \tau^0_* \setminus A_f \quad (4)$$

For the second member of 3, since the derivative is uniformly bounded below by a positive constant, we can find α such that $\forall (x,y,z,f) \in [0;1]^3 \times B_{d_1}(f;\alpha)$, we have:

$$\left|\frac{f'(x)}{\sqrt{f'(y)f'(z)}} - \frac{g'(x)}{\sqrt{g'(y)g'(z)}}\right| \leq \frac{\epsilon}{2} \quad (5)$$

By replacing 4 and 5 (applied to $x = x^g_{p,q}$, $y = p$ and $z = q$) in 3, one obtains:

$$\left|\frac{\Lambda(f)(e)-\Lambda(g)(e)}{\sqrt{2}}\right| < \epsilon \qquad \forall (g,e) \in B_{d_1}(f;\alpha) \times \tau^0_* \setminus A_f \quad (6)$$

Since A_f is finite, the map $g \in \mathcal{G}_r \mapsto (\Lambda(g))_{e\in A_f} \in (\mathbb{R}^*_+)^{card(A_f)}$ is continuous with the metric d_1 on the domain. Therefore, up to decreasing α, inequality 6 holds for any $e \in \tau^0_*$, which proves 1 and the proposition. □

Though we are not able to prove that the two topologies agree, we have a good reason to believe it is true (see [Mar] section 2.4. for details).

4.2 Shift (or Horocyclic) Coordinates

Let $f \in \widehat{\mathcal{G}_r}$ and $x \in \widehat{\mathbb{Q}}$, a rational point. Recall from definition 2.2 that the shift $D_x(f)$ is a parabolic transformation fixing x, which measures locally at x how f fails to be a projective transformation. By transitivity, choose $\gamma \in PSL_2\mathbb{Z}$ such that $\gamma \cdot x = \infty$. The conjugate $\gamma D_x f \gamma^{-1}$ is of the form $z \in \widehat{\mathbb{R}} \mapsto z + d_x(f) \in \widehat{\mathbb{R}}$. Since the isomorphism $P_x \cong P_\infty$ is canonical, we have the following:

Definition 4.8. (Shift coordinate) If $x \in \widehat{\mathbb{Q}}$, the well-defined real number $d_x(f)$ is called the *x-shift coordinate* of f.

Notice that for $x \in \widehat{\mathbb{Q}}$, the function $d_x : \widehat{\mathcal{G}_r} \to \mathbb{R}$ is continuous. The proof (as well as some interesting properties of these coordinates) of the following theorem will be given at the end of the appendix of the paper, which deals with bracelets and horocyclic coordinates. Let us denote $\mathbb{R}^{*\,(\widehat{\mathbb{Q}})}_+ = \lim \mathbb{R}^{S_n}$ under the natural inclusions $\mathbb{R}^{S_n} \subset \mathbb{R}^{S_{n+1}}$.

Theorem 4.9. *The map $d : f \in \widehat{\mathbb{R}} \to (d_x(f))_{x\in\mathbb{R}^{*\,(\widehat{\mathbb{Q}})}_+} \in \mathbb{R}^{*\,(\widehat{\mathbb{Q}})}_+$ is a homeomorphism onto its image.*

It provides interesting global coordinates, which, as we shall see in 5, are well-adapted to the geometry of the group T.

5 Action of Thompson's Group T

The simple following result is the key lemma to understand the action of Thompson's group T on $\mathcal{G}_r$. Recall that $b^F = (h_x^F)_{x\in\widehat{\mathbb{Q}}}$ represents the set of all Ford circles. We call b^F the *Ford bracelet*. Since CPP acts to the left on $\mathcal{H}$ (proposition 2.3), the same holds for the group T.

Lemma 5.1. *The Ford bracelet b^F is globally invariant under the action of T, and we have $\gamma \cdot h_x^F = h_{\gamma(x)}^F$ for any pair $(x, \gamma) \in \widehat{\mathbb{Q}} \times T$.*

Proof: This follows from the fact that b^F is $PSL_2\mathbb{Z}$-invariant, by construction. □

In order to define the action of T on $\mathcal{G}_r$, remark that for any pair $(u, v) \in CPP^2$, we have $bk(u \circ v) \subset v^{-1}(bk(u)) \cup bk(v)$. Since $\widehat{\mathbb{Q}}$ is invariant under T, it follows that $\widehat{\mathcal{G}_r} \subset CPP$ is T-invariant by pre-composition. Hence the following action:

$$(f, \gamma) \in \mathcal{G}_r \times T \mapsto N(f \circ \gamma) = f \cdot \gamma \in \widehat{\mathcal{G}_r}$$

where $N(f \circ \gamma)$ is the normalization of $f \circ \gamma$ in $\mathcal{G}_r$, is well defined. By definition, we have $N(f \circ \gamma) = h \circ f \circ \gamma$, where h is the unique projective transformation such that $h^{-1}(0; \infty; -1) = f \circ \gamma(0; \infty; -1)$.

For now, this action is defined in the CPP geometry. We next study the action in both coordinates systems introduced in section 4.

5.1 Coordinate Transformations

We shall now study the behavior of global coordinates (both lambda and shift coordinates) given in section 4 under the action of T. We say that a geodesic $e = (x; y)$ is a *rational geodesic* if its endpoints x and y are elements of $\widehat{\mathbb{Q}}$. Let $\Delta_{\mathbb{Q}}$ denote the set of rational geodesics. T acts on $\Delta_{\mathbb{Q}}$ as follows: $\gamma \cdot (x; y) = (\gamma(x); \gamma(y))$, since $\widehat{\mathbb{Q}}$ is invariant under T. To each rational geodesic $e \in \Delta_{\mathbb{Q}}$ corresponds a parameter on the space $\widehat{\mathcal{G}_r}$ defined by:

$$f \in \widehat{\mathcal{G}_r} \mapsto \Lambda(f; e) \in \mathbb{R}_+^*$$

where $\Lambda(f; e)$ is the lambda-length of $f \cdot b^F$ along $f(e)$ (see definition 4.1). This parameter is obviously invariant under $PSL_2\mathbb{R}$.

Proposition 5.2. $\forall (f, \gamma, e) \in \widehat{\mathcal{G}_r} \times T \times \Delta_{\mathbb{Q}}$, *we have* $\Lambda(f; \gamma \cdot e) = \Lambda(f \circ \gamma; e)$.

Proof: This follows directly from definition 4.1 and lemma 5.1. □

A tesselation whose edges are rational geodesics is called a *rational tesselation*. Let $Tess'_{\mathbb{Q}}$ denote the subspace of $Tess'$ of rational tesselations. As for τ_*, there are global coordinates defined from any $\tau \in Tess'_{\mathbb{Q}}$ by the map $\Lambda^\tau : f \in \widehat{\mathcal{G}_r} \mapsto (\Lambda(f; e))_{e\in\tau} \in {\mathbb{R}_+^*}^\tau$. This time, the coordinates are carried by the edges of τ. For example Λ^{τ_*} corresponds to the map $\Lambda = E \circ s$ defined in theorem 4.4, up to $PSL_2\mathbb{R}$.

Thompson's group T acts on the left on the space $Tess'_{\mathbb{Q}}$. In terms of lambda-coordinates, this action is nothing more than coordinate changes. More precisely:

Lemma 5.3. $\forall (f, \gamma, e) \in \widehat{\mathcal{G}_r} \times T \times \tau_*$, *we have*

$$\Lambda^{\tau_*}(f \cdot \gamma)(e) = \Lambda^{\gamma\cdot\tau_*}(f)(\gamma \cdot e)$$

Proof: This follows from the definition of $\Lambda^{\gamma\cdot\tau_*}$, Λ^{τ_*}, lemma 5.1 and the fact that Λ^{τ_*} is $PSL_2\mathbb{R}$-invariant. □

We end this subsection with a result which relates the behavior of the shift-coordinates under the action of T.

Lemma 5.4. *For any pair* $(f, \gamma, x) \in \widehat{\mathcal{G}_r} \times T \times \widehat{\mathbb{Q}}$*, we have:*

$$d_x(f \circ \gamma) = d_{\gamma \cdot x}(f) + d_x(\gamma)$$

Proof: By definition, we have $D_x(f \circ \gamma) = (\gamma_x^r)^{-1} D_{\gamma \cdot x}(f) \gamma_x^l$. Now let $p_1 \in PSL_2\mathbb{Z}$ such that $p_1 \cdot x = \infty$ and $p_2 = p_1 \circ (\gamma_x^r)^{-1}$. The following equalities hold:

$$p_1 D_x(f \circ \gamma) p_1^{-1} = t_{d_x(f \circ \gamma)}, \quad p_2 D_{\gamma \cdot x}(f) p_2^{-1} = t_{d_{\gamma \cdot x}(f)} \quad \text{and} \quad p_1 D_x(\gamma) p_1^{-1} = t_{d_x(\gamma)}$$

where t_a denotes the translation $z \mapsto z + a$. By writing $D_x(f \circ \gamma) = D_x(\gamma)(\gamma_x^l)^{-1} D_{\gamma \cdot x}(f) \gamma_x^l$, we obtain, by uniqueness of the conjugations by modular transformations,

$$t_{d_x(f \circ \gamma)} = t_{d_{\gamma \cdot x}(f)} \circ t_{d_x(\gamma)}$$

which proves the lemma □

5.2 Orbits

We fix $g \in \mathcal{G}r_p$ once and for all in this subsection. Let $A = \{a_0 = \sqrt{2}; a_1; \cdots; a_k\}$ denote the set of all lambda coordinate values taken by g. It is a finite set since $\Lambda(\mathcal{G}r_p) \subset (\mathbb{R}_+^*)_{\sqrt{2}}^{\tau_*}$. Let us denote

$$N(A) = \left\{ \sum_{i=0}^{k} n_i a_i \mid (n_i)_{i \in <0;k>} \in (\mathbb{N})^{k+1} \setminus \{0\} \right\}$$

It is a discrete subset of $\mathbb{R}_+^*$. Finally, we introduce a discrete subset of $(\mathbb{R}_+^*)_{\sqrt{2}}^{\tau_*}$ with the direct topology, namely:

$$(\mathbb{R}_+^*)_A^{\tau_*} = \left\{ \Lambda \in (\mathbb{R}_+^*)_{\sqrt{2}}^{\tau_*} \mid \Lambda(e) \in N(A) \quad \forall e \in \tau_* \right\}$$

Notice that this is also a discrete subspace under the topology of the metric d^Λ defined in section 4.

To understand the goal of the preceding argumentation, we recall Ptolemy's formula. Let Q denote an ideal quadrilateral with sides a, b, c, d and diagonals e, f. The labelling of the sides is taken so that a and c are opposing sides. Let h be a decoration of the set of vertices of Q, so that $\Lambda(h; x)$ makes sense for any $x \in \{a; b; c; d; e; f\}$, according to definition 4.1.

Proposition 5.5. *(Ptolemy's formula [Pen2]) We have:*

$$\Lambda(h; e)\Lambda(h; f) = \Lambda(h; a)\Lambda(h; c) + \Lambda(h; b)\Lambda(h; d)$$

In the context of tesselations, Penner defines a group, which is generated by flips along tesselations. Roughly speaking, a flip consists to change a diagonal of one quadrilateral of the tesselation. This group is called the Ptolemy group. In [Imb], Imbert has shown that the Ptolemy group and Thompson's group T are isomorphic. Since Thompson's group fixes globally the Ford bracelet, the idea is to apply inductively (on the flips) Ptolemy's formula in order to control the lambda coordinates along a single orbit. This is done as follows:

Proposition 5.6. *We have the inclusion* $\Lambda(\mathcal{O}_T(g)) \subset (\mathbb{R}_+^*)_A^{\tau_*}$*. In particular, the orbits under* T *are discrete.*

Proof: Let $(\Delta_\mathbb{Q})_n = \{e = (x; y) \in \Delta_\mathbb{Q} \mid (x, y) \in S_n^2\}$. We shall prove by induction the following assertion:

$$\Lambda(g(b^F); g(e)) \in N(A) \quad \forall\, (e, n) \in (\Delta_\mathbb{Q})_n \times \mathbb{N}^*$$

Let us assume it is true for a moment. If $(e, \gamma) \in \tau_* \times T$, following lemma 5.3, we have $\Lambda(g \cdot \gamma)(e) = \Lambda(g(b^F); g \circ \gamma(e))$. Since $\gamma(e)$ lies in $(\Delta_\mathbb{Q})_n$, the assumption proves that the term $\Lambda(g(b^F); g \circ \gamma(e))$ lies in $N(A)$. This proves the proposition since Λ is a homeomorphism.

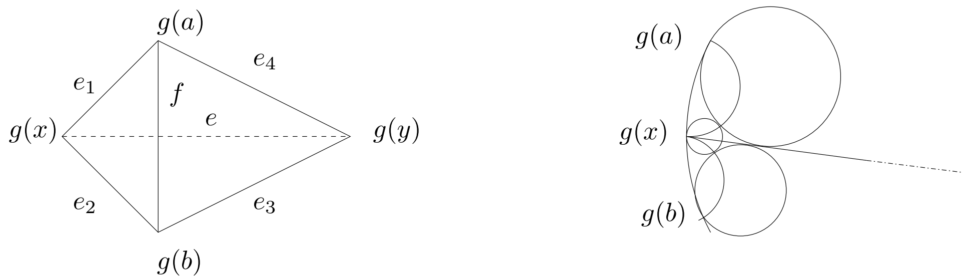


Figure 2: Ptolemy's formula

Thus, it remains to prove the assertion. Notice that it is already proved for any $n \leq p$ by definition of $N(A)$. Assume the assertion is true for some $n \geq p$. Let $e = (x; y) \in (\Delta_{\mathbb{Q}})_{n+1}$. Consider two cases whether or not both endpoints of e lie in $S_{n+1} \setminus S_n$.

• $\underline{x \in S_{n+1} \setminus S_n \text{ and } y \in S_n}$: Label the edges as on figure 2. Since $g \in \mathcal{G}r_p$ and $n \geq p$, we have $\Lambda(g(b^F); g(w)) = \sqrt{2}$ for any $w \in \{e_1; e_2; f\}$. By assumption, there are two elements $(n_i^3)_{i \in <o;k>}$, $(n_i^4)_{i \in <o;k>}$ in $(\mathbb{N})^{k+1} \setminus \{0\}$ such that $\Lambda(g(b^F); g(e_m)) = \sum_{i=0}^{k} n_i^m a_i$ for $m \in \{3; 4\}$. Ptolemy's formula applied to the quadrilateral on figure 2 gives $\Lambda(g(b^F); g(e)) = \sum_{i=0}^{k} (n_i^3 + n_i^4) a_i$.

• $\underline{x \in S_{n+1} \setminus S_n \text{ et } y \in S_{n+1} \setminus S_n}$: Since a and b lies in S_n, the former case says that there are two elements $(n_i^3)_{i \in <o;k>}$, $(n_i^4)_{i \in <o;k>}$ in $(\mathbb{N})^{k+1} \setminus \{0\}$ such that $\Lambda(g(b^F); g(e_m)) = \sum_{i=0}^{k} n_i^m a_i$ for $m \in \{3; 4\}$. Once again, applying Ptolemy's formula, we obtain $\Lambda(g(b^F); g(e)) = \sum_{i=0}^{k} (n_i^3 + n_i^4) a_i$, which concludes the proof of the proposition. □

Corollary 5.7. *T acts with discrete orbits on $\mathcal{G}_r$ with the $\mathcal{C}^1$-topology.*

Proof: This follows from the fact that $(\mathbb{R}_+^*)_A^{\tau_*}$ is a discrete subset of $(\mathcal{G}_r, d^\Lambda)$ and that $\Lambda : (\mathcal{G}_r, d_1) \to ((\mathbb{R}_+^*)_{\sqrt{2}}^{\tau_*}, d_\Lambda)$ is continuous (proposition 4.7). □

5.3 Stabilizers

The method employed to describe the stabilizers here is essentially the same as that used in [Gre2]. However, we avoid the use of a non trivial result from [Ghy-Ser] which shows that the rotation number of any element of T is rational. To do so, we first show:

Lemma 5.8. *Let $f \in \mathcal{G}_r$ and a torsion free element $\gamma \in Stab(f) \subset T$. Then, $D_x f$ is a modular transformation for any $x \in \widehat{\mathbb{Q}}$.*

Proof: By assumption, there exists a projective transformation h such that $h \circ f = f \circ \gamma$. Consider the finite γ-invariant subset $A = \bigcap_{n \in \mathbb{Z}} \gamma^n bk(f)$ of $\widehat{\mathbb{Q}}$. We proceed in two cases whether or not A is empty.

Suppose first that A is empty. Since $D_x f = (\gamma_x^r)^{-1} \circ D_{\gamma x} f \circ \gamma_x^l$ for all $x \in \widehat{\mathbb{Q}}$, it follows that there exists a sequence of pairs $(a_n, b_n)_{n \in \mathbb{Z}} \in (PSL_2\mathbb{Z})^{2\mathbb{Z}}$ such that $D_x f = a_n \circ D_{\gamma^n x} f \circ b_n$ for all $n \in \mathbb{Z}$. Since $A = \emptyset$, there exists $n_x \in \mathbb{Z}$ such that $x \in \gamma^{-n_x}(bk(f))$. Hence $D_{\gamma^{n_x} \cdot x} f = id$, which proves that $D_x f = a_{n_x} \circ b_{n_x}$ is indeed a modular transformation.

Suppose now that it is not empty. Since $h \circ f = f \circ \gamma$, $f(A)$ must be a finite subset invariant under h, and h is of infinite order since γ is torsion-free by assumption. These two facts combined imply first that h is either of parabolic or hyperbolic type and then that $card(A) \leq 2$. If $card(A) = 2$, then h must be of hyperbolic type. Thus, its derivative at both fixed-points is different from 1, and the same holds for γ since it is a conjugate of h by the $\mathcal{C}^1$-diffeomorphism f. Let x denote one of the two fixed-points of γ. Then the germs γ_x^r and γ_x^l must be of hyperbolic type with at least one rational fixed-point, x, but it contradicts the fact that they are in $PSL_2\mathbb{Z}$. Indeed, any modular transformation of hyperbolic type possesses only irrational fixed-points. Hence the set A must consist to a single element x. Up to $PSL_2\mathbb{R}$, we can assume that $f^l_x = id$. Let $bk(f) = (x_i)_{i\in<1;n>}$ where $x = x_1 < x_2 < \cdots < x_n < x_1$ on the circle and set $f_i = f^l_{x_i}$. Since $x_2 \notin A$, there exists $n_2 \in \mathbb{Z}$ such that $\gamma^{n_2}(x)$ is a breakpoint for f. By the same argument used in the case A is empty, we deduce that $D_{x_2}f = f_2^{-1} \circ f_1 = f_2^{-1}$ is a modular transformation. We repeat the argument on each x_i, since none of each belongs to A, to prove that $f_i \in PSL_2\mathbb{Z}$ for each $i \in< 1; n >$. In particular, it proves the proposition for the new normalization of f, but since $D_x(h \circ f) = D_x f$ for any $h \in PSL_2\mathbb{R}$, the same holds for the previous f. □

We are now ready to give the types of stabilizers.

Theorem 5.9. *Let $f \in \mathcal{G}_r$.*

1. *If $f \in \mathcal{O}_T(id)$, then $Stab(f)$ is conjugated to $PSL_2\mathbb{Z}$ in T.*

2. *If $f \notin \mathcal{O}_T(id)$, then $Stab(f)$ is a cyclic group with order bounded by the number of breakpoints for f.*

Proof: 1. It is clear that $Stab(id) = PSL_2\mathbb{Z}$. Hence, for $f = id \cdot \gamma \in \mathcal{O}_T(id)$, we have $Stab(f) = \gamma^{-1}PSL_2\mathbb{Z}\gamma$.
2. We first prove that any element of $Stab(f)$ has a finite order by contradiction. Assume it is not the case for some $\gamma \in Stab(f)$. By lemma 5.8, $D_x f$ is a modular transformation for all $x \in \widehat{\mathbb{Q}}$. Let $f' = h \circ f$, where $h \in PSL_2\mathbb{R}$, such that $f^l_0 = id$. It is easy to see that f' is an element of T, so that f belongs to $\mathcal{O}_T(id)$, hence a contradiction.

Now, we shall show that the order of any $\gamma \in Stab(f)$ is bounded by $card(bk(f))$. To do so, it suffices to prove that there exists x whose γ-orbit is contained in $bk(f)$. If it is not the case, then for all $x \in bk(f)$, there exists an integer k_x, verifying $\gamma^{k_x}(x) \notin bk(f)$. This implies $D_{f(x)}\gamma^{k_x} f^{-1} = D_{f(x)} f\gamma^{k_x} f^{-1} = id$. Hence $D_x f = (D_x\gamma^{k_x})^{-1}$ is a modular transformation for $x \in bk(f)$, but it contradicts the fact that f is not an element of $\mathcal{O}_T(id)$.

It suffices and remains to show that $Stab(f)$ is cyclic. Let $(\gamma, \gamma') \in Stab(f)^2$ and $(h, h') \in PSL_2\mathbb{R}^2$ such that $f \circ \gamma = h \circ f$ and $f \circ \gamma' = h' \circ f$. Thus, f respectively conjugates γ to h, γ' to h' and the commutator $[\gamma, \gamma']$ to $[h, h']$. Since they are of finite order, h, h' and $[h, h']$ are of elliptic type. By theorem 7.39.2 in [Bea], this implies that $[h, h']$ must be trivial. We deduce that f conjugates $Stab(f)$ to a subgroup of $\mathbb{S}^1 \cong PSO_2(\mathbb{R})$. Any subgroup of the circle is either dense or finite cyclic. Since the number of breakpoints of f bounds uniformly the order of elements of $Stab(f)$, $Stab(f)$ is finite cyclic, and this finishes the proof. □

5.4 T Acts Isometrically (as does the Mapping-Class Group)

Recall that theorem 4.9 provides $\mathcal{G}_r$ with the global shift coordinates. For each element, there is a finite number of non-zero coordinates. Hence, as for lambda coordinates, we can provide $\mathcal{G}_r$ with another metric:

$$d^b(f; g) = \sup_{x\in\widehat{\mathbb{Q}}} |d_x f - d_x g| \quad \forall (f, g) \in \mathcal{G}_r^2$$

Proposition 5.10. *T embeds in $Iso(\mathcal{G}_r, d^b)$*

Proof: From lemma 5.4, we have $d_x(f \circ \gamma) - d_x(g \circ \gamma) = d_{\gamma\cdot x} f - d_{\gamma\cdot x} g$ for all (x, f, g, γ). The proof of the proposition follows from the definition of d^b. □

As pointed out by Greenberg in the original preprint [Gre2], it may be useful for our intuition to consider the pair ($\mathcal{G}_r$, T) as a sort of infinite Teichmüller space with its associated Mapping-Class group. Indeed, we have seen that $\mathcal{G}_r$ is contractible (and even topologically a vector-space) and that T acts with finite stabilizers apart from one orbit. Although not finite on this orbit, it is essentially the "small" group $PSL_2\mathbb{Z}$. In the classical theory of Teichmüller spaces, an important result [Roy] says that the Mapping-Class group coincides with the full group of isometries under the Teichmüller metric. This result has been recently extended to the universal case in [Gar-Har]. The analogy with Teichmüller theory raises the natural question in our context:

Question: Is T the full group of isometries, relatively to the metric d^b?

Proposition 5.11. *T acts on $(\mathcal{G}_r, d^b)$ with discrete orbits.*

Proof: It follows directly from lemma 5.4 and the facts that for each $f \in \mathcal{G}_r$, the sequence $(d_x f)_{x \in \widehat{\mathbb{Q}}}$ takes on a finite number of values, and these values are integers whenever f lies in T. □

Corollary 5.12. *T acts on $\mathcal{G}_r$ (under the direct-limit topology) with the following property:*

$$\forall f \in \mathcal{G}_r,\ \exists V_f \text{ an open neighborhood of } f \mid V_f \cdot \gamma \cap V_f \neq \emptyset \Rightarrow \gamma \text{ fixes } f$$

Proof: Assume by contradiction that there exists $f \in \mathcal{G}_r$ that does not verify the property. For each $p \geq 1$, let $V_p = B_{d^b}(f; \frac{1}{p})$, which is an open neighborhood of f for the direct-limit topology, so there exists $\gamma_p \in T$ such that $f \cdot \gamma_p \neq f$ and $V_p \cap V_p \cdot \gamma_p \neq \emptyset$. Let f_p be an element of this intersection. It follows by a triangular inequality that $d^b(f; f \cdot \gamma_p) < \frac{2}{p}$, since d^b is invariant under T and $f_p \cdot \gamma_p \in V_p$. This contradicts proposition 5.11. □

5.5 Classifying Spaces for Thompson's Groups

In this subsection, we use the notation $\mathbb{S}^1$ instead of $\widehat{\mathbb{R}}$ to denote the circle. The set $\widetilde{Homeo^+}(\mathbb{S}^1)$ of orientation-preserving homeomorphisms of the real line which commute with the translation $t : x \mapsto x + 1$ is the universal covering of $Homeo^+(\mathbb{S}^1)$ and provides the central extension:

$$0 \to \mathbb{Z} \to \widetilde{Homeo^+}(\mathbb{S}^1) \xrightarrow{p} Homeo^+(\mathbb{S}^1) \to 1$$

where $\mathbb{Z}$ is the subgroup generated by t.

Now, we recall the definition of two other Thompson's groups related to T.
Let $F' = \{\gamma \in T \mid \gamma_0^r = \gamma_0^l = id\}$ and $\widetilde{T} = \left\{\tilde{\gamma} \in \widetilde{Homeo^+}(\mathbb{S}^1) \mid \gamma = p(\tilde{\gamma}) \in T\right\}$.

Notice that F' corresponds to the commutator subgroup of the group $F = Stab_T(0)$ and $\widetilde{T}$ is a central extension of T:

$$0 \to \mathbb{Z} \to \widetilde{T} \to T \to 1$$

As for T, let us consider the lift of $\widehat{\mathcal{G}_r}$, namely $\widetilde{\widehat{\mathcal{G}_r}} \subset \widetilde{Homeo^+}(\mathbb{S}^1)$.

Lemma 5.13. *$\widetilde{\widehat{\mathcal{G}_r}}$ is contractible. Moreover, the group $\widetilde{T}$ acts on it with quotient space homeomorphic to $\widehat{\mathcal{G}_r}/T$.*

Proof: By corollary 4.6 $\mathcal{G}_r$ is a contractible space. Thus, the fibration $PSL_2\mathbb{R} \to \widehat{\mathcal{G}_r} \to \mathcal{G}_r$ indicates that $\widehat{\mathcal{G}_r}$ is topologically the product $PSL_2\mathbb{R} \times \mathcal{G}_r$. Hence, $\widetilde{\widehat{\mathcal{G}_r}}$ is also contractible, since it is topologically the product $\widetilde{PSL_2\mathbb{R}} \times \mathcal{G}_r$. The composition map $\widetilde{\widehat{\mathcal{G}_r}} \to \widehat{\mathcal{G}_r} \to \widehat{\mathcal{G}_r}/T$ is continuous, onto, and factors to:

$$\phi : \tilde{f}[\widetilde{T}] \in \widetilde{\widehat{\mathcal{G}_r}}/\widetilde{T} \mapsto f[T] \in \widehat{\mathcal{G}_r}/T$$

with the obvious notations for equivalence classes. It remains to show that ϕ is one-to-one. If $\phi(\tilde{f}[\widetilde{T}]) = \phi(\tilde{g}[\widetilde{T}])$, then there exists $\gamma \in T$ such that $g = f \circ \gamma$. Let $\tilde{\gamma}$ be a lift of γ and $k \in \mathbb{Z}$ such that $\tilde{g} = \widetilde{f \circ \gamma} + k$. Since the last term is also $\tilde{f} \circ (\tilde{\gamma} + k)$, and $\tilde{\gamma} + k$ lies in T, we conclude that $\tilde{f}[\widetilde{T}] = \tilde{g}[\widetilde{T}]$. This ends the proof. □

Corollary 5.14. *One has the following:*

1. *$\mathcal{G}_r/F'$ is a $K(F'; 1)$.*

2. $\widehat{\mathcal{G}_r}/T$ *is a* $K(\widetilde{T};1)$.

3. $\widehat{\mathcal{G}_r}/T \times_{\mathbb{S}^1} E\mathbb{S}^1$ *is a* $K(T;1)$.

Proof: 1. Since the germ of any element of F' is the identity near 0, the group F' embeds in $Homeo^+(\mathbb{R})$ which has no torsion. It follows from theorem 5.9 that $Stab_{F'}(f) = \{e\}$ if $f \notin \mathcal{O}_T(id)$. Moreover, $PSL_2\mathbb{Z} \cap F' = \{e\}$, so the previous assertion on the stabilizer holds even if $f \in \mathcal{O}_T(id)$. We have proved that F' acts freely on $\mathcal{G}_r$. It follows from proposition 5.12 that it also acts properly discontinuously. Since $\mathcal{G}_r$ is contractible, $\mathcal{G}_r/F'$ is indeed a $K(F';1)$.

2. Since $\widetilde{T}$ acts freely and properly discontinuously on $\widetilde{\widehat{\mathcal{G}_r}}$, it follows from lemma 5.13 that $\widehat{\mathcal{G}_r}/T$ is a $K(\widetilde{T};1)$.

3. The long exact sequence of the $\mathbb{S}^1$-fibration

$$PSO_2(\mathbb{R}) = \mathbb{S}^1 \to \widehat{\mathcal{G}_r}/T \times E\mathbb{S}^1 \to \widehat{\mathcal{G}_r}/T \times_{\mathbb{S}^1} E\mathbb{S}^1$$

shows that $\pi_i\left(\widehat{\mathcal{G}_r}/T \times_{\mathbb{S}^1} E\mathbb{S}^1\right) = 0 \ \forall i \geq 2$, and gives the short exact sequence of groups:

$$0 \to \pi_1(\mathbb{S}^1) \to \widetilde{T} \to \pi_1\left(\widehat{\mathcal{G}_r}/T \times_{\mathbb{S}^1} E\mathbb{S}^1\right) \to 1$$

since $\pi_1\left(\widehat{\mathcal{G}_r}/T \times E\mathbb{S}^1\right) = \widetilde{T}$ by lemma 5.13. The generator of $\mathbb{Z}$ corresponds to the loop $PSO_2(\mathbb{R}) \subset \widehat{\mathcal{G}_r}$ and lifts to the path $\rho : s \in [0;1] \mapsto (x \mapsto x+s) \in \widehat{\mathcal{G}_r}$ via the tangent function. This generator is also the generator of the center of $\widetilde{T}$, and we conclude with the central extension $\widetilde{T}$ that $T = \pi_1\left(\widehat{\mathcal{G}_r}/T \times_{\mathbb{S}^1} E\mathbb{S}^1\right)$. Finally, $\widehat{\mathcal{G}_r}/T \times_{\mathbb{S}^1} E\mathbb{S}^1$ is a $K(T;1)$. □

Since T is a discrete group, $\widehat{\mathcal{G}_r}/T \times_{\mathbb{S}^1} E\mathbb{S}^1$ is a classifying space for T. It is worth notifying the reader that as an application of this version of BT, we have found (see [Mar] for further details) a simple geometric interpretation of the theorem in [Ghy-Ser] which gives an isomorphism between the homology of Thompson's group T and the homology of the free loop space over $\mathbb{S}^3$. This work will form the subject of a forthcoming paper by the author.

6 Appendix: horocyclic lengths

This appendix gives a brief account on horocyclic lengths, associated with bracelets, and links these lengths to the shift coordinates in order to prove theorem 4.9.

Let $(x_i)_{i\in<1;n>}$ a finite increasing sequence of the circle $x_1 < x_2 < \cdots < x_n < x_1$.

Definition 6.1. (Bracelet) An *n-bracelet* is a collection of horocycles $(h_{x_i})_{i\in<1;n>}$ such that each horocycle of the collection is tangent to its two neighbors.

Some spaces of bracelets are carefully studied in [Gre1] and more recently in [Mar]. By definition of the space $\mathcal{G}_{rn}$ (definition 2.4), for any $f \in \mathcal{G}_{rn}$, the collection $f \cdot b_n^F = (f(h_x^F))_{x\in S_n}$ is a $3\cdot 2^{n-1}$-bracelet. To a given n-bracelet $b = (h_{x_i})_{i\in<1;n>}$, we can associate n real numbers as followed:

Definition 6.2. (Horocyclic length) Let $l_i = l_i(b)$ denote the hyperbolic length of the segment in h_i whose endpoints are $h_{x_i} \cap (x_{i-1};x_i)$ and $h_{x_i} \cap (x_{i+1};x_i)$. l_i is called the i^{th} *horocyclic length* of the bracelet b.

From the remark preceding definition 6.2, there is a map: $q_n : f \in \mathcal{G}_{rn} \mapsto (l_x(f\cdot b_n^F))_{x\in S_n} \in (\mathbb{R}_+^*)^{S_n}$, where $l_x(f\cdot b_n^F)$ denotes the horocyclic length based on $f(x)$ for the bracelet $f\cdot b_n^F$. Using h-coordinates introduced in [Pen2], we can express the horocyclic lengths in terms of lambda lengths. Let $b = (h_{x_i})_{i\in<1;n>}$ be an n-bracelet and consider the set of geodesics $(e_i)_{i\in<3;n-1>}$ where $e_i = (x_1;x_i)$. Furthermore, let σ_i denote the lambda length of the decoration b along e_i divided by $\sqrt{2}$. The following lemma was originally conjectured in [Gre1].

Lemma 6.3. *1. If $n=3$, we have $l_1 = l_3 = \frac{2}{\sigma_3}$ et $l_2 = l_4 = \sigma_3$.*

2. In the general case:

- $l_1 = \frac{1}{\sigma_3} + \sum_{j=2}^{n-3} \frac{1}{\sigma_{j+1}\sigma_{j+2}} + \frac{1}{\sigma_{n-1}}$
- $l_2 = \sigma_3$
- $l_3 = \frac{1+\sigma_4}{\sigma_3}$
- $l_i = \frac{\sigma_{i-1}+\sigma_{i+1}}{\sigma_i}\ \forall i \in < 4, n-2 >$
- $l_{n-1} = \frac{1+\sigma_{n-2}}{\sigma_{n-1}}$
- $l_n = \sigma_{n-1}$

Proof: We prove directly the general case. Let P denote the hyperbolic n-gon defined by the collection of base-points of b. The sequence of geodesics $(e_i)_{i\in<2;n-1>}$ cuts P into $n-2$ triangles $T_1, \cdots, T_{n-2}$. Each horocycle is also cut into small horocyclic segments and for $\forall (i,j) \in < 1, n+1 > \times < 1, n-1 >$, we denote l_i^j the hyperbolic length of the segment $h_i \cap T_j$, so that $l_i = \sum_{j=1}^{k_i} l_i^j$. Now proposition 2.8. in [Pen1] applied to each triangle T_j gives:

- T_1: $l_1^1 = \frac{1}{\sigma_3}$, $l_2^1 = \sigma_3$, $l_3^1 = \frac{1}{\sigma_3}$
- T_j: $l_1^j = \frac{1}{\sigma_{j+1}\sigma_{j+2}}$, $l_{j+1}^j = \frac{\sigma_{j+2}}{\sigma_{j+1}}$, $l_{j+2}^j = \frac{\sigma_{j+1}}{\sigma_{j+2}}$
- T_{n-1}: $l_1^{n-1} = \frac{1}{\sigma_n}$, $l_n^{n-1} = \frac{1}{\sigma_n}$, $l_{n+1}^{n-1} = \sigma_n$

and the formulas of the lemma follow by summations. □

Let us consider the "coordinate changes" map:

$$\begin{array}{cccc} r_n : & {\mathbb{R}_+^*}^{n-3} & \rightarrow & {\mathbb{R}_+^*}^{n} \\ & (\sigma_3, \cdots, \sigma_{n-1}) & \mapsto & \left(\frac{1}{\sigma_3} + \sum_{j=2}^{n-3} \frac{1}{\sigma_{j+1}\sigma_{j+2}} + \frac{1}{\sigma_{n-1}}, \sigma_3, \frac{1+\sigma_4}{\sigma_3}, \cdot, \frac{\sigma_{i-1}+\sigma_{i+1}}{\sigma_i}, \cdot, \frac{1+\sigma_{n-2}}{\sigma_{n-1}}, \sigma_{n-1}\right) \end{array}$$

Proposition 6.4. *The map r_n is an embedding.*

Proof: The map is obviously of class $\mathcal{C}^1$. It is also one-to-one, since σ_3 depends uniquely on l_2, then σ_4 on l_3 and $l_2 = \sigma_3$, and so on. To see that it is an immersion, it suffices to cyclically permute the coordinates by the map $(l_1, \cdots, l_n) \mapsto (l_2, \cdots, l_n, l_1)$ and a simple calculation shows that the Jacobian is decomposed into an upper triangular $(n-3) \times (n-3)$ block, with diagonal $1, \sigma_3^{-1}, \ldots, \sigma_{n-2}^{-1}$ and a $(n-3) \times 3$ block. Henceforth, the Jacobian is one-to-one.

It remains to show that r_n is a proper map. It is equivalent to show that it maps any unbounded sequence σ^k to an unbounded sequence $\left(r_n(\sigma^k)\right)_{k\geq 0}$. Let us consider an unbounded sequence $\left(\sigma^k\right)_{k\geq 0}$ and let i_0 denote the smallest $i \in < 3; n-1 >$ such that the sequence $\left(\sigma_i^k\right)_{k\geq 0}$ is unbounded. We consider two cases:

- $\underline{\lim_{k\to\infty} \sigma_{i_0}^k = 0}$: If $i_0 = 3$, then the sequence $\left(l_2^k = \sigma_3^k\right)_{k\geq 0}$ tends to 0, hence the sequence $\left(r_n(\sigma^k)\right)_{k\geq 0}$ is unbounded. If $i_0 > 3$, then the sequence $\left(l_{i_0}^k = \frac{\sigma_{i_0-1}^k + \sigma_{i_0+1}^k}{\sigma_{i_0}^k}\right)_{k\geq 0}$ tends to ∞, by definition of i_0.
- $\underline{\lim_{k\to\infty} \sigma_{i_0}^k = +\infty}$: If $i_0 = 3$, then $\left(l_2^k\right)_{k\geq 0}$ tends to ∞. If $i_0 > 3$, this time, the sequence $\left(l_{i_0-1}^k = \frac{\sigma_{i_0-2}^k + \sigma_{i_0}^k}{\sigma_{i_0-1}^k}\right)_{k\geq 0}$ tends to ∞.

In both cases, it follows that $r_n(\sigma^k)$ is unbounded, hence r_n is proper. Finally, r_n is a one-to-one proper immersion. Thus, it is an embedding. □

Lemma 6.5. *For any $f \in \widehat{\mathcal{G}_{rn}}$ and $x \in S_n$, we have the equation: $d_x(f) = l_x\left(f \cdot b_n^F\right) - l_x\left(b_n^F\right)$*

Proof: Let $x \in S_n$ and denote x^-, x^+ the two neighbors of x in S_n, such that $x^- < x < x^+ < x^-$ with respect to the orientation of the circle. Consider $P \in PSL_2\mathbb{Z}$ and $Q \in PSL_2\mathbb{R}$ such that $P(x) = \infty$ and $Q(f(x)) = \infty$. Since $P \in PSL_2\mathbb{Z}$, QfP^{-1} belongs to $\widehat{\mathcal{G}_r}$. Hence, $D_\infty\left(QfP^{-1}\right)$ is well defined and equals to PD_xfP^{-1} since $Q \in PSL_2\mathbb{R}$. Thus, we have $D_\infty\left(QfP^{-1}\right) = t_{d_x(f)}$.

Moreover, QfP^{-1} fixes the point at infinity. Therefore, it is of the form $z \mapsto a^2z + d^+a$ (resp. $z \mapsto a^2z + d^-a$) on the interval $[\infty; P(x^+)]$ (resp. $[P(x^-); \infty]$). We compute explicitly that $D_\infty\left(QfP^{-1}\right) = t_{\frac{d^+-d^-}{a}}$, and it follows the equality $d_x(f) = \frac{d^+-d^-}{a}$.

Let us compute the x-horocyclic coordinate for f and for id. Since $P \in PSL_2\mathbb{Z}$, we have:

$$l_x\left(b_n^F\right) = l_\infty\left(P \cdot b_n^F\right) = \int_{Px^-}^{Px^+} \frac{dy}{h} = \frac{P\left(x^+\right) - P\left(x^-\right)}{h}$$

where h is the height of the horocycle $P \cdot h_x^F$ based at ∞. In the same manner, we have:

$$l_x\left(f \cdot b_n^F\right) = l_x\left(Qf \cdot b_n^F\right) = \int_{Qf(x^+)}^{Qf(x^-)} \frac{dy}{a^2h} = \frac{P\left(x^+\right) - P\left(x^-\right)}{h} + \frac{d^+ - d^-}{ah}$$

since $Q \in PSL_2\mathbb{R}$ and QfP^{-1} maps the horocycle with height h to the horocycle with height a^2h.

Thus, we obtain $l_x\left(f \cdot b_n^F\right) - l_x\left(b_n^F\right) = \frac{d_x(f)}{h}$. Finally, since $P \in PSL_2\mathbb{Z}$, $P\left(h_x^F\right)$ is a Ford circle, based at ∞, its height must be 1, and the result follows. □

We end the appendix by proving theorem 4.9.

Proof: (of theorem 4.9) Since both domain and range are given the direct-limit topology, it is sufficient to prove that each map $d_n : f \in \mathcal{G}_{rn} \mapsto (d_x(f))_{x \in S_n} \in (\mathbb{R}_+^*)^{S_n}$ is an embedding. By lemma 6.5, this is equivalent to showing that the map $q_n : f \in \mathcal{G}_{rn} \mapsto \left(l_x(f \cdot b_n^f)\right)_{x \in S_n} \in (\mathbb{R}_+^*)^{S_n}$ is an embedding. Label S_n as in subsection 2.2 and set $\sigma_i(f) = \frac{\Lambda(f \cdot b_n^F; (x_1^n; x_i^n))}{2}$ for each $i \in < 3; 3 \cdot 2^{n-1} - 1 >$. Denote $p_n : f \in \mathcal{G}_{rn} \mapsto (\sigma_i(f))_{i \in <3; 3 \cdot 2^{n-1} - 1>} \in (\mathbb{R}_+^*)^{3(2^{n-1}-1)}$ so that $q_n = r_{3 \cdot 2^{n-1}} \circ p_n$. We can adapt the proof of theorem 4.4 to prove that p_n is a homeomorphism. It follows from proposition 6.4 that q_n is an embedding and this concludes the proof of the theorem. □

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