Stochastic integral representations of second quantification operators

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Abstract

We give a necessary and sufficient condition for the second quantification operator $\Gamma(h)$ of a bounded operator h on $L^2(\mathbb{R}_+)$, or for its differential second quantification operator $\lambda(h)$, to have a representation as a quantum stochastic integral. This condition is exactly that h writes as the sum of a Hilbert-Schmidt operator and a multiplication operator. We then explore several extensions of this result. We also examine the famous counterexample due to Journé and Meyer and explain its representability defect. [1]

Introduction

Second quantification operators and differential second quantification operators on Fock spaces are the most basic operators appearing in the quantum theory of fields, after creation and annihilation operators. On the other hand, on Fock spaces of the form $\Phi = \Gamma(L^2(\mathbb{R}_+, \mathcal{K}))$, where $\mathcal{K}$ is a separable Hilbert space, an effective theory of quantum stochastic integration is now well-developed and has found numerous applications (such as the ergodic properties of dissipative quantum systems). One of the basic questions

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in that context is to characterize the operators on Φ which can be represented as a quantum stochastic integral. Many articles have been devoted to that problem (see for example [P-S], [At1], [Coq]), which is nevertheless far from being closed.

We study here two particular families of operators: the second quantification operators $\Gamma(h)$ and the differential second quantification operators $\lambda(h)$, for a bounded operator h on $L^2(\mathbb{R}_+, \mathcal{K})$. Rather surprisingly we find a necessary and sufficient condition for $\Gamma(h)$ and $\Gamma(h^*)$ to be represented as quantum stochastic integrals on the set of exponential vectors:

Theorem 1 *Let h be a bounded operator on $L^2(\mathbb{R}_+)$. The following properties are equivalent:*

1. *$\Gamma(h)$ and $\Gamma(h^*)$ have a quantum stochastic integral representation on the set $\mathcal{E}(L^2(\mathbb{R}_+))$*
2. *h is of the form*
$$h = K + \mathcal{M}_f$$
where K is a Hilbert-Schmidt operator and $\mathcal{M}_f$ is a multiplication by an essentially bounded function f.

Surprisingly also, the exact same theorem holds for differential second quantifications.

Theorem 2 *Let h be a bounded operator on $L^2(\mathbb{R}_+)$. The following properties are equivalent:*

1. *$\lambda(h)$ and $\lambda(h^*)$ have a quantum stochastic integral representation on the set $\mathcal{E}(L^2(\mathbb{R}_+))$*
2. *h is of the form*
$$h = K + \mathcal{M}_f$$
where K is a Hilbert-Schmidt operator and $\mathcal{M}_f$ is a multiplication by an essentially bounded function f.

We furthermore derive fully explicit formulas for the integrands in the integral representation in both cases (differential and nondifferential). We also give more general results for representability of such operators on particular subsets of the exponential domain, and prove various results concerning the obtained representations.

The paper is organized as follows: in section one we give all the necessary theoretical background and notations. The second section is the core of this article: the above quoted theorem for (non differential) second quantifications is proved. Characterizations of the fact that $\Gamma(h)$ defines a regular semimartingale (see [At1]) are also given and the counterexample of Journé and Meyer is discussed. In the third section we prove the counterpart for differential second quantification operators. Section four handles the extension of our characterizations and formulas to the case of Fock space of higher (possibly infinite) multiplicity. Section five gives criteria for the weaker properties of representability on subsets of the exponential domain. Section six gives simple sufficient conditions for representability of second quantifications and differential second quantifications of unbounded operators, a case which often arises in physical applications.

1 Preliminaries

We shall here recall briefly some necessary definitions and results from quantum stochastic calculus on regular Fock space. First of all, the Fock space is defined as the completion of $\bigotimes_{n\geq 0} L^2\left(\mathbb{R}_+\right)^{\circ n}$ (where $\circ$ denotes the symmetric tensor product); but thanks to Guichardet's interpretation it can be seen as $\Phi = L^2\left(\mathcal{P}\right)$, *i.e.* the set of functions on the set $\mathcal{P}$ of finite subsets of $\mathbb{R}_+$ when $\mathcal{P}$ is equipped with the measure such that the empty set is the only atom, of mass one, and the measure of a set of n-uples is equal to its n-dimensonal Lebesgue measure. The canonical variable will be denoted by σ, and the infinitesimal volume element by $d\sigma$. This means that the elements of Φ are the functions defined on all increasing simplexes $\Sigma_n = \{s_1 < \cdots < s_n\}$ of $\mathbb{R}_+$ such that

$$\sum_n \int_{\Sigma_n} |f(s_1, \cdots, s_n)|^2 \, ds_1 \cdots ds_n < +\infty. \tag{1.1}$$

If $u_1, \ldots, u_n$ are n functions on $\mathbb{R}_+$ we denote by $u_1 \circ \cdots \circ u_n$ the symmetrized product of $u_1, \ldots, u_n$; we will also denote by $f \circ g$ the symmetric function of $p+q$ variables obtained by symmetrization from two symmetric functions f and g of p and q variables respectively. The subset of Φ which is identified with the set of symmetric n-variables functions is called the n-th chaos of Φ. We shall label Φ_t the analogous set of functions defined on simplexes of $[0, t]$; Φ_t will be canonically included in Φ.

Abstract Ito calculus Let us consider for all t the element χ_t of Φ defined as follows:

$$\chi_t(\sigma) = \begin{cases} \mathbb{1}_{s<t} & \text{if } \sigma = \{s\} \\ 0 & \text{otherwise.} \end{cases}$$

One can define an integral of *adapted processes* $(f_t)_{t\geq 0}$ of elements of Φ (that is, such that $f_t \in \Phi_t$ for almost all t), with respect to the curve $(\chi_t)_{t\geq 0}$ (see [At3]), denoted

$$I(f) = \int f_t \, d\chi_t$$

and satisfying

$$\|I(f)\|^2 = \int_0^\infty \|f_t\|^2 dt \tag{1.2}$$

as soon as the latter real-valued integral is finite; the complete construction uses the isometry property (1.2) for step processes. This integral is called the (abstract) *Ito integral.*

There is an alternate construction for this integral:

$$I(f)(\sigma) = f_{\vee\sigma}(\sigma-) \tag{1.3}$$

where $\vee\sigma$ is the largest element in σ and $\sigma- = \sigma\backslash(\vee\sigma)$. The natural conditions for this to be well defined can be seen to be the same as above, namely the square-integrability of the process $(\|f_t\|)_{t\geq 0}$.

Let us define the two fundamental operators of abstract Ito calculus on Φ:

- the *adapted projection* P_t is, for all t, as the orthogonal projection onto Φ_t. Explicitly, for any $f \in \Phi$,
$$P_t f(\sigma) = \mathbb{1}_{\sigma<t} f(\sigma) \tag{1.4}$$
for almost all σ in $\mathcal{P}$,

- the *adapted gradient* by
$$D_t f(\sigma) = \mathbb{1}_{\sigma<t} f(\sigma \cup t) \tag{1.5}$$
for almost all σ in $\mathcal{P}$.

Substituting (1.5) in (1.3) yields immediately

$$f = f(\emptyset) + \int D_t f \, d\chi_t \tag{1.6}$$

and

$$\|f\|^2 = |f(\emptyset)|^2 + \int \|D_t f\|^2 dt, \tag{1.7}$$

that is, all elements of Φ have a *previsible representation* (1.6) together with the associated isometry formula (1.7).

Two relevent total sets We will mainly use in the sequel two subsets of Φ: the first is the classical exponential set. First let us define an exponential vector: for u in $L^2(\mathbb{R}_+)$ we denote by $\mathcal{E}(u)$ the element of Φ such that for all $\sigma \in \mathcal{P}$,

$$\mathcal{E}(u)(\sigma) = \prod_{s\in\sigma} u(s).$$

It is an element of Φ, as one can see from $\int_{\Sigma_n} |u(s_1)\cdots u(s_n)|^2 \, ds_1\cdots ds_n = \frac{\|u\|^{2n}}{n!}$ which implies that

$$\|\mathcal{E}(u)\|^2 = \exp(\|u\|^2).$$

The other set is the set $\mathcal{J}$ introduced by Coquio in [Coq]: it is generated by the vacuum vector $\mathbb{1}$ and all vectors of the form

$$j(v,u) = \int_0^\infty v(s)\mathcal{E}(u_s)\, d\chi_s$$

for v, u in $L^2(\mathbb{R}_+)$; note that here and later on we will denote by u_s, v_s... the functions $u\mathbb{1}_{[0,s]}$, $v\mathbb{1}_{[0,s]}$, *etc.* The set $\mathcal{J}$ strictly contains $\mathcal{E}(L^2(\mathbb{R}_+))$ since any $\mathcal{E}(u)$ satisfies

$$\mathcal{E}(u) = \mathbb{1} + j(u,u).$$

Quantum stochastic integrals We shall here define integrals

$$\int H_s da_s^\epsilon$$

with respect to the three quantum noises da_t^+, da_t°, da_t^- and to time, which, in the search for homogeneous notations, will be denoted $da_t^\times$ instead of dt.

The heuristics of the Attal-Meyer quantum stochastic calculus derive from the following demands:

- any da_t^ϵ acts only on $\Phi_{[t,t+dt]}$, which from (1.6) can be seen as "generated" by $\mathbb{1}$ and $d\chi_t$ and
- the da_t^ϵ are given by the following table:

$$\begin{array}{rcl}
da_t^+\mathbb{1} = d\chi_t & \text{and} & da_t^+ d\chi_t = 0 \\
da_t^-\mathbb{1} = 0 & \text{and} & da_t^- d\chi_t = dt\mathbb{1} \\
da_t^\circ\mathbb{1} = 0 & \text{and} & da_t^\circ d\chi_t = d\chi_t \\
da_t^\times\mathbb{1} = dt\mathbb{1} & \text{and} & da_t^\times d\chi_t = 0.
\end{array}$$

These heuristics allow us to define integrals $\int H_s da_s^\epsilon$ for *adapted processes* $(H_s)_{s\geq 0}$, that is, processes of operators such that for almost all s, all $f \in Dom\, H_s$,

$$P_s f \in Dom\, H_s,\ \ D_u f \in Dom\, H_s \text{ for a.a. } u \geq s \text{ and}$$

$$H_s P_s f = P_s H_s f \text{ and } H_s D_u f = D_u H_s f \text{ for a.a. } u \geq s.$$

In that case, a formal computation leads us to give the following definition: an adapted operator process $(T_t)_{t\geq 0}$ is said to be the integral process $(\int_0^t H_s da_s^\epsilon)_{t\geq 0}$ if the following equality holds for almost all $t \geq 0$:

$$T_t f = \int_0^\infty T_{t\wedge s} D_s f d\chi_s + \begin{cases} \int_0^t H_s P_s f d\chi_s & \text{if } \epsilon = + \\ \int_0^t H_s D_s f ds & \text{if } \epsilon = - \\ \int_0^t H_s D_s f d\chi_s & \text{if } \epsilon = \circ \\ \int_0^t H_s P_s f ds & \text{if } \epsilon = \times, \end{cases} \tag{1.8}$$

that is, f is in the common domain of the integrals if and only if *each* integral in the right-hand side is well defined and equality holds. An integral $\int_a^b H_s da_s^\epsilon$ is then simply the integral of the process equal to H_s for $s \in [a, b]$ and zero otherwise.

The fundamental operators a_f^+, a_f^- for f in $L^2(\mathbb{R}_+)$ are recovered as the integrals $\int_0^\infty f(s) da_s^+$, $\int_0^\infty \overline{f(s)} da_s^-$ in the above sense.

In the sequel we will be interested in representing operators of the Fock space as sums of three integral with respect to da_t^+, da_t^-, da_t° *only*, that is, we reject the integral with respect to time. The reason is that, if time integrals are allowed, the integral representation of a fixed operator H is not unique; besides, the conditional expectation at some time t of H can be deduced from an integral representation involving da^+, da^-, da°, only, but not from a representation involving $da^\times$. Therefore "a representation as quantum stochastic integrals" will from now on mean a representation as the sum of three integrals with respect to da^+, da^- and da°.

One naturally wonders which operators on Φ are representable in this way, at least on a reasonable domain; we will present in the next paragraph an example of an operator on Φ, and, what's more, a bounded one, that does not have such an integral representation on the exponential set.

Second quantification operators We define now the classes of operators in which we will be interested: *second quantification operators* and *differential second quantification operators*. For a bounded operator h on $L^2(\mathbb{R}_+)$, one defines an operator $\Gamma(h)$ on the n-th chaos by:

$$\Gamma(h)(u_1 \circ \cdots \circ u_n) = hu_1 \circ \cdots \circ hu_n,$$

(here $u_1, \dots, u_n$ belong to $L^2(\mathbb{R}_+)$) and an operator $\lambda(h)$ by

$$\lambda(h)(u_1 \circ \dots \circ u_n) = hu_1 \circ \dots \circ u_n + u_1 \circ hu_2 \circ \dots \circ u_n + \dots + u_1 \circ \dots \circ hu_n.$$

and these operators are extended by linearity and closure; one checks that their domain is therefore the set of vectors f in $L^2(\mathcal{P})$ such that, if f_n is the restriction of f to n-uples, one has $\sum_n \|\Gamma(h)f_n\|^2 < +\infty$ (respectively $\sum_n \|\lambda(h)f_n\|^2 < +\infty$). In particular, the set $\mathcal{J}$ is contained in the domain of $\Gamma(h)$ and $\lambda(h)$ for any bounded h (see [Coq]).

The action of these operators on exponential vectors is easily expressed:

$$\Gamma(h)\mathcal{E}(u) = \mathcal{E}(hu),$$

$$\lambda(h)\mathcal{E}(u) = a^+_{hu}\mathcal{E}(u).$$

These two types of operators are linked by the following formula which, can constitute a definition for differential second quantification operators, and explains the term differential: for any bounded h on $L^2(\mathbb{R}_+)$, any t in $\mathbb{R}$, one has

$$\Gamma(e^{ith}) = e^{it\lambda(h)}.$$

In the case of a selfadjoint h this reads as: the second quantification of a unitary semigroup with generator h on $L^2(\mathbb{R}_+)$ is a unitary semigroup on $L^2(\mathcal{P})$ with generator $\lambda(h)$.

We will mention in section four second quantifications of unbounded operators h. The definition is clear from the above; what we fail to formulate is the form of the domain.

The Journé-Meyer counterexample The question of whether all operators on Fock space are representable as quantum stochastic integrals up to simple domain assumptions was given a negative answer by Journé and Meyer in [J-M]: their counterexample consists of a bounded operator on $L^2(\mathcal{P})$ which is not representable on the whole of the exponential domain. The reason why we include this counterexample here is that it is a second quantification operator.

Journé and Meyer consider the second quantification operator $\Gamma(h)$ where h is the Hilbert transform on $L^2(\mathbb{R}_+)$. This application is a contraction of $L^2(\mathbb{R}_+)$ so that the associated $\Gamma(h)$ is a bounded operator, and yet it is not representable on the whole of $\mathcal{E}(L^2(\mathbb{R}_+))$ – not even on the subset $\mathcal{E}(L^2 \cap L^\infty(\mathbb{R}_+))$. Precisely Journé and Meyer prove that, if some u in $L^2 \cap L^\infty(\mathbb{R}_+)$ is such that $\mathcal{E}(u)$ is in the domain of some integral operator H, then $t \mapsto P_t H\mathcal{E}(u_t)$ has finite quadratic variation. Here they exhibit some u in $L^2 \cap L^\infty(\mathbb{R}_+)$ for which $t \mapsto P_t H\mathcal{E}(u_t)$ does not have finite quadratic variation.

The case of Fock space of higher multiplicity We present here briefly the counterpart of the above definitions in the case of Fock space; here the Fock space is constructed over $L^2(\mathbb{R}_+,\mathcal{K})$, where $\mathcal{K}$ is a separable Hilbert space. In that case we fix a hilbertian basis $(e_i)_{i\in\mathcal{I}}$ (where the set $\mathcal{I}$ is supposed not to contain the index zero) of $\mathcal{K}$; then the main difference lies therein, that notations are much more cumbersome. An element of the Fock space Φ will be a function of the variable $\sigma=\{(s_1,i_1),\cdots,(s_n,i_n)\}$ where the s_k's belong to $\mathbb{R}_+$ (indexed in increasing order) and the i_k's are elements of $\mathcal{I}$; the integrability condition will be that the sum of integrals of $\|f\|^2$ over all possible simplexes be finite.

The abstract Ito calculus uses now a set of curves χ_t^i and operators D_t^i: simply put, we define:

- for any f in Φ, $D_t^i f$ by,
$$D_t^i f(\sigma)=1\!\!1_{\sigma<t}f(\sigma\cup t,i) \text{ for } i\in\mathcal{I}$$
for almost all σ in $\mathcal{P}$
- and the integration with respect to χ_t^i of an adapted process $(f_t)_{t\geq 0}$ of elements of Φ by
$$\int f_t\,d\chi_t^i(\sigma)=\begin{cases} f_{s_n}(\sigma-) \text{ if } i_n=i,\\ \quad 0 \text{ otherwise}\end{cases}$$
for almost all $\sigma=\{(s_1,i_1),\cdots,(s_n,i_n)\}$ in $\mathcal{P}$, with notations as above.

To unify our notations we will denote by $d\chi_t^0$ the time differential dt, and by D_t^0 the adapted projection: $D_t^0 f(\sigma)=P_t f(\sigma)=1\!\!1_{\sigma<t}f(\sigma)$.
Exponential vectors are now constructed with respect to functions u in $L^2(\mathbb{R}_+,\mathcal{K})$ by

$$\mathcal{E}(u)(\{(s_1,i_1),\cdots,(s_n,i_n)\})=u(s_1,i_1)\cdots u(s_n,i_n),$$

that is, they have the previsible representation

$$\mathcal{E}(u)=1\!\!1+\sum_{i\in\mathcal{I}}\int_0^\infty u(s,i)\mathcal{E}(u_s)d\chi_s^i.$$

The set $\mathcal{J}$ of vectors $j(v,u)$ for v,u in $L^2(\mathbb{R}_+,\mathcal{K})$ is defined in an analogous way by

$$j(v,u)=1\!\!1+\sum_{i\in\mathcal{I}}\int_0^\infty v(s,i)\mathcal{E}(u_s)d\chi_s^i.$$

Second quantification and differential second quantification operators are defined as before, but now for operators h on $L^2(\mathbb{R}_+,\mathcal{K})$, so that once again $\Gamma(h)\mathcal{E}(u)=\mathcal{E}(hu)$ and $\lambda(h)\mathcal{E}(u)=a^+_{hu}\mathcal{E}(u)$.

Stochastic integrals of operators are now to be considered with respect to a set of quantum noises $da_t^{i,j}$ for i,j both in $\mathcal{I} \cup \{0\}$. The noise $da_t^{0,0}$ will represent the time differential dt Id; $da_t^{i,0}$, $da_t^{0,i}$ and $da_t^{i,i}$ represent respectively the creation, annihilation and conservation operators at site $i \in \mathcal{I}$, the remaining ones $da_t^{i,j}$ for $i \neq j$ representing *exchange* operators.

Now an integral

$$H = \lambda Id + \sum_{i,j \in \mathcal{I}\cup\{0\}} \int_0^\infty H_s^{i,j} da_s^{i,j}$$

is defined on $f \in \Phi$ by

$$Hf = \lambda f(\emptyset) + \sum_{i \in \mathcal{I}} \int_0^\infty H_s D_s^i f \, d\chi_s^i + \sum_{i,j \in \mathcal{I}\cup\{0\}} \int_0^\infty H_s^{i,j} D_s^j f \, d\chi_s^i,$$

if all terms are well defined and the sum of the norms in Φ of *each* integral is finite. This definition meets the one for Fock space of multiplicity one if $\mathcal{I}$ is made of a single element.

Note that, as before, we will be interested in representing operators as sums of integrals that exclude integration with respect to time: only the noises $da_s^{i,j}$ for $(i,j) \neq (0,0)$ will be considered.

2 Second quantification operators

2.1 A representation theorem

The main theorem to be proved in this section is the following. The explicit formulas for the integrands to appear in the representation are given along the proof, in (2.9), (2.10).

Theorem 2.1 *Let h be a bounded operator on $L^2(\mathbb{R}_+)$. The following properties are equivalent:*

1. *$\Gamma(h)$ and $\Gamma(h^*)$ have a stochastic integral representation on the set $\mathcal{E}(L^2(\mathbb{R}_+))$*
2. *h is of the form*

$$h = K + \mathcal{M}_\gamma$$

where K is a Hilbert-Schmidt operator and $\mathcal{M}_\gamma$ is a multiplication by an essentially bounded function γ.

Besides, if one of these holds, then the representation holds on the set $\mathcal{J}$.

The proof will be decomposed as follows:

- we prove in Proposition 2.2 below that *1.* implies a seemingly weaker set of conditions **(C)**,
- we prove the equivalence of **(C)** and *2.* in Lemma 2.6,
- we prove the implication *2.* $\Rightarrow$ *1.*,
- we extend a representation to the domain $\mathcal{J}$.

Proposition 2.2 *If $\Gamma(h)$ and $\Gamma(h^*)$ have a stochastic integral representation on the set $\mathcal{E}(L^2(\mathbb{R}_+))$ then h satisfies the following set of conditions **(C)**:*

$$
(\mathbf{C})\begin{cases}
-\textit{for a.a. } t, \textit{ the function } h\mathbb{1}_{[t,t+\epsilon]}/\epsilon \textit{ converges in } L^2([0,t]) \textit{ to a function } \alpha_t \textit{ as } \epsilon\to 0,\\
-\textit{for a.a. } t, \textit{ the function } h^*\mathbb{1}_{[t,t+\epsilon]}/\epsilon \textit{ converges in } L^2([0,t]) \textit{ to a function } \beta_t \textit{ as } \epsilon\to 0,\\
-\textit{the functions } t\mapsto \|\alpha_t\|_{L^2([0,t])},\ t\mapsto\|\beta_t\|_{L^2([0,t])} \textit{ are square-integrable,}\\
-\textit{for a.a. } t, \textit{ the integral } \frac{1}{\epsilon}\int_t^{t+\epsilon} h\mathbb{1}_{[t,t+\epsilon]}(s)ds \textit{ converges to a scalar } \gamma(t) \textit{ as } \epsilon\to 0 \textit{ and}\\
-\textit{the function } t\mapsto\gamma(t) \textit{ is essentially bounded.}
\end{cases}
$$

Proof of Proposition 2.2. We suppose here that one can write the equality

$$\Gamma(h) = \lambda \mathrm{Id} + \int H_s^+\, da_s^+ + \int H_s^+\, da_s^- + \int H_s^\circ\, da_s^\circ \quad \text{on } \mathcal{E}(L^2(\mathbb{R}_+))$$

in the sense that, on $\mathcal{E}(L^2(\mathbb{R}_+))$, the integrals are well-defined and equality holds.

We make the equivalent assumption for $\Gamma(h^*)$ but will use this assumption only to transfer our proofs of properties of h to the case of h^* by dual arguments and do not need to make a choice of explicit notations.

First let us remark that $\Gamma(h)\mathbb{1} = \mathbb{1}$ and that this translates on the integral as:

$$\mathbb{1} = \lambda\mathbb{1} + \int H_s^+\mathbb{1}\, d\chi_s$$

so that $\lambda = 1$ (and $H_s^+\mathbb{1} = 0$ for almost all s, but we will not need this).

Now we compute $P_t\Gamma(h)\mathcal{E}(\mathbb{1}_{t+\epsilon]})$ and $P_t\Gamma(h)\mathcal{E}(\mathbb{1}_{t]})$:

$$\begin{aligned}\Gamma(h)\mathcal{E}(\mathbb{1}_{t+\epsilon]}) &= \mathbb{1} + \int_0^{t+\epsilon} H_s\mathcal{E}(\mathbb{1}_{s]})\,d\chi_s + \int_0^{+\infty} H_s^+\mathcal{E}(\mathbb{1}_{s\wedge t+\epsilon]})\,d\chi_s \\ &\qquad + \int_0^{t+\epsilon} H_s^-\mathcal{E}(\mathbb{1}_{s]})\,ds + \int_0^{t+\epsilon} H_s^\circ\mathcal{E}(\mathbb{1}_{s]})\,d\chi_s,\end{aligned}$$

so that

$$\begin{aligned}P_t\,\Gamma(h)\mathcal{E}(\mathbb{1}_{t+\epsilon]}) &= \mathbb{1} + \int_0^{t} H_s\mathcal{E}(\mathbb{1}_{s]})\,d\chi_s + \int_0^{t} H_s^+\mathcal{E}(\mathbb{1}_{s]})\,d\chi_s \\ &\qquad + \int_0^{t+\epsilon} P_tH_s^-\mathcal{E}(\mathbb{1}_{s]})\,ds + \int_0^{t} H_s^\circ\mathcal{E}(\mathbb{1}_{s]})\,d\chi_s,\end{aligned}$$

whereas

$$\begin{aligned}P_t\,\Gamma(h)\mathcal{E}(\mathbb{1}_{t]}) &= \mathbb{1} + \int_0^{t} H_s\mathcal{E}(\mathbb{1}_{s]})\,d\chi_s + \int_0^{t} H_s^+\mathcal{E}(\mathbb{1}_{s]})\,d\chi_s \\ &\qquad + \int_0^{t} H_s^-\mathcal{E}(\mathbb{1}_{s]})\,ds + \int_0^{t} H_s^\circ\mathcal{E}(\mathbb{1}_{s]})\,d\chi_s.\end{aligned}$$

Hence

$$\frac{1}{\epsilon}P_t\left(\Gamma(h)\mathcal{E}(\mathbb{1}_{t+\epsilon]}) - \Gamma(h)\mathcal{E}(\mathbb{1}_{t]})\right) = P_t\frac{1}{\epsilon}\int_t^{t+\epsilon} H_s^-\mathcal{E}(\mathbb{1}_{s]})\,ds.$$

But $s \mapsto \left\|H_s^-\mathcal{E}(\mathbb{1}_{s]})\right\|$ is locally integrable by definiteness of the da^- integral on $\mathcal{E}(L^2(\mathbb{R}_+))$, so that, for almost all t, the integral on the right-hand side above tends in norm to $H_t^-\mathcal{E}(\mathbb{1}_{t]})$. Applying P_t does not change the picture since it is a projection and the limit $H_t^-\mathcal{E}(\mathbb{1}_{t]})$ belongs to its image $\Phi_{t]}$.

The left-hand side, when restricted to the first chaos, is simply the restriction of $(h\mathbb{1}_{[t,t+\epsilon]})/\epsilon$ to $[0,t]$ so that we have proved that

$$\text{for almost all } t,\, (h\mathbb{1}_{[t,t+\epsilon]})/\epsilon \overset{L^2([0,t])}{\longrightarrow} P^1H_t^-\mathcal{E}(\mathbb{1}_{t]}) \text{ as } \epsilon \text{ tends to zero.} \tag{2.1}$$

where P^1 is the projection on the first chaos. We denote from now on $\alpha_t = P^1H_t^-\mathcal{E}(\mathbb{1}_t)$, and recall that $t \mapsto \|\alpha_t\|$ is a locally integrable function.

All that we have done here translates to the case of h^* and β so that we have proved the two first conditions of **(C)**. We have not proved the third, that is, the square-integrability of $\|\alpha_t\|$ and $\|\beta_t\|$, and postpone it until after the last proof of convergence. For now let us simply notice that, had we done the same computations with some u in $L^2(\mathbb{R}_+)$ instead of an indicator function, we would have shown that

$$\text{for almost all } t,\, \frac{h\,u_{[t,t+\epsilon]}}{\epsilon} \overset{L^2([0,t])}{\longrightarrow} u(t)P^1H_t^-\mathcal{E}(u_t) \text{ as } \epsilon \text{ tends to zero.} \tag{2.2}$$

We prove the fourth property of **(C)**. Let us consider $\Gamma(h)\mathcal{E}(u_{t+\epsilon})$ on the first chaos, for some u in $L^2(\mathbb{R}_+)$. For almost any $s \leq t$,

$$\Gamma(h)\mathcal{E}(u_{t+\epsilon})(s) = u(s)H_s\mathcal{E}(u_s)(\emptyset)+H_s^+\mathcal{E}(u_s)(\emptyset)+\int_s^t u(r)H_r^-\mathcal{E}(u_r)(s)\,dr+u(s)H_s^\circ\mathcal{E}(u_s),$$

where the lower bound s for the a^- integral arises because $H_r^-\mathcal{E}(u_r)(s)$ is zero for $r < s$ by adaptedness.

Thus one has, using the fact that $\Gamma(h)\mathcal{E}(u_{t+\epsilon})(s) = hu_{t+\epsilon}(s)$,

$$\begin{aligned}\frac{1}{\epsilon}\int_t^{t+\epsilon} hu_{t+\epsilon}(s)\,ds &= \frac{1}{\epsilon}\int_t^{t+\epsilon} u(s)ds + \frac{1}{\epsilon}\int_t^{t+\epsilon} H_s^+\mathcal{E}(u_s)(\emptyset)\,ds \\ &+\frac{1}{\epsilon}\int_t^{t+\epsilon}\int_s^t u(r)H_r^-\mathcal{E}(u_r)(s)\,dr\,ds + \frac{1}{\epsilon}\int_t^{t+\epsilon} u(r)H_r^\circ\mathcal{E}(u_r)(\emptyset)\,dr.\end{aligned} \tag{2.3}$$

For almost all t:

- the first term converges to $u(t)$,
- the second and fourth converge respectively to $H_t^+\mathcal{E}(u_t)(\emptyset)$ and $u(t)H_t^\circ\mathcal{E}(u_t)(\emptyset)$ because of the integrability properties of of $s \mapsto H_s^+\mathcal{E}(u_s)$, $s \mapsto H_s^\circ\mathcal{E}(u_s)$,
- the third term vanishes. Indeed, an application of Fubini's theorem shows that it is, in norm, dominated by $\frac{1}{\epsilon}\int_t^{t+\epsilon} |u(r)| \times \sqrt{\epsilon}\,\|H_r^-\mathcal{E}(u_r)\|\,dr$, which is $\sqrt{\epsilon}$ times a convergent term.

Thus we have shown that $\frac{1}{\epsilon}\int_t^{t+\epsilon} hu_{t+\epsilon}(s)\,ds$ converges as $\epsilon \to 0$. But then

$$\frac{1}{\epsilon}\int_t^{t+\epsilon} hu_t(s)\,ds = \langle (h^*\mathbb{1}_{[t,t+\epsilon]})/\epsilon\,,\,u_t\rangle, \tag{2.4}$$

so that it converges as $\epsilon \to 0$ by (2.2). By difference, one finally obtains that for all u in $L^2(\mathbb{R}_+)$,

$$\text{for almost all } t,\ \frac{1}{\epsilon}\int_t^{t+\epsilon} hu_{[t,t+\epsilon]}(s)\,ds \text{ has a limit as } \epsilon \text{ tends to zero.}$$

In the case where $u_{[t,t+\epsilon]}$ is equal to $\mathbb{1}_{[t,t+\epsilon]}$, we denote the limit $\gamma(t)$ and notice that, for every ϵ,

$$\begin{aligned}\left|\frac{1}{\epsilon}\int_t^{t+\epsilon} h\mathbb{1}_{[t,t+\epsilon]}(s)ds\right| &\leq \frac{1}{\sqrt{\epsilon}}\int \left|h\mathbb{1}_{[t,t+\epsilon]}\right|^2 \\ &\leq \|h\|^2\end{aligned}$$

so that the function γ is necessarily essentially bounded.

We now prove the square-integrability of α_t and β_t. The argument is the following: we prove in Lemma 2.4 below that

$$u(t)\alpha_t = u(t)P^1 H_t^- \mathcal{E}(u_t),$$

therefore showing that $t \mapsto u(t) \|\alpha_t\|_{L^2([0,t])}$ is integrable for any u in $L^2(\mathbb{R}_+)$, and conclude thanks to the following lemma:

Lemma 2.3 *Let $p \in [1, +\infty[$. Let v be a measurable function such that uv is integrable for every u is in $L^p(\mathbb{R}_+)$. Then v is in $L^q(\mathbb{R}_+)$, where $q \in]1, +\infty]$ is the conjugate index of p.*

Proof of Lemma 2.3

If we prove that $u \mapsto \int uv$ is bounded, then the classical duality theorem imply that $v \in L^q(\mathbb{R}_+)$. We can actually prove that $u \mapsto uv$ is continuous between the two Banachs $L^p(\mathbb{R}_+)$ and $L^1(\mathbb{R}_+)$ by application of the closed graph theorem: for that let us suppose that $u_n \to u$ in $L^p(\mathbb{R}_+)$ with $u_n v$ converging in $L^1(\mathbb{R}_+)$. Almost-everywhere convergence of subsequences holds in both cases, and this shows that the limit of $u_n v$ is uv.

□

The following lemma is therefore a crucial step:

Lemma 2.4 *For all $u \in L^2(\mathbb{R}_+)$, almost all t,*

$$u(t)P^1 H_t^- \mathcal{E}(u_t) = u(t)P^1 H_t^- \mathcal{E}(\mathbb{1}_{t]}). \tag{2.5}$$

Proof of Lemma 2.4

First notice two simple consequences of (2.2):

- for all u, v, one has $(u+v)(t)P^1 H_t^- \mathcal{E}(u_t+v_t) = u(t)P^1 H_t^- \mathcal{E}(u_t) + v(t)P^1 H_t^- \mathcal{E}(v_t)$,
- for any almost everywhere differentiable function u, the desired equality (2.5) holds for almost all t, as can be seen using a Taylor approximation of u.

Thanks to the first point, one can restrict the proof of Lemma 2.4 to the case of a positive function u. Observe now that any positive element u of $L^2(\mathbb{R}_+)$ can be approximated by an increasing sequence of almost everywhere differentiable and square-integrable functions with $\|u_n - u\|_\infty < \frac{1}{n}$: simply define u_n to be $\frac{k}{n}$ on the set $\{\frac{k}{n} \leq u < \frac{k+1}{n}\}$. Then, almost everywhere, equality (2.5) holds for any of these functions u_n.

To obtain equality (2.5) for u, we need some regularity of the application $u \mapsto u(t)H_t^- \mathcal{E}(u_t)$; this is obtained by the following lemma:

Lemma 2.5 *Let t be a positive real. The application which to u in $L^2(\mathbb{R}_+)$ associates the function*

$$s \mapsto \int_s^t u(r)H_r^- \mathcal{E}(u_r)(s)\,dr \text{ for } s \leq t$$

is continuous from $L^2(\mathbb{R}_+)$ to $L^1([0,t])$.

Proof of Lemma 2.5 Once again we write that, thanks to $\Gamma(h)\mathcal{E}(u_t) = \mathcal{E}(hu_t)$, one has for almost all $s < t$:

$$hu_t(s) = u(s) + H_s^+ \mathcal{E}(u_s)(\emptyset) + u(s)H_s^\circ \mathcal{E}(u_s)(\emptyset) + \int_0^t u(r)H_r^- \mathcal{E}(u_r)(s)\,dr,$$

but as a consequence of (2.3) and (2.4), that is

$$hu_t(s) = \langle \beta_s, u_s \rangle + \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_s^{s+\epsilon} hu_{[t,t+\epsilon]}(s)ds + \int_s^t u(r)H_r^- \mathcal{E}(u_r)(s)\,dr \tag{2.6}$$

Now notice that

- $s \mapsto hu_t(s)$ is a function with L^2 norm smaller than $\|h\|\,\|u\|$,
- the limit of $\frac{1}{\epsilon}\int_s^{s+\epsilon} hu_{[t,t+\epsilon]}(s)ds$ is smaller than $\|h\|\,|u(s)|$ so that it defines a function with L^2 norm smaller than $\|h\|\,\|u\|$ as well,
- the first term on the right has $L^1([0,t])$ norm smaller than $\int_0^t \|\beta_s\|\,ds\,\|u\|_{L^2}$,

so that, if we restrict s to $[0,t]$, those three terms define bounded linear functionals of u from $L^2(\mathbb{R}_+)$ to $L^1([0,t])$. Therefore the remaining term in the equality (2.6) has the same property.

□

Now if we recall that, with a sequence $(u_n)_{n\geq 0}$ chosen as above,

$$u_n(r)H_r^- \mathcal{E}((u_n)_r) = u_n(r)\alpha_r,$$

Lemma 2.5 yields

$$\int_0^t \left| \int_s^t \left(u(r)H_r^- \mathcal{E}(u_r) - u_n(r)\alpha_r\right)(s)\,dr \right| ds \leq \|u - u_n\|_{L^2}. \tag{2.7}$$

But then,

$$\begin{aligned}\int_0^t \left| \int_s^t (u-u_n)(r)\alpha_r(s)\,dr \right| ds &\leq \int_0^t |u-u_n|(r) \int_0^r |\alpha_r|(s)\,ds\,dr \\ &\leq \sqrt{t} \int_0^t (u-u_n)(r)\,\|\alpha_r\|\,dr,\end{aligned}$$

where the first inequality is obtained by application of Fubini's theorem. Since $\|u-u_n\|_\infty < \frac{1}{n}$ and $r \mapsto \alpha_r$ is locally integrable, the above integral tends to zero. Combined with the inequality (2.7), this implies that

$$\text{for almost all } s,\, t,\ \int_s^t u(r)\left(\alpha_r - H_r^- \mathcal{E}(u_r)\right)(s)\,dr = 0,$$

so that

$$\text{for almost all } t, u(t)\alpha_t = u(t)P^1 H_t^- \mathcal{E}(u_t),$$

and Lemma 2.4 is proved.

□

This concludes the proof of Proposition 2.2.

□

We now take on the second step of our proof, that is, the equivalence of conditions **(C)** and *2.*

Lemma 2.6 *For a bounded operator h, to satisfy the conditions **(C)** in Proposition 2.2 on h is equivalent to the existence of a Hilbert-Schmidt operator K such that*

$$h = K + \mathcal{M}_\gamma,$$

where $\mathcal{M}_f$ is the multiplication operator by γ.

Proof of Lemma 2.6 That h being of the mentioned form implies conditions **(C)** is straightforward (and unnecessary in our scheme). Now, to prove the converse, let us define a kernel κ by

$$\begin{cases} \kappa(s,t) &= \overline{\beta_t(s)} \quad \text{if } s<t \text{ and} \\ \kappa(s,t) &= \alpha_s(t) \quad \text{if } s>t. \end{cases} \tag{2.8}$$

Our assumptions on α and β show that

$$\int_{0<s<t} |\kappa(s,t)|^2 \, ds \, dt < +\infty$$

and

$$\int_{0<t<s} |\kappa(s,t)|^2 \, ds \, dt < +\infty$$

so that the kernel κ defines a Hilbert-Schmidt operator K which is in particular a bounded operator on $L^2(\mathbb{R}_+)$. The operator of multiplication by γ is also bounded.

Therefore we can consider $h - K - \mathcal{M}_\gamma$ in place of h and stick to showing that, for h to satisfy **(C)** with α, β, γ all null implies $h = 0$.

To prove that claim, observe that if $u \in L^2(\mathbb{R}_+)$, and $a < b$, then for almost every $t > b$:

$$\begin{aligned} h(u\mathbb{1}_{[a,b]})(t) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} h(u\mathbb{1}_{[a,b]})(s) \, ds \\ &= \lim_{\epsilon \to 0} \langle (h^* \mathbb{1}_{[t,t+\epsilon]})/\epsilon, u\mathbb{1}_{[a,b]} \rangle \end{aligned}$$

and the limit is zero since the restriction of $(h^*\mathbb{1}_{[t,t+\epsilon]})/\epsilon$ to $[0,t]$ converges to zero in L^2. Therefore $h(u\mathbb{1}_{[a,b]})$ is a.e. null on $[b, +\infty[$. The same property holds by symmetry for h^*.

For $c < d < a$,

$$\int_c^d h(u\mathbb{1}_{[a,b]})(s) \, ds = \langle h^*\mathbb{1}_{[c,d]} \, , \, u\mathbb{1}_{[a,b]} \rangle$$

which is zero by the previous step.

Therefore $h(u\mathbb{1}_{[a,b]})$ has support in $[a,b]$, so that for any $u \in L^2(\mathbb{R}_+)$, any interval I of $\mathbb{R}_+$, one has

$$h(u\mathbb{1}_I) = \mathbb{1}_I(hu.)$$

From this one deduces that h is a multiplication operator. It is straightforward from the last condition in **(C)** with $f = 0$ that this multiplication is null.

□

Proof of *2.* ⇒ *1.* of Theorem 2.1

Suppose that h is of the form

$$h = K + \mathcal{M}_\gamma$$

where K is a Hilbert-Schmidt operator with kernel κ. We then define for any $u \in L^2(\mathbb{R}_+)$

$$\begin{cases} H_t^+\mathcal{E}(u_t) = \int_0^t \kappa(s,t)u(s)\,ds\,\mathcal{E}((hu_t)_t) \\ H_t^-\mathcal{E}(u_t) = \mathcal{E}((hu_t)_t) \circ \int_0^t \kappa(t,s)\,d\chi_s \\ H_t^\circ\mathcal{E}(u_t) = \quad (\gamma(t)-1)\ \mathcal{E}((hu_t)_t), \end{cases} \tag{2.9}$$

and extend all three operators by adaptedness.

We should remark that this means that on the set $\mathcal{E}(L^2(\mathbb{R}_+))$,

$$\begin{cases} H_t^+P_t = \quad P_t\Gamma(h)\,a^-_{\overline{\kappa(.,t)}}P_t \\ H_t^-P_t = \quad P_t\,a^+_{\kappa(t,.)}\,\Gamma(h)P_t \\ H_t^\circ P_t = (\gamma(t)-1)\,P_t\,\Gamma(h)P_t. \end{cases} \tag{2.10}$$

The above equalities define the operators on all of $\mathcal{E}(L^2(\mathbb{R}_+))$ and one has

$$\begin{cases} \left\|H_t^+\mathcal{E}(u_t)\right\|^2 \leq \int_0^t |\kappa(s,t)|^2\,ds\,\|u_t\|^2\exp(\|h\|^2\|u_t\|^2) \\ \left\|H_t^-\mathcal{E}(u_t)\right\|^2 \leq \quad \int_0^t |\kappa(t,s)|^2\,ds\,\exp(\|h\|^2\|u_t\|^2) \\ \|H_t^\circ\mathcal{E}(u_t)\|^2 \leq \quad (\|\gamma\|_\infty+1)^2\exp(\|h\|^2\|u_t\|^2), \end{cases} \tag{2.11}$$

so that for all $u \in L^2(\mathbb{R}_+)$, $|u(t)|\left\|H_t^-\mathcal{E}(u_t)\right\|$ is integrable and $\left\|H_t^+\mathcal{E}(u_t)\right\|$, $|u(t)|\,\|H_t^\circ\mathcal{E}(u_t)\|$ are square-integrable.

The operator

$$H = \mathrm{Id} + \int H_s^+ da_s^+ + \int H_s^- da_s^- + \int H_s^\circ da_s^\circ$$

is well-defined on the set $\mathcal{E}(L^2(\mathbb{R}_+))$. One can check that the operator defined with h^* instead of h is adjoint to H on the exponential domain, so that H has a densely defined adjoint, and therefore is closable.

We now prove that H actually equals $\Gamma(h)$ on $\mathcal{E}(L^2(\mathbb{R}_+))$. Let u belong to $L^2(\mathbb{R}_+)$. Then

$$\begin{aligned}
H\mathcal{E}(u) &= \mathbb{1} + \int u(s)H_s\mathcal{E}(u_s)\,d\chi_s + \int H_s^+\mathcal{E}(u_s)\,d\chi_s \\
&\quad + \int u(s)H_s^-\mathcal{E}(u_s)\,ds + \int u(s)H_s^\circ\mathcal{E}(u_s)\,d\chi_s
\end{aligned}$$

and for almost all $t \in \mathbb{R}_+$,

$$D_tH\mathcal{E}(u) = u(t)H_t\mathcal{E}(u_t) + H_t^+\mathcal{E}(u_t) + D_t\int u(s)H_s^-\mathcal{E}(u_s)\,ds + u(t)H_t^\circ\mathcal{E}(u_t)$$

so that, for $\sigma < t$ one has

$$\begin{aligned}
D_tH\mathcal{E}(u)(\sigma) &= u(t)H_t\mathcal{E}(u_t)(\sigma) + \int_0^t \kappa(s,t)u(s)\,ds\,(hu_t)(\sigma) \\
&\quad + \int_t^{+\infty} u(s)H_s^-\mathcal{E}(u_s)(\sigma\cup t)\,ds + u(t)\,(\gamma(t)-1)\,(hu_t)(\sigma).
\end{aligned}$$

Where the a^- integral is restricted to $[t,+\infty[$ for reasons of adaptedness.

We will write $(hu_t)(\sigma)$ for $\prod_{a\in\sigma} hu_t(a)$ or the equivalent short notation for other functions, as is customary. We now use the fact that

$$hu_t(a) = \int_0^t \kappa(s,a)u(s)\,ds + u(a)\gamma(a)$$

and develop such expressions as $(hu_t)(\sigma)$.

$$\begin{aligned}
D_tH\mathcal{E}(u)(\sigma) &= u(t)\,(H_t\mathcal{E}(u_t) - \mathcal{E}(hu_t))\,(\sigma) + \sum_{\tau\subset\sigma}\prod_{a\in\tau\cup t}\left(\int_0^t \kappa(s,a)u(a)\,ds\right)(u\gamma)(\sigma\setminus\tau) \\
&+ \int_t^{+\infty} u(s)\sum_{b\in\sigma\cup t}\kappa(s,b)(hu_s)(\sigma\cup t\setminus b)\,ds + \sum_{\tau\subset\sigma}\prod_{a\in\tau}\left(\int_0^t \kappa(s,a)u(a)\,ds\right)(u\gamma)(\sigma\cup t\setminus\tau)
\end{aligned}$$

We now develop the a^- term, that is, the one with the $\int_t^{+\infty}$ integral. That term is equal to

$$\begin{aligned}
&\int_t^{+\infty} \kappa(s,t)u(s)\,ds \sum_{\tau\subset\sigma}\prod_{a\in\tau}\int_0^s \kappa(r,a)u(r)\,dr(u\gamma)(\sigma\setminus\tau)\,ds \\
&+ \sum_{\tau\cup\nu\cup\{b\}=\sigma}\int_t^{+\infty} \kappa(s,b)u(s)(u\gamma)(\nu+t)\prod_{a\in\tau}\int_0^s \kappa(r,a)u(r)\,dr\,ds \\
&+ \sum_{\tau\cup\nu\cup\{b\}=\sigma}\int_t^{+\infty} \kappa(s,b)u(s)\prod_{a\in\tau\cup t}\int_0^s \kappa(r,a)u(r)\,dr\,(u\gamma)(\tau)\,dt.
\end{aligned}$$

From this formulation one can see that the sum of these terms can be written as

$$\sum_{\tau\cup\nu\cup\{b\}=\sigma\cup\{t\}} \int_t^{+\infty} \kappa(s,b)u(s) \prod_{a\in\tau} \int_0^s \kappa(r,a)u(r)\,dr (u\gamma)(\tau)\,dt.$$

Thus we finally have

$$\begin{aligned} D_t H\mathcal{E}(u)(\sigma) &= u(t)\left(H_t\mathcal{E}(u_t) - \mathcal{E}(hu_t)\right)(\sigma) + \sum_{\tau+\nu=\sigma\cup\{t\}} \prod_{a\in\tau} \left(\int_0^{+\infty} \kappa(s,a)u(s)\,ds\right)(u\gamma)(\tau) \\ &= u(t)\left(H_t\mathcal{E}(u_t) - \mathcal{E}(hu_t)\right)(\sigma) + \mathcal{E}(hu)(\sigma\cup t), \end{aligned}$$

We have proved that

$$D_t H\mathcal{E}(u) - D_t\mathcal{E}(hu) = u(t)P_t\left(\mathcal{E}(hu) - \mathcal{E}(hu_t)\right),$$

and since $H\mathcal{E}(u)(\emptyset) = \mathcal{E}(hu)(\emptyset)$, the previsible representation isometry (1.7) yields

$$\begin{aligned} \|H\mathcal{E}(u) - \mathcal{E}(hu)\|^2 &= \int_0^\infty |u(t)|^2 \left\|P_t\left(H\mathcal{E}(u_t) - \mathcal{E}(hu_t)\right)\right\|^2 dt \\ &\leq \int_0^\infty |u(t)|^2 \left\|\left(H\mathcal{E}(u_t) - \mathcal{E}(hu_t)\right)\right\|^2 dt, \end{aligned}$$

and it is now easy to conclude that $H\mathcal{E}(u) = \mathcal{E}(hu)$ thanks to Gronwall's lemma and the closability of H. This ends the proof of the equivalence in Theorem 2.1.

Extension of the representation to $\mathcal{J}$

We consider the stochastic integral representation defined on $\mathcal{E}(L^2(\mathbb{R}_+))$ above and prove now that it holds on all of the domain $\mathcal{J}$. First of all, since $D_t j(g,f) = g(t)\mathcal{E}(f_t)$, the estimates (2.11) are enough to prove that $t \mapsto \left\|H_t^- D_t j(g,f)\right\|$ and $t \mapsto \|H_t^\circ D_t j(g,f)\|^2$ are integrable. Now one needs to show that H_t^+ can be extended to $j(g,f)$: one defines it through (2.10). To prove that it is well-defined it is enough to show, since the restriction of $\Gamma(h)$ to the n-th chaos has norm $\|h\|^n$, that

$$\sum_{,n\geq 0} \|h\|^{2n} \left\|P^n(a^-_{\overline{\kappa(.,t)}} P_t j(g,f))\right\|^2 < +\infty,$$

and this series would be a bound for $\left\|H_t^+ P_t j(g,f)\right\|$. One can see that, for almost all $s_1 < \cdots < s_n$,

$$\begin{aligned} a^-_{\overline{\kappa(.,t)}} P_t j(g,f)(s_1,\cdots,s_n) &= \int_0^{s_n} \kappa(r,t)f(r)\,dr\; j(g,f)(s_1,\cdots,s_n) \\ &\qquad + \int_{s_n}^t \kappa(r,t)g(r)\,dr\; \mathcal{E}(f_t)(s_1,\cdots,s_n), \end{aligned}$$

so that

$$\left\| P_n a^-_{\kappa(.,t)} P_t j(g,f)(s_1,\cdots,s_n) \right\|^2 \le 2\int_0^t |\kappa(r,t)|^2 \, dr \left(\|f\|^2 \|P^n j(g,f)\|^2 + \|g\|^2 \|P^n \mathcal{E}(f)\|^2 \right).$$

Since $\sum_{n\le 0} \|h\|^{2n} \|P^n j(g,f)\|^2$ and $\sum_{n\le 0} \|h\|^{2n} \|P^n \mathcal{E}(f)\|^2$ are finite, this proves both that H_t^+ is well defined on $\mathcal{J}$ and that $t \mapsto \left\| H_t^+ P_t j(g,f) \right\|$ is square-integrable. Therefore the considered quantum stochastic integral is defined on $\mathcal{J}$ and a proof that it actually equals $\Gamma(h)$ on that set would be similar to the proof of equality on $\mathcal{E}(L^2(\mathbb{R}_+))$. This ends the proof of Theorem 2.1.

□

2.2 The Journé-Meyer counterexample

First of all notice that this counterexample concerns the representability of the operator $\Gamma(h)$ and does not assume anything about $\Gamma(h^*)$. But then, if h is the Hilbert transform then $h^* = -h$, so that the conditions concerning convergence of $(h^* \mathbb{1}_{[t,t+\epsilon]})/\epsilon$ to β_t and the properties of β_t are straightforward. Here these conditions follow immediately from $h^* = -h$ and those on h. Therefore, up to a slight modification, our results apply and $\Gamma(h)$ is representable on $\mathcal{E}(L^2(\mathbb{R}_+))$ if and only if h is of the form given in Theorem 2.1. But here the operator is associated to the kernel $\kappa(s,t) = \frac{1}{\pi(s-t)}$, which is not even integrable in one variable; in particular, the integral $\int_0^t \kappa(s,t) d\chi_s$ is not a well-defined element of Fock space and our formulas do not define $H_t^+ \mathcal{E}(u_t)$ for all u; besides, our formulas define $H_t^- \mathcal{E}(u_t)$ for no u whereas our proof has shown that our formulas have to hold at least on the first chaos.

This is why a stochastic integral representation can be defined only in a weaker way: indeed the representation defined in Parthasarathy's response [Par] is probably the best possible one: $H_t^+ \mathcal{E}(u_t)$ is defined only for u's with some regularity (enough to make the integral $\int_0^t \frac{1}{t-s} u(s)\, ds$ meaningful) and H_t^- is only defined as a distribution, through the action of its adjoint.

Let us jump ahead to make an additional remark: the function u pointed out by Journé and Meyer is actually bounded with compact support, so that it belongs to all sets $L^p(\mathbb{R}_+)$; this proves that $\Gamma(h)$ and $\Gamma(h^*)$ are not representable even on the smaller subspaces $L^2 \cap L^p(\mathbb{R}_+)$. Characterizing representability on such a subspace will be the point of Theorem 5.3; here the representability defect is actually so strong (the kernel is not even square integrable in one variable) that that theorem does not teach us anything new about this particular case.

2.3 Second quantification operators as regular martingales

It is clear from formulas 2.9 or 2.10 that the boundedness of $\Gamma(h)$ is strongly linked to that of the operators H_t^+, H_t^-, H_t°. In fact we will show in the next proof that H_t^- being bounded implies that h is a contraction when restricted from $L^2([0,t])$ to $L^2(\mathbb{R}_+)$. This allows us to obtain a pleasant characterization of the operators of $\Gamma(h)$ that belong to one of Attal's semimartingale algebras (see [At1]). We recall the definition of the two semimartingale algebras: an operator H is an element of $\mathcal{S}'$ if it has a quantum stochastic integral representation with bounded integrands H_t^+, H_t^- and H_t° that moreover are such that $t \mapsto \|H_t^+\|, \|H_t^-\|$ are square integrable functions and $t \mapsto \|H_t^\circ\|$ is an essentially bounded function. An operator H' is an element of $\mathcal{S}$ if it is an element of $\mathcal{S}'$ and is a bounded operator.

Proposition 2.7 *For a second quantification operator $\Gamma(h)$ the following are equivalent:*

1. *$\Gamma(h)$ belongs to $\mathcal{S}'$,*
2. *$\Gamma(h)$ belongs to $\mathcal{S}$,*
3. *h is of the form $K + \mathcal{M}_f$ as in Theorem 2.1 and h is a contraction.*

Proof.
We first prove that *3.* implies that $\Gamma(h)$ is an element of $\mathcal{S}$. If h is a contraction then (2.11) shows that H_t^-, H_t° are bounded on $\mathcal{E}(L^2(\mathbb{R}_+))$, with norms smaller than $\sqrt{\int_0^t |\kappa(t,s)|^2\, ds}$, $(\|f\|_\infty + 1)$ respectively. That is, both can be extended as bounded operators on Φ, with $t \mapsto \|H_t^-\|$ and $t \mapsto \|H_t^\circ\|$ respectively square-integrable and essentially bounded. Now to prove that H_t^+ is bounded we need the following remark, which is proved by a simple computation: if one denotes by $\widetilde{H_t^-}$ the operator H_t^- defined from h^* instead of h (or equivalently if considering the kernel $\overline{\kappa(t,s)}$ instead of $\kappa(s,t)$), then $\widetilde{H_t^-} = (H_t^+)^*$ on $\mathcal{E}(L^2(\mathbb{R}_+))$. Therefore the adjoint of H_t^+ is contained in a bounded operator with norm $\sqrt{\int_0^t |\kappa(s,t)|^2\, dr}$, thanks to the fact that h^* is a contraction. Therefore the closure of H_t^+ is a bounded operator and $t \mapsto \|H_t^+\|^2$ is square-integrable; besides $\Gamma(h)$ itself is bounded so that $\Gamma(h) \in \mathcal{S}$.

Now let us prove that *1.* implies *3.*: we suppose that $\Gamma(h)$ has a representation as a quantum stochastic integral with the relevant boundedness assumptions on H_t^+, H_t^-, H_t°. Then the same property holds for $\Gamma(h^*)$ (see [At1]) and our theorem 2.1 applies, and shows that h is of the form $K + \mathcal{M}_f$ as before. Besides, since the integrands H_s^+, H_s^-, H_s° are bounded and therefore closable, Attal's uniqueness theorem applies (see [At2]) and shows that the integrands satisfy formulas (2.9). Therefore

$$\left\|H_t^- \mathcal{E}(u_t)\right\|^2 = \|\alpha_t\|^2 \exp \|(hu_t)_t\|^2,$$

and is to be smaller than a constant times $\exp\|u_t\|^2$ for every u. Denote by p_t the projection in $L^2(\mathbb{R}_+)$ of restriction to $[0,t]$; it is necessary that $u \mapsto p_t h p_t$ be a contraction. This being true for every t, one has for any u in $L^2(\mathbb{R}_+)$, any s, any t larger than s that

$$\|p_t h p_s u\| \leq \|u\|,$$

so that hp_s is a contraction for any s. The boundedness of h implies that h is itself a contraction.

That *2.* implies *1.* is always true, so that the proof is complete.

□

3 Differential second quantification operators

We consider now the case of differential second quantification operators and obtain a characterization which is exactly the same as in the previous case of of non differential second quantifications. The explicit formulas for the integrands in the obtained representations are given below in (3.1) and (3.2).

Theorem 3.1 *Let h be a bounded operator on $L^2(\mathbb{R}_+)$. The following properties are equivalent:*

1. *$\lambda(h)$ and $\lambda(h^*)$ have a stochastic integral representation on the set $\mathcal{E}(L^2(\mathbb{R}_+))$*
2. *h is of the form*
$$h = K + \mathcal{M}_f$$
where K is a Hilbert-Schmidt operator and $\mathcal{M}_f$ is a multiplication by a L^∞ function f,

and if one of these holds, then the stochastic integral representation holds on the set $\mathcal{J}$.

Note that this theorem is an improvement of one proved by Coquio in [Coq], where the representation of $\lambda(h)$ and $\lambda(h^*)$ in *1.* was assumed to hold on the set $\mathcal{J}$.

Proof.
Observe simply that our proof of Proposition 2.2 uses strictly equalities involving the stochastic integral operator and $\Gamma(h)\mathcal{E}(u)$ or $\Gamma(h^*)\mathcal{E}(u)$ on the chaoses of order zero and one, for functions u in $L^2(\mathbb{R}_+)$. But then $\lambda(h)\mathcal{E}(u)$ (respectively $\lambda(h^*)\mathcal{E}(u)$) coincides with $\Gamma(h)\mathcal{E}(u)$ (respectively $\Gamma(h^*)\mathcal{E}(u)$) on the chaos of order one, and is zero indepently of u on the chaos of order zero whereas $\Gamma(h)\mathcal{E}(u)$ (respectively $\Gamma(h)\mathcal{E}(u)$) is one indepently of u on that same chaos. Therefore it is easy to see that our proof of Proposition 2.2 would hold if we considered *differential* second quantification operators instead of the non differential ones. That *1.* implies *2.* is then proved by Lemma 2.6. The proofs for the converse and the extension to $\mathcal{J}$ are also similar to the former: if *2.* is assumed then we define H_t^+, H_t^-, H_t° by the following formulas:

$$\begin{cases} H_t^+\mathcal{E}(u_t) = \int_0^t \kappa(s,t)u(s)\,ds\,\mathcal{E}(u_t) \\ H_t^-\mathcal{E}(u_t) = \mathcal{E}(u_t) \circ \int_0^t \kappa(t,s)\,d\chi_s \\ H_t^\circ\mathcal{E}(u_t) = (f(t)-1)\;\mathcal{E}(u_t), \end{cases} \tag{3.1}$$

and the definiteness of the integral

$$H = \int H_s^+ da_s^+ + \int H_s^- da_s^- + \int H_s^\circ da_s^\circ$$

on $\mathcal{J}$ and the equality $D_t Hj(v,u) = D_t\lambda(h)j(v,u)$ are obtained as before.

□

Note that the action of the above integrand can be explicited on the exponential subset by:

$$\begin{cases} H_t^+ P_t = a^-_{\overline{\kappa(.,t)}} P_t \\ H_t^- P_t = a^+_{\kappa(t,.)} P_t \\ H_t^\circ P_t = (f(t)-1)\,P_t. \end{cases} \tag{3.2}$$

4 The case of Fock space of higher multiplicity

Once again the proofs in this section would be simple rewritings of the proof of Theorem 2.1; our task will be therefore to point out the similarities and to define a correct way of writing our conditions in a concise form.

Let us consider as in the preliminaries a fixed hilbertian basis $(e_i)_{i\in\mathcal{I}}$ of our multiplicity space $\mathcal{K}$. Let us define or recall the terms to appear below:

Definition 4.1 • *A Hilbert-Schmidt operator in $L^2(\mathbb{R}_+, \mathcal{K})$ is an operator K for which there exists a family $(\kappa_{i,j})_{i,j\in\mathcal{I}}$ of functions in $L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ such that*

$$\sum_{i,j\in\mathcal{I}} \int_{\mathbb{R}_+\times\mathbb{R}_+} |\kappa_{i,j}(s,t)|^2 \, ds\, dt < +\infty$$

and for all i in $\mathcal{I}$, almost all s in $\mathbb{R}_+$,

$$Kf(s,i) = \sum_{j\in\mathcal{I}} \int_0^\infty \kappa_{i,j}(r,s) f(r,j) dr.$$

- *A matrix multiplication operator is an operator M_γ for which there exists a set of functions $(\gamma_{i,j})_{i,j\in\mathcal{I}}$ on $\mathbb{R}_+$ such that for almost all s in $\mathbb{R}_+$, the quantity*

$$\|\gamma\|(s) = \left(\sum_{i,j\in\mathcal{I}} |\gamma_{i,j}(s)|^2\right)^{1/2}$$

is finite and

$$M_\gamma f(s,i) = \sum_{j\in\mathcal{I}} \gamma_{i,j}(s) f(s,j).$$

Here are Theorems 2.1 and 3.1 rolled into one in the case of Fock space of infinite multiplicity. The formulas for the integrands are given below, in (4.1), (4.2) for the integrands of $\Gamma(h)$ and (4.3), (4.4) for the integrands of $\lambda(h)$.

Theorem 4.2 *Let h be a bounded operator on $L^2(\mathbb{R}_+, \mathcal{K})$. The following properties are equivalent:*

1. *$\Gamma(h)$ and $\Gamma(h^*)$ have a stochastic integral representation on the set $\mathcal{E}(L^2(\mathbb{R}_+))$*
2. *$\lambda(h)$ and $\lambda(h^*)$ have a stochastic integral representation on the set $\mathcal{E}(L^2(\mathbb{R}_+))$*
3. *h is of the form*
$$h = K + \mathcal{M}_f$$
where K is a Hilbert-Schmidt operator and $\mathcal{M}_f$ is a multiplication by a matrix $(\gamma_{i,j})_{i,j\in\mathcal{I}}$ such that the function $\|\gamma\|$ is essentially bounded on $\mathbb{R}_+$.

Besides, if one of these holds, then the representations hold on the set $\mathcal{J}$.

Proof.
The proof is essentially the same as before; to deduce *3.* from *1.* or *2.* one can compute the action of the integral from any i-th to any j-th coordinate of $\mathcal{K}$. This gives the form of the "(i,j) coefficient" of h. That the matrix γ constructed in that way defines a bounded operator is straightforward from the estimates, so that the kernel operator K is also bounded; the theory of Hilbert-Schmidt operators (in $L^2(\mathbb{R}_+,\mathcal{K})$ for K, in $l^2(\mathcal{I})$ for $\gamma(s)$) implies the square-integrability assumptions. The proof that *3.* allows the construction of a well-defined integral which coincides with the desired operator is exactly the same as before. The formulas for integrands are the following:

Integrands for the representation of $\Gamma(h)$: the integrands in the representation of $\Gamma(h)$ are given by:

$$\begin{aligned} H_t^{i,0}\mathcal{E}(u_t) &= \sum_{j\in\mathcal{I}}\int_0^t \kappa_{i,j}(s,t)u(s,j)ds\ \mathcal{E}((hu_t)_t) \\ H_t^{0,j}\mathcal{E}(u_t) &= \mathcal{E}((hu_t)_t)\circ\sum_{i\in\mathcal{I}}\int_0^t \kappa_{i,j}(t,s)d\chi_s^i \\ H_t^{i,j}\mathcal{E}(u_t) &= (\gamma_{i,j}(t)-1)\ \mathcal{E}((hu_t)_t). \end{aligned} \tag{4.1}$$

Otherwise stated, they satisfy the following equalities on $\mathcal{E}(L^2(\mathbb{R}_+))$,

$$\begin{aligned} H_t^{i,0}P_t &= P_t\Gamma(h)P_t\sum_{j\in\mathcal{I}}\int_0^t \overline{\kappa_{i,j}(s,t)}da_s^{0,i} \\ H_t^{0,j}P_t &= \sum_{i\in\mathcal{I}}\int_0^t \kappa_{i,j}da_s^{j,0}\ P_t\Gamma(h)P_t \\ H_t^{i,j}P_t &= (\gamma_{i,j}(t)-1)\,P_t\Gamma(h)P_t, \end{aligned} \tag{4.2}$$

and these equalities can be extended to $\mathcal{J}$.

Integrands for the representation of $\lambda(h)$: the integrands in the representation of $\lambda(h)$ are given by:

$$\begin{aligned} H_t^{i,0}\mathcal{E}(u_t) &= \sum_{j\in\mathcal{I}}\int_0^t \kappa_{i,j}(s,t)u(s,j)ds\ \mathcal{E}(u_t) \\ H_t^{0,j}\mathcal{E}(u_t) &= \mathcal{E}(u_t)\circ\sum_{i\in\mathcal{I}}\int_0^t \kappa_{i,j}(t,s)d\chi_s^i \\ H_t^{i,j}\mathcal{E}(u_t) &= (\gamma_{i,j}(t)-1)\ \mathcal{E}(u_t). \end{aligned} \tag{4.3}$$

Otherwise stated, they satisfy the following equalities on $\mathcal{E}(L^2(\mathbb{R}_+))$,

$$\begin{aligned} H_t^{i,0} P_t &= \sum_{j\in\mathcal{I}} \int_0^t \overline{\kappa_{i,j}(s,t)} da_s^{0,i}\ P_t \\ H_t^{0,j} P_t &= \sum_{i\in\mathcal{I}} \int_0^t \kappa_{i,j} da_s^{j,0}\ P_t \\ H_t^{i,j} P_t &= (\gamma_{i,j}(t) - 1)\ P_t, \end{aligned} \tag{4.4}$$

and these equalities can be extended to $\mathcal{J}$.

5 Extensions of Theorem 2.1 and Theorem 3.1

In its above formulation, Theorem 2.1 contains a quite strong assumption, that is, the fact that the mentioned representation holds on all of the exponential set. It is nevertheless clear from the proof that this assumption is only needed to obtain, through Lemma 2.3, the strong integrability properties that make the kernel a Hilbert-Schmidt kernel as necessary conditions for representability. It is easy to obtain necessary conditions for representability on various subsets of the exponential domain. One can obtain for example the following:

Proposition 5.1 *Let $\mathcal{A}$ be a sub-vector space of $L^2(\mathbb{R}_+)$ such that*

- *if a function u is in $\mathcal{A}$, then so is $|u|$ and*
- *$\mathcal{A}$ contains the indicator functions of intervals of $\mathbb{R}_+$.*

*Then if $\Gamma(h)$ and $\Gamma(h^*)$ are representable on $\mathcal{E}(\mathcal{A})$ then conditions **(C)** hold, except that the third is replaced by*

$$t \mapsto \|\alpha_t\|\, u(t), t \mapsto \|\beta_t\|\, u(t) \text{ are integrable for every function } u \text{ in } \mathcal{A}.$$

This proposition lacks a clear formulation of the form of the operator h: we are not able to turn the last condition into an intrinsic characterization of h. Of course this proposition also lacks a "sufficiency" result, but as we mentioned in the introduction, we refer the reader to the last section of this paper for simple sufficient conditions for representability. The problem with such general cases is that the assumptions are actually too weak to even be sure that the kernel operators associated to α and β are well-defined.

We therefore restrict to the case where the set $\mathcal{A}$ is of the form $L^2(\mathbb{R}_+)\cap L^p(\mathbb{R}_+)$. In this case we obtain a concise sufficient and necessary condition for the case of differential second quantifications $\lambda(h)$. The case of operators $\Gamma(h)$ is not exactly as simple but minor adaptations to the proof of Theorem 5.3 allow one to handle many cases; see the remarks after the statement of the Theorem.

Let us fix the following notations and definitions: for any function κ on $\mathbb{R}_+ \times \mathbb{R}_+$ we denote by $\tilde{\kappa}$ the function defined by $\tilde{\kappa}(s,t) = \overline{\kappa(t,s)}$, and by $\|\kappa\|(t)$ the quantity

$$\|\kappa\|(t) = \left(\int_0^\infty |\kappa(s,t)|^2 \, ds\right)^{1/2}.$$

Definition 5.2 *Let q belong to $[1, +\infty[$. A $(2,q)$ kernel is a function κ on $\mathbb{R}_+ \times \mathbb{R}_+$ such that for almost all t, the functions $\|\kappa\|$ and $\|\tilde{\kappa}\|$ are almost everywhere finite and belong to $L^2(\mathbb{R}_+) + L^q(\mathbb{R}_+)$.*

We then have the following result:

Theorem 5.3 *Let h be a bounded operator on $L^2(\mathbb{R}_+)$, and let $p \leq 2$. Consider the following assumptions:*

1. *$\Gamma(h)$ and $\Gamma(h^*)$ have a stochastic integral representation on the set $\mathcal{E}(L^2 \cap L^p(\mathbb{R}_+))$,*
2. *$\lambda(h)$ and $\lambda(h^*)$ have a stochastic integral representation on the set $\mathcal{E}(L^2 \cap L^p(\mathbb{R}_+))$,*
3. *there exist a $(2,q)$ kernel κ and an essentially bounded function γ such that*
$$h = K_\kappa + \mathcal{M}_\gamma \text{ and } h^* = K_{\tilde{\kappa}} + \mathcal{M}_{\overline{\gamma}}$$
on $L^2 \cap L^p(\mathbb{R}_+)$.

Then

- 1. *or* 2. *imply* 3.
- 3. *implies* 1. *and* 2. *if one adds the assumption that*
$$\int_0^\infty \left|\int_0^s \kappa(r,s) u(r) dr\right|^2 ds \leq C \int_0^\infty |u(r)|^2 \, dr$$
and
$$\int_0^\infty \left|\int_0^s \kappa(s,r) u(r) dr\right|^2 ds \leq C \int_0^\infty |u(r)|^2 \, dr$$
for some positive constant C and all u in $L^2 \cap L^p(\mathbb{R}_+)$.

Besides, as soon as 3. *holds then the representations hold on the subset $\mathcal{J}(L^2 \cap L^p)$ of $\mathcal{J}$ made of vectors $j(g,f)$ with f and g in $L^2 \cap L^p(\mathbb{R}_+)$.*

Remarks

- One can explicit many cases in which *1.* and *2.* hold true. For example if h is positive then all three assumptions *1.*, *2.*, *3.* are equivalent; if $|h|$ satisfies the conditions in *3.* then $\lambda(h)$, $\lambda(h^*)$ are representable. We refer the reader to the remarks after the end of the proof which points out where adaptations can be made.

- The case for $p > 2$ can not be explicited in a symmetrical way; actually there is no reason why the function κ constructed in the proof should be such that $s \mapsto \kappa(s,t)$ and $s \mapsto \kappa(t,s)$ are square integrable, so that no analogue to Lemma 2.6 can be given.

We will need in the sequel an analogue of Lemma 2.3:

Lemma 5.4 *Let p belong to $[1,2]$. Let u be a measurable function such that uv is integrable for every function v in $L^2 \cap L^p(\mathbb{R}_+)$. Then u belongs to $L^2(\mathbb{R}_+) + L^q(\mathbb{R}_+)$, where $q \in [2, +\infty]$ is the conjugate index of p.*

Proof.
Suppose that u is positive and $p < 2$ – the case of $p = 2$ is settled by Lemma 2.3. We will show that:

- $u\mathbb{1}_{u\leq 1}v$ is integrable for any v in $L^p(\mathbb{R}_+)$ and
- $u\mathbb{1}_{u>1}v$ is integrable for any v in $L^2(\mathbb{R}_+)$,

To prove the first point, consider some positive function v in $L^p(\mathbb{R}_+)$; first of all

$$(v\mathbb{1}_{v\leq 1})^2 \leq (v\mathbb{1}_{v\leq 1})^p$$

so that $v\mathbb{1}_{v\leq 1}$ is in $L^2 \cap L^p$. By assumption $uv\mathbb{1}_{v\leq 1}$ is integrable; $u\mathbb{1}_{u\leq 1}v\mathbb{1}_{v\leq 1}$ also is.
Besides,

$$u\mathbb{1}_{u\leq 1}v\mathbb{1}_{v>1} \leq v\mathbb{1}_{v>1} \leq v^p$$

so that $u\mathbb{1}_{u\leq 1}v\mathbb{1}_{v>1}$ is integrable and finally $u\mathbb{1}_{u\leq 1}v$ is integrable.

Let us prove now the second point. By assumption, for every positive $v \in L^2 \cap L^p(\mathbb{R}_+)$, $u\mathbb{1}_{u>1}v$ is integrable; but then it is larger than $v\mathbb{1}_{u>1}$ which is therefore integrable also. This means that the space $L^2 \cap L^p$ of the set $\{u > 1\}$ is included in the L^1 space of that same set. Therefore $\{u > 1\}$ has finite Lebesgue measure. As a consequence, for every function v in $L^2(\mathbb{R}_+)$, $v\mathbb{1}_{u>1}$ is in $L^2 \cap L^p(\mathbb{R}_+)$ and $u_{u>1}v$ is integrable by assumption.

Now Lemma 2.3 give the desired conclusion.

□

Remark that this lemma can be extended for all p in $[1,+\infty]$ but we do not need this here.

Proof of Theorem 5.3: We will give a complete proof in the case of second quantifications $\Gamma(h)$, $\Gamma(h^*)$, that is, we prove the equivalence of *1.* and *3.*

Let us start with the proof that *1.* implies *3.*; we obtain just as before the following properties for h and h^*:

$$\begin{cases} -\text{for almost all } t, \text{ the function } h\mathbb{1}_{[t,t+\epsilon]}/\epsilon \text{ converges in } L^2\left([0,t]\right) \text{ to a function } \alpha_t, \\ -\text{for almost all } t, \text{ the function } h^*\mathbb{1}_{[t,t+\epsilon]}/\epsilon \text{ converges in } L^2\left([0,t]\right) \text{ to a function } \beta_t, \\ -\text{the functions } t \mapsto \left\|\alpha_t\right\|_{L^2([0,t])},\ t \mapsto \left\|\beta_t\right\|_{L^2([0,t])} \text{ belong to } L^2\left(\mathbb{R}_+\right) + L^q(\mathbb{R}_+), \\ -\text{for almost all } t, \text{ the integral } \frac{1}{\epsilon}\int_t^{t+\epsilon} h\mathbb{1}_{[t,t+\epsilon]}(s)\, ds \text{ converges to a scalar } \gamma(t) \\ -\text{the function } \gamma \text{ is essentially bounded.} \end{cases}$$

We define as in the proof of 2.6 κ from α, β.

Now consider some sequence $(t_n)_{n\geq 0}$ in $\mathbb{R}_+$ which increases to infinity. Let us denote again by p_t the projection in $L^2\left(\mathbb{R}_+\right)$ of restriction to $[0,t]$. Since $p_{t_n}ht_n$ converges strongly to h one has

$$\text{for all } u \in L^2\left(\mathbb{R}_+\right), \text{ a subsequence of } p_{t_n}hp_{t_n}u(s) \text{ converges to } u(s) \text{ for almost all } s.$$

Using the separability of $L^2\left(\mathbb{R}_+\right)$, the boundedness of h and a diagonal procedure, one obtains a subsequence of $(t_n)_{n\geq 0}$ for which

$$\text{for almost all } s, \text{ all } u \text{ in } L^2\left(\mathbb{R}_+\right), p_{t_n}hp_{t_n}u(s) \to hu(s).$$

We keep the notation $(t_n)_{n\geq 0}$ for the subsequence. When restricting to the bounded interval $[0,t_n]$, the set $L^2 \cap L^p$ is equal to the L^2 set; therefore the proof of Lemma 2.6 adapted to functions on $[0,t_n]$ shows that for almost all s, the following formula holds for every u in $L^2\left(\mathbb{R}_+\right)$ and large enough n:

$$hp_{t_n}u(s) = \int_0^{t_n} \kappa(r,s)u(r)\, dr + u(s)f(s).$$

The left-hand-side converges to $hu(s)$, and the second one in the right-hand-side is fixed; therefore the integral term converges as n goes to infinity and allowing a temporary abuse of notation, we still have for almost all s, all u:

$$hu(s) = \int_0^\infty \kappa(r,s)u(r)dr + u(s)f(s). \tag{5.1}$$

The abuse lies therein that we have not proved that the integral $\int_0^\infty \kappa(r,s)u(r)dr$ is convergent. To swap the choice of $(t_n)_{n\geq 0}$ and the choice of s and u would be enough; it is easier to notice that one can consider, instead of u, the function defined by

$$u(r) = \begin{cases} u(r) \times \frac{\overline{\kappa(r,s)}}{|\kappa(r,s)|} \text{ if } \kappa(r,s) \neq 0 \\ 0 \text{ otherwise.} \end{cases}$$

This shows that the above convergence holds with $|\kappa(r,s)u(r)|$ as integrand so that the integral is actually absolutely convergent.

The proofs concerning h^* or $r \mapsto \kappa(s,r)$ are exactly the same, so that the proof that *1.* implies *3.* is complete.

The proof that *3.* with the additional assumption *3'.* entails *1.* is strictly similar to that of Theorem 2.1: thanks to *3'.*, one is able to define $H_t^+\mathcal{E}(u_t)$, $H_t^-\mathcal{E}(u_t)$ through 2.9 and the algebraic computations are the same as before.

The extension to the mentioned subset of $\mathcal{J}$ is obtained in the same way as in Theorem 2.1.

This ends the proof.

$\square$

Remarks on the additional assumption *3'.* : Let us point out that adding the assumption *3'.* is necessary. Indeed, the fact that

$$\int_0^\infty \left| \int_0^\infty \kappa(r,s)u(r)dr \right|^2 ds \leq \|h\|^2 \|u\|^2,$$

which is obtained from the form of h, does not imply that

$$\int_0^\infty \left| \int_0^s \kappa(r,s)u(r)dr \right|^2 ds \leq \|h\|^2 \|u\|^2.$$

It does, actually, if h is a positive operator, which proves one of our remarks. Besides, if *3.* holds for $|h|$ then, of the four positive operators that appear in the decomposition of h:

$$h = h^+_{\Re} - h^-_{\Re} + ih^+_{\Im} - ih^-_{\Im};$$

each fulfills condition *3.* because all of them are bounded by $|h|$; but then, since

$$\lambda(h) = \lambda(h^+_{\Re}) - \lambda(h^-_{\Re}) + i\lambda(h^+_{\Im}) - i\lambda(h^-_{\Im})$$

and

$$\lambda(h^*) = \lambda(h^+_{\Re}) - \lambda(h^-_{\Re}) - i\lambda(h^+_{\Im}) + i\lambda(h^-_{\Im})$$

hold on $\mathcal{J}$, each term is representable and an equality of the same kind holds for integrals. Other extensions are, of course, possible.

6 The case of unbounded operators

This section is meant to handle the case where h is an unbounded operator. In this particular case, the lack of information about the domain of h strengthens the problems already mentioned after Proposition 5.1. One can still obtain necessary conditions for the representability of $\Gamma(h)$: for example, for any u in the domain of h, one has for almost all t

- $(hu_{t,t+\epsilon})/\epsilon$ converges in $L^2([0,t])$,
- $\frac{1}{\epsilon}\int_0^\epsilon hu_{[t,t+\epsilon]}(s)ds$ converges to some complex number.

But once again these conditions do not translate to a more satisfactory condition on h.

Nevertheless, sufficient conditions are very easy to obtain: the proof of Theorem 2.1 makes it clear that it is enough for $\Gamma(h)$ to be representable on $\mathcal{E}(L^2(\mathbb{R}_+))$ that the expressions in 2.9 and 3.1 be well-defined on the set $\mathcal{E}(Dom\, h)$ with estimates that make the quantum stochastic integrals also well-defined.

Proposition 6.1 *Let h be an operator on $L^2(\mathbb{R}_+)$ with domain $Dom\, h$. Assume that there exist a function $\kappa : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ and a function $f : \mathbb{R}_+ \to \mathbb{C}$ such that, for any u in $Dom\, h$, almost all s in $\mathbb{R}_+$,*

$$hu(s) = \int_0^\infty \kappa(r,s)u(r)dr + u(s)f(s).$$

Consider the following conditions:

1. *$t \mapsto \int_0^t |\kappa(s,t)|^2\, ds$ is integrable,*

2. *for any u in $\mathrm{Dom}\, h$, $t \mapsto |u(t)| \sqrt{\int_0^t |\kappa(t,s)|^2\, ds}$ and $t \mapsto |f(t)-1|\,|u(t)|$ are integrable,*

3. *for fixed u in $\mathrm{Dom}\, h$, $\|(hu_t)_t\|$ is uniformly bounded in t.*

If conditions 1. and 2. hold, then $\lambda(h)$ is representable as a quantum stochastic integral on $\mathcal{J}(\mathrm{Dom}\, h)$, that is, the set of vectors of the form $j(g,f)$ with g, f in $\mathrm{Dom}\, h$; if conditions 1.,2. and 3. hold, then $\Gamma(h)$ is representable as a quantum stochastic integral on $\mathcal{J}(\mathrm{Dom}\, h)$.

References

[At1] S.Attal: An algebra of noncommutative bounded semimartingales, square and angle quantum brackets, *Journal of functional analysis 124* (1994), Academic Press, pp. 292-332.

[At2] S.Attal: Problèmes d'unicité dans les représentations d'opérateurs sur l'espace de Fock, *Sém. de probabilités XXVI* (1993), Springer Verlag, pp. 619-632.

[At3] S.Attal: Classical and quantum stochastic calculus, *Quantum probability and related topics X* (1998), World Scientific, pp.1-52.

[A-M] S. Attal and P.A. Meyer: Interprétations probabilistes et extension des intégrales stochastiques non commutatives, *Sém. de probabilités XXVII*, Springer Verlag, LNM 1557 (1993), pp. 312-327.

[Coq] A.Coquio: Stochastic integral representations of unbounded operators in Fock spaces, *Journal of the London Mathematical Society*, to appear.

[Gui] A.Guichardet: Symmetric Hilbert spaces and related topics, Springer Verlag LNM 261 (1970).

[J-M] J.L. Journé and P.A. Meyer: Une martingale d'opérateurs bornés, non représentable en intégrale stochastique, *Sém. de probabilités XX* (1986), Springer Verlag, pp. 313-316.

[Mey] P.A.Meyer: Quantum probability for probabilists, Springer-Verlag LNM 1538 (1993).

[Par] K.R. Parthasarathy: A remark on the paper “Une martingale d’opérateurs bornés, non représentable en intégrale stochastique”, *Sém. de probabilités XX* (1986), Springer Verlag, pp. 317-320.

[P-S] K.R. Parthasarathy and K.B. Sinha: Representation of bounded quantum martingales in Fock space, *Journal of functional analysis 67* (1986), pp. 126-151.