ON SMOOTH MAPS WITH FINITELY MANY CRITICAL POINTS

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Abstract. We compute the minimal number of critical points over all smooth maps between two manifolds of small codimension and give some partial results for higher codimension spheres.

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1. Introduction

Set $\varphi(M, N)$ for the minimal number of critical points a smooth map $f : M \to N$ can have. One can consult [3] which surveys various features of this invariant (see also [20]). Generally one gave topological conditions ensuring that $\varphi(M, N)$ is infinite.

The aim of this note is to find when non-trivial $\varphi(M^m, N^n)$ can occur if the dimensions m and n satisfy $m \geq n \geq 2$. Non-trivial means here finite and non-zero. Our main result is the following:

Theorem 1.1. *Assume that $\varphi(M^m, N^n)$ is finite, where $0 \leq m - n \leq 3$ and $(m, n) \notin \{(2, 2), (4, 3), (4, 2), (5, 3), (5, 2), (6, 3), (8, 5)\}$. When $m - n = 3$ assume that the Poincaré conjecture in dimension 3 holds true.*
Then $\varphi(M^m, N^n) \in \{0, 1\}$ and $\varphi(M^m, N^n) = 1$ precisely when M^m is diffeomorphic to the connect sum $\widehat{N}\sharp\Sigma^m$ of a manifold $\widehat{N}$ fibering over N^n with an exotic sphere Σ^m but M^m is not a fibration itself.

Proof. The statement is a consequence of propositions 3.1, 4.1 and 5.1. □

Remark 1.2.
- In general it is not sufficient to ask M^m be diffeomorphic to the connect sum $\widehat{N}\sharp\Sigma^m$ and not diffeomorphic to $\widehat{N}$. In fact the exotic 7-spheres constructed by Milnor in [16] are all fibrations over S^4 (with fibre S^3) and pairwise not-diffeomorphic.
- However in the codimension 0 case we think that if M^m is diffeomorphic to the connect sum $\widehat{N}\sharp\Sigma^m$ and not diffeomorphic to $\widehat{N}$ then M^m is not a (smooth) covering of N. This is the case when N^n is hyperbolic, which provides lots of examples when φ equals 1.

  In fact Farrell and Jones ([11]) proved that any hyperbolic N^n has some finite covering (in fact one needs only large enough degree) $\widehat{N}$ such that $\widehat{N}\sharp\Sigma$ is not diffeomorphic to $\widehat{N}$. Then $\widehat{N}\sharp\Sigma$ has metrics of negative curvature although they cannot be hyperbolic. This shows that $\varphi(\widehat{N}\sharp\Sigma, N) = 1$.

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Conversely if $\varphi(M^n, N^n) = 1$ then $M^n = \widehat{N}\sharp\Sigma$, which must be not diffeomorphic to $\widehat{N}$. In particular, if the degree of the covering is large enough M^n supports a negatively curved metric (by Farrell-Jones' construction) but not a hyperbolic structure (since otherwise by Mostow rigidity M^n would be isometric hence difeomorphic to $\widehat{N}$.

- In most of the cases excluded in the hypothesis of the theorem one can find examples with non-trivial $\varphi(M^m, N^n) \geq 2$ (see below).
- One expects that for all (m, n) with $m-n \geq 4$ such examples abound. This is the situation for the local picture. A sample is any complex projective manifolds X admitting non-trivial morphisms into $\mathbf{CP}^1$.
- The case $n = 1$ was analyzed in [21] for h-cobordisms. Their results states that for non-trivial h-cobordisms $\varphi(M, [0, 1]) = 2$.

We compute then $\varphi(S^m, S^n)$ in a few cases and look for a more subtle invariant which would measure how far is a manifold from being a covering of another one. We will consider henceforth that all manifolds are closed, oriented and connected unless the opposite is stated.

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2. Elementary computations for surfaces

Necessary and sufficient conditions were given by Patterson in [19] for the existence of a covering of some surface with prescribed degree and ramification orders. Specifically his result can be stated as follows:

Proposition 2.1. *Let X be a Riemann surface of genus $g \geq 1$. Let $p_1, ..., p_k$ be distinct points of X and $m_1, ..., m_k$ strictly positive integers so that*

$$\sum_{i=1}^{k}(m_i - 1) = 0(mod\ 2).$$

Let d be an integer such that $d \geq max_{i=1,...,k} m_i$. Then there exists a Riemann surface Y and a holomorphic covering map $f : Y \to X$ of degree d such that there exist k points $q_1, ..., q_k$ in Y so that $f(q_j) = p_j$, f is ramified to order m_j at q_j and is unramified outside the set $\{q_1, ..., q_k\}$.

One observes now that a smooth map $f : Y \to X$ between surfaces has finitely many critical points if and only if it is a ramified covering. Furthermore $\varphi(Y, X)$ is the minimal number of ramification points of a covering $Y \to X$. Estimations could be obtained from the previous result. Denote by Σ_g the oriented surface of genus g. Set $[[r]]$ for the smallest integer greater than or equal to r. Our principal result in this section is the following:

Proposition 2.2. *Let Σ and Σ' be oriented surfaces of Euler characteristic χ and χ' respectively.*

1. *If $\chi' > \chi$ then $\varphi(\Sigma', \Sigma) = \infty$.*
2. *If $\chi \leq 0$ then $\varphi(\Sigma, S^2) = 3$.*
3. *If $\chi \leq -2$ then $\varphi(\Sigma, \Sigma_1) = 1$.*
4. *If $2 + 2\chi \leq \chi' < \chi \leq -2$ then $\varphi(\Sigma', \Sigma) = \infty$.*

(5) *If* $0 \leq |\chi| \leq \frac{|\chi'|}{2}$, *write* $|\chi'| = a|\chi| + b$ *with* $0 \leq b < |\chi|$*; then*

$$\varphi(\Sigma', \Sigma) = \left[\left[\frac{b}{a-1}\right]\right].$$

In particular if $G \geq 2(g-1)^2$ *then*

$$\varphi(\Sigma_G, \Sigma_g) = \begin{cases} 0 & \textit{if} \quad \frac{G-1}{g-1} \in \mathbf{Z}_+ \\ 1 & \textit{otherwise} \end{cases}$$

Proof. The first claim is obvious.
Further $\varphi(\Sigma, S^2) \leq 3$ because any surface is a branched cover of the 2-sphere in 3 points (from [2]). A deeper result is that the same inequality holds in the holomorphic framework. In fact Belyi's theorem states that any Riemann surface defined over a number field admits a meromorphic function on it with only three critical points (see e.g. [24]).
On the other hand, assume that $f : \Sigma \to S^2$ is a ramified covering with at most two critical points. Then, f induces a covering map $\Sigma - f^{-1}(E) \to S^2 - E$, where E is the set of critical values and its cardinality $|E| \leq 2$. Therefore one has an injective homomorphism $\pi_1(\Sigma - f^{-1}(E)) \to \pi_1(S^2 - E)$; but, as $\pi_1(E)$ is a quotient of $\pi_1(\Sigma - f^{-1}(E))$ and $\pi_1(S^2 - E)$ is trivial or infinite cyclic, this shows that $\Sigma = S^2$ necessarily.
Next, the unramified coverings of tori are tori; thus, any smooth map $f : \Sigma_G \to \Sigma_1$ with finitely many critical points must be ramified, hence $\varphi(\Sigma_G, \Sigma_1) \geq 1$ when $G \geq 2$. On the other hand, by Patterson's theorem, there exists a degree $d = 2G-1$ covering from a surface Σ into Σ_1, with a single ramification point of multiplicity $2G-1$; then, by Hurwitz' formula it follows that Σ has genus G, which shows that $\varphi(\Sigma_G, \Sigma_1) = 1$ precisely.

Lemma 2.3. $\varphi(\Sigma', \Sigma)$ *is the smallest integer* k *for which one has*

$$\left[\left[\frac{\chi' - k}{\chi - k}\right]\right] \leq \frac{\chi' + k}{\chi}$$

Proof. Suppose that Σ_G is a degree d covering of Σ_g ramified in k points with the multiplicities $m_i = d - \lambda_i$, where $0 \leq \lambda_i \leq d-2$. Set $\lambda = \sum_i \lambda_i$. One has

$$\lambda \leq k(d-2),$$

while Hurwitz' formula states that

$$d(k - \chi) = k - \chi' + \lambda.$$

Conversely, if there are solutions (k, λ, d) of the two equations above, with $k, \lambda \geq 0$ and $d \geq 1$, then one can find integers m_i, λ_i as above and construct, by Patterson's theorem, a ramified covering $\Sigma' \to \Sigma$ of degree d, with k ramification points of multiplicities m_i. So, $\varphi(\Sigma', \Sigma)$ is the least integer $k \geq 0$ for which there exists a solution $(k, \lambda, d) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N}_+$ of

$$0 \leq d(k - \chi) + \chi' - k = \lambda \leq k(d-2).$$

That is, for $\chi \leq -2$, $\varphi(\Sigma', \Sigma)$ is the least $k \in \mathbf{N}$ for which there exists a positive integer d satisfying

$$\frac{\chi' - k}{\chi - k} \leq d \leq \frac{\chi' + k}{\chi}$$

and this is clearly equivalent to what stated in Lemma 2.3. □

Assume now that $2+2\chi \leq \chi' < \chi \leq -2$. If $f : \Sigma' \to \Sigma$ was a ramified covering then, as $\frac{\chi'+k}{\chi} < 2$, by Lemma 2.3 one would deduce that $\chi' = \chi$, which is a contradiction. Therefore $\varphi(\Sigma', \Sigma) = \infty$ holds.

Finally, assume that $\frac{\chi'}{2} \leq \chi \leq -2$. One has to compute the minimal k fulfilling

$$\left[\left[\frac{a\chi - b - k}{\chi - k}\right]\right] \leq \frac{a\chi - b + k}{\chi},$$

or equivalently,

$$\left[\left[\frac{b + (1-a)k}{\chi - k}\right]\right] \geq \frac{b - k}{\chi}.$$

The smallest k for which the quantity in the brackets is non-positive is $k = \left[\left[\frac{b}{a-1}\right]\right]$, for which

$$\left[\left[\frac{b + (1-a)k}{\chi - k}\right]\right] \geq 0 \geq \frac{b - k}{\chi}.$$

For smaller k one has a strictly positive integer on the left hand side, which is therefore at least 1. But the right hand side is strictly smaller than 1, hence the inequality could not hold. This proves the claim. $\square$

3. Equidimensional case $n \geq 3$

The situation changes completely in dimension $n \geq 3$. The first who noticed that a smooth map $\mathbf{R}^n \to \mathbf{R}^n$ ($n \geq 3$) which has only an isolated critical point p is actually a local homeomorphism at p was H.Hopf according to ([8], II, p.535). Our result below is an easy application of this fact. We outline the proof for the sake of completeness.

Proposition 3.1. *Assume that M^n and N^n are compact manifolds. If $\varphi(M^n, N^n)$ is finite and $n \geq 3$ then $\varphi(M^n, N^n) \in \{0, 1\}$. Further $\varphi(M^n, N^n) = 1$ if and only if M^n is the connected sum of a finite covering $\widehat{N^n}$ of N^n with an exotic sphere and M^n is not a covering of N^n.*

Proof. There exists a smooth map $f : M^n \to N^n$ which is a local diffeomorphism on the preimage of the complement of a finite subset of points. Notice that f is a proper map.

Let $p \in M^n$ be a critical point and $q = f(p)$, B a small (closed) ball around q and U the connected component of $f^{-1}(B)$ containing p.

The restriction $f|_{U - f^{-1}(q)}$ is a proper submersion, since there are no other critical points than p. Thus, for small enough B, $f|_{U-\{p\}}$ will be a proper submersion if and only if the points of $f^{-1}(q) - \{p\}$ do not accumulate at p. Suppose that the contrary holds, so that there exist $p_i \in U \cap f^{-1}(q) - \{p\}$ converging to p. Since p_i is not critical, f is a diffeomorphism of a small neighborhood V_i of p_i onto a small disk around q. These neighborhoods can be chose to be disjoint. In particular there exist pairs of distinct points (r_i, r_j), with $r_i \in V_i - f^{-1}(q)$, $r_j \in V_j - f^{-1}(q)$ and $f(r_i) = f(r_j)$. However if r_i and r_j are closed enough to p then they belong to the same connected component of $f^{-1}(B - \{q\})$. The proper submersion $f|_{U-f^{-1}(q)}$ is a covering, hence its restriction to a connected component is a diffeomorphism. In particular this shows that f is injective in a neighborhood of p (except possibly at p), contradicting the existence of pairs (r_i, r_j) of double points accumulating at p.

Therefore $f|_{U-\{p\}}$ is a covering. The image $f(U-\{p\}) \subset B$ contains $B-\{q\}$. If the image was B then $U-\{p\}$ would be a ball (since it covers a ball) which is impossible. Therefore $f(U-\{p\}) = B-\{q\}$ and $U-\{p\}$ is a punctured ball (since in dimension $n \geq 3$ coverings of punctured balls are trivial) and $f|_{U-\{p\}} : U-\{p\} \to B-\{q\}$ is a diffeomorphism. One verifies then easily that the inverse of $f|_U : U \to B$ is continuous at q hence it is a homeomorphism. In particular U is homeomorphic to a ball. Since ∂U is a sphere the results of Smale (e.g. [25]) imply that U is diffeomorphic to the ball for $n \neq 4$.
One obtained that f is a local homeomorphism hence topologically a covering map. Thus M^n is homeomorphic to a covering of N^n. Let us show now that one can modify M^n by taking the connect sum with an exotic sphere in order to get a smooth covering of N^n.
By gluing a disk to U, using the identification $h : \partial U \to \partial B = S^{n-1}$, we obtain an exotic sphere $\Sigma_1 = U \cup_h B^n$. Set $M_0 = M - int(U)$, $N_0 = N - int(B)$. Given the diffeomorphisms $\alpha : S^{n-1} \to \partial U$ and $\beta : S^{n-1} \to \partial B$ one can form the manifolds

$$M(\alpha) = M_0 \cup_{\alpha : S^{n-1} \to \partial U} B^n, N(\beta) = N_0 \cup_{\beta : S^{n-1} \to \partial B} B^n.$$

Set $h = f|_{\partial U} : \partial U \to \partial B = S^{n-1}$. There is then a map $F : M(\alpha) \to N(h \circ \alpha)$ given by:

$$F(x) = \begin{cases} x & \text{if } x \in D^n, \\ f(x) & \text{if } x \in M_0. \end{cases}$$

The map F has the same critical points as $f|_{M_0}$, hence it has precisely one critical point less than $f : M \to N$.
One chooses $\alpha = h^{-1}$ and we remark that $M = M(h^{-1}) \sharp \Sigma_1$. We obtained above that $f : M = M_1 \sharp \Sigma_1 \to N$ decomposes as follows. The restriction of f to M_0 extends to M_1 without introducing extra critical points while the restriction to the homotopy ball corresponding to the holed Σ_1 has precisely one critical point.
Thus iterating this procedure one finds that there exist exotic spheres Σ_i so that $f : M = M_k \sharp \Sigma_1 \sharp \Sigma_2 ... \sharp \Sigma_k \to N$ decomposes as follows: the restriction of f the k-holed M has no critical points, and it extends to M_k without introducing any critical point. Each critical point of f corresponds to a (holed) exotic Σ_i. In particular M_k is a smooth covering of N.
Now the connected sum $\Sigma = \Sigma_1 \sharp \Sigma_2 ... \sharp \Sigma_k$ is also an exotic sphere. Let $\Delta = \Sigma - int(B^n)$ be the homotopy ball obtained by removing an open ball from Σ. We claim that there exists a smooth map $\Delta \to B^n$, extending any given diffeomorphism of the boundary and which has exactly one critical point. Then one builds up a smooth map $M_k \sharp \Sigma \to N$ having precisely one critical point, by putting together the obvious covering on the 1-holed M_k and $\Delta \to B^n$. This will show that $\varphi(M, N) \leq 1$.
The claim follows easily from the following two remarks. First the homotopy ball Δ is diffeomorphic to the standard ball by [25], when $n \neq 4$. Further any diffeomorphism $\varphi : S^{n-1} \to S^{n-1}$ extends to a smooth homeomorphism with one critical point $\Phi : B^n \to B^n$, by example

$$\Phi(z) = \exp\left(-\frac{1}{\| z \|^2}\right) \varphi\left(-\frac{z}{\| z \|}\right).$$

For $n = 4$ we need an extra argument. Each homotopy ball $\Delta_i^4 = \Sigma_i - int(B^4)$ can be chosen to be small enough (by construction) hence Δ_i^4 can be engulfed in a 4-ball. Therefore Δ^4 can be engulfed in S^4. Moreover Δ^4 is the closure of one connected component of the complement of $\partial \Delta^4 = S^3$ in S^4. The result of Huebsch and Morse

from [14] states that any diffeomorphism $S^3 \to S^3$ has a Schoenflies extension to a homeomorphism $\Delta^4 \to B^4$ which is a diffeomorphism everywhere but at one (critical) point. This proves the claim.
Remark finally that $\varphi(M^n, N^n) = 0$ if and only if M^n is a covering of N^n. Therefore if M^n is diffeomorphic to the connect sum $\widehat{N^n}\sharp\Sigma^n$ of a covering $\widehat{N^n}$ with an exotic sphere Σ^n, and if it is not diffeomorphic to a covering of N^n then $\varphi(M^n, N^n) \neq 0$. Now drill a small hole in $\widehat{N^n}$ and glue (differently) an n-disk B^n (respectively a homotopy 4-ball if $n = 4$) in order to get $\widehat{N^n}\sharp\Sigma^n$. The restriction of the covering $\widehat{N^n} \to N^n$ to the boundary of the hole extends (by the previous arguments) to a smooth homeomorphism with one critical point over Σ^n. Thus $\varphi(M^n, N^n) = 1$. □

Remark 3.2.
- One should stress that not all exotic structures can be obtained from a given structure by connect sum with an exotic sphere. On the other hand the connect sum with an exotic sphere does not necessarily change the diffeomorphism type. However results of Farrell and Jones [11] show that any hyperbolic manifold has finite coverings for which any connect sum with an exotic sphere changes the diffeomorphisms type.
- We don't know whether the condition that $M^n = \widehat{N^n}\sharp\Sigma$ be homeomorphic but not diffeomorphic to $\widehat{N^n}$ would be sufficient to imply that M^n is not a smooth covering of N^n. The results of Farrell-Jones above show that this is the case for large degree coverings of hyperbolic manifolds.

Corollary 3.1. In dimension $n \leq 6$, $n \neq 4$ φ is either 0 or ∞.

Proof. In fact two 3-manifolds which are homeomorphic are diffeomorphic and in dimensions 5 and 6 there are no exotic spheres. □

Remark 3.3. A careful analysis of open maps between manifolds of the same dimensions was carried out in [8]. In particular one proved that an open map of finite degree has a branch locus of codimension 2 (see [8], II) in the PL or TOP framework.

4. Local obstructions for higher codimension

Our main result in this section, less precise than that for codimension 0, is a simple consequence of the investigation of local obstructions. In fact the existence of analytic maps $\mathbf{R}^m \to \mathbf{R}^n$ with isolated singularities is rather exceptional in the context of smooth real maps (see [17]).

Proposition 4.1. *If $\varphi(M^m, N^n)$ is finite and either $m = n+1 \neq 4$, $m = n+2 \neq 4$, or $m = n+3 \notin \{5, 6, 8\}$ (when one assumes that the Poincaré conjecture to be true) then M is homeomorphic to a fibration of base N. In particular if $m = 3, n = 2$ then $\varphi(M^3, N^2) \in \{0, \infty\}$, except possibly for M^3 a non-trivial homotopy sphere and $N^2 = S^2$.*

Proof. One shows first:

Lemma 4.2. *Assume that $\varphi(M^m, N^n) \neq 0$ is finite for two manifolds M^m and N^n. Then there exists a polynomial map $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$ having an isolated singularity at the origin.*

Proof. The hypothesis implies the existence of a smooth map $f : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$ with one isolated singularity at the origin. We can assume that the critical point is not an isolated point on the fiber over 0 (see the remark 4.10 below). If the restriction $f|_{S^{m-1}}$ is of maximal rank then the construction goes at follows. One approximates up to the first derivative the restriction $f|_{S^{m-1}}$ to the unit sphere by a polynomial map $\tilde{\psi}$ (of some degree d) and one extends the later to all of $\mathbf{R}^m$ by $\psi(x) = |x|^d \tilde{\psi}\left(\frac{x}{|x|}\right)$. If the approximation is sufficiently closed then $\tilde{\psi}$ will be of maximal rank around the unit sphere hence ψ will have an isolated singularity at the origin. However some caution is needed when $f|_{S^{m-1}}$ is not of maximal rank. We consider then the restriction $f|_{B^m - B^m_{1-\delta}}$ to the annulus bounded by the spheres of radius 1 and $1-\delta$ respectively. We claim that:

Lemma 4.3. *There exists some $\delta > 0$ and a polynomial map $\tilde{\psi}$ (of some degree d) such that its extension $\psi(x) = |x|^d \tilde{\psi}\left(\frac{x}{|x|}\right)$ approximates $f|_{B^m - B^m_{1-\delta}}$ sufficiently closed.*

Proof. It suffices to see that $f^{-1}(0) \cap (B^m - B^m_{1-\delta})$ has a conical structure. Remark that the function $r(x) = |x|^2$ has finitely many critical values on $f^{-1}(0) \cap B^m - B^m_{1-\delta}$ since f is smooth and has no critical points in this range. Thus one can choose δ small enough δ so that r has no critical points. Then the proof of theorem 2.10 p.18 from [17] applies in this context. This implies the existence of a good approximation of conical type. □

In particular our approximating $\psi(x)$ has no critical points in the given annulus. However if $x_0 \neq 0$ is a critical point of $\psi(x)$ in the ball, then all points of the line $0x_0$ would be critical, since ψ is homogeneous. Therefore ψ has an isolated singularity at the origin.
Notice that one can choose the approximation so that $S^{m-1} \cap \psi^{-1}(0)$ is isotopic to $S^{m-1} \cap f^{-1}(0)$ and therefore non-void. Although ψ is not real analytic, each of its components are algebraic because they can be represented as $\psi_j(x) = P_j(x) + Q_j(x)|x|$, where $P_j(x)$ (resp. $Q_j(x)$) are polynomials of even (resp. odd) degree. One verifies easily that the curve selection lemma ([17], p.25) can be extended without difficulty to sets defined by equations like the ψ_j above. Then the proof of theorem 11.2 (and lemma 11.3) from [17], p.97-99 extends to the case of ψ. In particular there exists a Milnor fibration associated to ψ (the complement of the singular fiber $\psi^{-1}(0)$ in the unit sphere S^{m-1} fibers over S^{n-1}). Alternatively $(S^{m-1}, S^{m-1} \cap \psi^{-1}(0))$ is a Neuwirth-Stallings pair according to [15] and $S^{m-1} \cap \psi^{-1}(0)$ is non-empty. The main theorem from [15] provides then a polynomial map with an isolated singularity at the origin as required. □

Milnor (see [17]) called such an isolated singularity trivial if its local Milnor fiber is diffeomorphic to a disk. Then it was shown in ([7], p.151) that f is trivial if and only if f is locally topologically equivalent to the projection map $\mathbf{R}^m \to \mathbf{R}^n$, whenever the dimension of the fiber is $m - n \neq 4, 5$. We recall that the existence of polynomials with isolated singularities was (almost) settled in [7, 17]:

Proposition 4.4. *For $0 \leq m - n \leq 2$ non-trivial polynomial singularities exist precisely for $(2, 2)$, $(4, 3)$ and $(4, 2)$.*
For $m - n \geq 4$ non-trivial examples occur for all (m, n).

For $m-n=3$ non-trivial examples occur for $(5,2)$ and $(8,5)$. Morover, if the 3-dimensional Poincaré conjecture is false then there are non-trivial examples for all (m,n). Otherwise all examples are trivial except $(5,2)$, $(8,5)$ and possibly $(6,3)$.

We consider now a smooth map $f: M^m \to N^n$ where m,n are as in the hypothesis. For each critical point p there are open balls $B^m(p)$ and $B^n(f(p))$ for which the restriction $f|_{2B^m(p)}: 2B^m(p) \to 2B^n(f(p))$ has an isolated singularity at p. One identifies $B^m(p)$ with the ball of radius 1 in $\mathbf{R}^m$. In the proof of lemma 4.2 we approximated $f|_{\partial B^m(p)}$ by a polynomial map g with isolated singularities, both maps having isotopic links and being closed to each other. Assume for simplicity that f (hence g) is of maximal rank around this (unit) sphere. The general case follows along th same lines. Then there exists an isotopy f_t ($t \in [0,1]$) between $f|_{\partial B^m(p)}$ and $g|_{\partial B^m(p)}$, which is closed to the identity. In particular all f_t are of maximal rank around the unit sphere. Let $\rho: [0,4] \to [0,1]$ be a smooth decreasing function with $\rho(x) = 0$ if $x \geq 1$ and $\rho(x) = 1$ if $x \leq \frac{1}{2}$). Set $F: 2B^m(p) \to 2B^n(f(p))$ for the map defined by:

$$F(x) = \begin{cases} f_{\rho(|x|^2)}\left(\frac{x}{|x|}\right) & \text{if } |x| \geq \frac{1}{2}, \\ g(x) & \text{if } |x| \leq \frac{1}{2}. \end{cases}$$

If one replace $f_{2B^m(p)}$ by F one obtains a smooth function with an isolated singularity at p, which must be a topological submersion at p (by the previous proposition 4.4). An induction on the number of critical points yields a map $F: M^m \to N^n$ which is a topological fibration. □

Remark 4.5. Notice that not all real smooth maps f have a Milnor fibration, in order that the singularity be trivial in the sense which was used by Milnor. We don't know whether f itself is a topological submersion in this case, thus the need to replace it by another map (locally algebraic), in order to be able to apply proposition 4.4.

Remark 4.6. Therefore, within the range $0 \leq m-n \leq 3$, with the exception of $(2,2)$, $(4,3)$, $(4,2)$, $(5,2)$, $(8,5)$ and $(6,3)$ the non-triviality of φ is related to the exotic structures on fibrations.

One expects that in the case when non-trivial singularities can occur such examples abound.

Example 4.7. In the remaining cases we have:
$(m,n) \in \{(4,3),(8,5)\}$. We will prove below that $\varphi(S^4,S^3) = \varphi(S^8,S^5) = 2$.
$(m,n) = (4,2)$. Non-trivial examples come from Lefschetz fibrations X over a surface F. For instance X is an elliptic K3 surface and F is S^2.
$(m,n) = (2k,2)$. More generally than Lefschetz fibrations, complex projective k-manifolds admitting morphisms onto an algebraic curve.

Further one notices that this local obstructions ar far from being sufficient. In fact the maps arising from smooth maps between compact manifolds are more restrictive. For instance if $M = S^m$ then there is a restriction $\mathbf{R}^m \to \mathbf{R}^n$ which is proper and has finitely many isolated singularities. However adding extra conditions can further restrict the range of dimensions:

Proposition 4.8. *There are no proper smooth functions $f: (\mathbf{R}^m,0) \to (\mathbf{R}^n,0)$ with one isolated singularity at the origin if $m \leq 2n-3$.*

Proof. There is a direct proof similar to that of proposition 6.1. Otherwise let us show that the hypothesis implies that $\varphi(S^m, S^n) \leq 2$ and so proposition 6.1 yields the result.

Let $j_k : S^k \to \mathbf{R}^k$ denote the stereographic projection from the north pole ∞. There exists an increasing unbounded real function ρ such that $|f(x)| \geq \lambda(|x|)$ for all $x \in \mathbf{R}^m$, because f is proper.

We claim that there exists a real function ρ such that $\rho(|x|)f(x)$ extends to a smooth function $F : S^m \to S^n$. Specifically, we want that the function $F_\rho : S^m \to S^n$ defined by

$$F_\rho(x) = \begin{cases} j_n^{-1}(\rho(|j_m(x)|)f(j_m(x)) & \text{if } x \in S^m - \{\infty\}, \\ \infty & \text{otherwise .} \end{cases}$$

be smooth at ∞. This is easy to achieve by taking $\rho(x) > exp(|x|)\lambda^{-1}(|x|)$ for large $|x|$. Now the critical points of F_ρ consists of the two poles, and the claim is proved. □

Remark 4.9. Notice that proper maps like above for $m = 2n - 2$ exist only for $n \in \{2, 3, 5, 9\}$ (see below).

Remark 4.10. A special case is when the critical point p is an isolated point in the fiber $f^{-1}(f(p))$. This situation was settled in [7, 27] where it was shown that the dimensions (m, k) should be $(2, 2), (4, 3), (8, 5)$ or $(16, 9)$, and the map is locally the cone over the respective Hopf fibration.

5. The global structure for topological submersions

Roughly speaking the previous results state that in low codimension maps with finitely many critical points are topological submersions.

Proposition 5.1. *Assume that there exists a topological submersion $f : M^m \to N^n$ with finitely many critical points, and $m > n \geq 2$. Then $\varphi(M, N) \in \{0, 1\}$ and it equals 1 precisely when M is diffeomorphic to the connect sum of a fibration $\widehat{N}$ (over N) with an exotic sphere without being a fibration itself.*

Proof. The first step is to split off one critical point by localizing it within an exotic sphere. Set M_0 for the 1-holed manifold M.

Lemma 5.2. *There exists an exotic sphere Σ_1 and a map f_1 such that:*

1. *f_1 agrees with f on M_0 and f has one critical point in $M - M_0$.*
2. *the map f_1 splits as $f : M = M_1 \sharp \Sigma_1 \to N$, where the restriction $f_1|_{M_0}$ extends over M_1 without introducing new critical points.*
3. *and the restriction of f_1 to the holed exotic sphere $\Sigma_1 - D^m$ has precisely one critical point.*

Proof. Let p be a critical point of f, $q = f(p)$, δ be a small disk around q. We replace f by a map which is locally polynomial around the critical point p, as in the previous section. We show first that:

Lemma 5.3. *There exists a neigbourhood Z_p of p such that the following conditions are fulfilled:*

1. *Z_p is diffeomorphic to $D^n \times D^{m-n}$ (for $m \neq 4$).*
2. *$\partial Z_p = \partial^h Z \cup \partial^v Z_p$, where the restrictions $f : \partial^v Z_p \to D^n$ and $f : \partial^h Z \to \partial D^n$ are trivial fibrations, and $\partial^h Z \cap \partial^v Z_p = S^{n-1} \times S^{m-n-1}$.*

Proof. Let $B^m(p)$ be a sufficiently small ball around p and δ be such that $\delta \subset f(B^m(p))$. We claim that $Z_p = B^m(p) \cap f^{-1}(\delta)$ has the required properties.
One chooses a small ball containing p, $B_0^m(p) \subset Z_p$. Then one uses the argument from ([17], p.97-98) and derive that q is a regular value of the map:

$$f : Z_p - int(B_0^m(p)) \to \delta.$$

Therefore the latter is a fibration, hence a trivial fibration. In particular the manifold with corners ∂Z_p has a collar whose outer boundary is a smooth sphere. Further the manifold with boundary Z_p is homeomorphic to $D^n \times D^{m-n}$ and the boundary ∂Z_p is collared as above. The outer sphere bounds a smooth disk (by Smale) and so Z_p is diffeomorphic to $D^n \times D^{m-n}$. □

Now the proof goes on as in codimension 0. We excise Z_p and glue it back by another diffeomorphism in order that the restriction of f extends over the new ball, without introducing critical points. The gluing diffeomorphism respects the corner manifold structure. □

An inductive argument shows that $\varphi(M, N)$ is finite implies that the connect sum $M \sharp \Sigma$ with an exotic sphere is diffeomorphic to a fibration over N.
We want to prove now that one can find another map $M \to N$ having precisely one critical point. We have first to put all critical points together inside a standard neighborhood:

Lemma 5.4. *If $m > n \geq 2$ then the critical points of f are contained in some cylinder $Z^m \subset M$ which is diffeomorphic to $D^n \times D^{m-n}$ (respectively homeomorphic when $m = 4$, by a homeomorphism which is a diffeomeorphism on the boundary) such that the fibers of f are either transversal to the boundary (actually to the part $D^n \times \partial D^{m-n}$) or contained in $\partial D^n \times D^{m-n}$.*

Proof. Pick-up a regular point x_0 in M. Set U for the set of regular points which can be joined to x_o by an arc γ everywhere transversal to the fibers of f (which will be called transversal for short).
We show first that U is open. In a small neighborhood V of $x \in U$ the fibers can be linearized (by means of a diffeomorphism) and identified to parallel $(m-n)$-planes. Let $y \in V$. If x and y are not in the same fiber then the line joining them is a transversal arc. Otherwise use a helicoidal arc spining around the line, which can be constructed since the fibers have codimension at least 2.
In meantime U is closed in the complement of the critical set. In fact the previous arguments show that two regular points which are sufficiently closed to each other can be joined by a transversal arc with prescribed initial velocity (provided this tangent vector is also transversal to the fiber). Thus, if y_i converge to a regular point y and $y_i \in U$ then y can be joined to x_0 by joining first x_0 to y_i and further y_i to y (for large enough i) with some prescribed initial velocity, in order to insure the smootheness of the arc. This proves that U consists in the set of all regular points.
Further we consider the cylinders Z_{p_i} given by lemma 5.3. Let $f_i \subset \partial Z_{p_i}$ be some fibers in the boundary. Points $q_i \in f_i$ can be joined by everywhere transversal arcs. Since this is an open condition one can find disjoint tubes $T_{i,i+1}$ joining neighborhoods of the fibers f_i in ∂Z_{p_i} and f_{i+1} in $\partial Z_{p_{i+1}}$ and one built up this way a cylinder Z containing all critical points. □

Lemma 5.5. *There exists a smooth map $g : Z \to D^n$ having one critical point such that $g|_{\partial Z} = f|_{\partial Z}$.*

Proof. We know that each restriction $f : Z_i \to D^n$ is topologically a trivial fibration, whose restriction to a collar of the boundary sphere ∂D^n is also trivial as a smooth fibration. We claim that $f : Z \to D^n$ enjoys the same property. This follows from the fact that the restriction to the gluing tubes $f : T_{i,i+1} \to D^n$ are also trivial fiber bundles.
The restriction $f|_{\partial^v Z} : \partial^v Z \to D^n$ is a trivial fibration over the ball. Thus there exists a diffeomorphism $\partial^v f : \partial^v Z \to D^n \times \partial D^{m-n}$, which commutes with the trivial projection $\pi : D^n \times D^{m-n} \to D^n$, namely $\pi \circ \partial^v f = f|_{\partial^v Z}$.
Since $f|_{\partial^h Z} : \partial^v Z \to \partial D^n$ is a trivial fibration there exists a diffeomorphsim $\partial^h f : \partial^v Z \to \partial D^n \times D^{m-n}$ commuting with π. Moreover these two diffeomeorphsims can be chosen to agree on their common intersection, $\partial^h f|_{\partial^h Z \cap \partial^v Z} = \partial^v f|_{\partial^h Z \cap \partial^v Z}$.
We obtain therefore a diffeomorphism ∂f of manifolds with corners $\partial f : \partial Z \to \partial D^m$, defined by:

$$\partial f(x) = \begin{cases} \partial^h f(x) & \text{if } x \in \partial^h Z, \\ \partial^v f(x) & \text{if } x \in \partial^v Z. \end{cases}$$

Assume now that there exists a smooth homeomorphism $\Phi : Z \to D^n \times D^{m-n}$ having precisely one critical point, which extends ∂f, i.e. such that the diagram

$$\begin{array}{ccc} \partial Z & \stackrel{\partial f}{\to} & \partial(D^n \times D^{m-n}) \\ \downarrow & & \downarrow \\ Z & \stackrel{\Phi}{\to} & D^n \times D^{m-n} \end{array}$$

commutes. We set therefore $g(x) = \pi(\Phi(x))$. It is immediate that g has at most one critical point and g is an extension of $f_{\partial Z}$. Our claim in then a consequence of the following:

Lemma 5.6. *Any diffeomorphism of the sphere S^m, with the structure of a manifold with corners $\partial(D^n \times D^{m-n})$, extends to a smooth homeomorphism of $D^n \times D^{m-n}$ with at most one critical point.*

Proof. Instead of working for a direct proof remark that the trivializations leading to ∂f extend over a collar of $\partial(D^n \times D^{m-n})$. This collar is still a manifold with corners, but it contains a smoothly embedded sphere. We use then the standard result (see [14]) to extend further the diffeomorphism from the smooth sphere to the ball. □

Now lemma 5.5 follows. □

□

6. Maps between spheres

For spheres the situation is somewhat simpler due to the global obstructions of topological nature. Our main result settles the case when the codimension is smaller than the dimension of the base. Specifically, we can state:

Proposition 6.1.
(1) *The values of $m > n > 1$ for which $\varphi(S^m, S^n) = 0$ are exactly those arising in the Hopf fibrations i.e. $n \in \{2, 4, 8\}$ and $m = 2n-1$.*
(2) *One has $\varphi(S^4, S^3) = \varphi(S^8, S^5) = \varphi(S^{16}, S^9) = 2$.*
(3) *If $m \leq 2n-3$ then $\varphi(S^m, S^n) = \infty$.*

(4) *If* $\varphi(S^{2n-2}, S^n)$ *is finite then* $n \in \{2, 3, 5, 9\}$.

Proof. Remark first that the existence of the Hopf fibrations $S^3 \to S^2, S^7 \to S^4, S^{15} \to S^8$, shows that $\varphi(S^3, S^2) = \varphi(S^7, S^4) = \varphi(S^{15}, S^8) = 0$. The converse is already known (see Lemma 2.7 in [27] or Lemma 1 in [7]). We will give a slightly different proof below, on elementary grounds.
Using Serre's exact sequence for a $(n-1)$-connected basis one finds that the homology of the fiber F (of the fibration $f : S^m \to S^n$) agrees with that of S^{n-1} up to dimension $n-1$, and a subsequent application of the same sequence shows that F is a homology $(n-1)$-sphere. Therefore $m = 2n-1$. In particular we obtain that the transgression map $\tau^* : H^{n-1}(F) \to H^n(S^n)$ is an isomorphism.
Let $i_F : F \to S^{2n-1}$ and $j_F : S^{2n-1} \to (S^{2n-1}, F)$ denote the inclusion maps. We set $C^*(X)$ for the cochain complex of the space X.

Lemma 6.2. *The composition of maps*

$$C^{n-1}(S^{2n-1}) \xrightarrow{i_F^*} C^{n-1}(F) \xrightarrow{\tau^*} C^n(S^{2n}) \xrightarrow{f^*} C^n(S^{2n-1})$$

is the boundary operator $d : C^{n-1}(S^{2n-1}) \to C^n(S^{2n-1})$.

Proof. The transgression map τ^* can be identified (see for example [30], p.648-651) with the composition

$$H^{n-1}(F) \xrightarrow{\partial^*} H^n(S^{2n-1}, F) \xrightarrow{f^{*-1}} H^n(S^n),$$

where ∂^* is the boundary homomorphism in the long exact sequence of the pair (S^{2n-1}, F). One sees then that

$$C^n(S^{2n-1}, F) \xrightarrow{f^{*-1}} C^n(S^n) \xrightarrow{f^*} C^n(S^{2n-1})$$

agrees with j_F^*. Further the composition of maps from the statement of the lemma is equivalent to

$$C^{n-1}(S^{2n-1}) \xrightarrow{i_F^*} C^{n-1}(F) \xrightarrow{\partial^*} C^n(S^{2n-1}, F) \xrightarrow{j_F^*} C^n(S^{2n-1}),$$

which acts as the boundary operator d, as claimed. □

Let u be an $(n-1)$-form on S^{2n-1} such that $i_F^* u$ is a generator for $H^{n-1}(F, \mathbf{Z}) \subset H^{n-1}(F)$. Then

$$< i_F^* u, [F] >= \int_F u = 1.$$

Since τ^* is an isomorphism it follows that $\tau^* i_F^* u = v$, where v is the generator of $H^n(S^n, \mathbf{Z})$. Thus v is the volume form on S^n, normalized so that $\int_{S^n} v = 1$.
Let us recall the definition of the Hopf invariant $H(f)$. Consider any $(n-1)$-form w on S^{2n-1} satisfying $f^* v = dw$. Then

$$H(f) = \int_{S^{2n-1}} w \wedge dw = \int_{x \in S^n} \left(\int_{f^{-1}(x)} w \right) v.$$

According to the lemma one has $f^* v = du$. However it is clear that the function $x \to \int_{f^{-1}(x)} w$ is constant (more generally it is locally constant on the set of regular values for an arbitrary f), and this constant in our case is $\int_F u = 1$. Therefore $H(f) = 1$ and Adams's theorem (see [1]) implies the claim.

Remark 6.3. The result above holds true if one relaxes the assumptions by asking f to be a Serre fibration. One replaces the integral in the definition of the Hopf invariant by the intersection of chains (see e.g. [30], p.509-510).

Proof of 2. However we can suspend these maps and get examples of pairs with non-trivial φ. Choose a Hopf map $f : S^{2n-1} \to S^n$, and extend it to $B^{2n} \to B^{n+1}$ by taking the cone and smoothing it at the origin. Then glue together two copies of B^{2n} along the boundary. One gets a smooth map having two critical points. The previous result implies that:

$$1 \leq \varphi(S^4, S^3), \varphi(S^8, S^5), \varphi(S^{16}, S^9) \leq 2.$$

Let us introduce some notations: set $p_1, ..., p_r$ for the critical points of the map $f : S^m \to S^n$ under consideration, if there are finitely many. Let $F_{e_i} = f^{-1}(f(p_i))$ denote the singular fibers, $F_e = \cup_{i=1}^r F_{e_i}$ stand for their union, and F for the generic fibre which is a closed oriented $(m-n)$-manifold.

Lemma 6.4. *Each component of F_e is either an $(m-n)$-manifold except at the critical points p_i, or an isolated p_i.*

Proof. In fact f is a submersion at all points but p_i. □

Lemma 6.5. *If $m < 2n-1$ then $\varphi(S^m, S^n) \geq 2$.*

Proof. Assume that there is a map $f : S^m \to S^n$ with precisely one critical point p. Then $f : S^m - F_e \to B^n$ is a fibration, so that $S^m - F_e = B^n \times F$.
One rules out the case when the exceptional fiber is one point by observing that $H_{m-n}(F)$ is not trivial. Using an $(n-1)$-cycle linking once a component of F_{e_i} one shows that $H_{n-1}(S^m - F_e)$ is non-trivial. Since $n-1 > m-n$ the equality above is impossible, and the claim is proved. □

Now the equalities from the statement follow.

Remark 6.6. This might be used to construct other examples with finite φ in the respective dimensions. For instance one finds that $\varphi(\Sigma^8, S^5) = \varphi(\Sigma^{16}, \Sigma^9) = 2$, where Σ^n denotes an exotic n-sphere.

Proof of 3. Assume that there is a smooth map $f : S^m \to S^n$ with r critical points. We suppose, for simplicity, that the critical values q_i are distinct. One uses the Serre sequence for the fibration $S^m - F_e \to S^n - \{q_1, ..., q_r\}$ and derives that:

$$H_i(F) = H_i(S^m - F_e), \text{ if } i \leq n-3.$$

One has $H^{m-i}(F_e) = 0$ for $i \leq n-1$ because F_e has dimension at most $(m-n)$. Then Alexander's duality, $\widetilde{H}_{i-1}(S^m - F_e) = \widetilde{H}^{m-i}(F_e)$, and the previous equality imply that $H_i(F) = 0$ for all $i \leq n-3$. This is impossible because the fiber F is a compact $(m-n)$-manifold and $m-n \leq n-3$.
Proof of 4. As above, Serre's exact sequence shows that F is an $(n-2)$-homology sphere. Further the generalized Gysin sequence yields

$$\widetilde{H}^{2n-3-j}(F_e) = \widetilde{H}_j(S^{2n-2} - F_e) = \begin{cases} \mathbf{Z}^r & \text{if } j = 2n-3, \\ \mathbf{Z}^{r-2} & \text{if } j = n-1, \\ 0 & \text{otherwise} \end{cases}$$

Notice that $H_{n-2}(F_e)$ (or equivalently $H_{n-1}(S^{2n-2} - F_e)$) cannot be of rank $r-2$ unless some (actually two) exceptional fibre in F_e consists of one point. In fact if we

have q connected components of F_e of dimension $(n-2)$ then the rank of $H_{n-2}(F_e)$ would be at least q.
Furthermore such a critical point p is isolated in $f^{-1}(f(p))$. Then proposition 3.1 from [27] yields the claim. □

Remark 6.7. One notices that F^{m-n} is $(n-3)$-connected. In particular, if $m \leq 3n-6$ $(n \geq 5)$ then F is homeomorphic to S^{m-n}. In fact one can obtain S^m from the complement $S^m - int(N(F_e))$ of a neighborhhod of the exceptional fibers by adding cells of dimension $\geq n$, one $(n+i)$-cell for each i-cell of F_e. Therefore $\pi_j(S^m - int(N(F_e)) = 0 = \pi_j(S^m) = 0$, for $j \leq n-2$. The base space of the fibration $f|_{S^m - int(N(F_e))}$ is S^n with small open neighborhoods of the critical values deleted, thus it is homotopy equivalent to a bouquet of S^{n-1} (at least one critical value). The long exact sequence in homotopy shows then that the fiber is $(n-3)$-connected.

7. Remarks concerning a substitute for φ in dimension 3

One saw that $\varphi(M^n, N^n)$ is not an interesting invariant for $n \geq 3$. One would like to have an invariant of the pair (M^n, N^n) measuring the defect of M^n being an unramified covering of N^n. First one has to know whether there is a branched covering $M^n \to N^n$ and next if the branch locus could be empty.

Remark 7.1. Notice for instance that an old theorem of Alexander ([2]) states that any n-manifold is a branched covering of the sphere S^n. Moreover the ramification locus can be taken as the $(n-2)$-skeleton of the standard n-simplex.

Remark 7.2. There exists an obvious obstruction to the existence of a ramified covering $M^n \to N^n$, namely the existence of a map of non-zero degree $M^n \to N^n$. In particular a necessary condition is $\| M \| \geq \| N \|$, where $\| M \|$ denotes the simplicial volume of M (see [13, 18]).
However this condition is far from being sufficient. Take M with finite fundamental group and N with infinite amenable fundamental group (for instance of polynomial growth); then $\| M \| = \| N \| = 0$, while it is elementary that there does not exist a non-zero degree map $M \to N$.

A good candidate for replacing φ in dimension 3 is the ratio of simplicial volumes mod $\mathbf{Z}$, namely

$$v(M, N) = \frac{\| M \|}{\| N \|} (mod\, \mathbf{Z}) \in [0, 1),$$

which is defined for all M, N with positive Gromov norm.
Notice that for (closed manifolds M) the simplicial volume $\| M \|$ depends only on the fundamental group $\pi_1(M)$ of M. In particular it vanishes for simply connected manifolds, making it less useful in dimensions at least 4.

Remark 7.3. If M^n covers N^n then (see [13]) $v(M, N) = 0$. The converse holds true for surfaces of genus at least 2, from Hurwitz formula.

The norm ratio has been extensively studied for hyperbolic manifolds in dimension 3, where it coincides with the volume ratio, in connection with commensurability problems (see e.g. [28]). In particular the values $v(M^3, N^3)$ accumulate on 1 since the set of volumes of closed hyperbolic 3-manifolds has an accumulation point.
The simplicial volume is zero for a Haken 3-manifold iff the manifold is a graph

manifold (from [26]), and conjecturally the simplicial volume is the sum of (the hyperbolic) volumes of the hyperbolic components of the manifold.
However it seems that this invariant is not appropriate in dimensions higher than 3 (even if one restricts to aspherical manifolds). Here are two arguments in the favour of this claim:

Proposition 7.4. *Let us suppose that M^n is a ramified covering of N^n over the complex K^{n-2}. Assume that both the branch locus K^{n-2} and its preimage in M^n can be engulfed in a simply connected codimension one submanifold. Then $v(M^n, N^n) = 0$.*
Assume that there is a map $f : M^n \to N^n$ such that the kernel $\ker(f_* : \pi_1(M) \to \pi_1(N))$ is an amenable group. Then $v(M^n, N^n) = 0$.

Proof. One uses the fact that for any simply connected codimension one submanifold $A^{n-1} \subset M^n$ one has $\| M \| = \| M - A \|$ (see [13], p.10 and 3.5). The second part follows from ([23], Remark 3.5) which states that, under the amenability hypothesis on the kernel of f_*, one has $\| M \| = deg(f) \| N \|$. □

Remark 7.5. For $n \geq 4$ A.Sambusetti ([23]) constructed many examples of manifolds M^n and N^n satisfying the second condition, which are therefore far from being a fibration, though as the invariant v vanishes.
It seems that there are not such examples in dimension 3. At least for Haken hyperbolic N^3, any M^3 dominating N^3 with amenable kernel must be a covering, according to ([23], Remark 3.5) and to the rigidity result of Soma and Thurston (see [26]).

Remark 7.6. One could replace the simplicial volume by any other volume, as defined by Reznikov [22]. For instance the $\widetilde{SL(2,R)}$-volume is defined for Seifert fibered 3-manifolds and it behaves multiplicatively under finite coverings (compare to [29]). In particular one can define an appropiate $v(M, N)$ for graph manifolds using this volume.

Remark 7.7. If there is branched covering $f : M^n \to N^n$ then the branch locus is of codimension 2. This explains why $\varphi(M^n, N^n)$ is trivial in high dimensions. A possible extension of φ would have to take into account the (minimal) complexity of the branch locus, (e.g. its minimal Betti number) over all branched coverings. Given N this must be bounded from above (as it is for N a sphere). However it seems that such invariants are far from being explicitly computable.

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