The Hartogs phenomenon in hypersurfaces with constant signature

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Abstract

In this paper we study the $\overline{\partial}$-equation with zero Cauchy data along a hypersurface with constant signature. Applications to the solvability of the tangential Cauchy-Riemann equations for smooth forms with compact support and currents on the hypersurface are given. We also prove that the Hartogs phenomenon holds in weakly 2-convex-concave hypersurfaces with constant signature of Stein manifolds.

1 Introduction

It is well-known that if X is a Stein manifold of complex dimension $n \geq 2$ and K a compact subset of X with $X \setminus K$ connected, then every holomorphic function on $X \setminus K$ extends holomorphically to X. In fact it is sufficient that X satisfies $H_c^{0,1}(X) = 0$, which holds for example under the assumption that X is completely 1-convex in the sense of definition 5.1 in [He/Le]. This extension property of holomorphic functions is called the Hartogs phenomenon.

The Hartogs phenomenon has also been studied in so-called q-convex-concave hypersurfaces. These are hypersurfaces, whose Levi form has at least q positive and q negative eigenvalues at each point.

Indeed, it is known that the Hartogs phenomenon holds if M is a 2-convex-concave hypersurface in a Stein manifold or if M is 1-convex-concave and K sufficiently small (see [He] and [L-T]).

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On the other hand, the following example given in [Hi/Na2] shows that the Hartogs phenomenon fails to hold globally for 1-convex-concave hypersurfaces:
Set $M = \{z \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^2 - |z_3|^2 = 1\}$ and $K = \{z \in M \mid z_3 = 0\}$. Then the CR function $f(z) = \frac{1}{z_3}$ defined on $M \setminus K$ has no CR extension to M.

Here we will prove the following result on the Hartogs phenomenon in hypersurfaces:

Theorem 1.1
Let X be a Stein manifold and M a smooth, closed, connected hypersurface in X. Suppose that the signature of M is the same at each point and that M is weakly 2-convex-concave. Let K be a compact subset of M such that $M \setminus K$ is connected and globally minimal. Then every smooth CR function on $M \setminus K$ extends to a smooth CR function on M.

M being weakly 2-convex-concave signifies that the Levi form of M has at least 2 nonnegative and 2 nonpositive eigenvalues at each point. In particular, this class of hypersurfaces contains all Levi flat hypersurfaces of real dimension at least 5. Another interesting case is the one of signature $(1, 1, 1)$.

We remark that the assumption of global minimality is needed in order to assure that the weak analytic continuation principle holds for CR functions. However, this assumption is always satisfied as long as the Levi form is not identically zero (or if M is of finite bracket type).

Theorem 1.1 will be proved by showing that $H_c^{0,1}(M) = 0$. More precisely, if M has signature (p^-, p^0, p^+) at each point, we show that $H_c^{p,q}(M) = 0$ for $q \leq \min(p^-, p^+) + p^0 - 1$.

To show the vanishing of these tangential Cauchy-Riemann cohomology groups with compact support, we study the $\overline{\partial}$-equation with exact support in a domain $D = \Omega \cap \{\varrho < 0\}$ where Ω is a completely strictly pseudoconvex domain and ϱ is a smooth function such that the Levi form of ϱ has exactly p^- negative, p^0 zero and p^+ positive eigenvalues on $T_x^{1,0}\{\varrho = 0\}$ for every $x \in \{\varrho = 0\}$, $q \leq p^0 + p^+$. We prove that, under these conditions, for every $\overline{\partial}$-closed (p, q)-form f on X with supp $f \subset \overline{D}$, the equation

$$\begin{cases} \overline{\partial} u = f \\ \text{supp } u \subset \overline{D} \end{cases}$$

can be solved as long as $q \leq p^0 + p^+$.
This will be done by means of basic L^2-estimates on D with powers of the inverse of the boundary distance as weight functions. The regularity

of the minimal L^2-solutions follow from the Sobolev-estimates for elliptic operators whose coefficients can be controlled by some powers of the inverse of the boundary distance obtained in [Br].

By duality, we can then solve the $\overline{\partial}$-equation for extensible currents on D for bidegrees (p,q), $q \geq p^- + 1$. We then obtain the solvability of the $\overline{\partial}_b$-equation for currents of bidegree (p,q), on $\{\varrho = 0\} \cap \Omega$, $q \geq \max(p^-, p^+) + 1$.

2 Construction of a family of metrics

Let $\Omega \subset\subset X$ be a nonempty domain. We say that Ω is completely strictly pseudoconvex if there exists a function ϱ of class $\mathcal{C}^2$ in a neighborhood $U_{\overline{\Omega}}$ of $\overline{\Omega}$ such that $\Omega = \{z \in U_{\overline{\Omega}} \mid \varrho(z) < 0\}$ and such that $\mathcal{L}(\varrho, x)$ is positive definite for all $x \in U_{\overline{\Omega}}$.

The remainder of this paper is dedicated to the study of the $\overline{\partial}$-equation with exact support in a certain domain which is a transversal intersection of a completely strictly pseudoconvex domain with smooth boundary and a weakly q-convex domain with smooth boundary. In particular such a domain is piecewise smooth and is a q-convex manifold, i.e. it admits a $(q+1)$-convex exhaustion function.

Let Ω be a smooth bounded completely strictly pseudoconvex domain in a complex n-dimensional manifold X and M a real hypersurface of class $\mathcal{C}^\infty$ intersecting $\partial\Omega$ transversally such that $\Omega \setminus M$ has exactly two connected components. We suppose that $M = \{\varrho = 0\}$ where ϱ is a $\mathcal{C}^\infty$ function whose Levi form has exactly p^+ positive, p^0 zero and p^- negative eigenvalues on $T_x^{1,0}M$ for each $x \in M$, $p^- + p^0 + p^+ = n - 1$. We put $D = \Omega \cap \{\varrho < 0\}$.

As Ω is completely strictly pseudoconvex, there exists a neighborhood $U_{\overline{\Omega}}$ of $\overline{\Omega}$ in X and a strictly pseudoconvex smooth function ψ on $U_{\overline{\Omega}}$ such that $\Omega = \{z \in U_{\overline{\Omega}} \mid \psi(z) < 0\}$. We define $\omega_g = i\partial\overline{\partial}\psi$. ω_g is then a hermitian metric on $U_{\overline{\Omega}}$.

We can find a weakly $(p^0 + p^+)$-convex domain $\widetilde{\Omega} \subset\subset X$ with smooth boundary such that $M \cap \overline{\Omega} \subset \partial\widetilde{\Omega}$. Then by [Ma, Proposition7.1], there exists $c > 0$ and a smooth defining function δ_M for M, defined on a neighborhood V of $M \cap \overline{\Omega}$ such that for every $x \in V \cap \{\varrho < 0\}$, $\mathcal{L}(-\log \delta_M, x)$ has p^- negative eigenvalues less than or equal to $\frac{-c}{\delta_M(x)}$, p^0 positive eigenvalues greater or equal to c and $p^+ + 1$ positive eigenvalues greater than or equal to $\frac{c}{\delta_M(c)}$ with respect to ω_g. For later convenience, we set $V^- = V \cap \{\varrho < 0\}$.

The proof of the following lemma basically follows from the proof of

Proposition 2.3 in [Mi]. However, since we have made some adjustments and precisions, we include the complete proof.

Lemma 2.1
Fix $x_0 \in M \cap \overline{\Omega}$. Then there exists a neighborhood U of x_0 in X and a smooth orthonormal basis $(\zeta_1(x), \dots, \zeta_n(x))$ of $(T_x^{1,0}X)^*$ with respect to ω_g on U such that on $U \cap D$ we have

$$\begin{aligned}
\mathcal{L}(x) &:= -i\partial\overline{\partial}\log\delta_M(x) \\
&= \sum_{\mu,\nu=1}^{p^-} a^-_{\mu\nu}(x)\zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) + \sum_{\mu,\nu=p^-+1}^{p^-+p^0} a^0_{\mu\nu}(x)\zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) \\
&\quad + \sum_{\mu,\nu=p^-+p^0+1}^{n-1} a^+_{\mu\nu}(x)\zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) + a_n(x)\zeta_n(x)\wedge\overline{\zeta}_n(x) \\
&= \mathcal{L}^-(x)\oplus\mathcal{L}^0(x)\oplus\mathcal{L}^+(x)\oplus\mathcal{L}^n(x)
\end{aligned}$$

such that $\mathcal{L}^-(x)$ has p^- eigenvalues smaller than the p^0 eigenvalues of $\mathcal{L}^0(x)$, which in turn are smaller than the p^+ eigenvalues of $\mathcal{L}^+(x)$, and $a_n(x)$ is the biggest eigenvalue of $\mathcal{L}(x)$.

Moreover, if $(L_1(x), \dots, L_n(x))$ is the dual basis of $(\zeta_1(x), \dots, \zeta_n(x))$, we can arrange that

(i) $[L_\alpha, L_\beta](x) \in \mathrm{Span}(L_{p^-+1}(x), \dots, L_{p^-+p^0}(x))$ *for* $x \in M$ *and* $\alpha, \beta \in \{p^-+1, \dots, p^-+p^0\}$

(ii) $[L_\alpha, L_\beta](x) \in \mathrm{Span}(L_1(x), \dots, L_{n-1}(x))$ *for* $\alpha, \beta \in \{1, \dots, n-1\}$ *and* $x \in M$

(iii) $[L_\alpha, \overline{L}_\beta](x) \in \mathrm{Span}(L_{p^-+1}(x), \dots, L_{p^-+p^0}(x), \overline{L}_{p^-+1}(x), \dots, \overline{L}_{p^-+p^0}(x))$ *for* $x \in M$ *and* $\alpha, \beta \in \{p^-+1, \dots, p^-+p^0\}$

(iv) $[L_\alpha, \overline{L}_\beta](x) \in \mathrm{Span}(L_1(x), \dots, L_{n-1}(x), \overline{L}_1(x), \dots, \overline{L}_{n-1}(x))$ *for* $\alpha \in \{1, \dots, n-1\}$, $\beta \in \{p^-+1, \dots, p^-+p^0\}$ *and* $x \in M$

Proof: The Levi form of M at the point x is the bilinear map $\mathcal{L}_x : \{T_x^{1,0}M \oplus \overline{T_x^{1,0}M}\} \times \{T_x^{1,0}M \oplus \overline{T_x^{1,0}M}\} \longrightarrow \{T_xM \otimes \mathbb{C}\}/\{T_x^{1,0}M \oplus \overline{T_x^{1,0}M}\}$ defined by $\mathcal{L}_x(X_x, Y_x) = \frac{1}{2i}\pi_x[X_x, Y_x]_x$, where π_x is the projection $\{T_xM \otimes \mathbb{C}\} \longrightarrow \{T_xM \otimes \mathbb{C}\}/\{T_x^{1,0}M \oplus \overline{T_x^{1,0}M}\}$. Since, by hypothesis on M, the Levi form of M has exactly p^0 zero eigenvalues everywhere, $N^{1,0}M = \cup_{x\in M} N_x^{1,0}M$, where

$$N_x^{1,0}M = \{L_x \in T_x^{1,0}M \mid \mathcal{L}_x(L_x, Y_x) = 0 \;\forall\; Y_x \in T_x^{1,0}M\}$$

is the Levi null set at x, forms a subbundle of $T_x^{1,0}M$ of rank p^0. Moreover, it is easy to see (use the Jacobi identity, cf [Fr]) that $N^{1,0}M \oplus \overline{N}^{1,0}M$ is involutive.

Now fix $x_0 \in M$. We may then choose a subbundle $N = \cup_x N_x$ of rank p^0 of $T^{1,0}X$ on a neighborhood V of x_0 in X such that $N_x = N_x^{1,0}M$ for $x \in M \cap V$. Moreover, we may assume that N is itself a subbundle of $T = \mathrm{Ker}\partial\delta_M \cap T^{1,0}X$. Note that T is a subbundle of $T^{1,0}X$ of rank $(n-1)$ on V such that $T_x = T_x^{1,0}M$ for $x \in M \cap V$.

Let $\lambda_1^x \leq \ldots \leq \lambda_{n-1}^x$ be the eigenvalues of $\mathcal{M}(x) := i\partial\overline{\partial}\delta_M(x)_{|T_x}$. It is well known that the functions $x \mapsto \lambda_\nu^x$ are continuous on V. Using the assumptions on M, we have

$$\lambda_{p^-}^x < 0 = \lambda_{p^-+1}^x = \ldots = \lambda_{p^-+p^0}^x < \lambda_{p^-+p^0+1}^x$$

for every $x \in M \cap V$. For a small $\varepsilon > 0$, we therefore get a neighborhood W of x_0 in X such that for $x \in W$

$$\lambda_{p^-}^x < -\varepsilon,\ \lambda_{p^-+p^0+1}^x > \varepsilon,$$

$$\lambda_i^x \in (-\varepsilon, \varepsilon) \text{ for } i = p^- + 1, \ldots, p^- + p^0$$

Moreover, we can find $R > 0$ such that all the λ_ν^x are of absolute value smaller than R for each $x \in W$.

Intersecting the cercle of radius R centered at 0 with the lines $[-\varepsilon + i\mathbb{R}]$ and $[\varepsilon + i\mathbb{R}]$, we obtain three closed paths Γ^-, Γ^0 and Γ^+ such that for $x \in W$, none of the eigenvalues of $\mathcal{M}(x)$ lies on Γ^-, Γ^0 or Γ^+.

We may assume that W is small enough such that there exists a smooth orthonormal basis $X_1, \ldots, X_{n-1}$ of T on W such that $X_{p^-+1}, \ldots, X_{p^-+p^0}$ is a smooth basis of N on W.

We denote by $M(x)$ the matrix of $\mathcal{M}(x)$ in the basis $X_1, \ldots, X_{n-1}$ and by $(e_1, \ldots, e_{n-1})$ the standard basis of $\mathbb{C}^{n-1}$. We then have $\mathrm{Ker}M(x) = \mathrm{Span}(e_{p^-+1}, \ldots, e_{p^-+p^0})$ for every $x \in M \cap W$.

For $x \in W$, we may set

$$\begin{aligned}
\Pi^-(x) &= \frac{1}{2i\pi}\int_{\Gamma^-}(M(x) - z\mathrm{Id})^{-1}dz, \\
\Pi^0(x) &= \frac{1}{2i\pi}\int_{\Gamma^0}(M(x) - z\mathrm{Id})^{-1}dz, \\
\Pi^+(x) &= \frac{1}{2i\pi}\int_{\Gamma^+}(M(x) - z\mathrm{Id})^{-1}dz
\end{aligned}$$

Then Π^-, Π^0 and Π^+ are $\mathcal{C}^\infty$ mappings in a neighborhood of x_0 (e.g. Π^- is the composition of the $\mathcal{C}^\infty$ map $x \mapsto M(x)$ and the holomorphic mapping from the space of hermitian $(n-1)\times(n-1)$ matrices to itself given by $A \mapsto \frac{1}{2i\pi}\int_{\Gamma^-}(A - z\mathrm{Id})^{-1}dz$). It is easy to see that $\Pi^-(x)$, $\Pi^0(x)$ and $\Pi^+(x)$ are the orthogonal projections of $\mathbb{C}^{n-1}$ onto

$$\begin{aligned} E^-(x) &= \sum_{\nu=1}^{p^-} \mathrm{Ker}(M(x) - \lambda_\nu^x \mathrm{Id}), \\ E^0(x) &= \sum_{\nu=p^-+1}^{p^-+p^0} \mathrm{Ker}(M(x) - \lambda_\nu^x \mathrm{Id}) \qquad \text{and} \\ E^+(x) &= \sum_{\nu=p^-+p^0+1}^{n-1} \mathrm{Ker}(M(x) - \lambda_\nu^x \mathrm{Id}) \end{aligned}$$

For every $x \in M \cap W$ we have $E^0(x) = \mathrm{Span}(e_{p^-+1}, \ldots, e_{p^-+p^0})$. Therefore, if W is small enough, the vectors

$$\tilde{e}_{p^-+1}(x) := \Pi^0(x)(e_{p^-+1}), \ldots, \tilde{e}_{p^-+p^0} := \Pi^0(x)(e_{p^-+p^0})$$

form a basis for $E^0(x)$. After a permutation of some indices, we can also achieve that

$$\tilde{e}_1(x) := \Pi^-(x)(e_1), \ldots, \tilde{e}_{p^-}(x) := \Pi^-(x)(e_{p^-})$$

span $E^-(x)$ and that

$$\tilde{e}_{p^-+p^0+1}(x) := \Pi^+(x)(e_{p^-+p^0+1}), \ldots, \tilde{e}_{n-1}(x) := \Pi^+(x)(e_{n-1})$$

span $E^+(x)$. Due to the Gram-Schmidt orthonormalization procedure and the fact that eigenvectors associated to different eigenvalues are orthogonal, we may assume that $(\tilde{e}_1(x), \ldots, \tilde{e}_{n-1}(x))$ is an orthonormal basis for the standard scalar product on $\mathbb{C}^{n-1}$.

We define $l_{ij}(x)$ by $\tilde{e}_i(x) = \sum_{j=1}^{n-1} l_{ij}(x) e_j$ and set $L_i(x) = \sum_{j=1}^{n-1} l_{ij}(x) X_j(x)$. Then $(L_1(x), \ldots, L_{n-1}(x))$ is an orthonormal basis of T on W. Moreover, we have $N_x^{1,0}M = \mathrm{span}(L_{p^-+1}(x), \ldots, L_{p^-+p^0}(x))$ for $x \in W \cap M$.

Now we apply the same procedure as above to the hermitian form $\delta_M^2(x)\mathcal{L}(x) = -i\delta_M(x)\partial\overline{\partial}\delta_M + i\partial\delta_M \wedge \overline{\partial}\delta_M$ on $T^{1,0}X$. We observe that this hermitian form has $(n-1)$ eigenvalues which vanish on M as well as 1 eigenvalue which is positive on M. After possibly shrinking W, we then obtain a unitary vector $L_n \in T^{1,0}X$ on W, depending smoothly on x, which is an eigenvector of $\mathcal{L}(x)$ and which is orthogonal to $L_1(x), \ldots, L_{n-1}(x)$ with respect to ω_g.

Let $(\zeta_1(x), \ldots, \zeta_n(x)) \in (T^{1,0}X)^*$ be the dual basis of $(L_1(x), \ldots, L_n(x))$ on W. This basis then gives the desired decomposition of $\mathcal{L}(x)$ on W. The assertion (ii) follows because $T^{1,0}M$ is stable under $[\ ,\ ]$. Moreover, since $N^{1,0}M \oplus \overline{N^{1,0}M}$ is involutive, we get (i) and (iii). Finally, (iv) follows by definition of $N^{1,0}M$. □

Let $\delta_{D,g}$ be the boundary distance function of D with respect to ω_g. $\delta_{D,g}$ will not be smooth since D is only a Lipschitz domain. However, [St]) provides us with a regularized distance having essentially the same profile as $\delta_{D,g}$:

There exists a function $\Delta \in \mathcal{C}^\infty(D, \mathbb{R})$ satisfying

$$c_1\delta_{D,g}(x) \leq \Delta(x) \leq c_2\delta_{D,g}(x) \qquad \text{and}$$

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\Delta(x)\right| \leq B_\alpha(\delta_{D,g}(x))^{1-|\alpha|},$$

where $x = (x_1, \ldots, x_{2n})$ are local coordinates on X. B_α, c_1 and c_2 are independent of D.

We also need to define a regularized maximum function. For each $\beta > 0$, let χ_β be a fixed non negative real $\mathcal{C}^\infty$-function on $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, $\chi_\beta(x) = \chi_\beta(-x)$, $|x| \leq \chi_\beta(x) \leq |x| + \beta$, $|\chi'_\beta(x)| \leq 1$, $\chi''_\beta(x) \geq 0$ and $\chi_\beta(x) = |x|$ if $|x| \geq \frac{\beta}{2}$. We moreover assume that $\chi'_\beta(x) > 0$ if $x > 0$ and $\chi'_\beta(x) < 0$ if $x < 0$. We set $\max_\beta(t, s) = \frac{t+s}{2} + \chi_\beta(\frac{t-s}{2})$ for $t, s \in \mathbb{R}$.

We omit the proof of the following simple lemma:

Lemma 2.2
Let φ, ψ be two real-valued $\mathcal{C}^2$-functions on some real $\mathcal{C}^2$ manifold X. Then, for all $\beta > 0$, and $x \in X$, the following assertions hold:

(i) $\max(\varphi(x), \psi(x)) \leq \max_\beta(\varphi(x), \psi(x)) \leq \max(\varphi(x), \psi(x)) + \beta$

(ii) $\max_\beta(\varphi(x), \psi(x)) = \max(\varphi(x), \psi(x))$ *if* $|\varphi(x) - \psi(x)| \geq \beta$

(iii) There is a number $\lambda_x(\varphi, \psi)$ with $0 \leq \lambda_x(\varphi, \psi) \leq 1$, namely

$$\lambda_x(\varphi, \psi) = \frac{1}{2} + \frac{1}{2}\chi'_\beta\left(\frac{\varphi(x) - \psi(x)}{2}\right),$$

such that

$$\begin{aligned}\mathcal{L}(\max_\beta(\varphi, \psi), x) &= \lambda_x(\varphi, \psi)\mathcal{L}(\varphi, x) + (1 - \lambda_x(\varphi, \psi))\mathcal{L}(\psi, x) \\ &\quad + \frac{1}{4}\chi''_\beta\left(\frac{\varphi - \psi}{2}\right)\partial(\varphi - \psi) \wedge \overline{\partial}(\varphi - \psi)(x)\end{aligned}$$

Finally, we write $a \lesssim b$ (resp. $b \gtrsim a$) if there exists an *absolute* constant $C > 0$ such that $a \leq C \cdot b$ (resp. $b \geq C \cdot a$). We write $a \sim b$ if $a \lesssim b$ and $a \gtrsim b$.

For some $\beta > 0$, we define $\varphi = \max_\beta(-\log \delta_M, -\log(-\psi)) \in \mathcal{C}^\infty(D)$. Then φ is an exhaustion function for D and (i) of Lemma 2.2 implies

$$\max(-\log \delta_M, -\log(-\psi)) \leq \varphi \leq \max(-\log \delta_M, -\log(-\psi)) + \beta,$$

thus

$$e^{-\beta} \min(\delta_M, -\psi) \leq e^{-\varphi} \leq \min(\delta_M, -\psi).$$

Hence $e^{-\varphi} \sim \Delta$.

We set $D_j = \{z \in D \mid e^{-\varphi(x)} > \frac{1}{j}\}$.

The following technical lemma is the key point of our paper. It permits to obtain L^2-vanishing theorems on D with powers of the boundary distance as weight functions.

Lemma 2.3
There exists a hermitian metric ω_M on D and a family $(\omega_j)_{j\in\mathbb{N}}$ of complete hermitian metrics on D with the following properties:

(i) $\omega_j = \omega_M$ *on a neighborhood of* $\overline{D}_j$, $\omega_j \geq \omega_M$ *on* D.

(ii) Let $\gamma_1 \leq \ldots \leq \gamma_n$ *be the eigenvalues of* $i\partial\overline{\partial}\varphi$ *with respect to* ω_M. *There exists* $\sigma > 0$ *such that* $\gamma_1 + \ldots + \gamma_r > \sigma$ *for* $r \geq n - p^+ - p^0$.

(iii) There are constants $a, b > 0$ *such that* $a\ \omega_g \leq \omega_M \leq b\ \delta_M^{-2}\omega_g$ *for all* $j \in \mathbb{N}$.

(iv) There is a constant $C > 0$ *such that* $|\partial\omega_M|_{\omega_M} \leq C$.

(v) Let $\omega_M = i\sum_{\mu\nu}\omega_M^{\mu\nu} dz_\mu \wedge d\overline{z}_\nu$ *on* $U \cap D$, *where* U *is a neighborhood of* $x \in M$ *and* $(z_1, \ldots, z_n)$ *are local holomorphic coordinates on* U. *Then, for every multiindex* α, *there exists a constant* C_α *such that* $\sup_{\mu\nu} |D^\alpha\omega_M^{\mu\nu}(z)| \leq C_\alpha\ \delta_M^{-2-|\alpha|}(z)$ *for every* $z \in U \cap D$.

Proof: Let $A_g \in \mathcal{C}^\infty(\mathrm{End}T\Omega)$ be the hermitian endomorphism associated to the hermitian form $-i\partial\overline{\partial}\log\delta_M$ with respect to ω_g and let $\gamma_1^g \leq \ldots \leq \gamma_n^g$ be the eigenvalues of A_g.

We have $-i\partial\overline{\partial}\log\delta_M = \frac{i}{-\delta_M}\partial\overline{\partial}\delta_M + i\partial\log\delta_M \wedge \overline{\partial}\log\delta_M$. Thus there is a constant $c > 0$ such that

$$\begin{aligned}
&\gamma_1^g(x) \le -\frac{c}{\delta_M}, \dots,\ \gamma_{p^-}^g(x) \le -\frac{c}{\delta_M},\\
&\gamma_{p^-+1}^g(x) \ge c, \dots,\ \gamma_{p^-+p^0}^g(x) \ge c,\\
&\gamma_{p^-+p^0+1}^g(x) \ge \frac{c}{\delta_M}, \dots,\ \gamma_{n-1}^g(x) \ge \frac{c}{\delta_M}, \text{ and}\\
&\gamma_n^g(x) \ge c\ |\partial\log\delta_M|_g^2(x)
\end{aligned}$$

for every $x \in V^-$, after possibly shrinking V.

Moreover, we claim that there exists a constant $c' > 0$ such that

$$\gamma_{p^-+1}^g(x) \le c', \dots,\ \gamma_{p^-+p^0}^g(x) \le c'$$

This can be seen as follows:
Fix $x_0 \in M$. As in the proof of Lemma 2.1, there exists a neighborhood U of x_0 in X and a smooth extension T of $T^{1,0}M$ on U, such that for every $x \in U$

$$\mathcal{M}(x) := i\partial\overline{\partial}\delta_{M|T_x} = \mathcal{M}^-(x) \oplus \mathcal{M}^0(x) \oplus \mathcal{M}^+(x)$$

in a smooth orthonormal basis with respect to ω_g on U, such that the eigenvalues of $\mathcal{M}^-(x)$ are the p^- smallest eigenvalues of $\mathcal{M}(x)$ and those of $\mathcal{M}^+(x)$ are the p^+ biggest. Since M has exactly p^0 zero eigenvalues everywhere, this implies that for $x \in M \cap U$, $\mathcal{M}^0(x) \equiv 0$. Therefore the eigenvalues of $\mathcal{M}^0(x)$ are of absolute value smaller than $c'\delta_M(x)$ for some c', which proves the claim.

Choose a strictly positive function $\theta \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ such that

$$\theta(t) = \begin{cases} -nt & \text{for } t \le -c\\ c & \text{for } 0 \le t \le c'\\ t & \text{for } t \ge c'+1 \end{cases}$$

We use the following notation:
Let $\phi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. If A is a hermitian $n \times n$ matrix with eigenvalues $\lambda_1 \le \ldots \le \lambda_n$ and corresponding eigenvectors $v_1, \dots, v_n$, we define $\phi[A]$ as the hermitian matrix with eigenvalues $\phi(\lambda_j)$ and eigenvectors v_j, $1 \le j \le n$.

We let ω_M be the hermitian metric defined by the hermitian endomorphism $A(x) = \theta[A_g(x)]$. ω_M is then a smooth metric. By construction, the eigenvalues of $A(x)$ are $\sigma_\nu(x) = \theta(\gamma_\nu^g(x))$ and we have

$$\begin{aligned}
&\sigma_1(x) = n\ |\gamma_1^g(x)|, \dots,\ \sigma_{p^-}(x) = n\ |\gamma_{p^-}^g(x)|,\\
&\sigma_{p^-+1}(x) = c, \dots,\ \sigma_{p^-+p^0}(x) = c,\\
&\sigma_{p^-+p^0+1}(x) = \gamma_{p^-+p^0+1}^g(x), \dots,\ \sigma_n(x) = \gamma_n^g(x)
\end{aligned}$$

for every $x \in V^-$, after possibly shrinking V.

The eigenvalues of $-i\partial\overline{\partial}\log\delta_M$ with respect to ω_M are $\alpha_\nu(x) = \frac{\gamma_\nu^g(x)}{\sigma_\nu(x)}$. Thus we have for every $x \in V^-$ $\alpha_1(x) = -\frac{1}{n}$ and $\alpha_{n-p^+-p^0}(x) \geq 1$, hence

$$\alpha_1 + \ldots + \alpha_r \geq 1 - \frac{1}{n}(n - p^+ - p^0 - 1) \geq \frac{1}{n} \text{ for } r \geq n - p^+ - p^0 \qquad (2.1)$$

Let us now estimate $|\partial\omega_M|_{\omega_M}$.

Fix $x_0 \in M \cap \overline{\Omega}$. Using Lemma 2.1, there exists a neighborhood U of x_0 in X such that we have on $U \cap \Omega$

$$\begin{aligned}-i\partial\overline{\partial}\log\delta_M(x) &= \sum_{\mu,\nu=1}^{p^-} a_{\mu\nu}^-(x)\zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) + \sum_{\mu,\nu=p^-+1}^{p^-+p^0} a_{\mu\nu}^0(x)\zeta_\mu(x)\wedge\overline{\zeta}_\nu(x)\\ &+ \sum_{\mu,\nu=p^-+p^0+1}^{n-1} a_{\mu\nu}^+(x)\zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) + a_n(x)\zeta_n(x)\wedge\overline{\zeta}_n(x)\end{aligned}$$

where $(\zeta_1(x),\ldots,\zeta_n(x))$ is an orthonormal basis of $T_x^{1,0}X$ with respect to ω_g on U.

By construction of ω_M, we have

$$\begin{aligned}\omega_M &= \sum_{\mu,\nu=1}^{p^-} b_{\mu\nu}^-(x)\ \zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) + c\sum_{\nu=p^-+1}^{p^-+p^0}\zeta_\nu(x)\wedge\overline{\zeta}_\nu(x)\\ &+ \sum_{\mu,\nu=p^-+p^0+1}^{n-1} b_{\mu\nu}^+(x)\ \zeta_\mu(x)\wedge\overline{\zeta}_\nu(x) + a_n(x)\ \zeta_n(x)\wedge\overline{\zeta}_n(x)\end{aligned}$$

where $(\overset{\pm}{b}_{\mu\nu})_{\mu,\nu} = \theta[(\overset{\pm}{a}_{\mu\nu})_{\mu,\nu}]$. In order to get more condensed formulae, we extend $\overset{\pm}{b}_{\mu\nu}$ to all pairs $(\mu,\nu) \in \{1,\ldots,n\}\times\{1,\ldots,n\}$ by setting it equal to zero whenever it is not defined for such a pair.

Let $(L_1(x),\ldots,L_n(x))$ be the dual basis of $(\zeta_1(x),\ldots,\zeta_n(x))$. The well known Cartan formula for d implies that

$$\partial\zeta_\mu(L_\alpha,L_\beta) = L_\alpha(\zeta_\mu(L_\beta)) - L_\beta(\zeta_\mu(L_\alpha)) - \zeta_\mu([L_\alpha,L_\beta]) = -\zeta_\mu([L_\alpha,L_\beta]),$$

$$\partial\overline{\zeta}_\nu(L_\alpha,\overline{L}_\beta) = L_\alpha(\overline{\zeta}_\nu(\overline{L}_\beta)) - \overline{L}_\beta(\overline{\zeta}_\nu(L_\alpha)) - \overline{\zeta}_\nu([L_\alpha,\overline{L}_\beta]) = -\overline{\zeta}_\nu([L_\alpha,\overline{L}_\beta]).$$

Thus

$$\partial\zeta_\mu = \sum_{\alpha,\beta} c_{\alpha\beta}^\mu \zeta_\alpha\wedge\zeta_\beta,$$

$$\partial\overline{\zeta}_\nu = -\sum_{\alpha,\beta} d^\nu_{\alpha\beta}\zeta_\alpha \wedge \overline{\zeta}_\beta,$$

where the $c^\nu_{\alpha\beta}$ and $d^\nu_{\alpha\beta}$ are determined by the conditions

$$\begin{aligned}
[L_\alpha, L_\beta](x) &= -\sum_{\nu=1}^n c^\nu_{\alpha\beta}(x) L_\nu(x) \ \mathrm{mod}(\overline{L}_1(x), \ldots, \overline{L}_n(x)) \\
[L_\alpha, \overline{L}_\beta](x) &= \sum_{\nu=1}^n d^\nu_{\alpha\beta}(x) \overline{L}_\nu(x) \ \mathrm{mod}(L_1(x), \ldots, L_n(x))
\end{aligned}$$

(i)-(iv) of Lemma 2.1 therefore yield

$$c^\mu_{\alpha\beta} \sim \delta_M, \ d^\mu_{\alpha\beta} \sim \delta_M \tag{2.2}$$

for (α, β, μ) such that $\mu \notin \{p^- + 1, \ldots, p^- + p^0\}$, $\alpha, \beta \in \{p^- + 1, \ldots, p^- + p^0\}$ and

$$c^n_{\alpha\beta} \sim \delta_M, \ d^n_{\alpha\beta} \sim \delta_M \tag{2.3}$$

for (α, β) such that $\alpha \in \{1, \ldots, n-1\}$, $\beta \in \{p^- + 1, \ldots, p^- + p^0\}$.

Moreover, by definition of $c^\nu_{\alpha\beta}$ and $d^\mu_{\alpha\beta}$, we have

$$\partial\omega_M = \sum_{\alpha=1}^n \sum_{\mu,\nu=1}^{p^-} \sum_{\epsilon\in\{-,+\}} L_\alpha(b^\epsilon_{\mu\nu})(x) \ \zeta_\alpha \wedge \zeta_\mu \wedge \overline{\zeta}_\nu \tag{2.4}$$

$$+ \sum_{\alpha,\beta=1}^n \sum_{\mu,\nu=1}^{p^-} \sum_{\epsilon\in\{-,+\}} b^\epsilon_{\mu\nu}(x) c^\mu_{\alpha\beta}(x) \ \zeta_\alpha \wedge \zeta_\beta \wedge \overline{\zeta}_\nu \tag{2.5}$$

$$+ \sum_{\alpha,\beta=1}^n \sum_{\mu,\nu=1}^{p^-} \sum_{\epsilon\in\{-,+\}} b^\epsilon_{\mu\nu}(x) d^\nu_{\alpha\beta}(x) \ \zeta_\mu \wedge \zeta_\alpha \wedge \overline{\zeta}_\beta \tag{2.6}$$

$$+c \sum_{\alpha,\beta=1}^n \sum_{\nu=p^-+1}^{p^-+p^0} c^\nu_{\alpha\beta}(x) \ \zeta_\alpha \wedge \zeta_\beta \wedge \overline{\zeta}_\nu \tag{2.7}$$

$$+c \sum_{\alpha,\beta=1}^n \sum_{\nu=p^-+1}^{p^-+p^0} d^\nu_{\alpha\beta}(x) \ \zeta_\nu \wedge \zeta_\alpha \wedge \overline{\zeta}_\beta \tag{2.8}$$

$$+ \sum_{\alpha=1}^n L_\alpha(a_n)(x) \ \zeta_\alpha \wedge \zeta_n \wedge \overline{\zeta}_n \tag{2.9}$$

$$+ \sum_{\alpha,\beta=1}^n a_n(x) c^n_{\alpha\beta}(x) \ \zeta_\alpha \wedge \zeta_\beta \wedge \overline{\zeta}_n \tag{2.10}$$

$$+ \sum_{\alpha,\beta=1}^n a_n(x) d^n_{\alpha\beta}(x) \ \zeta_n \wedge \zeta_\alpha \wedge \overline{\zeta}_\beta \tag{2.11}$$

As A_g is the hermitian endomorphism associated to $-i\partial\overline{\partial}\log\delta_M = \frac{i}{-\delta_M}\partial\overline{\partial}\delta_M + i\partial\log\delta_M\wedge\overline{\partial}\log\delta_M$, it is easy to see that we have $b^{\pm}_{\mu\nu} = \frac{1}{\delta_M}\tilde{b}^{\pm}_{\mu\nu}$, where $\tilde{b}^{\pm}_{\mu\nu}$ is defined and positive definite on U. Moreover, we see that $a_n = \frac{1}{\delta_M^2}\tilde{a}_n$, where $\tilde{a}_n$ is also defined and positive on U. From this we conclude that

$$|\zeta_\nu|^2_{\omega_M} \sim \delta_M \text{ for } \nu \in \{1,\ldots,p^-, p^- + p^0 + 1,\ldots,n-1\}, \tag{2.12}$$

$$|\zeta_\nu|^2_{\omega_M} \sim 1 \text{ for } \nu \in \{p^- + 1,\ldots,p^- + p^0\} \text{ and } |\zeta_n|^2_{\omega_M} \sim \delta_M^2. \tag{2.13}$$

By construction of ω_M, we clearly have $\omega_M \gtrsim \partial\log\delta_M\wedge\overline{\partial}\log\delta_M$,so

$$|\partial\log\delta_M|^2_{\omega_M} \lesssim 1. \tag{2.14}$$

We have

$$\begin{aligned}\sum_{\alpha=1}^{n} L_\alpha(b^\epsilon_{\mu\nu})(x)\ \zeta_\alpha(x) &= (\partial b^\epsilon_{\mu\nu})(x) = (\partial(\frac{1}{\delta_M}\tilde{b}^\epsilon_{\mu\nu}))(x)\\ &= -b^\epsilon_{\mu\nu}(x)\partial\log\delta_M(x) + \frac{1}{\delta_M}\sum_{\alpha=1}^{n} L_\alpha(\tilde{b}^\epsilon_{\mu\nu})(x)\ \zeta_\alpha(x),\\ \sum_{\alpha=1}^{n} L_\alpha(a_n)(x)\zeta_\alpha(x) &= -2a_n(x)\partial\log\delta_M(x) + \frac{1}{\delta_M^2}\sum_{\alpha=1}^{n} L_\alpha(\tilde{a}_n)(x)\zeta_\alpha(x),\end{aligned}$$

therefore (2.4) and (2.9) are bounded with respect to ω_M by (2.14), (2.12) and (2.13).

(2.7) and (2.8) are bounded with respect to ω_M by (2.12) and (2.13). Finally, (2.2), (2.3), (2.12) and (2.13) imply that (2.5), (2.6), (2.10) and (2.11) are bounded with respect to ω_M.

It is also clear that (iii) and (v) of Lemma 2.3 are satisfied.

Let us now prove (ii). We assume $p^- \geq 1$ (the weakly pseudoconvex case $p^- = 0$ was settled in [Br]). We then have $r \geq 2$.

From Lemma 2.2, we get

$$i\partial\overline{\partial}\varphi \geq -\lambda i\partial\overline{\partial}\log\delta_M - (1-\lambda)i\partial\overline{\partial}\log\psi$$

where $\lambda = \frac{1}{2} + \frac{1}{2}\chi'_\beta(\frac{\log(-\psi)-\log\delta_M}{2})$. On the set where $\lambda \geq \frac{1}{2}$, the assertion (ii) is clear by (2.1). On the other hand, on $\{\lambda \leq \frac{1}{2}\}$, we have $-\psi \leq \delta_M$ (see the definition of χ_β), and thus by construction of ω_M we get $\omega_M \lesssim \frac{1}{\delta_M}\omega_g \leq \frac{1}{-\psi}\omega_g \lesssim -i\partial\overline{\partial}\log(-\psi)$ on $\mathrm{Ker}\partial\delta_M \cap T^{1,0}X$, which is a subbundle of rank $(n-1)$ of $T^{1,0}X$. If $0 < \beta_1 \leq \ldots \leq \beta_n$ are the eigenvalues of $-i\partial\overline{\partial}\log(-\psi)$ with respect to ω_M, we thus have $\beta_2 \geq 2\sigma$ on $\{\lambda \leq \frac{1}{2}\}$ for some $\sigma > 0$. Since $\alpha_1 + \ldots + \alpha_r > 0$ for $r \geq n - p^+ - p^0 \geq 2$, we then have $\gamma_1 + \ldots + \gamma_r \geq \frac{1}{2}(\beta_1 + \ldots + \beta_r) \geq \sigma$ on $\{\lambda \leq \frac{1}{2}\}$. This establishes (ii).

We define $\omega_j = \omega_M + i\theta_j\partial\varphi \wedge \overline{\partial}\varphi$ where $\theta_j \in \mathcal{C}^\infty(D)$ vanishes on a neighborhood of $\overline{D}_j$ and equals one on $D \setminus D_{j+1}$. Then $|\partial\varphi|_{\omega_j}$ is bounded (j is fixed!), thus ω_j is complete and has all the desired properties. □

3 The L^2 estimates

From now on, D will be equipped with the metric ω_M given by Lemma 2.3. Properties (ii) and (iv) will be used to obtain L^2-solutions of some $\overline{\partial}$-equation. Property (v) will yield regularity results for these solutions.

Let (E, h) be a hermitian vector bundle on X, and let $N \in \mathbb{Z}$. We denote by $L^2_{p,q}(D, E, N)$ the Hilbert space of (p, q)-forms u on D with values in E which satisfy

$$\|u\|_N^2 := \int_D |u|^2_{\omega_M,h}\Delta^N \mathrm{dV}_{\omega_M} < +\infty.$$

Here dV_{ω_M} is the canonical volume element associated to the metric ω_M, and $|\ |_{\omega_M,h}$ is the norm of (p, q)-forms induced by ω_M and h.

Proposition 3.1
Let $N \gg 1$. Suppose $f \in L^2_{n,r}(D, E, N) \cap \mathrm{Ker}\overline{\partial}$, $r \geq n - p^+ - p^0$. Then there exists $u \in L^2_{n,r-1}(D, E, N)$ such that $\overline{\partial}u = f$ and $\|u\|_N \leq \|f\|_N$.

Proof: We have already seen that $\Delta \sim e^{-\varphi}$. Also $\Delta^N \sim e^{-N\varphi}$ for $N \in \mathbb{N}$. Thus it suffices to prove the statement with Δ^N replaced by $e^{-N\varphi}$ in the definition of the spaces $L^2_{p,q}(D, E, N)$.

For $j \in \mathbb{N}$, let us denote by $L^2_{p,q}(D, E, N, j)$ the Hilbert space of (p, q)-forms u on D with values in E which satisfy

$$\|u\|_{N,j}^2 := \int_{D_j} |u|^2_{\omega_j,h} e^{-N\chi_j(\varphi)} \mathrm{dV}_{\omega_j} < +\infty.$$

where $\chi_j \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ with $\chi_j(t) = t$ if $t \leq \log j$, $\chi_j(t) \geq t$ for all $t \in \mathbb{R}$.

Let $\overline{\partial}^*_{N,j}$ be the Hilbert adjoint of $\overline{\partial}$ with respect to the canonical scalar product $\langle\!\langle\ ,\ \rangle\!\rangle_{N,j}$ of (p,q)-forms with values in E induced by $\|\ \|_{N,j}$.

Nakano's inequality (see [De2], [Oh]) yields

$$\begin{aligned}\frac{3}{2}\,(\|\overline{\partial}u\|^2_{N,j}+\|\overline{\partial}^*_{N,j}u\|^2_{N,j}) &\geq \langle\!\langle[i\Theta(E_{N,j}),\Lambda_j]u,u\rangle\!\rangle_{N,j} \\ &\quad -\frac{1}{2}(\|\tau_j u\|^2_{N,j}+\|\tau_j^* u\|^2_{N,j}+\|\overline{\tau}_j u\|^2_{N,j}+\|\overline{\tau}_j^* u\|^2_{N,j})\end{aligned} \tag{3.1}$$

where $\Theta(E_{N,j})$ is the curvature of the bundle $E_{N,j}=(E,e^{-N\chi_j(\varphi)}h)$, Λ_j is the adjoint of multiplication by ω_j and $\tau_j=[\Lambda_j,\partial\omega_j]$. ω_j is the metric given by Lemma 2.3.

As $i\Theta(E_{N,j}) = iN\partial\overline{\partial}\chi_j(\varphi)\otimes \mathrm{Id}_E + i\Theta(E)$, a standard calculation (cf [De2]) yields

$$\begin{aligned}[i\Theta(E_{N,j}),\Lambda_j] &= N[i\partial\overline{\partial}\chi_j(\varphi)\otimes\mathrm{Id}_E,\Lambda_j]+[i\Theta(E),\Lambda_j] \\ &\geq N\chi_j'(\varphi)(\gamma_1^j+\ldots+\gamma_r^j)\otimes\mathrm{Id}_E+[ic(E),\Lambda_j]\end{aligned}$$

when this curvature tensor acts on (n,r)-forms. Here γ_l^j are the eigenvalues of $i\partial\overline{\partial}\varphi$ with respect to ω_j.

For $r\geq n-p^+-p^0$, we have $\gamma_1+\ldots+\gamma_r\geq\sigma$ on D_j. Since $|\partial\omega_M|_{\omega_M}$ is bounded on D by (iv) of Lemma 2.3 and $\omega_j=\omega_M$ on D_j, the pointwise norms $|\tau_j u|_{\omega_j}$, $|\overline{\tau}_j u|_{\omega_j}$, $|\tau_j^* u|_{\omega_j}$ and $|\overline{\tau}_j^* u|_{\omega_j}$ are uniformly bounded with respect to j by some constant times $|u|_{\omega_M}$ on D_j. Thus, choosing N big enough and χ_j sufficiently rapidly increasing on $\{t>\log j\}$, the right hand side of (3.1) can be made $\geq \frac{3}{2}\|u\|^2_{N,j}$.

Let $f\in L^2_{n,r}(D,E,N)\cap\mathrm{Ker}\overline{\partial}$, $r\geq n-p^+-p^0$. Since f is of bidegree (n,r) and $\chi_j(\varphi)\geq\varphi$, a standard calculation (see [De1]) yields $\|f\|_{N,j}\leq\|f\|_N$. By standard L^2-theory we then get $u_j\in L^2_{n,r-1}(D,E,N,j)$ satisfying $\overline{\partial}u_j=f$ and $\|u_j\|_{N,j}\leq\|f\|_{N,j}\leq\|f\|_N$. Therefore the solutions u_j are uniformly bounded in L^2 norm on every compact subset of D. Since the unit ball of a Hilbert space is weakly compact, we can extract a subsequence $u_{\ell_j}\to u\in L^2_{\mathrm{loc}}$ converging weakly in L^2 on any compact subset $K\subset D$, for some $\ell_j\to+\infty$. By the weak continuity of differentiation, we get again in the limit $\overline{\partial}u=f$. Also, since $\chi_j(\varphi)=\varphi$ on D_j, we have

$$\int_{D_j}|u|^2_{\omega_M}e^{-N\varphi}dV_{\omega_M}\leq\liminf_{j\to+\infty}\int_{D_j}|u_{\ell_j}|^2_{\omega_{\ell_j},h}e^{-N\chi_j(\varphi)}dV_{\omega_{\ell_j}}\leq\|f\|^2_N,$$

hence $\|u\|^2_N\leq\|f\|^2_N$. □

Proposition 3.2
Let $N \gg 1$. Suppose $f \in L^2_{0,r}(D,E,-N) \cap \mathrm{Ker}\overline{\partial}$, $r \leq p^+ + p^0$. Then there exists $u \in L^2_{0,r-1}(D,E,-N+2)$ such that $\overline{\partial}u = f$ and $\|u\|_{-N+2} \leq \|f\|_{-N}$. Moreover, $\mathrm{Im}(\overline{\partial} : L^2_{0,p^++p^0}(D,E,-N+2) \to L^2_{0,p^++p^0+1}(D,E,-N))$ is closed in $L^2_{0,p^++p^0+1}(D,E,-N)$.

Proof. Suppose $r \leq p^+ + p^0$ and let $f \in L^2_{0,r}(D,E,-N) \cap \mathrm{Ker}\overline{\partial}$, $N \gg 1$. We define the linear operator

$$L_f: \quad \overline{\partial}L^2_{n,n-r}(D,E^*,N-2) \longrightarrow \mathbb{C}$$
$$\overline{\partial}g \longmapsto \int_D f \wedge g$$

Note that the integral on the right hand side is finite, since

$$|\int_D f \wedge g|^2 \leq (\int_D |f|^2_{\omega_M} \Delta^{-N} dV_{\omega_M}) \cdot (\int_D |g|^2_{\omega_M} \Delta^N dV_{\omega_M}) \leq \|f\|^2_{-N} \|g\|^2_{N-2}.$$

Let us first show that L_f is well defined.

Indeed, let $g_1,\ g_2 \in L^2_{n,n-r}(D,E^*,N-2)$ such that $\overline{\partial}g_1 = \overline{\partial}g_2$. Then $\overline{\partial}(g_1 - g_2) = 0$ and by Proposition 3.1, since $n - r \geq n - p^+ - p^0$, there exists $\alpha \in L^2_{n,n-r-1}(D,E^*,N-2)$ such that $\overline{\partial}\alpha = g_1 - g_2$. But then

$$\begin{aligned} \int_D f \wedge (g_1 - g_2) &= \int_D f \wedge \overline{\partial}\alpha \\ &= \lim_{\varepsilon \to 0} (-1)^r \int_{\partial D_\varepsilon} f \wedge \alpha \\ &= -\lim_{\varepsilon \to 0} \int_{D \setminus D_\varepsilon} f \wedge \overline{\partial}\alpha \\ &= -\lim_{\varepsilon \to 0} \int_{D \setminus D_\varepsilon} f \wedge (g_1 - g_2) \end{aligned}$$

with $(D_\varepsilon)_\varepsilon$ an exhaustion of D by smooth open sets such that $D_\varepsilon \supset \{z \in D \mid \Delta(z) > \varepsilon\}$. Here we have used Stoke's theorem several times. The third equality is obtained as follows: Fix $\varepsilon < 0$ and choose for each large $j > \frac{2}{\varepsilon}$ a $\mathcal{C}^\infty$ function χ_j such that $\chi_j \equiv 1$ on $D_{\frac{2}{j}}$, $\chi_j \equiv 0$ on $D_{\frac{1}{j}}$, $0 \leq \chi_j \leq 1$, $|D\chi_j| \leq Cj$, and set $\alpha_j = \chi_j \alpha \in \mathcal{D}^{n,n-q-1}(D)$. Then we have

$$\int_{D \setminus D_\varepsilon} f \wedge \overline{\partial}\alpha_j = \int_{D \setminus D_\varepsilon} \chi_j f \wedge \overline{\partial}\alpha + \int_{D \setminus D_\varepsilon} f \wedge \overline{\partial}\chi_j \wedge \alpha$$

and

$$\begin{aligned} |\int_{D \setminus D_\varepsilon} f \wedge \overline{\partial}\chi_j \wedge \alpha|^2 &\leq C \int_{D \setminus D_\varepsilon} |f|^2_{\omega_M} \Delta^{-N} dV_{\omega_M} \cdot \int_{D \setminus D_{\frac{2}{j}}} j^2 |\alpha|^2_{\omega_M} \Delta^N dV_{\omega_M} \\ &\leq C \|f\|^2_{-N} \|\alpha\|^2_{N-2}. \end{aligned}$$

Hence the dominated convergence theorem gives

$$\begin{aligned}\int_{D\setminus D_\varepsilon} f\wedge\overline{\partial}\alpha &= \lim_j \int_{D\setminus D_\varepsilon} f\wedge\overline{\partial}\alpha_j = (-1)^q \lim_j \int_{D\setminus D_\varepsilon} \overline{\partial}(f\wedge\alpha_j)\\ &= -(-1)^q \lim_j \int_{\partial D_\varepsilon} f\wedge\alpha_j = -(-1)^q \int_{\partial D_\varepsilon} f\wedge\alpha.\end{aligned}$$

Moreover,

$$|\int_{D\setminus D_\varepsilon} f\wedge(g_1-g_2)| \le (\int_{D\setminus D_\varepsilon} |f|^2_{\omega_M}\Delta^{-N})^{1/2}\ (\int_{D\setminus D_\varepsilon} |g_1-g_2|^2_{\omega_M}\Delta^{N})^{1/2}$$

$$\longrightarrow_{\varepsilon\to 0} 0.$$

(note that $(\int_{D\setminus D_\varepsilon} |g_1-g_2|^2_{\omega_M}\Delta^{N})^{1/2} \lesssim \varepsilon \int_{D\setminus D_\varepsilon} |g_1-g_2|^2_{\omega_M}\Delta^{N-2})^{1/2} \le \varepsilon\|g_1-g_2\|_{-N-2} \to 0$ as $\varepsilon\to 0$ since $g_1,g_2\in L^2_{n,n-r}(D,E^*,N-2)$).

Thus $L_f(g_1)=L_f(g_2)$.

Now let $g\in \mathrm{Dom}(\overline{\partial} : L^2_{n,n-r}(D,E^*,N-2)\to L^2_{n,n-r+1}(D,E^*,N-2))$. Applying Proposition 3.1, there exists $\widetilde{g}\in L^2_{n,n-r}(D,E^*,N-2)$ satisfying $\overline{\partial}\widetilde{g}=\overline{\partial}g$ and $\|\widetilde{g}\|_{N-2}\le\|\overline{\partial}g\|_{N-2}$. This yields

$$|L_f(\overline{\partial}g)| = |L_f(\overline{\partial}\widetilde{g})| = |\int_D f\wedge\widetilde{g}| \le \|f\|_{-N}\|\widetilde{g}\|_N$$

$$\le \|f\|_{-N}\|\widetilde{g}\|_{N-2} \le \|f\|_{-N}\|\overline{\partial}g\|_{N-2}.$$

Thus L_f is a continuous linear operator of norm $\le\|f\|_{-N}$ and therefore, using the Hahn-Banach theorem, L_f extends to a continuous linear operator with norm $\le\|f\|_{-N}$ on the Hilbert space $L^2_{n,n-r+1}(D,E^*,N-2)$. By the theorem of Riesz, there exists $u\in L^2_{0,r-1}(D,E,-N+2)$ with $\|u\|_{-N+2}\le\|f\|_{-N}$ such that for every $g\in L^2_{n,n-r}(D,E^*,N-2)$ we have

$$(-1)^r\int_D u\wedge\overline{\partial}g = L_f(g) = \int_D f\wedge g,$$

i.e. $\overline{\partial}u=f$.

To prove the last assertion, we show that

$$\mathrm{Im}(\overline{\partial} : L^2_{0,p^++p^0}(D,E,-N+2) \longrightarrow L^2_{0,p^++p^0+1}(D,E,-N)) =$$

$$\{g\in L^2_{0,p^0+p^++1}(D,E,-N) \mid \int_D g\wedge h = 0\ \forall h\in L^2_{n,n-p^0-p^+-1}(D,E^*,N-2)\}.$$

Suppose $f \in \mathrm{Im}(\overline{\partial} : L^2_{0,p^+ + p^0}(D,E,-N+2) \longrightarrow L^2_{0,p^+ + p^0+1}(D,E,-N))$. Then there exists $\alpha \in L^2_{0,p^+ + p^0}(D,E,-N+2)$ such that $\overline{\partial}\alpha = f$. Thus we get for every $h \in L^2_{n,n-p^0-p^+-1}(D,E^*,N-2)$

$$\begin{aligned}
\int_D f \wedge h &= \int_D \overline{\partial}\alpha \wedge h \\
&= \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \alpha \wedge h \\
&= -\lim_{\varepsilon \to 0} \int_{D \setminus D_\varepsilon} \overline{\partial}\alpha \wedge h \\
&= -\lim_{\varepsilon \to 0} \int_{D \setminus D_\varepsilon} f \wedge h
\end{aligned}$$

with $(D_\varepsilon)_\varepsilon$ an exhaustion of D by smooth open sets such that $D_\varepsilon \supset \{z \in D \mid \Delta(z) > \varepsilon\}$ and

$$\begin{aligned}
\Big|\int_{D \setminus D_\varepsilon} f \wedge h\Big| &\leq \Big(\int_{D \setminus D_\varepsilon} |f|^2_{\omega_M} \Delta^{-N}\Big)^{1/2} \Big(\int_{D \setminus D_\varepsilon} |h|^2_{\omega_M} \Delta^{N}\Big)^{1/2} \\
&\leq \varepsilon \|f\|_{-N} \|h\|_{N-2} \longrightarrow_{\varepsilon \to 0} 0,
\end{aligned}$$

which shows the inclusion $\subset$ (the justification of the third equality is as above).

Conversely, we show that for every $f \in \{g \in L^2_{0,p^0+p^++1}(D,E,-N) \mid \int_D g \wedge h = 0 \ \forall h \in L^2_{n,n-p^0-p^+-1}(D,E^*,N-2)\}$, there exists $u \in L^2_{0,p^+ + p^0}(D,E,-N+2)$ satisfying $\overline{\partial}u = f$. Again, we define the linear operator

$$\begin{aligned}
L_f : \quad \overline{\partial} L^2_{n,n-p^+-p^0-1}(D,E^*,N-2) &\longrightarrow \mathbb{C} \\
\overline{\partial} g &\longmapsto \int_D f \wedge g
\end{aligned}$$

Here we write $\overline{\partial} L^2_{n,n-p^+-p^0-1}(D,E^*,N-2)$ for $\mathrm{Im}(\overline{\partial} : L^2_{n,n-p^+-p^0-1}(D,E^*,N-2) \to L^2_{n,n-p^+-p^0}(D,E^*,N-2))$. L_f is well defined because of the moment conditions imposed on f. We then show the existence of the desired u as in the first part of the proof. $\square$

Let $U \subset X$ be an open set and E a holomorphic vector bundle on X. For $k \in \mathbb{N} \cup \{+\infty\}$, we define

$$\mathcal{C}^k_{p,q}(X, \overline{U}, E) = \{f \in \mathcal{C}^k_{p,q}(X,E) \mid \mathrm{supp}\ f \subset \overline{U}\}$$

From [Br] we get a regularity theorem for $\square_{-N}$ (cf also Theorem 4.1 in [Br]).
Here $\square_{-N} = \overline{\partial}\overline{\partial}^*_{-N} + \overline{\partial}^*_{-N}\overline{\partial}$ where $\overline{\partial}^*_{-N}$ is the Von Neumann adjoint of $\overline{\partial} : L^2_{p,q}(D,E,-N+2) \to L^2_{p,q+1}(D,E,-N)$.

Theorem 3.3
If $u \in L^2_{p,q}(D,E,-N)$ satisfies $\overline{\partial}u = f$ and $\overline{\partial}^*_{-N}u = 0$ with $f \in \mathcal{C}^N_{p,q}(X,\overline{D},E) \cap \mathcal{C}^\infty_{p,q}(D,E)$, then $u \in \mathcal{C}^{s(N)}_{p,q}(X,\overline{D},E) \cap \mathcal{C}^\infty_{p,q}(D,E)$ where $s(N) \sim \sqrt{N}$ for all $N \gg 1$.

4 The $\overline{\partial}$-equation with exact support

Let Ω be a smooth bounded completely strictly pseudoconvex domain in a complex n-dimensional manifold X and M a real hypersurface of class $\mathcal{C}^\infty$ intersecting $\partial\Omega$ transversally such that $\Omega \setminus M$ has exactly two connected components. We suppose that $M = \{\varrho = 0\}$ where ϱ is a $\mathcal{C}^\infty$ function whose Levi form has exactly p^+ positive, p^0 zero and p^- negative eigenvalues on $T^{1,0}_x M$ for each $x \in M$, $p^- + p^0 + p^+ = n-1$. We put $D = \Omega \cap \{\varrho < 0\}$.

In this section, we will show some vanishing and separation theorems for the $\overline{\partial}$-cohomology groups with values in a holomorphic vector bundle E supported in $\overline{D}$:

$$H^{p,q}(X,\overline{D},E) = \mathcal{C}^\infty_{p,q}(X,\overline{D},E) \cap \mathrm{Ker}\overline{\partial}/\overline{\partial}(\mathcal{C}^\infty_{p,q-1}(X,\overline{D},E))$$

Theorem 4.1
Let E be a holomorphic vector bundle on X. Then we have
$$H^{p,q}(X,\overline{D},E) = 0 \qquad \text{for } 1 \leq q \leq p^0 + p^+$$
and
$$H^{p,p^0+p^++1}(X,\overline{D},E) \text{ is separated for the usual } \mathcal{C}^\infty\text{-topology.}$$

Proof: Replacing the vector bundle E by $\Lambda^p(T^{1,0}X)^* \otimes E$, it is no loss of generality to assume $p = 0$.

We will begin by proving the following claim:
Let $f \in \mathcal{C}^k_{0,q}(X,\overline{D},E) \cap \mathcal{C}^\infty_{0,q}(D,E) \cap \mathrm{Ker}\overline{\partial}$, $1 \leq q \leq p^0 + p^+$, $k \gg 1$. Then there exists $u \in \mathcal{C}^{s(k)}_{0,q-1}(X,\overline{D},E) \cap \mathcal{C}^\infty_{0,q-1}(D,E)$ such that $\overline{\partial}u = f$ with $s(k) \sim \sqrt{k}$.

Proof of the claim: Let $f \in \mathcal{C}^k_{0,q}(X,\overline{D},E) \cap \mathrm{Ker}\overline{\partial}$, $1 \leq q \leq p^0 + p^+$, $k \gg 1$. General results on Lipschitz domains (see e.g. [Gr, Theorem 1.4.4.4]) show that $f \in L^2_{0,q}(D,E,-2k)$ if we keep in mind property (iii) of Lemma 2.3. Proposition 3.2 implies that there exists $u \in L^2_{0,q-1}(D,E,-2k+2)$ such that $\overline{\partial} u = f$ in D and $\|u\|_{-2k+2} \leq \|f\|_{-2k}$. Moreover, choosing the minimal solution, we may assume $\overline{\partial}^*_{-2k} u = 0$. From Theorem 3.3 we get that $u \in \mathcal{C}^{s(k)+1}_{0,q-1}(X,\overline{D},E)$ with $s(k) \sim \sqrt{k}$.

Let us now prove the theorem.

$H^{0,1}(X,\overline{D},E) = 0$ follows immediately from the above claim and the hypoellipticity of $\overline{\partial}$ in bidegree $(0,1)$ if $1 \leq p^0 + p^+$.

Now assume $1 < q \leq p^0 + p^+$ and let $f \in \mathcal{C}^\infty_{0,q}(X,\overline{D},E) \cap \mathrm{Ker}\overline{\partial}$. By induction, we will construct $u_k \in \mathcal{C}^k_{0,q-1}(X,\overline{D},E) \cap \mathcal{C}^\infty_{0,q-1}(D,E)$ such that $\overline{\partial} u_k = f$ and $|u_{k+1} - u_k|_{s(k)-1} < 2^{-k}$. It is then clear that $(u_k)_{k\in\mathbb{N}}$ converges to $u \in \mathcal{C}^\infty_{0,q-1}(X,\overline{D},E)$ such that $\overline{\partial} u = f$.

Suppose that we have constructed $u_1, \ldots, u_k$. By the above claim, there exists $\alpha_{k+1} \in \mathcal{C}^{k+1}_{0,q-1}(X,\overline{D},E) \cap \mathcal{C}^\infty_{0,q-1}(D,E)$ such that $f = \overline{\partial}\alpha_{k+1}$. We have $\alpha_{k+1} - u_k \in \mathcal{C}^k_{0,q-1}(X,\overline{D},E) \cap \mathcal{C}^\infty_{0,q-1}(D,E) \cap \mathrm{Ker}\overline{\partial}$. Once again by the above claim, there exists $g \in \mathcal{C}^{s(k)}_{0,q-2}(X,\overline{D},E) \cap \mathcal{C}^\infty_{0,q-2}(D,E)$ satisfying $\alpha_{k+1} - u_k = \overline{\partial} g$.

Since $\mathcal{C}^\infty_{0,q-2}(X,\overline{D},E)$ is dense in $\mathcal{C}^{s(k)}_{0,q-2}(X,\overline{D},E)$, there exists $g_{k+1} \in \mathcal{C}^\infty_{0,q-2}(X,\overline{D},E)$ such that $|g - g_{k+1}|_{s(k)} < 2^{-k}$.

Define $u_{k+1} = \alpha_{k+1} - \overline{\partial} g_{k+1} \in \mathcal{C}^{k+1}_{0,q-1}(X,\overline{D},E) \cap \mathcal{C}^\infty_{0,q-1}(D,E)$. Then $\overline{\partial} u_{k+1} = f$ and $|u_{k+1} - u_k|_{s(k)-1} = |\overline{\partial} g - \overline{\partial} g_{k+1}|_{s(k)-1} \leq |g - g_{k+1}|_{s(k)} < 2^{-k}$. Thus u_{k+1} has the desired properties.

The last assertion is proved similarly, using the "moreover" statement in Proposition 3.2 and the fact that the $\mathcal{C}^\infty$ topology is stronger that the L^2 topologies. □

The results of this section will allow us to solve the $\overline{\partial}$-equation for extensible currents by duality.

We recall the notations. A current T defined on D is said to be *extensible*, if T is the restriction to D of a current defined on X.

It was shown in [Ma] that, since D satisfies $\overset{\circ}{\overline{D}}$, the vector space $\check{\mathcal{D}}'^{p,q}_D(X)$ of extensible currents on D of bidegree (p,q) is the topological dual of $\mathcal{C}^\infty_{n-p,n-q}(X,\overline{D})$.

Theorem 4.2
Let $T \in \check{\mathcal{D}}'^{p,q}_D(X)$ be an extensible current on D of bidegree (p,q), $q \geq n - p^0 - p^+$ such that $\overline{\partial}T = 0$ in D. Then there exists $S \in \check{\mathcal{D}}'^{p,q-1}_D(X)$ satisfying $\overline{\partial}S = T$ in D.

Proof: Let $T \in \check{\mathcal{D}}'^{p,q}_D(X)$ be an extensible current on D of bidegree (p,q), $q \geq n - p^0 - p^+$, such that $\overline{\partial}T = 0$ in D.

Consider the operator

$$\begin{array}{cccc} L_T : & \overline{\partial}\mathcal{C}^\infty_{n-p,n-q}(X,\overline{D}) & \longrightarrow & \mathbb{C} \\ & \overline{\partial}\varphi & \longmapsto & < T, \varphi > \end{array}$$

We first notice that L_T is well-defined. Indeed, let $\varphi \in \mathcal{C}^\infty_{n-p,n-q}(X,\overline{D})$ be such that $\overline{\partial}\varphi = 0$.
If $q = n$, the analytic continuation principle for holomorphic functions yields $\varphi = 0$, so $< T, \varphi >= 0$.
If $n - 1 \geq q \geq n - p^0 - p^+$, one has $\varphi = \overline{\partial}\alpha$ with $\alpha \in \mathcal{C}^\infty_{n-p,n-q-1}(X,\overline{D})$ by Theorem 4.1. As $\mathcal{D}^{n-p,n-q-1}(D)$ is dense in $\mathcal{C}^\infty_{n-p,n-q-1}(X,\overline{D})$, there exists $(\alpha_j)_{j\in\mathbb{N}} \in \mathcal{D}^{n-p,n-q-1}(D)$ such that $\overline{\partial}\alpha_j \underset{j\to+\infty}{\longrightarrow} \overline{\partial}\alpha$ in $\mathcal{C}^\infty_{n-p,n-q}(X,\overline{D})$.
Hence $< T, \varphi >=< T, \overline{\partial}\alpha >= \lim_{j\to+\infty} < T, \overline{\partial}\alpha_j >= 0$, because $\overline{\partial}T = 0$.

By Theorem 4.1, $\overline{\partial}\mathcal{C}^\infty_{n-p,n-q}(X,\overline{D})$ is a closed subspace of $\mathcal{C}^\infty_{n-p,n-q+1}(X,\overline{D})$, thus a Fréchet space. Using Banach's open mapping theorem, L_T is in fact continuous, so by the Hahn-Banach theorem, we can extend L_T to a continuous linear operator $\tilde{L}_T : \mathcal{C}^\infty_{n-p,n-q+1}(X,\overline{D}) \longrightarrow \mathbb{C}$, i.e. $\tilde{L}_T$ is an extensible current on D satisfying

$$< \overline{\partial}\tilde{L}_T, \varphi >= (-1)^{p+q} < \tilde{L}_T, \overline{\partial}\varphi >= (-1)^{p+q} < T, \varphi >$$

for every $\varphi \in \mathcal{C}^\infty_{n-p,n-q}(X,\overline{D})$. Therefore $T = (-1)^{p+q}\overline{\partial}\tilde{L}_T$. □

Remark. Analogous results have been obtained in [Sa1] for completely strictly q-convex domains with smooth boundary. These are domains of the form $\Omega = \{z \in U_{\overline{\Omega}} \mid \psi(z) < 0\}$ where ψ is a smooth function defined on an open neighborhood $U_{\overline{\Omega}}$ of $\overline{\Omega}$ whose Levi form has at least $q+1$ positive eigenvalues everywhere. Sambou shows that for such a domain the $\overline{\partial}$-equation is solvable for extensible currents of bidegree (p,r), $r \geq n - q$. In [Sa2], also the strictly q-concave case is discussed.

5 Applications to hypersurfaces with constant signature

If M is a CR manifold, then we denote by $H_c^{p,q}(M)$ the $\overline{\partial}_M$-cohomology groups for smooth forms with compact support in M. We have the following result:

Theorem 5.1
Let X be a Stein manifold of complex dimension $n \geq 2$ and M a smooth, closed, connected hypersurface in X. Suppose that M has signature (p^-, p^0, p^+) at each point. Then $H_c^{p,q}(M) = 0$ for $0 \leq p \leq n$, $0 \leq q \leq \min(p^-, p^+) + p^0 - 1$.

Proof: Let $f \in \mathcal{C}^\infty_{p,q}(M) \cap \mathrm{Ker}\overline{\partial}_M$ such that $\mathrm{supp}\, f \subset K$, where K is a compact subset of M. Since X is Stein, there exists a smooth bounded completely strictly pseudoconvex domain Ω such that $K \subset \Omega$, $\Omega \setminus M$ has exactly two connected components D^+ and D^-, and M intersects $\partial\Omega$ transversally.

Next, we can find $\tilde{f} \in \mathcal{C}^\infty_{p,q}(X)$ such that $\tilde{f}_{|M} = f$, $\mathrm{supp}\, \tilde{f} \subset\subset \Omega$ and $\overline{\partial}\tilde{f}$ vanishes to infinite order on M.

Applying Theorem 4.1, we conclude that $H^{p,q+1}(X, \overline{D}^{\pm}) = 0$ for $q+1 \leq p^0 + \min(p^-, p^+)$. Therefore there exists a solution $u \in \mathcal{C}^\infty_{p,q}(X)$ to the equation $\overline{\partial}u = \overline{\partial}\tilde{f}$ in such a way that u vanishes on $M \cup (X \setminus \overline{\Omega})$. $F = \tilde{f} - u$ is then $\overline{\partial}$-closed in X and we have $F_{|M} = f$, $\mathrm{supp}\, F \subset \overline{\Omega}$.

If $q = 0$, the analytic continuation principle yields $F \equiv 0$, thus $f \equiv 0$, proving $H_c^{p,0}(M) = 0$.

Now let $q \geq 1$. Ω being completely strictly pseudoconvex, there exists an open set $\tilde{\Omega} \supset \Omega$ which is also completely strictly pseudoconvex. Then the $\overline{\partial}$-cohomology groups with compact support in $\tilde{\Omega}$, $H_c^{p,q}(\tilde{\Omega})$, vanish for $q \geq 1$. Thus we can find $U \in \mathcal{C}^\infty_{p,q-1}(X)$, supp $U \subset\subset \tilde{\Omega}$ satisfying $\overline{\partial}U = F$. We then have $\overline{\partial}_M(U_{|M}) = f$, which proves the theorem. $\square$

It is clear that Theorem 1.1 is an immediate consequence of Theorem 5.1, if we keep in mind that global minimality assures the validity of the weak analytic continuation principle for CR function. Let us just remind that $M \setminus K$ is globally minimal if any two points $p, q \in M \setminus K$ can be joined by a piecewise smooth curve $\gamma = \gamma_1 \cup \ldots \cup \gamma_r$, $\gamma_i : [0,1] \longrightarrow M \setminus K$, such that $\gamma_i'(t) \in T_{\gamma_i(t)}M \cap JT_{\gamma_i(t)}M$ for all $t \in (0,1)$; here J denotes the complex structure on X.

Using the results of the previous section we can also prove a result on the solvability of the $\overline{\partial}_M$-equation for currents on hypersurfaces with constant signature.

Theorem 5.2
Let X be a complex manifold of dimension n and M a smooth, closed, connected hypersurface in X. Suppose that M has signature (p^-, p^0, p^+) at each point. Let $\Omega \subset\subset X$ be a smooth bounded completely strictly pseudoconvex domain in X such that $\Omega \setminus M$ has exactly two connected components and M intersects $\partial\Omega$ transversally. Then $H^{p,q}_{cur}(M \cap \Omega) = 0$ for $0 \leq p \leq n$, $q \geq n - \min(p^-, p^+) - p^0 + 1$.
Moreover, let Ω' be any open set which is relatively compact in Ω. Then for $q = n - \min(p^-, p^+) - p^0$, the restriction mapping

$$H^{p,q}_{cur}(M \cap \Omega) \longrightarrow H^{p,q}_{cur}(M \cap \Omega')$$

is the zero mapping.

Proof: We denote by D^+ and D^- the connected components of $\Omega \setminus M$.

Let $H^{p,q}(\check{\mathcal{D}}'_{D^+}(\Omega))$ (resp. $H^{p,q}(\check{\mathcal{D}}'_{D^-}(\Omega))$) the $\overline{\partial}$-cohomology groups of currents on D^+ (resp. D^-) which are extendable to Ω. Moreover, we consider the $\overline{\partial}$-cohomology groups $H^{p,q}(M \cap \Omega, \mathcal{D}'_M)$ of currents on Ω with support on $M \cap \Omega$.

We then have the following long exact sequence (cf [Hi/Na1], [Na/Va])

$$\ldots \to H^{p,q}_{cur}(\Omega) \to H^{p,q}(\check{\mathcal{D}}'_{D^+}(\Omega)) \oplus H^{p,q}(\check{\mathcal{D}}'_{D^-}(\Omega)) \to H^{p,q+1}(M \cap \Omega, \mathcal{D}'_{M\cap\Omega}) \to H^{p,q+1}_{cur}(\Omega) \to \ldots$$

Since Ω is completely strictly pseudoconvex, we have $H^{p,q}_{cur}(\Omega) = 0$ for all $q \geq 1$. Moreover, it follows from [Hi/Na2] and [Na/Va] that we have a natural isomorphism $H^{p,q}_{cur}(M \cap \Omega) \to H^{p,q+1}(M \cap \Omega, \mathcal{D}'_{M\cap\Omega})$. Hence $H^{p,q}_{cur}(M \cap \Omega) \simeq H^{p,q}(\check{\mathcal{D}}'_{D^+}(\Omega)) \oplus H^{p,q}(\check{\mathcal{D}}'_{D^-}(\Omega))$. The theorem is now a consequence of the following lemma (for the case $q = n - \min(p^-, p^+) - p^0$, note that all diagrams induced by the restriction mapping are commutative). □

Lemma 5.3
For $0 \leq p \leq n$ and $q \geq n - \min(p^-, p^+) - p^0 + 1$ we have $H^{p,q}(\check{\mathcal{D}}'_{D^+}(\Omega)) = H^{p,q}(\check{\mathcal{D}}'_{D^-}(\Omega)) = 0$.
Moreover, let Ω' be any relatively compact domain in Ω. Then for $q = n - \min(p^-, p^+) - p^0$, the restriction mappings

$$H^{p,q}(\check{\mathcal{D}}'_{D^+}(\Omega)) \longrightarrow H^{p,q}(\check{\mathcal{D}}'_{D^+\cap\Omega'}(\Omega')),$$

$$H^{p,q}(\check{\mathcal{D}}'_{D^-}(\Omega)) \longrightarrow H^{p,q}(\check{\mathcal{D}}'_{D^-\cap\Omega'}(\Omega'))$$

are the zero mappings.

Proof. Let $(\Omega_j)_{j\in\mathbb{N}}$ be an exhaustion of Ω by smooth bounded strictly pseudoconvex domains such that M intersects $\partial\Omega_j$ transversally.

Let $T \in \check{\mathcal{D}}'^{p,q}_{D^+\cap\Omega}(\Omega)$ satisfy $\overline{\partial}T = 0$ in $D^+ \cap \Omega$, $q \geq n - \min(p^-, p^+) - p^0$. It follows from Theorem 4.2 that there exists $S_j \in \check{\mathcal{D}}'^{p,q-1}_{D^+}(\Omega)$ satisfying $\overline{\partial}S_j = T$ in $D^+ \cap \Omega_j$. The same holds true of course with D^+ replaced by D^-. This proves the assertion of the lemma for $q = n - \min(p^-, p^+) - p^0$.

Now let $q \geq n - \min(p^-, p^+) - p^0 + 1$. We have $\overline{\partial}(S_{j+1} - S_j) = 0$ in $D^+ \cap \Omega_j$. Hence, again by Theorem 4.2, there exists $H \in \check{\mathcal{D}}'^{p,q-2}_{D^+}(\Omega)$ satisfying $\overline{\partial}H = S_{j+1} - S_j$ in $D^+ \cap \Omega_j$. Setting $\widetilde{S}_{j+1} = S_{j+1} - \overline{\partial}H$, we have $\overline{\partial}\widetilde{S}_{j+1} = T$ in $D^+ \cap \Omega_{j+1}$ and $\widetilde{S}_{j+1} = S_j$ in $D^+ \cap \Omega_j$. Thus we can find a sequence $(G_j)_{j\in\mathbb{N}}$, $G_j \in \check{\mathcal{D}}'^{p,q-1}_{D^+}(\Omega)$ satisfying $\overline{\partial}G_j = T$ in $D^+ \cap \Omega_j$ and $G_{j+1} = G_j$ in $D^+ \cap \Omega_j$. Hence (G_j) converges to $G \in \check{\mathcal{D}}'^{p,q-1}_{D^+}(\Omega)$ satisfying $\overline{\partial}G = T$ in $D^+ \cap \Omega$. Since the same holds true also for D^-, we have proved the lemma. □

We remark that Theorem 5.2 gives a Poincaré lemma for currents on a certain type of hypersurfaces. Combining this with the results of [Na/Va], [An/Hi], [Mi] and [Hi/Na3], we obtain the following corollary:

Corollary 5.4
Let X be a smooth hypersurface in $\mathbb{C}^n$ and suppose that M has signature (p^-, p^0, p^+) at each point in a neighborhood of $x_0 \in M$. Then the Poincaré lemma holds for smooth forms and for currents of bidegree (p, q) at the point x_0 if $1 \leq q \neq p^-, p^+$, i.e. each smooth form (resp. current) of bidegree (p, q), $1 \leq q \neq p^-, p^+$, which is $\overline{\partial}$-closed on some open neighborhood of x_0 is $\overline{\partial}$-exact on some open neighborhood of x_0.
The Poincaré lemma fails to hold at x_0 for smooth forms and currents of bidegree (p, p^-) and (p, p^+).

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