EFFECTIVE DIOPHANTINE APPROXIMATION ON ABELIAN VARIETIES I

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ABSTRACT. We prove new measures of linear independence of logarithms on an abelian variety defined over $\overline{\mathbb{Q}}$, which are *totally explicit* in function of the invariants of the abelian variety. Besides, except an extra-hypothesis on the algebraic point considered, we improve on earlier results (in particular David's lower bound). We also introduce into the main theorem two algebraic subgroups that lead to a great variety of different lower bounds. An important feature of the proof is the implementation of J.-B. Bost's slopes method and some results of Arakelov geometry naturally associated with it.

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Notation. For the sake of notational consistency, we denote (in general) with a slanting letter a mathematical object over a field, with a calligraphic letter an object over the ring of integers of a number field and with a bold letter elements of product set (*e.g.* $\mathbf{Y} = (Y_1, \ldots, Y_g)$). In this way, we have the usual notation $\mathbf{i}! = i_1! \cdots i_g!$ (*resp.* the length $|\mathbf{i}| = i_1 + \cdots + i_g$) for $\mathbf{i} = (i_1, \ldots, i_g) \in \mathbb{N}^g$. In a general way, D is a differential operator; if $\mathbf{x} = (x_1, \ldots, x_g) \in \mathbb{C}^g$ then $\mathrm{D}_{\mathbf{x}} := x_1 \frac{\partial}{\partial z_1} + \cdots + x_g \frac{\partial}{\partial z_g}$ and if $\mathbf{w} = (w_1, \ldots, w_g)$ is a basis of $\mathbb{C}^g$ then $\mathrm{D}_{\mathbf{w}}^{\mathbf{t}} = \mathrm{D}_{w_1}^{t_1} \cdots \mathrm{D}_{w_g}^{t_g}$ where $\mathbf{t} = (t_1, \ldots, t_g) \in \mathbb{N}^g$. Let $x \in \mathbb{R}$. We define $\log^+(x) := \log \max\{1, x\}$ and $[x]$ the integral part of x.

Let $\mathcal{E}$ be a module over a ring R. We denote by $\mathbb{V}(\mathcal{E})$ (*resp.* $\mathbb{P}(\mathcal{E})$) the spectrum of the symmetric algebra $\mathbf{S}(\mathcal{E})$ (*resp.* the projective scheme $\operatorname{Proj} \mathbf{S}(\mathcal{E})$) over $\operatorname{Spec}(R)$ (following Grothendieck's convention). We set $\mathcal{E}^{\mathsf{v}}$ the dual module $\operatorname{Hom}_R(\mathcal{E}, R)$. When $\mathcal{E} = R^{N+1}$, we write $\mathbb{P}_R^N$ instead.

For a group scheme $\mathbf{G}$ over $\operatorname{Spec}(R)$ we denote by $t_{\mathbf{G}}$ its tangent space at the origin. Let $\mathbf{G}$ be a complex Lie group and $\exp_{\mathbf{G}} : \operatorname{Lie}(\mathbf{G}) \to \mathbf{G}$ its exponential map. For a point $\mathbf{p} \in \mathbf{G}(\mathbb{C})$, a *logarithm* of $\mathbf{p}$ is an element $\mathbf{u} \in \operatorname{Lie}(\mathbf{G})$ such that $\exp_{\mathbf{G}}(\mathbf{u}) = \mathbf{p}$. For a nonnegative integer S, we shall denote by $\Gamma_{\mathbf{p}}(S)$ the set of points $\{0_{\mathbf{G}}, \mathbf{p}, \ldots, S.\mathbf{p}\}$. If k is a number field, we denote by $\mathcal{O}_k$ the ring of integers of k. Let $\mathbf{A}$ be an abelian variety over k (number field) and L be an ample line bundle over $\mathbf{A}$. We denote by $\mathrm{h}^0(\mathbf{A}, L)$ the dimension of the space of global sections $L(\mathbf{A})$, by $t_{\mathbf{A}}$ the tangent space at the origin of $\mathbf{A}$, by $h_F(\mathbf{A})$ the Faltings height of $\mathbf{A}$ and, for $\mathbf{p} \in \mathbf{A}(k)$, the real number $\widehat{h}_L(\mathbf{p})$ is the *Néron-Tate* height (relative to L) of $\mathbf{p}$. Besides, for an archimedean place σ of number field k, we write $\mathbf{A}_\sigma(\mathbb{C})$ the complex points of the abelian variety $\mathbf{A} \times_{\operatorname{Spec} \sigma(\mathcal{O}_k)} \operatorname{Spec} \mathbb{C}$ and $L_\sigma \to \mathbf{A}_\sigma(\mathbb{C})$ is the complex line bundle induced by L.

Furthermore, when α is an algebraic number, we denote by $h(\alpha)$ the (Weil) absolute logarithmic height of α (see for instance [46], § 3.2). Lastly, $e := \exp(1) = 2.718\ldots$ (do not confuse with $\mathfrak{e}$, which will occur later).

1. Introduction

The *theory of linear forms in logarithms* over a commutative algebraic group (defined over $\overline{\mathbb{Q}}$) with an abelian part (*i.e.*, not reduced to linear groups) begins with the works of D. Masser [32, 33] and S. Lang [31] in the seventies. The first theorems valid for an unspecified algebraic group are due to P. Philippon and M. Waldschmidt in 1987 [40, 41]. Their theorems were improved in the last ten years [28, 29, 1, 17, 22] but, in these works, appeared a constant depending on the algebraic group that was not explicit (even if theoretically computable). Till now, only a result of S. David [15] provided this constant in the particular case of a product of elliptic curves and an affine group $\mathbb{G}_a$*. The measure of linear independence in logarithms he achieved was applied successfully to find integral points on a wide range of diophantine equations through the so-called *Elliptic Logarithm Method* [23, 43] (see also [44]). David's result relies on Baker's method (as developed in [40]) mixed with *concrete* choice of Weierstrass models for these elliptic curves and explicit calculations on the Weierstrass $\wp$-functions. One could think that the generalization to an abelian variety would have involved some important technical difficulties induced by explicit calculations with theta functions (see the thesis [25] of P. Graftieaux to get an idea of this assertion).

In this paper, we show how techniques arising from diophantine approximation (Baker's method, Hirata's trick, Chudnovsky's process of variable change) can be integrated in a wholly geometric way into the Arakelovian *method of slopes* of J.-B. Bost. That simplifies and conceptualizes calculations and then we achieve new measures of linear independence of logarithms (over a product of abelian varieties) which take into account *all* the parameters of the problem (in particular Faltings' heights and degrees of the (polarized) abelian varieties) while remaining comparable to the ones given in [22]. For a discussion about the theoretical aspect of the measures of this article (parameters $\log b$, $\log a_i$), we refer the reader to *op.cit.* For instance, a corollary representative of the main theorem stated in § 2.3 is the following result.

Theorem 1.1. *Let* $\mathbf{A}$ *be an abelian variety of dimension* g *defined over a number field* k *of degree* D *over* $\mathbb{Q}$ *and embedded in* $\mathbb{C}$ *by* σ_0. *Let* L *be an ample symmetric line bundle over* $\mathbf{A}$.

Let W_0 *be a subspace of* $t_{\mathbf{A}}$, *of codimension* t. *We endow every complex tangent space* $t_{\mathbf{A}_\sigma}(\mathbb{C})$, $\sigma : k \hookrightarrow \mathbb{C}$, *with the hermitian metric* $\|.\|_\sigma$ *given by the Riemann form of* $L_\sigma \to \mathbf{A}_\sigma(\mathbb{C})$. *We denote by* $\check{h}(W_0)$ *the Arakelov degree of the hermitian vector bundle* $t_{\mathbf{A}}/W_0$ *endowed with quotient metrics deduced from* $\|.\|_\sigma$ *(see § 4.3.1).*

Let us consider also a point $\mathbf{p} \in \mathbf{A}(k)$ *and* $\mathbf{u} \in t_{\mathbf{A}}(\mathbb{C})$ *a logarithm of* $\mathbf{p}$. *Let* a, b, $\mathfrak{e}$ *be some positive real numbers verifying the following inequalities:*

$$\mathfrak{e} \geqslant e, \qquad \log a \geqslant \max\left\{\widehat{h}_L(\mathbf{p}),\, \frac{(\mathfrak{e}\|\mathbf{u}\|_{\sigma_0})^2}{D}\right\}, \qquad \log b \geqslant D\,\check{h}(W_0)$$

Define the integer

$$\mathfrak{a} := \left[\frac{D}{\log \mathfrak{e}} \max\left\{1,\, h_F(\mathbf{A}),\, \log \mathrm{h}^0(\mathbf{A},\, L),\, \log^+\left(\frac{D}{\log \mathfrak{e}}\right),\, \log^+ \log a\right\}\right] + 1$$

Let us assume the point $\mathbf{p}$ *is of* infinite *order modulo any proper abelian subvariety of* $\mathbf{A}$. *Then* $\mathbf{u} \notin W_0 \otimes_{\sigma_0} \mathbb{C}$ *and, if we denote by* d *the distance on* $t_{\mathbf{A}_{\sigma_0}}(\mathbb{C})$ *associated to* $\|.\|_{\sigma_0}$, *we have*

$$\log \mathrm{d}(\mathbf{u},\, W_0) \geqslant -(20g)^{60g^2/t}\mathfrak{a}^{1/t}\,(\mathfrak{a}\log \mathfrak{e} + \log b)\left(1 + \frac{D\mathfrak{a}}{\log \mathfrak{e}}\log a\right)^{g/t}. \tag{1}$$

If we fix a basis of $t_{\mathbf{A}_{\sigma_0}}(\mathbb{C})$, we can choose equations of W_0, namely $\beta_{i,1}z_1 + \cdots + \beta_{i,g}z_g = 0$, $1 \leqslant i \leqslant t$, $\beta_{i,j} \in k$, and by denoting $(u_1, \ldots, u_g)$ components of $\mathbf{u}$, the distance $\mathrm{d}(\mathbf{u}, W_0)$ is proportional to

$$\max_{1\leqslant i\leqslant t}\{|\beta_{i,1}u_1 + \cdots + \beta_{i,g}u_g|\}$$

(up to a constant depending on the choice of the basis of $t_{\mathbf{A}_{\sigma_0}}(\mathbb{C})$ and on β_i's), the latter quantity appearing in the literature under the name "simultaneous approximation of linear forms in logarithms of algebraic numbers (points)". A direct proof of this corollary along the lines we propose for the main theorem would require the sharpest lemmas of this paper.

Theorem 1.2. *With the notation and hypotheses of theorem 1.1, let us assume moreover that* W_0 *is the tangent space at the origin of an abelian subvariety* $\mathbf{B}$ *of* $\mathbf{A}$. *Then we have*

$$\log \mathrm{d}(\mathbf{u},\, t_{\mathbf{B}}) \geqslant -(20g)^{60g}\,(\mathfrak{a}\deg \mathbf{B})^{1/\operatorname{codim}\mathbf{B}}\,(\mathfrak{a}\log \mathfrak{e} + \log b)\left(1 + \frac{D\mathfrak{a}}{\log \mathfrak{e}}\log a\right). \tag{2}$$

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*Its value is about $(10g)^{10g^2}$ where g is the number of elliptic curves.

Estimate (2) is an avatar of Liouville's inequality when applied to a rational linear form in (usual) logarithms of algebraic numbers, namely, if we consider $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0\}$ and $b_1, \ldots, b_n \in \mathbb{Z}$ then

$$\log|b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| \geqslant -D|b_1|\, h(\alpha_1) - \cdots - D|b_n|\, h(\alpha_n) \tag{3}$$

(here D is the degree $[\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \mathbb{Q}]$). Inequality (3) – whose proof is very easy (there is a stronger variant in [2], Lemma 3) – means that for a rational subspace it is possible to have a *linear* dependence of the measure in the height of $\mathbf{p} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ if one agrees to have an *exponential* dependence in the height of the subspace* defined by linear form $b_1z_1 + \cdots + b_nz_n$ (note that, from the Tangent Space Lemma [34], $\log\deg \mathbf{B}$ is greater than $\check{h}(t_{\mathbf{B}})$). Here, inequality (2) is not linear in $\log a$ but there is also $\log\log a$ just because we did not take into account the particular hypothesis on W_0 during the demonstration of the main theorem. We refer the reader to part II of this work for a detailed study of this so-called "rational case".

Another corollary of our main theorem deals with the minimal norm of a logarithm of an algebraic point of an abelian variety. That has already been considered for instance by D. Bertrand [3], N. Hirata-Kohno [30] and F. Pellarin [38] in relation to Siegel's theorem on integral points. None of these results has precised the constant depending on the abelian variety.

Theorem 1.3. *With the notation and hypotheses of theorem 1.1, we have*

$$\log\|\mathbf{u}\|_{\sigma_0} \geqslant -(60g)^{60g} D \max\{1,\, h_F(\mathbf{A}),\, \log \mathrm{h}^0(\mathbf{A}, L),\, \widehat{h}_L(\mathbf{p})\}\ . \tag{4}$$

We shall explain the main arguments of the proof in detail at the end of § 2.3. However, let us already indicate that the *method of slopes* has been presented and developed by J.-B. Bost in [8, 9, 10]. It was also used by P. Graftieaux [26, 27] and some missing details (and improvements) of [8] form the object of the thesis of E. Viada [45].

Acknowledgements. A first and weaker version of main theorem 2.1 can be found in the second part of my thesis [20], supervised by G. Diaz, S. David and M. Waldschmidt and refereed by J.-B. Bost. I thank them a lot for their help and their interest in this work. I am also very grateful to D. Bertrand and P. Philippon for many encouragements and stimulating discussions (in particular about the possibility for parameters D_i's to be zero). Besides, I thank again J.-B. Bost, G. Diaz and G. Rémond for numerous helpful suggestions and comments on the original draft of this paper. Finally, I would like to express my warmest thanks to the Number Theory Research Group of Saint-Étienne (France) for the hospitality and the generous financial support while I was preparing my thesis.

2. Data and result

2.1. **Data.** Let n be a positive integer. For every $i \in \{1, \ldots, n\}$, let $\mathbf{A}_i$ be an abelian variety, of dimension g_i, defined over the field of algebraic numbers $\overline{\mathbb{Q}}$, and let us consider also an ample symmetric line bundle L_i over $\mathbf{A}_i$. We put $\mathbf{A} := \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$, $g := \dim \mathbf{A}$ and L the external tensor product $L_1 \boxtimes \cdots \boxtimes L_n$. For $i \in \{1, \ldots, n\}$, let $\mathbf{u}_i$ be an element of $t_{\mathbf{A}_i}(\mathbb{C})$ such that $\mathbf{p}_i := \exp_{\mathbf{A}(\mathbb{C})}(\mathbf{u}_i)$ belongs to $\mathbf{A}_i(\overline{\mathbb{Q}})$. Let W_0 be a (proper) subspace of $t_{\mathbf{A}}(\overline{\mathbb{Q}})$ of codimension t and let $\mathbf{u}_0$ be a vector of $t_{\mathbf{A}}(\overline{\mathbb{Q}})/W_0$. Let $\mathbf{G}_0 := \mathbb{V}\left((t_{\mathbf{A}}/W_0)^{\vee}\right)$ be the affine group scheme (over $\mathrm{Spec}(\overline{\mathbb{Q}})$) associated to the vector space $t_{\mathbf{A}}/W_0$. We denote by $\mathbf{G}$ the algebraic group $\mathbf{G}_0 \times \mathbf{A}$ and by $\mathbf{p} := (\mathbf{u}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n)$ the induced algebraic point of $\mathbf{G}$. Let λ be the canonical projection $t_{\mathbf{A}} \to t_{\mathbf{A}}/W_0$ and let us consider W the subspace of $t_{\mathbf{G}} = (t_{\mathbf{A}}/W_0) \oplus t_{\mathbf{A}}$ defined as the graph of λ that is,

$$W := \{\lambda(y) \oplus z,\ z \in t_{\mathbf{A}}\}\ . \tag{5}$$

We fix a subfield k of $\overline{\mathbb{Q}}$ of finite degree D on which there exists an MB-model † of $(\mathbf{A}, L, \{\mathbf{p}\})$ (see § 4.2.2). We write $\sigma_0 : k \hookrightarrow \mathbb{C}$ for the particular embedding deduced from $k \subseteq \overline{\mathbb{Q}}$ and we shall assume also W_0 be defined over k. We denote by ι the embedding $\mathbf{G} \hookrightarrow \mathbb{P} := \mathbf{X}_0 \times \prod_{i=1}^n \mathbb{P}\left(\mathrm{H}^0(\mathbf{A}_i, L_i^{\otimes 3})\right)$ where $\mathbf{X}_0 := \mathbb{P}\left(k \oplus (t_{\mathbf{A}}/W_0)^{\vee}\right)$. Geometric degrees of subgroups of $\mathbf{G}$ will be

*The too rough aspect of estimate (3) explains why it is (very often) useless for applications. However, the same lower bound with a very slight improvement in the b_i's (let us say $b_i^{1-\varepsilon}$ instead of b_i) would have numerous consequences for some diophantine equations (Catalan', Fermat's equations, see [19, 46] for some recent overviews of these questions).

†Initials MB are the ones of L. Moret-Bailly, whose work [36, 37] are at the root of the notion of MB-model, introduced by J.-B. Bost in [7], § 4.3. A construction of such a subfield k is explained at the beginning of the demonstration of theorem 4.10 of *ibid.*, p. 58.

relative to ι. For instance, we have $\deg_\iota \mathbf{G} = 3^g \prod_{i=1}^n \{g_i!\, \mathrm{h}^0(\mathbf{A}_i,\, L_i)\}$. Besides, for any embedding $\sigma : k \hookrightarrow \mathbb{C}$, we consider the hermitian metric $\|.\|_\sigma$ on $t_{\mathbf{A}_\sigma}(\mathbb{C})$ obtained from the *Riemann form* of $L_\sigma \to \mathbf{A}_\sigma(\mathbb{C})$. By quotient, we can extend that metric (keeping on the same notation) to $t_{\mathbf{G}_\sigma}(\mathbb{C}) = (t_{\mathbf{A}}/W_0)_\sigma(\mathbb{C}) \overset{\perp}{\oplus} t_{\mathbf{A}_\sigma}(\mathbb{C})$. Let d be the distance on $t_{\mathbf{A}_{\sigma_0}}(\mathbb{C})$ associated to $\|.\|_{\sigma_0}$.

Moreover, let us indicate that the term $h_{\overline{\mathcal{O}_{\mathbf{X}_0}(1)}}(m\mathbf{u}_0)$ which will appear in the lower bound (8) of $\log b$ below is the Faltings height of $1 \oplus m\mathbf{u}_0 \in \mathbf{X}_{0,\sigma_0}(\mathbb{C})$ (that is, more or less, the Weil height of $(1,\, m\mathbf{u}_0)$ (writing $\mathbf{u}_0$ in a basis of $t_{\mathbf{A}}/W_0$)). We refer the reader to § 4.3, p. 9, for a more precise definition of this object.

2.2. **Hypotheses and supplementary data.** For the positive integer $\mathfrak{a}$ defined below by (7), we shall *suppose* either

À For any integer $m \in \{1, \dots, (20g)^{5g+10}\,\mathfrak{a}\}$, the point $m\mathbf{p}$ is not a k-rational point of any proper algebraic subgroup $\mathbf{G}'$ of $\mathbf{G}$.

or

` The preceding condition is true only for subgroups $\mathbf{G}'$ such that $W + t_{\mathbf{G}'} \neq t_{\mathbf{G}}$.

Let us emphasize that the hypothesis À can be satisfied *only* when $t = 1$ (otherwise we could take $\mathbf{G}' = \mathbb{V}((k.\mathbf{u}_0)^\vee) \times \mathbf{A}$). Then we define an integer $y \in \{0,\, 1\}$ by

$$y = \begin{cases} 0 & \text{in case À}, \\ 1 & \text{in case ` }. \end{cases} \tag{6}$$

Let us consider some (proper) algebraic subgroups $\mathbf{G}_1 = \mathbf{G}_{01} \times \mathbf{B}_1$ and $\mathbf{G}_2 = \mathbf{G}_{02} \times \mathbf{B}_2$ of $\mathbf{G}$ such that $\lambda(t_{\mathbf{B}_1}) \subseteq t_{\mathbf{G}_{01}} \neq t_{\mathbf{A}}/W_0$ and $t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}) \neq t_{\mathbf{A}}/W_0$. Let δ_1 (*resp.* δ_2) be the smallest integer $i \in \{1, \dots, n\}$ such that $\dim \mathbf{B}_1 < g_1 + \cdots + g_{\delta_1}$ (*resp.* $\dim \mathbf{B}_2 < g_1 + \cdots + g_{\delta_2}$).

2.3. **Result.** By now, we are in a position to set out the main theorem of this paper, whose purpose is to give *lower bounds* for the *distance* $\mathrm{d}(\mathbf{u},\, W_{\mathbb{C}})$ between $\mathbf{u} := \mathbf{u}_0 \oplus \cdots \oplus \mathbf{u}_n \in t_{\mathbf{G}_{\sigma_0}}(\mathbb{C})$ and $W_{\mathbb{C}} := W \otimes_{\sigma_0} \mathbb{C}$.

Theorem 2.1. *With the previous notations, let $a_1 \geqslant \cdots \geqslant a_n$ and $\mathfrak{e}$ be some positive real numbers such that $\mathfrak{e} \geqslant e$ and*

$$\forall i \in \{1, \dots, n\}, \quad \log a_i \geqslant \max\left\{\widehat{h}_{L_i}(\mathbf{p}_i),\ \frac{(\mathfrak{e}\|\mathbf{u}_i\|_{\sigma_0})^2}{D}\right\}.$$

One define the integer

$$\mathfrak{a} := \left[\frac{D}{\log \mathfrak{e}} \max\left\{1,\, \log^+\left(\frac{D}{\log \mathfrak{e}}\right),\, h_F(\mathbf{A}_1),\, \dots,\, h_F(\mathbf{A}_n),\, \log \mathrm{h}^0(\mathbf{A},\, L),\, \log \deg_\iota \mathbf{G}_1,\, \log^+ \log(a_{\delta_1} \cdots a_n)\right\}\right] + 1 \tag{7}$$

and let b be a positive real number satisfying

$$\log b \geqslant D \max_{0 \leqslant m \leqslant (20g)^{5g+10}\,\mathfrak{a}} \left\{h_{\overline{\mathcal{O}_{\mathbf{X}_0}(1)}}(m\mathbf{u}_0)\right\} + D\check{h}(W_0) + \log^+ \min\{\mathfrak{e},\ \mathfrak{e}\mathfrak{a}\|\mathbf{u}_0\|_{\sigma_0}\}\,. \tag{8}$$

Last, let U be

(9)

$$(\mathfrak{a} \deg_\iota \mathbf{G}_2)^{\frac{1}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))}} \left(\frac{1}{\mathfrak{a} \log \mathfrak{e}}\right)^{\frac{\dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2})) - \dim \mathbf{G}_{02}}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))}} (\mathfrak{a}^y \log \mathfrak{e} + \log b)^{\frac{t - \dim \mathbf{G}_{02}}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))}}$$

$$\times \left(\left(1 + \frac{D\mathfrak{a}}{\log \mathfrak{e}} \log a_n\right)^{g_n} \cdots \left(1 + \frac{D\mathfrak{a}}{\log \mathfrak{e}} \log a_{\delta_2+1}\right)^{g_{\delta_2+1}} \left(1 + \frac{D\mathfrak{a}}{\log \mathfrak{e}} \log a_{\delta_2}\right)^{g_1 + \cdots + g_{\delta_2} - \dim \mathbf{B}_2}\right)^{\frac{1}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))}}.$$

Then $\mathbf{u}$ does not belong to $W_{\mathbb{C}}$ *and we have the following* estimate

$$\log \mathrm{d}(\mathbf{u},\, W_{\mathbb{C}}) \geqslant -c_1 \left(1 + \frac{\log^+ \|\mathbf{u}\|_{\sigma_0}}{\mathfrak{a} \log \mathfrak{e}}\right) U, \tag{10}$$

with $c_1 := (20g)^{30g \frac{\mathrm{codim}_{\mathbf{G}}(\mathbf{G}_2)}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))}}$.

The hypothesis on $\mathbf{p}$ is exactly what we need to extrapolate over its *multiples* during the proof. This type of assumption has been usual in works related to the Lehmer's problem (see for instance [16]). Moreover the condition $a_1 \geqslant \cdots \geqslant a_n$ is of no consequence on the general nature of the theorem. It allows us to simplify the presentation of the parameter U.

The high numerical value of the constant c_1 appearing in estimate (10) arises first from a Siegel-type condition (lemma 5.7 below) which imposes the extrapolation process to be carried out with sufficiently many points (namely, with the notation of the proof, $S/S_0 \gtrsim (20g)^{7g}$). Now, more or less, the right hand side of (10) is $-TS_0 \log \mathfrak{e}$ (arising from an "approached" Schwarz lemma) and it must be lower than $D_i S^2 \left(D\widehat{h}_{L_i}(\mathbf{p}_i) + (\mathfrak{e}\|\mathbf{u}_i\|_{\sigma_0})^2 \right)$, for any $i \in \{1, \ldots, n\}$. So the ratio between S and S_0 has direct repercussions on the measure (10). Moreover the dependence in g of the constant c_1 (*i.e.*, $\simeq g^{g^2}$ if we choose $\mathbf{G}_2 = \{0\}$) has not been improved in comparison with the result of S. David, although we take care of it during the demonstration (see the discussion about that point in [15], p. 12). In the current state of the art, it seems difficult to improve this point as Matveev [35] does it for commutative linear groups (replacing n^n by e^n). Besides, the dependence on Faltings heights of $\mathbf{A}_i$'s can be slightly improved. Indeed, we can take $h_F(\mathbf{A})$ instead of $h_F(\mathbf{A}_i)$, $1 \leqslant i \leqslant n$, in the definition of $\mathfrak{a}$ on condition that we add $-10g^4 \max\{h_F(\mathbf{A}_1), \ldots, h_F(\mathbf{A}_n)\}$ in the right hand side of (10). We do not do it by concern of readability of the proof.

Let us mention also that the theorem 2.1 can be extended without any particular difficulty to a semi-abelian variety *except for* the constants $(20g)^{5g+10}$ and c_1 which are in that case unspecified (because of auxiliary proposition 4.3). However, A. Chambert-Loir points out to me that proposition 4.3 should be (easily) generalized to semi-abelian varieties. We shall integrate his remark into a next version of this paper.

Before explaining the proof of this result in a more detailed way, let us verify quickly how theorems 1.1, 1.2 and 1.3 can be deduced from it.

Proof of theorem 1.1. In theorem 2.1, let us take $n = 1$, $\mathbf{A}_1 = \mathbf{A}$, $\mathbf{u}_0 = 0$, $\mathbf{G}_1 = \mathbf{G}_2 = \{0\}$ (so $\delta_1 = \delta_2 = 1$). From § 4.3.3, we have $h_{\overline{\mathcal{O}_{\mathbb{P}(k\oplus(t_{\mathbf{A}}/W_0)^\vee)}(1)}}(0) = 0$ and thanks to the hypothesis on $\mathbf{p}$, we are in case ` that is, $y = 1$. We can also notice

$$2\mathfrak{a}\log\mathfrak{e} \geqslant D\log^+\left(\frac{D\log a}{\log\mathfrak{e}}\right) \geqslant D\log^+\left\{\frac{(\mathfrak{e}\|\mathbf{u}\|_{\sigma_0})^2}{\log\mathfrak{e}}\right\} \geqslant \log^+\|\mathbf{u}\|_{\sigma_0}$$

because $\mathfrak{e} \geqslant e$ (in fact, as soon as $n = 1$ and $\mathbf{u}_0 = 0$ we can remove the term

$$(\log^+\|\mathbf{u}\|_{\sigma_0})/(\mathfrak{a}\log\mathfrak{e})$$

from (10)). Then inequality (1) is a simple consequence of (10). □

Proof of theorem 1.2. We choose here $n = 1$, $\mathbf{A}_1 = \mathbf{A}$, $\mathbf{u}_0 = 0$, $\mathbf{u}_1 = \mathbf{u}$, $\mathbf{G}_1 = \{0\}$, $\mathbf{G}_2 = \{0\} \times \mathbf{B}$ and the result follows from (10). □

Proof of theorem 1.3. In the previous theorem we take $\mathbf{B} = \{0\}$,

$$\log b = 2D\max\{1,\ h_F(\mathbf{A}),\ \log \mathrm{h}^0(\mathbf{A},\ L)\}$$

and we choose $\log\mathfrak{e} = D\max\{1,\ \widehat{h}_L(\mathbf{p}),\ h_F(\mathbf{A}),\ \log\mathrm{h}^0(\mathbf{A},\ L)\}$. Then if $\mathfrak{e}\|\mathbf{u}\|_{\sigma_0} \geqslant 1$ theorem 1.3 is trivial and, if not, we can take $\log a = (\log\mathfrak{e})/D$, $\mathfrak{a} = 1$ and from (2) we get the expected lower bound for $\log\|\mathbf{u}\|_{\sigma_0}$. □

3. Description of the proof of the main theorem

The demonstration of the theorem 2.1 relies both on slope inequality (73) (see § 5.8) which makes up the heart of the so-called *method of slopes* and on Baker's method* which guides the choice of the filtration necessary to apply (73). This framework being fixed, the rest of the proof consists in giving an estimate for each term appearing in the slope inequality, namely the Arakelov slope of an hermitian vector bundle $\overline{\mathcal{E}}$, the maximal slope of quotients $\overline{\mathcal{G}}_i$, ultrametric and archimedean norms of morphisms φ_i. It is worth noticing that the distance $\mathrm{d}(\mathbf{u},\ W)$ between the logarithm $\mathbf{u}$ and the subspace W occurs *only* in the upper bound of the σ_0-norm of φ. So the slope inequality gives immediatly a lower bound for $\mathrm{d}(\mathbf{u},\ W)$.

However, the quality of the measure of $\mathrm{d}(\mathbf{u},\ W)$ depends highly on the previous estimates concerning terms in the slope inequality. In this manner, N. Hirata-Kohno showed in [28, 29]† how to refine the problem to establish a much better measure with respect to the height of the subspace W. In this paper, we use her new "trick" framework: geometrically, it consists in working with the group $\mathbf{G} = \mathbf{G}_0 \times \mathbf{A}$ (instead of $\mathbf{A}$) and in considering the transversal subspace $W \subseteq t_{\mathbf{G}} = (t_{\mathbf{A}}/W_0) \oplus t_{\mathbf{A}}$

*Let us recall that the principle of Baker's method consists in using the vanishing of some jets of order $2T$ of a section s of a line bundle (over a compact manifold X) in (at least) one point of X in order to get some informations (most often in upper bound form) about jets of order T of s at other points of X.

†Her works were thorough and made best possible (for the parameter $h(W)$) in [17] and [22].

obtained from the *graph* of the projection $t_{\mathbf{A}} \to t_{\mathbf{A}}/W_0$ (that is why theorem 2.1 is slightly more general than theorem 1.1 as regards the considered group). One asset of this transformation is that there is an additional parameter D_0 associated to the affine group $\mathbf{G}_0$, on which it is possible to concentrate the weight of derivations (of order $\leqslant T$) along W. In this way, the expected quantity $T\,h(W)$ is replaced by $\min\{D_0,\, T\}\,h(W)$. Moreover, as in [17, 22], this trick allows us to use Chudnovsky's process of variable change in order to get a better estimate ($\min\{D_0,\, T\}^T$ instead of $T!$) for the denominator of some Taylor coefficients which appear during the proof (see § 5.5). Both these processes lead to a best possible measure in the height of the subspace W_0 (height bounded from above by $\log b$ here). Once we have fixed the new data $\mathbf{G}$ and W (instead of the original ones), calculations about Arakelovian objects needed during the method of slopes are *naturally* simpler and more conceptual (see, for instance, propositions 5.3 and 5.6).

Finally, as a counterbalance to the numerous advantages brought by the method of slopes in this type of problems, let us point out that the extra-hypothesis on the point $\mathbf{p}$ (it must not be a torsion point modulo some subgroups) in the theorems above results from the need to be within the scope of the "strict" Baker's method, namely an extrapolation on points and not on derivations. I have not yet seen how to remove that condition but it is known from the usual transcendence method it could be.

The paper is organized as follows.

Firstly, in the next section 4, we gather various elementary notions and properties about Arakelovian objects used during the demonstration of theorem 2.1. It does not contain any new result but it may clarify some technical points of the proof while introducing notation. Section 5 is devoted to the demonstration of theorem 2.1. As it has often been the case in transcendence proofs, the first step (§ 5.1) consists in fixing parameters x, T, S, $D_0, \ldots, D_n$. To apply the slope inequality, we need a certain linear map to be injective and this property is equivalent here to proposition 5.2. The choice of x (from equality (33)) is totally determined by this purpose*. In § 5.2, we define the space $\mathcal{E}$ of sections considered in the following and after endowing it with a structure of hermitian vector bundle we compute its slope. Then (§ 5.3), we define the vector space F of jets, the linear map $\varphi : \mathcal{E}_k \to F$ we mentioned above and the filtration $(F_i)_i$ of F. We give an estimate for the maximal slope of $\overline{F_i/F_{i+1}}$, noteworthy inasmuch as the height of W_0 is combined with parameter D_0 on $\mathbf{G}_0$ and not with the order of derivation T, as, for instance, it was the case with estimates (5.29) and (5.30) in [8]. It is made possible thanks to Hirata's trick. In § 5.4, we prove a technical lemma, more or less a rewriting of a Siegel condition. Finally, in order to estimate the height of morphisms φ_i, we give successively upper bounds for ultrametric norms (§ 5.5) and archimedean norms (§ 5.6). In § 5.7, under the assumption that the distance $\mathrm{d}(\mathbf{u},\, W)$ is "very small" (inequality (63) makes this precise), we give a sharp estimate for $\|\varphi_i\|_{\sigma_0}$. To conclude (§ 5.8), we apply the slope inequality (73) and that leads to a contradiction with the hypothesis "very small".

4. Toolbox

Each part of this section can be read independently from the rest of the text. It contains some more or less elementary facts about Arakelovian objects, put together here in view of setting some notation, conventions and recalling some results that will be used during the proof of theorem 2.1. We assume familiarity with the notion of *hermitian vector bundles* over $\mathrm{Spec}(\mathcal{O}_k)$ (k number field) as well as their properties mentioned in the appendix of [8] or in § 4.1 of [10].

4.1. **Symmetric powers.** Let $\overline{\mathcal{E}} = (\mathcal{E},\, (\|.\|_\sigma)_{\sigma:k\hookrightarrow\mathbb{C}})$ be an hermitian vector bundle over $\mathrm{Spec}(\mathcal{O}_k)$ of rank $N+1$ and ℓ be a positive integer. Let us recall that for a finite place $\mathfrak{p}$ of k with valuation ring $\mathcal{O}_\mathfrak{p}$, the module $\mathcal{E}_\mathfrak{p} := \mathcal{E}\otimes\mathcal{O}_\mathfrak{p}$ is *free* of rank $N+1$. If we consider a basis $(e_0, \ldots, e_N)$ over $\mathcal{O}_\mathfrak{p}$ of $\mathcal{E}_\mathfrak{p}$, then, the $\mathfrak{p}$-adic norm $\|.\|_{\overline{\mathcal{E}},\mathfrak{p}}$ on $\mathcal{E}_\mathfrak{p}$ is defined by $\|\sum_{i=0}^N x_ie_i\|_{\overline{\mathcal{E}},\mathfrak{p}} = \max_{0\leqslant i\leqslant N}\{|x_i|_\mathfrak{p}\}$, where the $\mathfrak{p}$-adic absolute value $|.|_\mathfrak{p}$ on $\mathcal{O}_\mathfrak{p}$ is normalized by $|\varpi|_\mathfrak{p} = \mathrm{card}\,(\mathcal{O}_k/\mathfrak{p})^{-1}$, ϖ being a uniformizing parameter of $\mathcal{O}_\mathfrak{p}$.

*Because of the structure of algebraic subgroups of $\mathbf{G}$, we have been able to avoid introducing the (usual) subgroup (called $\widetilde{\mathbf{G}}$ in the literature, see proposition 5.2 of [40]) which allows us to manage (a part of) the obstruction subgroups appearing in the zero estimate [39].

For each place $\sigma : k \hookrightarrow \mathbb{C}$, we endow the ℓ-th symmetric power $S^\ell(\mathcal{E}_\sigma)$ with the quotient metric $\|.\|_{\ell,\sigma}$ deduced from the canonical surjective map $\mathcal{E}_\sigma^{\otimes \ell} \twoheadrightarrow S^\ell(\mathcal{E}_\sigma)$. If we take $(e_0, \ldots, e_N)$ an orthonormal basis of $\mathcal{E}_\sigma$, the square norm $\|\mathsf{e}\|_{\ell,\sigma}^2$ of an element $\mathsf{e} := \sum x_{\mathbf{i}} e_1^{i_0} \cdots e_N^{i_N}$ equals $\sum |x_{\mathbf{i}}|^2 \frac{\mathbf{i}!}{\ell!}$. We have a natural isomorphism Θ_ℓ between $S^\ell(\mathcal{E}^\vee) \otimes k$ and $\left(S^\ell(\mathcal{E})\right)^\vee \otimes k$.

Lemma 4.1. *The k-linear map Θ_ℓ is an isometry at ultrametric places of k. For each archimedean place $\sigma : k \hookrightarrow \mathbb{C}$, the operator norms satisfy*

$$\|\Theta_\ell\|_\sigma \leqslant \max_{\substack{\mathbf{i}\in\mathbb{N}^{N+1}\\ |\mathbf{i}|=\ell}} \left\{\frac{\ell!}{\mathbf{i}!}\right\} \quad \textit{and} \quad \|\Theta_\ell^{-1}\|_\sigma \leqslant 1 \cdot \tag{11}$$

The very elementary proof is left to the reader.

4.2. **Slope computations.** The *normalized slope* of an hermitian vector bundle is the quotient of the (normalized) Arakelov degree by its rank:

$$\widehat{\mu}(\overline{\mathcal{E}}) := \frac{\widehat{\deg}_{\mathrm{n}}\overline{\mathcal{E}}}{\mathrm{rk}(\mathcal{E})} \cdot \tag{12}$$

It turns tensor products into sums: $\widehat{\mu}\left(\overline{\mathcal{E}} \otimes \overline{\mathcal{F}}\right) = \widehat{\mu}\left(\overline{\mathcal{E}}\right) + \widehat{\mu}\left(\overline{\mathcal{F}}\right)$. We shall give here the calculation of $\widehat{\mu}(\overline{\mathcal{E}})$ for some hermitian vector bundle $\overline{\mathcal{E}}$ arising from algebraic geometry. But before, let us recall that the *maximal slope* $\widehat{\mu}_{\max}(\overline{\mathcal{E}})$ is the maximum of the slopes $\widehat{\mu}(\overline{\mathcal{E}'})$ over (non zero) hermitian vector subbundle $\overline{\mathcal{E}'}$ of $\overline{\mathcal{E}}$.

4.2.1. Let $\overline{\mathcal{E}}$ be an hermitian vector bundle of rank $N+1$. We endow the canonical line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ with the Fubini-Study metrics (that is, quotient metrics $\pi^*\mathcal{E}_\sigma \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E}_\sigma)}(1)$, $\pi : \mathbb{P}(\mathcal{E}) \to \mathrm{Spec}(\mathcal{O}_k)$) at infinite places of k. Thus, for a positive integer n, the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)^{\otimes n}$ is metrized and there is a structure of hermitian vector bundle on the sheaf of global sections $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)$. We can notice that the latter metrics are the same as the ones on $\overline{S^n(\mathcal{E})}$ (defined in the previous §), twisted by $\binom{n+N}{n}^{-1/2}$ (see for instance lemma 4.3.6 of [11]). The following result is a variant of proposition 4.2.8 of Randriambololona's thesis [42].

Proposition 4.2. *Denote by $\gamma_{N,n}$ the real number*

$$\left\{\prod_{\substack{\mathbf{i}\in\mathbb{N}^{N+1}\\ |\mathbf{i}|=n}} \binom{n}{\mathbf{i}}\right\}^{\frac{1}{\binom{n+N}{n}}} . \tag{13}$$

The slope of the hermitian vector bundle $\overline{\Gamma\left(\mathbb{P}(\mathcal{E}),\, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)\right)}$ is

$$\widehat{\mu}\left(\overline{\Gamma\left(\mathbb{P}(\mathcal{E}),\, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)\right)}\right) = \frac{1}{2}\log\left\{\binom{n+N}{n}\gamma_{N,n}\right\} + n\,\widehat{\mu}\left(\overline{\mathcal{E}}\right) \cdot \tag{14}$$

Sketch of proof. After a field extension, we can suppose that $\mathcal{E}$ is a *free* $\mathcal{O}_k$-module. Let us choose an isomorphism $q : \mathcal{O}_k^{N+1} \to \mathcal{E}$. By endowing $\mathcal{O}_k^{N+1}$ with "trivial" metrics, we can estimate archimedean norms of the n^{th} symmetric power $S^n q : S^n(\mathcal{O}_k^{N+1}) \xrightarrow{\sim} S^n(\mathcal{E})$, from which we deduce the slope of $\overline{S^n(\mathcal{E})}$ (namely, $\widehat{\mu}\left(\overline{S^n\mathcal{E}}\right) = n\widehat{\mu}\left(\overline{\mathcal{E}}\right) + \frac{1}{2}\log\gamma_{N,n}$) and formula (14) *via* $S^n(\mathcal{E}) \simeq \Gamma\left(\mathbb{P}(\mathcal{E}),\, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)\right)$ (we control the change of norms as we mentioned above). More details can be found in *op.cit.* □

4.2.2. We are now going to estimate the Arakelov slope of sections bundle of an ample symmetric line bundle over an abelian variety. Let $\mathbf{A}$ be an abelian variety over $\overline{\mathbb{Q}}$ of dimension g and L an ample symmetric line bundle on $\mathbf{A}$. We consider $(\mathcal{A}, \overline{\mathcal{L}})$ an MB-model of $(\mathbf{A}, L)$. Definition and properties of this notion are given in § 4.3 of [7]. Nevertheless, let us recall that $\mathcal{A}$ is a semistable (smooth) group scheme over $\mathrm{Spec}\,\mathcal{O}_k$, whose generic fiber $\mathcal{A}_k$ is isomorphic to $\mathbf{A}$ and $\overline{\mathcal{L}} \to \mathcal{A}$ is a cubist line bundle (with $\mathcal{L}_k \simeq L$). We endow the space of global sections $\Gamma(\mathcal{A}, \mathcal{L})$ with the hermitian metrics induced by the cubist structure on $\overline{\mathcal{L}}$ (definition (22)).

Proposition 4.3. *The slope of the hermitian vector bundle $\overline{\Gamma(\mathcal{A},\, \mathcal{L})}$ is*

$$\widehat{\mu}\left(\overline{\Gamma(\mathcal{A},\, \mathcal{L})}\right) = -\frac{1}{2}h_F(\mathbf{A}) + \frac{1}{4}\log\frac{\mathrm{h}^0(\mathbf{A},\, L)}{(2\pi)^g} \cdot \tag{15}$$

Remark 4.4. By ampleness, the dimension $\mathrm{h}^0(\mathbf{A},\,L)$ is also the Euler-Poincaré characteristic $\chi(\mathbf{A},\,L)$ of L. Therefore, for any positive integer n, we have $\mathrm{h}^0(\mathbf{A},\,L^{\otimes n}) = n^g\,\mathrm{h}^0(\mathbf{A},\,L)$ and we deduce the slope of $\overline{\Gamma(\mathcal{A},\,\mathcal{L}^{\otimes n})}$ by adding $\frac{g}{4}\log(n)$ in the right hand side of (15).

This proposition has been established by J.-B. Bost [7]. It relies on work by L. Moret-Bailly [36, 37] as well as on the arithmetic Riemann-Roch theorem of H. Gillet and C. Soulé [24] (see also [6] for a more complicated exact formula when L is not supposed to be symmetric).

4.2.3. We use the same notation $(\mathbf{A},\,L)$ as in the preceding paragraph. Let k be a number field of definition of $(\mathbf{A},\,L)$. At each infinite place $\sigma : k \hookrightarrow \mathbb{C}$, the Riemann form of the complex line bundle $L_\sigma \to \mathbf{A}_\sigma(\mathbb{C})$ induces a metric $\|.\|_{L,\sigma}$ on the tangent space at the origin $t_{\mathbf{A}_\sigma}(\mathbb{C}) = t_{\mathbf{A}}(k)\otimes_\sigma \mathbb{C}$ and thus we get the hermitian tangent bundle $\overline{t_{\mathcal{A}}} = (t_{\mathcal{A}},\,(\|.\|_{L,\sigma})_{\sigma:k\hookrightarrow\mathbb{C}})$. The following statement is proposition D.1 of [8].

Proposition 4.5. *The slope of the dual hermitian vector bundle* $\overline{t_{\mathbf{A}}^{\vee}}$ *is*

$$\widehat{\mu}\left(\overline{t_{\mathbf{A}}^{\vee}}\right) = \frac{h_F(\mathbf{A})}{g} + \frac{1}{2g}\log \mathrm{h}^0(\mathbf{A},\,L) - \frac{1}{2}\log\pi\ . \tag{16}$$

As we shall see in the following section, this result can also be viewed as a height computation.

4.3. **Heights.** Throughout this paper occurs the terminology of *height* applied to various objects such that vector subspaces, k-linear maps and rational points of an arithmetic variety. We specify here definitions and properties related to this notion. For a much more global vision, the interested reader can refer to [5, 11] for instance.

4.3.1. We keep notation of § 4.2.3 (in particular the MB-model $(\mathcal{A},\,\overline{\mathcal{L}})$ of $(\mathbf{A},\,L)$). Let V be a vector subspace of $t_{\mathbf{A}}(k)$. Denote by $\mathcal{V} := V\cap t_{\mathcal{A}}$ the saturated $\mathcal{O}_k$-module obtained from V. That defines a subbundle of $t_{\mathcal{A}}$ and the *co-height* $\check{h}(V)$ of V (relative to $(\mathbf{A},\,L)$) is the normalized Arakelov degree of $\overline{t_{\mathcal{A}}/\mathcal{V}}$, where $\mathcal{V}$ is endowed with the restricted metrics of $\overline{t_{\mathcal{A}}}$. That terminology (quite personal for the moment!) is justified given the definition [8] who rather considered $h(V) := \widehat{\deg}_{\mathrm{n}}\overline{\mathcal{V}^{\vee}}$. So we have $\check{h}(V) = h(V) + \widehat{\deg}_{\mathrm{n}}\overline{t_{\mathcal{A}}}$ and this last degree may be computed with proposition 4.5. The interest of our choice is that it is the "natural" quantity which appears during the demonstration of theorem 2.1.

4.3.2. Now, consider some hermitian vector bundles $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ over $\operatorname{Spec}\mathcal{O}_k$ and $f : \mathcal{E}_k \to \mathcal{F}_k$ a k-linear map. At each place v of k, the vector spaces $\mathcal{E}\otimes\overline{k_v}$ and $\mathcal{F}\otimes\overline{k_v}$ over an algebraic closure $\overline{k_v}$ of k_v are metric spaces (when v is ultrametric, see beginning of § 4.1); let us denote by $\|.\|_{\overline{\mathcal{E}},v}$, $\|.\|_{\overline{\mathcal{F}},v}$ these metrics; then the *height* $h(\overline{\mathcal{E}},\,\overline{\mathcal{F}},\,f)$ (simplified in $h(f)$) *of* f with respect to $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ is defined by the formula

$$h(f) := \frac{1}{[k:\mathbb{Q}]}\sum_{v\text{ place of }k}\log\sup_{x\in\mathcal{E}_v\setminus\{0\}}\left\{\frac{\|f(x)\|_{\overline{\mathcal{F}},v}}{\|x\|_{\overline{\mathcal{E}},v}}\right\}.$$

4.3.3. In the same spirit, given an arithmetic variety $\mathcal{X}$ endowed with a metrized line bundle $\overline{\mathcal{M}}$, the *height of a rational point* $x\in\mathcal{X}(\overline{\mathbb{Q}})$ is the Arakelov degree of $\overline{x^*\mathcal{M}}$:

$$h_{\overline{\mathcal{M}}}(x) := \widehat{\deg}_{\mathrm{n}}\overline{x^*\mathcal{M}}\ . \tag{17}$$

If $(\mathcal{X},\,\overline{\mathcal{M}})$ is an MB-model of an abelian variety $(\mathbf{A},\,L)$, this definition coincides with the Néron-Tate height (relative to L) of x (see chap. III of [36]).

Now, let us assume $(\mathcal{X},\,\overline{\mathcal{M}})$ is $\left(\mathbb{P}(\mathcal{E}),\,\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}\right)$. First fact we would mention here regards the change of $h_{\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}}(x)$ when $\overline{\mathcal{E}}$ is varying, at constant rank. Consider $\overline{\mathcal{E}}$, $\overline{\mathcal{F}}$ some hermitian vector bundles over $\mathcal{O}_k$ of the same rank and $\phi : \mathcal{E}_k \to \mathcal{F}_k$ an isomorphism. Let $x\in\mathbb{P}(\mathcal{E}_k)(k)$ and V_x the hyperplane of $\mathcal{E}_k$ corresponding to x. We denote by $\widetilde{\phi} : \mathcal{E}_k/V_x \to \mathcal{F}_k/\phi(V_x)$ the map deduced from ϕ and by $\phi(x)\in\mathbb{P}(\mathcal{F}_k)(k)$ the k-point of $\mathbb{P}(\mathcal{F}_k)$ corresponding to the hyperplane $\phi(V_x)$ of $\mathcal{F}_k$. Then

$$h_{\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}}(x) = h_{\overline{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}}(\phi(x)) + h\left(\widetilde{\phi}\right)\ . \tag{18}$$

Indeed it suffices to notice that we have isometric isomorphisms

$$\mathcal{E}_k/V_x \simeq x^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \quad\text{and}\quad \mathcal{F}_k/\phi(V_x) \simeq \phi(x)^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$$

and to apply definition (17). As a corollary, we find that

$$h_{\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}}(\phi(x)) \leqslant h_{\overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}}(x) + h\left(\phi^{-1}\right) \cdot \tag{19}$$

For instance, when $\overline{\mathcal{E}} = \overline{\mathcal{O}_k^{N+1}}$ and $\phi((x_0, \ldots, x_N)) = (x_0, mx_1, \ldots, mx_N)$ (multiplication by an integer m), that gives the well-known inequality: $h_{\overline{\mathcal{O}_{\mathbb{P}^N}(1)}}(mx) \leqslant h_{\overline{\mathcal{O}_{\mathbb{P}^N}(1)}}(x) + \log m$. On the other hand, still in the case $\overline{\mathcal{E}} = \overline{\mathcal{O}_k^{N+1}}$, given $x = (x_0 : \cdots : x_N) \in \mathbb{P}^N(k)$, one can show that (see for instance [11], p. 49)

$$h_{\overline{\mathcal{O}_{\mathbb{P}^N}(1)}}(x) = \frac{1}{[k:\mathbb{Q}]} \log \frac{\prod_{\sigma : k \hookrightarrow \mathbb{C}} \left(\sum_{i=0}^N |\sigma(x_i)|^2\right)^{1/2}}{\mathrm{N}_{k|\mathbb{Q}}(x_0\mathcal{O}_k + \cdots + x_N\mathcal{O}_k)} \tag{20}$$

where, if I is a fractional ideal of $\mathcal{O}_k$, the rational number $\mathrm{N}_{k|\mathbb{Q}}(I)$ is the *norm* of I. This formula leads to a comparison with the *absolute logarithmic height* of x:

$$h_{\mathrm{Weil}}(x) \leqslant h_{\overline{\mathcal{O}_{\mathbb{P}^N}(1)}}(x) \leqslant h_{\mathrm{Weil}}(x) + \frac{1}{2}\log(1+N) \cdot \tag{21}$$

Finally, it is worth noting that the height relative to $\overline{\mathcal{O}_{\mathbb{P}(\mathcal{O}_k \oplus \mathcal{E})}(1)}$ of the ("zero") point $1 \oplus 0 \in \mathcal{O}_k \oplus \mathcal{E}$ equals 0.

4.4. **Gromov-type lemma.** Terminology "Gromov inequality" refers to the comparison between the sup-norm and the L^2-norm of a section of an hermitian vector bundle over a (compact) complex manifold. The archetype of such an inequality is at § 5.2.3 of [24].

Let $\pi : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_k)$ be a projective arithmetic variety (in the sense of [7]) and $\overline{\mathcal{M}} \to \mathcal{X}$ an hermitian line bundle over $\mathcal{X}$. We suppose that $\mathcal{M}$ is ample. For each $\sigma : k \to \mathbb{C}$, we consider a probability measure $\mathrm{d}\mu_\sigma$ on the complex manifold $\mathcal{X}_\sigma(\mathbb{C})$. Then one can define a hermitian structure $\|.\|_{\overline{\pi_*\mathcal{M}},\sigma}$ on $\mathrm{H}^0(\mathcal{X}_\sigma(\mathbb{C}), \mathcal{M}_\sigma)$:

$$\forall s \in \mathrm{H}^0(\mathcal{X}_\sigma(\mathbb{C}), \mathcal{M}_\sigma), \quad \|s\|^2_{\overline{\pi_*\mathcal{M}},\sigma} := \int_{\mathcal{X}_\sigma(\mathbb{C})} \|s(x)\|^2_{\overline{\mathcal{M}},\sigma}\, \mathrm{d}\mu_\sigma(x) \tag{22}$$

and there are also sup norms:

$$\forall s \in \mathrm{H}^0(\mathcal{X}_\sigma(\mathbb{C}), \mathcal{M}_\sigma), \quad \|s\|_{\infty,\sigma} := \sup_{x \in \mathcal{X}_\sigma(\mathbb{C})} \|s(x)\|_{\overline{\mathcal{M}},\sigma} \cdot \tag{23}$$

The quantity that interests us here is

$$\Xi[(\mathcal{X}, \overline{\mathcal{M}})] := \frac{1}{[k:\mathbb{Q}]} \sum_{\sigma : k \hookrightarrow \mathbb{C}} \log \sup_{\substack{s_\sigma \in \mathrm{H}^0(\mathcal{X}_\sigma(\mathbb{C}), \mathcal{M}_\sigma) \\ s_\sigma \neq 0}} \left\{ \frac{\|s_\sigma\|_{\infty,\sigma}}{\|s_\sigma\|_{\overline{\pi_*\mathcal{M}},\sigma}} \right\} \cdot \tag{24}$$

From Cauchy-Schwarz inequality, we see that Ξ is an "almost" additive function:

$$0 \leqslant \Xi[(\mathcal{X}_1 \times \cdots \times \mathcal{X}_m, \overline{\mathcal{M}_1} \boxtimes \cdots \boxtimes \overline{\mathcal{M}_m})] - \sum_{i=1}^m \Xi[(\mathcal{X}_i, \mathcal{M}_i)] \leqslant \frac{1}{2} \sum_{i=1}^m \log \mathrm{h}^0(\mathcal{X}_i, \mathcal{M}_i) \cdot \tag{25}$$

The first basic statement for an estimate of $\Xi[(\mathcal{X}, \overline{\mathcal{M}})]$ concerns projective spaces.

Lemma 4.6. *Let $\overline{\mathcal{E}}$ an hermitian vector bundle over* $\mathrm{Spec}(\mathcal{O}_k)$ *of rank $N+1$. Then, for any positive integer m, we have*

$$\Xi\left[\left(\mathbb{P}(\mathcal{E}), \overline{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(m)}\right)\right] \leqslant \frac{1}{2}\log\binom{N+m}{N} \cdot \tag{26}$$

The following statement for abelian varieties is more difficult to establish.

Lemma 4.7. *Let $(\mathbf{A}, L)$ be a polarized abelian variety and $(\mathcal{A}, \overline{\mathcal{L}})$ be an MB-model of $(\mathbf{A}, L)$. Then, for any positive integer m, we have*

$$\Xi\left[(\mathcal{A}, \overline{\mathcal{L}^{\otimes m}})\right] \leqslant c(g) \max\left\{1, h_F(\mathbf{A}), \log \mathrm{h}^0(\mathbf{A}, L), \log m\right\} \tag{27}$$

where $c(g)$ depends only on the dimension of the abelian variety. Furthermore, if the polarization is principal, *we may take $c(g) = 4g^4$.*

The proof of this lemma relies on the effective form of the "matrix lemma" of D. Masser given in [6, 8] as well as estimates on theta functions. The constant has been computed by P. Graftieaux [26], proposition 2.11.

Remark 4.8. Estimates (26) and (27) will be very useful to give an upper bound of the analytic part of $h(\varphi_i)$, the height of linear maps φ_i considered during the proof (see § 5.6 and 5.7). Actually, quantity (24) occurs through most of analytic estimates required to achieve some sharp upper bound for (maximal) slopes of hermitian vector bundles, for it is often simpler to work with the sup norm than the original hermitian metric! For example, J.-B. Bost described in [8], § 5.3.4, how to provide quickly an estimate of $\widehat{\mu}_{\max}\left(\overline{\mathcal{V}^{\mathsf{v}}}\right)$ (where $\overline{\mathcal{V}} \subseteq \overline{t_{\mathcal{A}}}$) from the "Shimura map" $\Sigma : \Gamma\left(\mathcal{A},\, \mathcal{L}^{\otimes 3}\right)^{\otimes 2} \to t^{\mathsf{v}}_{\mathcal{A}}$, $\Sigma(s_1 \otimes s_2) = \varepsilon^*\left(s_2^{\otimes 2} \otimes \mathrm{d}\left(s_1/s_2\right)\right)$ (where $\varepsilon : \operatorname{Spec}\mathcal{O}_k \to \mathcal{A}$ is the zero section of $\mathcal{A}$) and from slopes inequalities (proposition 4.3, *ibid.*). As it were, the main issue is to evaluate the height $h(\Sigma)$ of Σ (in the sense of § 4.3.2). Now, one can see at once from the definitions (cf. the proof of proposition 2.14 of [26]) that

$$h(\Sigma) \leqslant \log(6g) + 2\,\Xi\left[\left(\mathcal{A},\, \overline{\mathcal{L}^{\otimes 3}}\right)\right] \cdot$$

Hence (lemma 4.7 above) one deduces the following inequalities:

$$\widehat{\mu}_{\max}\left(\overline{t_{\mathcal{A}}}\right) \leqslant 5g + 2\,\Xi\left[\left(\mathcal{A},\, \overline{\mathcal{L}^{\otimes 3}}\right)\right] \leqslant \mathsf{c}_1(g)\,\max\left\{1,\, h_F(\mathbf{A}),\, \log \mathrm{h}^0(\mathbf{A},\, L)\right\} \tag{28}$$

and

$$\widehat{\mu}_{\max}\left(\overline{\mathcal{V}^{\mathsf{v}}}\right) \leqslant \widehat{\deg}_{\mathrm{n}}\overline{\mathcal{V}^{\mathsf{v}}} + (\dim\mathcal{V} - 1)\,\widehat{\mu}_{\max}\left(\overline{t_{\mathcal{A}}}\right) \leqslant \mathsf{c}_2(g)\,\max\left\{1,\, h_F(\mathbf{A}),\, \log \mathrm{h}^0(\mathbf{A},\, L),\, h(\mathcal{V})\right\} \tag{29}$$

by using also the lower bound $h_F(\mathbf{A}) \geqslant -\frac{g}{2}\log(2\pi)$ mentioned in [6] (here c_i, $i = 1, 2$, are constants that depend only on $g = \dim\mathbf{A}$ and $\mathsf{c}_1 = 14g^4$ (*resp.* $\mathsf{c}_2(g) = 15g^5$) if L is principal).

We conclude thus general points and it is time presently to look into the proof of the main theorem.

5. Proof of the main theorem

Preliminary remark . For the demonstration, we shall assume the *polarization* L of $\mathbf{A}$ to be *principal* that is, $\mathrm{h}^0(\mathbf{A},\, L) = 1$. Indeed, it is possible to reduce the original problem to this case by using isogenies $\varsigma_i : (\mathbf{A}_i,\, L_i) \to (\mathbf{A}'_i,\, L'_i)$, of degree $\mathrm{h}^0(\mathbf{A}_i,\, L_i)$, for all $i \in \{1,\, \ldots,\, n\}$. So $(\mathbf{A}'_i,\, L'_i)$ is a principally polarized abelian variety and then we must know how each quantity in the theorem 2.1 has been modified when working with the new system of data

$$\left\{\mathbf{A}'_i,\, L'_i,\, \varsigma(\mathbf{G}_1),\, \varsigma(\mathbf{G}_2),\, \mathrm{d}\varsigma_i.(\mathbf{u}_i),\, \mathrm{d}\varsigma.(W_0),\, \mathrm{d}\varsigma.(\mathbf{u}_0)\right\},$$

"image" of the original one by the isogeny $\varsigma := \varsigma_1 \times \cdots \times \varsigma_n$ between $\mathbf{A}$ and $\mathbf{A}' := \mathbf{A}'_1 \times \cdots \times \mathbf{A}'_n$ (and $\mathrm{d}\varsigma : t_{\mathbf{A}} \to t_{\mathbf{A}'}$ is the differential map at the origin). We have

$$h_F(\mathbf{A}') \leqslant h_F(\mathbf{A}) + \frac{1}{2}\log \mathrm{h}^0(\mathbf{A},\, L),$$
$$\deg \varsigma(\mathbf{G}_i) = \deg(\mathbf{G}_i) \ (i = 1, 2),$$
$$\widehat{h}_{L'_i}(\varsigma_i(\mathbf{p}_i)) = \widehat{h}_{L_i}(\mathbf{p}_i) \ (i = 1, \ldots, n)$$

and the choice of (cubist) metrics does not change the distance between $\mathbf{u}$ and W (nor norms of $\mathbf{u}_i$'s). We need $(\mathbf{A},\, L)$ to be principal because of inequality (42) of proposition 5.3 which relies on the totally effective version of lemma 4.7.

5.1. **Choice of parameters and zero estimate.** We are going to give in an abrupt way the values of parameters used in the course of the proof. Nevertheless, in order to attempt to convince the reader they are best possible (given the constraints and apart from the absolute constant), we shall explain locally during the demonstration the reasons of our choice.

We use again notation of § 2.1 (with, as explained above, the hypothesis of principality) and, in all the proof, the Hilbert-Samuel polynomial of $\mathbf{G}$ will be relative to the canonical embedding ι considered in that paragraph. We fix algebraic subgroups $\mathbf{G}_1$ and $\mathbf{G}_2$ considered in theorem 2.1. Let us define $S_0 := (10g)^8\mathfrak{a}$ and $S := (20g)^{5g+2}S_0 - 1$. We define the real number U_0 by

$$\begin{aligned} U_0 :=\ & (20g)^{\frac{(11g+14)\,\mathrm{codim}_{\mathbf{G}}(\mathbf{G}_2)}{t-\dim(t_{\mathbf{G}_{02}}+\lambda(t_{\mathbf{B}_2}))}}\,\left(S_0\deg_\iota \mathbf{G}_2\right)^{\frac{1}{t-\dim(t_{\mathbf{G}_{02}}+\lambda(t_{\mathbf{B}_2}))}} \\ & \times\left(\frac{1}{S_0\log\mathfrak{e}}\right)^{\frac{\dim(t_{\mathbf{G}_{02}}+\lambda(t_{\mathbf{B}_2}))-\dim\mathbf{G}_{02}}{t-\dim(t_{\mathbf{G}_{02}}+\lambda(t_{\mathbf{B}_2}))}} \times \left(\log b + S_0^y\log\mathfrak{e}\right)^{\frac{t-\dim\mathbf{G}_{02}}{t-\dim(t_{\mathbf{G}_{02}}+\lambda(t_{\mathbf{B}_2}))}} \\ & \times\left(\left(1+\frac{DS_0}{\log\mathfrak{e}}\log a_n\right)^{g_n}\cdots\left(1+\frac{DS_0}{\log\mathfrak{e}}\log a_{\delta_2+1}\right)^{g_{\delta_2}+1}\right)^{\frac{1}{t-\dim(t_{\mathbf{G}_{02}}+\lambda(t_{\mathbf{B}_2}))}} \\ & \times\left(\left(1+\frac{DS_0}{\log\mathfrak{e}}\log a_{\delta_2}\right)^{g_1+\cdots+g_{\delta_2}-\dim\mathbf{B}_2}\right)^{\frac{1}{t-\dim(t_{\mathbf{G}_{02}}+\lambda(t_{\mathbf{B}_2}))}} \end{aligned} \tag{30}$$

Let $\widetilde{D}_1, \ldots, \widetilde{D}_n, \widetilde{T}$ be positive real numbers given by

$$\widetilde{T} := \frac{U_0}{S_0 \log \mathfrak{e}}\ , \quad \widetilde{D}_i := \frac{U_0}{(10g)^{6g+11}(S_0 \log \mathfrak{e})\left(1 + \frac{DS_0}{\log \mathfrak{e}} \log a_i\right)}\ . \tag{31}$$

Then define

$$\widetilde{D}_0 := \frac{U_0}{(12g)^g \left(\log b + S_0^y \log \mathfrak{e}\right)} \tag{32}$$

where $y \in \{0, 1\}$ was introduced in § 2.2. Furthermore, we recall that a connected subgroup $\mathbf{G}^\star$ of $\mathbf{G}$ can be written $\mathbf{G}_0^\star \times \mathbf{A}^\star$ with $\mathbf{G}_0^\star \subseteq \mathbf{G}_0$ (of dimension $t^\star$) and $\mathbf{A}^\star \subseteq \mathbf{A}$ (*resp.* $g^\star$). Then, for such a subgroup $\mathbf{G}^\star$ with $t^\star < t$, we consider

$$x(\mathbf{G}^\star) := \frac{\left(\widetilde{T}\right)^{\frac{\dim(W + t_{\mathbf{G}^\star}) - \dim \mathbf{G}^\star}{t - t^\star}}}{\widetilde{D}_0 \left(\widetilde{D}_n^{g_n} \cdots \widetilde{D}_{\delta^\star+1}^{g_{\delta^\star+1}} \widetilde{D}_{\delta^\star}^{g_1 + \cdots + g_{\delta^\star} - \dim \mathbf{A}^\star}\right)^{\frac{1}{t - t^\star}}} \times \left(\frac{(S+1)\deg_\iota \mathbf{G}^\star}{(24)^g (g!)^3}\right)^{1/(t - t^\star)} \tag{33}$$

where $\delta^\star = \delta^\star(\mathbf{G}^\star)$ is the smallest integer in $\{1, \ldots, n\}$ such that $\dim \mathbf{A}^\star < g_1 + \cdots + g_{\delta^\star}$. Let x be the real number $x := \inf\{x(\mathbf{G}^\star)\}$ where the lower bound is taken on the algebraic subgroups $\mathbf{G}^\star$ of $\mathbf{G}$ such that $t_{\mathbf{G}^\star} + W \subsetneq t_{\mathbf{G}}$ (so for these subgroups we have necessarily $t^\star < t$). Besides, it is worth noting that for such subgroups, we have $\mathrm{codim}_{\mathbf{G}}\, \mathbf{G}^\star - \mathrm{codim}_W(W \cap t_{\mathbf{G}^\star}) = t - \dim(t_{\mathbf{G}_0^\star} + \lambda(t_{\mathbf{A}^\star}))$ and with definitions (31) and (32), we get $x \leqslant x(\mathbf{G}_2) \leqslant 1$.

Denote $T := \left[\widetilde{T}\right]$, $D_i := \left[\widetilde{D}_i\right]$ for $1 \leqslant i \leqslant n$, and $D_0 := \left[x\widetilde{D}_0\right]$. In the following, the D_i's will be (partial) degrees of polynomials, T will be an "order of derivation", S a "number of points" for the extrapolation process and x a fitting variable for the zero estimate. In the following lemma, we sum up some important properties of these parameters.

Lemma 5.1. *We have the following inequalities.*

(i) $T \geqslant (10g)^{5g+11} \max\{D_1 + 1, \ldots, D_n + 1\}$.
(ii) $D_0 \geqslant 1$ *and* $D_n \geqslant 1$.
(iii) $T \geqslant (6g)^g D_0 / S_0^{1-y}$.
(iv) $2(g+t)T \log(4D_0) \leqslant U_0/D$.

Proof. First assertion is a straightforward consequence of the choice of parameters $\widetilde{T}$ and $\widetilde{D}_i$'s (31). For D_0, it must be noted that for subgroups $\mathbf{G}^\star$ which occur in the definition of x we have

$$\dim(W + t_{\mathbf{G}^\star}) - \dim \mathbf{G}^\star \geqslant g - \dim \mathbf{A}^\star$$

and so, as $\widetilde{T} \geqslant \max\{\widetilde{D}_1, \ldots, \widetilde{D}_n\}$, we get the lower bound

$$x\widetilde{D}_0 \geqslant \inf_{\mathbf{G}^\star} \left\{ \left(\frac{(S+1)\deg_\iota \mathbf{G}^\star}{(24)^g (g!)^3}\right)^{\frac{1}{t - t^\star}} \right\} \geqslant 1.$$

As for D_n, we bound U_0 from below, using $a_i \geqslant a_n$ for all $i \in \{1, \ldots, n\}$ and $\log b + S_0^y \log \mathfrak{e} \geqslant S_0^y \log \mathfrak{e}$,

$$\begin{aligned} U_0 \geqslant{} & (20g)^{6g+14} \times S_0^{\frac{1}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))}} \left(\frac{1}{S_0 \log \mathfrak{e}}\right)^{\frac{\dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2})) - \dim \mathbf{G}_{02}}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))}} \\ & \times S_0^{\frac{y(t - \dim \mathbf{G}_{02})}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))}} (\log \mathfrak{e})^{\frac{t - \dim \mathbf{G}_{02}}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))}} \times \left(1 + \frac{DS_0}{\log \mathfrak{e}} \log a_n\right)^{\frac{g - \dim \mathbf{B}_2}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))}}\ . \end{aligned}$$

And then we get

$$\widetilde{D}_n \geqslant S_0^{\frac{1 + \dim \mathbf{G}_{02} + y(t - \dim \mathbf{G}_{02}) - t}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))}} \times \left(1 + \frac{DS_0}{\log \mathfrak{e}} \log a_n\right)^{\frac{g - \dim \mathbf{B}_2}{t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))} - 1}\ . \tag{34}$$

It is easy to verify $1 + \dim \mathbf{G}_{02} + y(t - \dim \mathbf{G}_{02}) \geqslant t$ using the definition of y. Likewise, we have $g - \dim \mathbf{B}_2 \geqslant t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2}))$ because

$$\begin{aligned} t - \dim(t_{\mathbf{G}_{02}} + \lambda(t_{\mathbf{B}_2})) &\leqslant \dim(t_{\mathbf{A}}/W_0) - \dim(t_{\mathbf{B}_2} + W_0)/W_0 \\ &\leqslant \dim(t_{\mathbf{A}}/(t_{\mathbf{B}_2} + W_0)) \leqslant g - \dim \mathbf{B}_2\ . \end{aligned}$$

So inequality (34) implies $D_n \geqslant 1$. As for (iii), it arises directly from the definitions of $\widetilde{T}$ and $\widetilde{D}_0$ as well as property $x \leqslant 1$. Finally, as regards estimate (iv), we get an upper bound for $\log D_0$ from the definition of x by taking $\mathbf{G}^\star = \mathbf{G}_1$:

$$\log D_0 \leqslant 25g^3 \max\left\{1,\, \log S_0,\, \log \deg_\iota \mathbf{G}_1,\, \log^+\left(\frac{D}{\log \mathfrak{e}} \log(a_{\delta_1} \cdots a_n)\right)\right\}$$
$$\leqslant 30g^3\, \frac{\mathfrak{a} \log \mathfrak{e}}{D}$$

and then we deduce (iv) from the definition of $\widetilde{T}$. □

Remarks .

(1) Let us emphasize that hypothesis $\lambda(t_{\mathbf{B}_1}) \subseteq t_{\mathbf{G}_{01}}$ on $\mathbf{G}_1$ (*a priori* the condition $t_{\mathbf{G}_{01}} + \lambda(t_{\mathbf{B}_1}) \neq t_{\mathbf{A}}/W_0$ would have been enough) is justified here to get rid of U_0 from the upper bound of $x\widetilde{D}_0$ (U_0 is raised to the power $(\dim(W + t_{\mathbf{G}_1}) - \dim \mathbf{G}_{01} - g)/(t - \dim \mathbf{G}_{01}) = 0)$. Without that precaution, the upper bound of $\log D_0$ would have depended on $\log b$ and the final measure (10) would have not been best possible in the height of W_0 (there would have been a supplementary $\log\log b$ in definition (9) of U, as there was for instance in theorem 2.1 of [28]). Besides, the presence of the term $\log^+ \log(a_{\delta_1} \cdots a_n)$ in the definition of $\mathfrak{a}$ is justified here by inequality (iv) that is, as we shall show in § 5.5, by the *arithmetic part* of the proof.

(2) Integers D_i, $1 \leqslant i \leqslant n-1$, are not necessarily positives. So, for instance, in the next definition (35), we should rather consider $\mathcal{O}_{\mathbf{A}_i}$ instead of $L_i^{\otimes 3D_i}$ if $D_i = 0$ and we should introduce κ the least integer $\geqslant 1$ such that $D_\kappa \geqslant 1$ etc. However, for the sake of notational simplicity, we shall do this abusive writing which does not disrupt further calculations.

We denote by M the line bundle

$$\mathcal{O}_{\mathbf{X}_0}(D_0) \boxtimes L_1^{\otimes 3D_1} \boxtimes \cdots \boxtimes L_n^{\otimes 3D_n} \tag{35}$$

over the Zariski closure $\overline{\mathbf{G}}$ of $\iota(\mathbf{G})$.

Proposition 5.2. *No nonzero element $s \in \mathrm{H}^0\left(\overline{\mathbf{G}},\, M\right)$ vanishes along W at order $(g+t)T+1$ in all points of $\{0_{\mathbf{G}},\, \mathbf{p},\, 2\mathbf{p},\, \ldots,\, (g+t)S\mathbf{p}\}$.*

Proof. We suppose that there exists such a section $s \neq 0$. For every $1 \leqslant i \leqslant n$, the complex line bundle $L_{i,\sigma_0}^{\otimes 3}$ (over $\mathbf{A}_{i,\sigma_0}(\mathbb{C})$) is *normally generated* and, if we define $D_i' := \max\{1,\, D_i\}$, the restriction map

$$\mathrm{H}^0\left(\mathbb{P}_{\mathbb{C}}^{N_i},\, \mathcal{O}(D_i')\right) \to \mathrm{H}^0\left(\mathbf{A}_{i,\sigma_0}(\mathbb{C}),\, L_{i,\sigma_0}^{\otimes 3D_i'}\right)$$

(where $N_i := \mathrm{h}^0(\mathbf{A}_i,\, L_i^{\otimes 3}) - 1 = 3^{g_i} - 1$) is surjective (see [4], chap. 7, Theorem (3.1)). Therefore, the section s arises from a polynomial $P(\mathbf{X}_0,\, \ldots,\, \mathbf{X}_n)$, homogeneous in $\mathbf{X}_i = (X_{i,0},\, \ldots,\, X_{i,N_i})$, with complex coefficients and of degree $\leqslant D_i'$, that has the same property as s (in terms of vanishing). From the zero estimate of P. Philippon [39], there exists a proper connected subgroup $\mathbf{G}^\star$ of $\mathbf{G}$ such that

$$T^{\operatorname{codim}_W(W \cap t_{\mathbf{G}^\star})} \operatorname{card}\left(\frac{\Gamma_{\mathbf{p}}(S) + \mathbf{G}^\star(k)}{\mathbf{G}^\star(k)}\right) \mathscr{H}\left(\mathbf{G}^\star\,;\, D_0',\, \ldots,\, D_n'\right) \leqslant 2^g g!\, \mathscr{H}(\mathbf{G}\,;\, D_0',\, \ldots,\, D_n')\,. \tag{36}$$

Like in remark 2 (before proposition 5.1), let κ be the least integer in $\{1,\, \ldots,\, n\}$ such that $D_\kappa \geqslant 1$. We define the projection map $\pi_\kappa : \mathbf{A} \to \mathbf{A}_\kappa \times \cdots \times \mathbf{A}_n$ and $\mathbf{G}_\kappa^\star$ the (connected) algebraic subgroup $\mathbf{G}_0^\star \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{\kappa-1} \times \pi_\kappa(\mathbf{A}^\star)$ (where, as before, we have split $\mathbf{G}^\star$ into $\mathbf{G}_0^\star \times \mathbf{A}^\star$, $\mathbf{G}_0^\star \subseteq \mathbf{G}_0$, $\mathbf{A}^\star \subseteq \mathbf{A}$). As $\mathbf{G}^\star \subseteq \mathbf{G}_\kappa^\star$ and by definition of κ, inequality (36) is still true if we replace $\mathbf{G}^\star$ by $\mathbf{G}_\kappa^\star$. We also note that

$$\mathscr{H}\left(\mathbf{G}_\kappa^\star\,;\, D_0',\, \ldots,\, D_n'\right) \leqslant 3^g g!\, \mathscr{H}\left(\mathbf{G}^\star\,;\, D_0',\, \ldots,\, D_n'\right)$$

and since $\operatorname{codim}_W(W \cap t_{\mathbf{G}^\star}) > 0$, $T > 6^g (g!)^2$, we deduce from (36) that $\mathbf{G}_\kappa^\star \neq \mathbf{G}$. Now, let us show that (36) cannot happen and this will give a contradiction with the assumption on s. We shall distinguish several cases according to $\mathbf{G}_\kappa^\star$.

First, let us suppose that $t_{\mathbf{G}_\kappa^\star} + W = t_{\mathbf{G}}$. Then $\operatorname{codim}_W(W \cap t_{\mathbf{G}_\kappa^\star}) = \operatorname{codim}_{\mathbf{G}}(\mathbf{G}_\kappa^\star)$ and (36) implies

$$T^{\operatorname{codim}_{\mathbf{G}}(\mathbf{G}_\kappa^\star)} \operatorname{card}\left(\frac{\Gamma_{\mathbf{p}}(S) + \mathbf{G}_\kappa^\star(k)}{\mathbf{G}_\kappa^\star(k)}\right)$$
$$\leqslant 18^g (g!)^3\, D_0^{t-t^\star} \max\{1,\, D_1,\, \ldots,\, D_n\}^{g - \dim(\mathbf{A}_1 \times \cdots \times \mathbf{A}_{\kappa-1} \times \pi_\kappa(\mathbf{A}^\star))}\,.$$

Since $\max\{1,\, D_1,\, \ldots,\, D_n\}$ does not exceed $T/(10g)^{5g+11}$ (lemma 5.1, (i)), we obtain

$$T^{t-t^\star} \operatorname{card}\left(\frac{\Gamma_{\mathbf{p}}(S) + \mathbf{G}_\kappa^\star(k)}{\mathbf{G}_\kappa^\star(k)}\right) < D_0^{t-t^\star}\,.$$

So by lemma 5.1 (iii), we have necessarily $y=0$ which implies $t=1$. But then hypothesis À on $\mathbf{p}$ involves

$$\operatorname{card}\left(\frac{\Gamma_{\mathbf{p}}(S)+\mathbf{G}^\star_\kappa(k)}{\mathbf{G}^\star_\kappa(k)}\right)=S+1 \tag{37}$$

and there is a contradiction yet with lemma 5.1 (iii). Hence we have $t_{\mathbf{G}^\star_\kappa}+W\subsetneq t_{\mathbf{G}}$. Then on noting that (37) is still true because of hypothesis À or ` on $\mathbf{p}$ and that $D_1\leqslant\cdots\leqslant D_n$, we get from (36):

$$T^{\operatorname{codim}_W(W\cap t_{\mathbf{G}^\star_\kappa})}(S+1)\deg\mathbf{G}^\star_\kappa\leqslant 18^g(g!)^3(x\widetilde{D}_0)^{t-t^\star}D_{\kappa'}^{g_{\kappa'}+\cdots+g_\kappa-\dim\pi_\kappa(\mathbf{A}^\star)}D_{\kappa'+1}^{g_{\kappa'+1}}\cdots D_n^{g_n} \tag{38}$$

where κ' is the smallest integer $\geqslant\kappa$ such that $g_{\kappa'}+\cdots+g_\kappa\geqslant\dim\pi_\kappa(\mathbf{A}^\star)$. Since κ' is also the least integer such that $g_{\kappa'}+\cdots+g_1\geqslant\dim(\mathbf{A}_1\times\cdots\times\mathbf{A}_{\kappa-1}\times\pi_\kappa(\mathbf{A}^\star))$, we have $\kappa'=\delta^\star(\mathbf{G}^\star_\kappa)$ (by definition of $\delta^\star$ next (33)). Then, using $T\geqslant\widetilde{T}/2$, for T is a positive integer, inequality (38) is contradicting with the definition of x.

Finally, the subgroup $\mathbf{G}^\star$ cannot exist and the proposition has been proved. □

5.2. **The hermitian vector bundle.** In this paragraph, we define the hermitian vector bundle that is the starting point of the construction of all the objects necessary to apply the slope inequality.

By definition of the number field k, and for each $i\in\{1,\ldots,n\}$, there exists an MB-model

$$\left(\pi_i:\mathcal{A}_i\to\operatorname{Spec}(\mathcal{O}_k),\,\overline{\mathcal{L}}_i,\,\{\varepsilon_{m\mathbf{p}_i}:\operatorname{Spec}(\mathcal{O}_k)\to\mathcal{A}_i\}_{m\in\{0,\ldots,(g+t)S\}}\right) \tag{39}$$

of $(\mathbf{A}_i,\,L_i,\,\Gamma_{\mathbf{p}_i}((g+t)S))$. Such models, studied in [7], possess exactly the properties we need for this work and we refer the reader to *op.cit.*, theorem 4.10, for more detail. We put $\mathcal{A}=\mathcal{A}_1\times\cdots\times\mathcal{A}_n$ and $\mathcal{W}_0$ the saturated $\mathcal{O}_k$-module $t_{\mathcal{A}}\cap W_0$. Let $\mathcal{G}$ be the smooth group scheme $\mathbb{V}\left((t_{\mathcal{A}}/\mathcal{W}_0)^\vee\right)\times\mathcal{A}$ over $\operatorname{Spec}(\mathcal{O}_k)$, whose generic fiber identifies with $\mathbf{G}$. Denote by $\mathcal{X}$ the scheme $\mathbb{P}\left(\mathcal{O}_k\oplus(t_{\mathcal{A}}/\mathcal{W}_0)^\vee\right)\times\mathcal{A}$ over $\operatorname{Spec}(\mathcal{O}_k)$, and to simplify some future expressions, we put $\mathcal{X}_0:=\mathbb{P}\left(\mathcal{O}_k\oplus(t_{\mathcal{A}}/\mathcal{W}_0)^\vee\right)$. The k-rational points $m\mathbf{p}\in\mathbf{G}(k)$, $0\leqslant m\leqslant(g+t)S$, extend to $\varepsilon_{m\mathbf{p}}\in\mathcal{X}(\mathcal{O}_k)$, thanks to the $(\varepsilon_{m\mathbf{p}_i})_{1\leqslant i\leqslant n}$. Finally, consider $\mathcal{W}$ the *graph of the canonical projection* $\lambda:t_{\mathcal{A}}\to t_{\mathcal{A}}/\mathcal{W}_0$ that is, the $\mathcal{O}_k$-submodule of $t_{\mathcal{G}}=(t_{\mathcal{A}}/\mathcal{W}_0)\oplus t_{\mathcal{A}}$ made of elements $\lambda(z)\oplus z$, $z\in t_{\mathcal{A}}$. Its generic fiber $W=\mathcal{W}\otimes k$ is the subvector space of $t_{\mathbf{G}}$ (of dimension g) considered in § 2.1. On the scheme $\mathcal{X}$, we can consider the line bundle

$$\mathcal{M}:=\mathcal{O}_{\mathcal{X}_0}(D_0)\boxtimes\mathcal{L}_1^{\otimes 3D_1}\boxtimes\cdots\boxtimes\mathcal{L}_n^{\otimes 3D_n}\ . \tag{40}$$

It can be endowed with hermitian metrics at infinite place $\sigma:k\to\mathbb{C}$, obtained from the cubist metrics on each $\overline{\mathcal{L}_i}$ and from the Fubini-Study metrics on $\overline{\mathcal{O}_{\mathcal{X}_0}(1)}$. The $\mathcal{O}_k$-module $\mathcal{E}:=\mathrm{H}^0(\mathcal{X},\,\mathcal{M})$ of global sections of $\mathcal{M}$ is locally free and finitely generated. Its rank is

$$\operatorname{rk}\mathcal{E}=\binom{D_0+t}{t}3^g D_1^{g_1}\cdots D_n^{g_n}$$

(we used here the ampleness and the principality of L_i). For each archimedean embedding $\sigma:k\hookrightarrow\mathbb{C}$, the $\mathbb{C}$-vector space $\mathcal{E}_\sigma:=\mathcal{E}\otimes_\sigma\mathbb{C}$ is endowed with an hermitian metric $\|.\|_{\overline{\mathcal{E}},\sigma}$ defined by

$$\forall\ s\in\mathcal{E}_\sigma,\quad \|s\|^2_{\overline{\mathcal{E}},\sigma}:=\int_{\mathcal{X}_\sigma(\mathbb{C})}\|s(x)\|^2_{\overline{x^*\mathcal{M}},\sigma}\,\mathrm{d}\mu_\sigma(x)$$

where $\mathrm{d}\mu_\sigma$ is the probability measure on the (compact) set $\mathcal{X}_\sigma(\mathbb{C})$, invariant under the action of the unitary group $\mathbf{U}_{t+1}(\mathbb{C})$ and the group of translations of $\mathbf{A}_\sigma(\mathbb{C})$. The *hermitian vector bundle* $\overline{\mathcal{E}}=(\mathcal{E},\,(\|.\|_{\overline{\mathcal{E}},\sigma})_\sigma)$ made in this way has a normalized Arakelov slope $\widehat{\mu}(\overline{\mathcal{E}})$. The following proposition is the explicit calculation of this quantity.

Proposition 5.3. *The (normalized) Arakelov slope of* $\overline{\mathcal{E}}$ *is*

$$\widehat{\mu}(\overline{\mathcal{E}})=\frac{1}{2}\log\left\{\binom{D_0+t}{t}\gamma_{t,D_0}\right\}-\frac{1}{t+1}D_0\,\check{h}(W_0)+\sum_{i=1}^n\left\{-\frac{1}{2}h_F(\mathbf{A}_i)+\frac{g_i}{4}\log\left(\frac{3D_i}{2\pi}\right)\right\} \tag{41}$$

where γ_{t,D_0} *has been defined in proposition 4.2. In particular, with the notations of § 4.4, we have*

$$\widehat{\mu}(\overline{\mathcal{E}})-\Xi\left[(\mathcal{X},\,\overline{\mathcal{M}})\right]\geqslant-10g^4\left(\max\{1,\,h_F(\mathbf{A}_1),\,\ldots,\,h_F(\mathbf{A}_n)\}+\log\left(D_0\cdots D_n\right)\right)-D_0\,\check{h}(W_0)\ \cdot \tag{42}$$

Proof. Equality (41) is a straightforward application of the results of § 4.2. As for inequality (42), we use first § 4.4 to get an upper bound for $\Xi\left[(\mathcal{X},\,\overline{\mathcal{M}})\right]$:

$$\begin{aligned}\Xi\left[(\mathcal{X},\,\overline{\mathcal{M}})\right] &\leqslant \frac{1}{2}\log \mathrm{h}^0(\mathcal{X},\,\mathcal{M}) + \Xi\left[\left(\mathbb{P}\left(\mathcal{O}_k\oplus(t_{\mathcal{A}}/\mathcal{W}_0)^{\vee}\right),\,\overline{\mathcal{O}(D_0)}\right)\right] + \sum_{i=1}^{n}\Xi\left[\left(\mathcal{A}_i,\,\overline{\mathcal{L}_i^{\otimes 3D_i}}\right)\right]\\ &\leqslant \frac{1}{2}\log \operatorname{rk}\mathcal{E} + \frac{1}{2}\log\binom{D_0+t}{t} + \sum_{i=1}^{n}4g_i^4\max\left\{1,\,h_F(\mathbf{A}_i),\,\log(3D_i)\right\}\end{aligned}$$

from which we deduce easily (42) using (41). □

Remark 5.4. Even if we shall not need the following property for the demonstration, let us point out that the hermitian vector bundle $\overline{\mathcal{E}}$ is ***semi-stable*** that is, $\widehat{\mu}(\overline{\mathcal{E}}) = \widehat{\mu}_{\max}\left(\overline{\mathcal{E}}\right)$. Indeed, from proposition A.3 of [8], it suffices to show there exists a group made of isometric automorphisms of $\overline{\mathcal{E}}$, whose action on $\mathcal{E}$ is *irreducible*. The group product of unitary matrices $t\times t$ by Mumford groups $K\left(L_i^{\otimes 3D_i}\right)$ fulfils this condition (for more details, see [8], p. 12).

5.3. **Choice of the space of jets and its filtration.** The analytic part of the proof rests on Baker's method, which determines in this way the choice of the filtration. As in all proofs using method of slopes until now, the morphism $\varphi : E\to F$ we need will be an evaluation map. To a section of the line bundle M, we associate jets of order $2(g+t)T$ (*resp.* $(g+t)T$) along W at the points $m\mathbf{p}$ for $m\in\{0,\,\dots,\,S_0\}$ (*resp.* $m\in\{S_0+1,\,\dots,\,(g+t)S\}$). With a suggestive writing, we want $\varphi(s) = \left(\left(\mathrm{jet}_W^{\mathbf{t}}\,s\right)(m\mathbf{p})\right)$ where $(m,\,\mathbf{t})$ belongs to

$$\nabla := \left\{(m,\,\mathbf{t});\quad \begin{array}{c}\mathbf{t}\leqslant 2(g+t)T \text{ and } 0\leqslant m\leqslant S_0 \text{ or}\\ \mathbf{t}\leqslant (g+t)T \text{ and } S_0+1\leqslant m\leqslant (g+t)S\end{array}\right\}. \tag{43}$$

Obviously, the vector space F will be built in an appropriate way to "welcome" all these jets. It will be isomorphic to $\bigoplus_{(m,\,\mathbf{t})\in\nabla} k$. The filtration F_i of F will be defined by vanishing the first components up to i of F (the set ∇ being endowed with the lexicographical order):

$$\text{if } i=(m,\,\mathbf{t}) \quad\text{then}\quad F_i := \bigoplus_{(m',\,\mathbf{t}')\geqslant(m,\mathbf{t})} k\,. \tag{44}$$

The notion of ***infinitesimal neighbourhood*** will enable us to give a precise mathematical statement for these considerations.

Given a nonnegative integer ℓ, we denote by $\mathbf{G}_W^{(\ell)}$ the infinitesimal neighbourhood of order ℓ along W of the zero section of (the group scheme) $\mathbf{G}$ that is, the closed subscheme of $\mathcal{X}_k$ whose ideal sheaf $I_{\mathbf{G}_W^{(\ell)}}$ is defined by

$$s\in I_{\mathbf{G}_W^{(\ell)}} \iff \left(\left(\exp_{\mathbf{G}_{\sigma_0}(\mathbb{C})}\right)^* s\right)(z_1w_1+\cdots+z_gw_g)\in(z_1,\,\dots,\,z_g)^{\ell+1} \tag{45}$$

for any basis $(w_1,\,\dots,\,w_g)$ of W_{σ_0} and local coordinates $z_1,\,\dots,\,z_g$ near 0. For $\mathbf{g}\in\mathbf{G}(k)$, we denote by $\tau_{\mathbf{g}} : \mathbf{G}_k\to\mathbf{G}_k$ the translation map. Let us consider the non-reduced closed subscheme of $\mathcal{X}_k$

$$\mathbf{T} := \bigsqcup_{m=0}^{S_0}\tau_{m\mathbf{p}}\left(\mathbf{G}_W^{(2(g+t)T)}\right)\sqcup\bigsqcup_{m=S_0+1}^{(g+t)S}\tau_{m\mathbf{p}}\left(\mathbf{G}_W^{((g+t)T)}\right) \tag{46}$$

and F the k-vector space of sections of the restriction of the line bundle M to $\mathbf{T}$:

$$F := \mathrm{H}^0\left(\mathbf{T},\,M_{|\mathbf{T}}\right)\,. \tag{47}$$

We consider then $\varphi : E\to F$ the restriction morphism.

Lemma 5.5. *The linear map φ is* injective.

It is a direct consequence of proposition 5.2. Define the integer $\mathcal{N} := ((g+t)S+S_0+2)(g+t)T+1$.

Fact .

‹ For every integer $i\in\{1,\,\dots,\,2(S_0+1)(g+t)T\}$, there exists a unique couple $(m,\,\ell)\in\{0,\,\dots,\,S_0\}\times\{0,\,\dots,\,2(g+t)T\}$ such that $i=2m(g+t)T+\ell+1$.

› For every integer $i\in\{2(S_0+1)(g+t)T+1,\,\dots,\,\mathcal{N}\}$, there exists a unique couple $(m,\,\ell)\in\{S_0+1,\,\dots,\,(g+t)S\}\times\{0,\,\dots,\,(g+t)T\}$ such that $i=(S_0+m+1)(g+t)T+\ell+1$.

We shall say that $(m,\,\ell)$ is *associated* to the index $i\in\{1,\,\dots,\,\mathcal{N}\}$.

For such an integer i, we let

$$\mathbf{T}_i := \bigsqcup_{m'=0}^{m-1} \tau_{m'\mathbf{p}}\left(\mathbf{G}_W^{(2(g+t)T)}\right) \sqcup \tau_{m\mathbf{p}}\left(\mathbf{G}_W^{(\ell)}\right) \tag{48}$$

if $1 \leqslant i \leqslant 2(S_0+1)(g+t)T$, and

$$\mathbf{T}_i := \bigsqcup_{m'=0}^{S_0} \tau_{m'\mathbf{p}}\left(\mathbf{G}_W^{(2(g+t)T)}\right) \sqcup \bigsqcup_{m'=S_0+1}^{m-1} \tau_{m'\mathbf{p}}\left(\mathbf{G}_W^{((g+t)T)}\right) \sqcup \tau_{m\mathbf{p}}\left(\mathbf{G}_W^{(\ell)}\right) \tag{49}$$

if $2(S_0+1)(g+t)T \leqslant i \leqslant \mathcal{N}$. By convention, we put $\mathbf{T}_0 := \emptyset$. We have $\mathbf{T}_{\mathcal{N}} = \mathbf{T}$ and, moreover, if $\pi : \mathcal{X}_k \to \mathrm{Spec}(k)$ denotes the structural morphism of $\mathcal{X}_k$, there are (surjective) morphisms (k-linear maps)

$$q_i : \pi_*\left(M_{|\mathbf{T}}\right) \twoheadrightarrow \pi_*\left(M_{|\mathbf{T}_i}\right).$$

We consider the k-vector space $F_i := \ker q_i$. This defines a filtration of F:

$$\{0\} = F_{\mathcal{N}} \subseteq F_{\mathcal{N}-1} \subseteq \cdots \subseteq F_0 = F \tag{50}$$

whose intermediary quotients $G_i := F_i/F_{i+1}$ also identify with the kernels of the maps $\pi_*(M_{|\mathbf{T}_i}) \twoheadrightarrow \pi_*(M_{|\mathbf{T}_{i-1}})$ (use for instance the snake lemma to see this). The vector space G_i is isomorphic to $S^\ell(W^\vee) \otimes (m\mathbf{p})^* M$ (where $m\mathbf{p} : \mathrm{Spec}(k) \to \mathcal{X}_k$ refers to the point of $\mathbf{G}(k)$ extended to $\mathcal{X}_k(k)$ by $\mathbf{G} \hookrightarrow \mathcal{X}_k$). In this way, it has an underlying *integral* structure given by

$$\mathcal{G}_i := S^\ell(\mathcal{W}^\vee) \otimes \varepsilon_{m\mathbf{p}}^* \mathcal{M}. \tag{51}$$

The $\mathcal{O}_k$-module $\mathcal{G}_i$ has then a natural structure of hermitian vector bundle $\overline{\mathcal{G}}_i$. Indeed, for each infinite place $\sigma : k \hookrightarrow \mathbb{C}$, the tangent space $t_{\mathcal{X}_\sigma}(\mathbb{C}) = (t_{\mathbf{A}}/W_0)_\sigma \oplus t_{\mathbf{A}_\sigma}(\mathbb{C})$ can be endowed with the Riemann metric on $t_{\mathbf{A}_\sigma}(\mathbb{C})$ (coming from the line bundle $(L_1 \boxtimes \cdots \boxtimes L_n)_\sigma$) and the quotient metric on $(t_{\mathbf{A}}/W_0)_\sigma$. The inclusion $\mathcal{W} \hookrightarrow t_{\mathcal{X}}$ gives by restriction a metric on W_σ, and so on the dual $W_\sigma^\vee$ then on the tensor product $(W_\sigma^\vee)^{\otimes \ell}$and, at last, by quotient, there is a (hermitian) metric on $S^\ell(W_\sigma^\vee)$. Fubini-Study and cubist metrics give a structure of metrized line bundle on $\mathcal{M}$, which provides an hermitian line bundle structure on $\varepsilon_{m\mathbf{p}}^*\mathcal{M}$. We can then give an estimate for the maximal slope of $\overline{\mathcal{G}}_i$.

Proposition 5.6. *For every integer $i \in \{1, \ldots, \mathcal{N}\}$, the maximal slope of the hermitian vector bundle $\overline{\mathcal{G}}_i$ is smaller than $U_0/(3D)$.*

Proof. By using formulae (4.5) and (4.6) of [10], we get

$$\widehat{\mu}_{\max}\left(\overline{\mathcal{G}}_i\right) \leqslant \ell\left(\widehat{\mu}_{\max}\left(\overline{\mathcal{W}^\vee}\right) + 2g\log g\right) + h_{\overline{\mathcal{M}}}(\varepsilon_{m\mathbf{p}}).$$

Contrary to [8] and thanks to the choice of the hermitian vector bundle $\overline{\mathcal{W}}$, the maximal slope of $\overline{\mathcal{W}^\vee}$ admits an upper bound *that does not depend on* the co-height of W_0. This point is crucial to have the best estimate (that is, linear) in $\check{h}(W_0)$ for measure (10). The proof of this statement relies on a simple slope inequality. First we note the injective map $\mathcal{W} \hookrightarrow t_{\mathcal{A}}$, $\lambda(y) \oplus y \mapsto y$ has v-adic norms $\leqslant 1$ (v is any place of k) and then we deduce $\widehat{\mu}_{\min}\left(\overline{t_{\mathcal{A}}}\right) \leqslant \widehat{\mu}_{\min}\left(\overline{\mathcal{W}}\right)$ and so $\widehat{\mu}_{\max}\left(\overline{\mathcal{W}^\vee}\right) \leqslant \widehat{\mu}_{\max}\left(\overline{t_{\mathcal{A}}^\vee}\right)$. Since $(\mathbf{A}, L)$ is principally polarized, we can apply inequality (29) to $\overline{\mathcal{V}} = \overline{t}_{\mathcal{A}}$ (or also prop. 2.14 of [26], which is more precise for the numerical constant), that gives an upper bound for the latter maximal slope and we get:

$$\widehat{\mu}_{\max}\left(\overline{\mathcal{W}^\vee}\right) \leqslant 15g^5 \max\{1, h_F(\mathbf{A})\}.$$

Moreover the Arakelov degree of $\overline{\varepsilon_{m\mathbf{p}}^*\mathcal{M}}$ breaks up into a sum

$$h_{\overline{\mathcal{M}}}(\varepsilon_{m\mathbf{p}}) = D_0\, h_{\overline{\mathcal{O}_{\mathcal{X}_0}(1)}}(m\mathbf{u}_0) + 3m^2\left(D_1\widehat{h}_{L_1}(\mathbf{p}_1) + \cdots + D_n\widehat{h}_{L_n}(\mathbf{p}_n)\right).$$

So we get

$$\widehat{\mu}_{\max}\left(\overline{\mathcal{G}}_i\right) \leqslant 5(g+t)^6\left\{T\max\{1, h_F(\mathbf{A})\} + S^2\sum_{i=1}^n D_i\widehat{h}_{L_i}(\mathbf{p}_i)\right\} + D_0 \max_{0\leqslant m\leqslant (g+t)S}\left\{h_{\overline{\mathcal{O}_{\mathcal{X}_0}(1)}}(m\mathbf{u}_0)\right\} \tag{52}$$

and the proposition follows from the choice of parameters. □

5.4. **Lower bound for a rank.** Following the example of M. Laurent's method of interpolation determinants, the method of slopes does not require the construction of an auxiliary function as it has been usual in transcendence's proofs. Nevertheless, there is no miracle (!) and Siegel condition (number of unknowns versus number of equations), even transformed, has to be present somewhere in the demonstration. The following lemma gives an account of this type of constraint.

Lemma 5.7. *Let i_0 be the integer $2(S_0+1)(g+t)T+1$. Then $\operatorname{rk}\mathcal{E}_{i_0} \geqslant (\operatorname{rk}\mathcal{E})/2$.*

Proof. First, by choice of i_0, we notice that $s \in \mathcal{E}_{i_0}$ when s vanishes along W at order $2(g+t)T$ in all points 0, $\mathbf{p}$, $\ldots$, $S_0\mathbf{p}$. Using the surjective map

$$\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D_0, \ldots, D_n)) \twoheadrightarrow \Gamma(\mathcal{X}, \mathcal{M})$$

(see the proof of proposition 5.2), whose kernel is made of polynomials vanishing on $\mathbf{G}$, it amounts to considering multihomogeneous polynomials of degree $(D_0, \ldots, D_n)$. Thus the rank of $\mathcal{E}_{i_0}$ equals the maximal number of polynomials P, linearly independent (over k), and not identically zero on $\mathbf{G}$, such that $P \circ \iota \circ \exp_{\mathbf{G}(\mathbb{C})}$ vanishes along W at order $2(g+t)T$ in 0, $\mathbf{u}$, $\ldots$, $S_0\mathbf{u}$. In other words, we have to estimate the rank ρ of the linear system

$$\forall x \in \{0, \mathbf{u}, \ldots, S_0\mathbf{u}\},\ \mathbf{t} \in \mathbb{N}^g,\ |\mathbf{t}| \leqslant 2(g+t)T, \quad \mathrm{D}_{\mathsf{w}}^{\mathbf{t}}\left(P \circ \iota \circ \exp_{\mathbf{G}(\mathbb{C})}\right)(x) = 0 \tag{53}$$

where $\mathsf{w} = (w_1, \ldots, w_g)$ is a basis of W and $\frac{1}{\mathbf{t}!}\mathrm{D}_{\mathsf{w}}^{\mathbf{t}}\left(P \circ \iota \circ \exp_{\mathbf{G}(\mathbb{C})}\right)(x)$ is the $\mathbf{t}$-th Taylor coefficient of $(P\circ\iota\circ\exp_{\mathbf{G}(\mathbb{C})})(x+z_1w_1+\cdots+z_gw_g)$. Then the continuation of the proof has been classical since the basic work of P. Philippon and M. Waldschmidt [40]. As we need an effective estimate, the more precise calculations in § 6.3 of [14] suit better for our situation. Let us consider a subgroup $\widetilde{\mathbf{G}}$ of $\mathbf{G}$ that reaches the minimum in the definition of x (p. 12). Then, recalling that $D_i' = \max\{1, D_i\}$, we have

$$\rho \leqslant \binom{2(g+t)T + \operatorname{codim}_W(W \cap t_{\widetilde{\mathbf{G}}})}{\operatorname{codim}_W(W \cap t_{\widetilde{\mathbf{G}}})}(S_0+1)\left(\frac{\tilde{d}(\tilde{d}+1)}{2} + (\tilde{d}+1)\mathscr{H}(\widetilde{\mathbf{G}}\,;\,D_0', \ldots, D_n')\right)$$

where $\tilde{d} := \dim\widetilde{\mathbf{G}}$. Choices of x and $\widetilde{\mathbf{G}}$ imply

$$\mathscr{H}(\widetilde{\mathbf{G}}\,;\,D_0', \ldots, D_n') \leqslant \frac{(5g)^{3g}\mathscr{H}(\mathbf{G}\,;\,D_0', \ldots, D_n')}{(S+1)\,T^{\operatorname{codim}_W(W\cap t_{\widetilde{\mathbf{G}}})}}$$

so $\rho \leqslant (10g)^{3g}(g+t)^{g+2}\frac{S_0+1}{S+1}\mathscr{H}(\mathbf{G}\,;\,D_0', \ldots, D_n')$. Since $\frac{1}{t!}D_0^t \leqslant \binom{D_0+t}{t}$, we have

$$\mathscr{H}(\mathbf{G}\,;\,D_0', \ldots, D_n') \leqslant t!\operatorname{rk}\mathcal{E}$$

and so

$$\operatorname{rk}\mathcal{E}_{i_0} \geqslant \left(1 - (10g)^{3g}(g+t)^{g+2}t!\,\frac{S_0+1}{S+1}\right)\operatorname{rk}\mathcal{E}\ . \tag{54}$$

Now lemma 5.7 is an immediate consequence of the definition of S. □

5.5. **Ultrametric estimate.** In this part, we are going to give an upper bound for the $\mathfrak{p}$-adic norm of each φ_i. For a section $s \neq 0$ of M, we have to find a denominator for the jet $\varphi_i(s)$ that is, a positive integer m such that $m\varphi_i(s)$ extends to an element of $\mathcal{G}_i$. When s is an integral section of $\mathcal{M}$, a naïve solution would be $\ell!$ (to see this, you can do a simple local calculation). Unfortunately, that leads to an extra logarithm of the height of W_0 in the final measure (10) (see for instance [28]). As we showed in the general case of a commutative algebraic group (not only an abelian variety) [22], we have to be shrewder to avoid such a situation and to obtain a linear measure in the height of W_0.

Let ℓ, h be positive integers. We define the integer

$$\delta_\ell(h) := \operatorname{lcm}\{i_1\cdots i_{h'}\ ;\ 1 \leqslant h' \leqslant h,\ i_j \in \mathbf{N}^*,\ i_1+\cdots+i_{h'} \leqslant \ell\}\ . \tag{55}$$

Prime number theorem implies there exists an absolute constant c such that

$$\log\delta_\ell(h) \leqslant \ell\log(ch)$$

S. Bruiltet [12] showed $c = 4$ did suit.

Proposition 5.8. *For every integer $i \in \{1, \ldots, \mathcal{N}\}$ and every finite place $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_k) \setminus \{(0)\}$, we have $\|\delta_\ell(D_0)\varphi_i\|_{\mathfrak{p}} \leqslant 1$.*

Proof. We adapt the demonstration of lemma 3.1 given in [21]. It slightly simplifies because we do not resort to explicit addition formulae on the abelian variety.

Fix an integer $i \in \{1, \ldots, \mathcal{N}\}$ and an element $s \neq 0$ of $\Gamma(\mathcal{X}, \mathcal{M})$. We must control the denominator of $\varphi_i(s) \in S^\ell(W^\vee) \otimes (m\mathbf{p})^* M$. By considering the pull-back of s by the translation map $\tau_{\varepsilon_{m\mathbf{p}}} : \mathcal{G} \to \mathcal{G}$, we can suppose that $m = 0$. Let $\mathfrak{p}$ be a non zero prime ideal of $\mathcal{O}_k$ and $\mathcal{O}_\mathfrak{p} (\subseteq k_\mathfrak{p})$ its valuation ring. Consider the scheme $\mathcal{X}_\mathfrak{p} = \mathcal{X} \underset{\operatorname{Spec} \mathcal{O}_k}{\times} \operatorname{Spec} \mathcal{O}_\mathfrak{p}$ and $\widehat{\mathcal{X}}_\mathfrak{p}$ the formal group over $\mathcal{O}_\mathfrak{p}$ obtained by completion of $\mathcal{X}_\mathfrak{p}$ along the neutral element ε of $\mathcal{G}_\mathfrak{p}$ (extended to $\mathcal{X}_\mathfrak{p}$). As $\mathcal{X}_\mathfrak{p} \to \operatorname{Spec} \mathcal{O}_\mathfrak{p}$ is *smooth*, there exist formal parameters $X_1, \ldots, X_t, Y_1, \ldots, Y_g$ such that $\mathcal{O}_{\widehat{\mathcal{X}}_\mathfrak{p}}$ is isomorphic (over $\mathcal{O}_\mathfrak{p}$) to $\mathcal{O}_\mathfrak{p}[[X_1, \ldots, Y_g]]$. Moreover we can choose them in a compatible way with the "splitting" $\mathcal{G} = \mathbb{V}\left((t_\mathcal{A}/\mathcal{W}_0)^\vee\right) \times \mathcal{A}$ (as we have in fact suggested in the notation of parameters). We develop the section $s \otimes 1$ of $\mathcal{M} \otimes \mathcal{O}_\mathfrak{p}$ in terms of these parameters:

$$s \otimes 1 = \sum \theta_{\mathbf{i},\mathbf{j}} \mathbf{X}^\mathbf{i} \mathbf{Y}^\mathbf{j} \quad \text{with } \theta_{\mathbf{i},\mathbf{j}} \in \mathcal{O}_\mathfrak{p}. \tag{56}$$

And, as s is a polynomial map of degree $\leqslant D_0$ on the first components $X_1, \ldots, X_t$, we have $\theta_{\mathbf{i},\mathbf{j}} = 0$ when $|\mathbf{i}| \geqslant D_0 + 1$. The invariant differential forms $\{dz_i = \varepsilon^* dY_i\}_{1 \leqslant i \leqslant g}$ on the cotangent bundle $\omega^1_{\mathcal{A}|\mathcal{O}_k} \otimes \mathcal{O}_\mathfrak{p}$ infer local coordinates $z_1, \ldots, z_g$ on the Lie algebra $t_\mathcal{A} \otimes \mathcal{O}_\mathfrak{p}$. The relationship between z_i's and Y_i's is given by the *formal logarithm* $z_i = \ell_i(Y_1, \ldots, Y_g) \in k_\mathfrak{p}[[Y_1, \ldots, Y_g]]$ whose property

$$d\ell_i \in \sum_{i=1}^{g} \mathcal{O}_\mathfrak{p}[[Y_1, \ldots, Y_g]]\, dY_i \tag{57}$$

implies some arithmetic informations on the coefficients $a^{(i)}_\mathbf{n}$ of ℓ_i; namely for all $\mathbf{n} = (n_1, \ldots, n_g) \in \mathbb{N}^g$ and $i \in \{1, \ldots, g\}$, there exists an index $j = j(i, \mathbf{n}) \in \{1, \ldots, g\}$ such that $(n_j + 1) a^{(i)}_\mathbf{n} \in \mathcal{O}_\mathfrak{p}$. If we denote by $(\lambda_{l,j}) \in \operatorname{Mat}_{t,g}(\mathcal{O}_\mathfrak{p})$ the matrix of the map $\lambda_\mathfrak{p} : t_{\mathcal{A}_\mathfrak{p}} \to (t_\mathcal{A}/\mathcal{W}_0) \otimes \mathcal{O}_\mathfrak{p}$ (note this last module is torsion free on a local ring and so it is a free module) with respect to coordinates $(z_1, \ldots, z_g)$ and $(X_1, \ldots, X_t)$, the equations defining $\mathcal{W}$ are

$$X_l = \sum_{j=1}^{g} \lambda_{l,j} z_j, \quad l \in \{1, \ldots, t\} \tag{58}$$

and so the ones defining the formal Lie subgroup $\widehat{\exp}_{\mathcal{G}_\mathfrak{p}} \mathcal{W}$ of $\widehat{\mathcal{X}}_\mathfrak{p}$ are

$$X_l = \sum_{j=1}^{g} \lambda_{l,j} \ell_j(\mathbf{Y}) \cdot$$

Since s vanishes up to order ℓ along $\mathcal{W}$ at the unit section, the change of variables $(X_l, z_j) \rightsquigarrow (X_l, Y_j)$ does not modify the ℓ-th jet of s along $\mathcal{W}$ at 0. Thus $\varphi_i(s)$ is the part of degree ℓ of

$$s_{|\widehat{\exp}\mathcal{W}} = \sum \theta_{\mathbf{i},\mathbf{j}} \mathbf{Y}^\mathbf{j} \prod_{l=1}^{t} \left(\sum_{j=1}^{g} \lambda_{l,j} \ell_j(\mathbf{Y}) \right)^{i_l} \cdot \tag{59}$$

We develop this expression using the multinomial formula and we find that $\delta_\ell(D_0)\varphi_i(s)$ belongs to $\mathcal{O}_\mathfrak{p}[Y_1, \ldots, Y_g]_\ell$ and so it belongs to $\mathcal{O}_\mathfrak{p}[z_1, \ldots, z_g]_\ell = S^\ell(\mathcal{W}^\vee) \otimes_{\mathcal{O}_k} \mathcal{O}_\mathfrak{p}$. □

5.6. **General upper bound for archimedean norms of evaluation maps.** In this section, for any fixed embedding $\sigma : k \hookrightarrow \mathbb{C}$, we give an upper bound for the operator norm $\|\varphi_i\|_\sigma$. As we have already underlined, an important feature of inequality (10) is that it is *best possible* in $\log b$ and it is crucial that the factorial of the order of derivation do not appear anywhere else during the proof*. To be on the safe side and in order that there be no doubt on this (critical) point, we describe very carefully morphisms φ_i and associated operator norms $\|.\|_\sigma$, before stating the result. Details we mention here for the estimate of $\|\varphi_i\|_\sigma$ are very close to those of the proof of proposition 2.13 of [26].

Let $\exp_\sigma : t_{\mathbf{G}_\sigma}(\mathbb{C}) \to \mathcal{X}_\sigma$ be the exponential map of the complex Lie group $\mathbf{G}_\sigma(\mathbb{C})$, extended to $\mathcal{X}_\sigma(\mathbb{C})$. Then the vector bundle $\exp^*_\sigma M_\sigma$ over $t_{\mathbf{G}_\sigma}(\mathbb{C})$ is trivial. Let us choose an isomorphism $\nu : \exp^*_\sigma M_\sigma \to \mathcal{O}_{t_{\mathbf{G}_\sigma}}(\mathbb{C})$ such that for any $s \in \mathcal{E}_\sigma(\mathbb{C})$, the holomorphic function $\nu(\exp^*_\sigma s)$ on $t_{\mathbf{G}_\sigma}(\mathbb{C})$ satisfies

$$\forall z = z_0 \oplus \cdots \oplus z_n \in t_{\mathbf{G}_\sigma}(\mathbb{C}) = (t_\mathbf{A}/W_0)_\sigma(\mathbb{C}) \overset{\perp}{\oplus} t_{\mathbf{A}_{1,\sigma}}(\mathbb{C}) \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} t_{\mathbf{A}_{n,\sigma}}(\mathbb{C}) \quad \text{and} \quad x = \exp_\sigma(z),$$

*Besides, that technical difficulty has also been at the center of theorem 3.4 of [10] (see beginning of § 4) and results of P. Graftieaux [26, 27].

$$\|s(x)\|_{\overline{x^*\mathcal{M}},\sigma} = \frac{|\nu(\exp_\sigma^* s)(z)|}{(1+\|z_0\|_\sigma^2)^{D_0/2}} \exp\left\{-\frac{3\pi}{2}\left(D_1\|z_1\|^2_{\overline{L_1},\sigma}+\cdots+D_n\|z_n\|^2_{\overline{L_n},\sigma}\right)\right\}. \tag{60}$$

This equality enables us to work with "authentic" holomorphic functions and all estimates involving σ-norm of $s(x)$ (or s) will be made with $\nu(\exp_\sigma^* s)$.

Fix an integer $i \in \{1, \ldots, \mathcal{N}\}$ and let us consider a non-zero element $s \in E_i \otimes_\sigma \mathbb{C}$. The image of s by φ_i belongs to $S^\ell(W^\vee) \otimes (m\mathbf{p})^* M \otimes_\sigma \mathbb{C}$, which is isomorphic to $\mathrm{Hom}_{\mathbb{C}}\left(S^\ell W_\sigma, (m\mathbf{p})^* M_\sigma\right)$. However this isomorphism is *not* isometric as we saw in lemma 4.1. Thus, with the notation Θ_ℓ of this lemma, we can consider the map $\Theta_\ell\left(\varphi_i(s)\right)_\sigma : S^\ell W_\sigma \to (m\mathbf{p})^* M_\sigma$, $\mathrm{D} \mapsto (\mathrm{D}s)(m\mathbf{p})$ where D is viewed as a derivation (of order ℓ) along W_σ. So

$$\begin{aligned}\|\varphi_i(s)\|_\sigma &\leqslant \|\Theta_\ell\left(\varphi_i(s)\right)\|_\sigma \\ &\leqslant \sup_{\substack{\mathrm{D}\in S^\ell(W_\sigma)\\ \mathrm{D}\neq 0}} \left\{\frac{\|\mathrm{D}s(m\mathbf{p})\|_{\overline{\varepsilon^*_{m\mathbf{p}}\mathcal{M}},\sigma}}{\|\mathrm{D}\|_{\overline{S^\ell(\mathcal{W})},\sigma}}\right\}.\end{aligned}$$

By writing $\mathrm{D} = \sum d_{\mathbf{i}}\, w_1^{i_1} \otimes \cdots \otimes w_g^{i_g}$ with $(w_1, \ldots, w_g)$ an orthonormal basis of W_σ, we have

$$\|\mathrm{D}\|^2_{\overline{S^\ell(\mathcal{W})},\sigma} = \sum_{|\mathbf{i}|=\ell} |d_{\mathbf{i}}|^2 \frac{\mathbf{i}!}{\ell!} \geqslant \left(\sum_{\mathbf{i}} |d_{\mathbf{i}}|\right)^2 \times g^{-\ell}$$

and so

$$\begin{aligned}\|\varphi_i\|_\sigma &\leqslant g^{\ell/2} \max_{|\mathbf{i}|=\ell}\left\{\left\|w_1^{i_1}\otimes\cdots\otimes w_g^{i_g} s(m\mathbf{p})\right\|_{\overline{\varepsilon^*_{m\mathbf{p}}\mathcal{M}},\sigma}\right\} \\ &\leqslant \frac{g^{\ell/2} e^{-\frac{3\pi}{2}m^2(D_1\|\mathbf{u}_1'\|^2_\sigma+\cdots+D_n\|\mathbf{u}_n'\|^2_\sigma)}}{(1+\|m\mathbf{u}_0\|^2_\sigma)^{D_0/2}} \max_{|\mathbf{i}|=\ell}\left\{\left|\frac{1}{\mathbf{i}!}\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\mathbf{i}} \nu(\exp_\sigma^* s)(\mathbf{u}'+z_1w_1+\cdots+z_gw_g)_{|\mathbf{z}=0}\right|\right\}\end{aligned}$$

where $\mathbf{u}' := m\mathbf{u}_0 \oplus \mathbf{u}_1' \oplus \cdots \oplus \mathbf{u}_n'$ and $\mathbf{u}_j'$, $1 \leqslant j \leqslant n$, is a logarithm of $m\sigma(\mathbf{p}_j)$ of norm $\leqslant 2\sqrt{g_j}$ (it is possible by Minkowski theorem). We can bound from above this last term with the Cauchy inequality applied to the holomorphic function $\nu(\exp_\sigma^* s)$. We have

$$\begin{aligned}\max_{|\mathbf{i}|=\ell}\left\{\left|\frac{1}{\mathbf{i}!}\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\mathbf{i}} \nu(\exp_\sigma^* s)(\mathbf{u}'+z_1w_1+\cdots+z_gw_g)_{\mathbf{z}=0}\right|\right\} &\leqslant \frac{1}{r^\ell}\sup_{\substack{x\in W_\sigma\\ \|x\|_\sigma\leqslant r}} \left|\nu(\exp_\sigma^* s)(\mathbf{u}'+x)\right| \\ &\leqslant \frac{1}{r^\ell}\sup_{\substack{x\in t_{\mathbf{G}_\sigma}(\mathbb{C})\\ \|x\|_\sigma\leqslant r}} \left|\nu(\exp_\sigma^* s)(\mathbf{u}'+x)\right|\end{aligned} \tag{61}$$

so, with equality (60), we deduce

$$\|\varphi_i\|_\sigma \leqslant \frac{1}{r^\ell} g^{\ell/2} e^{\frac{3\pi}{2}r(D_1(2\|\mathbf{u}_1'\|_\sigma+r)+\cdots+D_n(2\|\mathbf{u}_n'\|_\sigma+r))}\left(\frac{1+(\|m\mathbf{u}_0\|_\sigma+r)^2}{1+\|m\mathbf{u}_0\|^2_\sigma}\right)^{D_0/2} \times \|s\|_{\infty,\sigma}$$

Let us choose $r = \sqrt{g}$ (that is not the best possible choice but it involves the disappearance of ℓ from the upper bound). Then we get

$$\|\varphi_i\|_\sigma \leqslant (3g)^{D_0/2}\, e^{\frac{9\pi}{2}ng\max\{D_1,\ldots,D_n\}} \times \max_{\substack{s\in\mathcal{E}_\sigma\\ s\neq 0}}\left\{\frac{\|s\|_{\infty,\sigma}}{\|s\|_{\overline{\mathcal{E}},\sigma}}\right\}$$

(we bounded $\frac{1+(\|m\mathbf{u}_0\|_\sigma+r)^2}{1+\|m\mathbf{u}_0\|^2_\sigma} \leqslant 1+r+r^2 \leqslant 3g$) and from choice of parameters, we obtain the following general estimate.

Proposition 5.9. *For all archimedean places $\sigma : k \hookrightarrow \mathbb{C}$ and all integers $i \in \{1, \ldots, \mathcal{N}\}$, we have*

$$\|\varphi_i\|_\sigma \leqslant e^{U_0/(3D)} \times \max_{\substack{s\in\mathcal{E}_\sigma\\ s\neq 0}}\left\{\frac{\|s\|_{\infty,\sigma}}{\|s\|_{\overline{\mathcal{E}},\sigma}}\right\}. \tag{62}$$

5.7. **Precise upper bound for some evaluation maps.** In the previous section, we did not use all the information concerning properties of element s of E_i. As a matter of fact, the membership of s to E_i has been only convenient to define $\varphi_i(s)$, but the vanishing of first jets (up to ℓ) of s along W at (only) $m\mathbf{p}$ would have been sufficient to define $\varphi_i(s)$. Here, we are going to be more careful, using a so-called *interpolation lemma.* By and large, all the "essence" of Baker's method holds in estimates of this paragraph.

In all this section, the archimedean embedding considered is $\sigma_0 : k \hookrightarrow \mathbb{C}$. We keep notation of the last section. Fix an integer $i \in \{1, \ldots, \mathcal{N}\}$, a non zero element s of E_{i,σ_0} and denote by ϑ the entire function $\nu(\exp^*_{\sigma_0} s)$ defined over $t_{\mathbf{G}_{\sigma_0}}(\mathbb{C})$. We also consider an orthonormal basis $\mathrm{w} := (w_1, \ldots, w_g)$ of W_{σ_0}. So, knowing

$$\vartheta(m'\mathbf{u} + z_1 w_1 + \cdots + z_g w_g) \in (z_1, \ldots, z_g)^{2(g+t)T+1} \quad \text{for} \quad m' \in \{0, \ldots, S_0\},$$

we want to give a (precise) upper bound for derivatives of ϑ along W at order ℓ in $m\mathbf{p}$. Let $\mathbf{w}$ be a vector of W_{σ_0} such that $\|\mathbf{u} - \mathbf{w}\|_{\sigma_0} = \mathrm{d}(\mathbf{u}, W)$. We prove theorem 2.1 by contradiction and, from now on, we *assume*

$$\log \mathrm{d}(\mathbf{u}, W) \leqslant -128 \left(1 + \frac{\log(1 + \|\mathbf{u}\|_{\sigma_0})}{\mathfrak{a} \log \mathfrak{e}}\right) U_0 \,. \tag{63}$$

Let us stress that, strictly speaking, this hypothesis is *not* necessary to continue the demonstration but it is a convenient way to present it.

First of all, we shall need the following comparison lemma.

Lemma 5.10. *For any $\mathbf{t} \in \mathbb{N}^g$ of length $\leqslant 2(g+t)T$ and any $m' \in \{0, \ldots, (g+t)S\}$, we have*

$$\log \left| \frac{1}{\mathbf{t}!} \mathrm{D}^{\mathbf{t}}_{\mathrm{w}} \vartheta(m'\mathbf{u}) - \frac{1}{\mathbf{t}!} \mathrm{D}^{\mathbf{t}}_{\mathrm{w}} \vartheta(m'\mathbf{w}) \right| \leqslant -64 \left(1 + \frac{\log(1 + \|\mathbf{u}\|_{\sigma_0})}{\mathfrak{a} \log \mathfrak{e}}\right) U_0 \,. \tag{64}$$

Proof. It relies on the mean value theorem applied to the real variable function $[0, 1] \to \mathbb{C}$, $x \mapsto \frac{1}{\mathbf{t}!}(\mathrm{D}^{\mathbf{t}}_{\mathrm{w}}\vartheta)(m'\mathbf{u} + xm'(\mathbf{w} - \mathbf{u}))$. We use also hypothesis (63) (in particular $2(g+t)S\|\mathbf{u} - \mathbf{w}\|_{\sigma_0} \leqslant 1$) and the Cauchy upper bound (61) with $r = 1$. □

Let $\mathbf{t} \in \mathbb{N}^g$ be of length ℓ (this integer is the one corresponding to i, see the Fact p. 15). In particular, we have $|\mathbf{t}| \leqslant (g+t)T$. Let f be the analytic function of one complex variable defined by

$$\mathrm{f}(z) := \frac{1}{\mathbf{t}!} \left(\mathrm{D}^{\mathbf{t}}_{\mathrm{w}} \vartheta\right)(z\mathbf{w}) \,. \tag{65}$$

Let us write $\mathbf{w} = x_1 w_1 + \cdots + x_g w_g$. For any integer $l \geqslant 0$, derivation formula of composed maps gives

$$\frac{1}{l!}\mathrm{f}^{(l)}(z) = \sum_{|\mathbf{j}|=l} \binom{\mathbf{t}+\mathbf{j}}{\mathbf{j}} x_1^{j_1} \cdots x_g^{j_g} \frac{\mathrm{D}^{\mathbf{t}+\mathbf{j}}_{\mathrm{w}} \vartheta}{(\mathbf{t}+\mathbf{j})!}(z\mathbf{w})$$

from which we deduce the upper bound

$$\max_{\substack{0 \leqslant h \leqslant S_0 \\ 0 \leqslant l \leqslant (g+t)T}} \left\{ \frac{1}{l!} \left| \mathrm{f}^{(l)}(h) \right| \right\} \leqslant (g + |x_1| + \cdots + |x_g|)^{(g+t)T} \times \max_{\substack{0 \leqslant h \leqslant S_0 \\ |\mathbf{j}| \leqslant 2(g+t)T}} \left\{ \frac{1}{\mathbf{j}!} \left| \mathrm{D}^{\mathbf{j}}_{\mathrm{w}} \vartheta(h\mathbf{w}) \right| \right\} \,. \tag{66}$$

On one hand, the real number $|x_1| + \cdots + |x_g|$ is bounded by $\sqrt{g}\|\mathbf{w}\|_{\sigma_0}$, and then by $1 + \sqrt{g}\|\mathbf{u}\|_{\sigma_0}$. On the other hand, for $0 \leqslant h \leqslant S_0$ and $|\mathbf{j}| \leqslant 2(g+t)T$, derivatives $\mathrm{D}^{\mathbf{j}}_{\mathrm{w}}\vartheta(h\mathbf{u})$ vanish and from lemma 5.10 we get the upper bound

$$\log \max_{\substack{0 \leqslant h \leqslant S_0 \\ 0 \leqslant l \leqslant (g+t)T}} \left\{ \frac{1}{l!} |\mathrm{f}^{(l)}(h)| \right\} \leqslant -32 \left(1 + \frac{\log(1 + \|\mathbf{u}\|_{\sigma_0})}{\mathfrak{a} \log \mathfrak{e}}\right) U_0 \,. \tag{67}$$

Remark 5.11. It is only at this place that $\log(1 + \|\mathbf{u}\|_{\sigma_0})$ shows itself, which has repercussions on measure (10) (despite its very discreet appearance). However, if we take $\mathbf{G}_1 = \{0\}$ (as it has been usual) then $\delta_1 = 1$ and $\mathfrak{a} \log \mathfrak{e} \geqslant \log^+ \|\mathbf{u}\|_{\sigma_0}$ and so, in that particular case, it is possible to remove the term $\left(1 + \frac{\log^+ \|\mathbf{u}\|_{\sigma_0}}{\mathfrak{a} \log \mathfrak{e}}\right)$ in (10).

Here is a last technical lemma before the extrapolation process.

Lemma 5.12. *For any integer $m \in \{0, \ldots, (g+t)S\}$, we have*

$$\sup_{\substack{z \in \mathbb{C} \\ |z| \leqslant 4m\mathfrak{e}}} \{|\mathrm{f}(z)|\} \leqslant e^{U_0} \times \|s\|_{\infty, \sigma_0} \,. \tag{68}$$

Proof. It is an immediate consequence of Cauchy inequality

$$|f(z)| \leqslant 2^{|\mathbf{t}|} \sup_{\substack{a \in W_{\sigma_0} \\ \|a\| \leqslant 1/2}} \{|\vartheta(z\mathbf{w} + a)|\}$$

and of inequality $8(g+t)S\mathfrak{e}\|\mathbf{u} - \mathbf{w}\|_{\sigma_0} \leqslant 1$ deducible from (63). □

The precise estimate for $\|\varphi_i\|_{\sigma_0}$ mentioned in the title of this section relies on the following "approached Schwarz lemma".

Lemma 5.13 (Lemma 2 of [13]). *Let f be an holomorphic function on the disc centered at 0 and of radius R. Let S_1 be an integer $\geqslant 2$. We suppose $R \geqslant 2S_1$ and consider a real number $r \in [S_1, R/2]$. Let also T_1 be a positive integer. Then*

$$\sup_{|z|\leqslant 2r} |f(z)| \leqslant 2\left(\frac{4r}{R}\right)^{T_1 S_1} \times \sup_{|z|\leqslant R} |f(z)| + 5\left(\frac{18r}{S_1}\right)^{T_1 S_1} \times \max_{\substack{0\leqslant h\leqslant S_1\\ 0\leqslant l\leqslant T_1}} \left\{\frac{1}{l!}\left|f^{(l)}(h)\right|\right\} \cdot \tag{69}$$

We can now state the main result of this section, which is at the heart of the extrapolation process in Baker's method.

Proposition 5.14. *For $i \in \{i_0, \ldots, \mathcal{N}\}$, we have*

$$\log \|\varphi_i\|_{\sigma_0} \leqslant -8\left(1 + \frac{\log(1+\|\mathbf{u}\|_{\sigma_0})}{\mathfrak{a}\log\mathfrak{e}}\right) U_0 + \log \max_{\substack{s\in\mathcal{E}_{\sigma_0}\\ s\neq 0}} \left\{\frac{\|s\|_{\infty,\sigma_0}}{\|s\|_{\overline{\mathcal{E}},\sigma_0}}\right\} \tag{70}$$

Proof. After the preparatory phase which came before the statement of the proposition, the demonstration of this proposition is a straightforward application of lemma 5.13. We use it with $R := 4m\mathfrak{e}$ (note that $m > S_0$ because of the choice of i), $T_1 := (g+t)T$, $S_1 := S_0$, $r := m$ and f defined by (65). In this way, we achieve an upper bound for $\frac{1}{\mathbf{t}!}|\mathrm{D}^{\mathbf{t}}_{\mathrm{w}}\vartheta(m\mathbf{w})|$ and a second use of lemma 5.10 enables us to bound $\frac{1}{\mathbf{t}!}|\mathrm{D}^{\mathbf{t}}_{\mathrm{w}}\vartheta(m\mathbf{u})|$. Proposition 5.14 immediately follows. □

5.8. **End of the proof.** We are synthesizing the series of results we got in the following statement.

Proposition 5.15. *For all $i \in \{0, \ldots, \mathcal{N}\}$, we have*

$$\widehat{\mu}_{\max}\left(\overline{\mathcal{G}}_i\right) + h(\varphi_i) \leqslant \frac{U_0}{D} + \Xi\left[(\mathcal{X}, \overline{\mathcal{M}})\right] \cdot \tag{71}$$

For all $i \in \{i_0, \ldots, \mathcal{N}\}$, we have

$$\widehat{\mu}_{\max}\left(\overline{\mathcal{G}}_i\right) + h(\varphi_i) \leqslant -4\left(1 + \frac{\log(1+\|\mathbf{u}\|_{\sigma_0})}{\mathfrak{a}\log\mathfrak{e}}\right)\frac{U_0}{D} + \Xi\left[(\mathcal{X}, \overline{\mathcal{M}})\right] \tag{72}$$

Proof. Let $i \in \{0, \ldots, \mathcal{N}\}$. From propositions 5.6, 5.8 and 5.9, we deduce

$$\widehat{\mu}_{\max}\left(\overline{\mathcal{G}}_i\right) + h(\varphi_i) \leqslant \frac{U_0}{3D} + \left(\frac{U_0}{3D} + \Xi\left[(\mathcal{X}, \overline{\mathcal{M}})\right]\right) + \log\delta_\ell(D_0) \cdot$$

Then (71) follows from the estimate of S. Bruiltet for $\delta_\ell(D_0)$ (p. 17) as well as from inequality (iv) of lemma 5.1. As for the second inequality, we use also proposition 5.14 (true for $i \geqslant i_0$). □

As φ is an injective map (lemma 5.5), we can apply the general slope inequality (proposition 4.6 of [10]):

$$\widehat{\deg}_{\mathrm{n}}\overline{\mathcal{E}} \leqslant \sum_{i=0}^{\mathcal{N}} (\operatorname{rk}\mathcal{E}_i - \operatorname{rk}\mathcal{E}_{i+1})\left(\widehat{\mu}_{\max}\left(\overline{\mathcal{G}}_i\right) + h(\varphi_i)\right), \tag{73}$$

which becomes after an obvious simplification:

$$\widehat{\mu}\left(\overline{\mathcal{E}}\right) \leqslant \left(1 - \frac{\operatorname{rk}\mathcal{E}_{i_0}}{\operatorname{rk}\mathcal{E}}\right) \max_{0\leqslant i\leqslant i_0-1}\left\{\widehat{\mu}_{\max}\left(\overline{\mathcal{G}}_i\right) + h(\varphi_i)\right\} + \frac{\operatorname{rk}\mathcal{E}_{i_0}}{\operatorname{rk}\mathcal{E}} \max_{i_0\leqslant i\leqslant\mathcal{N}}\left\{\widehat{\mu}_{\max}\left(\overline{\mathcal{G}}_i\right) + h(\varphi_i)\right\} \cdot$$

As $\operatorname{rk}\mathcal{E}_{i_0} \geqslant \frac{\operatorname{rk}\mathcal{E}}{2}$ (lemma 5.7) and by using proposition 5.15, we obtain

$$\widehat{\mu}\left(\overline{\mathcal{E}}\right) - \Xi\left[(\mathcal{X}, \overline{\mathcal{M}})\right] \leqslant \frac{U_0}{D} - 2\left(1 + \frac{\log(1+\|\mathbf{u}\|_{\sigma_0})}{\mathfrak{a}\log\mathfrak{e}}\right)\frac{U_0}{D}$$

and this is a contradiction with lower bound (42) (p. 14). So the assumption (63) is false and we achieve the lower bound (10) for the distance between $\mathbf{u}$ and $W_{\mathbb{C}}$ by replacing S_0 by its value in the definition of U_0.

References

[1] M. Ably. Formes linéaires de logarithmes de points algébriques sur une courbe elliptique de type CM. *Ann. Inst. Fourier (Grenoble)*, 50(1):1–33, 2000.

[2] A. Baker and G. Wüstholz. Logarithmic forms and group varieties. *J. reine angew. Math*, 442:19–62, 1993.

[3] D. Bertrand. Transcendental methods in arithmetic geometry. In *Analytic Number Theory (Tokyo 1988)*, volume 1434 of *Lecture Notes in Maths.*, pages 31–44. Springer, 1990.

[4] C. Birkenhake and H. Lange. *Complex abelian varieties*, volume 302 of *Grundlehren Der Mathematischen Wissenschaften*. Springer-Verlag Berlin, 1992.

[5] J.-B. Bost. Semi-stability and heights of cycles. *Invent. Math.*, 118(2):223–253, 1994.

[6] J.-B. Bost. Arakelov geometry of abelian varieties. In *Conference on Arithmetical Geometry*, volume 96-51, Max Planck Institut für Mathematik Bonn, mars 1996. Manuscript notes.

[7] J.-B. Bost. Intrinsic heights of stable varieties. Application to abelian varieties. *Duke Math. J.*, 82:21–70, 1996.

[8] J.-B. Bost. Périodes et isogénies des variétés abéliennes sur les corps de nombres (d'après D. Masser et G. Wüstholz). In *Séminaire Bourbaki*, volume 237 of *Astérisque*, pages 115–161. Société Mathématique de France, 1996.

[9] J.-B. Bost. DEA lectures at the Institut Henri Poincaré (Paris), 1997 and 1999. Manuscript notes.

[10] J.-B. Bost. Algebraic leaves of algebraic foliations over number fields. *Publications Mathématiques de l'IHÉS*, 93:161–221, 2001.

[11] J.-B. Bost, C. Soulé, and H. Gillet. Heights of projective varieties and positive Green forms. *J. Amer. Math. Soc.*, 7(4):903–1027, 1994.

[12] S. Bruiltet. D'une mesure d'approximation simultanée à une mesure d'irrationalité : le cas de $\Gamma(1/4)$ et $\Gamma(1/3)$. *Acta Arith.*, 104(3):243–281, 2002.

[13] P. L. Cijsouw and M. Waldschmidt. Linear forms and simultaneous approximations. *Compos. Math.*, 34:173–197, 1977.

[14] S. David. Minorations de hauteurs sur les variétés abéliennes. *Bull. Soc. Math. France*, 121:509–544, 1993.

[15] S. David. *Minorations de formes linéaires de logarithmes elliptiques*, volume 62 of *Mémoire de la Société Mathématique de France*. S. M. F., 1995.

[16] S. David and M. Hindry. Minoration de la hauteur de Néron-Tate sur les variétés abéliennes de type C. M. *J. reine angew. Math.*, 529:1–74, 2000.

[17] S. David and N. Hirata-Kohno. Linear forms in elliptic logarithms. *Preprint in preparation*, 2002. A weaker version has been presented in [18].

[18] S. David and N. Hirata-Kohno. Recent progress on linear forms in elliptic logarithms. In *A Panorama in Number Theory*. Cambridge Univ. Press, September 2002. Ed. by G. Wüstholz.

[19] N.I. Fel'dman and Yu.V. Nesterenko. *Number Theory IV. Transcendental numbers*, volume 44 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag Berlin, 1998. A.N. Parshin, I.R. Shafarevich (Eds).

[20] É. Gaudron. *Mesure d'indépendance linéaire de logarithmes dans un groupe algébrique commutatif.* Thèse de doctorat, Université Jean Monnet de Saint-Étienne, décembre 2001. http://theses-EN-ligne.in2p3.fr/documents/archives0/00/00/11/65/index_fr.html.

[21] É. Gaudron. Mesure d'indépendance linéaire de logarithmes dans un groupe algébrique commutatif. *C. R. Acad. Sci*, I 333:1059–1064, 2001.

[22] É. Gaudron. Mesures d'indépendance linéaire de logarithmes dans un groupe algébrique commutatif. Preprint, February 2002.

[23] J. Gebel, A. Pethö, and H.G. Zimmer. Computing integral points on elliptic curves. *Acta. Arith.*, 68:171–192, 1994.

[24] H. Gillet and C. Soulé. An Arithmetic Riemann-Roch Theorem. *Invent. Math.*, 110:473–543, 1992.

[25] P. Graftieaux. *Groupes formels et critères d'isogénie.* Thèse de Doctorat, Université Paris VI, mars 1998.

[26] P. Graftieaux. Formal Groups and Isogeny Theorem. *Duke Math. J.*, 106:81–121, 2001.

[27] P. Graftieaux. Formal subgroups of abelian varieties. *Invent. Math.*, 145:1–17, 2001.

[28] N. Hirata-Kohno. Formes linéaires de logarithmes de points algébriques sur les groupes algébriques. *Invent. Math.*, 104:401–433, 1991.

[29] N. Hirata-Kohno. Approximations simultanées sur les groupes algébriques commutatifs. *Compos. Math.*, 86:69–96, 1993.

[30] N. Hirata-Kohno. Une relation entre les points entiers sur une courbe algébrique et les points rationnels de la jacobienne. In *Advances in number theory (Kingston, ON, 1991)*, pages 421–433. Oxford Univ. Press, New York, 1993.

[31] S. Lang. Diophantine approximation on abelian varieties with complex multiplications. *Advances in Math.*, 17(3):281–336, 1975.

[32] D. Masser. *Elliptic functions and transcendence.*, volume 437 of *Lecture Notes in Maths.* Springer-Verlag, 1975.

[33] D. Masser. Linear forms in algebraic points of abelian functions. I.(II.). *Math. Proc. Cambridge Philos. Soc.*, 77 (79):499–513 (55–70), 1975 (1976).

[34] D. Masser and G. Wüstholz. Periods and minimal abelian subvarieties. *Annals of Mathematics*, 137:407–458, 1993.

[35] E.M. Matveev. An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers I. II. *Izv. Ross. Akad. NaukSer. Math*, 62,(*resp.* 64)(4,(*resp.* 6)):81–136,(*resp.* 125–180), 1998,(*resp.* 2000).

[36] L. Moret-Bailly. *Pinceaux de variétés abéliennes*, volume 129 of *Astérisque*. Société Mathématique de France, 1985.

[37] L. Moret-Bailly. Sur l'équation fonctionnelle de la fonction thêta de Riemann. *Compositio Math.*, 75(2):203–217, 1990.

[38] F. Pellarin. Sur la distance d'un point algébrique à l'origine dans les variétés abéliennes. *J. Number Theory*, 88:241–262, 2001.

[39] P. Philippon. Lemme de zéros dans les groupes algébriques commutatifs. *Bull. Soc. Math. France*, 114:355–383, 1986. Errata et Addenda, id.115 (1987).

[40] P. Philippon and M. Waldschmidt. Formes linéaires de logarithmes sur les groupes algébriques commutatifs. *Illinois J. Math.*, 32(2):281–314, 1988.

[41] P. Philippon and M. Waldschmidt. Formes linéaires de logarithmes simultanées sur les groupes algébriques commutatifs. In *Séminaire de Théorie des Nombres Paris* 1986 – 87, volume 75 of *Progress in Mathematics*, pages 313–347. Birkhäuser Boston, Inc., 1989. Édité par Catherine Goldstein.

[42] H. Randriambololona. *Hauteurs de sous-schémas, géométrie d'Arakelov des schémas de Hilbert et exemples d'utilisation de méthodes arakeloviennes en théorie de l'approximation diophantienne.* Thèse de doctorat, Université Paris XI, Janvier 2002.

[43] R.J. Stroeker and N. Tzanakis. Solving elliptic diophantine equations by estimating linear forms in elliptic logarithms. *Acta. Arith.*, 67:177–196, 1994.

[44] R.J. Stroeker and N. Tzanakis. Computing all integer solutions of a general elliptic equation. In *ANTS-IV*, volume 1838 of *Lecture Notes in Computer Science*, pages 551–561. Springer-Verlag, 2000. W. Bosma (Ed).

[45] E. Viada-Aehle. *Elliptic Isogenies and Slopes.* Dissertation, Swiss Federal Institute of technology Zurich, 2001. Number 14195.

[46] M. Waldschmidt. *Diophantine Approximation On Linear Algebraic Groups*, volume 326 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, 2000.

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