Plane curves and their fundamental groups: Generalizations of Uludag's Construction

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Abstract

Dans ce travail, nous utilisons la méthode d'Uludag pour construire de nouvelles courbes dont le groupe fondamental est extension centrale de celui de la courbe initiale par un groupe fini cyclique.

Dans la première partie, nous généralisons la méthode d'Uludag pour obtenir de nouvelles familles de courbes dont le groupe fondamental est controlé. Dans la deuxième partie, nous examinons les propriétés de ces groupes préservées par cette construction. Nous décrivons enfin précisément les familles de courbes obtenues par cette construction appliquée à divers types de courbes planes.

Keywords: fundamental groups, plane curves, Zariski pairs, Hirzebruch surfaces, central extension.

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1 Introduction

The fundamental group of complements of plane curves is a very important topological invariant with many different applications. This invariant was

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used by Chisini [5], Kulikov [11] and Kulikov-Teicher [12] in order to distinguish between connected components of the moduli space of surfaces of general type. Moreover, Zariski-Lefschetz hyperplane section theorem (see [13]) showed that

$$\pi_1(\mathbb{P}^N - S) \cong \pi_1(H - H \cap S)$$

where S is an hypersurface and H is a 2-plane. Since $H \cap S$ is a plane curve, this invariant can be used also for computing the fundamental group of complements of hypersurfaces in $\mathbb{P}^N$.

A different direction for the need of fundamental groups' computations is for getting more examples of Zariski pairs ([32],[33]). A pair of plane curves is called *a Zariski pair* if they have the same singularities, but their complements have non-isomorphic fundamental groups. Several families of Zariski pairs were presented by Artal-Bartolo [1],[2], Degtyarev [6], Oka [15] and Shimada [22],[23],[24]. Tokunaga and his coauthors thoroughly investigated Zariski pairs of curves of degree 6 (see [3], [4], [25], [26] and [27]). Some candidates for Zariski pairs can also be found in [10], where any pair of arrangements with the same signature but with different lattices can serve as a candidate for a Zariski pair (it is only needed to be checked that the pair of arrangements have non-isomorphic fundamental groups).

It is also interesting to explore new finite non-abelian groups which are serving as fundamental groups of complements of plane curves.

Uludag ([28],[29]) presents a way to obtain new curves whose fundamental groups are central extensions of the fundamental group of a given curve. Using his method, one can produce a family of examples of Zariski pairs from a given Zariski pair (see also Section 6 here). His main result is:

Theorem 1.1 (Uludag). *Let C be a plane projective curve and $G = \pi_1(\mathbb{P}^2 - C)$. Then for any $n \in \mathbb{N}$, there is a curve $\tilde{C} \subset \mathbb{P}^2$ birational to C such that $\tilde{G} = \pi_1(\mathbb{P}^2 - \tilde{C})$ is a central extension of G by $\mathbb{Z}/(n+1)\mathbb{Z}$:*

$$1 \to \mathbb{Z}/(n+1)\mathbb{Z} \to \tilde{G} \to G \to 1$$

In particular, if C is irreducible so is $\tilde{C}$ as well.

A natural question is which curves and fundamental groups can be obtained by this method. Also, one might ask if this method can be generalized, and what will be the effect of the general method on the relation between the fundamental groups of the original curve and the resulting curve.

In this paper we first generalize Uludag's method to get new families of curves whose fundamental groups are controlled by the original curve in the same manner. Precisely, instead of using only two fibers for performing the elementary transformations between Hirzebruch surfaces F_n, we allow any finite number of different fibers. Afterwards, we list properties of groups which are preserved by the methods. Also, we describe the curves obtained by the application of these methods to several type of plane curves. Then we present some infinite families of new Zariski pairs which can be obtained by the application of these methods. In the last part of the paper, we suggest a different construction based on Uludag's method. In this case, the obtained group is an extension of the fundamental group of the original curve by a finite cyclic group. Since the extension is not central, we lose some of the group properties preserved by the previous constructions.

Between the interesting results in this paper is the exploring of families of curves with deep singularities which yet have cyclic groups as the fundamental group of the complement (see the beginning of Section 4). Also, using the general construction, one can construct more curves with finite and non-abelian fundamental groups and more Zariski pairs (see Section 6).

The paper is organized as follows. In Section 2 we present Uludag's original construction. In Section 3, we present some generalizations of Uludag's construction, and we prove that also in the general constructions, the obtained curve has a fundamental group which is a central extension of the original curve's fundamental group by a finite cyclic group. Section 4 deals with properties of groups which are preserved while applying the constructions. In Section 5, we describe precisely the families of plane curves which can be obtained by the general constructions and when we calculate the degrees of the new curves. At the end of this section, we describe some specific families of curves obtained by applying the constructions to several types of plane curves. In Section 6 we discuss and present new examples of Zariski pairs obtained by these constructions. Section 7 deals with a different generalization for Uludag's construction in which the obtained curve has a fundamental group which is only an extension of the original curve's fundamental group by a finite cyclic group.

2 Uludag's method

The idea of the method is the following (it was partially introduced by Degtyarev [6], and the sequel was developed by Uludag [28],[29]). If a curve C_2 is obtained from a curve C_1 by means of a Cremona transformation $\psi : \mathbb{P}^2 \to \mathbb{P}^2$, then ψ induces an isomorphism

$$\mathbb{P}^2 - (C_2 \cup A) \cong \mathbb{P}^2 - (C_1 \cup B)$$

where A and B are certain line arrangements. Hence there is an induced isomorphism between their fundamental groups. The fundamental groups of the curves themselves are easy to compute by adding the relations which correspond to the arrangements.

Now, if we start with a curve C_1 whose fundamental group is known, one can find a curve C_2 whose fundamental group has not yet been known as being a fundamental group of a plane curve.

For these Cremona transformations one can use Hirzebruch surfaces F_n. In principle, the Hirzebruch surfaces are $\mathbb{P}^1$-bundles over $\mathbb{P}^1$. It is known that two Hirzebruch surfaces can be distinguished by the self-intersection of the exceptional section (for F_n the self-intersection of its exceptional section is $-n$).

There are two types of elementary transformations, one transforms F_n to F_{n+1} for all positive n, and the other transforms F_n to F_{n-1} for all positive n. The first transformation blows up a point O on the exceptional section, and then blows down the proper transform of the fiber passed through O. The second transformation blows up a point Q on one of the fibers F, outside the exceptional section, and then blows down the proper transform of the fiber F. The two transformations are schematically presented in Figures 1 and 2.

Uludag used a special type of Cremona transformations which can be described as follows. We start with a curve C in $\mathbb{P}^2$, and an additional line Q which intersects the curve transversally. We choose another line P which also intersects the curve transversally, and meets Q outside the curve C. Then we blow up the intersection point of the two lines. This yields the Hirzebruch surface F_1. Then we apply n elementary transformations of the first type each time on the same fiber (which is the image of the line Q) to reach F_{n+1}. Then we apply n elementary transformations of the second type each time on the same fiber (which is the image of the line P) to return back to F_1. Then we blow down the exceptional section (whose self-intersection

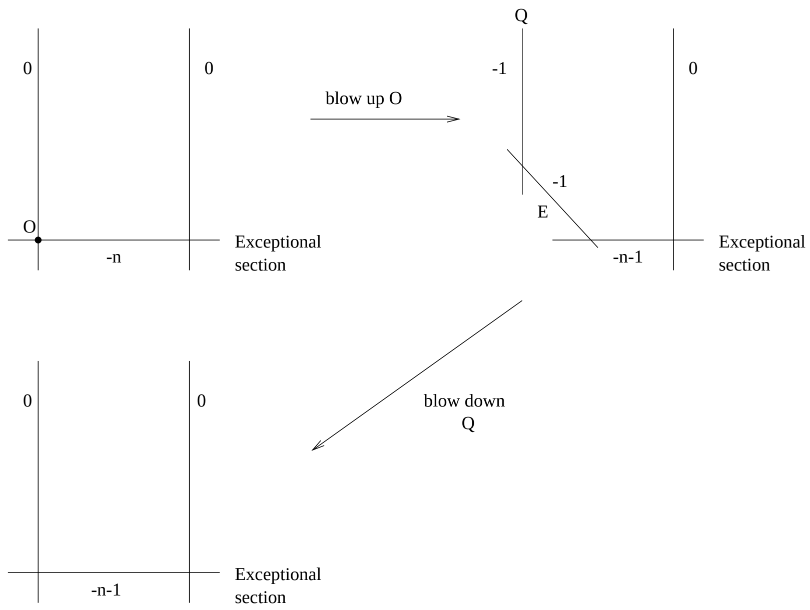

Figure 1: Elementary transformation from F_n to F_{n+1}

is now -1), and we get again $\mathbb{P}^2$. This process defines a family of Cremona transformations from $\mathbb{P}^2$ to $\mathbb{P}^2$.

For each n, we get a different Cremona transformation. Uludag has shown in [28] that applying to a curve C a Cremona transformation whose process reaches F_{n+1}, yields a new curve $\tilde{C}$ such that its fundamental group $\pi_1(\mathbb{P}^2 - \tilde{C})$ is a central extension of $\pi_1(\mathbb{P}^2 - C)$ by a cyclic group of order $n+1$.

3 Generalizations of Uludag's method

In this section we present some generalizations for Uludag's method. These generalizations yield new ways to construct curves with deep singularities whose fundamental groups can be controlled, though they produce no more new groups as fundamental groups than the original method of Uludag.

In the first step, we generalize Uludag's method in the simplest way: instead of using the same fiber all the time to perform the elementary transformations of the first type, we will use two different fibers for performing them, n times with the first fiber and m times with the second one (Subsection 3.3). Afterwards, we will generalize the method to an arbitrary finite

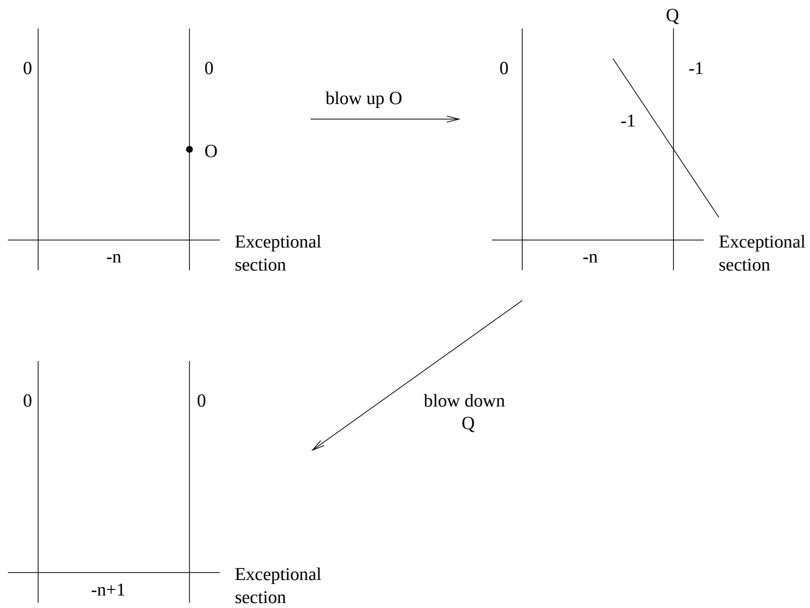

Figure 2: Elementary transformation from F_n to F_{n-1}

number of different fibers (Subsection 3.4). In Subsection 3.5, we generalize our construction even more, and we allow also an arbitrary finite number of fibers for performing the elementary transformations of the second type. In Subsection 3.6 we discuss a particularly interesting special case, where we perform all the transformations (of both types) on the same fiber.

Before we pass to the generalizations of the methods and their proofs, we first have to introduce *meridians* and prove a lemma which we need in the sequel (Subsection 3.1). Also, we have to understand what happens to the fundamental group when we glue one line back to $\mathbb{P}^2$ (Subsection 3.2).

3.1 Meridians and a generalization of Fujita's lemma

As in Uludag's proof [28], in order to find the relations induced by the additional lines, we have to calculate the *meridians* of these lines. We first recall the definition of a meridian of a curve C at a point p (see [28],[29]): Let Δ be a smooth analytical branch meeting C transversally at p and let $x_0 \in \mathbb{P}^2 - C$ be a base point. Take a path ω joining x_0 to a boundary point of Δ, and define *the meridian of C at p* to be the loop $\mu_p = \omega \cdot \delta \cdot \omega^{-1}$, where δ is the boundary of Δ, oriented in the positive sense (see Figure 3).

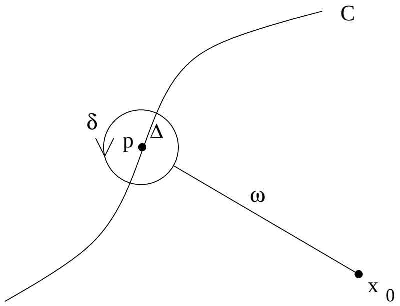

Figure 3: A meridian of a curve C at a point p

For computing the meridians in our case, one has to use some rules. The first rule deals with the connection between the meridian of the curve C at a point $p \in C$ before the blow up and the meridian of the exceptional section created by the blow up (see [28, p. 5]):

Claim 3.1. *Let $\sigma_p : X \to \mathbb{P}^2$ be a blow up of the point $p \in C$, and let $E \subset X$ be the exceptional section. Let $C' = \sigma_p^{-1}(C)$. Then, the loop $\sigma_p^{-1}(\mu_p)$ is the meridian of E at a point $q \in E - C'$.*

The second rule deals with the meridian at a nodal point:

Lemma 3.2 (Fujita [8, p. 540, Lemma 7.17]). *Let B be a ball centered at the origin O of $\mathbb{C}^2$, and consider the curve C defined by $x^2 - y^2 = 0$. C has an ordinary double point at the origin and $\pi_1(B - C) = \mathbb{Z}^2$. Take meridians α and β of C on the branches $x = y$ and $x = -y$ respectively. Then $\alpha\beta$ is a meridian of C at the node O (see Figure 4).*

For our generalizations, we need the following more general version of this lemma:

Lemma 3.3. *Let B be a ball centered at the origin O of $\mathbb{C}^2$, and consider the curve C defined by $\prod_{i=1}^{k}(y - m_i x) = 0$ where the m_i are some complex numbers and $k \geq 3$. C has an intersection of k lines at the origin and $\pi_1(B - C) = \mathbb{Z} \oplus \mathbb{F}_{k-1}$, where $\mathbb{F}_{k-1}$ is the free group on $k-1$ generators. Take meridians α_i of C on the branches $y = m_i x$ respectively. Then $\alpha_1 \alpha_2 \cdots \alpha_k$ is a meridian of C at the intersection point O up to orientation.*

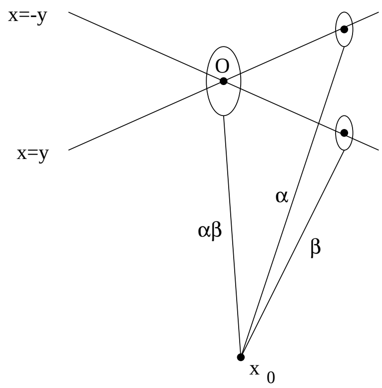

Figure 4: The situation of Fujita's lemma

Proof. The fact that $\pi_1(B-C) = \mathbb{Z} \oplus \mathbb{F}_{k-1}$ is from [18] or [9]: Using van Kampen's method [30], the presentation of the group $\pi_1(B-C)$ is:

$$\langle \alpha_1, \alpha_2, \cdots, \alpha_k \mid \alpha_1\alpha_2 \cdots \alpha_k = \alpha_2 \cdots \alpha_k\alpha_1 = \cdots = \alpha_k\alpha_1 \cdots \alpha_{k-1} \rangle$$

which can be written also as:

$$\langle \alpha, \alpha_2, \cdots, \alpha_k \mid \alpha\alpha_2 = \alpha_2\alpha, \alpha\alpha_3 = \alpha_3\alpha, \cdots, \alpha\alpha_k = \alpha_k\alpha \rangle$$

where $\alpha = \alpha_1\alpha_2 \cdots \alpha_k$ (see [9]). This presentation can be written also as:

$$\langle \alpha \rangle \oplus \langle \alpha_2, \cdots, \alpha_k \rangle \cong \mathbb{Z} \oplus \mathbb{F}_{k-1}$$

and hence the generator of the cyclic group (which is also the center of the group) is $\alpha = \alpha_1\alpha_2 \cdots \alpha_k$.

In order to prove the lemma, we have to show that this generator is indeed the meridian of the exceptional section E after we blow up the point O. When we blow the point O up, we get Hirzebruch surface $F_1 \to \mathbb{P}^1$. After deleting the exceptional section and another disjoint section (which corresponds to the line at infinity), we also throw away k fibers corresponding to the k lines which pass through O before the blow up. The resulting **affine surface** can be decomposed as a product: $(\mathbb{C} - \{\text{pt}\}) \times (\mathbb{P}^1 - \{\text{k points}\})$. Hence, its fundamental group can be decomposed into a direct sum too:

$$\pi_1(\mathbb{C} - \{\text{pt}\}) \oplus \pi_1(\mathbb{P}^1 - \{\text{k points}\}) \cong \mathbb{Z} \oplus \mathbb{F}_{k-1}.$$

Now, since the cyclic group $\mathbb{Z}$ is in the center of the group $\mathbb{Z} \oplus \mathbb{F}_{k-1}$, its generator corresponds indeed to the meridian of the base E. Due to the fact that the generator of the cyclic group in the center is $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ (up to orientation, as we have shown above), therefore since both the meridian and the generator $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ generate the infinite cyclic group which is the center of the group, this generator is a meridian of the exceptional section E (up to orientation), and a meridian of the intersection point O too. □

3.2 The effect on the fundamental group while gluing back a line

In this short subsection, we prove a simple but useful lemma about the effect on the fundamental group when we glue a line back to $\mathbb{P}^2$.

Zaidenberg [31] has proved the following lemma:

Lemma 3.4 (Zaidenberg [31, Lemma 2.3(a)]). *Let D be a closed hypersurface in a complex manifold M. Then the group* $\mathrm{Ker}\{i_* : \pi_1(M - D) \to \pi_1(M)\}$ *is generated by the vanishing loops of D. In particular, if D is irreducible, then, as a normal subgroup, this group is generated by any of these loops.*

Let C be a plane curve. Substituting $\mathbb{P}^2 - C$ for M and a line L for D, we get that $\mathrm{Ker}\{i_* : \pi_1(\mathbb{P}^2 - (C \cup L)) \to \pi_1(\mathbb{P}^2 - C)\}$ is generated by the vanishing loops (=meridians) of L. Since L is a line, we have:

$$\mathrm{Ker}\{i_* : \pi_1(\mathbb{P}^2 - (C \cup L)) \to \pi_1(\mathbb{P}^2 - C)\} = \langle\langle \mu \rangle\rangle$$

where μ is a meridian of L.

Therefore, it is easy to deduce the following lemma:

Lemma 3.5. *Let C be a plane projective curve and let L be a line. Let μ be a meridian of L. Then:*

$$\pi_1(\mathbb{P}^2 - (C \cup L))/\langle \mu \rangle \cong \pi_1(\mathbb{P}^2 - C)$$

Note that it is not necessary that L will be transversal to C.

3.3 The case of three fibers

The first generalized construction can be described as follows. Let C be the initial plane curve, and let m, n be two given natural numbers. Instead of one line Q in Uludag's original construction, here we have two lines, Q_1 and Q_2, which both meet C transversally. They intersect in a point O outside C. The other line P passes through O too and intersects C transversally. Now blow up the point O, in order to get Hirzebruch surface F_1. Then apply n elementary transformations of the first type on the proper image of Q_1. After that, apply m elementary transformations of the first type on the proper image of Q_2. After this step, we have reached F_{n+m+1}. Now, we apply $n+m$ elementary transformations of the second type on the proper image of P. Then, we reach back F_1. At last, we blow down the exceptional section (with self-intersection -1), and we get again $\mathbb{P}^2$. This defines a family of Cremona transformations from $\mathbb{P}^2$ to $\mathbb{P}^2$. Note that this construction is indeed a generalization of Uludag's method, since when we set $m=0$, we return to the original construction of Uludag.

For any pair of natural numbers n and m, we obtain a Cremona transformation, denoted by $T_{n,m}$. We will show that the new curve $\tilde{C} = T_{n,m}(C)$ has a fundamental group $\pi_1(\mathbb{P}^2 - \tilde{C})$ which is a central extension of $\pi_1(\mathbb{P}^2 - C)$ by a cyclic group of order $n+m+1$.

Remark 3.6. *Before formulating the result, we note that the curve obtained by $T_{n,m}$ can not be obtained by two successive applications of Uludag's method, since two appropriate applications will yield an extension of order $n+m+2$ whereas the extension of the Cremona transformation $T_{n,m}$ is of order $n+m+1$. Also the obtained singularities will be different (see Section 5).*

Theorem 3.7. *Let C be a plane projective curve and $G = \pi_1(\mathbb{P}^2 - C)$. Then for any $n, m \in \mathbb{N}$, the curve $\tilde{C} = T_{n,m}(C)$ is birational to C and its fundamental group $\tilde{G} = \pi_1(\mathbb{P}^2 - \tilde{C})$ is a central extension of G by $\mathbb{Z}/(n+m+1)\mathbb{Z}$:*

$$1 \to \mathbb{Z}/(n+m+1)\mathbb{Z} \to \tilde{G} \to G \to 1$$

In particular, if C has r irreducible components so is $\tilde{C}$.

Proof. We start with precising the notations. Let Q_1, Q_2 and P be the three lines which meet in the point O outside C. After the blow up of the point O, we denote the proper image of Q_1, Q_2, P, C under this blow up by Q_1^1, Q_2^1, P^1, C^1 respectively. After the first elementary transformation of the first type (blow up of the point $q_1^1 = Q_1^1 \cap E$ where E is the exceptional section and blow down of Q_1^1), we denote by Q_1^2 the blow up of the point $q_1^1 = Q_1^1 \cap E$, and by Q_2^2, P^2, C^2 the proper image of Q_2^1, P^1, C^1 by this elementary transformation (see Figure 5, where we omit the images of C).

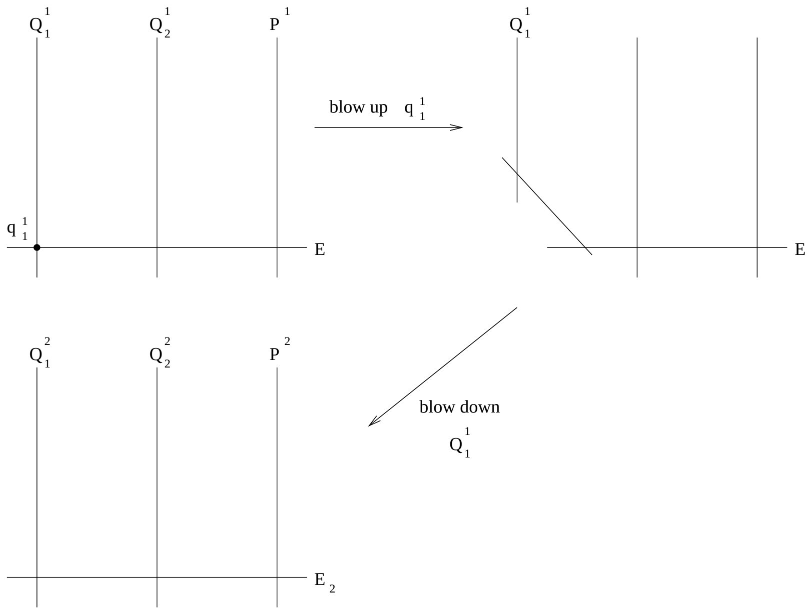

Figure 5: Elementary transformations of the first type

Applying $n-1$ more elementary transformations of the first type in this manner, we get that the images of Q_1^2, Q_2^2, P^2, C^2 are Q_1^{n+1}, Q_2^{n+1}, P^{n+1}, C^{n+1} respectively. Now we perform m elementary transformations of the first type in a slightly different manner (blow up of the point $q_2^{n+1} = Q_2^{n+1} \cap E_{n+1}$ where E_{n+1} is the corresponding exceptional section and blow down of Q_2^{n+1}, see Figure 6), to get after the applications of these transformations that the images of $Q_1^{n+1}, Q_2^{n+1}, P^{n+1}, C^{n+1}$ are $Q_1^{n+m+1}, Q_2^{n+m+1}, P^{n+m+1}$, C^{n+m+1} respectively.

After these two sequences of elementary transformations of the first type, we have reached Hirzebruch surface F_{n+m+1}, which has an exceptional section E_{n+m+1} with self-intersection $-(n+m+1)$. Now, we start to perform $n+m$ elementary transformations of the second type in order to return to F_1. After the first elementary transformation of the second type (blow up of a point $p_{n+m+1} \in P^{n+m+1} - (E_{n+m+1} \cup C^{n+m+1})$ and blow down of P^{n+m+1}), we denote by P^{n+m+2} the fiber replacing P^{n+m+1} which is the blow up of the point p_{n+m+1} and by $Q_1^{n+m+2}, Q_2^{n+m+2}, C^{n+m+2}$ the proper images of $Q_1^{n+m+1}, Q_2^{n+m+1}, C^{n+m+1}$ by this elementary transformation (see Figure 7).

After performing all the elementary transformations of the second type,

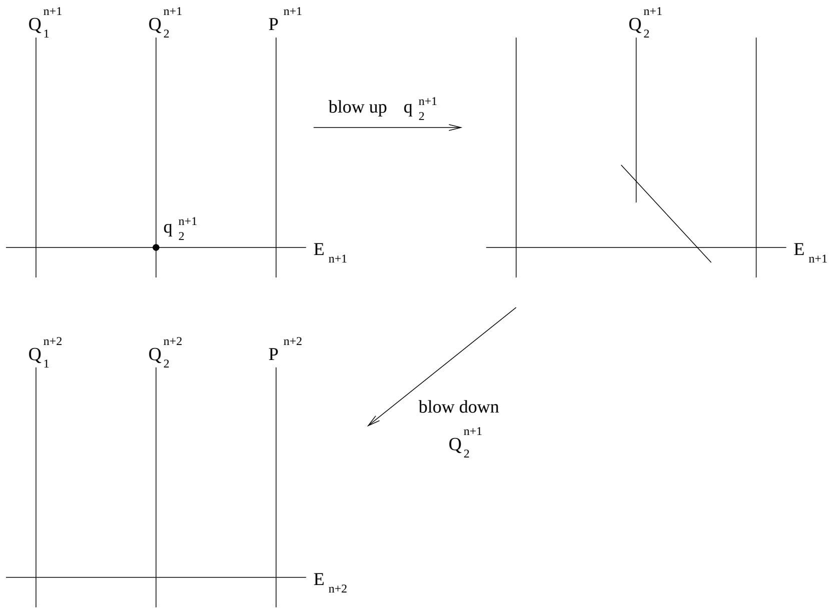

Figure 6: Elementary transformations of the first type

we are back in F_1, and the images of $Q_1^{n+m+2}, Q_2^{n+m+2}, P^{n+m+2}, C^{n+m+2}$ are $Q_1^{2n+2m+1}, Q_2^{2n+2m+1}, P^{2n+2m+1}, C^{2n+2m+1}$ respectively. At last, we blow down the exceptional section (whose self-intersection is now -1), and the images of $Q_1^{2n+2m+1}, Q_2^{2n+2m+1}, P^{2n+2m+1}, C^{2n+2m+1}$ under this map are $\tilde{Q}_1, \tilde{Q}_2, \tilde{P}, \tilde{C}$ respectively.

The composition of all the above transformations yields a Cremona transformation from $\mathbb{P}^2$ to $\mathbb{P}^2$ (which was denoted by $T_{n,m}$). At this point we remark that since the blow up and the blow down transformations are birational transformations, then $\tilde{C} = T_{n,m}(C)$ is birational to C, and if the curve C has r irreducible components, $\tilde{C}$ has r irreducible components too.

This Cremona transformation defines an isomorphism:

$$\mathbb{P}^2 - (C \cup P \cup Q_1 \cup Q_2) \cong \mathbb{P}^2 - (\tilde{C} \cup \tilde{P} \cup \tilde{Q}_1 \cup \tilde{Q}_2)$$

which induces an isomorphism of the corresponding fundamental groups:

$$\pi_1(\mathbb{P}^2 - (C \cup P \cup Q_1 \cup Q_2)) \cong \pi_1(\mathbb{P}^2 - (\tilde{C} \cup \tilde{P} \cup \tilde{Q}_1 \cup \tilde{Q}_2))$$

In order to compute $\pi_1(\mathbb{P}^2 - \tilde{C})$, we just have to add the relations which correspond to gluing back the lines $\tilde{P}, \tilde{Q}_1$ and $\tilde{Q}_2$.

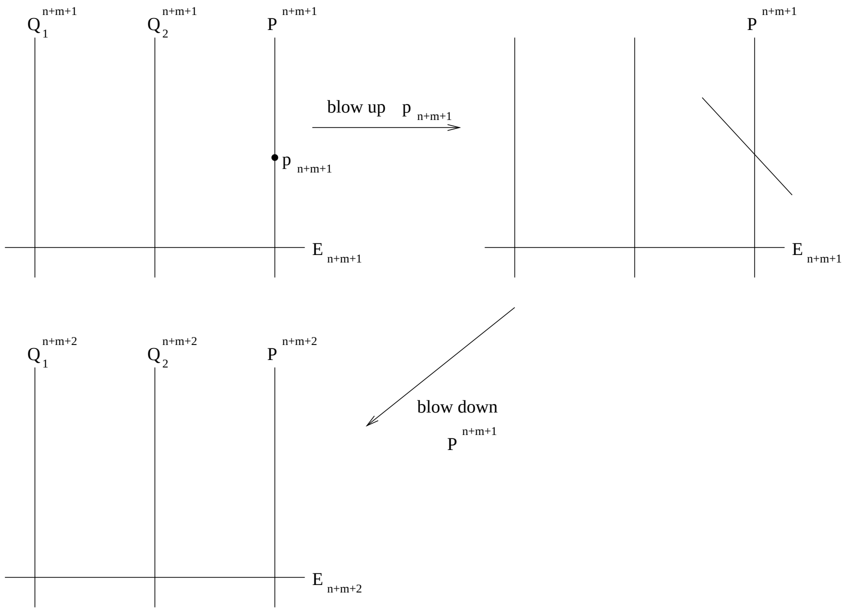

Figure 7: Elementary transformations of the second type

Let α, β and γ be the meridians of P,Q_1 and Q_2 respectively. Using Lemma 3.3 for $k = 3$, we get that the meridian of the curve $C \cup P \cup Q_1 \cup Q_2$ at the point O in our case (which is the intersection of P, Q_1 and Q_2) is $\alpha\beta\gamma$, and hence the meridian of E, which is the blow up of this point, is $\alpha\beta\gamma$ too (see Figure 8).

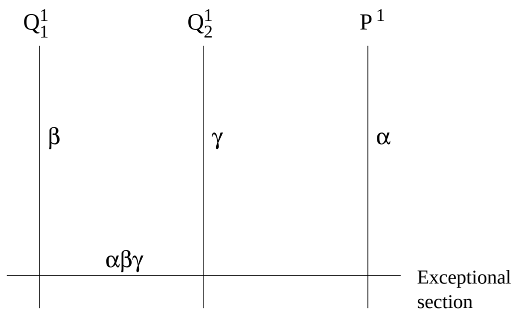

Figure 8: The meridians of the lines after the first blow up

When we apply the first n elementary transformations of the first type using the fibers $Q_1^i, 1 \leq i \leq n$, we use Fujita's lemma (Lemma 3.2) each time we blow up the intersection point between the fiber and the exceptional section, to get at last that the meridian of Q_1^{n+1} is $(\alpha\beta\gamma)^n \cdot \beta$ (see Figure 9 for the effect of one elementary transformation of the first type). At the same time the meridians of Q_2^i and P^i do not change and remain γ and α respectively.

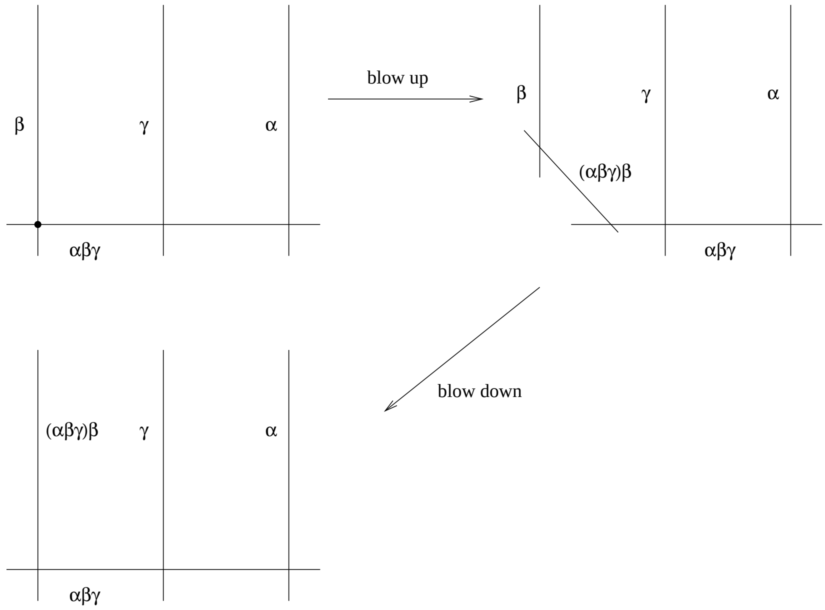

Figure 9: The effect of an elementary transformation of the first type on the meridians

In the second step, when we apply m elementary transformations of the first type using the fibers $Q_2^i, n+1 \leq i \leq n+m$, we use Fujita's lemma (Lemma 3.2) in the same way to get that the meridian of Q_2^{n+m+1} is $(\alpha\beta\gamma)^m \cdot \gamma$. In the same time the meridians of Q_1^i and P^i do not change and remain $(\alpha\beta\gamma)^n \cdot \beta$ and α respectively. In the last two steps, when we apply $n+m$ elementary transformations of the second type using the fibers $P^i, n+m+1 \leq i \leq 2n+2m$, and the last blow down (to return to $\mathbb{P}^2$), all the meridians do not change (see Figure 10 for the effect of one elementary transformation of the second type). Hence, the meridians of $\tilde{P}, \tilde{Q}_1$ and $\tilde{Q}_2$ are $\alpha, (\alpha\beta\gamma)^n \cdot \beta$ and $(\alpha\beta\gamma)^m \cdot \gamma$ respectively.

Therefore, by Lemma 3.5, we conclude that:

$$\pi_1(\mathbb{P}^2 - \tilde{C}) \cong \pi_1(\mathbb{P}^2 - (\tilde{C} \cup \tilde{P} \cup \tilde{Q}_1 \cup \tilde{Q}_2))/\langle \alpha, (\alpha\beta\gamma)^n\beta, (\alpha\beta\gamma)^m\gamma \rangle$$

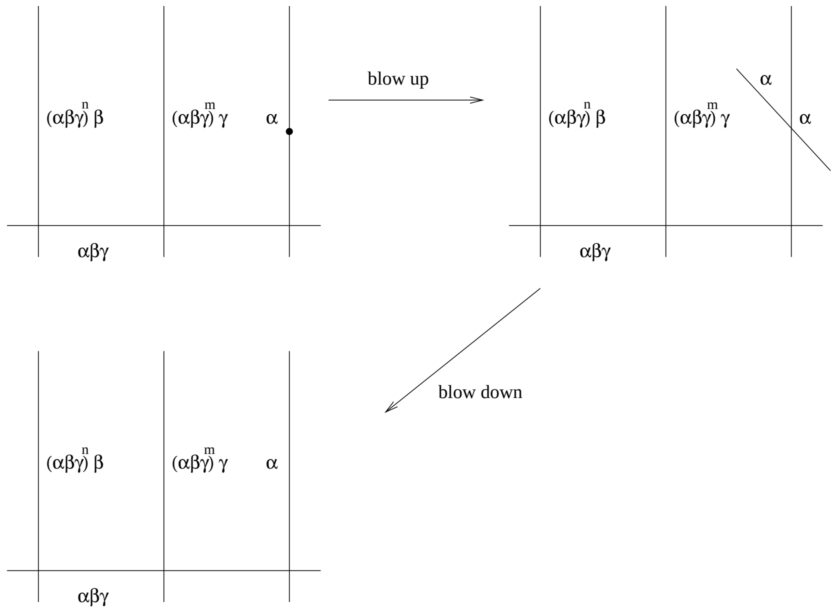

Figure 10: The effect of an elementary transformation of the second type on the meridians

which is equivalent to:

$$\pi_1(\mathbb{P}^2 - \tilde{C}) \cong \pi_1(\mathbb{P}^2 - (\tilde{C} \cup \tilde{P} \cup \tilde{Q}_1 \cup \tilde{Q}_2))/\langle \alpha, (\beta\gamma)^n\beta, (\beta\gamma)^m\gamma \rangle,$$

since the connecting relations between α, β and γ are

$$\alpha\beta\gamma = \beta\gamma\alpha = \gamma\alpha\beta,$$

because all the three lines intersect in one point O (see the proof of Lemma 3.3). Since the Cremona transformation defines an isomorphism between $\mathbb{P}^2 - (C \cup P \cup Q_1 \cup Q_2)$ and $\mathbb{P}^2 - (\tilde{C} \cup \tilde{P} \cup \tilde{Q}_1 \cup \tilde{Q}_2)$, we can also write:

$$\pi_1(\mathbb{P}^2 - \tilde{C}) \cong \pi_1(\mathbb{P}^2 - (C \cup P \cup Q_1 \cup Q_2))/\langle \alpha, (\beta\gamma)^n\beta, (\beta\gamma)^m\gamma \rangle$$

When we pass to the quotient by $\langle \alpha \rangle$ which corresponds to gluing the line P back into $\mathbb{P}^2$ in the original configuration (by Lemma 3.5), we get that:

$$\pi_1(\mathbb{P}^2 - \tilde{C}) \cong \pi_1(\mathbb{P}^2 - (C \cup Q_1 \cup Q_2))/\langle (\beta\gamma)^n\beta, (\beta\gamma)^m\gamma \rangle$$

While moving to this quotient, we get that β and γ commute since in the quotient $\alpha = 1$, and hence the relation $\alpha\beta\gamma = \gamma\alpha\beta$ becomes $\beta\gamma = \gamma\beta$. Hence, we get:

$$\pi_1(\mathbb{P}^2 - \tilde{C}) \cong \pi_1(\mathbb{P}^2 - (C \cup Q_1 \cup Q_2))/\langle \beta^{n+1}\gamma^n, \beta^m\gamma^{m+1}\rangle$$

Now, since we choose Q_1 and Q_2 to be both transversal to C, then their meridians are central elements in $\pi_1(\mathbb{P}^2 - (C \cup Q_1 \cup Q_2))$. By Lemma 3.5, when we take the quotient of $\pi_1(\mathbb{P}^2 - (C \cup Q_1 \cup Q_2))$ by the group $\langle \beta, \gamma \rangle$ generated by these meridians, we get $\pi_1(\mathbb{P}^2 - C)$. Hence, the following extension is central:

$$1 \to \mathbb{Z} \oplus \mathbb{Z} \cong \langle \beta, \gamma \mid \beta\gamma = \gamma\beta \rangle \to \pi_1(\mathbb{P}^2 - (C \cup Q_1 \cup Q_2)) \to \pi_1(\mathbb{P}^2 - C) \to 1$$

Now, it is easy to see that:

$$\frac{\pi_1(\mathbb{P}^2 - (C \cup Q_1 \cup Q_2))/\langle \beta^{n+1}\gamma^n, \beta^m\gamma^{m+1}\rangle}{\langle \beta, \gamma \mid \beta\gamma = \gamma\beta \rangle/\langle \beta^{n+1}\gamma^n, \beta^m\gamma^{m+1}\rangle} \cong \pi_1(\mathbb{P}^2 - (C \cup Q_1 \cup Q_2))/\langle \beta, \gamma \rangle$$

and therefore:

$$\frac{\pi_1(\mathbb{P}^2 - \tilde{C})}{\langle \beta, \gamma \mid \beta\gamma = \gamma\beta \rangle/\langle \beta^{n+1}\gamma^n, \beta^m\gamma^{m+1}\rangle} \cong \pi_1(\mathbb{P}^2 - C)$$

which can be written as the following extension:

$$1 \to \langle \beta, \gamma \mid \beta\gamma = \gamma\beta \rangle/\langle \beta^{n+1}\gamma^n, \beta^m\gamma^{m+1}\rangle \to \pi_1(\mathbb{P}^2 - \tilde{C}) \to \pi_1(\mathbb{P}^2 - C) \to 1$$

In order to finish the proof, we have to show two more things:

1. $\langle \beta, \gamma \mid \beta\gamma = \gamma\beta, \beta^{n+1}\gamma^n, \beta^m\gamma^{m+1}\rangle \cong \mathbb{Z}/(n+m+1)\mathbb{Z}$.
2. This extension is central.

Since $\beta^{n+1}\gamma^n \cdot \beta^m\gamma^{m+1} = \beta^{n+m+1}\gamma^{n+m+1}$, we can change the presentation of the subgroup in the denominator a bit:

$$\langle \beta^{n+1}\gamma^n, \beta^m\gamma^{m+1} \mid \beta\gamma = \gamma\beta \rangle \cong \langle \beta^{n+1}\gamma^n, \beta^{n+m+1}\gamma^{n+m+1} \mid \beta\gamma = \gamma\beta \rangle$$

On the other hand, one can see that the following is a presentation of $\mathbb{Z} \oplus \mathbb{Z}$ too:

$$\mathbb{Z} \oplus \mathbb{Z} \cong \langle \beta^{n+1}\gamma^n, \beta\gamma \mid \beta\gamma = \gamma\beta \rangle$$

since both β and γ can be achieved by the new pair of generators. Using these two new presentations, it can be easily seen that the quotient of these two groups is $\mathbb{Z}/(n+m+1)\mathbb{Z}$, since the first generator in both presentations is the same, and the second generator of the denominator is the $(n+m+1)$-th power of its corresponding generator in the numerator, and hence (1) is proved.

About the centrality of the extension, due to the centrality of the extension

$$1 \to \langle \beta, \gamma \rangle \to \pi_1(\mathbb{P}^2 - (C \cup Q_1 \cup Q_2)) \to \pi_1(\mathbb{P}^2 - C) \to 1$$

it follows that:

$$\langle \beta, \gamma \rangle \leq Z(\pi_1(\mathbb{P}^2 - (C \cup Q_1 \cup Q_2)))$$

where $Z(G)$ is the center of the group G. Therefore, also the following holds:

$$\frac{\langle \beta, \gamma \rangle}{\langle \beta^{n+1}\gamma^n, \beta^m\gamma^{m+1} \rangle} \leq Z\left(\frac{\pi_1(\mathbb{P}^2 - (C \cup Q_1 \cup Q_2))}{\langle \beta^{n+1}\gamma^n, \beta^m\gamma^{m+1} \rangle}\right)$$

which means that:

$$\mathbb{Z}/(n+m+1)\mathbb{Z} \leq Z(\pi_1(\mathbb{P}^2 - \tilde{C}))$$

and therefore the extension is central, (2) is proved, and we have proved Theorem 3.7. $\square$

3.4 The general case

The general construction is the following: Let C be the initial plane curve, and let $n_1, \cdots, n_k$ be k given natural numbers. In this construction, we have k different lines $Q_1, \cdots, Q_k$ which all meet C transversally. They intersect in a point O outside C, in such a way that locally they are organized counterclockwise around O. The additional line P passes via O too and intersects C transversally. Now we blow up the point O, in order to get Hirzebruch surface F_1. Then we apply n_i elementary transformations of the first type on the proper image of Q_i for all $i = 1, \ldots, k$. After this step, we have reached Hirzebruch surface $F_{(\sum_{i=1}^k n_i)+1}$. Now, we apply $\sum_{i=1}^k n_i$ elementary transformations of the second type on the proper image of P. Then, we reach back F_1. At last, we blow down the exceptional section, and we get again $\mathbb{P}^2$. As before, this defines a family of Cremona transformations from $\mathbb{P}^2$ to $\mathbb{P}^2$.

For any k-tuple $(n_1, \cdots, n_k) \in \mathbb{N}^k$, we get a Cremona transformation, denoted by $T_{(n_1,\cdots,n_k)}$. We will show that the new curve $\tilde{C} = T_{(n_1,\cdots,n_k)}(C)$ has a fundamental group $\pi_1(\mathbb{P}^2 - \tilde{C})$ which is a central extension of $\pi_1(\mathbb{P}^2 - C)$ by a cyclic group of order $(\sum_{i=1}^k n_i) + 1$. As before, the curves obtained by the general construction can not be obtained by successive applications of Uludag's original method (see Remark 3.6).

Theorem 3.8. *Let C be a plane projective curve and $G = \pi_1(\mathbb{P}^2 - C)$. Then for any k-tuple $(n_1, \cdots, n_k) \in \mathbb{N}^k$, the curve $\tilde{C} = T_{(n_1,\cdots,n_k)}(C)$ is birational to C, and its fundamental group $\tilde{G} = \pi_1(\mathbb{P}^2 - \tilde{C})$ is a central extension of G by $\mathbb{Z}/((\sum_{i=1}^k n_i) + 1)\mathbb{Z}$:*

$$1 \to \mathbb{Z}/((\sum_{i=1}^k n_i) + 1)\mathbb{Z} \to \tilde{G} \to G \to 1$$

Moreover, if C has r irreducible components so is $\tilde{C}$.

The proof of the general case is similar to the proof of the case of three fibers (Theorem 3.7). Hence, we will focus only on the differences between the two proofs.

Proof. As in the previous case, since the blow up and the blow down transformations are birational transformations, then $\tilde{C}$ is birational to C, and if the curve C has r irreducible components, also $\tilde{C}$ has r irreducible components.

Let $\tilde{P} = T_{(n_1,\cdots,n_k)}(P)$ and $\tilde{Q}_i = T_{(n_1,\cdots,n_k)}(Q_i)$ for $1 \leq i \leq k$. This Cremona transformation defines an isomorphism:

$$\mathbb{P}^2 - (C \cup P \cup (\bigcup_{i=1}^k Q_i)) \cong \mathbb{P}^2 - (\tilde{C} \cup \tilde{P} \cup (\bigcup_{i=1}^k \tilde{Q}_i))$$

which induces an isomorphism of the corresponding fundamental groups:

$$\pi_1(\mathbb{P}^2 - (C \cup P \cup (\bigcup_{i=1}^k Q_i))) \cong \pi_1(\mathbb{P}^2 - (\tilde{C} \cup \tilde{P} \cup (\bigcup_{i=1}^k \tilde{Q}_i)))$$

In order to compute $\pi_1(\mathbb{P}^2 - \tilde{C})$, we have to add the relations correspond to gluing back the lines $\tilde{P}$ and $\tilde{Q}_i, 1 \leq i \leq k$.

Let β and α_i be the meridians of the lines P and Q_i respectively. Using Lemma 3.3, we get that the meridian of the curve $C \cup P \cup (\bigcup_{i=1}^k Q_i)$ at the point O in our case (which is the intersection of $k+1$ lines: $P, Q_1, \cdots, Q_k$) is $\beta\alpha_1\cdots\alpha_k$, and hence the meridian of E, which is the blow up of this point, is $\beta\alpha_1\cdots\alpha_k$ too.

As before, by Fujita's lemma (Lemma 3.2), one can easily see that the meridian of $\tilde{Q}_i$ is $(\beta\alpha_1\cdots\alpha_k)^{n_i}\cdot\alpha_i$ for all $1\leq i\leq k$, and the meridian of $\tilde{P}$ remains β.

Therefore, by Lemma 3.5, we conclude that:

$$\pi_1(\mathbb{P}^2-\tilde{C}) \cong \pi_1(\mathbb{P}^2-(\tilde{C}\cup\tilde{P}\cup(\bigcup_{i=1}^{k}\tilde{Q}_i)))/\langle\beta, (\beta\alpha_1\cdots\alpha_k)^{n_1}\alpha_1, \cdots, (\beta\alpha_1\cdots\alpha_k)^{n_k}\alpha_k\rangle$$

which is equivalent to:

$$\pi_1(\mathbb{P}^2-\tilde{C}) \cong \pi_1(\mathbb{P}^2-(\tilde{C}\cup\tilde{P}\cup(\bigcup_{i=1}^{k}\tilde{Q}_i)))/\langle\beta, (\alpha_1\cdots\alpha_k)^{n_1}\alpha_1, \cdots, (\alpha_1\cdots\alpha_k)^{n_k}\alpha_k\rangle$$

since the connecting relations between $\beta, \alpha_1, \cdots, \alpha_k$ are

$$\beta\alpha_1\cdots\alpha_k = \alpha_1\cdots\alpha_k\beta = \cdots = \alpha_k\beta\alpha_1\cdots\alpha_{k-1}$$

(see proof of Lemma 3.3), because all the $k+1$ lines intersect in one point O.

Since

$$\mathbb{P}^2-(C\cup P\cup(\bigcup_{i=1}^{k}Q_i)) \cong \mathbb{P}^2-(\tilde{C}\cup\tilde{P}\cup(\bigcup_{i=1}^{k}\tilde{Q}_i)),$$

we can also write:

$$\pi_1(\mathbb{P}^2-\tilde{C}) \cong \pi_1(\mathbb{P}^2-(C\cup P\cup(\bigcup_{i=1}^{k}Q_i)))/\langle\beta, (\alpha_1\cdots\alpha_k)^{n_1}\alpha_1, \cdots, (\alpha_1\cdots\alpha_k)^{n_k}\alpha_k\rangle$$

When we pass to the quotient by $\langle\beta\rangle$ which corresponds to gluing the line P back into $\mathbb{P}^2$ in the original configuration (by Lemma 3.5), we get that:

$$\pi_1(\mathbb{P}^2-\tilde{C}) \cong \pi_1(\mathbb{P}^2-(C\cup(\bigcup_{i=1}^{k}Q_i)))/\langle(\alpha_1\cdots\alpha_k)^{n_1}\alpha_1, \cdots, (\alpha_1\cdots\alpha_k)^{n_k}\alpha_k\rangle$$

While moving to this quotient, we get the following k cyclic relations for the denominator:

$$\alpha_1\alpha_2\cdots\alpha_k = \alpha_2\cdots\alpha_k\alpha_1 = \cdots = \alpha_k\alpha_1\alpha_2\cdots\alpha_{k-1}$$

since in the quotient $\beta = 1$. These relations can be presented also as the following set of relations, where $\alpha = \alpha_1\alpha_2\cdots\alpha_k$:

$$\{\alpha\alpha_2 = \alpha_2\alpha, \alpha\alpha_3 = \alpha_3\alpha, \cdots, \alpha\alpha_k = \alpha_k\alpha\}$$

Now, since we choose $Q_1, \cdots, Q_k$ to be all transversally intersected with C, then their meridians commute with the meridians of C. By Lemma 3.5, when we take the quotient of $\pi_1(\mathbb{P}^2 - (C \cup (\bigcup_{i=1}^k Q_i)))$ by the normal subgroup generated by the meridians of $Q_1, \cdots, Q_k$, we get $\pi_1(\mathbb{P}^2 - C)$. Due to the fact that all the lines are intersected at O, then the subgroup generated by the meridians $Q_1, \cdots, Q_k$ is of the form (see the proof of Lemma 3.3):

$$\mathbb{Z} \oplus \mathbb{F}_{k-1} \cong \langle \alpha, \alpha_2, \cdots, \alpha_k \mid \alpha\alpha_i = \alpha_i\alpha, 2 \leq i \leq k \rangle$$

where $\alpha = \alpha_1 \cdots \alpha_k$.

As before, it is easy to see that:

$$\frac{\pi_1(\mathbb{P}^2 - (C \cup (\bigcup_{i=1}^k Q_i)))/\langle \alpha^{n_1}\alpha_1, \cdots, \alpha^{n_k}\alpha_k \rangle}{\langle \alpha, \alpha_2, \cdots, \alpha_k \mid \alpha\alpha_i = \alpha_i\alpha, 2 \leq i \leq k \rangle / \langle \alpha^{n_1}\alpha_1, \cdots, \alpha^{n_k}\alpha_k \rangle} \cong$$

$$\cong \pi_1(\mathbb{P}^2 - (C \cup (\bigcup_{i=1}^k Q_i)))/\langle \alpha, \alpha_2, \cdots, \alpha_k \rangle$$

where $\alpha = \alpha_1 \cdots \alpha_k$, and therefore:

$$\frac{\pi_1(\mathbb{P}^2 - \tilde{C})}{\left(\frac{\langle \alpha, \alpha_2, \cdots, \alpha_k \mid \alpha\alpha_i = \alpha_i\alpha, 2 \leq i \leq k \rangle}{\langle \alpha^{n_1}\alpha_1, \cdots, \alpha^{n_k}\alpha_k \rangle} \right)} \cong \pi_1(\mathbb{P}^2 - C)$$

which can be written as the following extension:

$$1 \to \frac{\langle \alpha, \alpha_2, \cdots, \alpha_k \mid \alpha\alpha_i = \alpha_i\alpha, 2 \leq i \leq k \rangle}{\langle \alpha^{n_1}\alpha_1, \cdots, \alpha^{n_k}\alpha_k \rangle} \to \pi_1(\mathbb{P}^2 - \tilde{C}) \to \pi_1(\mathbb{P}^2 - C) \to 1$$

The centrality of the extension in this case is a little bit tricky. Although α_i does not commute with α_j ($2 \leq i, j \leq k$, $i \neq j$), the generators of $\tilde{G} =$

$\pi_1(\mathbb{P}^2 - \tilde{C})$ are the meridians of C and only one more generator - α (the other generators - $\alpha_2, \cdots, \alpha_k$ - corresponding to the lines $Q_2, \cdots, Q_k$ in the bigger group $\pi_1(\mathbb{P}^2 - (C \cup (\bigcup_{i=1}^k Q_i)))$ disappear in $\tilde{G}$ by the additional relations $\alpha^{n_2}\alpha_2, \cdots, \alpha^{n_k}\alpha_k$). This generator indeed commutes with the meridians of C in $\pi_1(\mathbb{P}^2 - C)$ (as it is equal to the multiplication of all the α_i's which commute with the meridians of C in $\pi_1(\mathbb{P}^2 - C)$ as mentioned above), and hence we get also in this case that the extension is central.

Therefore, it remains to show that:

$$\frac{\langle \alpha, \alpha_2, \cdots, \alpha_k \mid \alpha\alpha_i = \alpha_i\alpha, 2 \leq i \leq k \rangle}{\langle \alpha^{n_1}\alpha_1, \cdots, \alpha^{n_k}\alpha_k \rangle} \cong \mathbb{Z}/((\sum_{i=1}^k n_i) + 1)\mathbb{Z}$$

Since

$$\alpha^{n_1}\alpha_1 \cdots \alpha^{n_k}\alpha_k = \alpha^{n_1+\cdots+n_k}(\alpha_1 \cdots \alpha_k) = \alpha^{n_1+\cdots+n_k}\alpha = \alpha^{n_1+\cdots+n_k+1}$$

we can change the presentation of the subgroup in the denominator a bit:

$$\langle \alpha^{n_1+\cdots+n_k+1}, \alpha^{n_2}\alpha_2, \cdots, \alpha^{n_k}\alpha_k \mid \alpha\alpha_i = \alpha_i\alpha, 2 \leq i \leq k \rangle$$

On the other hand, one can see that the following is a presentation of $\mathbb{Z} \oplus \mathbb{F}_{k-1}$ too:

$$\mathbb{Z} \oplus \mathbb{F}_{k-1} \cong \langle \alpha, \alpha^{n_2}\alpha_2, \cdots, \alpha^{n_k}\alpha_k \mid \alpha\alpha_i = \alpha_i\alpha, 2 \leq i \leq k \rangle$$

since all the $\alpha_i, 2 \leq i \leq k$ can be achieved by the new set of generators. Using these two new presentations, it can be easily shown that the quotient of these two groups is $\mathbb{Z}/((\sum_{i=1}^k n_i)+1)\mathbb{Z}$, as we have shown it in the previous case, and hence we are done, and Theorem 3.8 is proved. $\square$

3.5 A slightly more general construction

During the proof, we have shown that elementary transformations of the second type do not affect the meridians of the fibers which we perform the transformations on. Therefore, we can even generalize our construction to the following one: instead of performing all the elementary transformations of the second type on the same fiber P, we can apply them on several fibers $P_1, \cdots, P_t$, with the condition that the total number of applications of elementary transformations of the second type will be equal to the total number of applications of elementary transformations of the first type.

Using this observation, we can describe a slightly more general construction: Let C be the initial plane curve, and let $n_1, \cdots, n_k$ and $m_1, \cdots, m_l$ be two sets of k and l given natural numbers, such that $\sum n_i = \sum m_j$. We start with $k + l$ different lines $Q_1, \cdots, Q_k$ and $P_1, \cdots, P_l$ which all meet C transversally. They all intersect in a point O outside C, in such a way that locally they are organized counterclockwise around O. Now we blow up the point O, in order to get Hirzebruch surface F_1. Then we apply n_i elementary transformations of the first type on the proper image of Q_i for all $i = 1, \ldots, k$. After this step, we have reached Hirzebruch surface $F_{(\sum_{i=1}^k n_i)+1}$. Now, we apply m_j elementary transformations of the second type on the proper image of P_j for all $j = 1, \ldots, l$. Since $\sum n_i = \sum m_j$, we reach back F_1. At last, we blow down the exceptional section (whose self-intersection is now -1), and we get again $\mathbb{P}^2$. This defines a family of Cremona transformations from $\mathbb{P}^2$ to $\mathbb{P}^2$.

For any $(k + l)$-tuple $(n_1, \cdots, n_k, m_1, \cdots, m_l) \in \mathbb{N}^{k+l}$ such that $\sum n_i = \sum m_i$, we get a Cremona transformation, denoted by $T_{(n_1,\cdots,n_k;m_1,\cdots,m_l)}$. Then we can state:

Corollary 3.9. *Let C be a plane projective curve and $G = \pi_1(\mathbb{P}^2 - C)$. Then for any $(k+l)$-tuple $(n_1, \cdots, n_k, m_1, \cdots, m_l) \in \mathbb{N}^{k+l}$ such that $\sum n_i = \sum m_i$, the curve $\tilde{C} = T_{(n_1,\cdots,n_k;m_1,\cdots,m_l)}(C)$ is birational to C and its fundamental group $\tilde{G} = \pi_1(\mathbb{P}^2 - \tilde{C})$ is a central extension of G by $\mathbb{Z}/((\sum_{i=1}^k n_i) + 1)\mathbb{Z}$:*

$$1 \to \mathbb{Z}/((\sum_{i=1}^{k} n_i) + 1)\mathbb{Z} \to \tilde{G} \to G \to 1$$

Moreover, if C has r irreducible components so is $\tilde{C}$.

3.6 An interesting special case

Just before finishing the section of the constructions, we want to concentrate on an interesting special construction. In this construction, we perform all the elementary transformations from both types on the same fiber.

First, we define this construction precisely, and then we prove that the fundamental group of the obtained curve is again a central extension of the original curve, as we had in the previous constructions. We have to prove it, since the proof is slightly different from the proofs of the previous cases.

We start with a curve C and one additional line L in $\mathbb{P}^2$ which intersects C transversally. Blow up a point O on L (which does not belong to C) in order to reach F_1. Then perform n elementary transformations of the first type on the proper image of L. Hence we reach F_{n+1}. Now, perform n elementary transformations of the second type again on the proper image of L. Now, blow down the exceptional section (which now has self-intersection -1) in order to return to $\mathbb{P}^2$. This construction defines a Cremona transformation from $\mathbb{P}^2$ to $\mathbb{P}^2$, which we denote by T_n. Then we have the following result:

Proposition 3.10. *Let C be a plane projective curve and $G = \pi_1(\mathbb{P}^2 - C)$. Then for any natural number $n \in \mathbb{N}$, the curve $\tilde{C} = T_n(C)$ is birational to C and its fundamental group $\tilde{G} = \pi_1(\mathbb{P}^2 - \tilde{C})$ is a central extension of G by $\mathbb{Z}/(n+1)\mathbb{Z}$:*

$$1 \to \mathbb{Z}/(n+1)\mathbb{Z} \to \tilde{G} \to G \to 1$$

Moreover, if C has r irreducible components so is $\tilde{C}$.

Proof. As before, since the blow up and the blow down transformations are birational transformations, then $\tilde{C}$ is birational to C, and if the curve C has r irreducible components, $\tilde{C}$ has r irreducible components too.

Let $\tilde{L} = T_n(L)$. T_n defines an isomorphism:

$$\mathbb{P}^2 - (C \cup L) \cong \mathbb{P}^2 - (\tilde{C} \cup \tilde{L})$$

which induces an isomorphism of the corresponding fundamental groups:

$$\pi_1(\mathbb{P}^2 - (C \cup L)) \cong \pi_1(\mathbb{P}^2 - (\tilde{C} \cup \tilde{L}))$$

In order to compute $\pi_1(\mathbb{P}^2 - \tilde{C})$, we have to add the relation correspond to gluing back the line $\tilde{L}$. Let α be the meridian of L. By Claim 3.1, we get that the meridian of the exceptional section E, which is the blow up of the point $O \in L$, is again α.

After we apply the sequence of n elementary transformations of the first type using the fiber L and its images, we have by Fujita's lemma (Lemma 3.2) that the meridian of the image of L is α^{n+1}. As before, the applications of elementary transformations of the second type and the final blow down do not change this meridian.

Therefore, by Lemma 3.5, we conclude that:

$$\pi_1(\mathbb{P}^2 - \tilde{C}) \cong \pi_1(\mathbb{P}^2 - (\tilde{C} \cup \tilde{L}))/\langle \alpha^{n+1} \rangle$$

Since
$$\mathbb{P}^2 - (C \cup L) \cong \mathbb{P}^2 - (\tilde{C} \cup \tilde{L}),$$
we can also write:
$$\pi_1(\mathbb{P}^2 - \tilde{C}) \cong \pi_1(\mathbb{P}^2 - (C \cup L))/\langle \alpha^{n+1} \rangle$$

Now, since L intersects C transversally, its meridian α commutes with the meridians of C. By Lemma 3.5,
$$\pi_1(\mathbb{P}^2 - (C \cup L))/\langle \alpha \rangle = \pi_1(\mathbb{P}^2 - C).$$

As before, since:
$$(\pi_1(\mathbb{P}^2 - (C \cup L))/\langle \alpha^{n+1} \rangle)/(\langle \alpha \rangle) \cong \pi_1(\mathbb{P}^2 - (C \cup L))/\langle \alpha \rangle$$
we have:
$$\pi_1(\mathbb{P}^2 - \tilde{C})/\langle \alpha \rangle \cong \pi_1(\mathbb{P}^2 - C)$$
which can be written as the following extension:
$$1 \to (\langle \alpha \rangle / \langle \alpha^{n+1} \rangle) \to \pi_1(\mathbb{P}^2 - \tilde{C}) \to \pi_1(\mathbb{P}^2 - C) \to 1$$

Obviously, $\langle \alpha \rangle / \langle \alpha^{n+1} \rangle \cong \mathbb{Z}/(n+1)\mathbb{Z}$, and since α commutes with the generators of $\pi_1(\mathbb{P}^2 - C)$, the extension is central. $\square$

4 Properties of groups preserved by the constructions

In this section, we indicate some properties of the fundamental group which are preserved by the constructions of the previous section.

We start with an interesting property about the splitness of the central extension we have in the constructions. Using this property, we will show the following important property of the fundamental group: if we start with an irreducible curve which has a cyclic group as the fundamental group (such as smooth irreducible curves), then the resulting fundamental group will be cyclic too. The importance of this property is that although the constructions add to the curve deep singularities (as is proved in Proposition 5.3), the fundamental group of the curve is still cyclic. Hence, these constructions may yield families of plane curves which have some deep singularities but have cyclic fundamental groups.

Proposition 4.1. *Let C be a plane curve with r irreducible components. Let $n \in \mathbb{N}$. Let $\tilde{C}$ be the curve whose fundamental group $\tilde{G}$ is obtained from $G = \pi_1(\mathbb{P}^2 - C)$ by a central extension by $\mathbb{Z}/(n+1)\mathbb{Z}$. If the abelian group $H_1(\mathbb{P}^2 - C)$ has r direct summands, then the extension does not split, i.e. $\tilde{G} \ncong G \oplus \mathbb{Z}/(n+1)\mathbb{Z}$.*

Proof. On the contrary, assume that $\tilde{G} \cong G \oplus \mathbb{Z}/(n+1)\mathbb{Z}$. As $H_1(X)$ is the abelinization of $\pi_1(X)$, we have that

$$H_1(\mathbb{P}^2 - \tilde{C}) \cong \mathrm{Ab}(\tilde{G}) \cong \mathrm{Ab}(G \oplus \mathbb{Z}/(n+1)\mathbb{Z}) \cong H_1(\mathbb{P}^2 - C) \oplus \mathbb{Z}/(n+1)\mathbb{Z}.$$

Hence, $H_1(\mathbb{P}^2 - \tilde{C})$ has $r+1$ direct summands. This contradicts the fact that $\tilde{C}$ has only r irreducible components, as the number of irreducible components is preserved by the constructions. □

Proposition 4.2. *Let C be an irreducible plane curve with a cyclic fundamental group $\mathbb{Z}/r\mathbb{Z}$. Let $n \in \mathbb{N}$. Let $\tilde{C}$ be the curve whose fundamental group $\tilde{G}$ is obtained from $G = \pi_1(\mathbb{P}^2 - C)$ by a central extension by $\mathbb{Z}/(n+1)\mathbb{Z}$. Then $\tilde{G} = \pi_1(\mathbb{P}^2 - \tilde{C})$ is also cyclic of order $r(n+1)$.*

Proof. As $H_1(X)$ is the abelinization of $\pi_1(X)$, we have that $G = H_1(\mathbb{P}^2 - C)$. Since G is cyclic, $H_1(\mathbb{P}^2 - C)$ has one direct summand, which equals the number of irreducible components in C (one too). By the previous proposition, the extension does not split, and $H_1(\mathbb{P}^2 - \tilde{C})$ is cyclic too.

Since the extension is central, we have that $\tilde{G}$ is abelian of order $r(n+1)$. Hence, $\tilde{G} = H_1(\mathbb{P}^2 - \tilde{C})$. Therefore, $\tilde{G}$ is cyclic of order $r(n+1)$. □

Remark 4.3. *The condition that the curve is irreducible is essential, since if we take a reducible curve with a cyclic fundamental group, it is not guaranteed that the resulting curve will have a cyclic fundamental group. For example, if we start with a curve C consists of two intersecting lines whose fundamental group $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}$ is cyclic, and we apply on it Uludag's method for $n = 1$, we get that the resulting curve $\tilde{C}$ has a fundamental group $\pi_1(\mathbb{P}^2 - \tilde{C}) = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ (see after Proposition 5.11) which is not cyclic.*

Let p be a prime number. Then:

Remark 4.4. *Let C be a plane curve with a fundamental group G which is a p-group. Let $n \in \mathbb{N}$. Let $\tilde{C}$ be the curve whose fundamental group $\tilde{G}$ is obtained from $G = \pi_1(\mathbb{P}^2 - C)$ by a central extension by $\mathbb{Z}/(n+1)\mathbb{Z}$. Then: if $n+1 = p^l$ for some l, then $\tilde{G} = \pi_1(\mathbb{P}^2 - \tilde{C})$ is also a p-group.*

Proof. From the extension, we get that $\tilde{G}/(\mathbb{Z}/(n+1)\mathbb{Z}) \cong G$. Since $n+1 = p^l$, $\mathbb{Z}/(n+1)\mathbb{Z}$ is a p-group, and since G is also a p-group, then $\tilde{G}$ is a p-group too. □

Remark 4.5. *Let C be a plane curve with a finite fundamental group G. Let $n \in \mathbb{N}$. Let $\tilde{C}$ be the curve whose fundamental group $\tilde{G}$ is obtained from $G = \pi_1(\mathbb{P}^2 - C)$ by a central extension by $\mathbb{Z}/(n+1)\mathbb{Z}$. If $(|G|, n+1) = 1$, then the fundamental group $\tilde{G}$ of the resulting curve is a direct sum of G and $\mathbb{Z}/(n+1)\mathbb{Z}$:*

$$\tilde{G} \cong G \oplus \mathbb{Z}/(n+1)\mathbb{Z}$$

Proof. Use Theorem 7.77 of [20] that "if Q is a finite group, K is a finite abelian group, and $(|K|, |Q|) = 1$, then an extension G of Q by K is a semidirect product of K and Q", and the fact that if the extension is central, semidirect products become direct products. □

In the following proposition, we will list some more properties of the fundamental group which are preserved by the constructions. Before stating it, we remind some definitions. A group is called *polycyclic* if it has a subnormal series with cyclic factors. A group is called *supersolvable* if it has a normal series with cyclic factors. We say that a group G is *nilpotent* if its lower central series reaches 1 (see for example [20]).

Proposition 4.6. *Let C be a plane curve with a fundamental group G. Let $n \in \mathbb{N}$. Let $\tilde{C}$ be the curve whose fundamental group $\tilde{G}$ is obtained from $G = \pi_1(\mathbb{P}^2 - C)$ by a central extension by $\mathbb{Z}/(n+1)\mathbb{Z}$. Then if G has one of the following properties, the fundamental group $\tilde{G}$ of the resulting curve has this property too:*

1. *Finite.*
2. *Non-abelian.*
3. *Solvable.*
4. *Supersolvable.*
5. *Polycyclic.*
6. *Nilpotent.*
7. *Finitely presented.*

Proof. (1-2) Trivial.
(3) From the extension, we get that $\tilde{G}/(\mathbb{Z}/(n+1)\mathbb{Z}) \cong G$. But it is known [20, Theorem 5.17] that if $\mathbb{Z}/(n+1)\mathbb{Z}$ and G are both solvable, then $\tilde{G}$ is solvable too.
(4) From the extension, we have that $\tilde{G}/(\mathbb{Z}/(n+1)\mathbb{Z}) \cong G$. It is easy to show (very similar to the solvable case, see [20, Theorem 5.17]) that if $\mathbb{Z}/(n+1)\mathbb{Z}$ and G are both supersolvable, then $\tilde{G}$ is supersolvable too.
(5) Same proof as (4).
(6) Since $\tilde{G}$ is a central extension of G by $\mathbb{Z}/(n+1)\mathbb{Z}$ for a given n, we get that $\tilde{G}/(\mathbb{Z}/(n+1)\mathbb{Z}) \cong G$ where $\mathbb{Z}/(n+1)\mathbb{Z} \leq Z(\tilde{G})$. But it is easy to see that if $\mathbb{Z}/(n+1)\mathbb{Z} \leq Z(\tilde{G})$ and G is nilpotent, then $\tilde{G}$ is nilpotent too (see for example [20, p. 117, Exercise 5.38]).
(7) From the extension, we get that $\tilde{G}/(\mathbb{Z}/(n+1)\mathbb{Z}) \cong G$. Using a theorem of Hall (see for example [19, Theorem 2.2.4]) that "if N and G/N are finitely presented, then G is finitely presented", we are done. $\square$

We note here that if G is a nilpotent group of class c (means that the last non-zero term of the lower central series is the c-th term), then $\tilde{G}$ is a nilpotent group of class c or $c+1$. It will be of class c if and only if the 2-cocycle defining the extension is symmetric (see [21]).

Here we indicate one more family of group properties which are preserved by the constructions.

Remark 4.7. *Let C be a plane curve with a fundamental group G which has a subgroup N of finite index with a special property (for example: finite, solvable, nilpotent, etc.). Then the fundamental group $\tilde{G}$ of the resulting curve $\tilde{C}$ has a subgroup $\tilde{N}$ of finite index with the same special property too.*

In particular, if G is virtually-finite, virtually-nilpotent or virtually-solvable, then $\tilde{G}$ has the same property as well.

Proof. Let $n \in \mathbb{N}$. Since $\tilde{G}$ is an extension of G by $\mathbb{Z}/(n+1)\mathbb{Z}$, then $\tilde{G}/(\mathbb{Z}/(n+1)\mathbb{Z}) \cong G$. $N \leq G \cong \tilde{G}/(\mathbb{Z}/(n+1)\mathbb{Z})$, therefore there exists $\tilde{N} \leq \tilde{G}$ such that $\tilde{N}/(\mathbb{Z}/(n+1)\mathbb{Z}) \cong N$. Obviously $[\tilde{G} : \tilde{N}] = [G : N] < \infty$.

Since $\tilde{G}$ is a central extension, one can show that also $\tilde{N}$ is a central extension of N by $\mathbb{Z}/(n+1)\mathbb{Z}$ (since $\mathbb{Z}/(n+1)\mathbb{Z}$ is a subgroup of the intersection of $Z(\tilde{G})$ and $\tilde{N}$, and hence in $Z(\tilde{N})$).

Therefore, one can apply Proposition 4.6 to show that the properties of N are moved to $\tilde{N}$. $\square$

Remark 4.8. *Note that all the results of this section hold also for the constructions of Oka [15] and Shimada [22], since also in their constructions the fundamental group of the resulting curve is a central extension of the fundamental group of the original curve by a cyclic group.*

5 The curves obtained by the constructions

In this section we investigate the curves which can be obtained using Uludag's original construction and the general constructions we have presented in the previous sections. In the first subsection, we will describe the types of singularities which are added by the constructions. In Subsection 5.2 we compute the degrees of the resulting curves. In the next subsections, we describe some families of curves which can be obtained if we apply the constructions on several different types of curves.

5.1 The types of singularities which are added to the curves

At the beginning of this subsection, we want to fix a notation for singular points. We follow the notations of Flenner and Zaidenberg [7]. Any singular point P has a resolution by a sequence of s blow ups. We denote by $t_i > 1$ $(1 \leq i \leq s)$ the multiplicity of the curve at P before the i-th blow up. Then $[t_1, \cdots, t_s]$ is called *the type of the singularity.* If we have a sequence of r equal multiplicities l, we abbreviate it by l_r. For example, $[2, 2, 2] = [2_3]$ corresponds to a ramphoid cusp, which has to be blown up three times (each time of multiplicity 2) for smoothing it. We note that in general it is possible that a blow up can split a singular point into two or more singular points, and then we have to do two or more different resolutions for each singular point, and it indeed happens in the last generalization (Corollary 3.9), but except for this case, in all the other cases such a situation cannot occur, since we choose for the construction fibers which meet the curve not in singular points.

To simplify the description, we also introduce the notion of a *d-tacnode.*

Definition 5.1. d**-tacnode** *is a singular point where d smooth branches of the curve are tangented to the same line in the same point.*

For example, the usual tacnode is 2-tacnode.

Here we describe the singularities which are added to the curve during these constructions:

Proposition 5.2. *Let C be a curve of degree d. Let $n \in \mathbb{N}$. Let $\tilde{C}$ be the curve which is obtained by Uludag's original construction for this n (see Section 2).*

Then $\tilde{C}$ has two additional singularities to those of C: a d-tacnode of order $n-1$, and another singular point which is a blow down of a d-tacnode of order $n-1$ (i.e., the curve has the following additional singularities: $[d_n]$ and $[dn, d_n]$).

Proof. The first blow-up (from $\mathbb{P}^2$ to F_1) does not change the curve. The first elementary transformation of the first type (from F_1 to F_2) creates one intersection point (of d branches) on one of the fibers. The second elementary transformation of the first type (from F_2 to F_3) converts it into a d-tacnode of order 1. Another $n-2$ elementary transformations of the first type convert it into a d-tacnode of order $n-1$. Now, the first elementary transformation of the second type (from F_{n+1} to F_n) creates another intersection point (of d branches) which is located on the exceptional section. The second elementary transformation of the second type (from F_n to F_{n-1}) converts it into a d-tacnode of order 1. Another $n-2$ elementary transformations of the second type convert it into a d-tacnode of order $n-1$, which is located on the exceptional section. Therefore, when we blow down this section in order to return to $\mathbb{P}^2$, this d-tacnode of order $n-1$ is blown down into a more complicated singular point. □

Now we describe the corresponding situation for the general construction:

Proposition 5.3. *Let C be a curve of degree d. Let $(n_1, \cdots, n_k) \in \mathbb{N}^k$ be a k-tuple. Let $\tilde{C} = T_{(n_1,\cdots,n_k)}(C)$ (see Theorem 3.8).*

Then $\tilde{C}$ has $k+1$ additional singularities to those of C: k d-tacnodes of order $n_i - 1$ $(1 \leq i \leq k)$, and another singular point which is a blow down of a d-tacnode of order $n_1 + \cdots + n_k - 1$ (i.e., the curve has the following additional singularities: $[d_{n_1}], \cdots, [d_{n_k}]$ and $[d(n_1 + \cdots + n_k), d_{(n_1+\cdots+n_k)}]$).

Proof. As in Uludag's original construction, the first blow-up (from $\mathbb{P}^2$ to F_1) does not change the curve. Each sequence of n_i elementary transformations of the first type on the fiber Q_i^1 and its images creates one d-tacnode of order $n_i - 1$, which is located at the image of Q_i^1, outside the exceptional section.

The sequence of $n_1 + \cdots + n_k$ elementary transformations of the second type (from $F_{n_1+\cdots+n_k+1}$ to F_1) creates another d-tacnode of order $n_1 + \cdots + n_k - 1$, which is now located on the exceptional section. Hence, when we blow down this section in order to return to $\mathbb{P}^2$, this d-tacnode of order $n_1 + \cdots + n_k - 1$ is blown down too to a more complicated singular point. $\square$

Here we describe the situation concerning the special case (Proposition 3.10).

Proposition 5.4. *Let C be a curve of degree d. Let $n \in \mathbb{N}$. Let $\tilde{C} = T_n(C)$ (see Proposition 3.10).*

Then $\tilde{C}$ has one additional singularity to those of C: a blow down of a d-tacnode of order $2n-1$ (i.e., the curve has the following additional singularity: $[2nd, d_{2n}]$).

Proof. As before, the first blow-up does not change the curve. The sequence of n elementary transformations of the first type creates one d-tacnode of order $n-1$. Since we apply the second sequence of n elementary transformations of the second type on the same fiber, we continue to deepen this singularity into a d-tacnode of order $2n-1$ which is now located also on the exceptional section. Hence, when we blow down this section in order to return to $\mathbb{P}^2$, this d-tacnode of order $2n-1$ is blown down too to a more complicated singular point. $\square$

The description of the curves obtained by Corollary 3.9 is a little bit more complicated: In this case we indeed have a singular point which is splitted after the first blow-up into several singular points.

Proposition 5.5. *Let C be a curve of degree d. Let $(n_1, \cdots, n_k, m_1, \cdots, m_l) \in \mathbb{N}^{k+l}$ be a $(k+l)$-tuple such that $\sum n_i = \sum m_j$. Let $\tilde{C} = T_{(n_1,\cdots,n_k;m_1,\cdots,m_l)}(C)$ (see Corollary 3.9).*

Then $\tilde{C}$ has $k+1$ additional singularities to those of C: k d-tacnodes of order $n_i - 1$ $(1 \le i \le k)$, and another singular point which is a blow down of l d-tacnodes of order $m_j - 1$ $(1 \le j \le l)$ on the exceptional section of that blow down (i.e., the curve has the following additional singularities: $[d_{n_1}], \cdots, [d_{n_k}]$ and $[d(n_1 + \cdots + n_k), ([d_{m_1}], \cdots, [d_{m_l}])]$).

Proof. The proof is similar to the proof of the previous proposition. As before, the elementary transformations of the first type create k d-tacnodes of order $n_i - 1$.

The sequences of m_j $(1 \leq j \leq l)$ elementary transformations of the second type on the fibers $P_1, \cdots, P_l$ create l d-tacnodes of order $m_j - 1$, which are all located on the exceptional section. Hence, when we blow down this section in order to return to $\mathbb{P}^2$, these l d-tacnodes are blown down together to a complicated singular point. $\square$

5.2 Change of the degree of the curve

In this subsection, we compute the degree of the resulting curve:

Proposition 5.6. *Let C be a plane projective curve of degree d. Let $(n_1, \cdots, n_k) \in \mathbb{N}^k$. Let $\tilde{C} = T_{(n_1,\cdots,n_k)}(C)$ (see Theorem 3.8).*

Then the degree of the resulting curve is $d(n_1 + \cdots + n_k + 1)$.

Proof. Let $\tilde{d}$ be the degree of $\tilde{C}$. We have to show that $\tilde{d} = d(n_1+\cdots+n_k+1)$. When we blow up once one of the singularities, say P, in order to resolve it, we have to decrease the self-intersection of the original curve $\tilde{C}$ by $(\mathrm{mult}_{\tilde{C}} P)^2$ (where $\mathrm{mult}_{\tilde{C}} P$ is the local multiplicity of $\tilde{C}$ at P) to get the self-intersection of the curve $\tilde{C}$ after the blow-up. Since this is the data which is given by the types of the singularities, one can compute easily the change in the self-intersection.

So, we start with $\tilde{C}$ whose self-intersection is $\tilde{d}^2$, since $\tilde{C}$ is in $\mathbb{P}^2$. For all $1 \leq i \leq k$, the n_i blow-ups of the singular point $[d_{n_i}]$ yield a decreasing of the self-intersection by $n_i \cdot d^2$, since the multiplicity of the curve at the singular point is d. The $n_1 + \cdots + n_k + 1$ blow-ups of the singular point of the type $[d(n_1 + \cdots + n_k), d_{n_1+\cdots+n_k}]$ yield an additional decreasing of the self-intersection by $(d(n_1+\cdots+n_k))^2+(n_1+\cdots+n_k)d^2$, since the multiplicity of the curve at the singular point in the first blow up is $d(n_1 + \cdots + n_k)$ and in the other blow ups it is again d. After all these blow-ups, we reach the original curve C in $\mathbb{P}^2$ and hence its self-intersection is d^2. Therefore, we have the following equation:

$$\tilde{d}^2 - \sum_{i=1}^{k}(n_i \cdot d^2) - ((d(n_1 + \cdots + n_k))^2 + (n_1 + \cdots + n_k)d^2) = d^2$$

and hence $\tilde{d}^2 = d^2(n_1 + \cdots + n_k + 1)^2$, which gives us $\tilde{d} = d(n_1 + \cdots + n_k + 1)$ as needed. $\square$

One can perform the same computations also for the curves obtained by the constructions presented in Corollary 3.9 and Proposition 3.10 (see Propositions 5.5 and 5.4 respectively for the descriptions of the additional singularities). Hence, Proposition 5.6 holds for those curves too.

5.3 Families of curves obtained by starting with smooth irreducible curves

In this subsection we describe the families of curves which are obtained by Uludag's original construction and its generalizations if we apply them to a smooth irreducible curve C of degree d (and therefore $\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/d\mathbb{Z}$, see Zariski [32]).

Proposition 5.7. *Let C be a smooth irreducible curve of degree d. Let $n \in \mathbb{N}$. Let $\tilde{C}$ be the curve obtained by Uludag's construction in such a way that its fundamental group is a central extension of $G = \pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/d\mathbb{Z}$ by $\mathbb{Z}/(n+1)\mathbb{Z}$.*

Then for $n = 1$, $\tilde{C}$ has an intersection point of d smooth branches and one d-tacnode (i.e., the curve has the singularities: $[d]$ and $[d_2]$).

For $n \geq 2$, $\tilde{C}$ has a d-tacnode of order $n-1$, and another singular point which is a blow down of a d-tacnode of order $n-1$ (i.e., the curve has the following singularities: $[d_n]$ and $[dn, d_n]$).

The degree of the resulting curve is $d(n+1)$.

Proof. Since a smooth curve has no singularities, then the only singularities of the resulting curve are those which were created by Uludag's construction (see Proposition 5.2). Therefore the curve has only the singularities described in Proposition 5.2.

The degree of the curve is computed directly by Proposition 5.6. □

For the particular case $d = 2$ and $n = 1$, we indeed get a quadric with a node and a tacnode, and its equation can be found in [14, p. 147, case 2]: $(x^2 + y^2 - 3x)^2 = 4x^2(2 - x)$.

Using Proposition 4.2, we have that

$$\pi_1(\mathbb{P}^2 - \tilde{C}) \cong \mathbb{Z}/(d(n+1))\mathbb{Z}.$$

Now, we describe the family of curves which are obtained by the general construction (Subsection 3.4).

Proposition 5.8. *Let C be a smooth irreducible curve. Let $(n_1, \cdots, n_k) \in \mathbb{N}^k$ be a k-tuple. Let $\tilde{C} = T_{(n_1,\cdots,n_k)}(C)$ (see Theorem 3.8).*

For every $1 \leq i \leq k$, $\tilde{C}$ has a d-tacnode of order $n_i - 1$, and another singular point which is a blow down of a d-tacnode of order $n_1 + \cdots + n_k - 1$ (i.e., the curve has the following singularities: $[d_{n_1}], \cdots, [d_{n_k}]$ and $[d(n_1 + \cdots + n_k), d_{n_1+\cdots+n_k}]$).

The degree of the resulting curve is $d(n_1 + \cdots + n_k + 1)$.

Proof. Similar to the proof of the previous proposition, but here we use the results of Proposition 5.3. □

Using Proposition 4.2 again, we have that

$$\pi_1(\mathbb{P}^2 - \tilde{C}) \cong \mathbb{Z}/(d(n_1 + \cdots + n_k + 1))\mathbb{Z}.$$

5.4 Families of curves obtained by starting with line arrangements

In this subsection we describe the families of curves and their groups which are obtained by Uludag's original construction and its generalizations if we apply them to some types of line arrangements.

Proposition 5.9. *Let $\mathcal{L}$ be a line arrangement consists of m lines intersecting in one point. Let $n \in \mathbb{N}$. Let $\tilde{\mathcal{L}}$ be the curve obtained by Uludag's construction in such a way that its fundamental group is a central extension of $G = \pi_1(\mathbb{P}^2 - \mathcal{L}) = \mathbb{F}_{m-1}$ by $\mathbb{Z}/(n+1)\mathbb{Z}$.*

Then for $n = 1$, $\tilde{\mathcal{L}}$ has two intersection points of m smooth branches and one m-tacnode (i.e., the curve has the following singularities: $[m]$,$[m]$ and $[m_2]$).

For $n \geq 2$, $\tilde{\mathcal{L}}$ has one intersection points of m smooth branches, one m-tacnode of order $n - 1$, and another singular point which is a blow down of a m-tacnode of order $n - 1$ (i.e., the curve has the following singularities: $[m]$,$[m_n]$ and $[mn, m_n]$).

The degree of the resulting curve is $m(n + 1)$.

Proof. Since $\mathcal{L}$ has one intersection point of m smooth branches, then the singularities of the resulting curve are those which were created by Uludag's construction (see Proposition 5.2) and an additional singularity which was in $\mathcal{L}$.

The degree of the curve is computed directly by Proposition 5.6. □

Since $H^2(\mathbb{F}_{m-1}, \mathbb{Z}/(n+1)\mathbb{Z})$ is trivial, then we get that:

$$\pi_1(\mathbb{P}^2 - \tilde{\mathcal{L}}) \cong \mathbb{F}_{m-1} \oplus \mathbb{Z}/(n+1)\mathbb{Z}.$$

Now, we describe the family of curves which are obtained by the general construction (Subsection 3.4).

Proposition 5.10. *Let $\mathcal{L}$ be a line arrangement consists of m lines intersecting in one point. Let $(n_1, \cdots, n_k) \in \mathbb{N}^k$ be a k-tuple. Let $\tilde{\mathcal{L}} = T_{(n_1,\cdots,n_k)}(\mathcal{L})$ (see Theorem 3.8).*

Then: in addition to the original intersection point of $\mathcal{L}$, for every $1 \le i \le k$, $\tilde{\mathcal{L}}$ has a m-tacnode of order $n_i - 1$, and another singular point which is a blow down of a m-tacnode of order $n_1+\cdots+n_k-1$ (i.e., the curve has the following singularities: $[m], [m_{n_1}], \cdots, [m_{n_k}]$ and $[m(n_1+\cdots+n_k), m_{n_1+\cdots+n_k}]$).

The degree of the resulting curve is $m(n_1 + \cdots + n_k + 1)$.

Proof. Similar to the proof of the previous proposition, but here we use the results of Proposition 5.3. □

As before, since $H^2(\mathbb{F}_{m-1}, \mathbb{Z}/(n_1 + \cdots + n_k + 1)\mathbb{Z})$ is trivial, then we get that:

$$\pi_1(\mathbb{P}^2 - \tilde{\mathcal{L}}) \cong \mathbb{F}_{m-1} \oplus \mathbb{Z}/(n_1 + \cdots + n_k + 1)\mathbb{Z}.$$

Now we deal with another important type of line arrangements: lines in a general position, which means that there is no intersection of more than two lines in a point. We describe the family of curves which are obtained by the general construction (Subsection 3.4).

Proposition 5.11. *Let $\mathcal{L}$ be a line arrangement consists of m lines in a general position. Let $(n_1, \cdots, n_k) \in \mathbb{N}^k$ be a k-tuple. Let $\tilde{\mathcal{L}} = T_{(n_1,\cdots,n_k)}(\mathcal{L})$ (see Theorem 3.8).*

Then: in addition to $\binom{m}{2}$ nodal points of $\mathcal{L}$, for every $1 \le i \le k$, $\tilde{\mathcal{L}}$ has a m-tacnode of order $n_i - 1$, and another singular point which is a blow down of a m-tacnode of order $n_1 + \cdots + n_k - 1$ (i.e., the curve has the following singularities: $[m_{n_1}], \cdots, [m_{n_k}]$, $[m(n_1 + \cdots + n_k), m_{n_1+\cdots+n_k}]$ and $\binom{m}{2}$ singularities of the type $[2]$).

The degree of the resulting curve is $m(n_1 + \cdots + n_k + 1)$.

Proof. Since $\mathcal{L}$ has $\binom{m}{2}$ nodal points, then the singularities of the resulting curve are those which were created by the general construction (see Proposition 5.3) and $\binom{m}{2}$ nodal points.

The degree of the curve is computed directly by Proposition 5.6. □

Since a central extension of $\mathbb{Z}^{m-1}$ by $\mathbb{Z}/(n_1+\cdots+n_k+1)\mathbb{Z}$ is not unique, it is interesting to know which group is indeed obtained in this case. For this, we perform a direct computation for finding a presentation for the fundamental group of the complement of $\tilde{\mathcal{L}}$, to get that

$$\pi_1(\mathbb{P}^2 - \tilde{\mathcal{L}}) \cong \mathbb{Z}^{m-1} \oplus \mathbb{Z}/(n_1 + \cdots + n_k + 1)\mathbb{Z}$$

This result is mainly achieved due to the commutative relations induced by the $\binom{m}{2}$ nodal points.

6 An application to Zariski pairs

As already mentioned, we call *a Zariski pair* to a pair of plane curves which have the same singularities, but their complements have non-isomorphic fundamental groups. In the same manner, we call *a Zariski triple* or *Zariski k-tuple* to a triple of curves or to a k-tuple of curves respectively, which have the same singularities, but their complements have pairwise non-isomorphic fundamental groups.

In this short section, we want to use the above constructions to produce new Zariski pairs.

Not every Zariski pair (C_1, C_2) can produce a family of Zariski pairs by our construction, since even if $G_1 = \pi_1(\mathbb{P}^2 - C_1)$ and $G_2 = \pi_1(\mathbb{P}^2 - C_2)$ are different, it is not guaranteed that $\tilde{G}_1 = \pi_1(\mathbb{P}^2 - \tilde{C}_1)$ and $\tilde{G}_2 = \pi_1(\mathbb{P}^2 - \tilde{C}_2)$ will be still different, as there are several ways to construct the same group by central extensions. Therefore, we have to characterize Zariski pairs which induce such families.

A possible characterization is the following:

Proposition 6.1. *Let (C_1, C_2) be a Zariski pair of two irreducible curves. If $\pi_1(\mathbb{P}^2 - C_1)$ is a cyclic group and $\pi_1(\mathbb{P}^2 - C_2)$ is not a cyclic group, then $(\tilde{C}_1, \tilde{C}_2)$ is a Zariski pair.*

Proof. Since (C_1, C_2) is a Zariski pair, then by definition C_1 and C_2 have the same degree and the same singularities. Therefore, using the results of Section 5, $\tilde{C}_1$ and $\tilde{C}_2$ have the same degree and the same singularities too.

Since $\pi_1(\mathbb{P}^2 - C_1)$ is cyclic and C_1 is irreducible, then $\pi_1(\mathbb{P}^2 - \tilde{C}_1)$ is cyclic too (by Proposition 4.2). On the other hand, a central extension of non-cyclic group can never be cyclic and hence $\pi_1(\mathbb{P}^2 - \tilde{C}_2)$ is not cyclic. Therefore, $(\tilde{C}_1, \tilde{C}_2)$ is a Zariski pair too. $\square$

The examples of Zariski ([32],[33]), Oka [17] and Shimada [23] satisfy the conditions of Proposition 6.1, and hence produce families of new examples of Zariski pairs.

7 Another generalization of Uludag's method

In this section we generalize Uludag's method in a different way. Now, we are not starting with lines that are transversally intersected with the curve, and we permit also tangent lines or lines which are intersected with the curve in its singular points. The rest of the construction is the same as in the previous constructions.

As in the previous constructions, we start with a curve C and $k+l$ lines (which are not transversal anymore to the curve) and a $(k+l)$-tuple $(n_1,\cdots,n_k,m_1,\cdots,m_l)\in\mathbb{N}^{k+l}$ such that $\sum n_i=\sum m_j$, and perform a similar construction to the one described in Subsection 3.5. We denote the resulting curve by $\tilde{C}=\tilde{T}_{(n_1,\cdots,n_k;m_1,\cdots,m_l)}(C)$.

In this situation, we get a weaker result, but even though we loose only few of the properties of groups which were preserved by the previous constructions.

Theorem 7.1. *Let C be a plane projective curve and $G=\pi_1(\mathbb{P}^2-C)$. Then for any $(k+l)$-tuple $(n_1,\cdots,n_k,m_1,\cdots,m_l)\in\mathbb{N}^{k+l}$ such that $\sum n_i=\sum m_j$, the curve $\tilde{C}=\tilde{T}_{(n_1,\cdots,n_k;m_1,\cdots,m_l)}(C)$ is birational to C and its fundamental group $\tilde{G}=\pi_1(\mathbb{P}^2-\tilde{C})$ is an extension of G by $\mathbb{Z}/((\sum_{i=1}^k n_i)+1)\mathbb{Z}$:*

$$1\to\mathbb{Z}/((\sum_{i=1}^k n_i)+1)\mathbb{Z}\to\tilde{G}\to G\to 1$$

Moreover, if C is irreducible so is $\tilde{C}$.

Proof. If we go through the proofs of the previous constructions (Theorem 3.8 and Corollary 3.9), we can see that the transversality of the lines to the curve is actually needed only for proving that the extension is central, but all the rest of the proof does not need it (the quotient equations hold even for non-transversal lines, see Section 3.2). So, although we lose the centrality of the extension, but the extension itself is still proved, and hence we have the result. □

Despite the weaker version of the result, most of the properties of groups which are preserved by the previous constructions (as was indicated in Section 4) hold also for this construction. The properties which are not preserved anymore are cyclicity, nilpotency, virtually-nilpotency and Remark 4.5. For all the other properties, there is no use of the extension's centrality in the proofs of Section 4.

The results of Section 5 do not hold in this case, since their proofs are based very much on the assumption that the lines intersect the curve transversally. Also, the singularities depend very much on the way the lines intersect the curve.

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