ON PRESENTATIONS OF SURFACE BRAID GROUPS

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Abstract. We give presentations of braid groups and pure braid groups on orientable surfaces and we show some properties of surface pure braid groups.

1. Introduction

1.1. **Braids on surfaces.** Let F be an orientable surface and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a set of n distinct points of F. A *geometric braid* on F based at $\mathcal{P}$ is an n-tuple $\Psi = (\psi_1, \dots, \psi_n)$ of paths $\psi_i : [0,1] \to F$ such that

- $\psi_i(0) = P_i,\ i = 1 \dots, n;$
- $\psi_i(1) \in \mathcal{P},\ i = 1 \dots, n;$
- $\psi_1(t), \dots, \psi_n(t)$ are distinct points of F for all $t \in [0,1]$.

The usual product of paths defines a group structure on the set of braids up to homotopies among braids. This group, denoted $B(n,F)$, does not depend on the choice of $\mathcal{P}$ and it is called the braid group on n strings on F. On the other hand, let be $F_nF = F^n \setminus \Delta$, where Δ is the big diagonal, i.e. the n-tuples $x = (x_1, \dots x_n)$ for which $x_i = x_j$ for some $i \neq j$. There is a natural action of Σ_n on F_nF by permuting coordinates. We call the orbit space $\hat{F}_nF = F_nF/\Sigma_n$ *configuration space*. Then the braid group $B(n,F)$ is isomorphic to $\pi_1(\hat{F}_nF)$. We recall that the pure braid group $P(n,F)$ on n strings on F is the kernel of the natural projection of $B(n,F)$ in the permutation group Σ_n. This group is isomorphic to $\pi_1(F_nF)$.

The first aim of this article is to give (new) presentations for braid groups on orientable surfaces.

A p-punctured surface of genus $g \geq 1$ is the surface obtained by deleting p points on a closed surface of genus $g \geq 1$.

Theorem 1.1. *Let F be an orientable p-punctured surface of genus $g \geq 1$. The group $B(n,F)$ admits the following presentation (see also section 2.2):*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_g, b_1, \dots, b_g, z_1, \dots, z_{p-1}$.
- *Relations:*
  - *Braid relations, i.e.*

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}\,; \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \textit{for } |i-j| \geq 2\,. \end{aligned}$$

  - *Mixed relations:*

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$$
\begin{array}{ll}
(R1) & a_r\sigma_i = \sigma_i a_r \quad (1 \le r \le g;\ i \ne 1)\,; \\
 & b_r\sigma_i = \sigma_i b_r \quad (1 \le r \le g;\ i \ne 1)\,; \\
(R2) & z_j\sigma_i = \sigma_i z_j \quad (i \ne n-1, j = 1, \dots, p-1)\,; \\
(R3) & \sigma_1^{-1} z_i \sigma_1 a_r = a_r \sigma_1^{-1} z_i \sigma_1 \quad (1 \le r \le g;\ i = 1, \dots, p-1;\ n > 1)\,; \\
 & \sigma_1^{-1} z_i \sigma_1 b_r = b_r \sigma_1^{-1} z_i \sigma_1 \quad (1 \le r \le g;\ i = 1, \dots, p-1;\ n > 1)\,; \\
(R4) & \sigma_1^{-1} z_j \sigma_1 z_l = z_l \sigma_1^{-1} z_j \sigma_1 \quad (j = 1, \dots, p-1,\ j < l)\,; \\
(R5) & \sigma_1^{-1} z_j \sigma_1^{-1} z_j = z_j \sigma_1^{-1} z_j \sigma_1^{-1} \quad (j = 1, \dots, p-1)\,; \\
(R6) & \sigma_1^{-1} a_r \sigma_1^{-1} a_r = a_r \sigma_1^{-1} a_r \sigma_1^{-1} \quad (1 \le r \le g)\,; \\
 & \sigma_1^{-1} b_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} b_r \sigma_1^{-1} \quad (1 \le r \le g)\,; \\
(R7) & \sigma_1^{-1} a_s \sigma_1 a_r = a_r \sigma_1^{-1} a_s \sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1} b_s \sigma_1 b_r = b_r \sigma_1^{-1} b_s \sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1} a_s \sigma_1 b_r = b_r \sigma_1^{-1} a_s \sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1} b_s \sigma_1 a_r = a_r \sigma_1^{-1} b_s \sigma_1 \quad (s < r)\,; \\
(R8) & \sigma_1^{-1} a_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} a_r \sigma_1 \quad (1 \le r \le g)\,.
\end{array}
$$

Theorem 1.2. *Let F be a closed orientable surface of genus $g \ge 1$. The group $B(n, F)$ admits the following presentation:*

- *Generators: $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_g\, b_1, \dots, b_g$.*
- *Relations:*
  - *Braid relations as in Theorem 1.1.*
  - *Mixed relations:*

$$
\begin{array}{ll}
(R1) & a_r\sigma_i = \sigma_i a_r \quad (1 \le r \le g;\ i \ne 1)\,; \\
 & b_r\sigma_i = \sigma_i b_r \quad (1 \le r \le g;\ i \ne 1)\,; \\
(R2) & \sigma_1^{-1} a_r \sigma_1^{-1} a_r = a_r \sigma_1^{-1} a_r \sigma_1^{-1} \quad (1 \le r \le g)\,; \\
 & \sigma_1^{-1} b_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} b_r \sigma_1^{-1} \quad (1 \le r \le g)\,; \\
(R3) & \sigma_1^{-1} a_s \sigma_1 a_r = a_r \sigma_1^{-1} a_s \sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1} b_s \sigma_1 b_r = b_r \sigma_1^{-1} b_s \sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1} a_s \sigma_1 b_r = b_r \sigma_1^{-1} a_s \sigma_1 \quad (s < r)\,; \\
 & \sigma_1^{-1} b_s \sigma_1 a_r = a_r \sigma_1^{-1} b_s \sigma_1 \quad (s < r)\,; \\
(R4) & \sigma_1^{-1} a_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} a_r \sigma_1 \quad (1 \le r \le g)\,; \\
(TR) & [a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}] = \sigma_1 \sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2 \sigma_1\,,
\end{array}
$$

where $[a, b] := aba^{-1}b^{-1}$.

We may assume that Theorem 1.1 provides also a presentation for $B(n, F)$, when F an orientable surface with p boundary components. We recall that a subsurface E of a surface F is the closure of an open set of F. In order to avoid pathology, we assume that E is connected and that every boundary component of E either is a boundary component of F or lies in the interior of F. We suppose also that E contains $\mathcal{P}$. It is known [16] that the natural map $\psi_n : B(n, E) \to B(n, F)$ induced by the inclusion $E \subseteq F$ is injective if and only if $\overline{F \setminus E}$ does not contain a disk. We may provide an analogous characterisation about surjection.

Proposition 1.1. *Let F be an orientable surface of genus $g \geq 1$ with p boundary components, and let E be a subsurface of F. The natural map $\psi_n : B(n,E) \to B(n,F)$ induced by the inclusion $E \subseteq F$ is surjective if and only if $\overline{F \setminus E} = \amalg D^2$.*

Proof: Remark that the natural morphism

$$\psi_1 : \pi(E, P_1) \to \pi(F, P_1)$$

is a surjection if and only if $\overline{F \setminus E} = \amalg D^2$. Now consider a pure braid $p \in P(n,F)$ as a n-tuple of paths $(p_1, \ldots, p_n)$ and let $\chi : P(n,F) \to \pi(F)^n$ be the map defined by $\chi(p) = (p_1, \ldots, p_n)$. We have the following commutative diagram

$$\begin{array}{ccc} P(n,E) & \xrightarrow{\ \chi\ } & \pi(E)^n \\ \psi_n \downarrow & & \downarrow \psi_1 \times \cdots \times \psi_1 \\ P(n,F) & \xrightarrow{\ \chi\ } & \pi(F)^n \end{array}$$

Since χ is surjective we deduce that ψ_n is not surjective on $P(n,F)$ and thus on $B(n,F)$. The last part of the Proposition follows from previous Theorem and considerations on corresponding geometric braids. □

When F is a closed orientable surface, our presentations are equivalent to Gonzalez-Meneses' presentations. We recall also that the first presentations of braid groups on closed surfaces were found by Scott ([17]), afterwards revised by Kulikov and Shimada ([13]). At our knowledge, the case of punctured surfaces is new in the literature. Our proof is inspired by Morita's combinatorial proof for the classical presentation of Artin's braid group ([14]). This proof holds also for Sergiescu's presentations (see [18]). We will explain this approach while proving Theorem 1.1. After that we will show how to make this technique fit for obtaining Theorem 1.2.

1.2. **Residual properties of surface pure braids.** The last part of the article concerns the study of surface pure braids groups. When F is a closed surface of genus $g > 0$, we provide in Theorem 5.1 a homogeneous presentation for $P(n,F)$ with $2gn$ generators. Let $K_n(F)$ be the normal closure of classical pure braid group P_n in $P(n,F)$. This group has been introduced in [2] and it has been used in [10] in order to define Vassiliev invariants for surface braid groups. We show that $K_n(F)$ is isomorphic to $[P(n,F), P(n,F)]$ if and only if $F = T^2$. Otherwise we have the strict inclusion $K_n(F) \subset [P(n,F), P(n,F)]$. Let F be a surface of genus $g > 0$ with $p > 0$ boundary components. Consider the sub-surface E obtained removing g handles. Let $Y_n(F)$ be the normal closure of $P(n,E)$ in $P(n,F)$. We extend to $Y_n(F)$ some results shown in [10] on $K_n(F)$, i.e.

- $\bigcap_{d=0}^{\infty} I(Y_n(F))^d = \{0\}$;
- $I(Y_n(F))^d/(Y_n(F))^{d+1}$ is a free $\mathbb{Z}$-module for all $d \geq 0$,

where I^k means the k-th power of the augmentation ideal of the group ring of $Y_n(F)$.

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2. Preliminaries

2.1. **Fadell-Neuwirth fibrations.** The main tool one uses is the Fadell-Neuwirth fibration, with its generalisation and the corresponding exact sequences. As observed in [4], if F is a surface (closed or punctured, orientable or not), the map $\theta : F_nF \to F_{n-1}F$ defined by

$$\theta(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$$

is a fibration with fiber $F \setminus \{x_1, \ldots, x_{n-1}\}$. The exact homotopy sequence of the fibration gives us the exact sequence

$$\cdots \pi_2(F_nF) \to \pi_2(F_{n-1}F) \to \pi_1(F \setminus \{x_1, \ldots, x_{n-1}\})$$
$$\to P(n, F) \to P(n-1, F) \to 1.$$

Since a punctured surface (with at least one puncture) has the homotopy type of a one dimensional complex, we deduce

$$\pi_k(F_nF) \cong \pi_k(F_{n-1}F) \cong \cdots \cong \pi_k(F), \quad k \geq 3$$

and

$$\pi_2(F_nF) \subseteq \pi_2(F_{n-1}F) \subseteq \cdots \subseteq \pi_2(F)\,.$$

If F is an orientable surface and $F \neq S^2$, all higher homotopy groups are trivial. Thus, if F is an orientable surface different from the sphere we can conclude that there is an exact sequence

$$(PBS) \quad 1 \longrightarrow \pi_1(F \setminus \{x_1, \ldots, x_{n-1}\}) \longrightarrow P(n, F) \xrightarrow{\theta} P(n-1, F) \to 1,$$

where θ is the map that "forgets" the last path pointed at x_n.

The problem of the existence of a section for (PBS) has been completely solved in [11]. It is possible to show that θ admits a section, when $p > 0$. On the other hand, when F is a closed orientable surface of genus $g \geq 2$, (PBS) splits if and only if $n = 2$. An explicit section is shown in [2] in the case of the torus.

2.2. **Geometric interpretations of generators and relations.** Let F be an orientable surface. Let $\widetilde{B}(n, F)$ be the group with the presentation given in Theorem 1.1 or Theorem 1.2 respectively. The geometric interpretation for generators of $\widetilde{B}(n, F)$, when F is a closed surface of genus $g \geq 1$ is the same as in [8], except that we represent F as a polygon L of $4g$ sides with the standard identification of edges (see also section 6.2). We can consider braids as paths on L, which we draw with the usual "over and under" information at the crossing points. Figure 1 presents the generators of $\widetilde{B}(n, F)$ realized as braids on L.

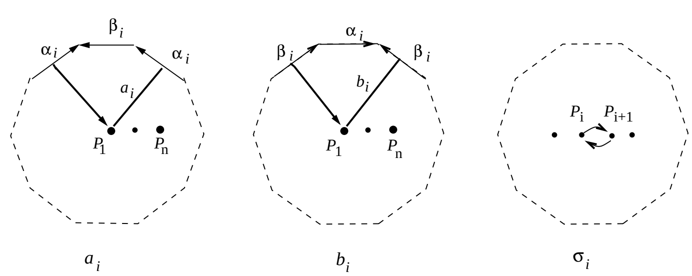

Figure 1. Generators as braids (for F an orientable closed surface).

Note that in the braid a_i (respectively b_i) the only non trivial string is the first one, which goes through the α_i-th wall (β_i-th wall). Remark also that $\sigma_1 \dots, \sigma_{n-1}$ are the classical braid generators on the disk.

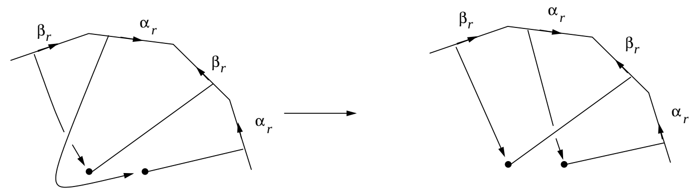

FIGURE 2. Geometric interpretation for relation (R4) in Theorem 1.1; homotopy between $\sigma_1^{-1}a_r\sigma_1^{-1}b_r$ (on the left) and $b_r\sigma_1^{-1}a_r\sigma_1$ (on the right).

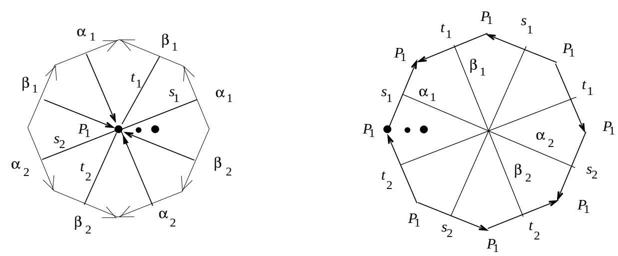

FIGURE 3. The fundamental domain L_1.

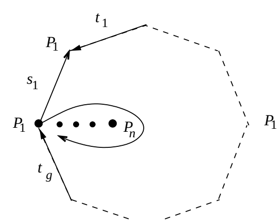

FIGURE 4. Braid $[a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}]$.

It is easy to check that the relations above hold in $B(n, F)$. The non trivial strings of a_r (b_r) and σ_i when $i \neq 1$, may be considered to be disjoint and then ($R1$) holds in $B(n, F)$. On the other hand, $\sigma_1^{-1}a_r\sigma_1^{-1}$ is the braid whose the only non trivial string is the second one, which goes through the r-th wall and disjoint from the corresponding non trivial string of a_r. Then $\sigma_1^{-1}a_r\sigma_1^{-1}$ and a_r commute. Similarly we have that $\sigma_1^{-1}b_r\sigma_1^{-1}$ and b_r commute and ($R2$) is verified. The case of ($R3$) is similar. Figure 2 presents a sketch of a homotopy between with $\sigma_1^{-1}a_r\sigma_1^{-1}b_r$ and $b_r\sigma_1^{-1}a_r\sigma_1$. Thus, ($R4$) holds in $B(n, F)$.

Let s_r (t_r) be the first string of a_r (b_r), for $r = 1, \dots, 2g$, and consider all the paths $s_1, t_1, \dots, s_g, t_g$. We cut L along them and we glue the pieces along the edges of L. We obtain a new fundamental domain (see Figure 3, for the case of a surface of genus 2), called L_1, with vertex P_1. On L_1 it is clear that $[a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}]$ is equivalent to the braid of Figure 4, equivalent to the braid $\sigma_1\sigma_2 \ldots \sigma_{n-1}^2 \cdots \sigma_2\sigma_1$ and then (TR) is verified in $B(n, F)$.

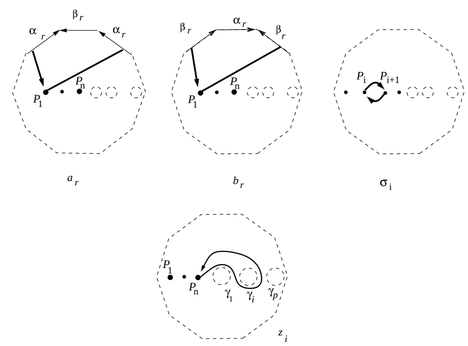

FIGURE 5. Generators as braids (for F an orientable surface with p boundary components).

There is an analogous geometric interpretation of generators of $\widetilde{B}(n, F)$, for F an orientable p-punctured surface. The definition of generators σ_i, a_j, b_j is the same as above. We only have to add generators z_i which represent a loop of the first string around the i-th boundary component (Figure 5), except the p-th one. As above, relations can be easily checked on corresponding paths. Remark that a loop of the first string around the p-th boundary component can be represented by the geometric braid corresponding to the element

$$[a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}]\sigma_1^{-1} \cdots \sigma_{n-1}^{-1} \cdots \sigma_1^{-1} z_1^{-1} \cdots z_{p-1}^{-1}.$$

Therefore, one has natural morphisms $\tilde{\phi}_n : \widetilde{B}(n, F) \to B(n, F)$. One further shows that $\tilde{\phi}_n$ are actually isomorphisms. From now on we replace the set of generators $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ with the set $\{c_1, \ldots, c_{2g}\}$, where $c_{2k-1} := a_k$ and $c_{2k} := b_k$.

3. Outline of the proof of Theorem 1.1

3.1. **The inductive assertion.** We outline the ideas of the proof for F a surface of genus g with one puncture. One applies an induction on the number n of strands. For $n = 1$, $\widetilde{B}(1, F) = \pi_1(F) = B(1, F)$, then ϕ_1 is an isomorphism.

Consider the subgroup $B^0(n, F) = \pi^{-1}(\Sigma_{n-1})$ and the map

$$\theta : B^0(n, F) \to B(n-1, F)$$

which "forgets" the last string. Now, let $\widetilde{B}^0(n, F)$ be the subgroup of $\widetilde{B}(n, F)$ generated by $c_1, \ldots, c_{2g}, \sigma_1, \ldots, \sigma_{n-2}, \tau_1, \ldots, \tau_{n-1}, \omega_1, \ldots, \omega_{2g}$, where

$$\begin{aligned}
\tau_j &= \sigma_{n-1} \cdots \sigma_j^2 \cdots \sigma_{n-1}^{-1} \qquad (\tau_{n-1} = \sigma_{n-1}^2);\\
\omega_r &= \sigma_{n-1}^{-1} \cdots \sigma_1^{-1} c_r \sigma_1 \cdots \sigma_{n-1} \quad r \leq \left[\tfrac{g+1}{2}\right];\\
\omega_r &= \sigma_{n-1} \cdots \sigma_1 c_r \sigma_1^{-1} \cdots \sigma_{n-1}^{-1} \quad r \geq \left[\tfrac{g+1}{2}\right] + 1.
\end{aligned}$$

We construct the following diagram:

$$\begin{array}{ccc}
\widetilde{B}^0(n,F) & \xrightarrow{\tilde{\theta}} & \widetilde{B}(n-1,F) \\
\Big\downarrow {\scriptstyle \phi_{n|\widetilde{B}^0(n,F)}} & & \Big\downarrow {\scriptstyle \phi_{n-1}} \\
B^0(n,F) & \xrightarrow{\theta} & B(n-1,F)
\end{array}$$

The map $\tilde{\theta}$ is defined as $\phi_{n-1}^{-1}\theta\phi_{n|B^0(n,F)}$. It is well defined, since ϕ_{n-1} is an isomorphism by the inductive assumption, and it is onto. In fact, $\tilde{\theta}(c_i) = c_i, \tilde{\theta}(\sigma_j) = \sigma_j$ for $j = 1, \ldots, n-2$.

3.2. **The existence of a section.** The morphism $\tilde{\theta}$ has got a natural section $s : \widetilde{B}(n-1,F) \to \widetilde{B}^0(n,F)$ defined as: $s(\sigma_j) = \sigma_j, s(c_i) = c_i$, for $j = 1, \ldots, n-2$ and $i = 1, \ldots, 2g$.

Definition 3.1. *Given a group G and a subset $\mathcal{G}$ of elements of G we set $\langle \mathcal{G} \rangle$ for the subgroup of G generated by $\mathcal{G}$ and $\langle\langle \mathcal{G} \rangle\rangle$ for the subgroup of G normally generated by $\mathcal{G}$.*

Lemma 3.1. *Let $\mathcal{G} = \{\tau_1, \ldots, \tau_{n-1}, \omega_1, \ldots, \omega_{2g}\}$. Then $Ker(\tilde{\theta}) = \langle \mathcal{G} \rangle$.*

Proof: We set $\beta = \tau_1 \cdots \tau_{n-1}$ and $\gamma = \tau_{n-1}^{-1} \cdots \tau_1 \cdots \tau_{n-1} = \sigma_{n-1}^{-1} \cdots \sigma_1^2 \cdots \sigma_{n-1}$. By construction we have $\langle \mathcal{G} \rangle \subset Ker(\tilde{\theta})$.

The existence of a section s implies that $Ker(\tilde{\theta}) = \langle\langle \mathcal{G} \rangle\rangle$. In fact, suppose that there is such $x \in Ker(\tilde{\theta})$ that $x \notin \langle\langle \mathcal{G} \rangle\rangle$. Thus, there is a word $x' \neq 1$ on generators $c_1, \ldots, c_{2g}, \sigma_1, \ldots, \sigma_{n-2}$, of $\widetilde{B}^0(n,F)$ such that $\tilde{\theta}(x') = 1$, because all other generators of $\widetilde{B}^0(n,F)$ are in $\langle \mathcal{G} \rangle$. This is false, since $x' = s(\tilde{\theta}(x'))$. To prove that $\langle \mathcal{G} \rangle$ is normal, we need to show that $h^g, {}^g h \in \langle \mathcal{G} \rangle$ for all generators g of $\widetilde{B}^0(n,F)$ and for all $h \in \mathcal{G}$.

i) Let g be one of the classical braid generators σ_j, $j = 1, \ldots, n-2$. It is clear that $\tau_i^{\sigma_j}$ and ${}^{\sigma_j}\tau_i$ $(i = 1, \ldots, n-1)$ belong to $\langle \tau_1, \ldots, \tau_{n-1} \rangle$, since it is already true in classical braid groups ([14], [18]). On the other hand, $\omega_i^{\sigma_j} = {}^{\sigma_j}\omega_i = \omega_i$ $(i = 1, \ldots, 2g)$.

ii) Let $g = c_r$ $(r = 1, \ldots, 2g)$. Commutativity relations imply $\tau_j^{c_r} = {}^{c_r}\tau_j = \tau_j$ $(r = 1, \ldots, 2g, j = 2, \ldots n-1)$. Note that

$$ {}^{c_r}\tau_1 = {}^{\beta\omega_r^{-1}}\gamma \quad \text{and} \quad \tau_1^{c_r} = {}^{\tau_1^{-1}\beta\omega_r}\gamma \quad \text{for} \quad r \leq \left[\tfrac{g+1}{2}\right]; $$
$$ {}^{c_r}\tau_1 = \tau_1^{\omega_r} \quad \text{and} \quad \tau_1^{c_r} = {}^{\tau_1^{-1}\omega_r}\tau_1 \quad \text{for} \quad r \geq \left[\tfrac{g+1}{2}\right] + 1. $$

We show only the first equation (the other ones are similar). By iterated application of $[c_r, \sigma_1 c_r^{-1} \sigma_1] = 1$ we obtain:

$$\begin{aligned}
{}^{c_r}\tau_1 &= \sigma_{n-1} \cdots \sigma_2 c_r \sigma_1 c_r^{-1} c_r \sigma_1 c_r^{-1} \sigma_1 \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{n-1}^{-1} = \\
&= \sigma_{n-1} \cdots \sigma_2 c_r \sigma_1 c_r^{-1} \sigma_1 c_r^{-1} \sigma_1 c_r \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{n-1}^{-1} = \\
&= \sigma_{n-1} \cdots \sigma_2 \sigma_1 c_r^{-1} \sigma_1 \sigma_1 c_r \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{n-1}^{-1} = {}^{\beta\omega_r^{-1}}\gamma.
\end{aligned}$$

Set $c_{2,s} = \sigma_1^{-1} c_s \sigma_1$ for $s \le \left[\frac{g+1}{2}\right]$ and $c_{2,s} = \sigma_1 c_s \sigma_1^{-1}$ for $s \ge \left[\frac{g+1}{2}\right] + 1$. In the same way as above we find that:

$$\begin{aligned}
(\sigma_1^2)^{c_r} &= {}^{c_{2,r}}(\sigma_1^2) \quad (1 \le r \le \left[\tfrac{g+1}{2}\right]);\\
{}^{c_r}(\sigma_1^2) &= (\sigma_1^2)^{c_{2,r}\sigma_1^{-2}} \quad (1 \le r \le \left[\tfrac{g+1}{2}\right]);\\
{}^{c_r}(\sigma_1^2) &= (\sigma_1^2)^{c_{2,r}} \quad (\left[\tfrac{g+1}{2}\right] + 1 \le r \le 2g);\\
(\sigma_1^2)^{c_r} &= {}^{\sigma_1^{-2}c_{2,r}}(\sigma_1^2) \quad (\left[\tfrac{g+1}{2}\right] + 1 \le r \le 2g).
\end{aligned}$$

Now, remark that relations $(R7)$ and $(R8)$ imply the following relations:

$$\begin{aligned}
&(R9) \quad c_r \sigma_1 c_s \sigma_1^{-1} = \sigma_1 c_s \sigma_1^{-1} c_r \quad (\left[\tfrac{r+1}{2}\right] + 1 \le s \le 2g);\\
&(R10) \quad c_r \sigma_1^{-1} c_{r-1} \sigma_1 = \sigma_1^{-1} c_{r-1} \sigma_1^{-1} c_r \quad (\text{even } r);
\end{aligned}$$

Previous relations combined with relations $(R6), \dots (R10)$ give:

$$\begin{aligned}
&{}^{c_r}c_{2,s} = c_{2,s}\\
&(1 \le r \le \left[\tfrac{g+1}{2}\right],\ s = 1, \dots, \left[\tfrac{r-1}{2}\right], \left[\tfrac{g+1}{2}\right] + 1, \dots, 2g;\\
&\left[\tfrac{g+1}{2}\right] + 1 \le r \le 2g,\ s = 1, \dots, \left[\tfrac{g+1}{2}\right], \left[\tfrac{r+1}{2}\right] + 1, \dots, 2g);\\
&c_{2,r}^{c_r} = {}^{c_{2,r}\sigma_1^{-2}}c_{2,r} \quad (1 \le r \le \left[\tfrac{g+1}{2}\right])\\
&{}^{c_r}c_{2,r} = {}^{\sigma_1^2}c_{2,r} \quad (1 \le r \le \left[\tfrac{g+1}{2}\right])\\
&{}^{c_r}c_{2,r} = c_{2,r}^{\sigma_1^{-2}c_{2,r}} \quad (\left[\tfrac{g+1}{2}\right] + 1 \le r \le 2g);\\
&c_{2,r}^{c_r} = c_{2,r}^{\sigma_1^2} \quad (\left[\tfrac{g+1}{2}\right] + 1 \le r \le 2g);\\
&c_{2,s}^{c_r} = {}^{[c_{2,r},\sigma_1^{-2}]}(c_{2,s}) \quad (s = \left[\tfrac{r+1}{2}\right] + 1, \dots, \left[\tfrac{g+1}{2}\right]);\\
&{}^{c_r}c_{2,s} = {}^{[c_{2,r}^{-1},\sigma_1^2]}(c_{2,s}) \quad (s = \left[\tfrac{g+1}{2}\right] + 1, \dots, \left[\tfrac{r-1}{2}\right]);\\
&{}^{c_r}c_{2,s} = {}^{[\sigma_1^2,c_{2,r}^{-1}]}(c_{2,s}) \quad (s = \left[\tfrac{r+1}{2}\right] + 1, \dots, \left[\tfrac{g+1}{2}\right]);\\
&c_{2,s}^{c_r} = {}^{[\sigma_1^{-2},c_{2,r}]}(c_{2,s}) \quad (s = \left[\tfrac{g+1}{2}\right] + 1, \dots, \left[\tfrac{r-1}{2}\right]);\\
&{}^{c_r}c_{2,r+1} = c_{2,r}^{-1}\sigma_1^2 c_{2,r} c_{2,r+1} \quad (r \text{ odd and } r \ge \left[\tfrac{g+1}{2}\right] + 1);\\
&c_{2,r+1}^{c_r} = \sigma_1^{-2} c_{2,r+1} \quad (r \text{ odd and } r \ge \left[\tfrac{g+1}{2}\right] + 1);\\
&c_{2,r+1}^{c_r} = (c_{2,r}\sigma_1^{-2}c_{2,r}^{-1}) c_{2,r+1} [\sigma_1^{-2}, c_{2,r}] \quad (r \text{ odd and } r \le \left[\tfrac{g+1}{2}\right]);\\
&{}^{c_r}c_{2,r+1} = \sigma_1^2 c_{2,r+1} [c_{2,r}^{-1}, \sigma_1^2] \quad (r \text{ odd and } r \le \left[\tfrac{g+1}{2}\right]);\\
&{}^{c_r}c_{2,r-1} = [c_{2,r}^{-1}, \sigma_1^2] c_{2,r-1} (c_{2,r}^{-1}\sigma_1^{-2}c_{2,r}) \quad (r \text{ even and } r > \left[\tfrac{g+1}{2}\right]);\\
&c_{2,r-1}^{c_r} = [\sigma_1^{-2}, c_{2,r}] c_{2,r-1} \sigma_1^2 \quad (r \text{ even and } r \ge \left[\tfrac{g+1}{2}\right] + 1);\\
&{}^{c_r}c_{2,r-1} = c_{2,r-1}\sigma_1^{-2} \quad (r \text{ even and } r \le \left[\tfrac{g+1}{2}\right]);\\
&c_{2,r-1}^{c_r} = c_{2,r-1} c_{2,r} \sigma_1^2 c_{2,r}^{-1} \quad (r \text{ even and } r \le \left[\tfrac{g+1}{2}\right]).
\end{aligned}$$

A consequence of these identities and relation $(R1)$ is that $\omega_i^{c_r}, {}^{c_r}\omega_i \in \langle \mathcal{G} \rangle$ $(i, r = 1, \dots, 2g)$. □

Lemma 3.2. *Set also* $\{\omega_1, \dots, \omega_{2g}, \tau_1, \dots \tau_{n-1}\}$ *in* $B^0(n, F)$ *for* $\{\phi_n(\omega_1), \dots, \phi_n(\omega_{2g}), \phi_n(\tau_1), \dots \phi_n(\tau_{n-1})\}$. *Then* $Ker(\theta)$ *is freely generated by* $\{\omega_1, \dots, \omega_{2g}, \tau_1, \dots, \tau_{n-1}\}$.

Proof: The diagram

$$\begin{array}{ccc} P(n,F) & \xrightarrow{\theta} & P(n-1,F) \\ \big\downarrow & & \big\downarrow \\ B^0(n,F) & \xrightarrow{\theta} & B(n-1,F) \end{array}$$

is commutative and the kernels of horizontal maps are the same. As stated in section 2.1, $Ker(\theta) = \pi_1(F \setminus \{P_1, \ldots, P_{n-1}\}, P_n)$. If the fundamental domain is changed as in Figure 6 and ω_j, τ_i are considered as loops of the fundamental group of $F \setminus \{P_1, \ldots, P_{n-1}\}$ based on P_n, it is clear that $\pi_1(F \setminus \{P_1, \ldots, P_{n-1}\}, P_n) = \langle \omega_1, \ldots, \omega_{2g}, \tau_1, \ldots \tau_{n-1} | \, \emptyset \, \rangle$. □

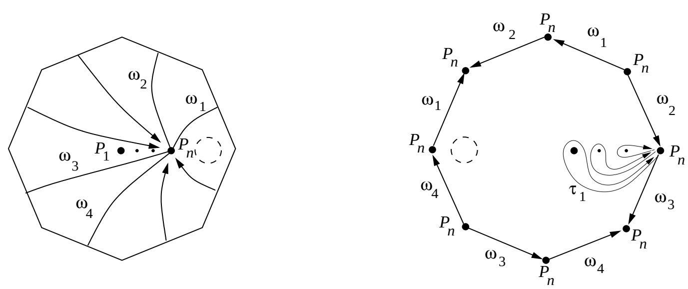

FIGURE 6. Interpretation of ω_j, τ_i as loops of the fundamental group.

Lemma 3.3. $\phi_{n|\widetilde{B}^0(n,F)}$ *is an isomorphism.*

Proof: From the previous Lemmas it follows that the map from $Ker(\tilde{\theta})$ to $Ker(\theta)$ is an isomorphism. The Five Lemma and the inductive assumption conclude the proof. □

3.3. **End of the proof.** In order to show that ϕ_n is an isomorphism, let us remark first that it is onto. In fact, from the previous Lemma the image of $\widetilde{B}(n,F)$ contains P_n and on the other hand $\widetilde{B}(n,F)$ surjects on Σ_n. Since the index of $B^0(n,F)$ in $B(n,F)$ is n, it is sufficient to show that $[\widetilde{B}(n,F) : \widetilde{B}^0(n,F)] = n$. Consider the elements $\rho_j = \sigma_j \cdots \sigma_{n-1}$ (we set $\rho_n = 1$) in $\widetilde{B}(n,F)$. We claim that $\bigcup_i \rho_i \widetilde{B}^0(n,F) = \widetilde{B}(n,F)$. We only have to show that for any (positive or negative) generator g of $\widetilde{B}(n,F)$ and $i = 1, \ldots, n$ there exists $j = 1, \ldots, n$ and $x \in \widetilde{B}^0(n,F)$ such that

$$g\rho_i = \rho_j x \ .$$

If g is a classical braid, this result is well-known ([3]). Other cases come almost directly from the definition of ω_j. Thus every element of $\widetilde{B}(n,F)$ can be written in the form $\rho_i \widetilde{B}^0(n,F)$. Since $\rho_i^{-1}\rho_j \notin \widetilde{B}^0(n,F)$ for $i \neq j$ we are done. □

The previous proof holds also for $p > 1$. This time $\widetilde{B}^0(n,F)$ is the subgroup of $\widetilde{B}(n,F)$ generated by $c_1, \dots, c_{2g}, \sigma_1, \dots, \sigma_{n-2}, \tau_1, \dots, \tau_{n-1}, \omega_1, \dots, \omega_{2g}, \zeta_1, \dots, \zeta_{p-1}$ where τ_j, ω_r are defined as above and $\zeta_j = \sigma_{n-1}^{-1} \cdots \sigma_1^{-1} z_j \sigma_1, \cdots, \sigma_{n-1}$. □

4. Proof of Theorem 1.2

4.1. **About the section.** The steps of the proof are the same. We set again $B^0(n,F) = \pi^{-1}(\Sigma_{n-1})$. This time $\widetilde{B}^0(n,F)$ is the subgroup of $\widetilde{B}(n,F)$ generated by $c_1, \dots, c_{2g}, \sigma_1, \dots, \sigma_{n-2}$, $\tau_1, \dots, \tau_{n-1}$, $\omega_1, \dots, \omega_{2g}$, where τ_j, ω_r are defined as above. Remark that $\tau_1 \in \langle \mathcal{G} \rangle$ since from (TR) relation, the following relation

$$\tau_1 = \gamma_1 \gamma_2 \, \tau_{n-1}^{-1} \cdots \tau_2^{-1} ,$$

holds in $\widetilde{B}^0(n,F)$, where $\gamma_1 = [\omega_{\left[\frac{g+1}{2}\right]+1}, \omega^{-1}_{\left[\frac{g+1}{2}\right]+2}] \cdots [\omega_{2g-1}, \omega_{2g}^{-1}]$ and $\gamma_2 = [\omega_1, \omega_2^{-1}] \cdots [\omega_{\left[\frac{g+1}{2}\right]-1}, \omega^{-1}_{\left[\frac{g+1}{2}\right]}]$. When F is a closed surface the corresponding $\tilde{\theta}$ has no section (see section 2.1). Nevertheless, we are able to prove the analogous of Lemma 3.1 (see section 4.2).

Lemma 4.1. *Let F be a closed surface. Then $Ker(\tilde{\theta})$ is generated by $\{\omega_1, \dots, \omega_{2g}, \tau_2, \dots \tau_{n-1}\}$.*

The following Lemma is analogous to Lemma 3.2.

Lemma 4.2. *Let F be a closed surface and set also $\{\omega_1, \dots, \omega_{2g}, \tau_2, \dots \tau_{n-1}\}$ in $B^0(n,F)$ for $\{\phi_n(\omega_1), \dots, \phi_n(\omega_{2g}), \phi_n(\tau_2), \dots \phi_n(\tau_{n-1})\}$. $Ker(\theta)$ is freely generated by $\{\omega_1, \dots, \omega_{2g}, \tau_2, \dots \tau_{n-1}\}$.*

Let $\rho_j = \sigma_j \cdots \sigma_{n-1}$ (where $\rho_n = 1$). We may conclude by checking that for any generator g of $\widetilde{B}(n,F)$ (or its inverse) and $i = 1, \dots, n$ there exists $j = 1, \dots, n$ and $x \in \widetilde{B}^0(n,F)$ such that

$$g \rho_i = \rho_j x ,$$

which is a sub-case of previous situation. □

4.2. **Proof of Lemma 4.1.** To conclude the proof of Theorem 1.2, we give the demonstration of Lemma 4.1. Let us begin with the following Lemma.

Lemma 4.3. *Let F be a closed surface and $\mathcal{G} = \{\tau_2, \dots, \tau_{n-1}, \omega_1, \dots, \omega_{2g}\}$. The subgroup $\langle \mathcal{G} \rangle$ is normal in $\widetilde{B}^0(n,F)$*

Proof: It suffices to consider relations in Lemma 3.1. Remark that from relations shown in Lemma 3.1, it follows also that the set

$$\{\gamma \tau_j \gamma^{-1} | j = 1, \dots n-1, \, \gamma \text{ word over } \{\omega_1^{\pm 1}, \dots, \omega_{2g}^{\pm 1}\}\} ,$$

is a system of generators for $\langle\langle \tau_1, \dots, \tau_{n-1} \rangle\rangle \equiv \langle\langle \tau_{n-1} \rangle\rangle$. □

In order to prove Lemma 4.1, let us consider the following diagram

$$\begin{array}{ccccc} Ker\tilde{\theta} & \xhookrightarrow[i]{} & \widetilde{B}^0(n,F) & \xrightarrow{\tilde{\theta}} & \widetilde{B}(n-1,F) \\ \downarrow t_n & & \downarrow q_n & \nearrow \tilde{\theta}' & \\ Ker\tilde{\theta}' & \xhookrightarrow[i']{} & \widetilde{B}^0(n,F)/\langle\langle \tau_{n-1}\rangle\rangle & & \end{array}$$

In this diagram q_n is the natural projection, $\tilde{\theta}'$ is defined by $\tilde{\theta}' \circ q_n = \tilde{\theta}$ and t_n is defined by $i' \circ t_n = q_n \circ i$. Since t_n is well defined and onto we deduce that $Ker(t_n) = \langle\langle \tau_{n-1}\rangle\rangle$. Now, $\tilde{\theta}'$ does have a natural section $s : \widetilde{B}(n-1,F) \to \widetilde{B}^0(n,F)/\langle\langle \tau_{n-1}\rangle\rangle$ defined as $s(c_i) = [c_j]$ and $s(\sigma_j) = [\sigma_j]$, where $[x]$ is a representative of $x \in \widetilde{B}^0(n,F)$ in $\widetilde{B}^0(n,F)/\langle\langle \tau_{n-1}\rangle\rangle$. Thus, using the same argument as in Lemma 3.1, we derive that $Ker(\tilde{\theta}') = \langle\langle \mathcal{K}\rangle\rangle$, where $\mathcal{K} = \{[\omega_1], \dots, [\omega_{2g}], [\tau_2], \dots [\tau_{n-1}]\}$. From Lemma 4.3 it follows that $\langle \mathcal{K}\rangle = \langle\langle \mathcal{K}\rangle\rangle$. Moreover, since $\tau_i \in \langle\langle \tau_{n-1}\rangle\rangle$ for $i = 1, \dots, n-2$, $Ker(\tilde{\theta}') = \langle [\omega_1], \dots, [\omega_{2g}]\rangle$.

From the exact sequence

$$1 \to \langle\langle \tau_{n-1}\rangle\rangle \to Ker(\tilde{\theta}) \to Ker(\tilde{\theta}') \to 1$$

it follows that a system of generators for $\langle\langle \tau_{n-1}\rangle\rangle$ and $\omega_1, \dots, \omega_{2g}$ form a system of generators for $Ker(\tilde{\theta})$. From the remark in Lemma 4.3 it follows that $Ker(\tilde{\theta}) = \langle \tau_2, \dots, \tau_{n-1}, \omega_1, \dots, \omega_{2g}\rangle$. □

5. Surface pure braid groups

5.1. **Presentations for surface pure braid groups.**

Theorem 5.1. *Let F be a closed, orientable surface of genus $g \geq 1$. $P(n,F)$ admits the following presentation (see also* [11] *for a similar result):*

- *Generators:* $\{b_{i,r}; 1 \leq i \leq n, 1 \leq r \leq 2g\}$.
- *Relations:*

$$(PR1) \quad \prod_{k=i+1}^{n} D_{i,k} = b_{i,1} b_{i,2}^{-1} \cdots b_{i,2g-1} b_{i,2g}^{-1} \left(\prod_{k=1}^{i-1} D_{k,i}\right)^{-1} b_{i,1}^{-1} b_{i,2} \cdots b_{i,2g-1}^{-1} b_{i,2g} \quad (1 \leq i \leq n)\,;$$

$$(PR2) \quad {}^{b_{i,s}}b_{j,s} = b_{j,s} \quad (1 \leq i < j \leq n; 1 \leq s \leq 2g)$$

$$(PR3) \quad {}^{b_{i,s}}b_{j,r} = U_{i,j} b_{j,r} \quad (1 \leq i < j \leq n; 1 \leq s < r \leq 2g)\,;$$

$$
\begin{array}{lll}
(PR4) & b_{j,r}^{b_{i,s}} = ({}^{b_{j,s}}(D_{i,j}^{-1}))b_{j,r} & (1 \leq i < j \leq n; 1 \leq s < r \leq 2g)\,; \\
(PR5) & {}^{b_{i,s}}b_{j,r} = b_{j,r}((U_{i,j}^{-1})^{b_{j,s}}) & (1 \leq i < j \leq n; 1 \leq r < s \leq 2g)\,; \\
(PR6) & b_{j,r}^{b_{i,s}} = b_{j,r}D_{i,j} & (1 \leq i < j \leq n; 1 \leq r < s \leq 2g)\,; \\
(PR7) & D_{i,j} = \prod_{k=j-1}^{i+1} {}^{U_{k,j}}U_{i,j}\,; & \\
(PR8) & U_{i,j} = D_{i,j}^{\prod_{k=i+1}^{j-1} D_{k,j}}\,; & \\
(PR9) & {}^{b_{k,s}}D_{i,j} = D_{i,j} & (1 \leq k < i < j \leq n \;\; and \;\; 1 \leq i < j < k \leq n; 1 \leq s \leq 2g)\,; \\
(PR10) & {}^{b_{i,s}}D_{i,j} = U_{i,j}^{b_{j,s}} & (1 \leq i < j \leq n; 1 \leq s \leq 2g)\,; \\
(PR11) & D_{i,j}^{b_{k,s}} = D_{i,j}^{D_{k,j}} & (1 \leq i < k < j \leq n; 1 \leq s \leq 2g)\,.
\end{array}
$$

Proof: Let $\widetilde{P}(n, F)$ be the group defined by the presentation. A closed orientable surface F of genus $g \geq 1$ may be represented as a polygon L of $4g$ sides, where opposite edges are identified. A geometric interpretation of generators $b_{j,s}$ and braids $D_{i,j}, U_{i,j}$ on this fundamental domain is provided in Figure 7. Drawing

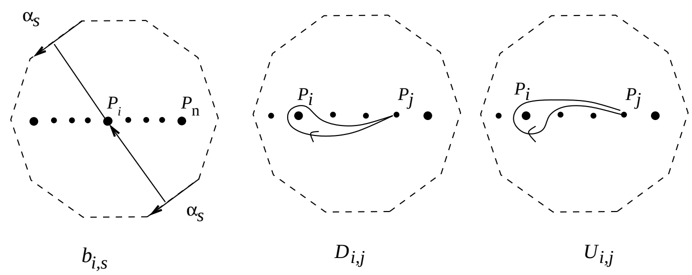

FIGURE 7. Geometric interpretation of $b_{i,s}$, $D_{i,j}$ and $U_{i,j}$.

corresponding braids one can verify that relations (PRi) hold in $P(n, F)$. As shown in [12], given an exact sequence

$$1 \to A \to B \to C \to 1\,,$$

and presentations $\langle G_A, R_A \rangle$ and $\langle G_C, R_C \rangle$, we can derive a presentation $\langle G_B, R_B \rangle$ for B with generators G_A and the coset representatives of C_A. The relations R_B are given by the union of three set. The first corresponds to relations R_A, and the second one to writing each relation in C in terms of corresponding coset representatives as an element of A. The last set corresponds to the fact that the action under conjugation of each coset representative of generators of C (and their inverses) on each generator of A is an element of A. We can apply this result on (PBS) sequence. The presentation is correct for $n = 1$. By induction, suppose that for $n-1$, $\widetilde{P}(n-1, F) \cong P(n-1, F)$. As shown in Figure 6 (here $\tau_i = D_{i,n}$ and $\omega_j = b_{n,j}$) the set of elements $D_{i,n}$, $b_{n,j}$ $(i = 2, \ldots, n-1,\ j = 1, \ldots, 2g)$ is a system of generators for $\pi_1(F \setminus \{P_1, \ldots, P_{n-1}\}, P_n)$. From the induction hypothesis and the fact that (PR4) holds for corresponding pure braids, the set $\{b_{i,r}; 1 \leq i \leq n, 1 \leq r \leq 2g\}$ is a system of generators for $P(n, F)$. Thus the natural morphism from $P(n, F)$ to $\widetilde{P}(n, F)$ is well defined and onto. To show that (PRi) is a complete system of relations for $P(n, F)$ it suffices to prove that relations $R_{P(n,F)}$ are a consequence of relations (PRi). Since $\pi_1(F \setminus \{P_1, \ldots, P_{n-1}\}, P_n)$ is a free group on the given generators, we just have to check the second and the third set of relations. Consider as coset representative for the generator $b_{i,j}$ in $P(n-1,F)$ the generator $b_{i,j}$ in $P(n,F)$. Except relation (PR1), relations lift directly to relations in $P(n,F)$. Relation (PR1) in $P(n-1,F)$ lifts to the following element:

$$(\prod_{k=i+1}^{n-1} D_{i,k})b_{i,2g}^{-1}b_{i,2g-1}\cdots b_{i,2}^{-1}b_{i,1}(\prod_{k=1}^{i-1} D_{k,i})b_{i,2g}b_{i,2g-1}^{-1}\cdots b_{i,2}b_{i,1}^{-1}$$

that is equal to $D_{i,n}$ from (PR1) in $P(n,F)$.

Now, remark that (PR8),(PR9) and (PR10) imply the following relations:

$$(PR12)\quad D_{i,j}^{b_{i,s}} = \prod_{k=i+1}^{j} {}^{D_{k,j}b_{j,s}}D_{i,j} \quad (1 \le i < j \le n; 1 \le s \le 2g);$$

$$(PR13)\quad {}^{b_{k,s}}D_{i,j} = {}^{b_{k,j}U_{k,j}b_{k,j}^{-1}}D_{i,j} \quad (1 \le i < k < j \le n; 1 \le s \le 2g);$$

Then, (PR2), ..., (PR13) imply that ${}^{b_{i,s}}b_{n,r}$, $b_{n,r}^{b_{i,s}}$, ${}^{b_{i,s}}D_{j,n}$ and $D_{j,n}^{b_{i,s}}$ belong to $\pi_1(F \setminus \{P_1,\ldots,P_{n-1}\},P_n)$, for $1 \le i \le n-1$, $2 \le j \le n-1$ and $1 \le r,s \le 2g$.

Thus, also the third set of relations of R_B is a consequence of (PRi). □

Corollary 5.1. *Let F be a closed, orientable surface of genus $g \ge 1$. $P(n,F)$ has the following presentation:*

- *Generators:* $\{b_{i,r}; 1 \le i \le n, 1 \le r \le 2g\}$.
- *Relations:*

$$(PR1)\quad \prod_{k=i+1}^{n}[b_{k,1}^{-1},b_{i,2}^{-1}] = b_{i,1}b_{i,2}^{-1}\cdots b_{i,2g-1}b_{i,2g}^{-1}(\prod_{k=1}^{i-1}[b_{i,1}^{-1},b_{k,2}^{-1}])^{-1}b_{i,1}^{-1}b_{i,2}\cdots b_{i,2g-1}^{-1}b_{i,2g} \quad (1 \le i \le n);$$

$$(PR2)\quad [b_{i,s},b_{j,s}] = 1 \quad (1 \le i < j \le n; 1 \le s \le 2g);$$

$$(PR3)\quad [b_{i,s},b_{j,r}] = [b_{i,1},b_{j,2}] \quad (1 \le i < j \le n; 1 \le s < r \le 2g);$$

$$(PR4)\quad [b_{i,s}^{-1},b_{j,r}] = {}^{b_{j,s}}[b_{i,2}^{-1},b_{j,1}^{-1}] \quad (1 \le i < j \le n; 1 \le s < r \le 2g);$$

$$(PR5)\quad [b_{i,s},b_{j,r}^{-1}] = [b_{i,1},b_{j,2}]^{b_{j,s}} \quad (1 \le i < j \le n; 1 \le r < s \le 2g);$$

$$(PR6)\quad [b_{i,s}^{-1},b_{j,r}^{-1}] = [b_{i,2}^{-1},b_{j,1}^{-1}] \quad (1 \le i < j \le n; 1 \le r < s \le 2g);$$

$$(PR7)\quad [b_{j,1}^{-1},b_{i,2}^{-1}]\prod_{k=j-1}^{i+1}[b_{k,1},b_{j,2}] = \prod_{k=j-1}^{i+1}[b_{k,1},b_{j,2}][b_{i,1},b_{j,2}];$$

$$(PR8)\quad \prod_{k=i+1}^{j-1}[b_{j,1}^{-1},b_{k,2}^{-1}][b_{i,1},b_{j,2}] = [b_{j,1}^{-1},b_{i,2}^{-1}]\prod_{k=i+1}^{j-1}[b_{j,1}^{-1},b_{k,2}^{-1}];$$

$$(PR9)\quad [b_{k,s},[b_{j,1}^{-1},b_{i,2}^{-1}]] = 1 \quad (1 \le k < i < j \le n);$$

$$(PR10)\quad [b_{i,s},[b_{j,1}^{-1},b_{i,2}^{-1}]] = [b_{j,s}^{-1}\prod_{k=j-1}^{i+1}[b_{k,2}^{-1},b_{j,1}^{-1}],[b_{j,1}^{-1},b_{i,2}^{-1}]] \quad (1 \le i < j \le n);$$

$$(PR11)\quad [b_{k,s}^{-1},[b_{j,1}^{-1},b_{i,2}^{-1}]] = [[b_{k,2}^{-1},b_{j,1}^{-1}],[b_{j,1}^{-1},b_{i,2}^{-1}]] \quad (1 \le i < k < j \le n).$$

A similar result holds for pure braid groups of a p punctured orientable surface. Relations may be expressed as commutators. We stress that these presentations have only "local" relations, i.e. there are no relations like as $(PR1)$ in Corollary 5.1.

5.2. **The normal closure of P_n in $P(n,F)$.** We recall that $\chi : P(n,F) \to \pi(F)^n$ is the map defined by $\chi(p) = (p_1, \ldots, p_n)$. Let $K_n(F)$ be the normal closure of P_n in $P(n,F)$. As corollary of previous presentations we give an easy proof of the well-known fact [7] that $Ker(\chi) = K_n(F)$.

Lemma 5.1. *Let F be a closed orientable surface of genus $g \geq 1$. Let $K_n(F)$ be the normal closure of P_n in $P(n,F)$. Then*

$$Ker(\chi) = K_n(F)\,.$$

Proof: Remark that

$$\pi(F)^n = \langle c_{1,1}, \ldots, c_{n,2g} | [c_{i,s}, c_{k,r}] = c_{i,1} \cdots c_{i,2g} c_{i,1}^{-1} \cdots c_{i,2g}^{-1} = 1 \rangle\,,$$

where $1 \leq i \neq j \leq n$ and $1 \leq r, s \leq 2g$. We have $\chi(b_{i,s}) = c_{i,s}$ when s is odd and $\chi(b_{i,s}) = c_{i,s}^{-1}$ when s is even. Recall that P_n is generated by $D_{i,j}$ $(1 \leq i \neq j \leq n)$ and it embeds in $P(n,F)$ (see [16]). It follows that $K_n(F) = \langle\langle D_{i,j} \rangle\rangle$. Since $D_{i,j} = [b_{j,1}^{-1}, b_{i,2}^{-1}]$, we deduce $Ker(\chi) \supseteq K_n(F)$. On the other hand, if $x \in Ker(\chi)$, then x is equivalent to a word $p_1 r_1 p_1^{-1} \cdots p_q r_q p_q^{-1}$, where r_l are commutators $[b_{i,r}^{\pm 1}, b_{j,s}^{\pm 1}]$ or $b_{i,1} b_{i,2}^{-1} \cdots b_{i,2g-1} b_{i,2g}^{-1} b_{i,1}^{-1} b_{i,2} \cdots b_{i,2g-1}^{-1} b_{i,2g}$, for some $1 \leq i \neq j \leq n$ and $1 \leq r, s \leq 2g$. Remark that from relation (PR1) in Theorem 5.1 one derives

$$(*)\ {}^{b_{i,1} b_{i,2}^{-1} \cdots b_{i,2g-1} b_{i,2g}^{-1}}\left(\prod_{k=1}^{i-1} D_{k,i}\right) \prod_{k=i+1}^{n} D_{i,k} = b_{i,1} b_{i,2}^{-1} \cdots b_{i,2g-1} b_{i,2g}^{-1} b_{i,1}^{-1} b_{i,2} \cdots b_{i,2g-1}^{-1} b_{i,2g}$$

The inclusion $Ker(\chi) \subseteq K_n(F)$ follows from relations (PRi) and (*). □

We remark that Lemma 5.1 may be generalised to F orientable surface with boundary.

Proposition 5.1. *Let F be a orientable surface of genus $g \geq 1$, possibly with boundary. Let $N_n(F)$ be the normal closure of the braid group on the disk B_n. The quotient $B(n,F)/N_n(F)$ is isomorphic to $H_1(F)$, the first homology group of the surface F.*

Proof: It suffices to replace all σ_j with 1 in relations in Theorems 1.1 and 1.2. □

Proposition 5.2. *Let F be a closed orientable surface of genus $g \geq 1$. For $g = 1$*

$$[P(n,F), P(n,F)] = K_n(F)\,.$$

Otherwise, for $g > 1$, the strict inclusion holds:

$$[P(n,F), P(n,F)] \supset K_n(F)\,.$$

Proof: The inclusion $K_n(F) \subset [P(n,F), P(n,F)]$ is clear.

Suppose that $[P(n,F), P(n,F)] = K_n(F) = Ker(\chi)$ for $g > 1$. It follows that $P(n,F)/Ker(\chi)$ is abelian. This is false since $\pi_1(F)^n$ is not abelian for $g > 1$.

When $F = T^2$, from $(*)$ in Lemma 5.1 it follows that

$$[b_{i,1}, b_{i,2}^{-1}] \in K_n(T^2) \quad (1 \leq i \leq n)\,,$$

and thus

$$(**) \quad [b_{i,1}, b_{i,2}] \in K_n(T^2) \quad (1 \le i \le n)\,.$$

Thus from relations (PRi) in Corollary 5.1 and $(**)$ we derive that $[a^{\pm 1}, b^{\pm 1}] \in K_n(F)$ for any pair of generators of $P(n, T^2)$. By well-known Witt-Hall identities we conclude that

$$[P(n, T^2), P(n, T^2)] = K_n(T^2).$$

□

Remark 5.1. *When $g > 1$ we can state that $K_n(F)$ is the normal closure of the subgroup generated by elements $[b_{i,r}, b_{j,s}]$, for $1 \le i \ne j \le n$ and $1 \le r, s \le 2g$. This is a straightforward consequence of Corollary 5.1. Remark also that when F is with boundary the inclusion $K_n(F) \subset [P(n,F), P(n,F)]$ is proper.*

5.3. Residual properties of subgroups of $P(n, F)$.

Theorem 5.2. *Let F be a surface of genus $g > 0$ with $p > 0$ boundary components. Consider the sub-surface E obtained removing g handles. Let $Y_n(F)$ be the normal closure of $P(n, E)$ in $P(n, F)$.*

- $\bigcap_{d=0}^{\infty} I(Y_n(F))^d = \{0\}$;
- $I(Y_n(F))^d / I(Y_n(F))^{d+1}$ *is a free $\mathbb{Z}$-module for all $d \ge 0$.*

Proof: To show the claim we have just to verify that hypotheses of following Theorem (see [10] or [15]) are fulfilled.

Theorem 5.3. *Let A, C be two groups. If C acts on A and the induced action on the abelianization of A is trivial,*

$$I(A \rtimes C)^m = \sum_{k=0}^{m} I(A)^k \otimes I(C)^{m-k} \quad \text{for all } m \ge 0\,.$$

We say that $A \rtimes C$ is the quasi-direct product *of A and C. Let B be a finitely iterated quasi-direct product of free groups, then*

- $\bigcap_{d=0}^{\infty} I(B)^d = \{0\}$;
- $I(B)^d / I(B)^{d+1}$ *is a free $\mathbb{Z}$-module for all $d \ge 0$.*

In order to simplify notations and computations we outline the proof for F orientable surface of genus $g \ge 1$ with one boundary component. Consider $B(n, F)$ with the presentation give in Theorem 1.1. For all $j = 2, \ldots, n$, we set $a_{j,s} = \sigma_{j-1}^{-1} \cdots \sigma_1^{-1} a_s \sigma_1 \cdots \sigma_{j-1}$ for $s \le \left[\frac{g+1}{2}\right]$ and $a_{j,s} = \sigma_{j-1} \cdots \sigma_1 a_s \sigma_1^{-1} \cdots \sigma_{j-1}^{-1}$ for $s > \left[\frac{g+1}{2}\right]$. Respectively we define $b_{j,s} = \sigma_{j-1}^{-1} \cdots \sigma_1^{-1} b_s \sigma_1 \cdots \sigma_{j-1}$ for $s \le \left[\frac{g+1}{2}\right]$ and $b_{j,s} = \sigma_{j-1} \cdots \sigma_1 b_s \sigma_1^{-1} \cdots \sigma_{j-1}^{-1}$ for $s > \left[\frac{g+1}{2}\right]$. This set of braids generates $P(n, F)$. As above, let $D_{i,j} = \sigma_{j-1} \cdots \sigma_i^2 \cdots \sigma_{j-1}^{-1}$ and $U_{i,j} = D_{i,j}^{\prod_{k=i+1}^{j-1} D_{k,j}} = \sigma_{j-1}^{-1} \cdots \sigma_i^2 \cdots \sigma_{j-1}$.

Lemma 5.2. *The group $Y_n(F)$ is the group normally generated by the following set:*

$$\{D_{i,j} | 1 \le i < j \le n\} \cup \{a_{i,k} | 1 \le i \le n,\ 1 \le k \le g\}$$

Proof: Consider the (injective) map $\psi_n : P(n,E) \to P(n,F)$ induced by the inclusion $E \subseteq F$. Thus, we may consider the following set of braids in $P(n,F)$,

$$\{D_{i,j} | 1 \leq i < j \leq n\} \cup \{a_{i,k} | 1 \leq i \leq n,\ 1 \leq k \leq g\} \cup \{b_{i,k}^{-1} a_{i,k}^{-1} b_{i,k} | 1 \leq i \leq n,\ 1 \leq k \leq g\}$$

as braids in $P(n,E)$ (see figure 8). This set generates $P(n,E)$ and the claim follows. Remark that this Lemma implies that the inclusion $K_n(F) \subset Y_n(F)$ is proper. □

From now on, let F_n the free group on n generators. Let $\{e_{i,j} | 1 = 1, \ldots, n,\ j = 1, \ldots, m\}$ be the generators of $\oplus_n F_m$. Let $\mu : \oplus_n F_{2g} \to \oplus_n F_g$ be the map defined by $\mu(e_{i,2k}) = e_{i,k}$ and $\mu(e_{i,2k+1}) = 1$. One can proceed as in Lemma 5.1 for showing that $Ker(\mu \circ \chi) = Y_n(F)$. Thus the following commutative diagram holds

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & F_{2g} & \longrightarrow & \oplus_n F_{2g} & \longrightarrow & \oplus_{n-1} F_{2g} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow {\scriptstyle \mu \circ \chi} & & \uparrow {\scriptstyle \mu \circ \chi} & & \\
1 & \longrightarrow & Ker(\theta) & \longrightarrow & P(n,F) & \xrightarrow{\theta} & P(n-1,F) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & G_n & \longrightarrow & Y_n(F) & \xrightarrow{\theta} & Y_{n-1}(F) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

where $G_n = Y_n(F) \cap Ker(\theta)$ is a free group.

Lemma 5.3. *The following set is a system of generators for* G_n.

$$\{\gamma D_{j,n} \gamma^{-1} | 1 \leq j < n\} \cup \{\gamma a_{n,k} \gamma^{-1} |\ 1 \leq k \leq g\}$$

where γ *is a word over* $\{a_{n,j}^{\pm 1} \cup b_{n,j}^{\pm 1} | 1 \leq j \leq g\}$.

Proof: Consider the vertical sequence

$$1 \longrightarrow G_n \longrightarrow Ker(\theta) \longrightarrow F_{2g} \longrightarrow 1\,.$$

Recall that $Ker(\theta) = \pi_1(F \setminus \{P_1, \ldots, P_{n-1}\}, P_n)$. A set of free generators for this group is given by

$$\{D_{j,n} | 1 \leq j < n\} \cup \{a_{n,j} | 1 \leq j \leq g\} \cup \{b_{n,j} | 1 \leq j \leq g\}\,.$$

The map $\mu \circ \chi$ sends $D_{n,j}$ in 1, $a_{n,k}$ in 1 and $b_{n,k}$ in $e_{1,k}$. It follows that G_n is the sub-group of $Ker(\theta)$ normally generated by the set

$$\{D_{j,n} | 1 \leq i < j \leq n\} \cup \{a_{n,k} |\ 1 \leq k \leq g\}$$

and the claim follows. □

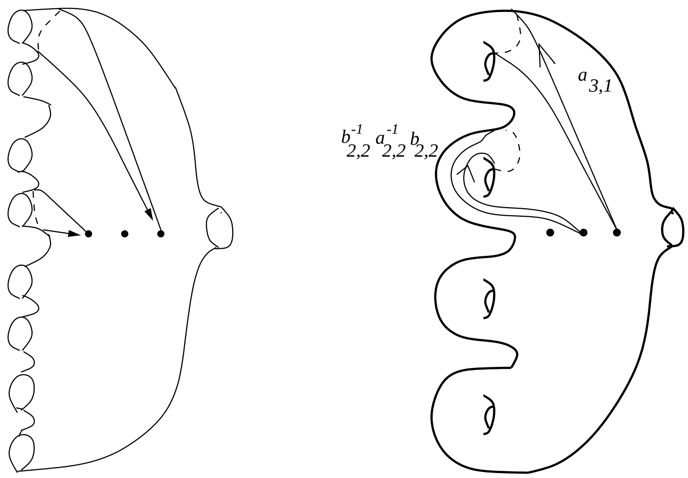

FIGURE 8. Braids of $P(n,F)$ as braids of $P(n,E)$

Recall that the existence of a section for θ implies that $Y_{n-1}(F)$ acts by conjugation on G_n and thus on the abelianization $G_n/[G_n,G_n]$.

Lemma 5.4. *The action of* $Y_{n-1}(F)$ *on* $G_n/[G_n,G_n]$ *is trivial.*

Proof: Let $t \in \{D_{i,j} | 1 \le i < j \le n-1\} \cup \{a_{i,k} | 1 \le i \le n-1,\ 1 \le k \le g\}$ and $f \in \{D_{j,n} | 1 \le j < n\} \cup \{a_{n,k} | 1 \le k \le g\}$. To show the Theorem it suffices to verify that each t acts trivially on $G_n/[G_n,G_n]$. It is evident that

$$^{D_{i,j}}a_{n,k} = a_{n,k} \quad (j = 2,\dots,n-1;\ 1 \le k \le g)\,,$$

and from classical pure braid relations it follows that

$$^{D_{i,j}}D_{1,n} \equiv D_{1,n} \quad (\mathrm{mod}\,[G_n,G_n])\,,$$

for $j = 2,\dots,n-1$. On the other hand, one can verify (drawing corresponding braids) that

$$(***) \quad a_{i,k} f a_{i,k}^{-1} \equiv f \quad (\mathrm{mod}\,[G_n,G_n])\,,$$

$$\begin{aligned}
^{a_{i,k}}D_{j,n} &= D_{j,n} \quad (1 \le i < j)\,;\\
^{a_{i,k}}D_{i,n} &= \textstyle\prod_{s=i}^{n-1} {}^{D_{s,n}a_{n,k}^{-1}}D_{i,n} \equiv D_{i,n} \quad (1 \le k \le \left[\frac{g+1}{2}\right])\,;\\
^{a_{i,k}}D_{i,n} &= D_{i,n}^{a_{n,k}} \equiv D_{i,n} \quad (\left[\frac{g+1}{2}\right] < k \le g)\,;\\
^{a_{i,k}}D_{j,n} &= D_{j,n} \quad (1 \le j < i;\ 1 \le k \le \left[\frac{g+1}{2}\right])\,;\\
^{a_{i,k}}D_{j,n} &= {}^{[D_{i,n}^{-1},a_{n,k}]}D_{j,n} \equiv D_{j,n} \quad (1 \le j < i;\ \left[\frac{g+1}{2}\right] < k \le g))\,;\\
^{a_{i,k}}a_{n,s} &= a_{n,s} \quad (s < k \le \left[\frac{g+1}{2}\right]\,;\\
&\ k \le \left[\frac{g+1}{2}\right] < s;
\end{aligned}$$

$$\left[\tfrac{g+1}{2}\right] < k < s;$$
$$s \le \left[\tfrac{g+1}{2}\right] < k);$$
$$^{a_{i,k}}a_{n,k} = {}^{U_{i,n}}a_{n,k} \equiv a_{n,k} \quad (1 \le k \le \left[\tfrac{g+1}{2}\right])\,;$$
$$^{a_{i,k}}a_{n,k} = a_{n,k}^{D_{i,n}^{-1}a_{n,k}} \equiv a_{n,k} \quad (\left[\tfrac{g+1}{2}\right] < k < 2g)\,;$$
$$^{a_{i,k}}a_{n,s} = {}^{[a_{n,k}^{-1},U_{i,n}]}(a_{n,s}) \equiv a_{n,s} \quad (\left[\tfrac{g+1}{2}\right] < s < k)\,;$$
$$^{a_{i,k}}a_{n,s} = {}^{[D_{i,n},a_{n,k}^{-1}]}(a_{n,s}) \equiv a_{n,s} \quad (k < s < \left[\tfrac{g+1}{2}\right])\,.$$

Now consider the action of each t on $b_{n,s}$, for $s = 1, \dots, g$. As above it is evident that $^{D_{i,j}}b_{n,s} = b_{n,s}$ for $s = 1, \dots, g$. One can verify that

$$(****) \quad a_{i,k}b_{n,s}a_{i,k}^{-1} = hb_{n,b} \quad (1 \le s \le g, 1 \le k \le g)\,,$$

where $h \in G_n$.

$$(1) \quad {}^{a_{i,k}}b_{n,s} = b_{n,s} \quad (s < k \le \left[\tfrac{g+1}{2}\right]\,;$$
$$k \le \left[\tfrac{g+1}{2}\right] < s;$$
$$\left[\tfrac{g+1}{2}\right] < k < s;$$
$$s \le \left[\tfrac{g+1}{2}\right] < k);$$
$$(2) \quad {}^{a_{i,k}}b_{n,s} = {}^{[a_{n,k}^{-1},D_{i,n}]}b_{n,s} = hb_{n,s} \quad (\left[\tfrac{g+1}{2}\right] < s < k)\,;$$
$$(3) \quad {}^{a_{i,k}}b_{n,s} = {}^{[U_{i,n},a_{n,k}^{-1}]}b_{n,s} = h'b_{n,s} \quad (k < s \le \left[\tfrac{g+1}{2}\right])$$
$$(4) \quad {}^{a_{i,k}}b_{n,k} = D_{i,n}b_{n,k}[a_{n,k}^{-1}, D_{i,n}] = h''b_{n,k}$$
$$(k \le \left[\tfrac{g+1}{2}\right])\,;$$
$$(5) \quad {}^{a_{i,k}}b_{n,k} = a_{n,k}^{-1}U_{i,n}a_{n,k}b_{n,k} = h'''b_{n,k}$$
$$(\left[\tfrac{g+1}{2}\right] < k)\,,$$

where $h, h', h'', h''' \in G_n$. Let γ be a word over $\{a_{n,j}^{\pm 1} \cup b_{n,j}^{\pm 1} | 1 \le j \le g\}$. From (***) and (****) it follows that, for each $t \in \{D_{i,j} | 1 \le i < j \le n-1\} \cup \{a_{i,k} | 1 \le i \le n-1,\ 1 \le k \le g\}$, $t\gamma f\gamma^{-1}t^{-1} \equiv \gamma f \gamma^{-1}$. □

Relations in Lemma 5.4 provide an other proof for Lemma 5.3. In fact, let $J_n = \langle \{\gamma D_{j,n}\gamma^{-1} | 1 \le j < n\} \cup \{\gamma a_{n,k}\gamma^{-1} |\ 1 \le k \le g\}\rangle$, γ word over $\{a_{n,j}^{\pm 1} \cup b_{n,j}^{\pm 1} | 1 \le j \le g\}$. Relations in Lemma 5.4 imply that for any $h \in J_n$ we have that $^{a_{i,k}}h \in J_n$. Similar computations for $b_{i,k}$ show that J_n is normal in $P(n, F)$. On the other hand the existence of a section for θ implies that $G_n = \langle\langle \{D_{j,n} | 1 \le j < n\} \cup \{a_{n,k} |\ 1 \le k \le g\}\rangle\rangle$ and thus we obtain $G_n = J_n$, since J_n is normal and $J_n \supseteq \langle \{D_{j,n} | 1 \le j < n\} \cup \{a_{n,k} |\ 1 \le k \le g\}\rangle$.

It seems that previous arguments hold also for the commutator subgroup of $P(n, F)$, but computations are very involved. A consequence of this result would be that $P(n, F)$ is residually solvable. We notice that classical techniques do not apply to the whole group $P(n, F)$. The main obstruction is that it seems that the action of $P(n-1, F)$ on the abelianisation of $\pi_1(F \setminus \{x_1, \dots, x_{n-1}\})$ is not trivial (see for instance relations (4) and (5) in Theorem 5.2).

Remark 5.2. *As $P(n, F)$ is a (normal) subgroup of the mapping class group of a pointed surface, it follows that $P(n, F)$ is residually finite.*

6. Appendix

6.1. **Braids on p-punctured spheres.** We recall that the exact sequence

$$1 \longrightarrow \pi_1(F \setminus \{P_1, \dots, P_{n-1}\}, P_n) \longrightarrow P(n,F) \xrightarrow{\theta} P(n-1,F) \to 1$$

holds also when $F = S^2$ ([5]). Thus, previous arguments may be repeated in the case of the sphere, to obtain a new proof for the well-known presentation of braid groups on the sphere as quotients of classical braid groups. On the other hand, when F is p-punctured sphere we have the following result.

Theorem 6.1. *Let F be an orientable p-punctured sphere. The group $B(n,F)$ admits the following presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, z_1, \dots, z_{p-1}$.
- *Relations:*
  - *Braid relations, i.e.*

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}\,; \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \textit{for}\ |i-j| \geq 2\,. \end{aligned}$$

  - *Mixed relations:*

$$\begin{array}{ll} (R1) & z_j \sigma_i = \sigma_i z_j \quad (i \neq 1, j = 1, \dots, p-1)\,; \\ (R2) & \sigma_1^{-1} z_j \sigma_1 z_l = z_l \sigma_1^{-1} z_j \sigma_1 \quad (j = 1, \dots, p-1,\ j < l)\,; \\ (R3) & \sigma_1^{-1} z_j \sigma_1^{-1} z_j = z_j \sigma_1^{-1} z_j \sigma_1^{-1} \quad (j = 1, \dots, p-1)\,; \end{array}$$

6.2. **Gonzalez-Meneses' presentations.** Let F be a closed orientable surface of genus $g \geq 1$. Using the same arguments outlined in sections 3 and 4 we may provide an other presentation for $B(n,F)$

Theorem 6.2. *Let F be a closed orientable surface of genus $g \geq 1$. The group $B(n,F)$ admits the following presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, b_1, \dots, b_{2g}$.
- *Relations:*
  - *Braid relations as in Theorem 1.1.*
  - *Mixed relations:*

$$\begin{array}{ll} (R1) & b_r \sigma_i = \sigma_i b_r \quad (1 \leq r \leq 2g; i \neq 1)\,; \\ (R2) & b_s \sigma_1^{-1} b_r \sigma_1^{-1} = \sigma_1 b_r \sigma_1^{-1} b_s \quad (1 \leq s < r \leq 2g)\,; \\ (R3) & b_r \sigma_1^{-1} b_r \sigma_1^{-1} = \sigma_1^{-1} b_r \sigma_1^{-1} b_r \quad (1 \leq r \leq 2g)\,; \\ (TR) & b_1 b_2^{-1} \dots b_{2g-1} b_{2g}^{-1} b_1^{-1} b_2 \dots b_{2g-1}^{-1} b_{2g} = \sigma_1 \sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2 \sigma_1\,. \end{array}$$

A closed orientable surface F of genus $g \geq 1$ is represented as a polygon L of $4g$ sides, where opposite edges are identified. Figure 7 gives a geometric interpretation of generators. Relations can be easily verified on corresponding braids.

The presentation in Theorem 6.2 is equivalent to Gonzalez-Meneses' presentation. More precisely, Gonzalez-Meneses [8] found the following presentation for $B(n,F)$, when F is a closed, orientable surface of genus $g \geq 1$.

- Generators: $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_{2g}$.
- Relations:

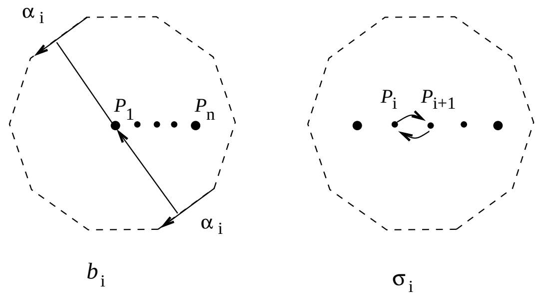

FIGURE 9. Generators as braids (for F an orientable closed surface).

$$
\begin{aligned}
&(1) \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\,, \\
&(2) \quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| \geq 2\,, \\
&(3) \quad [a_r, A_{2,s}] = 1 \quad (1 \leq r, s \leq 2g;\ r \neq s)\,, \\
&(4) \quad [a_r, \sigma_i] = 1 \quad (1 \leq r \leq 2g; i \neq 1)\,, \\
&(5) \quad [a_1 \dots a_r, A_{2,r}] = \sigma_1^2 \quad (1 \leq r \leq 2g)\,, \\
&(6) \quad a_1 \dots a_{2g} a_1^{-1} \dots a_{2g}^{-1} = \sigma_1\sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2\sigma_1\,,
\end{aligned}
$$

where $A_{2,r} = \sigma_1^{-1}(a_1 \dots a_{r-1} a_{r+1}^{-1} \dots a_{2g}^{-1})\sigma_1^{-1}$.

Consider the morphism ψ, from the presentation of Theorem 6.2 to Gonzalez-Meneses' presentation, defined as $\psi(\sigma_k) = \sigma_k$ for all $k = 1, \dots, n-1$, $\psi(b_j) = a_j$, when j is odd and $\psi(b_j) = a_j^{-1}$, when j is even. Tedious but simple computations show that this (surjective) morphism is well defined. On the other hand, we remark that

$$a_k = (A_{2,1}A_{2,2}^{-1} \cdots A_{2,k-2}A_{2,k-1}^{-1})(A_{2,k+1}A_{2,k+2}^{-1} \cdots A_{2,2g-1}^{-1}A_{2,2g}) \quad \text{if } k \text{ is odd,}$$

$$a_k = (A_{2,1}A_{2,2}^{-1} \cdots A_{2,k-2}^{-1}A_{2,k-1})(A_{2,k+1}A_{2,k+2}^{-1} \cdots A_{2,2g-1}A_{2,2g}^{-1}) \quad \text{if } k \text{ is even,}$$

Explicit calculations can prove that the relations in Theorem 6.2 are a consequence of Gonzalez-Meneses' relations. We remark that our presentations in Theorems 1.2 and 6.2 have less relations than Gonzalez-Meneses'ones.

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