On the symplectic connections with a parallel Ricci curvature and on the holonomy of the Einstein pseudo- and parakähler manifolds.

Charles BOUBEL

Institut Fourier, UMR 5582, B.P. 74, 38402 Saint Martin d'Hères Cedex, France

March 29, 2002

Prépublication de l'Institut Fourier n⁰ 562 (2002)

http://www-fourier.ujf-grenoble.fr/prepublications.html

Abstract. A symplectic connection on a symplectic manifold, unlike the Levi-Civita connection on a Riemannian manifold, is not unique. However, some spaces admit a canonical one (symmetric symplectic spaces, Kähler manifolds...); besides, some "preferred" symplectic connections can be defined in some situations (see [BC99]). Theses facts motivate a study of the symplectic connections *inducing a parallel Ricci tensor.* This paper gives the possible forms of the Ricci curvature on such manifolds and gives a decomposition theorem (linked with the holonomy decomposition) for them. As a first corollary of this result, we give then a classification of the Einstein non Ricci-flat manifolds, in relation with their holonomy group. This classification has a particular interest for the Einstein Kähler, pseudo-Kähler and parakähler manifolds.

Keywords: Symplectic connection, Ricci curvature, Holonomy, Kähler manifolds, parakähler manifolds, Einstein manifolds, pseudo-Riemannian manifolds.

AMS Classification: 53B05, 53B30, 53B35, 53C25, 53C55.

Introduction and motivation.

The goal of this work is twofold.

(a) On a Riemannian or pseudo-Riemannian manifold is defined the Levi-Civita connection. The symplectic analog is the following. Let $(\mathcal{M},\omega)$ be a symplectic manifold; a connection D on $\mathcal{M}$ is said to be symplectic when:

- D is torsion-free: for every vectorfields x and y, $D_x y - D_y x - [x,y] = 0$,
- the symplectic form is parallel for D : $D\omega = 0$.

To such a connection D is associated its (3,1)-curvature tensor R and its Ricci curvature tensor, here denoted by ric. Let us recall ric is the bilinear symmetric form defined on each tangent space by: $\mathrm{ric}(u,v) = \mathrm{tr}\, R(u,.)v$. Unlike in the (pseudo-)Riemannian situation, the set of symplectic connections is an affine space of infinite dimension (see 1**(b)** below).

In case $\mathcal{M}$ is a *symmetric* symplectic space (in the natural sense introduced in [Lo69]; see also [Bi98a] for the symplectic case), it has a *canonical* connection, which turns out to be symplectic. This connection is symmetric, so its Ricci curvature is in particular parallel.

On a general symplectic manifold, F.Bourgeois and M.Cahen have introduced in [BC99] a variational principle pointing out so-called "preferred" symplectic connections. The corresponding field equations are:

$$D_x \,\mathrm{ric}(y,z) + D_y \,\mathrm{ric}(z,x) + D_z \,\mathrm{ric}(x,y) = 0.$$

In particular, symplectic connections the Ricci curvature of which is parallel, *i.e.* such that $D\,\mathrm{ric} = 0$, are therefore preferred. More generally, they have a specific interest in this theory and were soon studied by M.Cahen and al. in [CGR00]. Note also that the canonical connection of a symmetric symplectic manifold is thus preferred.

Note too that, in case $\mathcal{M}$ is a (pseudo-)Riemannian manifold carrying a parallel symplectic form —*e.g.* a Kähler manifold—, the Levi Civita connection is symplectic with respect for it. We show a specific result in this framework (see **(b)** below).

Besides, Riemannian manifolds the Ricci curvature of which is parallel, shortly called here Ricci-parallel, are, at least locally, products of Einstein manifolds. Pseudo-Riemannian Ricci-parallel manifolds admit an analogous, though slightly different decomposition; see [BBB01]. We show here a similar result for Ricci parallel symplectic connections (theorem 1 p.5). It shall be noticed that the algebraic part of the result is the same as in the pseudo-Riemannian case, the geometrical consequence being weaker in general.

(b) If $(\mathcal{M}, g)$ is a (pseudo-)Riemannian manifold, theorem 1 enables then to *classify the possible structures of the algebra* $\mathfrak{so}(g)^{\mathfrak{h}}$ of the anti selfadjoint endomorphisms, stable under the action of the holonomy group, in case $(\mathcal{M}, g)$ is *Einstein non Ricci-flat.* This work is done by theorem 2 page 16. In particular, it classifies the *pseudo-Kähler* and *parakähler* Einstein non Ricci-flat manifolds.

The structure of the article is the following. After some lemmas and remarks given in section 1, theorem 1 is stated and commented in section 2, then proven in section 3. Section 4 gives a refinement of the decomposition obtained in theorem 1 and studies the subfactors. This leads to theorem 2, which is stated in section 5 and proven in section 6. Finally section 7 provides some examples and last remarks about both theorems.

Notations. On a symplectic manifold with a symplectic connection $(\mathcal{M}, \omega, D)$, we will denote by ric the Ricci tensor and by Ric the ω-anti selfadjoint endomorphism induced by ric, *i.e.* the endomorphism such that $\mathrm{ric}(.,.) = \omega(., \mathrm{Ric}\,.)$. We denote by H the holonomy group of $(\mathcal{M}, D)$, and classical Lie algebras by old German letters: $\mathfrak{so}$, $\mathfrak{sp}$, as well as the Lie algebra of the holonomy group: $\mathfrak{h}$.

1 Elementary facts about symplectic connections.

We need some basic facts in the following; pointing them out here together will also make symplectic connections more familiar.

(a) As hinted at above, the properties satisfied by a symplectic connection D are those that define the Levi-Civita connection of a Riemannian or pseudo-Riemannian metric g, if you replace ω by g. On any symplectic manifold, such a connection exists but it is *not* unique. The space of the symplectic connections associated with a given form ω is parametrized by $S^3\mathrm{T}^*\mathcal{M}$; let us remind the

Proposition 1 *let D be a symplectic connection on $(\mathcal{M}, \omega)$, then a connection Δ is symplectic* iff *:* $\omega(D.\cdot, \cdot) - \omega(\Delta.\cdot, \cdot) \in S^3\mathrm{T}^*\mathcal{M}$.

Proof. It is a straightforward remark, see [Li83] p.48.

(b) The curvature tensor R satisfies the usual algebraic properties :

- $R(x,y) = -R(y,x)$,
- $\omega(R(x,y)z,t) = \omega(R(x,y)t,z)$ *i.e.* all the $R(x,y)$ are ω-anti selfadjoint,
- $R(x,y).z + R(y,z).x + R(z,x).y = 0$ "Bianchi identity"

In the (pseudo-)Riemannian situation, an additional relation involving R and the metric g then follows:

$$g(R(x,y).z,t) = g(R(z,t).x,y) \tag{1}$$

It is *not* true with R and ω, ω being an *alternate* form. However, notice that, provided all the $R(x,y)$ for $x,y \in \mathrm{T}_p\mathcal{M}$ are anti selfadjoint with respect for a bilinear symmetric form g, we get (1) for R and g, whether g is degenerate or not. The proof does not need it, see [Mi63] p.54 for example.

In particular, we get the following little

Lemma 1 *If g is a parallel symmetric bilinear form on $(\mathcal{M},D)$, R and g satisfy (1).*

For example if ric is parallel, (1) holds for ric and more generally for all the bilinear symmetric forms $\omega(.,\mathrm{Ric}\,P(\mathrm{Ric}^2).)$ where P is a polynomial.

Note. Relation (1) between R and the metric g is one of the essential tools giving the pseudo-Riemannian result [BBB01]. So is it here: theorem 1 is based on the fact that R and ric satisfy (1).

(c) In the (pseudo-)Riemannian situation, ric is the only non-trivial invariant trace of R. In the symplectic case, there is *a priori* another one : $u,v \mapsto \mathrm{tr}_\omega[\omega(R(.,.)u,v)]$. However, it turns out that it is the same, up to a scalar; let us recall the (classical) little

Lemma 2 *If (M,ω,D) is a symplectic manifold with a symplectic connection:*

$$\mathrm{tr}_\omega[\omega(R(.,.)u,v)] = -2ric(u,v).$$

Proof. It follows from Bianchi identity; as the lemma will be useful here, let us recall its proof.

Let $2n$ be the dimension of $\mathcal{M}$, p be a point in $\mathcal{M}$ and $(e_i)_{i=1}^{2n}$ be a basis of $\mathrm{T}_p\mathcal{M}$ such that: $\omega = \sum_{i\leq n} e_i^* \wedge e_{n+i}^*$. For a and b in $\mathrm{T}_p\mathcal{M}$:

$$\begin{aligned}
\mathrm{tr}_\omega[\omega(R(\cdot,\cdot)a,b)] &= \textstyle\sum_{i\leq n} \omega(R(e_i,e_{n+i})a,b) - \omega(R(e_{n+i},e_i)a,b) \\
&= 2\textstyle\sum_{i\leq n} \omega(R(e_i,e_{n+i})a,b) \\
&= 2\left(\textstyle\sum_{i\leq n} \omega(R(a,e_{n+i})e_i,b) + \omega(R(e_i,a)e_{n+i},b)\right) \quad \text{(Bianchi Identity)} \\
&= 2\left(\textstyle\sum_{i\leq n} \omega(R(a,e_{n+i})b,e_i) - \omega(R(a,e_i)b,e_{n+i})\right) \\
&= -2\,\mathrm{tr}[R(a,\cdot)b] \\
&= -2\,\mathrm{ric}(a,b)
\end{aligned}$$

□

(d) The endomorphism Ric in this framework.

Eventually, a last preliminary work is necessary before stating the theorem. It is some classical linear algebra but has to be precisely stated here.

Let p be a point of $\mathcal{M}$; Ric being parallel, its minimal polynomial (*i.e.* the unitary generator of the ideal of the polynomials P of $\mathbb{R}[X]$ such that $P(\mathrm{Ric}) = 0$) is defined independently of the point. Now $\mathrm{Ric}_{|p} \in \mathfrak{sp}(\omega_{|p})$, so we can apply to the complexified endomorphism $\mathrm{Ric}^{\mathbb{C}}$ of $\mathrm{T}_p\mathcal{M} \otimes \mathbb{C}$ the following standard

Lemma 3 *Let (E, ω) be a complex vectorspace endowed with a nondegenerate alternate form ω and U in $\mathfrak{sp}(\omega)$. The minimal polynomial μ of U then satisfies: $\mu(X) = \pm\mu(-X)$. There thus exists an $L \subset \mathbb{C}$ such that $L \cup (-L) = \{$nonzero eigenvalues of $U\}$ and $L \cap (-L) = \emptyset$; with such a L:*

$$E = \ker U^{\alpha_0} \overset{\perp}{\oplus} \left(\underset{\lambda \in L}{\overset{\perp}{\oplus}} (\ker(U - \lambda\,\mathrm{Id})^{\alpha_\lambda} \oplus \ker(U + \lambda\,\mathrm{Id})^{\alpha_\lambda}) \right),$$

where α_λ is the common power of $(X - \lambda)$ and of $(X + \lambda)$ in μ. The decomposition is orthogonal with respect for ω and each space $\ker(U \pm \lambda\,\mathrm{Id})$ is ω-totally isotropic. Here α_0 may be zero.

Furthermore, Ric being real, its minimal polynomial μ is also invariant under complex conjugation; so taking for example $\Lambda = \{$ eigenvalues of Ric$\} \cap (\mathbb{R}^+ \times i\mathbb{R}^+) \subset \mathbb{C}$, we get:

$$\mu = \prod_{\lambda \in \Lambda} P_\lambda^{\alpha_\lambda} \quad \text{with:} \begin{cases} P_0 = X & \text{appearing if } 0 \in \Lambda \\ P_\lambda = (X-\lambda)(X+\lambda) & \text{if } \lambda \in \mathbb{R}^* \cup i\mathbb{R}^* \\ P_\lambda = (X-\lambda)(X+\lambda)(X-\overline{\lambda})(X+\overline{\lambda}) & \text{otherwise.} \end{cases} \tag{2}$$

and the corresponding decomposition of $\mathrm{T}_p\mathcal{M}$:

$$\mathrm{T}_p\mathcal{M} = \underset{\lambda \in \Lambda}{\overset{\perp}{\oplus}} \ker(P_\lambda^{\alpha_\lambda}(\mathrm{Ric})). \tag{3}$$

Remark. This the finest ω-orthogonal decomposition of $\mathrm{T}_p\mathcal{M}$ that is stable under the action of the centralisor of Ric. However, under this action and for example for $\lambda \in \mathbb{R}^*$:

- $\ker(P_\lambda^{\alpha_\lambda}(\mathrm{Ric})) = \ker(\mathrm{Ric} - \lambda\,\mathrm{Id})^{\alpha_\lambda} \oplus \ker(\mathrm{Ric} + \lambda\,\mathrm{Id})^{\alpha_\lambda}$; each factor being stable but ω-totally isotropic.
- $\ker(\mathrm{Ric} - \lambda\,\mathrm{Id})^{\alpha_\lambda}$ and $\ker(\mathrm{Ric} + \lambda\,\mathrm{Id})^{\alpha_\lambda}$ are irreducible *iff* $\alpha_\lambda = 1$.

2 The first theorem.

A Riemannian manifold with parallel Ricci curvature is, at least locally, a product of Einstein manifolds (its only a remark, see [BBB01], pp.2 and 3). Let us recall a manifold is said Einstein if ric is proportional to the metric. In our situation, this notion has no sense, since ω is alternate and ric symmetric.

Nevertheless, being a local product of Einstein Riemannian manifolds can be stated in other terms: ric is parallel and the minimal polynomial of Ric has simple roots in $\mathbb{C}$ (then necessarily in $\mathbb{R}$, g being positive definite). That statement has a sense in our symplectic situation. Is it true? Yes, except possibly for the root zero. It is the same result as for a pseudo-Riemannian connection, the proof being quite different: see [BBB01].

Remark. Before stating the theorem, let us recall a classical fact linking holonomy-stable subspaces with some foliations. If $(\mathcal{M}, D)$ is a manifold endowed with a torsion-free connection D, if p is a point of $\mathcal{M}$ and if the holonomy group stabilizes a subspace A of $\mathrm{T}_p\mathcal{M}$, then A can be extended by parallel transport to a (parallel) distribution on $\mathcal{M}$. The connection being torsion-free, this distribution is integrable; the leaves of the integral foliation are moreover totally geodesic.

Theorem 1 *Let (M, ω, D) be a symplectic manifold with a symplectic connection, the Ricci curvature* ric *of which is parallel. Let μ be the minimal polynomial of* Ric *and $\mu = \prod_{\lambda\in\Lambda} P_\lambda^{\alpha_\lambda}$ the decomposition (2) of μ given page 4. For simplicity of the statement, we set $0 \in \Lambda$ and authorize α_0 to be null. Let us also denote by M_λ the (parallel) distribution* $\ker(P_\lambda^{\alpha_\lambda}(\mathrm{Ric}))$ *and let us denote by $\mathcal{M}_\lambda$ the integral leaf of M_λ through p. Then:*

(i) *For each $\lambda \neq 0$, α_λ is equal to one and: $\alpha_0 \leq 2$.*

(ii) *Each $\omega_\lambda = \omega_{|\mathrm{T}\mathcal{M}_\lambda}$ is nondegenerate so, with $D_\lambda = D_{|\mathrm{T}\mathcal{M}_\lambda}$, $(\mathcal{M}_\lambda, \omega_\lambda, D_\lambda)$ is a symplectic manifold with a symplectic connection. Moreover, denoting by f the canonical local diffeomorphism $\prod_\lambda \mathcal{M}_\lambda \to \mathcal{M}$, defined in a neighbourhood of p, there exists a (2,1)-tensor S on this neighbourhood such that:*

$$f\ :\prod_\lambda(\mathcal{M}_\lambda, \omega_\lambda, D_\lambda) \to (\mathcal{M}, \omega, D - S).$$

If $((\mathcal{M}_\nu, \omega_\nu, D_\nu)_{\nu\in N}, f', S')$ is a triple satisfying the above isomorphism, and such that the minimal polynomial of Ric_ν *is $P_\nu^{\alpha_\nu}$, then $N = \Lambda$ and the triple is equal to $((\mathcal{M}_\lambda, \omega_\lambda, D_\lambda)_{\lambda\in\Lambda}, f, S)$, up to composition with automorphisms of the $(\mathcal{M}_\lambda, \omega_\lambda, D_\lambda)$.*

Moreover, S satisfies the following conditions:

- *$\omega(S(.,.),.)$ is completely symmetric,*
- *at all point q where it is defined, $S_{|q} = \pi_0^*(S^0_{|q})$ where $S^0_{|q}$ is a (2,1)-tensor on M_0 and π^0 the canonical projection $\mathrm{T}_q\mathcal{M} = \oplus_\lambda M_\lambda \to M_0$.*
- $\forall(x, y) \in \mathrm{T}_q\mathcal{M}$, $\mathrm{tr}[z \mapsto D_z S(x, y)] - \mathrm{tr}[z \mapsto S(x, S(y, z))] = 0$,
- $\mathrm{Im}\,\mathrm{Ric} \subset \ker S$,

the last property being a consequence of the third one.

Let us do some comments before proving the statement.

(a) The first point of the theorem is a purely "pointwise" consequence of the algebraic properties of the curvature tensor R.

The second one is a consequence of an adaptation of de Rham's decomposition theorem of Riemannian manifolds, see Proposition 2 below.

Point **(i)** will then give information on the factors $\mathcal{M}_\lambda$ given by point **(ii)**: see section 4, in particular Proposition 4 page 13.

Proposition 2 *Let $(\mathcal{M}, \omega, D)$ be a symplectic manifold with a symplectic connection and $p \in \mathcal{M}$. Suppose that the restricted holonomy group H^0 preserves an ω-orthogonal decomposition:*

$$\mathrm{T}_p\mathcal{M} = \overset{\perp}{\underset{0\leq i\leq k}{\oplus}} M_i$$

of $\mathrm{T}_p\mathcal{M}$. *Then for each* i, M_i *induces by parallel transport a parallel, thus integrable, distribution on* $\mathcal{M}$, *also denoted by* M_i.

Let $(\mathcal{M}_i)$ *be the integral manifold through* p *of the distribution* M_i. *Then:*

(i) *The* $(\mathcal{M}_i, \omega_i, D_i) = (\mathcal{M}_i, \omega_{|\mathrm{T}\mathcal{M}_i}, D_{|\mathrm{T}\mathcal{M}_i})$ *are symplectic manifolds with a symplectic connection.*

(ii) *The unique local diffeomorphism preserving the foliations induced by the* M_i *and equal to identity on the* $\mathcal{M}_i$ *identifies, on a suitable neighbourhood of* p, $\mathcal{M}$ *to* $\prod_i \mathcal{M}_i$. *On this neighbourhood:* $\omega = \prod_i \omega_i$.

(iii) *With this local identification* $\mathcal{M} \simeq \prod_i \mathcal{M}_i$, *there is* S *a (unique) (2,1)-tensor on* $\mathcal{M}$ *such that:* $D = (\prod_i D_i) + S$.

(iv) *The restricted holonomy group* H^0 *is the direct product:* $H^0 = \prod_i H_i^0$ *where* H_i^0 *is the subgroup of* H^0 *acting trivially on the* M_j *for* $j \neq i$.

Moreover, S *satisfies the following conditions:*

- $\omega(S(.,.),.)$ *is symmetric,*
- *at all point* q *where it is defined,* $S_{|q}$ *can be factored as* $S_{|q} = \sum_i (\pi_i)^*(S_q^i)$ *where* S_q^i *is a (2,1)-tensor on* M_i *and* π_i *the projection* $\mathrm{T}_q\mathcal{M} = \oplus_j M_j \to M_i$.
- *For each* i *and each* $q \in \mathcal{M}_i$, $S_q^i = 0$, i.e.*:* $S_{|(\mathrm{T}\mathcal{M}_i)^2}$ *is null on* $\mathcal{M}_i$.

This proposition is, adapted to a symplectic connection, the local (and easy) part of de Rham's theorem. Its proof, as well as that of Theorem 1, will be postponed to section 3 page 7. Two points of the Riemannian theorem fail here to be true:

- The result is weaker —and a little deceiving— because $(\mathcal{M}, D)$ is not a product for the affine structure: $\mathcal{M} \simeq \prod_i (\mathcal{M}_i, \omega_i)$ but $D \neq \prod_i D_i$. It follows from the non-uniqueness of a symplectic connection on a symplectic manifold.
- For a Riemannian manifold $\mathcal{M}$, $\mathrm{T}_p\mathcal{M}$ is the sum of a trivial subrepresentation of H and of a sum of irreducible subrepresentations; a consequence is the uniqueness of this decomposition. It is not the case here, since $\mathrm{T}_p\mathcal{M}$ may admit reducible-indecomposable factors. So in general there does not exist any *canonical* decomposition of $\mathrm{T}_p\mathcal{M}$ under the action of H (or of H^0).

Nevertheless, in case $(\mathcal{M}, \omega, D)$ is a *symmetric* symplectic space, a quite unexpected decomposition result holds, see [BCG97], theorems 2.3 and 2.12.

(b) In general, the local symplectomorphism f of the theorem is *not* an isomorphism of affine structure from $(\mathcal{M}, D)$ on $\prod_\lambda (\mathcal{M}_\lambda, D_\lambda)$. However, it *is* one in the case Ric is nondegenerate; the following decomposition holds:

Corollary 1 *Let* (M, ω, D) *be a symplectic manifold with a symplectic connection, the Ricci curvature* ric *of which is parallel and nondegenerate. Let* μ *be the minimal polynomial of* Ric *and* $\mu = \prod_{\lambda \in \Lambda} P_\lambda^{\alpha_\lambda}$ *the decomposition* (2) *of* μ *given in section 1 page 4. Then:*

(i) *For each* λ, α_λ *is equal to one (and* $0 \notin \Lambda$ *since* ric *is nondegenerate).*

(ii) *There exists a unique family* $((\mathcal{M}_\lambda, \omega_\lambda, D_\lambda))_{\lambda \in \Lambda}$ *of symplectic manifolds with a symplectic connection such that*

- *for each* λ*, the minimal polynomial of* $\mathrm{Ric}_{\mathcal{M}_\lambda}$ *is* P_λ*,*
- $(\mathcal{M},\omega,D)$ *is locally symplectomorphic and affinely equivalent to* $\prod_\lambda(\mathcal{M}_\lambda,\omega_\lambda,D_\lambda)$*.*

(iii) *If* $(\mathcal{M},\omega,D)$ *is moreover geodesically complete and simply connected, the same result holds globally.*

Proof. Points **(i)** and **(ii)** are simply the case "$\alpha_0 = 0$, $\mathcal{M}_0$ reduced to a point" of theorem 1: then $S = 0$, what gives the result. □

We can also easily understand autonomously the reason why it works. In that case indeed, $(\mathcal{M},\mathrm{ric})$ turns out to be a pseudo-Riemannian manifold (which is moreover Einstein with constant 1 by definition). Ric being parallel, the decomposition

$$\mathrm{T}_p\mathcal{M} = \overset{\perp}{\underset{\lambda\in\Lambda}{\oplus}} \ker(P_\lambda^{\alpha_\lambda}(\mathrm{Ric}))$$

is holonomy stable. Applying Wu's theorem, the pseudo-Riemannian generalization of de Rham's theorem (see [Wu67]), we get that $\mathcal{M}$ is isomorphic to the Riemannian product of the factors $\mathcal{M}_\lambda$. Besides, the symplectic connection D is torsion-free and satisfies $D\,\mathrm{ric} = 0$, so it is the Levi-Civita connection of the metric ric. Consequently, the Riemannian product is also a affine morphism $(\mathcal{M},D) \simeq \prod_\lambda(\mathcal{M}_\lambda,D_\lambda)$.

Point **(iii)** is an immediate consequence of the global part of Wu's theorem [Wu67], applied to the pseudo-Riemannian manifold $(\mathcal{M},\mathrm{ric})$. □

(c) Conversely, if $(\mathcal{M}_i,\omega_i,D_i)_{i=0}^k$ are symplectic manifolds with Ricci-parallel symplectic connections, with Ric_i nondegenerate except for $i = 0$, then a manifold of the type

$$\left(\textstyle\prod_i(\mathcal{M}_i,\omega_i), \prod_i D_i + S\right)$$

where S is as described in the theorem, is Ricci-parallel. It is an immediate consequence of proposition 1 combined with lemma 6 below.

3 Proof of Theorem 1.

Proof Proposition 2 page 5. We have to check that the Riemannian proof (see [Ko-No] pp.179 sq.) remains valid or can be adapted at each step. Let us do it for $k = 2$, the general case comes then by induction. We denote M_1, $\mathcal{M}_1$, M_2 and $\mathcal{M}_2$ by A, $\mathcal{A}$, B and $\mathcal{B}$ respectively. For another point q of $\mathcal{M}$, $\mathcal{A}_q$ (resp. $\mathcal{B}_q$) will stand for the integral leaf of A (resp. B) through q.

(i) At p, $\omega_{|A}$ and $\omega_{|B}$ are nondegenerate. Now $\mathcal{A}$ and $\mathcal{B}$ being integral leaves of *parallel* distributions and ω being parallel, $\omega_{|\mathrm{T}\mathcal{A}}$ and $\omega_{|\mathrm{T}\mathcal{B}}$ are nondegenerate; let us denote them by $\omega^{\mathcal{A}}$ and $\omega^{\mathcal{B}}$. $\mathcal{A}$ is totally geodesic, so the restriction $D^{\mathcal{A}}$ to $\mathrm{T}\mathcal{A}$ of the connection D has values in $\mathrm{T}\mathcal{A}$, so is the connection induced by D on the submanifold $\mathcal{A}$. Hence similarly for $\mathcal{B}$. Eventually, as $D\omega = 0$, $D^{\mathcal{A}}\omega^{\mathcal{A}} = 0$ and $(\mathcal{A},\omega^{\mathcal{A}},D^{\mathcal{A}})$ is (locally) a symplectic submanifold of $\mathcal{M}$, with a symplectic connection (hence also for $(\mathcal{B},\omega^{\mathcal{B}},D^{\mathcal{B}})$).

(ii) The fact that $\mathcal{M}$ is locally canonically diffeomorphic to $\mathcal{A}\times\mathcal{B}$ is obvious and purely differential, see [Ko-No], lemma p.182, for a formal proof. Note that, unlike for a Riemannian manifold, the diffeomorphism is not global in general since $(\mathcal{M}, D)$ may fail to be geodesically complete. We can then take local coordinates of $\mathcal{M}$ of the form $((a_i)_{i=1}^{d_\mathcal{A}}, (b_i)_{i=1}^{d_\mathcal{B}})$ such that, at every point q: $A_q = \mathrm{span}(\partial/\partial a_i)_{i=1}^{d_\mathcal{A}}$ and $B_q = \mathrm{span}(\partial/\partial b_i)_{i=1}^{d_\mathcal{B}}$.

Proving ω is equal to the product form $\omega^\mathcal{A} \times \omega^\mathcal{B}$ is showing: for each (i, j, k),

$$L_{\partial/\partial b_i}[\omega(\partial/\partial a_j, \partial/\partial a_k)] = 0.$$

It follows ([Ko-No], prop. 5.2 p.182), from the fact that D is torsion-free. Indeed for each (i, j): $D_{\partial/\partial b_i}(\partial/\partial a_j) = D_{\partial/\partial a_j}(\partial/\partial b_i)$. Now, as the distributions A and B are parallel:

$$D_{\partial/\partial b_i}(\partial/\partial a_j) \in A \quad \text{and:} \quad D_{\partial/\partial a_j}(\partial/\partial b_i) \in B,$$

so:

$$D_{\partial/\partial b_i}(\partial/\partial a_j) = D_{\partial/\partial a_j}(\partial/\partial b_i) = 0.$$

Then:

$$\begin{aligned} &L_{\partial/\partial b_i}[\omega(\partial/\partial a_j, \partial/\partial a_k)] = \\ &\quad \underbrace{(D_{\partial/\partial b_i}\omega)}_{=0}(\partial/\partial a_j, \partial/\partial a_k) + \omega(\underbrace{D_{\partial/\partial b_i}(\partial/\partial a_j)}_{=0}, \partial/\partial a_k) + \omega(\partial/\partial a_j, \underbrace{D_{\partial/\partial b_i}(\partial/\partial a_j)}_{=0}) = 0. \end{aligned}$$

(iii) and the properties of *S*. The product connection $D^\mathcal{A} \times \mathrm{D}^\mathcal{B}$ on $\mathcal{M}^\mathcal{A} \times \mathcal{M}^\mathcal{B}$ is a symplectic connection. Indeed, the local product structure of $(\mathcal{M}, \omega)$ induces a local diffeomorphism between $\mathcal{A}_p$ and $\mathcal{A}_q$ for each point q, preserving moreover ω and mapping $D^\mathcal{A}$ on $(D^\mathcal{A} \times \mathrm{D}^\mathcal{B})_{|\mathrm{T}\mathcal{A}_q}$ by definition of $D^\mathcal{A} \times \mathrm{D}^\mathcal{B}$. As $D^\mathcal{A}\omega = 0$ on $\mathcal{A}_p$, $(D^\mathcal{A} \times \mathrm{D}^\mathcal{B})_{|\mathrm{T}\mathcal{A}_q}\omega = 0$. So by proposition 1, there exists a (2,1)-tensor S on $\mathcal{A}_q$ such that:

- $D_{|\mathrm{T}\mathcal{A}_q} = (D^\mathcal{A} \times \mathrm{D}^\mathcal{B})_{|\mathrm{T}\mathcal{A}_q} + S$
- $\omega(S(.,.),.)$ is symmetric.

By construction, $S = 0$ if $q \in \mathcal{A}_p$. So for $\mathcal{B}_q$ and the result follows, with the announced properties of S.

(iv) It comes from a purely differential result, the "lasso lemma" ([KoNo69] p.281), combined ([KoNo69] p.183) with the fact that the foliations $\mathcal{A}$ and $\mathcal{B}$ are complementary, holonomy stable and orthogonal with respect for a parallel nondegenerate bilinear form (ω). The bilinear form may be supposed symmetric (as in [KoNo69]) or alternate (as here), it does not matter. [*Note:* that point can also be viewed as a consequence of Ambrose-Singer theorem.] □

In addition to proposition 2, we will also use three more lemmas. Point **(i)** of the theorem is a consequence of a technical lemma we can state autonomously.

Lemma 4 *Let p be a point of $\mathcal{M}$, $U \in \mathfrak{sp}(\omega_{|p})$ (*i.e.* U is an ω-anti selfadjoint endomorphism of $\mathrm{T}_p\mathcal{M}$), commuting with all the $R(x, y)$ for $x, y \in \mathrm{T}_p\mathcal{M}$. Let us take $a, b \in \mathrm{T}_p\mathcal{M}$ with $b \in \mathrm{Im}\, U$. The bilinear form $\omega(R(.,.)a, b)$ is skew-symmetric; let us denote by $A_{a,b}$ the ω-selfadjoint endomorphism such that: $\omega(R(.,.)a, b) = \omega(., A_{a,b}.)$. Then:*

$$A_{a,b} = -U \circ R(a, c) = -R(a, c) \circ U, \ \textit{where } c \textit{ is any antecedent of } b \textit{ by } U.$$

Proof of the lemma. Let us simply write here $A = A_{a,b}$ and let us take c such that $U.c = b$. As $U \in \mathfrak{sp}(\omega_{|p})$, the bilinear form $u : (x,y) \mapsto \omega(x, Uy)$ is symmetric; as all the $R(x,y)$ are supposed to commute with U, notice they are all u-anti selfadjoint:

$$\begin{aligned} u(R(x,y)z,t) &= \omega(R(x,y)z, Ut) \\ &= \omega(R(x,y)Ut, z) \\ &= \omega(UR(x,y)t, z) \\ &= -\omega(z, UR(x,y)t) \\ &= -u(z, R(x,y)t). \end{aligned}$$

Consequently, by Lemma 1 page 3, (1) holds for u:

$$\forall x,y,z,t,\ u(R(x,y).z,t) = u(R(z,t).x,y).$$

To prove the lemma it is sufficient to check: $\forall x,y \in \mathrm{T}_p\mathcal{M}$, $\omega(x, Ay) = \omega(x, U(R(a,c)y))$. Let x, y be any two vectors in $\mathrm{T}_p\mathcal{M}$. Then:

$$\begin{aligned} \omega(x, Ay) &= \omega(R(x,y)a, b) \\ &= \omega(R(x,y), a, Uc) \\ &= u(R(x,y)a, c) \quad \text{by definition of } u \\ &= u(R(a,c)x, y) \quad \text{by (1)} \\ &= -u(x, R(a,c)y) \quad R(a,c) \text{ being } u\text{-anti selfadjoint} \\ &= -\omega(x, UR(a,c)y) \quad \text{by definition of } u. \end{aligned}$$

□

Let us also recall the classical remark:

Lemma 5 *Let E be a real or complex vectorspace, $< .,. >$ a reflexive,* i.e. *symmetric or skew-symmetric form on E and U a $< .,. >$-anti selfadjoint endomorphism of E. Let $U = S + T$ be the decomposition of U into its semi-simple and nilpotent parts (unique such decomposition with $ST = TS$). Then S and T are $< .,. >$-anti selfadjoint.*

For point **(ii)** we will also need the following (classical) little

Lemma 6 *Let D and D' two symplectic connections on a symplectic manifold $(\mathcal{M}, \omega)$ and S the tensor such that $D' = D + S$. Let us denote by* ric *and* ric$'$ *the Ricci curvatures induced by D and D' respectively and by S_x the endomorphism $S(x,.)$. Then:*

$$\mathrm{ric}'(x,y) = \mathrm{ric}(x,y) - \mathrm{tr}[z \mapsto (D_z S)(x,y)] + \mathrm{tr}\, S_x S_y.$$

Proof. It is sufficient to do the proof with vectorfields which are coordinate-vectorfields for some normal coordinate system at some point p in $\mathcal{M}$. For two distinct such vectors u and v: $D_u v = D_v u$ and, at p, $D_u v = 0$. With such vectors:

$$R'(x,z)y = (D+S)_z(D+S)_x y - (D+S)_x(D+S)_z y \qquad \text{and:}$$

$$\begin{aligned} (D+S)_z(D+S)_x y &= D_z D_x y + D_x S_z y + S_x D_z y + S_x S_z y \\ &= D_z D_x y + (D_x S)_z y + S_{D_x z} y + S_z D_x y + S_x D_z y + S_x S_z y \\ &= D_z D_x y + (D_x S)_z y + S_x S_z y \end{aligned}$$

as $D_x z = D_x y = D_z y = 0$ at p. So:

$$R'(x,z)y = R(x,z)y + (D_x S)_z y - (D_z S)_x y + S_x S_z y - S_z S_x y, \qquad \text{thus:}$$

$$\mathrm{ric}'(x,y) = \mathrm{tr}[z \mapsto R'(x,z)y] = \mathrm{ric}(x,y) + \mathrm{tr}[z \mapsto (D_x S)_z y - (D_z S)_x y - S_x S_z y + S_z S_x y].$$

Now:

- $[z \mapsto S_z S_x y] = [z \mapsto S(z, S(x,y))] = S_{S(x,y)}$. But $\omega(S(.,.),.)$ is symmetric, so in particular: $\omega(S_u v, w) = \omega(S(u,v),w) = \omega(S(u,w),v) = -\omega(v, S(u,w)) = -\omega(v, S_u w))$ so the S_u are in $\mathfrak{sp}(\omega)$, thus trace-free. So $\mathrm{tr}[z \mapsto S_z S_x y] = 0$.
- For the same reason, $\mathrm{tr}(D_x S)_y = 0$. So by symmetry of S: $\mathrm{tr}[z \mapsto (D_x S)_z y] = \mathrm{tr}[(D_x S)_y] = 0$.
- By symmetry of S, $S_x S_z y = S_x S_y z$.

The result follows. □

We can now state the

Proof of the theorem.

(i) Let $\mathrm{Ric} = S + T$ be the decomposition of Ric into its semi-simple and nilpotent parts. As S and T are polynomials of Ric, they are themselves parallel. Let p be a point in $\mathcal{M}$ and $b \in \mathrm{Im}\, T_p$, say $b = T(c)$. By Lemma 5, $T_p \in \mathfrak{sp}(\omega_p)$; T being parallel, it commutes with all the $R(x,y)$ for $x, y \in \mathrm{T}_p\mathcal{M}$, we can therefore apply Lemma 4. Combined with Lemma 2 it gives:

$$\begin{aligned} \forall a \in \mathrm{T}_p\mathcal{M},\ \mathrm{ric}(a,b) &= \mathrm{tr}_\omega[R(.,.)a, b] \quad \text{(Lemma 2)} \\ &= -\frac{1}{2}\,\mathrm{tr}[R(a,c) \circ T_p] \quad \text{(Lemma 4).} \end{aligned}$$

But T is parallel so it commutes with $R(a,c)$; thus, T being nilpotent, so is $R(a,c) \circ T_p$. So $R(a,c) \circ T_p$ is trace-free, what means that: $\forall a \in \mathrm{T}_p\mathcal{M}$, $\mathrm{ric}(a,b) = 0$, that is to say: $b \in \ker \mathrm{Ric}_p$. So we get (at any point):

$$\mathrm{Im}\, T \subset \ker \mathrm{Ric}\,.$$

That is the wanted result. Indeed if μ is the minimal polynomial of Ric you can write:

$$\mu = X^{\alpha_0}.\prod_\lambda (X - \lambda)^{\alpha_\lambda}$$

where λ runs over the set of the nonzero eigenvalues of Ric and where α_0 is the –possibly null– power of X in μ. Then Ric is nondegenerate on $\ker[\prod_\lambda (\mathrm{Ric} - \lambda\,\mathrm{Id})^{\alpha_\lambda}]$ so on this space: $\mathrm{Im}\, T = \{0\}$ *i.e.* $T = 0$ *i.e.* all the α_λ for λ a nonzero eigenvalue of Ric are 1. On $\ker S = \ker \mathrm{Ric}^{\alpha_0}$, T is equal to Ric so $\mathrm{Im}\, T \subset \ker T$ *i.e.* $\alpha_0 \leq 2$.

(ii) The decomposition and the tensor S are given by Proposition 2. The uniqueness of the triple $((\mathcal{M}_\lambda, \omega_\lambda, D_\lambda), f, S)$ comes by from that of the decomposition $\mathrm{T}_p\mathcal{M} = \oplus_\lambda M_\lambda$.

The first property of S comes from Proposition 1. Let us prove the factorization of S. On the similar integral manifold $\mathcal{M}_\lambda^q$ through any q, for $\lambda \neq 0$, the nondegenerate bilinear form $\mathrm{ric}_{\mathcal{M}^q_\lambda}$ is parallel for the product connection $(\prod_\lambda D_\lambda)_{|\mathrm{T}\mathcal{M}^q_\lambda}$ and for the original connection D of $\mathcal{M}$. So these connections are both equal to the Levi-Civita connection of $\mathrm{ric}_{\mathcal{M}^q_\lambda}$. So $S_{|\mathrm{T}\mathcal{M}^q_\lambda} = 0$. This gives the factorization of S.

The third property of S comes from Lemma 6 page 9 and from the fact that D and $D - S$ induce the same Ricci curvature. Indeed, let us denote by ric' the Ricci curvature of the product connection $\prod_\lambda D_\lambda$. By definition of the product connection, $\mathrm{ric}' = \mathrm{ric}$ on each $\mathcal{M}_\lambda$. Now ric' is parallel by construction and ric is parallel by assumption, so by parallel transport, $\mathrm{ric} = \mathrm{ric}'$ everywhere. Note that actually, S satisfies the third property *iff* D and $D - S$ have the same Ricci curvature.

This implies finally that: $\mathrm{Im}\,\mathrm{Ric} \subset \ker S$. To see it, we show the following

Claim: Let D' a symplectic connection on some integral manifold $\mathcal{M}^q_0$ of $M_0 = \ker \mathrm{Ric}^2$ through some point q, inducing the same Ricci curvature as D and let S' be the tensor such that $D' = D + S'$. Then: $\mathrm{Im}\,\mathrm{Ric} \subset \ker S'$.

Let us indeed choose normal coordinates based at q. Then, for (x, y, z) any triple of coordinates-vectors and ric being parallel:

$$2\,\mathrm{ric}(D'_x y, z) = L_x \,\mathrm{ric}(y, z) + L_y \,\mathrm{ric}(x, z) + L_z \,\mathrm{ric}(x, y),$$

by the same computations than those that give the expression of the Levi Civita connection of a metric g. So $\mathrm{ric}(D'_x y, z)$ is fixed *i.e.* is equal to $\mathrm{ric}(D_x y, z)$. Therefore, $\mathrm{ric}(S'(.,.),.) = 0$ or, equivalently: $\omega(S'(.,.), \mathrm{Im}\,\mathrm{Ric}) = 0$ by definition of Ric. By symmetry of S', it is again equivalent to: $S'(\mathrm{Im}\,\mathrm{Ric}, .) = 0$. So the claim, what completes the proof. □

4 Ricci decomposition and holonomy decomposition.

4.1 A refinement of the decomposition given by theorem 1.

The decomposition of $(\mathcal{M}, \omega, D)$ appearing in theorem 1 may be refined. Let us introduce a definition.

Definition 1 *A pseudo-Riemannian manifold is said* weakly *irreducible if the holonomy group does not stabilize any* nondegenerate *proper subspace.*

Remark. Obviously, the holonomy representation is *weakly* irreducible *iff* it does not admit any decomposition into a direct *orthogonal* sum of stable subspaces.

De Rham's theorem on the decomposition of the Riemannian manifolds into a product of irreducible ones admits a pseudo-Riemannian generalization, in fact nearly the best that could be expected, *i.e.* the elementary factors are weakly irreducible. We recall the result of [Wu67], appendix 1 p.389.

Theorem (de Rham, Wu) *Let $(\mathcal{M}, g)$ be a geodesically complete, simply connected Riemannian or pseudo-Riemannian manifold and $p \in \mathcal{M}$. We suppose the maximal trivial subspace M^0_p of H in* $\mathrm{T}_p\mathcal{M}$ *is nondegenerate. Then:*

(i) $\mathrm{T}_p\mathcal{M}$ *admits a decomposition, unique up to order:* $\mathrm{T}_p\mathcal{M} = \overset{\perp}{\underset{0 \le i \le k}{\oplus}} M^i_p$, *and H the decomposition: $H \simeq \prod_{1 \le i \le k} H_i$, where each H_i acts weakly irreducibly on each M^i_p and* *trivially on the* M_p^j *for* $j \neq i$.

(ii) $\mathcal{M}$ *is isometric to the Riemannian product* $\prod_{0 \leq i \leq k} \mathcal{M}_i$*, where each* $\mathcal{M}_i$ *is the maximal integral leaf through* p *of the parallel distribution* M^i *generated by* M_p^i. $\mathcal{M}_0$ *is flat.*

If $(\mathcal{M}, g)$ *is not supposed to be geodesically complete and simply connected, the same result holds, for the full holonomy group* H *as well as for the restricted group* H^0*, except that the isometry of point* **(ii)** *is only local.*

A consequence of this theorem in our situation is the following

Proposition 3 *Let* $(\mathcal{M}, g)$ *be a Riemannian or pseudo-Riemannian manifold and* $p \in \mathcal{M}$. *We suppose the maximal trivial subspace* M_p^0 *of* H *in* $\mathrm{T}_p\mathcal{M}$ *is nondegenerate, and denote by* $(\mathcal{M}, g) \simeq \prod_{0 \leq i \leq k} (\mathcal{M}_i, g_i)$ *Wu's decomposition of* $\mathcal{M}$.

Suppose $(\mathcal{M}, g)$ *admits a parallel and nondegenerate symplectic form* ω. *Then* ω_i*, the restriction of* ω *to* $\mathrm{T}\mathcal{M}_i$*, is nondegenerate and :*

$$(\mathcal{M}, g, \omega) \simeq \prod_i (\mathcal{M}_i, g_i, \omega_i).$$

Proof. We use here the notations introduced in Wu's theorem above. It is sufficient to show that the M_p^i are in direct ω-orthogonal sum: the statement follows by parallel transport. Let us denote by Ω the element of $\mathfrak{so}(\mathrm{ric})$ such that: $\omega = g(., \Omega.)$. By definition:

$$M_p^0 = \{x \in \mathrm{T}_p\mathcal{M} \ ; \ H.x = \{x\} \ \}.$$

So, with $x \in M_p^0$:

$$\begin{aligned} H.\Omega(x) &= \Omega(H.x) \quad \text{as } \Omega \text{ belongs to } \mathfrak{so}(\mathrm{ric})^{\mathfrak{h}}\text{, so commutes with the action of } H, \\ &= \Omega(\{x\}) = \{\Omega(x)\}, \end{aligned}$$

therefore $\Omega(x) \in M_p^0$, that is: $\Omega(M_p^0) \subset M_p^0$, with equality as Ω is nondegenerate.

By point **(i)** of Wu's theorem, for $i \geq 1$:

$$M_p^i = (M_p^0)^{\perp} \cap \{x \in \mathrm{T}_p\mathcal{M} \ ; \ \forall j \neq i, H_j.x = \{x\} \ \}.$$

So similarly, for each $i \geq 1$: $\Omega(M_p^i) \subset M_p^0 \oplus M_p^i$. Now $\Omega \in \mathfrak{so}(\mathrm{ric})$ so:

$$g(\Omega(M_p^i), M_p^0) = -g(M_p^i, \Omega(M_p^0)) = -g(M_p^i, M_p^0) = \{0\},$$

so: $\Omega(M_p^i) \subset M_p^i$ (with equality). By definition of Ω, the wanted result follows. □

So Wu's holonomy decomposition provides a refinement of the Ricci decomposition given by theorem 1, at least a refinement of the decomposition of the factor on which ric is nondegenerate. Indeed, on this factor, ric, on the one hand, is parallel and nondegenerate, so is a (pseudo-)Riemannian metric, and, on the other hand, the trivial subspace of the action of the holonomy group is $\{0\}$, which is nondegenerate. So:

Corollary 2 *Let* $(\mathcal{M}, \omega, D)$ *a symplectic manifold with a symplectic connection* D *the Ricci curvature of which is parallel and nondegenerate. Then* $(\mathcal{M}, \omega, D)$ *admits a unique decomposition into a Riemannian product (with respect for* ric*, considered as a metric), such that each factor is weakly irreducible. Moreover, this decomposition holds also for* ω*:*

$$(\mathcal{M}, g, \omega) \simeq \prod_i (\mathcal{M}_i, g_i, \omega_i)$$

with g *standing here for* ric*, considered as the metric.*

Being unique and maximal, this decomposition is necessarily a refinement of that of theorem 1. Naturally, point **(i)** of theorem 1 still applies and Ric is semi-simple on each factor (in fact, of minimal polynomial one of the P_λ).

4.2 More about the weakly irreducible factors: they are parakähler or (pseudo-)Kähler manifolds.

Using theorem 1, we can now now give a more precise description of the weakly irreducible subfactors given by corollary 2. By the remark below, these factors are (pseudo-)Riemannian manifolds. We will also deal with paracomplex structures and related notions; so we recall their definitions. Their names are chosen by analogy with the corresponding complex structures.

Important remark. On these subfactors, as ric is parallel *and nondegenerate*, ric is a (pseudo-)Riemannian metric and D is its Levi-Civita connection. Moreover, such a manifold is obviously Einstein in that point of view, with Einstein constant 1. So in the following, symplectic manifolds with a symplectic connection such that ric is parallel and nondegenerate, will be viewed as Einstein non-Ricci flat manifolds endowed with a parallel symplectic form.

Definition 2 *A paracomplex structure on a manifold* $\mathcal{M}$ *of dimension* $2n$ *is an endomorphism field* L *on* $\mathcal{M}$*, integrable, satisfying* $L^2 = \mathrm{Id}$ *with* $\dim\ker(L-\mathrm{Id}) = \dim\ker(L+\mathrm{Id})$*. Equivalently, it is the data in* $\mathrm{T}\mathcal{M}$ *of two totally isotropic, integrable and complementary distributions of dimension* n *with zero intersection.*

If $(\mathcal{M}, g)$ *is pseudo-Riemannian, a paracomplex structure on* $\mathcal{M}$ *satisfying:* $g(Lx, y) = -g(x, Ly)$ *is said to be parahermitian. If moreover* $DL = 0$*, it is said to be parakähler.*

Remark. One checks easily that a pseudo-Riemannian metric admitting a paracomplex structure is of signature (n, n).

Vocabulary. Let us also recall that a pseudo-Kähler manifold is a pseudo-Riemannian manifold $(\mathcal{M}, g)$ admitting a g-orthogonal parallel complex structure J (in other words, a Kähler manifold with indefinite metric).

As announced in the introduction, a Riemannian or pseudo-Riemannian manifold admitting a parallel symplectic form is (pseudo-)Kähler or parakähler. This is the following proposition. The matrices of the different involved objects are also given, to make the situation clearer for the reader.

Notation. For each integer k, J_k will here denote the matrix: $\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$.

Proposition 4 *Let* $(\mathcal{M}, g)$ *be a weakly irreducible Einstein non Ricci-flat Riemannian or pseudo-Riemannian manifold and* $p \in \mathcal{M}$*. We suppose* $(\mathcal{M}, g)$ *admits a parallel symplectic form* ω*. Then, denoting* $\dim\mathcal{M}$ *by* $2n$*,* $\mathcal{M}$ *is in one of the three following situations:*

(i) $(\mathcal{M}, g)$ *has a parakähler structure* L *such that* $\omega = \lambda g(.,L.)$ *with some* λ *in* $\mathbb{R}^*$*. In that case,* g *is of signature* (n, n) *and there is a basis of* $\mathrm{T}_p\mathcal{M}$ *in which:*

$$\mathrm{Mat}(g) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \quad \mathrm{Mat}(L) = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}, \quad \mathrm{Mat}(\omega) = \lambda \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

(ii) *$(\mathcal{M},g)$ has a (pseudo-)Kähler structure J such that $\omega=\lambda g(.,J.)$ with some λ in $\mathbb{R}^*$. In that case, g is of signature $(2p,2q)$ with $p+q=n$ and there is a basis of $\mathrm{T}_p\mathcal{M}$ in which:*

$$\mathrm{Mat}(g)=\begin{pmatrix} I_{2p} & 0\\ 0 & -I_{2q}\end{pmatrix},\quad \mathrm{Mat}(J)=\begin{pmatrix} J_p & 0\\ 0 & J_q\end{pmatrix},\quad \mathrm{Mat}(\omega)=\lambda\begin{pmatrix} J_p & 0\\ 0 & -J_q\end{pmatrix}.$$

(iii) *$(\mathcal{M},g)$ has a pseudo-Kähler structure J and a parakähler structure L such that: $JL=LJ$ and that: $\omega=\alpha g(.,L.)+\beta g(.,J.)$ with $(\alpha,\beta)\in\mathbb{R}^{*2}$. In that case, n is even, g is of signature (n,n) and, setting $m=\frac{1}{2}n$, there is a basis of $\mathrm{T}_p\mathcal{M}$ in which:*

$$\mathrm{Mat}(g)=\begin{pmatrix} 0 & I_{2m}\\ I_{2m} & 0\end{pmatrix},\quad \mathrm{Mat}(L)=\begin{pmatrix} -I_{2m} & 0\\ 0 & I_{2m}\end{pmatrix},\quad \mathrm{Mat}(J)=\begin{pmatrix} J_m & 0\\ 0 & J_m\end{pmatrix},$$

$$\mathrm{Mat}(\omega)=\begin{pmatrix} 0 & \alpha I_{2m}+\beta J_m\\ -\alpha I_{2m}+\beta J_m & 0\end{pmatrix}.$$

Proof. After a possible rescaling, we may suppose that $g=\mathrm{ric}$. The decomposition (3) page 4 of $\mathrm{T}_p\mathcal{M}$ is stable under the action of H. So, by weak irreducibility of $\mathcal{M}$ and as ric is nondegenerate, the minimal polynomial of the endomorphism Ric is equal to a single factor $P_\nu^{\alpha_\nu}$ for some $\nu\in\mathbb{C}^*$ (with the definition given in (2) page 4). By point **(i)** of theorem 1 page 5, $\alpha_\nu=1$.

Let us discuss the situation for the different possible values of λ.

(i) If ν is real. Let us set $L=\frac{1}{\nu}\mathrm{Ric}$; L is a parallel endomorphism of $\mathfrak{so}(\mathrm{ric})$ with minimal polynomial $(X-1)(X+1)$, as $\alpha_\nu=1$. If $x,y\in\ker(L-\varepsilon\,\mathrm{Id})$ with $\varepsilon=\pm1$, $\mathrm{ric}(x,y)=\varepsilon\,\mathrm{ric}(x,Ly)=-\varepsilon\,\mathrm{ric}(Lx,y)=-\mathrm{ric}(x,y)$ so $\ker(L-\mathrm{Id})$ and $\ker(L+\mathrm{Id})$ are both ric-totally isotropic (that remark is also contained in lemma 3 page 4). As $\mathrm{T}_p\mathcal{M}=\ker(L-\mathrm{Id})\oplus\ker(L+\mathrm{Id})$ and ric is nondegenerate, these two spaces are of dimension n and ric is of signature (n,n); so L is a parakähler structure. Finally there is a basis of $\mathrm{T}_p\mathcal{M}$ as announced in the theorem, with $\lambda=\frac{1}{\nu}$, and $\omega=\mathrm{ric}(.,\mathrm{Ric}^{-1}.)=\lambda\,\mathrm{ric}(.,L^{-1}.)=\lambda\,\mathrm{ric}(.,L.)$ as $L=L^{-1}$.

(ii) If ν is purely imaginary. Let us set $J=-\frac{1}{|\nu|}\mathrm{Ric}$; J is a parallel endomorphism of $\mathfrak{so}(\mathrm{ric})$ with minimal polynomial $(X-i)(X+i)=X^2-1$, as $\alpha_\nu=1$, so $J^2=-\mathrm{Id}$ and J is a Kähler or pseudo-Kähler structure (whether ric is definite or not). By the same computation as above or by lemma 3, and extending ric to a bilinear complex form on $\mathrm{T}_p\mathcal{M}\otimes\mathbb{C}$: $\ker(J-i\,\mathrm{Id})$ and $\ker(J+i\,\mathrm{Id})$ are both ric-totally isotropic; let n be their dimension ($\mathcal{M}$ is then of dimension $2n$). The complex conjugation $e\mapsto\overline{e}$ being a linear isomorphism of $\ker(J-i\,\mathrm{Id})$ to $\ker(J+i\,\mathrm{Id})$ and ric being nondegenerate, the sesquilinear form $h\ :(e,e')\mapsto\mathrm{ric}(e,\overline{e'})$ is nondegenerate on $\ker(J-i\,\mathrm{Id})$ and on $\ker(J+i\,\mathrm{Id})$. Its signature on each of these spaces is the same, let us denote it by (p,q). So if $(e_i)_{i=1}^n$ is a h-(pseudo-)orthonormal basis of $\ker(J-i\,\mathrm{Id})$, and setting $\beta=((e_i)_{i=1}^n(\overline{e_i})_{i=1}^n)$:

$$\mathrm{Mat}_\beta(\mathrm{ric})=\begin{pmatrix} 0 & I_{p,q}\\ I_{p,q} & 0\end{pmatrix}\quad\text{and: }\mathrm{Mat}_\beta(J)=\begin{pmatrix} iI & 0\\ 0 & -iI\end{pmatrix}.$$

Now in the real basis $(f_i,f'_i)_{i=1}^n$ of $\mathrm{T}_p\mathcal{M}$ defined by $f_i=\frac{1}{\sqrt{2}}(e_i+\overline{e_i})$ and $f'_i=\frac{1}{i\sqrt{2}}(e_i-\overline{e_i})$, the matrices of ric, J and ω have the announced form, with $\lambda=\frac{1}{|\nu|}$. Besides, $\omega=\mathrm{ric}(.,\mathrm{Ric}^{-1}.)=\lambda\,\mathrm{ric}(.,J.)$ as $J=-J^{-1}$.

Otherwise. Let us set $L = \frac{1}{2\Re\nu}(\mathrm{Ric} + |\nu|^2 \mathrm{Ric}^{-1})$ and $J = \frac{1}{2\Im\nu}(\mathrm{Ric} - |\nu|^2 \mathrm{Ric}^{-1})$. Then:

$$\begin{aligned} L &= \frac{1}{\nu + \overline{\nu}} \mathrm{Ric}^{-1}(\mathrm{Ric}^2 + |\nu|^2 \mathrm{Id}) \\ &= \frac{1}{\nu + \overline{\nu}} \mathrm{Ric}^{-1}[(\mathrm{Ric} + \nu \mathrm{Id})(\mathrm{Ric} + \overline{\nu} \mathrm{Id}) - (\nu + \overline{\nu}) \mathrm{Ric}] \end{aligned}$$

so:

$$\begin{aligned} &(L + \mathrm{Id})(L - \mathrm{Id}) \\ =&\frac{1}{\nu + \overline{\nu}} \mathrm{Ric}^{-1}[(\mathrm{Ric} + \nu \mathrm{Id})(\mathrm{Ric} + \overline{\nu} \mathrm{Id}) + ((\mathrm{Ric} + \nu \mathrm{Id})(\mathrm{Ric} + \overline{\nu} \mathrm{Id}) - 2((\nu + \overline{\nu}) \mathrm{Ric}))] \\ =&\frac{1}{\nu + \overline{\nu}} \mathrm{Ric}^{-1}[(\mathrm{Ric} + \nu \mathrm{Id})(\mathrm{Ric} + \overline{\nu} \mathrm{Id}) + (\mathrm{Ric} - \nu \mathrm{Id})(\mathrm{Ric} - \overline{\nu} \mathrm{Id})] \\ =&P_\nu(\mathrm{Ric}) \qquad \text{as } \alpha_\nu = 1 \\ =&0 \end{aligned}$$

Similarly we obtain: $J^2 + \mathrm{Id} = 0$. Like in the previous point, using the nondegenerate hermitian form $h : e \mapsto \mathrm{ric}(e, \overline{e})$ of $\mathrm{T}_p\mathcal{M} \otimes \mathbb{C}$ and the fact J and L commute, and denoting by n the dimension of $\ker(L - \mathrm{Id}) \cap \ker(J - i\,\mathrm{Id})$, we obtain a basis $(e_i)_{i=1}^{2n}$ of $\ker(J - i\,\mathrm{Id})$ such that, setting $\beta = ((e_i)_{i=1}^n, (\overline{e_i})_{i=1}^n, (\overline{e_i})_{i=n+1}^{2n}, (e_i)_{i=n+1}^{2n})$:

$$\mathrm{Mat}_\beta(\mathrm{ric}) = \begin{pmatrix} 0 & I_{2n} \\ I_{2n} & 0 \end{pmatrix}, \quad \mathrm{Mat}_\beta(L) = \begin{pmatrix} -I_{2n} & 0 \\ 0 & I_{2n} \end{pmatrix} \quad \text{and:}$$

$$\mathrm{Mat}_\beta(J) = \begin{pmatrix} iI_n & 0 & 0 & 0 \\ 0 & -iI_n & 0 & 0 \\ 0 & 0 & -iI_n & 0 \\ 0 & 0 & 0 & iI_n \end{pmatrix}.$$

$\mathcal{M}$ is of dimension $4n$ and ric of signature $(2n, 2n)$. As L and J are moreover in $\mathfrak{so}(\mathrm{ric})^{\mathfrak{h}}$, they are then, respectively, a parakähler and a pseudo-Kähler structure on $(\mathcal{M}, \mathrm{ric})$. Note also they commute.

Now in the real basis $((f_i)_{i=1}^n, (f'_i)_{i=1}^n, (f_i)_{i=n+1}^{2n}, (f'_i)_{i=n+1}^{2n})$ of $\mathrm{T}_p\mathcal{M}$ defined by $f_i = \frac{1}{\sqrt{2}}(e_i + \overline{e_i})$ and $f'_i = \frac{1}{i\sqrt{2}}(e_i - \overline{e_i})$, the matrices of ric, L, J and ω have the announced form, with $\alpha + i\beta = \frac{1}{\nu}$. Besides, $\omega = \alpha\,\mathrm{ric}(., L.) + \beta\,\mathrm{ric}(., J.)$. □

5 A second theorem: Structure of $\mathfrak{so}(g)^{\mathfrak{h}}$ for Einstein non Ricci-flat manifolds.

Theorem 1 enables to say still more about the weakly irreducible Einstein non Ricci-flat factors. To be more precise, let us introduce the following

Question about pseudo-Riemannian manifolds. If $(\mathcal{M}, g)$ is a (weakly irreducible) pseudo-Riemannian manifold, what is the structure of the algebra $\mathfrak{so}(g)^{\mathfrak{h}}$ of the elements of $\mathfrak{so}(g)$ which commute with the action of the holonomy group ?

Now, point **(i)** of theorem 1, combined with some linear algebra computations and classical results on holonomy groups, gives a precise answer in case $(\mathcal{M}, g)$ is Einstein, non

Ricci-flat.

Remark. This question is little interesting for a Riemannian manifold, but nontrivial for a pseudo-Riemannian one.

Indeed, a weakly irreducible Riemannian manifold is irreducible, so cannot carry any degenerate nontrivial alternate parallel form. Now, it carries a parallel symplectic form *iff* it is *Kähler.* Now it is a classical fact that: if $(\mathcal{M}, g)$ is Einstein non Ricci-flat, Kähler or pseudo-Kähler, irreducible, $\mathfrak{so}(g)^{\mathfrak{h}} = \mathrm{span}(J)$ with J the complex structure.

Now, on a *weakly* irreducible, though reducible pseudo-Riemannian manifold, $\mathfrak{so}(g)^{\mathfrak{h}}$ may be *a priori* as great and complicated as you want; actually, only the additional assumption "Einstein non Ricci-flat" implies strong constraints on $\mathfrak{so}(g)^{\mathfrak{h}}$. It is the sense of theorem 2 below, which may be compared with the analog result for *symmetric* pseudo-Riemannian manifold, recalled in section 7.1 page 22. See also section 7.2 for some examples.

Theorem 2 *Let $(\mathcal{M}, g)$ be a weakly irreducible Einstein non Ricci-flat Riemannian or pseudo-Riemannian manifold.*

Then the non-invertible elements of $\mathfrak{so}(g)^{\mathfrak{h}}$ are of null square and the set $\mathfrak{a}$ of these elements is an associative algebra i.e. *: $\forall N, N' \in \mathfrak{a}, NN' \in \mathfrak{a}$. Besides:*

$$\forall K \in \mathfrak{so}(g)^{\mathfrak{h}} \cap GL(\mathrm{T}\mathcal{M}),\ K.\mathfrak{a} = \mathfrak{a} \quad \textit{and:} \quad \forall N \in \mathfrak{a}, KN = -NK.$$

Moreover, either

(0) *$(\mathcal{M}, g)$ admits no parallel symplectic form,* i.e.*: $\mathfrak{so}(g)^{\mathfrak{h}} \cap GL(\mathrm{T}\mathcal{M}) = \emptyset$,* i.e.*:*

$$\mathfrak{so}(g)^{\mathfrak{h}} = \mathfrak{a} \quad \textit{(with possibly } \mathfrak{a} = \{0\}\textit{)},$$

or $(\mathcal{M}, g)$ admits a parallel symplectic form ω, i.e. *: $\mathfrak{so}(g)^{\mathfrak{h}} \cap GL(\mathrm{T}\mathcal{M}) \neq \emptyset$. Then the product on $\mathfrak{a}$ is trivial: $\forall N, N' \in \mathfrak{a}, NN' = 0$ and $(\mathcal{M}, g)$ is in one of the three following cases:*

(I) *$(\mathcal{M}, g)$ does not admit any Kähler structure. In that case, g is of signature (n, n) and every element L of $\mathfrak{so}(g)^{\mathfrak{h}} \setminus \mathfrak{a}$ is, up to a multiplicative constant, a parakähler structure and satisfies:*

$$\mathfrak{so}(g)^{\mathfrak{h}} = \mathbb{R}.L \oplus \mathfrak{a}.$$

(II) *$(\mathcal{M}, g)$ does not admit any parakähler structure. In that case, g is of signature $(2p, 2q)$ with $p + q = n$ and every element J of $\mathfrak{so}(g)^{\mathfrak{h}} \setminus \mathfrak{a}$ is, up to a multiplicative constant, a (pseudo-)Kähler structure and satisfies:*

$$\mathfrak{so}(g)^{\mathfrak{h}} = \mathbb{R}.J \oplus \mathfrak{a}.$$

The algebra $\mathfrak{a}$ is necessarily $\{0\}$ if g is definite. More precisely:

$$\dim \left[+_{N \in \mathfrak{a}} \mathrm{Im}\, N\right] \leq 2 \min(p, q).$$

(III) *Otherwise, n is even, g is of signature (n, n) and $(\mathcal{M}, g)$ admits at least one parakähler structure L and one pseudo-Kähler structure J. With any two such structures:*

$$\mathfrak{so}(g)^{\mathfrak{h}} = \mathbb{R}.L \oplus \mathbb{R}.J \oplus \mathfrak{a}.$$

Besides, each parakähler (resp. pseudo-Kähler) element L (resp. J) of $\mathfrak{so}(g)^{\mathfrak{h}} \setminus \mathfrak{a}$ admits, up to sign, a unique pseudo-Kähler (resp. parakähler) element J (resp. L) s.t. $LJ = JL$.

Remark. In particular:

in case **(I)**, normalizing L so that $L^2 = \mathrm{Id}$, the set of the parakähler structures of $(\mathcal{M}, g)$ is: $\mathfrak{so}(g)^{\mathfrak{h}} \cap SL(\mathrm{T}\mathcal{M}) = \{\pm L + N \ / \ N \in \mathfrak{a}\}$.

in case **(II)**, normalizing J so that $J^2 = -\mathrm{Id}$, the set of the (pseudo-)Kähler structures of $(\mathcal{M}, g)$ is: $\mathfrak{so}(g)^{\mathfrak{h}} \cap SL(\mathrm{T}\mathcal{M}) = \{\pm J + N \ ; \ N \in \mathfrak{a}\}$.

in case **(III)**, the set of the parakähler structures of $(\mathcal{M}, g)$ is: $\{\pm L + N \ ; \ N \in \mathfrak{a}\}$ and the set of its pseudo-Kähler structures: $\{\pm J + N \ ; \ N \in \mathfrak{a}\}$.

Remark. The set of the parallel symplectic forms, *i.e.* of the symplectic forms admitting D as a symplectic connection, is: $g(., (\mathfrak{so}(g)^{\mathfrak{h}} \setminus \mathfrak{a}).)$.

Remark on the case (II). If $p \neq q$, setting $g = \pm \,\mathrm{ric}$ with the adequate sign, we get a (pseudo-)Riemannian metric g of signature $(\max(p,q), \min(p,q))$, admitting the same Levi-Civita connection. In particular, in case ric is negative definite, we get a real, *i.e.* positive definite, Kähler metric. This gives a sense to the sign of the Einstein constant: positive if $\mathrm{ric} = g$, negative if $\mathrm{ric} = -g$. In the case the signature of ric is "neutral": (n, n), the sign of the Einstein constant is meaningless.

6 Proof of theorem 2.

Let us prove the theorem. We will need a little lemma, which is a direct application of a classical result about holonomy; we also state then another technical lemma and the theorem about Ricci-parallel metrics on which it is based.

Lemma 7 *Let $\mathcal{M}$ be a manifold with holonomy group H, p a point in M, E a subspace of $\mathrm{T}_p\mathcal{M}$ and b and b' two reflexive nondegenerate bilinear forms on E (reflexive = symmetric or skew-symmetric). Let us suppose that H acts on E and preserves b and b'. Then, if :*

$$E = E_1 \oplus E_2,$$

the sum being b-orthogonal, and E_1 and E_2 being b'-totally isotropic, H acts trivially on E.

Proof. By point **(iv)** of proposition 2 page 5 if b is skew-symmetric, or by the corresponding classical statement if b is symmetric (see [KoNo69] p.183), the holonomy group H is equal to the direct product $H_1 \times H_2$ where $H_i = \{h \in H \ ; \ h \text{ acts trivially on } E_{3-i}\}$. But E_1 and E_2 are b'-totally isotropic and their sum is nondegenerate, so E_2 is identified to E_1^* by b', thus the action of H on E_2 is the dual of that on E_1. One then checks easily that each H_i acts trivially on E_i, so the holonomy acts trivially on E. □

Lemma 8 *Let $(\mathcal{M}, g)$ be a non Ricci-flat (pseudo-)Riemannian manifold with* ric *parallel and nondegenerate. Let A and B in $\mathfrak{so}(\mathrm{ric})^{\mathfrak{h}}$ with $A^2 = \pm \mathrm{Id}$ and $B^2 = \pm \mathrm{Id}$. Let $A^{-1}B = S + N$ be the decomposition of $A^{-1}B$ into its semi-simple and its nilpotent parts. We suppose the spectrum of S is $\{\nu, \overline{\nu}\}$ for some $\nu \in \mathbb{C}$ with $|\nu| = 1$. Then:*

(i) *<u>If $A^2 = B^2$</u> then: $\nu \neq \pm 1 \Rightarrow N = 0$.*

(ii) *<u>If $A^2 = -B^2$ and $S^2 = -\mathrm{Id}$</u>, then $S = \frac{1}{2}(A^{-1}B + BA^{-1})$ and $N = \frac{1}{2}[A^{-1}, B]$.*

That lemma is a consequence of the pseudo-Riemannian analog of theorem 1 (see [BBB01]), which will be used also in the proof of theorem 2. Let us recall it partially.

We suppose $(\mathcal{N}, g)$ is a Riemannian or pseudo-Riemannian manifold the Ricci curvature of which is parallel. There then exists a parallel g- (and ric-)selfadjoint endomorphism U s.t.: $\mathrm{ric}(.,.) = g(.,U.)$.

Theorem ([BBB01], partial version) *If* ric *is nondegenerate,* U *is semi-simple.*

Remark Under the above assumptions, ric can be taken as the metric of $\mathcal{N}$ and the result stated as follows: if $(\mathcal{N}, g)$ is a non Ricci-flat Einstein (pseudo-)Riemannian manifold and T a parallel and nondegenerate g-selfadjoint endomorphism on $\mathrm{T}\mathcal{N}$, then T is semi-simple. Similarly, the same statement with T a g-anti selfadjoint endomorphism follows from theorem 1 page 5. These forms of both theorems will be used in this section.

Proof of lemma 8. For a small enough $\varepsilon > 0$, $(A^{-1} + \varepsilon B)^2$ is invertible; now as it is ric-selfadjoint it is semi-simple by the above theorem. Developing that square:

$$A^{-1}B + BA^{-1} = \tfrac{1}{\varepsilon}\left[(A^{-1} + \varepsilon B)^2 - A^2 - \varepsilon^2 B^2\right]. \tag{4}$$

Let us now study cases **(i)** and **(ii)**.

(i) If $A^2 = B^2$, $(A^{-1}B)(BA^{-1}) = \mathrm{Id}$ so $BA^{-1} = (A^{-1}B)^{-1}$; then the decomposition of BA^{-1} into its semi-simple and nilpotent parts is of the form: $S^{-1} + \widetilde{N}$. Note that, as $A^{-1}B$ and BA^{-1} commute, they are simultaneously triangulable (over $\mathbb{C}$), so $N + \widetilde{N}$ is nilpotent. Besides, N and $\widetilde{N}$ commute with S and S^{-1}, so $(S + S^{-1}) + (N + \widetilde{N})$ is the decomposition of $A^{-1}B + BA^{-1}$. Now replacing in **(4)**:

$$(S + S^{-1}) + (N + \widetilde{N}) = A^{-1}B + BA^{-1} = \tfrac{1}{\varepsilon}\left[(A^{-1} + \varepsilon B)^2 \pm (1 + \varepsilon^2)\,\mathrm{Id}\right].$$

So $(S + S^{-1}) + (N + \widetilde{N})$ is null or semi-simple so:

$$N + \widetilde{N} = 0.$$

Now $[A^{-1}, B] = (S - S^{-1}) + (N - \widetilde{N}) = (S - S^{-1}) + 2N$, with $(S - S^{-1})$ and $2N$, respectively, its semismiple and nilpotent parts. By theorem 1, as $[A^{-1}, B]$ is g-anti selfadjoint, so $2N = 0$ as soon as $S - S^{-1} \neq 0$. It is the case if $\nu \neq \pm 1$.

(ii) If $A^2 = -B^2$, $(A^{-1}B)(BA^{-1}) = -\mathrm{Id}$ so $BA^{-1} = -(A^{-1}B)^{-1}$; then the decomposition of BA^{-1} into its semi-simple and nilpotent parts is of the form: $-S^{-1} + \widetilde{N}$. Note again that $(S - S^{-1}) + (N + \widetilde{N})$ is the decomposition of $A^{-1}B + BA^{-1}$.

By the same argument, $S + N - S^{-1} + \widetilde{N}$ is null or semi-simple so $\widetilde{N} = -N$. Consequently, as $S^2 = -Id$:

$$\begin{cases} A^{-1}B = S + N \\ BA^{-1} = -S^{-1} - N = S - N \end{cases}$$

So the result. □

Proof of theorem 2. We need to show first of all that all the elements of $\mathfrak{a}$ are of null square.

It follows from the theorem of [BBB01] recalled page 18: let N be in $\mathfrak{a}$; being degenerate, it admits the eigenvalue zero. By lemma 3 and as $N \in \mathfrak{so}(g)^{\mathfrak{h}}$, the decomposition (3) of $\mathrm{T}_p\mathcal{M}$, given page 4, is stable under the action of H. So, by weak irreducibility of $\mathcal{M}$, zero is the only eigenvalue of N, which is then nilpotent. Therefore, $\mathrm{Id} + N^2$ is nondegenerate and N^2 is its nilpotent part. Now N^2 is g-selfadjoint, so $\mathrm{Id} + N^2$ too and by [BBB01], it is semi-simple, *i.e.* $N^2 = 0$.

The following is based on the

Claim. If N and N' belong to $\mathfrak{a}$, $\ker N$ and $\ker N'$ have a nontrivial intersection.

Indeed, $N^2 = N'^2 = 0$ so both $\ker N$ and $\ker N'$ are of dimension greater or equal to $\frac{1}{2}\dim \mathcal{M}$. So, supposing $\ker N \cap \ker N' = \{0\}$, we have necessarily:

$$\dim \ker N = \dim \ker N' = \mathrm{rk}\, N = \mathrm{rk}\, N' = \tfrac{1}{2}\dim \mathcal{M} \quad \text{and: } \mathrm{T}_p\mathcal{M} = \mathrm{Im}\, N \oplus \mathrm{Im}\, N'. \qquad (5)$$

Besides, N and N' being g-anti selfadjoint and of null square, their images are both g-totally isotropic. Now, by equality of dimensions, $\ker N = \mathrm{Im}\, N$, so $g(.,N.)$ defines on $\mathrm{T}_p\mathcal{M}$ a bilinear skew-symmetric form of kernel $\mathrm{Im}\, N$; by **(5)**, that form is nondegenerate on $\mathrm{Im}\, N'$. Symmetrically, $g(.,N'.)$ is of kernel $\mathrm{Im}\, N'$ and is nondegenerate on $\mathrm{Im}\, N$. Finally, $g(.,(N+N').)$ defines on $\mathrm{T}_p\mathcal{M}$ a nondegenerate skew-symmetric bilinear form with respect for which $\mathrm{Im}\, N$ and $\mathrm{Im}\, N'$ are in direct orthogonal sum. The action of the holonomy group preserves it, so by lemma 7 page 17 with $E = \mathrm{T}_p\mathcal{M}$, $b = g(.,(N+N').)$, $b' = g$, $E_1 = \mathrm{Im}\, N$ and $E_2 = \mathrm{Im}\, N'$, this action is trivial, what is excluded. So the claim.

The first immediate consequence of the claim is that $\mathfrak{a}$ is stable by sum: if N and N' are in $\mathfrak{a}$, $\ker N \cap \ker N' \supsetneq \{0\}$ so $N + N'$ is degenerate *i.e.* $N + N' \in \mathfrak{a}$.

Let now K be in $\mathfrak{so}(g)^{\mathfrak{h}} \cap GL(\mathrm{T}\mathcal{M})$. Let us prove that $KN = -NK$. By proposition 4 page 13, K can be written $\alpha L + \beta J$, where L and J belong to $\mathfrak{so}(g)^{\mathfrak{h}}$ and: $L^2 = -J^2 = \mathrm{Id}$. To prove N anticommutes with K, it is sufficient to prove that it anticommutes with L and J. Replacing K by L, then by J, we thus may suppose that: $K^2 = \pm \mathrm{Id}$. Let us set $N_\pm = \frac{1}{2}(N \pm K^{-1}NK)$, then $N = N_+ + N_-$ with $KN_+ = N_+K$ and $KN_- = -N_-K$. As K^2 is supposed to be $\pm \mathrm{Id}$, one checks that $K^{-1}NK$ belongs to $\mathfrak{so}(g)^{\mathfrak{h}}$, so to $\mathfrak{a}$ as it is degenerate. Now $\mathfrak{a}$ is stable by sum, so N_+ and N_- belong to $\mathfrak{a}$. As N_+ commutes with K, N_+ is then the nilpotent part of $K + N_+$. Now by theorem 1 page 5, $K + N_+$ being g-anti selfadjoint and nondegenerate, it is semi-simple *i.e.* $N_+ = 0$. Consequently, $N = N_-$ *i.e.* $NK = -KN$.

The other announced fact: $\forall K \in \mathfrak{so}(g)^{\mathfrak{h}} \cap GL(\mathrm{T}\mathcal{M})$, $K.\mathfrak{a} = \mathfrak{a}$ follows then directly: as $NK = -KN$, $KN \in \mathfrak{so}(g)^{\mathfrak{h}}$, so $KN \in \mathfrak{a}$ as it is degenerate. So $K.\mathfrak{a} \subset \mathfrak{a}$, with equality as K is invertible.

Let now N' be in $\mathfrak{a}$ and let us finally show that $NN' \in \mathfrak{a}$ and $\mathfrak{so}(ric)^{\mathfrak{h}} \cap GL(\mathrm{T}^a st\mathcal{M}) \neq \emptyset \Rightarrow NN' = 0$. As $\mathfrak{a}$ is stable by sum, $N + N' \in \mathfrak{a}$ so $(N+N')^2 = 0$; now $(N+N')^2 = NN' + N'N$ so $NN' = -N'N$. Besides it follows again from the claim that $[N,N']$ is degenerate; as $[N,N'] \in \mathfrak{so}(g)^{\mathfrak{h}}$, $[N,N'] \in \mathfrak{a}$; so it comes: $NN' = \frac{1}{2}[N,N'] \in \mathfrak{a}$.

In the case $\mathfrak{so}(ric)^{\mathfrak{h}}$ has an invertible element, let A be a such one. As shown above, as $[N,N'] \in \mathfrak{a}$, $A[N,N'] = -[N,N']A$; besides one has also: $AN = -NA$ and $AN' = -N'A$, so A commutes with $[N,N']$. Finally, $A[N,N'] = 0$ so $[N,N'] = 0$, that is to say $NN' = 0$.

The general statements about $\mathfrak{a}$ are proven; nothing more has to be proven in the case $\mathfrak{so}(ric)^{\mathfrak{h}} \cap GL(\mathrm{T}\mathcal{M}) = \emptyset$. If $\mathfrak{so}(ric)^{\mathfrak{h}} \cap GL(\mathrm{T}\mathcal{M}) \neq \emptyset$, three cases appear:

Case (I): $(\mathcal{M}, g)$ admits no (pseudo-)Kähler structure.
Case (II): $(\mathcal{M}, g)$ admits no parakähler structure.
Case (III): Otherwise.

Let us examine each case.

Case (I) Suppose K is an invertible element of $\mathfrak{so}(\mathrm{ric})^{\mathfrak{h}}$. By proposition 4 page 13, K can be written $\alpha L + \beta J$ where L is a parakähler structure and J a (pseudo-)Kähler structure on $(\mathcal{M}, g)$. So by assumption of case **(I)**, K is here proportional to some parakähler structure L. In the following, replacing possibly K by $\frac{1}{\rho}K$, we suppose $K = L$.

Case (II) Symmetrically, swapping the role of para- and pseudo-Kähler structures, one shows that every element J' of $\mathfrak{so}(\mathrm{ric})^{\mathfrak{h}}$ is, up to a scalar, a pseudo-Kähler structure; so by rescaling we now suppose: $J'^2 = -\mathrm{Id}$.

In both first cases: We now want to get: $\mathfrak{so}(\mathrm{ric})^{\mathfrak{h}} = \mathbb{R}.K \oplus \mathfrak{a}$. So let us take K' an invertible elements of $\mathfrak{so}(\mathrm{ric})^{\mathfrak{h}}$. Like for K, after a possible multiplication by a suitable constant, we may suppose $K^2 = K'^2 = \pm \mathrm{Id}$.

The endomorphism $K^{-1}K'$ is ric-orthogonal:

$$\begin{aligned}
\mathrm{ric}(K^{-1}K'.,K^{-1}K'.) &= -\mathrm{ric}((K^{-1})^2K'.,K'.) \\
&= \mathrm{ric}(K'(K^{-1})^2.,K'.) \\
&= \mathrm{ric}(K'^2(K^{-1})^2.,.) \\
&= \mathrm{ric}(.,.).
\end{aligned}$$

So if ν is an eigenvalue of $K^{-1}K'$, so is $1/\nu$ and if ν and ν' are eigenvalues of $K^{-1}K'$, the corresponding characteristic subspaces are orthogonal *iff* $\nu \neq 1/\nu'$ (otherwise they are totally isotropic and their sum is nondegenerate). So, by weak irreducibility of $\mathcal{M}$, the spectrum of $K^{-1}K'$ is necessarily of the form: $\{\nu, 1/\nu, \overline{\nu}, 1/\overline{\nu}\}$ with $\nu \in \mathbb{C}^*$.

One can be even more precise. Indeed, if $\{\nu, \overline{\nu}\} \neq \{1/\nu, 1/\overline{\nu}\}$, let us denote by E_ν the characteristic subspace associated to $\{\nu, \overline{\nu}\}$ and $E_{1/\nu}$ the characteristic subspace associated to $\{1/\nu, 1/\overline{\nu}\}$. Let us also notice that $K^{-1}K'$ is ω-selfadjoint. Indeed:

$$\begin{aligned}
\omega(.,K^{-1}K'.) &= \lambda\,\mathrm{ric}(.,K'.) && \text{by definition of } \lambda \text{ and } K, \\
&= -\lambda\,\mathrm{ric}(K'.,.) && \text{as } K' \in \mathfrak{so}(\mathrm{ric}), \\
&= -\omega(K'.,K^{-1}.) \\
&= -\omega(K^{-1}K'.,.) && \text{as } K \in \mathfrak{sp}(\omega).
\end{aligned}$$

Consequently, the characteristic subspaces of $K^{-1}K'$ are in direct ω-orthogonal sum; so is it here with E_ν and $E_{1/\nu}$. Now by lemma 7 page 17 with $E = \mathrm{T}_p\mathcal{M}$, $b = \omega$ and $b' = \mathrm{ric}$, the holonomy acts trivially on $\mathrm{T}_p\mathcal{M}$, what is impossible, ric being nondegenerate. So $\nu = 1/\nu$ or $\nu = 1/\overline{\nu}$ and the spectrum of $K^{-1}K'$ is of the form:

$$\{\nu, \overline{\nu}\} \text{ with } |\nu| = 1.$$

Now actually, $\nu = \pm 1$. Indeed, let us take $K^{-1}K' = S + N$ the decomposition of $K^{-1}K'$ into its semi-simple and nilpotent parts and suppose $\nu \neq \pm 1$. Then there exists an S_1 such that: $S = \Re(\nu)\,\mathrm{Id} + \Im(\nu) S_1$ and $S_1^2 = -\,\mathrm{Id}$; besides, by point **(i)** of lemma 8 page 17, $N = 0$. Consequently:

$$\begin{cases} K^{-1}K' = \Re(\nu)\,\mathrm{Id} + \Im(\nu) S_1 \\ K'K^{-1} = (K^{-1}K')^{-1} = \Re(\nu)\,\mathrm{Id} - \Im(\nu) S_1 \end{cases} \quad \text{so: } S_1 = \frac{1}{2}[A^{-1}, B].$$

So: $S_1 \in \mathfrak{so}(\mathrm{ric})^{\mathfrak{h}}$, so finally S_1 is a pseudo-Kähler structure, satisfying moreover: $KS_1 = -S_1K$. This implies $(\mathcal{M}, g)$ is Ricci-flat, what contradicts the hypothesis. Indeed $(\mathcal{M}, g, S_1)$ is Kähler or pseudo-Kähler, and can be considered as a complex manifold of dimension n. The holonomy group of $(\mathcal{M}, g, S_1)$ is included in $U(p, q)$ where $(2p, 2q)$ is the (real) signature of g. Now, the complex alternate form $\omega_{\mathbb{C}}$ defined by $\omega_{\mathbb{C}} = g(., K.) - ig(., KS_1.)$ is, for this structure, a *complex* alternate form as $KS_1 = -S_1K$ (notice that therefore, n is necessarily even). Consequently, $\omega_{\mathbb{C}}^{\wedge m}$ where $m = \frac{n}{2}$ is a complex volume form on $(\mathcal{M}, g, S_1)$, the holonomy of which is consequently included in $SU(p, q)$. By a classical calculation (see for example [Iw50]), $(\mathcal{M}, g)$ is then Ricci-flat.

Remarks. The above reasoning is nothing but the classical proof "hyperkähler or pseudo-hyperkähler ⇒ Ricci-flat", slightly adapted. A manifold is said (pseudo-)hyperkähler if it is (pseudo-)Riemannian and admits two complex parallel structures J and J' in $\mathfrak{so}(g)$, which anticommute. Here S_1 is a such structure and plays the role of J; K plays the role of J': it may be parakähler instead of (pseudo-)Kähler but it does not affect the proof, as its only role is to provide a complex alternate form.

In case **(I)**, one could also propose a far simpler argument: the only existence of S_1, a pseudo-Kähler structure, is excluded by assumption. Nevertheless, the above proof ensures that $\nu = \pm 1$, independently of the assumption of case **(I)**: it will be useful below.

So necessarily $\nu = \pm 1$, *i.e.* $S = \pm\,\mathrm{Id}$. Consequently: $K' = \pm K + KN$, and therefore $KN \in \mathfrak{so}(g)^{\mathfrak{h}}$. As KN is degenerate, it belongs to $\mathfrak{a}$ and the result follows.

Additional claim in case (II). Any N in $\mathfrak{a}$ being ric-anti selfadjoint and of null square, its image is totally isotropic, so reduced to $\{0\}$ if ric is definite; so in that case, $\mathfrak{a} = \{0\}$. More precisely, as the product of any two elements of $\mathfrak{a}$ is null, $\dim\,[+_{N\in\mathfrak{a}} \operatorname{Im} N]$ is bounded by the dimension of the greatest totally isotropic subspace of $\mathrm{T}_p\mathcal{M}$, that is to say $\min(2p, 2q)$ if $\mathrm{sign}(\mathrm{ric}) = (2p, 2q)$.

Case (III) By assumption, there exists on $(\mathcal{M}, g)$ at least one parakähler structure L and one pseudo-Kähler structure J. Let us take two such structures and K in $\mathfrak{so}(g)^{\mathfrak{h}} \setminus \mathfrak{a}$. By proposition 4 page 13, K can be written: $K = L' + J'$, where L' is proportional to some parakähler structure and J' proportional to some pseudo-Kähler structure (one among L' and J' is possibly null). By the same proof as done just above, there exists then N_1 and N_2 in $\mathfrak{a}$ s.t.: $L' = \alpha L + N_1$ and $J' = \beta J + N_2$ with some (α, β) in $\mathbb{R}^2$. As $\mathfrak{a}$ is stable by sum, $N = N_1 + N_2 \in \mathfrak{a}$ so: $K = \alpha L + \beta J + N$, what is the wanted result.

Let us then prove that, fixing L (resp. J), we can find a pseudo-Kähler structure $\widetilde{J}$ (resp. parakähler structure $\widetilde{L}$) such that: $L\widetilde{J} = \widetilde{J}L$ (resp. $\widetilde{L}J = J\widetilde{L}$). Let us set $K = L$ and $K' = J$; the same reasoning about $K^{-1}K'$ as above still holds, except that $K^{-1}K' = L^{-1}J$ is this time ric-*anti* orthogonal. So its spectrum is of the form: $\{\nu, -1/\nu, \overline{\nu}, -1/\overline{\nu}\}$ with $\nu \in \mathbb{C}^*$. Moreover, the two characteristic subspaces E_ν and $E_{-1/\nu}$ of $K^{-1}K' = L^{-1}J$, respectively associated to the set of eigenvalues $\{\nu, \overline{\nu}\}$ and $\{-1/\nu, -1/\overline{\nu}\}$, are both ric-totally isotropic if they are distinct.

Now similarly, $L^{-1}J \in \mathfrak{sp}(\omega_K)$ with $\omega_K = \mathrm{ric}(.,K.)$, so E_ν and $E_{-1/\nu}$ are in direct ω_K-orthogonal sum if they are distinct. So, again by lemma 7, $E_\nu = E_{-1/\nu}$ *i.e.* $\nu = -1/\nu$ or $\nu = -1/\overline{\nu}$. So $\nu = \pm i$ and the spectrum of $L^{-1}J$ is $\{i, -i\}$.

Then by point **(ii)** of lemma 8 page 17, and with S the semi-simple part of $L^{-1}J$: $S = \frac{1}{2}(L^{-1}J + JL^{-1})$. Consequently, S commutes with L and J and is g-selfadjoint. Therefore, the endomorphism $\widetilde{J} = LS \in \mathfrak{so}(g)^{\mathfrak{h}}$ (resp. $\widetilde{L} = JS \in \mathfrak{so}(g)^{\mathfrak{h}}$) is as wanted.

Finally, such a $\widetilde{J}$ (resp. $\widetilde{L}$) is unique. Let us soon relabel this endomorphism so that $LJ = JL$ and suppose for example that we have a pseudo-Kähler structure J' such that: $LJ' = J'L$. By the first part of the result, there exists an N in $\mathfrak{a}$ such that: $J' = J + N$. So $N = J' - J$ also commutes with L. But $N \in \mathfrak{a}$ so $LN = -NL$, as shown above. So $LN = 0$, so $N = 0$ *i.e.* $J = J'$. □

7 Some remarks and examples.

7.1 The case where $(\mathcal{M}, g)$ is (locally) symmetric.

The result of theorem 2 has to be compared with the situation where not only ric, but R is supposed parallel, and ric still supposed nondegenerate. In that case, $(\mathcal{M}, D)$ is then a *simple* symmetric space —*cf* [Ber00] p. 99 for definition, equivalent to "ric is nondegenerate (thus symmetric) and $(\mathcal{M}, \mathrm{ric})$ weakly irreducible" in the words of this paper. The structure of $\mathrm{End}(\mathrm{T}\mathcal{M})^{\mathfrak{h}}$ is well known on such spaces; the first basic result can be found for example in [Ber00], theorem V.1.10 p.101. Let us recall the part of it dealing with $\mathfrak{so}(g)^{\mathfrak{h}}$.

Theorem. *Let $(\mathcal{M}, D)$ be a simple locally symmetric space, then $\mathfrak{so}(g)^{\mathfrak{h}}$ is of one of the four types:*

- **(0)** $\mathfrak{so}(g)^{\mathfrak{h}} = \{0\}$,
- **(I)** $\mathfrak{so}(g)^{\mathfrak{h}} = \mathbb{R}.L$ *where L is parakähler* ($L^2 = \mathrm{Id}$),
- **(II)** $\mathfrak{so}(g)^{\mathfrak{h}} = \mathbb{R}.J$ *where J is (pseudo-)Kähler* ($J^2 = -\mathrm{Id}$),
- **(III)** $\mathfrak{so}(g)^{\mathfrak{h}} = \mathbb{R}.L + \mathbb{R}.J$ *where L is parakähler, J pseudo-Kähler and $JL = LJ$.*

So the weaker assumption "Einstein, non Ricci-flat" let only the subalgebra $\mathfrak{a}$ of the degenerate elements of $\mathfrak{so}(g)^{\mathfrak{h}}$ be nontrivial (with the restrictions mentioned in theorem 2); the form of $\mathfrak{so}(g)^{\mathfrak{h}}/\mathfrak{a}$ is the same.

Besides, a representation is said totally reducible if every stable subspace admits a stable complement. Then for a (pseudo-)Riemannian manifold:

locally symmetric, simple	$\Rightarrow$	Einstein non Ricci-flat with totally reducible holonomy	$\Rightarrow$	Einstein non Ricci-flat

and the medium condition soon implies $\mathfrak{a} = \{0\}$. Indeed, each space $\ker N$ or $\operatorname{Im} N$ for $N \in \mathfrak{a}$ is stable under the action of the holonomy group, but admits no stable complement.

7.2 Theorem 2 deals with a typical problem in *pseudo*-Riemannian geometry.

We saw page 16 that for a Riemannian metric, $\mathfrak{so}(g)^{\mathfrak{h}}$ is semi-simple so studying it is easier. In particular, theorem 2 says nothing more than the classical result "hyperkähler ⇒ Ricci-flat": the semi-simpleness of $\mathfrak{so}(g)^{\mathfrak{h}}$ immediately implies $\mathfrak{a} = \{0\}$; besides only case **(II)** is possible.

For a pseudo-Riemannian manifold now, $\mathfrak{so}(g)^{\mathfrak{h}}$ is *not* semi-simple in general. The reason is that the holonomy group is not semi-simple in general. For instance, you can easily build a weakly irreducible pseudo-Riemannian manifold with a holonomy group preserving the isotropic subspace you want, and no complement of it. For example on $\mathbb{R}^{r+s}$, any metric of signature (r, s) with $r \geq s$ can be written, at some point $p \in \mathbb{R}^{r+s}$, in a suitable basis $\beta = (e_i)_{i=1}^{r+s}$ and with $t \leq s$:

$$\mathrm{Mat}(g_{|p}) = \begin{pmatrix} 0 & 0 & I_t \\ 0 & I_{r-t,s-t} & 0 \\ I_t & 0 & 0 \end{pmatrix}.$$

Then, denoting by $(x_i)_{i=1}^{r+s}$ the coordinates of $\mathbb{R}^{r+s}$ corresponding to β, any metric on $\mathbb{R}^{r+s}$ of the following form:

$$\mathrm{Mat}_\beta(g) = \begin{pmatrix} 0 & 0 & I_t \\ 0 & G & 0 \\ I_t & 0 & 0 \end{pmatrix} \quad \text{with} \begin{cases} {}^tG I_{r-t,s-t} G^{-1} = I_{r-t,s-t} \\ G = G((x_i)_{i=t+1}^{r+s}) \end{cases},$$

generates an holonomy group acting trivially on $\mathrm{span}((e_i)_{i=1}^{t+1})$. Besides if $G((x_i)_{i=t+1}^{r+s})$ is sufficiently generic, the action is weakly irreducible on $\mathrm{T}_p\mathcal{M}$. Note that $\mathrm{span}((e_i)_{i=1}^{t+1})^\perp = \mathrm{span}((e_i)_{i=1}^{r+s-t})$ is also holonomy-stable. Similarly, you can build indecomposable metrics admitting any number of (partially or totally) isotropic subspaces, stable by holonomy.

Consequently, whereas the holonomy groups of the *irreducible* pseudo-Riemannian manifolds belong, as in the Riemannian case, to a finite known list of families of groups, the situation is far more complicated for those of the *weakly irreducible* ones. This motivates theorems providing restrictions about the nilpotent part of the holonomy group (or of $\mathfrak{so}(g)^{\mathfrak{h}}$, like here).

An example. Let us go back to our problem. The above family of metrics immediately provides, for example, weakly irreducible Lorentzian (*i.e.* pseudo-Riemannian of signature $(n-1, 1)$) manifolds, admitting a nilpotent parallel g-anti selfadjoint endomorphism of nilpotence index 3, *i.e.* an N in $\mathfrak{so}(g)^{\mathfrak{h}}$ with $N^3 = 0$ and $N^2 \neq 0$.

Taking $(r, s) = (2, 1)$ and $t = 1$, we get on $\mathcal{M} = \mathbb{R}^3$ a metric g defined by the following matrix in the canonical coordinates (x_1, x_2, x_3):

$$\mathrm{Mat}(g) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & a & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{with } a = a(x_2, x_3) \text{ and } a(0,0) = 1.$$

Let us denote by (X_1, X_2, X_3) the coordinate-vectors associated to (x_1, x_2, x_3). The vector X_1 is stable by parallel transport, thus by holonomy. So, written in $\mathrm{T}_{(0,0,0)}\mathbb{R}^3$, the Lie algebra of the holonomy group is included in the following algebra:

$$\mathfrak{h} \subset \mathfrak{n} = \left\{ \begin{pmatrix} 0 & l & 0 \\ 0 & 0 & -l \\ 0 & 0 & 0 \end{pmatrix} \; ; \; l \in \mathbb{R} \right\} = \mathfrak{so}(g_{|0}) \cap \mathrm{Stab}(X).$$

For a sufficiently generic, for example $a(x_2, x_3) = x_2.x_3$, $\mathfrak{h} = \mathfrak{n}$ so $(\mathcal{M}, g)$ is weakly irreducible. Now, the endomorphism N defined at $(0,0,0)$ by $N(X_1) = 0$, $N(X_2) = X_1$ and $N(X_3) = -X_2$ commutes with $\mathfrak{h}$, is g-anti selfadjoint and nilpotent of index three, so is as required.

7.3 Concerning theorem 1: an example with $\mathrm{Ric}^2 = 0$ and $\mathrm{Ric} \neq 0$.

Theorem 1 requires that ric is nondegenerate to ensure that Ric has no nilpotent part. This assumption is necessary; it can be seen on a very simple example borrowed from [CGR00] p.40. Take $(\mathcal{M}, \omega) = (\mathbb{R}^2, \mathrm{d}x \wedge \mathrm{d}y)$ and, denoting the coordinate vectors by X and Y, the connection defined by

$$D_X X = D_Y X = D_X Y = 0, \quad D_Y Y = xX.$$

In particular, X is stable by holonomy. By definition, D is torsion-free and we check:

$$\begin{aligned} (D_{aX+bY}\omega)(X,Y) &= a(D_X\omega)(X,Y) + b(D_Y\omega)(X,Y) \\ &= a[L_X(\omega(X,Y)) - \omega(X, D_X Y)] + b[L_Y(\omega(X,Y)) - \omega(X, D_Y Y)] \\ &= 0 \end{aligned}$$

so D is symplectic. Now $R(X,Y)X = 0$ and $R(X,Y)Y = -X$, so $\mathrm{ric}(Y,Y) = -1$, $\ker \mathrm{ric} = \mathrm{span}(X)$ and $D\,\mathrm{ric} = 0$. Actually, $DR = 0$ *i.e.* $(\mathcal{M}, \omega, D)$ is even symmetric. Now, $\mathrm{Ric}(X) = 0$ and $\mathrm{Ric}(Y) = X$ so $\mathrm{Ric} \neq 0$ and $\mathrm{Ric}^2 = 0$.

Remark. Besides, examples where the minimal polynomial P_λ of Ric corresponds to a λ in $\mathbb{R}^*$, $i\mathbb{R}^*$ or $\mathbb{C} \setminus (\mathbb{R}^* \cup i\mathbb{R}^*)$ are numerous. They are the parakähler and (pseudo-)kähler manifolds, see prop. 4. Next subsection gives symmetric examples of the three types.

7.4 The low-dimensional cases in theorem 1 and 2.

Let us recall the following fact:

Proposition 5 *Let $(\mathcal{M}, g)$ be a Riemannian or pseudo-Riemannian manifold. Let us suppose that* $\dim \mathcal{M} \leq 3$*, or that $\mathcal{M}$ has a complex structure and is of complex dimension equal or less than 3. Then the curvature tensor R is determined by* ric.

A proof can be found in [Bes87] pp. 47–49. Consequently a Ricci-parallel manifold of low enough dimension, as required in the above proposition, is locally symmetric. So in theorem 1, the weakly indecomposable subfactors of the factor on which ric is nondegenerate are (locally) symmetric as soon as:

(i) they are of dimension two,

(ii) or they are of dimension four or six and admit a (pseudo-)Kähler structure, *i.e.* are of type **(II)** or **(III)** in the terms of theorem 2.

Then Berger's list —you can find its restriction to the symplectic case, with which we deal here, in [Bi98b] pp.267-268— provides the list of the relevant simply connected symmetric spaces. In dimension 2 and 4, it can be (see [Bi98a] p.315):

Space	Dimension	Type, in the sense of th.2	sign(ric)
$SL(2,\mathbb{R})/\mathbb{R}^*$	2	**(I)**	$(1,1)$
$SU(2)/SO(2)$	2	**(II)**	$(2,0)$
$SL(2,\mathbb{R})/SO(2)$	2	**(II)**	$(0,2)$
$SU(3)/(SU(2)\times SO(2))$	4	**(II)**	$(4,0)$
$SU(1,2)/(SU(2)\times SO(2))$	4	**(II)**	$(0,4)$
$SU(1,2)/(SU(1,1)\times SO(2))$	4	**(II)**	$(2,2)$
$SL(2,\mathbb{C})/\mathbb{C}^*$	4	**(III)**	$(2,2)$

The (pseudo-)Kähler symmetric spaces of dimension six are more numerous so we have not quoted them here. To obtain the full list of the simply connected, simple, symplectic symmetric spaces of dimension 4 or less, one has to add the only non-(pseudo)Kähler one of dimension 4:

$SL(3,\mathbb{R})/(SL(2,\mathbb{R})\times\mathbb{R}^*)$	4	**(I)**	$(2,2)$

References

[BBI97] Lionel Bérard-Bergery et Aziz Ikemakhen. Sur l'holonomie des variétés pseudo-riemanniennes de signature (n,n). *Bulletin de la Société Mathématique de France*, 125 n°1, 93–114, 1997.

[Ber00] W. Bertram, *The geometry of Jordan and Lie structures*, Lecture Notes in Mathematics n.1754, Springer Verlag — *Berlin, Heidelberg, New York*, 2000.

[Bes87] Arthur L. Besse. *Einstein Manifolds.* Springer Verlag — *Berlin, Heidelberg*, 1987.

[Bi98a] P. Bieliavsky, Four-Dimensional Simply Connected Symplectic Symmetric Spaces. Geom. Dedicata 69 (1998), no. 3, 291–316.

[Bi98b] P. Bieliavsky, Semi-Simple symplectic symmetric spaces. Geom. Dedicata 73 (1998), no. 3, 245–273.

[BCG97] P. Bieliavsky, M. Cahen and S. Gutt, A class of homogeneous symplectic manifolds, in: Geometry and nature, Nencka, Hanna (Ed.); Bourguignon, Jean-Pierre (Ed.), Contemporary mathematics 203, American Mathematical Society , Providence RI, 1997, 241–255.

[BBB01] Ch. Boubel and L. Bérard Bergery, On pseudo-Riemannian manifolds whose Ricci tensor is parallel. Geom. Dedicata 86 (2001), no. 1-3, 1–18.

[BC99] F. Bourgeois and M. Cahen, A variationnal principle for symplectic connections. *J. Geometry and Physics* **30** (1999), no. 3, pp. 233–265.

[CGR00] M. Cahen, S. Gutt and J. Rawnsley, Symplectic connections with parallel Ricci tensor, in: Poisson geometry (Warsaw, 1998), 31–41, Banach Center Publ., 51, Polish Acad. Sci., Warsaw, 2000.

[Iw50] H. Iwamoto, On the structure of Riemannian spaces whose holonomy fix a null system. *Tôhoku Math. J.* 1:109-135, 1950.

[KoNo69] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol I., Interscience Publ., 1969.

[Li83] A. Lichnerowicz, Quantum mechanics and deformations of geometrical dynamics, in: Quantum theory, groups, fields and particles, Reidel, 1983, 3–82.

[Lo69] O. Loos, Symmetric spaces, Benjamin, New York, 1969.

[Mi63] J. Milnor, Morse theory. Princeton University Press, 1963.

[Wu67] H. Wu. Holonomy groups of indefinite metrics. *Pacific Journal of Mathematics*, 20:351–392, 1967.