

TOPOLOGICAL GROUPS OF KAC-MOODY TYPE, FUCHSIAN TWINNINGS AND THEIR LATTICES

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ABSTRACT.— *This paper deals with a class of totally disconnected groups acting on buildings, among which are certain Kac-Moody groups. The apartments of our buildings are hyperbolic planes tiled by right-angled polygons. We discuss linearity properties for the groups, as well as an analogy with semisimple groups over local fields of positive characteristic. Looking for counter-examples to this analogy leads to the construction of Moufang twinings «with several ground fields» .*

Introduction

J. Tits' definition of Kac-Moody groups over fields [T2] can be seen as the achievement of a long process, which involved many people and many points of view. Among previous constructions, are important works by V. Kac and D. Peterson [KP] and by O. Mathieu [Mat]. Whereas O. Mathieu made use of delicate properties from algebraic geometry to derive representation-theoretic results (via Schubert varieties), the article [KP] proposed a combinatorial structure (refining that of a BN -pair) to study Kac-Moody groups over fields of characteristic 0 (seen as automorphism groups of Lie algebras). V. Kac and D. Peterson obtained some defining relations which reappeared in the generalized Steinberg presentation used by J. Tits [T2]. Later J. Tits gave a list of group-theoretic axioms well-suited to the Kac-Moody situation [T5], which he called *twin root data*; its geometric counterpart is the theory of *twin buildings*.

The work in [T2] describes Kac-Moody groups Λ generated by root groups, like semisimple algebraic groups, except that the Weyl group W is infinite. The positive (resp. negative) root groups, along with a torus, generate a subgroup B_+ (resp. B_-), but B_+ and B_- are not conjugate because W fails to contain an element of longest length performing the conjugation. As a result, there are two buildings Δ_+ and Δ_- , whose chambers can be regarded as conjugates of B_+ and B_- respectively. Two chambers in the same building cannot be opposite, but chambers in one can be opposite chambers in the other. This was the starting point for the theory of twin buildings, due initially to J. Tits and the second author.

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As a special case, consider the Kac-Moody group $\Lambda = \mathrm{SL}_n(k[t, t^{-1}])$ over the field k . The Weyl group is of affine type \tilde{A}_{n-1} , and the subgroups $\mathrm{SL}_n(k[t])$ and $\mathrm{SL}_n(k[t^{-1}])$ stabilize vertices in Δ_+ and Δ_- respectively. When k is a finite field \mathbf{F}_q , the buildings Δ_+ and Δ_- are locally finite, and the facet stabilizers of Δ_- act discretely on Δ_+ . On the other hand if we forget about Δ_- , then in its action on Δ_+ , Λ has a natural topological closure G , namely $\mathrm{SL}_n(\mathbf{F}_q((t)))$. Thus the twin root datum that generates Λ gives rise to lattices such as $\mathrm{SL}_n(\mathbf{F}_q[t^{-1}])$ inside the topological group $G = \mathrm{SL}_n(\mathbf{F}_q((t)))$. This production of lattices applies to all twin root data for which the buildings are locally finite. The data may or may not come from a Kac-Moody group. If it does, we call its closure G a *topological Kac-Moody group*, and if not then a *topological group of Kac-Moody type*.

QUESTION.— *To what extent is a topological group of Kac-Moody type analogous to a semisimple group over a local field?*

If the group G is a topological Kac-Moody group, the following theorem, stated more precisely in 1.C.2, gives one analogy:

THEOREM.— *A (locally compact, totally disconnected) topological Kac-Moody group admits a refined Tits system. This Tits system gives rise to a building in which any (spherical) facet-fixator is, up to finite index, a pro- p group.*

A *refined Tits system* is the structure proposed by V. Kac and D. Peterson to study Kac-Moody groups Λ [KP]. It is defined in 1.A.5, and unlike a twin root datum, defined in 1.A.1, is asymmetric with respect to positive and negative root groups, because it is concerned with only one building rather than two. This building is identified with its nonpositively curved realization [D], where only spherical residues appear as facets. The fixators of those facets are analogues of the parahoric subgroups of Bruhat-Tits theory, and their finite index pro- p subgroups are analogues of congruence subgroups.

The above analogy between semisimple groups over local fields and topological Kac-Moody groups can be extended to the more general setting of topological groups of Kac-Moody type. We do this by constructing twin root data that do not arise from Kac-Moody groups. This gives exotic twin buildings. In view of the classification work [Mü, MR] on 2-spherical twin buildings (meaning that their rank 2 residues have finite (dihedral) Weyl groups), the twin buildings we construct are not 2-spherical. One consequence of our work is the existence of Moufang twin buildings (i.e. twin buildings admitting root groups), of any desired finite rank, that do not arise from Kac-Moody groups. The apartments in these buildings can be realised as tilings of the hyperbolic plane by right-angled r -gons ($r \geq 5$). The rank of such a building is r , and the Weyl group is generated by reflections across the faces of a fundamental domain R in the hyperbolic plane. Buildings with such apartments are called *Fuchsian* by M. Bourdon [Bou3], and he proves existence and uniqueness results for them. Uniqueness [loc.cit.], along with the construction given in the present paper, proves the following theorem, described in more detail in 4.E.2:

THEOREM.— *A right-angled Fuchsian building belongs to a Moufang twinning whenever its thicknesses at panels are cardinalities of projective lines.*

The group combinatorics obtained by standard arguments [A] from these buildings are not Kac-Moody as soon as we choose two thicknesses «of different characteristics». As already mentioned, these buildings are non-2-spherical, since the reflections across any two non-intersecting

edges of the polygon R generate an infinite dihedral group. Hence, they are not covered by the Local to Global theorem [MR] that applies in the 2-spherical case, and is leading to a classification [Mü]. On the other hand, techniques from $\text{CAT}(-1)$ -geometry lead to deep rigidity and amenability theorems for (lattices of) Fuchsian buildings [BM, BP]. From the point of view of lattices of $\text{CAT}(-1)$ geometries, we can formulate the following consequence of our construction (5.B).

THEOREM.— *There exist groups with twin root data such that the associated buildings are Fuchsian, and in which any Borel subgroup of a given sign is a non-uniform lattice of the building of opposite sign. Moreover any group homomorphism from such a lattice to a product of linear algebraic groups (possibly over different fields) has infinite kernel.*

In particular, the above groups with twin root data are not linear over any field. This shows that the remaining question concerning linearity of Kac-Moody groups – to find a Kac-Moody group not linear over any field [Ré3] – is solved in the wider (easier) context of groups with twin root data. Discussing linearity, even in the smaller class of Kac-Moody groups, leads to surprising situations, as shown by our last result (5.C).

PROPOSITION.— *There exist topological Kac-Moody groups over \mathbf{F}_q which admit both non-uniform lattices which cannot be linear over any field of characteristic prime to q , and uniform lattices which have convex-cocompact embeddings into real hyperbolic spaces.*

The existence of the latter lattices and of their linear embeddings is due to M. Bourdon [Bou1], who proves that the limit sets of the embeddings often have Hausdorff dimension > 2 .

This paper is organized as follows. In the first section, we recall some combinatorial facts on twin root data (1.A). Then we define the topological groups of Kac-Moody type and topological Kac-Moody groups (1.B). Subsect. 1.C provides arguments to see the latter class of groups as a generalization of algebraic groups over local fields of positive characteristic. Sect. 2 sketches a construction of twin root data, and recalls some relations in SL_2 . Sect. 3 applies this construction to the case of trees. It is a special case of the construction of Moufang twin trees due to J. Tits [T4], but given in a very down-to-earth way. The details we supply in Sect. 3 allow us to avoid computation in Sect. 4 where we concentrate on the geometry of Fuchsian buildings. Finally, in Sect. 5 we discuss linearity properties for lattices of topological groups of Kac-Moody type: this is relevant to the study of Kac-Moody lattices as generalizations of arithmetic groups over function fields [Ré3].

Let us state a convention we will use for group actions. If a group G acts on a set X , the (pointwise) stabilizer of a subset $Y \subset X$ will be called its *fixator*, and will be denoted by $\text{Fix}_G(Y)$. The classical (global) stabilizer will be denoted by $\text{Stab}_G(Y)$. When Y is a facet of a building X , and G is a type preserving group of automorphisms, the two notions of course coincide. Finally the notation $G|_X$ means the group obtained from G by factoring out the kernel of the action on X .

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1. Closures of groups with twin root data

The purpose of this section is to introduce a family of totally disconnected topological groups, called *topological groups of Kac-Moody type* – see 1.B. This requires some basic properties of group combinatorics introduced by J. Tits and by V. Kac and D. Peterson, which we describe in Subsect. 1.A. In the case of topological Kac-Moody groups, some precise structure results are available, which are proved in 1.C. We exhibit analogies with an algebraic group over a local field of characteristic p , as mentioned above.

1.A Combinatorial framework. — In this subsection, we present all the combinatorial material we will need in the sequel. For references, see for instance [A, KP, Ré1, Ro1-2, T2-5].

1.A.1 Let (W, S) be the Coxeter system

$$W = \langle s \in S \mid (st)^{M_{st}} = 1 \text{ whenever } M_{st} < \infty \rangle,$$

where M is a Coxeter matrix. The existence of an abstract simplicial complex $\Sigma = \Sigma(W, S)$ acted upon by W is the starting point for the definition of buildings of type (W, S) in terms of apartment systems. This complex is the *Coxeter complex* associated to W [Ro1 §2]. Set-theoretically, Σ is the union of the translates $wW_J := w\langle J \rangle$ for $J \subset S$ and $w \in W$. It is partially ordered by reverse inclusion. The elements $w\langle \emptyset \rangle$, which are simply the elements of W , have maximum dimension and are called *chambers*. The *root system* of (W, S) is defined by means of the length function $\ell : W \rightarrow \mathbf{N}$ with respect to S [T2 §5]. The set W admits a W -action via left translations. *Roots* are distinguished halves of W , regarded as a W -set.

- DEFINITION.— (i) *The simple root of index $s \in S$ is the half $a_s := \{w \in W \mid \ell(sw) > \ell(w)\}$.*
(ii) *A root of W is a translate wa_s , $w \in W, s \in S$. The set of all roots will be denoted by Φ . It admits an obvious W -action by left translations.*
(iii) *A root is positive if it contains 1; otherwise, it is negative. We denote by Φ_+ (resp. Φ_-) the set of positive (resp. negative) roots.*
(iv) *The complement of a root a , denoted $-a$, is also a root, called the opposite of a .*
(v) *The boundary (or wall) ∂a of a root a is the union of the closed panels having exactly one chamber in a . This is the same as the intersection of the closures of a and $-a$.*

The following definitions are needed for the group combinatorics.

- DEFINITION.— (i) *A pair of roots $\{a; b\}$ is called prenilpotent if both intersections $a \cap b$ and $(-a) \cap (-b)$ are nonempty.*
(ii) *Given a prenilpotent pair of roots $\{a; b\}$, the interval $[a; b]$ is by definition the set of roots c with $c \supset a \cap b$ and $(-c) \supset (-a) \cap (-b)$.*

We can now turn to groups.

DEFINITION.— *Let Λ be an abstract group containing a subgroup H . Suppose Λ is endowed with a family $\{U_a\}_{a \in \Phi}$ of subgroups indexed by the set of roots Φ , and define the subgroups $U_+ := \langle U_a \mid a \in \Phi_+ \rangle$ and $U_- := \langle U_a \mid a \in \Phi_- \rangle$. We say that the triple $(\Lambda, \{U_a\}_{a \in \Phi}, H)$ is a twin root datum for Λ (or satisfies the (TRD) axioms) if the following conditions are fulfilled.*

- (TRD0) *Each U_a is nontrivial and normalized by H .*
(TRD1) *For each prenilpotent pair of roots $\{a; b\}$, the commutator subgroup $[U_a, U_b]$ is contained in the subgroup $\langle U_c : c \in [a; b] - \{a; b\} \rangle$.*

(TRD2) For each $s \in S$ and $u \in U_{a_s} \setminus \{1\}$, there exist uniquely defined $u', u'' \in U_{-a_s} \setminus \{1\}$ such that $m(u) := u'uu''$ conjugates U_b onto $U_{s,b}$ for every root b . Moreover $m(u)H = m(v)H$ for all $u, v \in U_{a_s} \setminus \{1\}$.

(TRD3) For each $s \in S$, we have $U_{a_s} \not\subset U_-$ and $U_{-a_s} \not\subset U_+$.

(TRD4) The group Λ is generated by H and the U_a 's.

EXAMPLE. — The special linear groups $\mathrm{SL}_n(\mathbf{F}_q[t, t^{-1}])$ over Laurent polynomials satisfy these axioms for W affine of type \tilde{A}_{n-1} . More generally, the groups $\mathbb{G}(\mathbf{F}_q[t, t^{-1}])$ where \mathbb{G} is a simply connected Chevalley group, satisfy these axioms when W is a suitable affine Coxeter group [A §1]. In these cases, the root groups are naturally defined as subgroups of the spherical root groups by using polynomials of the form at^r for fixed r as the off-diagonal entries.

1.A.2 Here are the main properties of such a group Λ . Define the *standard Borel subgroup* of sign $\epsilon = \pm$ to be HU_ϵ . The subgroup $N < \Lambda$ is by definition generated by H and the $m(u)$'s of axiom (TRD2). Then (Λ, HU_+, N, S) and (Λ, HU_-, N, S) are *BN-pairs* sharing the same Weyl group $W = N/H$ [T3-4], and HU_+ and HU_- are conjugate if and only if W is finite.

A formal consequence of the existence of a *BN-pair* is a Bruhat decomposition for the group, and using the root groups above this can be done as follows. For each $w \in W$, define the subgroups $U_w := U_+ \cap wU_-w^{-1}$ and $U_{-w} := U_- \cap wU_+w^{-1}$. In their action on the buildings defined by the *BN-pairs* above, the groups U_w and U_{-w} are freely transitive on the sets of chambers at distance w from the (base) chambers fixed by HU_+ and HU_- respectively.

For each $w \in W$, define the (finite) sets of roots

$$\Phi_w := \Phi_+ \cap w^{-1}\Phi_- \quad \text{and} \quad \Phi_{-w} := \Phi_- \cap w^{-1}\Phi_+.$$

In terms of buildings, if c_+ represents the chamber fixed by HU_+ in the apartment stabilised by N , then Φ_w is the set of roots in that apartment containing c_+ but not containing $w^{-1}c_+$. Similarly for Φ_{-w} with respect to the chamber fixed by HU_- . The group U_w (resp. U_{-w}) is in bijection with the set-theoretic product of the root groups indexed by $\Phi_{w^{-1}}$ (resp. $\Phi_{-w^{-1}}$) for a suitable (cyclic) ordering [T4, proposition 3]. Note that $U_w < U_{w'}$ as soon as $w \leq w'$ for the Bruhat ordering, because $\Phi_{w^{-1}}$ is a subset of $\Phi_{w'^{-1}}$ in this case.

The *refined Bruhat decompositions* are [KP, Proposition 3.2]:

$$\Lambda = \bigsqcup_{w \in W} U_w w HU_+ \quad \text{and} \quad \Lambda = \bigsqcup_{w \in W} U_{-w} w HU_-.$$

In these decompositions, the element in the left factor $U_{\pm w}$ is uniquely determined.

A third kind of decomposition involves both signs and is used to define the so-called *twin structures* (twin buildings, twin *BN-pairs*...). The *refined Birkhoff decompositions* are [KP, Proposition 3.3]:

$$\Lambda = \bigsqcup_{w \in W} (U_+ \cap wU_+w^{-1})wHU_- = \bigsqcup_{w \in W} (U_- \cap wU_-w^{-1})wHU_+.$$

1.A.3 The theory of (*Moufang*) *twin buildings* is the geometric side of the group combinatorics above [T3-T5]. The definition of a twin building is quite similar to that of a building in terms of W -distance [T5, A §2, Ro2 §2].

DEFINITION.— A twin building of type (W, S) consists of two buildings (Δ_+, d_+) and (Δ_-, d_-) of type (W, S) endowed with a (W) -codistance, namely a map

$$d^* : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \rightarrow W$$

satisfying the following conditions for each sign ϵ and all chambers x_ϵ in Δ_ϵ and $y_{-\epsilon}, y'_{-\epsilon}$ in $\Delta_{-\epsilon}$.

(TW1) $d^*(y_{-\epsilon}, x_\epsilon) = d^*(x_\epsilon, y_{-\epsilon})^{-1}$.

(TW2) If $d^*(x_\epsilon, y_{-\epsilon}) = w$ and $d_{-\epsilon}(y_{-\epsilon}, y'_{-\epsilon}) = s \in S$ with $\ell(ws) < \ell(w)$, then $d^*(x_\epsilon, y'_{-\epsilon}) = ws$.

(TW3) If $d^*(x_\epsilon, y_{-\epsilon}) = w$ then for each $s \in S$, there exists $z_{-\epsilon} \in \Delta_{-\epsilon}$ with $d_{-\epsilon}(y_{-\epsilon}, z_{-\epsilon}) = s$ and $d^*(x_\epsilon, z_{-\epsilon}) = ws$.

Two chambers are *opposite* if they are at codistance 1; two facets are *opposite* if they belong to opposite chambers and have the same type. Given an apartment \mathbb{A}_ϵ of sign ϵ , an *opposite* of it is an apartment $\mathbb{A}_{-\epsilon}$ such that each chamber of \mathbb{A}_ϵ is opposite exactly one chamber in $\mathbb{A}_{-\epsilon}$. An apartment has at most one opposite, and a pair of opposite apartments is called a *twin apartment* [A §2, Ro2, T3-5]. Note that every pair of opposite chambers lies in exactly one twin apartment.

Set-theoretically, the set of chambers of the building of sign ϵ attached to a twin root datum $(\Lambda, \{U_a\}_{a \in \Phi}, H)$ is $\Lambda/HU_\epsilon = \{gHU_\epsilon\}_{g \in \Lambda}$. The Λ -action is the natural one by left translations, and it is transitive on the pairs of opposite chambers – see [A §2] for details.

EXAMPLE. — When Λ is $\mathbb{G}(\mathbf{F}_q[t, t^{-1}])$ where \mathbb{G} is a Chevalley group, the buildings Δ_+ and Δ_- are the Bruhat-Tits buildings of $\mathbb{G}(\mathbf{F}_q((t)))$ and $\mathbb{G}(\mathbf{F}_q((t^{-1})))$ respectively, on which Λ acts diagonally.

1.A.4 We state now the generalized Levi decompositions of parabolic subgroups (which are by definition the fixators of facets). Let F be a spherical facet. Then for every choice of a twin apartment \mathbb{A} containing F , the parabolic subgroup $\text{Fix}_\Lambda(F)$ admits a semi-direct product decomposition [R 1, 6.2]:

$$\text{Fix}_\Lambda(F) = M(F, \mathbb{A}) \ltimes U(F).$$

The group $M(F, \mathbb{A})$ is the fixator of $F \cup -F$, where $-F$ is the opposite of F in \mathbb{A} ; it is generated by $H = \text{Fix}_\Lambda(\mathbb{A})$ and the root groups U_a with $\partial a \supset F$. Moreover it satisfies the (TRD) axioms for the finite root system attached to F . The group $U(F)$ only depends on F , and is the normal closure in $\text{Fix}_\Lambda(F)$ of the root groups U_a for which $a \supset F$.

1.A.5 We end this subsection by introducing some group combinatorics defined by V. Kac and D. Peterson, which we will need for the definition of topological groups of Kac-Moody type.

DEFINITION.— A refined Tits system for a group G is a sextuple (G, N, U_+, U_-, H, S) , where N, U_+, U_- are subgroups satisfying the following axioms:

(RT1) We have $G = \langle N, U_+ \rangle$, $H \triangleleft N$, $H < N_G(U_+) \cap N_G(U_-)$, $W := N/H$ and (W, S) is a Coxeter system.

(RT2) For each $s \in S$, we set $U_s := U_+ \cap s^{-1}U_-s$; and for any $w \in W$ and $s \in S$ we require:

(RT2a) $s^{-1}U_s s \neq \{1\}$ and $s^{-1}(U_s \setminus \{1\})s \subset U_s s H U_s$,

(RT2b) either $w^{-1}U_s w \subset U_+$ or $w^{-1}U_s w \subset U_-$,

(RT2c) $U_+ = U_s(U_+ \cap s^{-1}U_+s)$.

(RT3) If $u_- \in U_-$, $u_+ \in U_+$ and $n \in N$ are such that $u_- n u_+ = 1$, then $u_- = u_+ = n = 1$.

Many properties can be derived from these axioms: refined Bruhat decomposition, definition of subgroups as limits of natural inductive systems, etc. The reference is the original paper [KP]. We will see in 1.C why this framework concerns Lie groups over non-Archimedean local

fields of *positive* characteristic, but not the characteristic 0 case. The relation between twin root data and refined Tits systems is the following – see [Ré1, 1.5.4] for a proof.

THEOREM.— *If $(\Lambda, \{U_a\}_{a \in \Phi}, H)$ is a twin root datum indexed by the Coxeter system (W, S) , then $(\Lambda, N, U_+, U_-, H, S)$ and $(\Lambda, N, U_-, U_+, H, S)$ are refined Tits system with Weyl group W . Moreover we have $U_+ \cap s^{-1}U_-s = U_{a_s}$ and $U_- \cap s^{-1}U_+s = U_{-a_s}$ for any $s \in S$. \square*

1.B Topological groups of Kac-Moody type. — In this subsection, we use the buildings of groups with twin root data to define related topological groups. This requires to see buildings as metric spaces.

1.B.1 We assume we are given a group Λ endowed with a twin root datum $(\Lambda, \{U_a\}_{a \in \Phi}, H)$. We only deal with the positive building of Λ , which we simply denote by Δ . The notation Δ actually denotes the metric realization of the building in the sense of Moussong-Davis [D, Mou]. Let us list some assumptions and discuss their consequences.

ASSUMPTION 1.— *All root groups U_a are finite.*

We can thus set

$$q_{\min} := \min_{a \in \Phi} |U_a| = \min_{s \in S} |U_{a_s}| \quad \text{and} \quad q_{\max} := \max_{a \in \Phi} |U_a| = \max_{s \in S} |U_{a_s}|.$$

By the Moufang property [Ro1 §6], the number of chambers on each panel is then bounded by $1 + q_{\max}$. More generally, the fact that only spherical facets appear in the Moussong-Davis realization of a building implies that Δ is a locally finite cell complex.

ASSUMPTION 2.— *The Weyl group W is infinite; hence so is the root system Φ .*

This implies by [D] that the building Δ is a non-positively curved metric space, which means that it satisfies the so-called CAT(0)-property [BH, II.1.1]: geodesic triangles are at least as thin as in the Euclidean plane.

Our main motivation in defining topological groups of Kac-Moody type is geometric group theory, in which groups are studied via their actions on suitable spaces. The following lemma shows that moding out by the kernel of the Λ -action on Δ is harmless for the group combinatorics.

LEMMA.— *Let $(\Lambda, \{U_a\}_{a \in \Phi}, H)$ be a twin root datum with associated Coxeter system (W, S) and associated buildings Δ_{\pm} . Then the kernel K of the Λ -action on Δ_+ or Δ_- consists of the elements in H centralizing all root groups. The groups U_{\pm} embed in Λ/K , hence so do the root groups, and $(\Lambda/K, \{U_a\}_{a \in \Phi}, H/K)$ is again a twin root datum with the same associated Coxeter system and twinning.*

Proof. We use theorem 1.A.5, which allows us to use properties of refined Tits systems for Λ . We argue with Δ_+ , the negative case being completely similar. The kernel K fixes the chamber fixed by HU_+ and the apartment stabilized by N ; hence it lies in $HU_+ \cap N$. According to [KP, lemma 3.1], the latter intersection is H . That K must centralize each root group comes from the Moufang property, since each root group is in bijective correspondence with the set of chambers sharing a panel with a given chamber [Ro1, 6.4]. By (RT3) we have $U_{\pm} \cap H = \{1\}$, hence U_+ and U_- embed in Λ/K , which implies axioms (TRD0) and (TRD1) (a prenilpotent pair of roots can always be transformed into a pair of positive roots by a suitable element of W). Axiom (TRD4) is clear and (TRD3) for Λ/K is a consequence of (RT3) for Λ . Axiom (TRD2) for Λ/K is a consequence of axiom (TRD2) for Λ , the uniqueness part coming from

the fact that, when all the other axioms are satisfied, the elements u' and u'' in (TRD2) are uniquely determined by u [T5, p.257]. The set-theoretical definition of the positive (resp. negative) building as Λ/HU_+ (resp. Λ/HU_-) and the definition of the W -distances and of the codistance by means of the Bruhat and Birkhoff decompositions show that the associated twinings are the same. \square

REMARK. — When Λ is a Kac-Moody group, the kernel K is the center $Z(\Lambda)$ [R 1, 9.6.2], which is finite when the ground field is.

EXAMPLE. — For $SL_n(\mathbf{K}[t, t^{-1}])$, the buildings of Λ are Euclidean \tilde{A}_{n-1} -buildings, and the kernel K is the group $\mu_n(\mathbf{F}_q)$ of n -th roots of unity in \mathbf{F}_q .

The lemma leads us to make the following convention.

CONVENTION.— *Until the end of 1.B, Λ denotes the image of a group with twin root datum as above under the group homomorphism corresponding to the action on the building Δ . Consequently, $\Lambda < \text{Aut}(\Delta)$ admits a twin root datum with finite root groups and infinite Weyl group.*

1.B.2 We can now define topological groups of Kac-Moody type. Given an automorphism $g \in \text{Aut}(\Delta)$ and a finite union C of facets, the subset of $\text{Aut}(\Delta)$

$$O_C(g) := \{h \in \text{Aut}(\Delta) : h|_C = g|_C\}$$

is by definition an open neighborhood of g . The group Λ is not discrete on the single building Δ because each facet fixator is transitive on opposite facets in Δ_- , of which there are infinitely many.

DEFINITION.— *Let Λ be a group as in convention 1.B.1, with finite root groups and infinite Weyl group.*

(i) *We call the closure*

$$G := \overline{\Lambda}^{\text{Aut}(\Delta)}$$

a topological group of Kac-Moody type. If Λ is a Kac-Moody group over a finite field, we call G a topological Kac-Moody group.

(ii) *We call the fixator $\text{Fix}_G(F)$ of a facet F the parahoric subgroup of G associated to F , and we denote it by G_F . We call a chamber fixator an Iwahori subgroup.*

(iii) *We denote by U_F the closure $\overline{U(F)}^{\text{Aut}(\Delta)}$ of the «unipotent radical» of $\text{Fix}_\Lambda(F)$.*

REMARK. — This definition implies that an automorphism of Δ lies in G if and only if it coincides with an element of Λ on any finite set of facets.

EXAMPLE. — The image of $SL_n(\mathbf{F}_q[t, t^{-1}])$ under the action on its positive Euclidean building is the group $SL_n(\mathbf{F}_q[t, t^{-1}])/\mu_n(\mathbf{F}_q)$ – see 1.B.1. Then the «completion» G is the ultrametric Lie group $SL_n(\mathbf{F}_q((t)))/\mu_n(\mathbf{F}_q)$. This is close to being $\text{Aut}(\Delta)$ when $n > 2$, but when $n = 2$, Δ is a tree and G is far from being all of $\text{Aut}(\Delta)$. (Locally, the difference appears in the action on vertex stars: in G one has $\text{PSL}_2(\mathbf{F}_q)$, but in $\text{Aut}(\Delta)$ one has the symmetric group \mathcal{S}_{q+1} .)

The extension of the classical terminology of parahoric subgroups, supported by the above example, suggests an analogy with algebraic groups over local fields. One of the main questions in this work is to understand to what extent the analogy is relevant to this more general setting.

1.B.3 We now consider discrete subgroups of the topological group G . This will make the analogy deeper, bringing in a certain class of lattices generalizing some arithmetic groups over function fields. Fix a chamber c_- in Δ_- , and let Γ denote its stabilizer. Let \mathbb{A} be a twin apartment containing c_- . The combinatorial properties derived from the (TRD) axioms prove the following results:

- The positive apartment \mathbb{A}_+ is a fundamental domain for the Γ -action on Δ . More generally, if F_- is a facet in \mathbb{A}_- , we denote by F_+ its unique opposite in \mathbb{A}_+ . Given a chamber c_+ containing F_+ in its closure, each wall of \mathbb{A}_+ containing F_+ bounds a root containing c_+ . The intersection of these half-spaces is a fundamental domain for the action of $\text{Fix}_\Lambda(F_-)$ on Δ . See [A §3] for details.
- The group Λ acts diagonally as a discrete group on the product of buildings $\Delta_+ \times \Delta_-$. Consequently, the group Γ , hence the fixator (in Λ) of any negative facet is a discrete subgroup of G [CG, Ré2].
- If q_{\min} is large enough, then the (spherical) parabolic subgroups of a given sign ϵ are lattices of the locally finite CAT(0)-building $\Delta_{-\epsilon}$ [CG, Ré2].

In the same spirit, we can also prove a result about commensurators. Recall that two subgroups Γ and Γ' of a given group G are *commensurable* if they share a finite index subgroup. The *commensurator* of Γ in G is the group

$$\text{Comm}_G(\Gamma) := \{g \in G : \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable}\}.$$

LEMMA.— *Let F be a negative (spherical) facet.*

- (i) *For any other negative (spherical) facet F' , the fixators $\text{Fix}_\Lambda(F)$ and $\text{Fix}_\Lambda(F')$ are commensurable.*
- (ii) *The group Λ lies in the commensurator $\text{Comm}_{\text{Aut}(\Delta)}(\text{Fix}_\Lambda(F))$ of the facet fixator $\text{Fix}_\Lambda(F)$.*

Proof. We keep our negative Borel subgroup $\Gamma = \text{Fix}_\Lambda(c_-)$. The arguments for both points are based on finiteness of root groups and refined Bruhat decomposition. Recall that in 1.A.2 we attached a finite set of roots Φ_w to any element of the Weyl group $w \in W$. According to [KP, proposition 3.2] and [T4, proposition 3], for any $z \in W$ we have:

$$(\star) \quad \Gamma = \left(\prod_{a \in \Phi_{z^{-1}}} U_{-a} \right) \cdot (\Gamma \cap z\Gamma z^{-1}) \quad \text{hence} \quad (\star\star) \quad \Gamma z\Gamma = \left(\prod_{a \in \Phi_{z^{-1}}} U_{-a} \right) z\Gamma,$$

where the product of the root groups is $\{1\}$ if $z = 1$ (in which case $\Phi_{z^{-1}}$ is the empty set). Notice that in both equalities the first factor on the right is finite.

(i). For any spherical facet F , choose a chamber whose closure contains it. By the Bruhat decomposition for parabolics and $(\star\star)$, the Borel subgroup corresponding to this chamber is of finite index in $\text{Fix}_\Lambda(F)$. Hence we are reduced to proving that any two Borel subgroups are commensurable. By transitivity of Λ on pairs of chambers at given W -distance, it is enough to consider Γ and $w\Gamma w^{-1}$ for $w \in W$. Setting $z = w$ in equality (\star) shows that $\Gamma \cap w\Gamma w^{-1}$ is of finite index in Γ . Similarly setting $z = w^{-1}$, and conjugating the result by w shows that $\Gamma \cap w\Gamma w^{-1}$ is of finite index in $w\Gamma w^{-1}$.

(ii). Since Λ is transitive on chambers and the fixator of any spherical facet contains a Borel subgroup of finite index, it is enough to show that Λ lies in the commensurator of Γ . Let $g \in \Lambda$. By $(\star\star)$, $g = u_{-w}w\gamma$ for some $w \in W$, for $\gamma \in \Gamma$ and for u_{-w} in the (finite) group U_{-w} defined in 1.A.2. We have then $g\Gamma g^{-1} \cap \Gamma = u_{-w}(w\Gamma w^{-1} \cap \Gamma)u_{-w}^{-1}$ because $u_{-w} \in \Gamma$. By (i), $w\Gamma w^{-1} \cap \Gamma$

is of finite index in Γ , and since $u_{-w} \in \Gamma$, we have $[\Gamma : g\Gamma g^{-1} \cap \Gamma] < \infty$. In other words, if $g \in \Lambda$, then $g\Gamma g^{-1} \cap \Gamma$ is of finite index in Γ . Replacing g by g^{-1} and conjugating by g shows that $g\Gamma g^{-1} \cap \Gamma$ is of finite index in $g\Gamma g^{-1}$. \square

EXAMPLE. — Let us consider again the example of the group $\mathrm{SL}_n(\mathbf{F}_q[t, t^{-1}])/\mu_n(\mathbf{F}_q)$. Its closures in the automorphism groups of the positive and negative buildings are respectively $\mathrm{SL}_n(\mathbf{F}_q((t)))/\mu_n(\mathbf{F}_q)$ and $\mathrm{SL}_n(\mathbf{F}_q((t^{-1})))/\mu_n(\mathbf{F}_q)$. A negative vertex fixator in $\mathrm{SL}_n(\mathbf{F}_q((t^{-1})))/\mu_n(\mathbf{F}_q)$ is isomorphic to $\mathrm{SL}_n(\mathbf{F}_q[[t^{-1}]])/\mu_n(\mathbf{F}_q)$; consequently, the lattices we get by taking negative facet fixators in Λ are $\{0\}$ -arithmetic groups commensurable with $\mathrm{SL}_n(\mathbf{F}_q[t^{-1}])/\mu_n(\mathbf{F}_q)$. The commensurator of $\mathrm{SL}_n(\mathbf{F}_q[t^{-1}])/\mu_n(\mathbf{F}_q)$ contains $\mathrm{SL}_n(\mathbf{F}_q(t))/\mu_n(\mathbf{F}_q)$.

1.C The special case of Kac-Moody groups. — Let Λ be an almost split Kac-Moody group. According to J. Tits, a split Kac-Moody group is defined by generators and Steinberg relations, once a generalized Cartan matrix A and a ground field \mathbf{K} are fixed [T2] – see also 2.E.1. An almost split Kac-Moody group is the group of fixed points of a suitable Galois action on a split group – see [Ré1, 11.3] for a precise definition. The basic result we use about almost split Kac-Moody groups is that their rational points satisfy the (TRD) axioms [Ré1, 12.6.3].

1.C.1 We make the following convention until the end of the section.

CONVENTION.— *The group Λ is the image of the rational points of an almost split Kac-Moody group over the ground field \mathbf{F}_q (of cardinality q and characteristic p) in the full automorphism group of the positive building Δ . We denote its twin root datum by $(\Lambda, \{U_a\}_{a \in \Phi}, T)$, and assume that the Weyl group W is infinite. We denote by $G = \overline{\Lambda}^{\mathrm{Aut}(\Delta)}$ the corresponding topological Kac-Moody group defined in 1.B.2, and by \mathbb{A} the apartment of Δ stabilized by N .*

REMARKS. — 1) By [Ré1, 12.5.4], for each root a the group U_a is isomorphic to the \mathbf{F}_q -points of the unipotent radical of a Borel subgroup in a rank one finite group of Lie type. The finiteness assumption on the ground field implies the first assumption of 1.B.1, and we have the lower bound $|U_a| \geq q$ for every root a .

2) We use the notation T instead of H for the twin root datum because T is the group of rational points of an \mathbf{F}_q -torus [Ré1, 13.2.2].

EXAMPLES. — 1) The group $\mathrm{SL}_2(\mathbf{F}_q[t, t^{-1}])$ is a split Kac-Moody group over \mathbf{F}_q with infinite dihedral Weyl group. The associated buildings are homogeneous trees of valencies $1+q$ because the root groups of the affine twin Tits systems all have order q .

2) The group $\mathrm{SU}_3(\mathbf{F}_q[t, t^{-1}])$ is an almost split Kac-Moody group over \mathbf{F}_q with infinite dihedral Weyl group, too, but the associated buildings are semihomogeneous trees of valencies $1+q$ and $1+q^3$ since the root groups have order q or q^3 [Ré4, 3.5].

In order to get lattices by 1.B.3, the condition requiring that q_{\min} be large enough, simply means that q is large enough. In the case of split Kac-Moody groups, this assumption admits a sharp quantitative version relating the growth series of the Weyl group and q . Once the Haar measure of the full automorphism group of a building is normalized in such a way that a chamber fixator has measure 1, the covolume of the Borel subgroup Γ is $\sum_{n \geq 0} \frac{d_n}{q^n} = \sum_{w \in W} q^{-\ell(w)}$,

where d_n is the number of elements in W of length n [Ré2].

REMARKS. — 1) In the affine case, Kac-Moody groups are special cases of arithmetic groups over function fields. For this class of groups, a classical theorem by H. Behr and G. Harder

implies that the assumption on q is empty [Mar, I.3.2.4]. Therefore an affine Kac-Moody group over a finite field is always a lattice of the product of its two Euclidean buildings.

2) Let us consider a Kac-Moody group whose associated buildings have hyperbolic right-angled regular r -gons as chambers – see 2.E and [Ré4, 4.1] for the existence of such groups. Then the apartments are tilings of the hyperbolic plane \mathbb{H}^2 and the Weyl group W has exponential growth. More precisely, the growth series of W is $1 + rt + \sum_{n \geq 2} r(r-2)^{n-1}t^n$, and the finiteness

of $\sum_n d_n q^{-n}$ amounts to $q \geq r-1$. Therefore the Borel subgroup of such a Kac-Moody group is not always a lattice.

1.C.2 The inclusion $\Lambda < G$ implies that G also admits a BN -pair, since it is strongly transitive on the building Δ [Ro1 §5]. We will prove a stronger result stressing the analogy between G and semisimple groups over local fields of positive characteristic.

THEOREM.— *Let Λ and G be groups as in 1.C.1.*

(i) *The sextuple $(G, N, U_c, \Gamma, T, S)$ defines a structure of refined Tits system for the associated topological Kac-Moody group G .*

(ii) *Let F be a facet in the apartment \mathbb{A} . Then the group $M(\mathbb{A}, F)$ of 1.A.4 is finite of Lie type, U_F is a pro- p group, and the parahoric subgroup $G_F = \text{Fix}_G(F)$ decomposes as $G_F = M(\mathbb{A}, F) \rtimes U_F$.*

The proof will be given by a series of lemmas in 1.C.4 to 1.C.6.

Since the group Λ acts by type-preserving isometries on Δ , the stabilizer of a facet F in Λ is also its fixator. We will denote it by $\Lambda(F)$; it is contained in G_F but has no topological structure. As explained in 1.A.4, any twin apartment \mathbb{A} containing F gives rise to a Levi decomposition $\Lambda(F) = M(\mathbb{A}, F) \rtimes U(F)$. Here the subgroup $M(\mathbb{A}, F)$ is abstractly isomorphic to the Kac-Moody group associated to a submatrix of the generalized Cartan matrix defining Λ . Since all facets in the metric buildings are spherical, the walls in \mathbb{A} containing F are finite in number; in fact, $M(\mathbb{A}, F)$ is a finite group of Lie type over \mathbf{F}_q .

REMARK. — It follows from theorem 1.C.2 that the group $\text{SL}_n(\mathbf{F}_q((t)))$ satisfies the axioms of a refined Tits system, and that its parahoric subgroups admit semidirect product decompositions. This splitting of facet fixators is not true for algebraic groups over local fields of characteristic 0. It is related to the existence of finite root groups with nice properties, which geometrically amounts to the *Moufang property* – see [Ro1, 6.4]. Other arguments (involving torsion, for instance) explaining why the analogy with the characteristic 0 case is *not* relevant, will be given in 5.A and 5.B.

1.C.3 We choose a facet F in \mathbb{A} , and define an exhaustion $\{E_n\}_{n \geq 1}$ of Δ , with respect to F . We work inductively outwards from F . The first term E_1 is (the closure of) $\text{st}(F)$, the star of F in Δ . To define further terms, suppose that E_n is already defined. Write $\mathbb{A}_n := E_n \cap \mathbb{A}$ and choose a chamber c_n of \mathbb{A} sharing a panel with the boundary of \mathbb{A}_n . Then we define \mathbb{A}_{n+1} to be the convex hull of c_n and \mathbb{A}_n , and E_{n+1} to be $\Lambda(F) \cdot \mathbb{A}_{n+1}$, that is the set of all $\Lambda(F)$ -transforms of \mathbb{A}_{n+1} . By the Bruhat decomposition, for any chamber $c \in \text{st}(F)$, \mathbb{A} is a complete set of representatives for the action of $U(c) < \Lambda(F)$ on Δ . Hence $\{E_n\}_{n \geq 1}$ exhausts Δ . For each $n \geq 1$, the set E_n is $\Lambda(F)$ -stable. We can write the groups G_F and U_F defined in 1.B.2 as projective limits:

$$G_F = \varprojlim_{n \geq 1} \Lambda(F)|_{E_n} \quad \text{and} \quad U_F = \varprojlim_{n \geq 1} U(F)|_{E_n}.$$

We turn now to the proof of the theorem in 1.C.2. We shall first prove, in 1.C.4, that U_F is a pro- p group, then in 1.C.5 we prove the assertion about semi-direct products for parahoric subgroups, and finally in 1.C.6 we deal with the refined Tits system for G .

1.C.4 Recall that each root group $U_a < \Lambda$ is a p -group (1.C.1).

LEMMA.— *Let Λ and G be groups as in 1.C.1. Then the group U_F is pro- p .*

Proof. Recall that if a group G acts on a set S , the notation $G|_S$ means the group obtained from G by factoring out the kernel of the action on S . For each $n \geq 1$, the group $U(F)|_{E_{n+1}}$ when restricted to E_n has a kernel K_n . This is represented by the following exact sequence:

$$1 \longrightarrow K_n \longrightarrow U(F)|_{E_{n+1}} \longrightarrow U(F)|_{E_n} \longrightarrow 1.$$

The kernel K_n is the fixator $\text{Fix}_{U(F)|_{E_{n+1}}}(E_n)$ of E_n in the restricted group $U(F)|_{E_{n+1}}$. Since $U(F)$ fixes $\text{st}(F)$, we have $U(F)|_{E_1} = \{1\}$. By definition $U_F = \varprojlim_{n \geq 1} U(F)|_{E_n}$, so in order to show that U_F is pro- p , it is enough, by induction, to show that K_n is a p -group for each $n \geq 1$.

Let us fix $n \geq 1$, and $u \in K_n$. Let Π_n be the panel of c_n in the boundary of \mathbb{A}_n , and let d_n be the other chamber of \mathbb{A} having Π_n as a panel. Then $d_n \in \mathbb{A}_n$, so u fixes d_n , and by the Moufang property there exists $v \in U_a$ such that $v^{-1}u.c_n = c_n$. The element $v^{-1}u$ is in the «unipotent radical» of the Borel subgroup $\text{Fix}_\Lambda(d_n)$ and fixes $c_n \cup d_n$: hence it must fix the whole star of Π_n . Consequently, u and v coincide on $\text{st}(\Pi_n)$ and thus $u^{q_{\max}}$ acts trivially on it. In particular, $u^{q_{\max}}$ fixes $\mathbb{A}_n \cup \{c_n\}$ hence \mathbb{A}_{n+1} by convexity.

Every chamber of $E_{n+1} \setminus E_n$ is of the form $v.d$ for some d in the convex hull of \mathbb{A}_n and c_n , and $v \in \Lambda(F)$. Then $u^{q_{\max}}(vd) = v(v^{-1}u^{q_{\max}}v.d) = v((v^{-1}uv)^{q_{\max}}.d)$. Recall that $U(F)$ is normalized by $\Lambda(F)$, so that applying the result of the above paragraph to $v^{-1}uv$ instead of u , we get $(v^{-1}uv)^{q_{\max}}.d = d$, and thus $u^{q_{\max}}(vd) = vd$. This shows that the order of each element $u \in K_n$ divides q_{\max} , so K_n is a p -group. Therefore U_F is pro- p , as required. \square

1.C.5 The following simple lemma will also be useful for the last part of the proof.

LEMMA.— *Let Λ and G be groups as in 1.C.1.*

- (i) *We have $\Lambda \cap U_F = U(F)$, hence $M(\mathbb{A}, F) \cap U_F = \{1\}$. In particular, for any chamber c in \mathbb{A} , we have $T \cap U_c = \{1\}$.*
- (ii) *The group G_F decomposes as $G_F = M(\mathbb{A}, F) \times U_F$.*

Proof. (i). First $\Lambda \cap U_F < \text{Fix}_\Lambda(F) = \Lambda(F)$. Thus if g is in $\Lambda \cap U_F$, the Levi decomposition of $\Lambda(F)$ gives $g = mu$, with $m \in M(\mathbb{A}, F)$ and $u \in U(F)$. Since U_F fixes $\text{st}(F)$, the set of chambers containing F , we see that both g and u , hence also m , fix $\text{st}(F)$. Therefore m is central in the finite (reductive) group $M(\mathbb{A}, F)$ of Lie type; this implies that m lies in a torus, so its order is not divisible by p . But m is also a torsion element in the pro- p group U_F , and is hence trivial. The other assertions come from the trivial intersection $M(\mathbb{A}, F) \cap U(F) = \{1\}$ for any F .

(ii). For each $n \geq 1$, E_n is $\Lambda(F)$ -stable by definition, hence $M(\mathbb{A}, F)$ -stable. Since $M(\mathbb{A}, F)$ normalizes $U(F)$, it normalizes $U(F)|_{E_n}$ in $\Lambda(F)|_{E_n}$ for each $n \geq 1$. Hence U_F is normalized by $M(\mathbb{A}, F)$.

Let $g \in \Lambda(F)$. It can be written as $g = \varprojlim_{n \geq 1} g_n$, with $g_n \in \Lambda(F)|_{E_n}$. According to the Levi decomposition of $\Lambda(F)$, we have $g_n = m_n u_n$ with $m_n \in M(\mathbb{A}, F)|_{E_n}$ and $u_n \in U(F)|_{E_n}$. The group $M(\mathbb{A}, F)$ is finite, so up to extracting a subsequence, we may (and do) assume that m_n

is a constant element m in $M(\mathbb{A}, F)$. This enables to write $g = m \cdot (\varprojlim_{n \geq 1} u_n)$, and proves $G_F = M(\mathbb{A}, F) \cdot U_F$.

Finally (i) implies the trivial intersection $M(\mathbb{A}, F) \cap U_F = \{1\}$, hence the semidirect product decomposition $G_F = M(\mathbb{A}, F) \rtimes U_F$. \square

1.C.6 We can now complete the proof of the theorem in 1.C.2.

LEMMA.— *Let Λ and G be groups as in 1.C.1. The sextuple $(G, N, U_c, \Gamma, T, S)$ is a refined Tits system as defined by the (RT) axioms in 1.A.5.*

Proof. We already know that G admits a structure of BN -pair by strong transitivity of the Λ -action on Δ . The main result we use for the verification below is [Ré1, 1.5.4]. This theorem says that if $(\Lambda, \{U_a\}_{a \in \Phi}, T)$ is a twin root datum indexed by the Coxeter system (W, S) , then $(\Lambda, N, U_+, U_-, T, S)$ is a refined Tits system with Weyl group W . Moreover we have $U_+ \cap s^{-1}U_-s = U_{a_s}$ for any $s \in S$.

Axiom (RT1). The group T normalizes $U(c)$, and passing to the projective limit it normalizes U_c . The other statements in axiom (RT1) are either clear or true from the refined Tits system axioms for Λ .

Axioms (RT2). Define $U(s) := U_c \cap s^{-1}\Gamma s$. The first point of the lemma in 1.C.5 implies $U(s) < U(c) \cap s^{-1}\Gamma s$, hence $U(s)$ is the root group U_{a_s} according to the relation of a refined Tits system and the twin root datum of a Kac-Moody group [Ré1, 1.6]. Axioms (RT2a) and (RT2b) are then clear because they just involve subgroups of Λ . For axiom (RT2c), we need $U_c = U_{a_s} \cdot (U_c \cap s^{-1}U_c s)$. An element $u \in U_c$ fixes c , hence permutes the chambers sharing their panel of type s with c . Since U_{a_s} fixes c and is (simply) transitive on the set of chambers $\neq c$ of the latter type, there exists an element $v \in U_{a_s}$ such that $v^{-1}u$ fixes both c and sc . This proves (RT2c).

Axiom (RT3). Suppose now we have $\gamma n u_c = 1$, with $\gamma \in \Gamma$, $n \in N$ and $u_c \in U_c$. Then $\gamma n = u_c^{-1}$ belongs to $U_c \cap \Lambda$, which is $U(c)$ by the last lemma. Then all the factors are in Λ and we just have to apply axiom (RT3) from the refined Tits system structure of the latter group. \square

This completes the proof of the theorem in 1.C.2.

REMARKS. — 1) The groups U_F are analogues of congruence subgroups in the classical case. 2) In the context of refined Tits systems, we have a refined Bruhat decomposition. Hence, we could use the same arguments as in 1.B.3 to prove

LEMMA.— *All parahoric subgroups of G are commensurable.* \square

Subsect. 1.C supports the analogy between topological Kac-Moody groups and semisimple algebraic groups over non-Archimedean local fields of positive characteristic. The analogy with the wider class of topological groups of Kac-Moody type is postponed to the last (fifth) section. In the next three sections, we develop the construction of twin root data exotic enough to show that the analogy is strictly weaker in the wider context.

2. Sketch of construction. Auxiliary computations

We give here a summary of the construction, and introduce notation in SL_2 in order to state relations to be used later.

2.A *Sketch of construction.* — This construction will apply to trees as well as to some two-dimensional hyperbolic buildings. We assume we are given the tiling of \mathbf{R} by the segments defined by the integers, or a tiling of the hyperbolic plane \mathbb{H}^2 by regular right-angled r -gons. Here are the main steps.

- 1) Define the «Borel subgroup» as the limit of an inductive system described by the tiling.
- 2) To each type of vertex associate a «unipotent radical» and a «Levi factor» .
- 3) Show that the «Levi factor» admits an action on the «unipotent radical» .
- 4) In the two-dimensional case, define the other parabolic subgroups.
- 5) Amalgamate these groups along the inclusions given by the closure of a chamber.
- 6) Verify the twin root datum axioms for the so-obtained amalgam Λ .

In order to apply this procedure, we will use relations in SL_2 over a field. The rest of the section introduces notation and relations in this group.

REMARK. — Step 5) corresponds to defining Λ as the fundamental group of a complex of groups in the sense of Hæffliger [BH, III.C] – see 4.E.1 for more detailed explanations.

2.B *Notation in SL_2 .* — We use Tits' convention for the parametrization of the negative root group, and set:

$$\begin{aligned} u_i(r) &:= \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad u_{-i}(r) := \begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix} \\ m_i(\lambda) &:= u_{-i}(\lambda^{-1})u_i(\lambda)u_{-i}(\lambda^{-1}) = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} \\ m_{-i}(\lambda) &:= u_i(\lambda^{-1})u_{-i}(\lambda)u_i(\lambda^{-1}) = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix} \\ m_i &:= m_i(1) \quad \text{and} \quad t_i(\lambda) := m_i(\lambda)m_i^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \end{aligned}$$

2.C *Some straightforward relations.* — The verifications are straightforward.

$$\begin{aligned} m_i(\lambda)^{-1} &= m_i(-\lambda), \quad m_{-i}(\lambda) = m_i(\lambda^{-1}) \quad \text{and} \quad m_i^2 = -\mathrm{id}; \\ m_i(\lambda)u_i(r)m_i(\lambda)^{-1} &= u_{-i}(\lambda^{-2}r) \quad \text{and} \quad m_i(\lambda)u_{-i}(r)m_i(\lambda)^{-1} = u_i(\lambda^2r); \\ m_i(\lambda)t_i(\mu)m_i(\lambda)^{-1} &= t_i(\mu^{-1}) = t_i(\mu)^{-1} \quad \text{and} \quad m_i(\lambda)m_i(\mu) = t_i(-\lambda\mu^{-1}); \\ t_i(\lambda)u_i(r)t_i(\lambda)^{-1} &= u_i(\lambda^2r). \end{aligned}$$

2.D *The product formula.* — We want to describe the product of two elements g and h in SL_2 in terms of the Bruhat decomposition.

CONVENTION.— When g is in the big cell $\mathrm{SL}_2 \begin{pmatrix} * & * \\ \neq 0 & * \end{pmatrix}$, we write it $g := u_i(r)m_i(\lambda)u_i(r')$; when it is in the Borel subgroup $\mathrm{SL}_2 \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, we write it $g := u_i(r)t_i(\lambda)$. Similarly, when h

is in the big cell, we write it $h := u_i(s)m_i(\mu)u_i(s')$; when it is in the Borel subgroup, we write it $h := u_i(s)t_i(\mu)$.

2.D.1 First case: «big cell · big cell ∈ big cell», that is $r' + s \neq 0$. Then:

$$gh = u_i(r - \lambda^2(r' + s)^{-1})m_i(-\lambda\mu(r' + s)^{-1})u_i(s' - \mu^2(r' + s)^{-1}).$$

2.D.2 Second case: «big cell · big cell ∈ Borel», that is $r' + s = 0$. Then:

$$gh = u_i(r)t_i(-\lambda\mu^{-1})u_i(s') = u_i(r + \lambda^2\mu^{-2}s')t_i(-\lambda\mu^{-1}).$$

2.D.3 Third case: «Borel · big cell ∈ big cell». Then: $gh = u_i(r + \lambda^2s)m_i(\lambda\mu)u_i(s')$.

2.D.4 Fourth case: «big cell · Borel ∈ big cell». Then: $gh = u_i(r)m_i(\lambda\mu^{-1})u_i(\mu^{-2}(r' + s))$.

2.D.5 Fifth case: «Borel · Borel ∈ Borel». Then: $gh = u_i(r + \lambda^2s)t_i(\lambda\mu)$.

2.E *The examples from Kac-Moody theory.* — In this subsection, we recall some basic facts on Kac-Moody groups, and we present a very specific case of Kac-Moody groups which admit a well-defined ground field. Our aim, later in sect. 4, is to construct groups of the same type (i.e. having the same Weyl group) but using several different ground fields.

2.E.1 A *generalized Cartan matrix* is an integral matrix $A = [A_{ij}]_{i,j \in I}$ such that $A_{ii} = 2$, $A_{ij} \leq 0$ whenever $i \neq j$ and $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$. A *Kac-Moody root datum* with generalized Cartan matrix A is a 5-tuple $\mathcal{D} = (I, A, X^*, \{a_i\}_{s \in I}, \{h_i\}_{s \in I})$, where X^* and X_* are free \mathbf{Z} -modules \mathbf{Z} -dual to one another and such that the elements a_i of X^* and h_j of X_* satisfy $a_i(h_j) = A_{ij}$ for all $i, j \in I$ [T2, introduction]. The datum \mathcal{D} is called *simply connected* if the elements h_i freely generate X_* . A Kac-Moody root datum is a short way to encode the defining relations of a Kac-Moody group functor as defined by J. Tits [T2, 3.6]. When A is a (classical) Cartan matrix, such a functor is the functor of points of a split reductive group scheme over \mathbf{Z} , and X^* (resp. X_*) is the group of characters (resp. cocharacters) of a maximal split torus.

Let us denote by Λ the value over a field \mathbf{K} of the Kac-Moody group functor associated to a Kac-Moody root datum \mathcal{D} . We mentioned in the introduction of 1.C that Λ admits a twin root datum $(\Lambda, \{U_a\}_{a \in \Phi}, T)$. We describe it more precisely now, and refer to [R el  8] for proofs or references. The group T is the split torus $T = \text{Hom}(\mathbf{K}[X^*], \mathbf{K})$, with cocharacter group X_* . For each $i \in I$, there is an involution s_i of the \mathbf{Z} -module $\bigoplus_{i \in I} \mathbf{Z}a_i$ defined by $s_i(a_j) = a_j - A_{ij}a_i$.

The group generated by these involutions is a Coxeter group W , called the *Weyl group* of Λ . In the Kac-Moody case, the root system Φ of W , as defined in 1.A.1, is in bijection with the elements of the form wa_i . The roots $a \in \Phi$ index a family of isomorphisms $u_a : (\mathbf{K}, +) \simeq U_a$, where U_a is a root group of Λ . In the case of a simply connected root datum \mathcal{D} , the standard torus T is the product $\prod_{i \in I} h_i(\mathbf{K}^\times)$ of multiplicative one-parameter subgroups. The coordinates of a root a give the powers of the multiplicative parameters by which T operates on the root group U_a . Writing $t = \prod_{i \in I} h_i(\lambda_i) \in T$, we have:

$$t \cdot u_a(r) \cdot t^{-1} = u_a\left(\prod_{i \in I} \lambda_i^{a(h_i)} r\right).$$

The commutation relations between root groups can be made quite explicit by computations «à la Steinberg» in a \mathbf{Z} -form of the universal enveloping algebra of the Lie algebra attached to \mathcal{D} . This is done for instance in [R  1   9] in order to define an adjoint representation for Kac-Moody groups.

2.E.2 We now turn to the special cases we are interested in. Pick a prime power $q \geq 3$ and consider the generalized Cartan matrix A indexed by \mathbf{Z}/r ($r \geq 5$) in which $A_{i,i+1} = 0$ and $A_{ij} = 1 - q$ for $j \neq i, i \pm 1$. Then the associated Weyl group is the hyperbolic reflection group arising from the tiling of the hyperbolic plane by regular right-angled r -gons [R  1   13]. The root system of the tiling is studied geometrically in 4.C, where it is shown in particular that roots have a well-defined type: for instance, wa_i has type i .

We choose the ground field to be \mathbf{F}_q , in which case the choice of off-diagonal coefficients makes the action of tori on root groups as trivial as possible (using the fact that the $(q - 1)$ -th power of any element in \mathbf{F}_q^\times is 1). By induction each root wa_i can be written $\pm a_i + (q - 1) \times$ a linear combination of simple roots, and a multiple of $q - 1$ leads to a trivial action. Therefore the action of the standard torus T on the root group U_a is simply multiplication by the square (or the inverse of the square) of the multiplicative parameter having the same type as a .

We now turn to commutation relations between root groups. Since $q \geq 3$, J.-Y. H  e's work [Cho, 5.8] shows that a pair of roots where one contains the other leads to a trivial commutation relation. In the tiling of \mathbb{H}^2 any two walls are parallel or orthogonal. When two roots have orthogonal walls, the corresponding root groups commute, so the only possibility to have two non-commuting root groups is when the two roots (or their two opposites) intersect along a strip in \mathbb{H}^2 . In that case, there is no relation at all, because the pair is not prenilpotent [T4]: the free product $\mathbf{F}_q * \mathbf{F}_q$ of the root groups injects in the Kac-Moody group Λ .

REMARK. — Our coefficients in the generalized Cartan matrix were suitably chosen with respect to the characteristic of the ground-field. This trick is also the starting point of Ree-Suzuki torsions, which were defined for Kac-Moody groups by J.-Y. H  e [H].

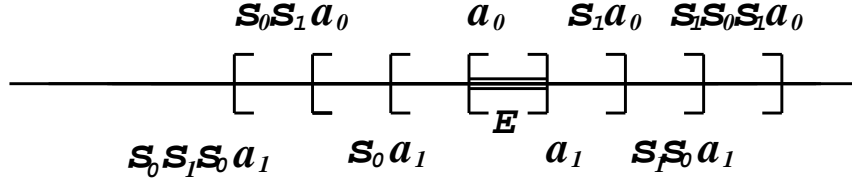
The above special cases of Kac-Moody groups show the existence of automorphism groups of Moufang twinings with particularly simple commutation relations (between root groups) and actions of tori on root groups. In these cases, root groups and multiplicative one-parameter subgroups have a well-defined type, and when the types are different, commutation relations are trivial. This is our model for the constructions in sections 3 and 4.

3. Twinning trees

We now apply procedure 2.A to trees. The computations for step 3) will be done in detail, enabling us to concentrate on more conceptual arguments in the two-dimensional case of Fuchsian buildings – see Sect. 4.

3.A Geometric description of the root system. — In this section, the Coxeter complex we are interested in is the tiling of the real line \mathbf{R} by the length 1 segments whose vertices are integers. We denote by E the segment $[0, 1]$, and by s_0 (resp. s_1) the reflection with respect to 0 (resp. 1). This provides a geometric realization of the Coxeter complex of the infinite dihedral group D_∞ , seen as the group generated by s_0 and s_1 . The roots are the half-lines defined by the integers, the positive roots are by definition those which contain E . We denote by a_0 (resp. a_1) the positive root having boundary 0 (resp. 1).

PICTURE. —



Each vertex has type 0 or 1, with the obvious notation, and the boundary of a root is called its *vertex*. The *type* of a root a is the type of its vertex, and is denoted by $\iota(a)$.

The notion of a prenilpotent pair of roots in a Coxeter complex is defined in 1.A.1, and will be used later. In this section, with the Coxeter complex as defined above, a positive root a is prenilpotent with a_i if and only if it contains a_i . Each positive root contains either a_0 or a_1 , but not both, so the positive roots fall into four disjoint subsets $\mathcal{P}(a_i, j)$ for $i, j \in \{0, 1\}$, where $\mathcal{P}(a_i, j)$ is the set of positive roots of type j which contain a_i . Note that these subsets of positive roots can also be defined as follows:

$$\begin{aligned} \mathcal{P}(a_0, 0) &:= \{(s_0 s_1)^m a_0\}_{m \geq 0}, & \mathcal{P}(a_0, 1) &:= \{(s_0 s_1)^m s_0 a_1\}_{m \geq 0}, \\ \mathcal{P}(a_1, 0) &:= \{(s_1 s_0)^m s_1 a_0\}_{m \geq 0}, & \mathcal{P}(a_1, 1) &:= \{(s_1 s_0)^m a_1\}_{m \geq 0}. \end{aligned}$$

We shall use $\mathcal{P}(a_i)$ to mean $\mathcal{P}(a_i, 0) \cup \mathcal{P}(a_i, 1)$, in other words the set of positive roots containing a_i (i.e. prenilpotent with a_i).

3.B The Borel subgroup. Unipotent subgroups. — We pick two fields \mathbf{K}_0 and \mathbf{K}_1 . To each positive root a of type i is attached a copy of the additive group $(\mathbf{K}_i, +)$. We denote it by $U_a := \{u_a(k) : k \in \mathbf{K}_i\}$ (u_a is the chosen isomorphism between the root group and its field).

DEFINITION.— (i) Let the abelian group A_i be the direct sum of the root groups associated to the roots in $\mathcal{P}(a_i)$.

(ii) Let the group U_+ be the free product of A_0 and A_1 , that is $U_+ := A_0 * A_1$.

(iii) Let the group T be the product $\mathbf{K}_0^\times \times \mathbf{K}_1^\times$ of the multiplicative groups of the chosen fields.

For a (positive) root a of type i , we define $\epsilon_a \in \{\pm 1\}$ to be $(-1)^m$, where m is the exponent appearing in 3.A. It is the parity of the number of vertices of type i in the interior of the segment joining the middle of E and the vertex ∂a . Using the notation of Sect. 2, T is the maximal torus $\{t_i(\lambda)t_{1-i}(\mu) : \lambda, \mu \text{ invertible}\}$ of $\mathrm{SL}_2(\mathbf{K}_0) \times \mathrm{SL}_2(\mathbf{K}_1)$, and we define the action of T on U_+ by:

$$(3B1) \quad t_j(\lambda)u_a(k)t_j(\lambda)^{-1} := u_a(\lambda^{2\epsilon_a\delta_{j\iota(a)}}k),$$

where the Kronecker symbol $\delta_{j\iota(a)}$ means that the element $t_j(\lambda)$ induces multiplication of k by $\lambda^{2\epsilon_a}$ only if the type of the root a is j . In particular, $t_j(\lambda)$ centralizes each root group of type $\neq j$.

DEFINITION.— *The (positive) Borel subgroup B is the semi-direct product $B := T \times U_+$.*

Let V^i be the subgroup of U_+ generated by all the positive root groups except U_i ; it is not normal in U_+ .

DEFINITION.— *The group U^i is the normal closure of V^i in U_+ .*

In view of the free product structure of U_+ , U^i is generated by $u_a(k)$ for $a \in \mathcal{P}(a_i) \setminus \{a_i\}$ and $u_i(r)u_a(k)u_i(r)^{-1}$ for $a \in \mathcal{P}(a_{1-i})$, with $r \in \mathbf{K}_i$.

3.C *Actions of Levi factors on unipotent radicals.* — Let us start with the definition of another subgroup.

DEFINITION.— *The (standard) Levi factor of type i is the direct product*

$$L_i := \mathrm{SL}_2(\mathbf{K}_i) \times \mathbf{K}_{1-i}^\times.$$

In order to define a parabolic subgroup as a semi-direct product $L_i \times U^i$, we must define an action of L_i on the group U^i . We define actions of $\mathrm{SL}_2(\mathbf{K}_i)$ and \mathbf{K}_{1-i}^\times which are easily seen to commute with one another. Hence we can deal with each factor separately.

The action of the torus \mathbf{K}_{1-i}^\times on U^i is that obtained as a subgroup of T . This makes sense because T obviously stabilizes U^i . We turn now to the action of the factor $\mathrm{SL}_2(\mathbf{K}_i)$. In this subsection, we define the actions of the generators $u_i(r)$ and $m_i(\lambda)$ given in 2.B.

For the root groups, we set for k in \mathbf{K}_0 or \mathbf{K}_1 , and r, s in \mathbf{K}_i :

$$(3C1) \quad u_i(r)u_a(k)u_i(r)^{-1} := u_a(k) \quad \text{for } a \in \mathcal{P}(a_i) \setminus \{a_i\},$$

$$(3C2) \quad u_i(s)(u_i(r)u_a(k)u_i(r)^{-1})u_i(s)^{-1} := u_i(r+s)u_a(k)u_i(r+s)^{-1} \quad \text{for } a \in \mathcal{P}(a_{1-i}).$$

The elements $m_i(\lambda)$, lifting the Weyl group reflections (see Sect. 2), have a conjugation action on the positive root group elements $u_a(k)$, defined by:

$$(3C3) \quad m_i(\lambda)u_a(k)m_i(\lambda)^{-1} := u_{s_i a}(\lambda^{-2\epsilon_a\delta_{i\iota(a)}}k).$$

We must also define a conjugation action of $m_i(\lambda)$ on elements $u_i(r)u_a(k)u_i(r)^{-1}$ in the free product of U_a and U_i , whenever a is a positive root containing a_{1-i} , and $u_i(r) \neq 1$. We set:

$$(3C4) \quad m_i(\lambda)(u_i(r)u_a(k)u_i(r)^{-1})m_i(\lambda)^{-1} := u_i\left(\frac{-\lambda^2}{r}\right)u_a\left(\left(\frac{-\lambda}{r}\right)^{2\epsilon_a\delta_{i\iota(a)}}k\right)u_i\left(\frac{-\lambda^2}{r}\right)^{-1}.$$

REMARK. — The Kronecker symbol $\delta_{i\iota(a)}$ in the exponents $^{2\epsilon_a\delta_{i\iota(a)}}$ involves the types of roots i and $\iota(a)$. It simply means that the element $m_i(\lambda)$ induces a multiplication of the additive parameter k in $u_a(k)$ by a factor $\lambda^{-2\epsilon_a}$ or $\left(\frac{-\lambda}{r}\right)^{2\epsilon_a}$ only if the type of the root a is i .

3.D *Checking the product relation.* — In this subsection, we make sure that the individual actions above define an action of L_i on U^i . We must show that given any two elements g, h of $\mathrm{SL}_2(\mathbf{K}_i) \times \mathbf{K}_{1-i}$ and a generator v of U^i , we always have

$$g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1}.$$

This equality involves the product formula for $g \cdot h$, so we must take into account the five cases of 2.D. The form of the generator v of U^i will also play a role. A generator $v = u_i(t)u_a(k)u_i(t)^{-1}$ with a a root containing a_{1-i} (hence not prenilpotent with a_i), will be referred to as a generator *of the first type*. A generator $v = u_a(k)$, with a a root containing a_i (hence prenilpotent with a_i), will be referred to as a generator *of the second type*.

3.D.1 «big cell \cdot big cell \in big cell» – see 2.D.1, and recall that

$$g = u_i(r)m_i(\lambda)u_i(r') \quad \text{and} \quad h = u_i(s)m_i(\mu)u_i(s').$$

We introduce the notation $R = r' + s$ and $S = s' + t$ for the remainder of section 3.D. In 3.D.1, as in 2.D.1, we have $R \neq 0$ and

$$gh = u_i\left(r - \frac{\lambda^2}{R}\right)m_i\left(\frac{-\lambda\mu}{R}\right)u_i\left(s' - \frac{\mu^2}{R}\right).$$

Let us deal with a generator v of the first type. By (3C2) $hvh^{-1} = [u_i(s)m_i(\mu)u_i(S)]u_a(k)[\dots]^{-1}$, and assuming that $S \neq 0$, we have by (3C4) and (3C2):

$$\begin{aligned} hvh^{-1} &= u_i(s - \mu^2 S^{-1})u_a((- \mu S^{-1})^{2\epsilon_a \delta_{iu(a)}}k)u_i(\dots)^{-1}, \\ g(hvh^{-1})g^{-1} &= [u_i(r)m_i(\lambda)u_i(R - \mu^2 S^{-1})]u_a((\mu^2 S^{-1})^{2\epsilon_a \delta_{iu(a)}}k)[\dots]^{-1}. \end{aligned}$$

Under the further assumption that $R - \mu^2 S^{-1} \neq 0$, (3C4) and (3C2) give:

$$\begin{aligned} g(hvh^{-1})g^{-1} &= u_i\left(r - \frac{\lambda^2}{R - \mu^2 S^{-1}}\right) \cdot u_a\left(\left(\frac{-\lambda}{R - \mu^2 S^{-1}}\right)^{2\epsilon_a \delta_{iu(a)}}(-\mu S^{-1})^{2\epsilon_a \delta_{iu(a)}}k\right) \cdot u_i(\dots)^{-1} \\ &= u_i\left(r - \frac{\lambda^2 S}{RS - \mu^2}\right) \cdot u_a\left(\left(\frac{\lambda\mu}{RS - \mu^2}\right)^{2\epsilon_a \delta_{iu(a)}}k\right) \cdot u_i(\dots)^{-1}. \end{aligned}$$

Now assuming $(R - \mu^2 S^{-1}) \cdot S \neq 0$, we use (3C4) and (3C2) to conjugate v by gh :

$$\begin{aligned} (gh)v(gh)^{-1} &= [u_i(r - \lambda^2 R^{-1})m_i(-\lambda\mu R^{-1})u_i(S - \mu^2 R^{-1})]u_a(k)[\dots]^{-1} \\ &= u_i\left(r - \frac{\lambda^2}{R} - \frac{\lambda^2 \mu^2}{R^2(S - \mu^2 R^{-1})}\right) \cdot u_a\left(\left(\frac{-\lambda\mu}{R(S - \mu^2 R^{-1})}\right)^{2\epsilon_a \delta_{iu(a)}}k\right) \cdot u_i(\dots)^{-1} \\ &= u_i\left(r - \frac{\lambda^2 S}{RS - \mu^2}\right) \cdot u_a\left(\left(\frac{\lambda\mu}{RS - \mu^2}\right)^{2\epsilon_a \delta_{iu(a)}}k\right) \cdot u_i(\dots)^{-1}, \end{aligned}$$

which proves $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1}$ when $S \neq 0$ and $R - \mu^2 S^{-1} \neq 0$.

Now, suppose $S \neq 0$ but $R - \mu^2 S^{-1} = 0$. Then the first equation above for $g(hvh^{-1})g^{-1}$ simplifies, and using (3C3) and (3C1) (which implies that u_i commutes with $u_{s_i a}$) we obtain:

$$g(hvh^{-1})g^{-1} = u_i(r)u_{s_i a}\left(\left(\frac{\mu}{\lambda S}\right)^{2\epsilon_a \delta_{iu(a)}}k\right)u_i(r)^{-1} = u_{s_i a}\left(\left(\frac{\mu}{\lambda S}\right)^{2\epsilon_a \delta_{iu(a)}}k\right),$$

and by (3C3) and (3C1) again:

$$(gh)v(gh)^{-1} = [u_i(r - \lambda^2 R^{-1})m_i(-\lambda\mu R^{-1})]u_a(k)[\dots]^{-1} = u_{s_i a}\left(\left(\frac{R}{\lambda\mu}\right)^{2\epsilon_a \delta_{iu(a)}}k\right).$$

Using $R = \frac{\mu^2}{S}$, we obtain the desired equality $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1}$ in this case.

Finally if $S = 0$, by (3C3) and (3C1) we have:

$$hvh^{-1} = [u_i(s)m_i(\mu)]u_a(k)[\dots]^{-1} = u_i(s)u_{s_i a}\left(\frac{k}{\mu^{2\epsilon_a \delta_{iu(a)}}}\right)u_i(s)^{-1} = u_{s_i a}\left(\frac{k}{\mu^{2\epsilon_a \delta_{iu(a)}}}\right).$$

By (3C3), $m_i(\lambda)u_{s_i a}(k)m_i(\lambda)^{-1} = u_a(\lambda^{2\epsilon_a \delta_{i(a)}})$.

Using this and (3C2):

$$g(hvh^{-1})g^{-1} = [u_i(r)m_i(\lambda)]u_{s_i a}(\mu^{-2\epsilon_a \delta_{i(a)}}k)[\dots]^{-1} = u_i(r) \cdot u_a\left(\left(\frac{\lambda}{\mu}\right)^{2\epsilon_a \delta_{i(a)}}k\right) \cdot u_i(r)^{-1}.$$

Again with $S = 0$, conjugation of v by gh gives by (3C4) and (3C2):

$$\begin{aligned} (gh)v(gh)^{-1} &= [u_i(r - \lambda^2 R^{-1})m_i(-\lambda\mu R^{-1})u_i(-\mu^2 R^{-1})]u_a(k)[\dots]^{-1} \\ &= [u_i(r - \frac{\lambda^2}{R})u_i(\frac{-\lambda^2 \mu^2 R^{-2}}{-\mu^2 R^{-1}})]u_a\left(\left(\frac{\lambda\mu R^{-1}}{-\mu^2 R^{-1}}\right)^{2\epsilon_a \delta_{i(a)}}k\right)[\dots]^{-1} \\ &= u_i(r) \cdot u_a\left(\left(\frac{\lambda}{\mu}\right)^{2\epsilon_a \delta_{i(a)}}k\right) \cdot u_i(r)^{-1}, \end{aligned}$$

hence $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1}$.

The case of a generator $v = u_a(k)$ of the second type is simpler, because $a \in \mathcal{P}_i - \{a_i\}$, so by (3C1) u_i commutes with u_a . Using (3C2), (3C3) and (3C4) one obtains:

$$g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_i\left(r - \frac{\lambda^2}{R}\right) \cdot u_{s_i a}\left(\left(\frac{R}{\lambda\mu}\right)^{2\epsilon_a \delta_{i(a)}}\right) \cdot u_i(\dots)^{-1}.$$

3.D.2 «big cell · big cell ∈ Borel» – see 2.D.2: with g and h as above, we now have $R = 0$, and $gh = u_i\left(r + \frac{\lambda^2}{\mu^2}s'\right)t_i\left(\frac{-\lambda}{\mu}\right)$. Let us consider a generator v of the first type. For $g(hvh^{-1})g^{-1}$, we compute as in 3.D.1 (with $R = 0$), the case $S \neq 0$ but $R - \mu^2 S^{-1} = 0$ being excluded. We obtain in any case: $g(hvh^{-1})g^{-1} = u_i\left(r + \frac{S\lambda^2}{\mu^2}\right) \cdot u_a\left(\left(\frac{\lambda}{\mu}\right)^{2\epsilon_a \delta_{i(a)}}k\right) \cdot u_i(\dots)^{-1}$, which equals $(gh)v(gh)^{-1}$ by (3C2) and (3B1).

For a generator of the second type, we have: $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_a\left(\left(\frac{\lambda}{\mu}\right)^{2\epsilon_a \delta_{i(a)}}k\right)$.

3.D.3 «Borel · big cell ∈ big cell» – see 2.D.3, and recall that

$$g = u_i(r)t_i(\lambda), \quad h = u_i(s)m_i(\mu)u_i(s') \quad \text{and} \quad gh = u_i(r + \lambda^2 s)m_i(\lambda\mu)u_i(s').$$

For a generator of the first type, we have for $S \neq 0$:

$$g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_i\left(r + \lambda^2 s - \frac{\lambda^2 \mu^2}{S}\right)u_a\left(\left(\frac{\lambda\mu}{S}\right)^{2\epsilon_a \delta_{i(a)}}k\right)u_i(\dots)^{-1},$$

and for $S = 0$: $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_{s_i a}\left(\left(\frac{1}{\mu\lambda}\right)^{2\epsilon_a \delta_{i(a)}}k\right)$.

For a generator of the second type, we have:

$$g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_i\left(r + \lambda^2 s\right)u_{s_i a}\left(\left(\frac{1}{\mu\lambda}\right)^{2\epsilon_a \delta_{i(a)}}k\right)u_i(\dots)^{-1}.$$

3.D.4 «big cell · Borel ∈ big cell» – see 2.D.4, and recall that

$$g = u_i(r)m_i(\lambda)u_i(r'), \quad h = u_i(s)t_i(\mu) \quad \text{and} \quad gh = u_i(r)m_i\left(\frac{\lambda}{\mu}\right)u_i\left(\frac{R}{\mu^2}\right).$$

Write $T = R + \mu^2 t$, and note that $R \neq 0$ in this case.

Let us deal with a generator of the first type. Using (2C) and (3B1) we have:

$$hvh^{-1} = u_i(s + \mu^2 t) \cdot u_a(\mu^{2\epsilon_a \delta_{ii(a)}} k) \cdot u_i(\dots)^{-1}.$$

Hence under the assumption that $T \neq 0$, we get by (3C4) and (3C2):

$$g(hvh^{-1})g^{-1} = u_i(r - \frac{\lambda^2}{T}) \cdot u_a\left(\left(\frac{\lambda\mu}{T}\right)^{2\epsilon_a \delta_{ii(a)}} k\right) \cdot u_i(\dots)^{-1},$$

which equals $(gh)v(gh)^{-1}$, since by (3C4) and (3C2) we have:

$$(gh)v(gh)^{-1} = [u_i(r)m_i(\lambda\mu^{-1})u_i\left(\frac{T}{\mu^2}\right)]u_a(k)[\dots]^{-1} = [u_i(r)u_i\left(\frac{-\lambda^2}{T}\right)]u_a\left(\left(\frac{\lambda\mu}{T}\right)^{2\epsilon_a \delta_{ii(a)}} k\right)[\dots]^{-1}.$$

When $T = 0$, we simply have: $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_{s_i a}\left(\left(\frac{\mu}{\lambda}\right)^{2\epsilon_a \delta_{ii(a)}} k\right)$.

For a generator of the second type, we have in any case:

$$g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_i(r) \cdot u_{s_i a}\left(\left(\frac{\mu}{\lambda}\right)^{2\epsilon_a \delta_{ii(a)}} k\right) \cdot u_i(\dots)^{-1}.$$

3.D.5 «Borel · Borel ∈ Borel» – see 2.D.5, and recall that $g = u_i(r)t_i(\lambda)$, $h = u_i(s)t_i(\mu)$ and $gh = u_i(r + \lambda^2 s)t_i(\lambda\mu)$. The common value of $g(hvh^{-1})g^{-1}$ and of $(gh)v(gh)^{-1}$ is:

$$\begin{array}{ll} u_i(r + \lambda^2 s + (\lambda\mu)^2 t) \cdot u_a\left((\lambda\mu)^{2\epsilon_a \delta_{ii(a)}} k\right) \cdot u_i(\dots)^{-1} & \text{for a generator of the first type,} \\ u_a\left((\lambda\mu)^{2\epsilon_a \delta_{ii(a)}} k\right) & \text{for a generator of the second type.} \end{array}$$

This concludes the computations and enables us to introduce the following

DEFINITION.— *The (standard) parabolic subgroup of type i is the semidirect product*

$$P_i := L_i \ltimes U^i.$$

We can now turn to combinatorial considerations.

3.E *Group combinatorics.* — This subsection is dedicated to our main constructive result about non Kac-Moody Moufang twin trees.

3.E.1 We can now define the group we are interested in.

DEFINITION.— *The (abstract) group of Kac-Moody type associated to the choices of fields above is the amalgam $\Lambda := P_0 *_B P_1$.*

This amalgam makes sense because we have $P_i = L_i \ltimes U^i$, with $L_i = \mathrm{SL}_2(\mathbf{K}_i) \times \mathbf{K}_{1-i}^\times$. In view of the actions of the subgroups $T := \mathbf{K}_i^\times \times \mathbf{K}_{1-i}^\times$ and U_i of L_i on U^i , the group $TU_i \ltimes U^i$ is isomorphic to (and identified with) B . By the Bruhat decomposition for SL_2 , each parabolic subgroup admits the following decomposition:

$$(\star) \quad P_i = B \sqcup U_i m_i B \quad (i \in \{0; 1\}).$$

Consequently, an element γ of Λ can be written

$$\gamma = u_1 m_{i_1} u_2 m_{i_2} \dots u_k m_{i_k} \bar{\gamma}, \text{ with } i_j \in \{0; 1\}, u_j \in U_{i_j} \text{ and } \bar{\gamma} \in B,$$

or also: $\gamma = u_1(m_{i_1} u_2 m_{i_1}^{-1}) \dots (m_{i_1} m_{i_2} \dots m_{i_{k-1}} u_k m_{i_{k-1}}^{-1} \dots m_{i_2}^{-1} m_{i_1}^{-1})(m_{i_1} m_{i_2} \dots m_{i_k}) \bar{\gamma}$, which can be geometrically interpreted in terms of galleries in a building.

3.E.2 We can now prove the existence of groups with twin root data different from those provided by Kac-Moody theory.

THEOREM.— *The group Λ satisfies the axioms of a twin root datum for the family of root groups $\{U_a = u_a(\mathbf{K}_{\iota(a)})\}_{a \in \Phi}$ above. As a consequence, there exists a semi-homogeneous Moufang twin tree of valencies $1 + |\mathbf{K}_0|$ and $1 + |\mathbf{K}_1|$ for any choice of two fields \mathbf{K}_0 and \mathbf{K}_1 .*

Proof. The group Λ being defined as an amalgam, it acts on a semihomogeneous tree Δ of valencies $[P_0 : B]$ and $[P_1 : B]$. This follows from Bass-Serre theory [S, I.4 Theorem 7]. In view of the decompositions (\star) in 3.E.1, these valencies are $1 + |\mathbf{K}_0|$ and $1 + |\mathbf{K}_1|$. A fundamental domain for this action is given by the closure of an edge. The stabilizer of an edge (of a vertex of type 0, resp. of type 1), is isomorphic to B (to P_0 , resp. P_1). In particular, the action is not discrete. We have the identification of Λ -sets:

$$\Delta \simeq \Lambda/B \sqcup \Lambda/P_0 \sqcup \Lambda/P_1.$$

The edges (the vertices of type 0, resp. of type 1) are in bijection with Λ/B (with Λ/P_0 , resp. Λ/P_1). We recover the simplicial structure of the tree Δ thanks to the (reversed) inclusion relation on stabilizers. Let us set $t := m_1 m_0$. The set of edges $\{t^n B\}_{n \in \mathbf{Z}} \sqcup \{t^n m_1 B\}_{n \in \mathbf{Z}}$ defines a geodesic in the tree: this is our standard apartment \mathbb{A} , containing the standard edge B .

We now check the (TRD) axioms for Λ , as given in 1.A.1, with T playing the role of H . We also need root groups. Recall that the positive roots are described in 3.A. The root groups indexed by positive roots are the groups of the form $t^{-n} U_0 t^n$, $t^n m_1 U_0 m_1^{-1} t^{-n}$, $t^n U_1 t^{-n}$ or $t^{-n} m_0 U_1 m_0^{-1} t^n$ for $n \geq 0$. The root groups indexed by negative roots are defined similarly. In the context of trees, as mentioned in 3.A, a pair of roots $\{a; b\}$ is prenilpotent if $a \supset b$ or $b \supset a$. Using conjugation by a suitable power of t enables us to see prenilpotent pairs of roots as pairs of positive roots. Since the group U_+ is defined in such a way that the commutation relations are trivial for all prenilpotent pairs, we get axiom (TRD1). By definition, the group Λ is generated by P_0 and P_1 , hence by the positive root groups, by T and by the two root groups indexed by the opposite of the simple roots: this is axiom (TRD4). Axiom (TRD2) follows from relations in SL_2 defining the elements $m_i(\lambda)$, and from the definition of the root groups. From the definition of B , we have that T normalizes the root groups, which is the second half of axiom (TRD0). By definition of Δ , the simple group U_i is simply transitive on the edges whose closure contains the vertex P_i and different from the standard edge: this proves the first assertion of axiom (TRD0) and the second one in axiom (TRD3). We are reduced to prove the first assertion of axiom (TRD3), but as noted by P. Abramenko [A §1 remark 2], this follows from what has already been proved above. \square

REMARK. — A basic consequence of the axioms of a twin root datum, is the existence of BN -pairs in the group. For the positive BN -pair in Λ , this can be proved concretely as follows. Since $\mathrm{Stab}_\Lambda(\mathbb{A})$ contains the subgroup $\langle m_0(\lambda_i), m_1(\lambda_1) : \lambda_i \in \mathbf{K}_i^\times \rangle$, it is transitive on the set of edges of \mathbb{A} . Let $\{e'; e''\}$ be a pair of edges in Δ . By definition, we have $e' = \gamma' B$ and $e'' = \gamma'' B$ for $\gamma', \gamma'' \in \Lambda$. As in the end of the previous subsection, the element $\gamma'^{-1} \gamma''$ can be written $\gamma'^{-1} \gamma'' = (v_1 v_2 \dots v_k)(m_{i_1} m_{i_2} \dots m_{i_k}) \bar{\gamma}$, with $v_j = m_{i_1} m_{i_2} \dots m_{i_{j-1}} u_j m_{i_{j-1}}^{-1} \dots m_{i_2}^{-1} m_{i_1}^{-1} \in U_+$. Hence we have $(v_1 v_2 \dots v_k) \gamma'^{-1} \cdot \{e'; e''\} = \{B; m_{i_1} m_{i_2} \dots m_{i_k} B\}$. In other words, we proved in this paragraph the strong transitivity of Λ on the tree Δ . Consequently, the group Λ admits a BN -pair $(B, \mathrm{Stab}_\Lambda(L))$ – see [Ro1, Theorem 5.2].

This result (about trees) is stated at a higher level of generality and abstraction in [T4]. The down-to-earth computations we used here will be useful in the next case, of dimension two and arbitrarily large rank.

4. Twinning Fuchsian buildings

The computations of the previous section will now be used to generate two-dimensional Mofang twin buildings of hyperbolic type and arbitrarily large rank. We will apply the procedure of 2.A, but first we need to examine the root system.

4.A Geometric description of the root system. — The starting point is the tiling of the hyperbolic plane \mathbb{H}^2 by regular right-angled r -gons. We choose such an r -gon R and label its edges (open segments) $\{E_i\}_{i \in \mathbf{Z}/r}$ in a natural cyclic order. Each E_i supports a geodesic with associated orthogonal reflection r_i . By the Poincaré theorem [Mas, IV.H.11], the reflections $\{r_i\}_{i \in \mathbf{Z}/r}$ give rise to a Coxeter system $(W, \{r_i\}_{i \in \mathbf{Z}/r})$ and $W \cdot \overline{R} = \mathbb{H}^2$ is our tiling. This tiling is a metric realization of the Coxeter complex associated to $(W, \{r_i\}_{i \in \mathbf{Z}/r})$, which represents the spherical facets. We will use freely the terminology of Coxeter complexes – see [Ro1 §2] and 1.A.1 here.

LEMMA/DEFINITION.— *Given a root a , the panels on its boundary wall all have the same type, and we shall refer to this as the type of the root a , and denote it by $\iota(a) \in \mathbf{Z}/r$. If a_i is the simple root of type i , then we have*

$$\text{Stab}_W(\partial a_i) = \langle r_i \rangle \times \langle r_{i-1}, r_{i+1} \rangle \simeq \mathbf{Z}/2 \times D_\infty \quad \text{and} \quad \text{Stab}_W(a_i) = \langle r_{i-1}, r_{i+1} \rangle \simeq D_\infty.$$

Hence, we have a bijection between the set of roots of type i and $W/\langle r_{i-1}, r_{i+1} \rangle$.

Proof. Let us fix a type i and remark that since we are working with a right-angled tiling, the simple reflections r_{i-1} and r_{i+1} stabilize the wall ∂a_i , and in fact the root a_i . Moreover $\langle r_{i-1}, r_{i+1} \rangle$ is transitive on panels contained in ∂a_i . In particular, ∂a_i is a union of closures of panels of type i . This proves our assertion on the type of a root. Assume now we are given $w \in \text{Stab}_W(\partial a_i)$. The standard panel $E_i \subset \overline{R}$ of type i is thus sent by w on a panel in ∂a_i , which writes $w'E_i$, with $w' \in \langle r_{i-1}, r_{i+1} \rangle$: $w'^{-1}w$ fixes E_i , hence is trivial or equal to r_i . If w stabilizes the root a_i , so does w' and we have $w = w'$. \square

For a root a , we define $\epsilon_a \in \{\pm 1\}$ to be $(-1)^m$, where m is the number of walls of type $\iota(a)$ meeting the interior of the geodesic segment from the barycenter of R to the wall ∂a .

4.B The Borel subgroup. Unipotent radicals of parabolic subgroups. — The definitions are completely analogous to that of the tree case. For each $i \in \mathbf{Z}/r$, we pick a field \mathbf{K}_i . To each positive root a of type i is attached a copy of the additive group $(\mathbf{K}_i, +)$. We denote it by $U_a := \{u_a(k) : k \in \mathbf{K}_i\}$. As in 1.A.2 we define, for each $w \in W$, a group U_w . It can be expressed as a product $\prod_{a \in \Phi_{w^{-1}}} U_a$, and using the Bruhat ordering of W , these groups $\{U_w\}_{w \in W}$ form an inductive system with $U_w < U_{w'}$ when $w < w'$.

DEFINITION.— (i) *The standard torus T is the direct product $T := \prod_{i \in \mathbf{Z}/r} \mathbf{K}_i^\times$ of the multiplicative groups of the chosen fields.*

(ii) *The group U_+ is the limit of the inductive system described above: $U_+ := \varinjlim_{w \in W} U_w$.*

Using the notation of Sect. 2, we view T as the maximal torus $\left\{ \prod_{i \in \mathbf{Z}/r} t_i(\lambda_i) : \lambda_i \in \mathbf{K}_i^\times \right\}$ of $\prod_{i \in \mathbf{Z}/r} \text{SL}_2(\mathbf{K}_i)$. As in 3.B.1, we make T act on U_+ by:

$$t_j(\lambda) u_a(k) t_j(\lambda)^{-1} := u_a(\lambda^{2\epsilon_a \delta_{j\iota(a)}} k).$$

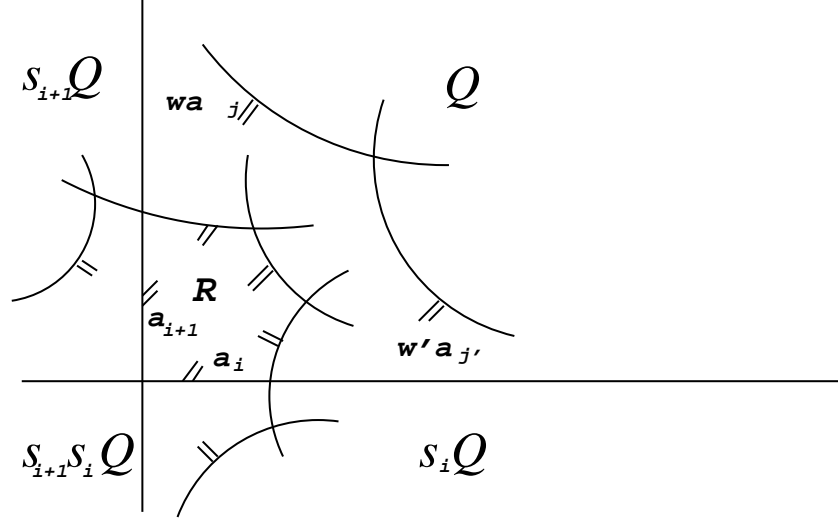
In particular, $t_j(\lambda)$ centralizes each root group of type $\neq j$.

DEFINITION.— *The (positive) Borel subgroup B is the semi-direct product $B := T \ltimes U_+$.*

Let us turn now to the construction of «unipotent radicals». Call $x_{i,i+1}$ the vertex of type $i, i+1$ in the closure \overline{R} of the standard chamber. As in 3.A we must classify positive roots according to whether they are prenilpotent with the simple root a_i and/or the simple root a_{i+1} . The point $x_{i,i+1}$ is the vertex of the sector $\mathcal{Q} := a_i \cap a_{i+1}$. Among the four connected components of $\mathbb{H}^2 \setminus (\partial a_i \cup \partial a_{i+1})$, this sector is the one containing R . We have (see the picture below):

$$\overline{a}_i = \overline{\mathcal{Q}} \cup s_{i+1}\overline{\mathcal{Q}} \quad \text{and} \quad \partial a_i = (\overline{\mathcal{Q}} \cup s_{i+1}\overline{\mathcal{Q}}) \cap s_i(\overline{\mathcal{Q}} \cup s_{i+1}\overline{\mathcal{Q}}).$$

PICTURE. —



DEFINITION.— (i) The group $V^{i,i+1}$ is the subgroup of U_+ generated by all the positive root groups except U_i and U_{i+1} .

(ii) The group $U^{i,i+1}$ is the normal closure of $V^{i,i+1}$ in U_+ .

According to the defining relations of U_+ , the group $U^{i,i+1}$ admits the following infinite set of generators (satisfying the obvious induced relations), where a is a positive root.

$$u_i(v)u_{i+1}(t)u_a(k)u_{i+1}(t)^{-1}u_i(v)^{-1} \text{ for } \partial a \subset \mathcal{Q}.$$

These generators will be referred to as the generators of the first type.

$$u_i(v)u_a(k)u_i(v)^{-1} \quad \begin{cases} \text{for } \partial a \subset s_{i+1}\mathcal{Q}, \\ \text{or if } \partial a \cap \partial a_{i+1} \text{ is a point in } s_{i+1}\overline{\mathcal{Q}}. \end{cases}$$

These generators will be referred to as the generators of the second type.

$$u_{i+1}(t)u_a(k)u_{i+1}(t)^{-1} \quad \begin{cases} \text{for } \partial a \subset s_i\mathcal{Q}, \\ \text{or if } \partial a \cap \partial a_i \text{ is a point in } s_i\overline{\mathcal{Q}}. \end{cases}$$

These generators will be referred to as the generators of the third type.

$$u_a(k) \quad \begin{cases} \text{for } \partial a \subset s_i s_{i+1}\mathcal{Q}, \\ \text{if } \partial a \cap \partial a_i \text{ is a point in } s_i s_{i+1}\overline{\mathcal{Q}}, \\ \text{or if } \partial a \cap \partial a_{i+1} \text{ is a point in } s_i s_{i+1}\overline{\mathcal{Q}}. \end{cases}$$

These generators will be referred to as the generators of the fourth type.

REMARK. — The parameters v , t and k are in the fields \mathbf{K}_i , \mathbf{K}_{i+1} and $\mathbf{K}_{\iota(a)}$, respectively.

4.C Actions of Levi factors on unipotent radicals. — We first define the Levi factors.

DEFINITION.— *The (standard) Levi factor of type $i, i + 1$ is the direct product*

$$L_{i,i+1} := \mathrm{SL}_2(\mathbf{K}_i) \times \mathrm{SL}_2(\mathbf{K}_{i+1}) \times \prod_{j \neq i, i+1} t_j(\mathbf{K}_j^\times).$$

By definition, this group contains the torus T ; it is generated by the toric elements $t_j(\lambda_j)$ ($j \neq i, i + 1$), the unipotent elements $u_i(r)$ ($r \in \mathbf{K}_i$) and $u_{i+1}(s)$ ($s \in \mathbf{K}_{i+1}$), and by the elements $m_i(\lambda_i)$ ($\lambda_i \in \mathbf{K}_i^\times$) and $m_{i+1}(\lambda_{i+1})$ ($\lambda_{i+1} \in \mathbf{K}_{i+1}^\times$).

Let us turn now to the action of such a group $L_{i,i+1}$ on $U^{i,i+1}$. We shall specify that:

– an element $u_i(r)$ centralizes the generators of the third and fourth types, and changes the conjugating element $u_i(v)$ into $u_i(r + v)$ for the generators of the first and second types, as in (3C1) and (3C2).

– an element $u_{i+1}(s)$ centralizes the generators of the second and fourth types, and changes the conjugating element $u_{i+1}(t)$ into $u_{i+1}(s + t)$ for the generators of the first and third types, as in (3C1) and (3C2).

– an element $t_j(\lambda_j)$ centralizes all the root groups indexed by roots of type $\neq j$. On a root group U_a with $\iota(a) = j$, it acts by multiplication of the additive parameter by $\lambda_j^{2\epsilon_a}$, as in (3B1).

Using (3C3) and (3C4) for trees, the action of the elements $m_i(\lambda_i)$ and $m_{i+1}(\lambda_{i+1})$ is defined as follows. For generators of the first type and when $v \neq 0$, we set:

$$\begin{aligned} m_i(\lambda) (u_i(v) u_{i+1}(t) u_a(k) u_{i+1}(t)^{-1} u_i(v)^{-1}) m_i(\lambda)^{-1} \\ := u_i\left(\frac{-\lambda^2}{v}\right) u_{i+1}(t) u_a\left(\left(\frac{-\lambda}{v}\right)^{2\epsilon_a \delta_{i\iota(a)}} k\right) u_{i+1}(t)^{-1} u_i\left(\frac{-\lambda^2}{v}\right)^{-1}; \end{aligned}$$

whereas when $v = 0$, we set:

$$m_i(\lambda) (u_{i+1}(t) u_a(k) u_{i+1}(t)^{-1}) m_i(\lambda)^{-1} := u_{i+1}(t) u_{s_i a}(\lambda^{-2\epsilon_a \delta_{i\iota(a)}} k) u_{i+1}(t)^{-1}.$$

For generators of the second type and when $v \neq 0$, we set:

$$m_i(\lambda) (u_i(v) u_a(k) u_i(v)^{-1}) m_i(\lambda)^{-1} := u_i\left(\frac{-\lambda^2}{v}\right) u_a\left(\left(\frac{-\lambda}{v}\right)^{2\epsilon_a \delta_{i\iota(a)}} k\right) u_i\left(\frac{-\lambda^2}{v}\right)^{-1};$$

whereas when $v = 0$, we set:

$$m_i(\lambda) u_a(k) m_i(\lambda)^{-1} := u_{s_i a}(\lambda^{-2\epsilon_a \delta_{i\iota(a)}} k).$$

For generators of the third type, we set:

$$m_i(\lambda) (u_{i+1}(t) u_a(k) u_{i+1}(t)^{-1}) m_i(\lambda)^{-1} := u_{i+1}(t) u_{s_i a}(\lambda^{-2\epsilon_a \delta_{i\iota(a)}} k) u_{i+1}(t).$$

For generators of the fourth type, we set:

$$m_i(\lambda) u_a(k) m_i(\lambda)^{-1} := u_{s_i a}(\lambda^{-2\epsilon_a \delta_{i\iota(a)}} k).$$

REMARK. — As in 3.C, $\delta_{i\iota(a)}$ in the exponents $^{2\epsilon_a \delta_{i\iota(a)}}$ means that the element $m_i(\lambda)$ induces a multiplication of the additive parameter k in $u_a(k)$ by a factor $\lambda^{-2\epsilon_a}$ or $\left(\frac{-\lambda}{v}\right)^{2\epsilon_a}$ only if the type $\iota(a)$ of the root a is i ; otherwise, it doesn't change k .

This defines the action of the factor $\mathrm{SL}_2(\mathbf{K}_i)$; that of $\mathrm{SL}_2(\mathbf{K}_{i+1})$ is defined *mutatis mutandis*.

4.D *Checking the product relation.* — According to the definition of the Levi factor of type $i, i+1$ as a direct product, it is enough to check separately the defining relations for each toric factor $t_j(\mathbf{K}_j^\times)$ ($j \neq i, i+1$) and each of both factors $\mathrm{SL}_2(\mathbf{K}_i)$ and $\mathrm{SL}_2(\mathbf{K}_{i+1})$, whose actions obviously commute to one another.

Let us consider the $\mathrm{SL}_2(\mathbf{K}_i)$ -action, and check the product relation $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1}$. We consider a generator $v = u_i(v)u_a(k)u_i(v)^{-1}$ of $U^{i,i+1}$ of the second type. Then the root a is not prenilpotent with a_i and, in view of the definition of the actions in 4.C, the computations to check the product relation reduce to the computation made in 3.D for a generator of the first type in the sense of trees (3.D). Similarly, the computations to check the product relation for a generator of the fourth type in the sense of Fuchsian buildings (4.B) reduce to the computations made in 3.D for a generator of the second type in the sense of trees (3.D). Finally, by the definitions in 4.C, the actions of $u_i(r)$ and of $m_i(\lambda_i)$ commute with conjugation by the elements $u_{i+1}(t)$. So, up to the conjugation by $u_{i+1}(t)$, checking the product relation for a generator of the third (resp. first) type in the sense of Fuchsian buildings amounts to checking the product relation for a generator of the second (resp. first) type in the sense of trees.

Consequently, we can introduce the following definition.

DEFINITION.— (i) *The (standard) parabolic subgroup of type $i, i+1$ is the semidirect product*

$$P_{i,i+1} := L_{i,i+1} \ltimes U^{i,i+1}.$$

(ii) *The (standard) parabolic subgroup of type i is the subgroup P_i of $P_{i,i+1}$ generated by $L_i := \langle T, \mathrm{SL}_2(\mathbf{K}_i) \rangle$ and U_+ .*

The groups P_i have a Levi decomposition $P_i = L_i \ltimes U^i$, where U^i is the normal closure in U_+ of all the positive root groups except U_i , and L_i is of course $\langle T, \mathrm{SL}_2(\mathbf{K}_i) \rangle$. Besides for each $i \in \mathbf{Z}/r$, we have chains of inclusions: $B \hookrightarrow P_i \hookrightarrow P_{i,i+1}$ and $B \hookrightarrow P_i \hookrightarrow P_{i,i-1}$. This gives rise to an inductive system of group homomorphisms for which we will have a geometric interpretation in the next subsection. As for trees, we will amalgamate the parabolic subgroups in order to define a group Λ endowed with a twin root datum.

4.E *Group combinatorics.* — The verification of the axioms of a twin root datum for Λ will of course follow the lines of 3.E. Nevertheless, we first need to present some geometric notions which are used to prove the existence of a building acted upon by Λ .

4.E.1 Let us recall some facts about complexes of groups [BH III.C], as well as an application to hyperbolic buildings due to D. Gaboriau and F. Paulin [GP]. The use of this theory should not be surprising, because it is a higher-dimensional generalization of Bass-Serre theory about group actions on trees, and we used Bass-Serre theory in the proof of 3.E.2. We will only use the simpler notion of *polytope of groups*, namely the datum of a compact convex polytope R , of a group G_σ for each face σ of R and of an injective group homomorphism $G_\sigma \hookrightarrow G_\tau$ for each inclusion of faces $\tau \subset \bar{\sigma}$. We require the commutativity of the diagram of group homomorphisms given by all the (reversed) inclusions of faces. This diagram is an inductive system indexed by the barycentric subdivision of R . The inductive limit is the *fundamental group* of the polytope of groups [BH III.C.3].

The connection with 4.D is that $(R, \{B \hookrightarrow P_i \hookrightarrow P_{i,i+1}\}_{i \in \mathbf{Z}/r})$ is a polytope of groups. The group B (resp. P_i , resp. $P_{i,i+1}$) is attached to the 2-dimensional cell of R (resp. to the edge of type $\{i\}$, resp. to the vertex of type $\{i, i+1\}$).

DEFINITION.— *The group Λ is the limit of the finite inductive system described in 4.D. In other words, it is the fundamental group of the above complex of groups.*

In the case of graphs, Bass-Serre theory provides a tree on which the fundamental group acts, with the groups G_σ as prescribed stabilizers. In the case of polytopes of groups, this kind of existence result is not immediate, because non-positive curvature arguments come into play [BH, 4.17]. Under non-positive curvature assumptions, we know that Λ acts on a simply connected cell complex with the groups G_σ as prescribed stabilizers. In our situation, this will be even better, since we can apply the result of D. Gaboriau and F. Paulin alluded to above. We follow their terminology:

DEFINITION.— *If R is a hyperbolic polyhedron providing a Poincaré tiling of the hyperbolic space \mathbb{H}^n , a hyperbolic building of type R is a piecewise polyhedral cell complex, covered by a family of subcomplexes – the apartments – all isomorphic to this tiling and satisfying the following incidence axioms.*

- (i) *Two points are always contained in an apartment.*
- (ii) *Two apartments are isomorphic by a polyhedral arrow fixing their intersection.*

A building whose apartments are tilings of the hyperbolic plane \mathbb{H}^2 will be called Fuchsian.

REMARK. — A hyperbolic building carries a natural CAT(−1) metric [GP, Proposition 1.5].

Let us go back to our construction. According to [GP, Theorem 0.1], there exists a hyperbolic building Δ of type R such that we have the identification of Λ -sets:

$$\Delta \simeq \Lambda/B \sqcup \bigsqcup_{i \in \mathbf{Z}/r} \Lambda/P_i \sqcup \bigsqcup_{i \in \mathbf{Z}/r} \Lambda/P_{i,i+1}.$$

The building structure is given by the (reversed) inclusion relation on stabilizers. For $\gamma \in \Lambda$, a translate γB (resp. γP_i , $\gamma P_{i,i+1}$) is a *chamber* (resp. an *edge of type i* , a *vertex of type $i, i + 1$*). A fundamental domain for this action is given by the closure of R . Moreover it is shown in [Bou3] that given R and $\underline{q} := \{q_i\}_{1 \leq i \leq r}$ a sequence of integers ≥ 2 , there exists a unique Fuchsian building $I_{r,1+\underline{q}}$ with apartments isomorphic to the tiling of \mathbb{H}^2 by R , and such that the link at any vertex of type $i, i + 1$ is the complete bipartite graph of parameters $(1 + q_i, 1 + q_{i+1})$. Recall that the *link* at a point is a sufficiently small sphere centered at this point; in our two-dimensional context, it is seen as a graph. Uniqueness implies that when we choose finite fields for our construction, the building Δ above is $I_{r,\underline{q}+1}$, for \underline{q} a sequence of prime powers. When the q_i 's are all equal to a given prime power q , the building comes from a (non-unique) Kac-Moody group; we denote it by $I_{r,1+q}$.

REMARK. — Let Λ be a Kac-Moody group over \mathbf{F}_q whose Weyl group is the group W associated to the hyperbolic tiling of \mathbb{H}^2 we are considering. Such a group exists: choose any generalized Cartan matrix $A = [A_{i,j}]_{i,j \in \mathbf{Z}/r}$ such that $A_{i,i} = 2$, $A_{i,i+1} = 0$ and $A_{i,j}A_{j,i} \geq 4$ for $j \neq i, i + 1$. In view of its combinatorial structure (positive BN -pair) and of Tits' amalgam theorem [T1 §14], Λ is the limit of the inductive system given by the inclusions of spherical parabolic subgroups of rank ≤ 2 . That is, the fundamental group of a complex of groups with the barycentric subdivision of a regular right-angled r -gon as indexing polytope. In the construction we give, in which the base field can vary from one panel of the base chamber to another, the parabolic subgroups are designed to have a similar Levi decomposition to these Kac-Moody examples.

4.E.2 Here is now our main constructive result about twinings in the two-dimensional case of Fuchsian buildings.

THEOREM.— *The group Λ defined in 4.E.1 satisfies the axioms of a twin root datum for the above family of root groups $\{U_a = u_a(\mathbf{K}_{\iota(a)})\}_{a \in \Phi}$. In particular, given any regular right-angled r -gon R , and a set $\underline{q} := \{q_i\}_{1 \leq i \leq r}$ of prime powers, the above defined Fuchsian building $I_{r,1+\underline{q}}$ belongs to a Moufang twinning.*

Proof. The aim of the first paragraph of the proof of 3.E.2 was to make a bit more explicit the use of Bass-Serre theory. In the higher-dimensional case of Fuchsian buildings, the analogous work was done in 4.E.1, where we presented what we need from Hæffliger's theory of complexes of groups. The verification of the axioms of a twin root datum is then the same as in 3.E.2. The final statement follows from the uniqueness of the Fuchsian building [Bou3] with the given local data. \square

REMARK. — As in remark 3.E.2, we can see directly that $(B, \text{Stab}_\Lambda(\mathbb{A}))$ is a BN -pair in Λ .

5. Non-linearities

A Kac-Moody group defined over a finite ground-field \mathbf{F}_q has no reason to be linear over \mathbf{F}_q merely because it is defined by a presentation involving subgroups that are linear over \mathbf{F}_q . Still, in case it *is* linear over a field, then the characteristic of the latter field must be that of its ground field (5.A). In Sect. 3 and 4 we constructed groups of Kac-Moody type with more than one ground field, and in 5.B we show that if they involve fields with two different characteristics, then they cannot be linear.

5.A Negative results for Kac-Moody groups. — In this subsection, we use the notation of 1.C, because we are specifically working with a topological Kac-Moody group G . We denote by $(G, N, U_c, \Gamma, T, S)$ its refined Tits system (1.C.2). The appropriate axioms, given in 1.A.5, can be proved for the affine BN -pairs associated to Chevalley groups over local fields of equal characteristic to their residue fields, but not in the case of local fields of characteristic 0. The following result can be found in [Ré3].

PROPOSITION.— *Assume Λ is an infinite Kac-Moody group over \mathbf{F}_q . Then the groups Λ and G cannot be linear over any field of characteristic different from p .* □

5.B The wider context of twin root data. — This is the point where Moufang twinings not arising from Kac-Moody theory provide lattices of hyperbolic buildings which are not linear at all. The result below should be seen as a consequence of the constructive theorem 4.E.2.

THEOREM.— *Let $r \geq 5$ be an integer and $\{\mathbf{K}_i\}_{i \in \mathbf{Z}/r}$ be a family of r fields. Assume there exist two distinct indices in \mathbf{Z}/r such that the corresponding fields have different positive characteristics. Let Λ be the group defined as in 4.E.2 from these fields, and Γ be the fixator of a chamber in the associated Moufang Fuchsian twinning. Then, any group homomorphism*

$$\rho : \Gamma \rightarrow \prod_{\alpha \in A} \mathbb{G}_\alpha(\mathbf{F}_\alpha)$$

has infinite kernel, whenever the index set A is finite and \mathbb{G}_α is a linear algebraic group defined over the field \mathbf{F}_α for each $\alpha \in A$.

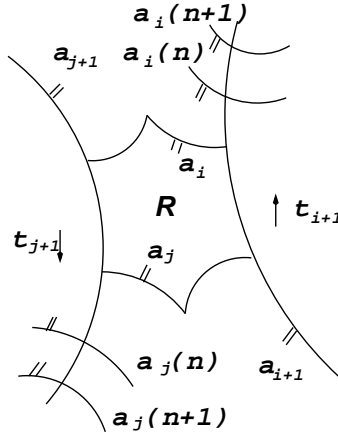
REMARK. — In particular, there exist twin root data for Fuchsian twinings that admit non-linear lattices (since for large enough thicknesses, a chamber-fixator is a lattice in the full automorphism group of the building of sign opposite the chamber).

Proof. Since any two chamber-fixators are isomorphic, we can deal with the standard positive chamber R of 4.A. Then, the corresponding chamber-fixator is the Borel subgroup $B = T \times U_+$ of subsect. 4.B, and for the proof we shall use the notation B rather than Γ . As before, \mathbb{A} is the apartment stabilized by the group N generated by the elements $t_i(\lambda)$ and $m_i(\lambda)$ when i ranges over \mathbf{Z}/r and λ ranges over \mathbf{K}_i^\times .

The element $m_i(1) \circ m_{i+2}(1)$ is a hyperbolic translation t_{i+1} in \mathbb{A} along the wall ∂a_{i+1} . The sequence $\{a_i(n) := t_{i+1}^n \cdot a_i\}_{n \geq 0}$ of positive roots has the property that $a_i(n+1) \supset a_i(n)$. Moreover for any $n, n' \geq 0$, the root groups $U_{a_i(n)}$ and $U_{a_i(n')}$ commute with one another, so the group $V_i := \langle U_{a_i(n)} : n \geq 0 \rangle$ is isomorphic to the additive group of the polynomial ring $\mathbf{K}_i[X]$. In particular, V_i is an infinite group of exponent $p_i := \text{char}(\mathbf{K}_i)$.

The assumption $r \geq 5$ allows us to choose fields \mathbf{K}_i and \mathbf{K}_j with characteristics $p_i \neq p_j$ such that the panels of R having types i and j support parallel walls. This defines as above an infinite group V_j of exponent p_j .

PICTURE. —



We set $A_i := \{\alpha \in A : \text{char}(\mathbf{F}_\alpha) = p_i\}$ and $A_j := \{\alpha \in A : \text{char}(\mathbf{F}_\alpha) = p_j\}$. For each $\alpha \in A$, we denote by $\text{pr}_\alpha : \prod_{\alpha \in A} \mathbb{G}_\alpha(\mathbf{F}_\alpha) \rightarrow \mathbb{G}_\alpha(\mathbf{F}_\alpha)$ the natural projection. By [Mar, VIII.3.7] for each $\alpha \notin A_i$ the group $(\text{pr}_\alpha \circ \rho)(V_i)$ is finite, so there is a finite index normal subgroup $N_i \triangleleft V_i$ such that $\prod_{\alpha \notin A_i} (\text{pr}_\alpha \circ \rho)(N_i)$ is trivial, and similarly for j replacing i .

The subgroup generated by the positive and negative root groups of type i and j satisfies the axioms of a twin root datum with infinite dihedral Weyl group, and by applying [KP, proposition 3.5 (c)] to the corresponding refined Tits system (see 1.A.5), one has a free product decomposition of the subgroup generated by the positive root groups of type i and j . This shows that $V_i * V_j$ injects in B , and if we pick $v \in N_i \setminus \{1\}$ and $v' \in N_j \setminus \{1\}$, then vv' has infinite order. But since $v \in N_i$ and $v' \in N_j$ the images $\rho(v)$ and $\rho(v')$ commute with one another and both have finite order. We have found a subgroup of B isomorphic to \mathbf{Z} and with finite image under ρ . This proves the theorem. \square

REMARK. — This result shows that closures of arbitrary groups with twin root data are too wide a framework for linearity problems. Hence, the last linearity problem to be solved is that of remark 5.A, the equal characteristic case for Kac-Moody groups over finite fields.

5.C Analogy with trees. — In this final subsection we specialize to the case of Kac-Moody groups, where all the fields \mathbf{K}_i are equal to the finite field of q elements.

PROPOSITION.— *For any prime power $q \geq 3$, there exists a Kac-Moody group Λ over \mathbf{F}_q whose building is isomorphic to the right-angled Fuchsian building $I_{r,1+q}$ and such that its natural image in $\text{Aut}(I_{r,1+q})$ contains a uniform lattice $\Gamma_{r,1+q}$ abstractly defined by the presentation*

$$\Gamma_{r,1+q} = \langle \{\gamma_i\}_{i \in \mathbf{Z}/r} : \gamma_i^{q+1} = 1 \text{ and } [\gamma_i, \gamma_{i+1}] = 1 \rangle.$$

REMARK. — The above uniform lattices $\Gamma_{r,1+q}$ (as well as the buildings $I_{r,1+q}$ themselves) were introduced by M. Bourdon [Bou2].

For the proof below, it is good to keep in mind the basic facts on Kac-Moody groups recalled in 2.E.1.

Proof. We fix the finite field \mathbf{F}_q and consider the generalized Cartan matrix A indexed by \mathbf{Z}/r and defined as in 2.E.2 by $A_{i,i} = 2$, $A_{i,i+1} = 0$ and $A_{i,j} = 1 - q$ for $j \neq i, i \pm 1$. In order to define a full Kac-Moody root datum, we need \mathbf{Z} -lattices. In our case, we start by the lattice of cocharacters, which we define as $X_* := \bigoplus_{j \in \mathbf{Z}/r} \mathbf{Z}h_j \oplus \mathbf{Z}\xi_j$. The \mathbf{Z} -lattice X^* is by definition the

\mathbf{Z} -dual of X_* , and the elements a_i are those in X^* which satisfy $a_i(h_j) = A_{ij}$ and $a_i(\xi_j) = -\delta_{ij}$ for all i and j in \mathbf{Z}/r . The Kac-Moody root datum $(\mathbf{Z}/r, A, X^*, \{a_i\}_{i \in \mathbf{Z}/r}, \{h_i\}_{i \in \mathbf{Z}/r})$ and the finite field \mathbf{F}_q define a Kac-Moody group Λ with maximal split torus $T = \text{Hom}(\mathbf{F}_q[X^*], \mathbf{F}_q)$. For any $j \in \mathbf{Z}/r$, let us denote by Λ_j the subgroup generated by T and the root groups $U_{\pm a_j}$ indexed by the simple root a_j and its opposite. This group is the Kac-Moody group with Kac-Moody «rank-one sub-root datum» $(j, [2], X^*, a_j, h_j)$ and is isomorphic to $\text{GL}_2(\mathbf{F}_q) \times (\mathbf{F}_q^\times)^{2(r-1)}$. Since it is the Levi factor of the standard parabolic subgroup of type $\{j\}$ [Ré1, 6.2.2], we deduce that the Levi factor of a spherical parabolic subgroup of type J is isomorphic to $(\mathbf{F}_q^\times)^{2r}$, to the direct product $\text{GL}_2(\mathbf{F}_q) \times (\mathbf{F}_q^\times)^{2(r-1)}$ or to $(\text{GL}_2(\mathbf{F}_q) \times \text{GL}_2(\mathbf{F}_q)) \times (\mathbf{F}_q^\times)^{2(r-2)}$, according to whether $|J|$ equals 0, 1 or 2.

By [Ré1, 9.6.2], the center $Z(\Lambda)$ of the group Λ is the subgroup of T whose group of cocharacters is $\bigoplus_{i \in \mathbf{Z}/r} \mathbf{Z}(h_i + 2\xi_i)$. It is the subgroup of T which centralizes all root groups, and by the argument of Lemma 1.B.1, $Z(\Lambda)$ is also the kernel of the homomorphism $\varphi : \Lambda \rightarrow \text{Aut}(I_{r,1+q})$ attached to the action of Λ on any of its two buildings. We have $Z(\Lambda) \simeq (\mathbf{F}_q^\times)^r$, and for each $i \in \mathbf{Z}/r$, the subgroup $Z(\Lambda)$ intersects the factor isomorphic to $\text{GL}_2(\mathbf{F}_q)$ of Λ_i along its torus of scalar matrices. Consequently, the Levi factors of the parabolic subgroups of $\varphi(\Lambda)$ are isomorphic to $(\mathbf{F}_q^\times)^r$, to $\text{PGL}_2(\mathbf{F}_q) \times (\mathbf{F}_q^\times)^{r-1}$ and to $(\text{PGL}_2(\mathbf{F}_q) \times \text{PGL}_2(\mathbf{F}_q)) \times (\mathbf{F}_q^\times)^{r-2}$.

On the one hand, as noted in 4.E.2, it is a consequence of Tits' theorem [T1 §14] that $\varphi(\Lambda)$ is the fundamental group of a complex of groups indexed by (the barycentric subdivision of) a regular right-angled r -gon R . The involved groups are the standard spherical parabolic subgroups of rank ≤ 2 , seen as facet fixators. On the other hand, the group $\Gamma_{r,1+q}$ is also the fundamental group of a complex of groups indexed by the same r -gon R . The group attached to the edge of type i (resp. the vertex of type $i, i+1$) is $\mathbf{Z}/(q+1)$ (resp. $\mathbf{Z}/(q+1) \times \mathbf{Z}/(q+1)$). The group attached to R itself is $\{1\}$. Hence, in order to see $\Gamma_{r,1+q}$ as a subgroup of $\varphi(\Lambda)$, it is enough to prove the existence of a morphism from the latter inductive system of finite groups to the former inductive system of parabolic subgroups, since taking the limit will then show that $\Gamma_{r,1+q} < \varphi(\Lambda)$.

But as in [Ch, Proposition 5], we can see $\mathbf{Z}/(q+1)$ as a subgroup of $\text{PGL}_2(\mathbf{F}_q)$ which is simply transitive on the chambers of the corresponding building $\mathbb{P}^1 \mathbf{F}_q$. The inclusions $\{1\} < (\mathbf{F}_q^\times)^r$, $\mathbf{Z}/(q+1) < \text{PGL}_2(\mathbf{F}_q) \times (\mathbf{F}_q^\times)^{r-1}$ and $\mathbf{Z}/(q+1) \times \mathbf{Z}/(q+1) < (\text{PGL}_2(\mathbf{F}_q) \times \text{PGL}_2(\mathbf{F}_q)) \times (\mathbf{F}_q^\times)^{r-2}$ then provide the morphism of inductive systems we are looking for. \square

REMARKS. — 1) The group $\Gamma_{r,1+q}$ is a uniform lattice of the building $I_{r,1+q}$. Indeed, by the very definition of a fundamental group of a complex of groups, attaching $\{1\}$ to the 2-cell R implies that $\Gamma_{r,1+q}$ is simply transitive on the chambers of the buildings. Finiteness of all facet fixators implies discreteness.

2) When the number r of edges is even and satisfies $\sin(\pi/r) < 1/\sqrt{q+1}$, the group $\Gamma_{r,1+q}$ can be embedded as a convex cocompact discrete group of isometries of the real hyperbolic space of dimension $2q$ [Bou1].

COROLLARY.— *Whenever q is large enough and r is even and large enough with respect to q , the topological Kac-Moody group G defined from the above group Λ as in 1.B.2, has virtually pro- p maximal compact subgroups, and contains both uniform lattices which are linear in characteristic 0 and non-uniform lattices such that the only possible characteristic of linearity is p (dividing q).*

Proof. By the non-positive curvature property and the Bruhat-Tits fixed point lemma, the maximal compact subgroups are the vertex-fixators, hence the claim on virtual pro- p -ness of these groups by 1.C.2 (ii). Assume q large enough, so that the spherical parabolic subgroups of positive (resp. negative) sign are lattices of the negative (resp. positive) building – see 1.B.3 and 1.C.1. It then remains to assume that r (even) satisfies $\sin(\pi/r) < 1/\sqrt{q+1}$ and apply Bourdon’s embedding result quoted in remark 2) above. \square

References

- [A] P. ABRAMENKO, *Twin buildings and applications to S -arithmetic groups*. Lecture Notes in Mathematics **1641**, Springer, 1997
- [BH] M. BRIDSON, A. HÆFLIGER, *Metric spaces of non-positive curvature*. Grund. der Math. Wiss. **319**, Springer, 1999
- [Bou1] M. BOURDON, *Sur la dimension de Hausdorff de l’ensemble limite d’une famille de sous-groupes convexes cocompacts*. C. R. Acad. Sci. Paris **325** (1997) 1097-1100
- [Bou2] M. BOURDON, *Immeubles hyperboliques, dimension conforme et rigidité de Mostow*. GAFA **7** (1997) 245-268
- [Bou3] M. BOURDON, *Sur les immeubles fuchsien et leur type de quasi-isométrie*. Erg. Th. and Dynam. Sys. **20** (2000), 343-364
- [BM] M. BURGER, S. MOZES, *CAT(-1)-spaces, divergence groups and their commensurators*. J. of AMS **9** (1996) 57-93
- [BP] M. BOURDON, H. PAJOT, *Rigidity of quasi-isometries for some hyperbolic buildings*. Commentarii Math. Helv. **75** (2000), 701-736
- [CG] L. CARBONE, H. GARLAND, *Existence and deformations of lattices in groups of Kac-Moody type*. In preparation
- [Ch] F. CHOUCROUN, *Sous-groupes discrets des groupes p -adiques de rang un et arbres de Bruhat-Tits*. Isr. J. Math. **93** (1996) 195-219
- [Cho] A. CHOSSON, *Isométries dans les immeubles jumelés et construction d’automorphismes de groupes de Kac-Moody*. PhD Univ. Amiens, 2000
- [D] M. DAVIS, *Buildings are CAT(0)*. In Geometry and Cohomology in Group Theory, P.H. Kropholler, G.A. Niblo, R. Stöhr eds, LMS Lecture Notes Series **252** (1997) 108-123
- [GP] D. GABORIAU, F. PAULIN, *Sur les immeubles hyperboliques*. To appear in Geometriae Dedicata
- [H] J.-Y. HÉE, *Construction de groupes tordus en théorie de Kac-Moody*, C. R. Acad. Sc. Paris **310** (1990) 77-80
- [KP] V. KAC, D. PETERSON, *Defining relations for certain infinite-dimensional groups*. Astérisque Hors-Série (1984) 165-208
- [Mar] G.A. MARGULIS, *Discrete subgroups of semisimple Lie groups*. Springer, 1990
- [Mas] B. MASKIT, *Kleinian groups*. Grund. der Math. Wiss. **287**, Springer, 1987
- [Mat] O. MATHIEU, *Construction d’un groupe de Kac-Moody et applications*. Compositio Math. **69** (1989) 37-60
- [Mou] G. MOUSSONG, *Hyperbolic Coxeter groups*. PhD Ohio State Univ., 1988
- [MR] B. MÜHLHERR, M. RONAN, *Local to global structure in twin buildings*. Invent. Math. **122** (1995) 71-81
- [Mü] B. MÜHLHERR, *Locally split and locally finite buildings of 2-spherical type*. J. Reine angew. Math. **511** (1999) 119-143

- [Ré1] B. RÉMY, Groupes de Kac-Moody déployés et presque déployés. To appear in Astérisque, Société Mathématique de France, 2002
- [Ré2] B. RÉMY, *Construction de réseaux en théorie de Kac-Moody*. C. R. Acad. Sc. Paris **329** (1999) 475-478
- [Ré3] B. RÉMY, *Classical and non-linearity properties of Kac-Moody lattices*. To appear in Proc. «Rigidity in Dynamics and Geometry» (Newton Institute 2000), M. Burger and A. Iozzi eds, Springer, 2002
- [Ré4] B. RÉMY, *Kac-Moody groups: split and relative theories. Lattices*. To appear in Proc. «Groups: Geometric and Combinatorial Aspects» (Bielefeld 1999), London Math. Soc. Lecture Notes Series
- [Ro1] M. RONAN, Lectures on buildings. Academic press, 1989
- [Ro2] M. RONAN, *Local isometries of twin buildings*. Math. Zeitschrift **234** (2000) 435-455
- [RT] M. RONAN, J. TITS, *Building buildings*. Math. Ann. **278** (1987) 291-306
- [S] J.-P. SERRE, Arbres, amalgames, SL_2 . Astérisque **46**, 1977
- [T1] J. TITS, *Ensembles ordonnés, immeubles et sommes amalgamées*. Bull. Soc. Math. Belg. **38** (1986) 367-387
- [T2] J. TITS, *Uniqueness and presentation of Kac-Moody groups over fields*. Journal of Algebra **105** (1987) 542-573
- [T3] J. TITS, *Théorie des groupes*. Résumé de cours, annuaire Collège de France (1989), 81-95
- [T4] J. TITS, *Théorie des groupes*. Résumé de cours, annuaire Collège de France (1990), 87-103
- [T5] J. TITS, *Twin buildings and groups of Kac-Moody type*. In Groups, Combinatorics & Geometry (Proc. LMS Symposium on Groups and Combinatorics, Durham, July 1990), M. Liebeck and J. Saxl eds, LMS Lecture Notes Series **165** (1992), Cambridge UP, 249-286

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