Approximation of the quantum stochastic calculus by the toy Fock space and convergence of the quantum Ito table

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Abstract

In [1], Attal constructed an approximation of the Fock space by the infinite dimensional toy Fock space. In this article we show that that approximation also leads to a rigorous approximation of the quantum stochastic calculus. This is achieved in three steps. First we develop a quantum stochastic calculus on the toy Fock space which is analogous to the usual one. We then compute the projection of the usual quantum stochastic integral on the toy Fock space and identify them as discrete quantum stochastic integrals. We finally rigorously show that the discrete Ito formula converges to the continuous one. [1]

Introduction

The method defined by Attal in [1] to use the toy Fock space as an approximation of the regular Fock space opens many new approaches to classical problems; in particular, one can hope that the representation theorems obtained on toy Fock space in [11] lead to analogues in the Fock space. This calls for a rigorous treatment of quantum stochastic calculus on toy Fock space and a proof that it leads to a rigorous discrete approximation of the quantum stochastic calculus. In section 1 we develop a quantum stochastic calculus on the toy Fock space. As we prove it, there is an Ito formula for quantum stochastic integrals on toy Fock space: if we denote by $a^\epsilon(h)$ the discrete-time integral of a process (h_i) with respect to the noise a^ϵ, it is given by

$$a^\epsilon(h)a^\eta(k) = a^\epsilon(h_. a^\eta_.(k)) + a^\eta(a^\epsilon_.(h)k_.) + a^{\epsilon\cdot\eta}(h_.k_.),$$

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where $a^{\epsilon.\eta}$ is defined by the table

$\vec{\ }$	$-$	$\circ$	$+$	$\times$
$-$	0	a^-	$a^\times - a^\circ$	a^-
$\circ$	0	a°	a^+	a°
$+$	a°	0	0	a^+
$\times$	a^-	a°	a^+	$a^\times$

which we call the *discrete* Ito table. It is shown that on the Fock space, and under some analytical conditions, (see [2],[5], [6]) there is also an Ito formula, of similar form

$$A^\epsilon(H)A^\eta(K) = A^\epsilon(H_. A^\eta_.(K)) + A^\eta(A^\epsilon_.(H)K_.) + A^{\epsilon.\eta}(H_. K_.),$$

for the continuous time integrals, but here $A^{\epsilon.\eta}$ is given by

$\vec{\ }$	$-$	$\circ$	$+$	$\times$
$-$	0	A^-	$A^\times$	0
$\circ$	0	A°	A^+	0
$+$	0	0	0	0
$\times$	0	0	0	0

which we call the *continuous* Ito table.

In section 2 we show that the projections of the quantum stochastic integrals of the regular Fock space are discrete quantum stochastic integrals on the toy Fock space (with some rather surprising results). Finally, in section 3 we show that the continuous Ito table can be obtained as a limit of the discrete one. This last point is achieved in the following way: taking two quantum stochastic integrals on the Fock space, we project them on the toy Fock space. We compute the product of these projections with the discrete Ito table, and then, passing to the limit, show that one recovers the Ito formula as computed using the regular Ito table.

1 Stochastic calculus on toy Fock space

1.1 Definitions

The most natural ways to introduce the infinite dimensional toy Fock space $T\Phi$, are the following:

- as $L^2(\{0,1\}^{\mathbb{N}}, \mu)$ where μ is a probability measure turning the coordinate maps into i.i.d. Bernoulli random variables,
- as the antisymmetric Fock space over $l^2(\mathbb{N})$,

but as in the regular Fock space, Guichardet's interpretation allows one to identify $T\Phi$ with the easier to handle $l^2(\mathcal{P})$, that is, the space of all maps $f : \mathcal{P} \mapsto \mathbb{C}$, such that $\sum_{A\in\mathcal{P}} |f(A)|^2 < +\infty$, where $\mathcal{P}$ is the set of finite subsets of $\mathbb{N}$.

When $T\Phi$ is seen as $l^2(\mathcal{P})$, a natural basis arises, that of the indicators $\mathbb{1}_A$ of elements A of $\mathcal{P}$; we will denote by X_A these vectors, and by $\mathbb{1}$ the vector $X_\emptyset$, called the *vacuum vector*. Every vector $f \in T\Phi$ thus admits an orthogonal decomposition of the form

$$f = \sum_{A\subset\mathbb{N}} f(A)X_A.$$

The toy Fock space has the following important property of tensor product decomposition: for any partition $\cup N_i$ of $\mathbb{N}$, one has

$$\mathrm{T}\Phi = \bigotimes_i \mathrm{T}\Phi_{N_i}$$

where $\mathrm{T}\Phi_{N_i} = l^2(\mathcal{P}_{N_i})$ for $\mathcal{P}_{N_i}$ the set of finite subsets of N_i. Each of these $\mathrm{T}\Phi_{N_i}$ can be identified with the subset $\{f \in \mathrm{T}\Phi \text{ s.t. } f = \sum_{A\subset N_i} f(A)X_A\}$. The isomorphism above comes from the identification $X_A = \bigotimes_i X_{A\cap N_i}$, and we shall forget the $\bigotimes$ and write $X_A = \prod_i X_{A\cap N_i}$. The notation $\mathrm{T}\Phi_{N_i}$ will be simplified to $\mathrm{T}\Phi_{[i}$, $\mathrm{T}\Phi_{(i}$, etc. in the cases where, respectively $N_i = \{i, i+1, ..\}$, $\{i+1, ..\}$, etc.

One of the essential differences with the case of Fock space over $L^2(\mathbb{R}_+)$ is, of course, the atomicity of the measure. This will allow many simplifications; the only problem it raises concerns exponential vectors. In our framework, they should be defined for every $u \in l^2(\mathbb{N})$ by

$$e(u)(A) = \prod_{i\in A} u(i) \text{ for } A \in \mathcal{P}.$$

This defines a vector $e(u)$ in $\mathrm{T}\Phi$, as can be seen by the inequality

$$n! \sum_{|A|=n} \left|\prod_{i\in A} |u(i)|\right|^2 \leq \left(\sum_{i\geq 0} |u(i)|^2\right)^n ,$$

but this yields only $\|e(u)\|^2 \leq e^{\|u\|^2}$ and no (simple) formula for $< e(u), e(v) >$. Moreover, contrarily to the classical result in Fock space, a finite set of exponential vectors over distinct functions u can be linearly dependent, as can be seen with u_1 equal to the null sequence, $u_2 = (1, 0, ..)$ and $u_3 = (2, 0, ..)$, in which case the associated exponential vectors satisfy $e(u_1) - 2e(u_2) + e(u_3) = 0$. Linear independence of n exponential vectors $e(u_i)$ could be obtained if, for example, there were a set of $n-1$ points separating the family of linear combinations of the u_i's (for the original proof in the continuous case, see [10]); this is always the case in continuous time, where the fact that (equivalence classes of) functions in $L^2(\mathbb{R}_+)$ are distinct means much more than their being different on a finite, or even countable number of points of $\mathbb{R}_+$.

Fundamental operators on $\mathrm{T}\Phi$ one defines for all $i \in \mathbb{N}$ three operators by their action on the basis $\{X_A\}$:

$$a_i^+ X_A = \begin{cases} X_{A\cup\{i\}} & \text{if } i \notin A \\ 0 & \text{otherwise} \end{cases}$$

$$a_i^- X_A = \begin{cases} X_{A\setminus\{i\}} & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases}$$

$$a_i^\circ X_A = \begin{cases} X_A & \text{if } i \in A \\ 0 & \text{otherwise.} \end{cases}$$

And the more general operators a_M^+, a_M^-, a_M° for $M \in \mathcal{P}$ by

$$a_M^\epsilon = \prod_{i\in M} a_i^\epsilon,\ \epsilon \in \{+, -, \circ\}.$$

Their action is therefore given by

$$a^+_M X_A = \begin{cases} X_{A\cup M} & \text{if } M\cap A=\emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$a^-_M X_A = \begin{cases} X_{A\setminus M} & \text{if } M\subset A \\ 0 & \text{otherwise .} \end{cases}$$

$$a^\circ_M X_A = \begin{cases} X_{A} & \text{if } M\subset A \\ 0 & \text{otherwise} \end{cases}$$

We will simplify notations to $a^+_M X_A = X_{A+M}$, $a^-_M X_A = X_{A-M}$, using the convention that $M + A = M \cup A$ (respectively $M - A = M \setminus A$) if $M \cap A = \emptyset$ (respectively $M \subset A$) and that the associated quantity is null otherwise . This practically means that we restrict summations over subsets to the indices where the underlying assumptions hold true, and this shall lead to no ambiguity. One more word of notation is needed here for such manipulations on sets: an inequality of type $A < B$ means that all points of $i < j$ for all i in A, all j in B. Besides, we won't make a difference, in the writing of such quantities, between the set $\{i\}$ and the point i.

The above operators are closable, of bounded closures (with norm 1), and we will keep the same notations for their closures, which we call operators of *creation* ($a^+_\cdot$), *annihilation* ($a^-_\cdot$) and *conservation* ($a^\circ_\cdot$). One should remark that the operator of creation we just defined is not the usual operator of creation on an antisymmetric Fock space, but differs only up to a sign factor.

1.2 Ito calculus on TΦ

The main difference between Ito calculus on Toy Fock space and on regular Fock space is that previsibility should replace adaptability for a simpler transcription of the classical results. Therefore we define the (everywhere defined) *previsible* projection and gradient at time $i \in \mathbb{N}$, of a $f \in T\Phi$ by

$$p_i f(A) = 1\!\!1_{A<i}\, f(A)$$

$$d_i f(A) = 1\!\!1_{A<i}\, f(A \cup i)$$

where $1\!\!1_{A<i}$ is equal to one if $A < i$, and zero otherwise.

These operators are called *previsible* because for any $f \in T\Phi$, both $(p_\cdot f)$ and $(d_\cdot f)$ are *previsible processes*, that is, are sequences of vectors such that the i-th vector belongs to $T\Phi_{i)}$. In contrast with the continuous time case, there is no definition problem for the d_i's as individual operators. We will write, to simplify notations, $d_A = d_{i_1} \dots d_{i_n}$ if $A = \{i_1 < \cdots < i_n\}$, and $d_\emptyset = Id$.

The other essential tool for quantum Ito calculus is the abstract Ito integral:

Definition 1.1 *A previsible process of vectors $(f_i)_{i\geq 0}$ is said to be Ito-integrable if $\sum \|f_i\|^2 < +\infty$. One then defines its Ito integral as*

$$I(f_\cdot) = \sum_i f_i\, X_i.$$

Let us stress the fact that, in $f_i\,X_i$ the product is just a tensor product in $T\Phi_{i)} \otimes T\Phi_{[i}$ thanks to the previsiblity of the process: f_i belongs to $T\Phi_{i)}$, X_i belongs to $T\Phi_{[i}$. The Ito integral of a process can also be defined equivalently in a simpler, more algebraic way by

$$I(f_.)(A) = f_{sup\,A}(A - sup\,A) \text{ and } I(f_.)(\emptyset) = 0,$$

where $sup\,A$ denotes the largest element in the n-uple A. The condition for such a function on $\mathcal{P}$ to belong to $T\Phi$ is easily seen to be the above Ito-integrability condition.

Substituting the equality $d_i f = \sum_{A<i} f(A+i)X_A$ in the chaotic decomposition of a f yields the following results:

Proposition 1.2 *Any $f \in T\Phi$ admits a unique decomposition of the form*

$$f \;=\; f(\emptyset)\,\mathbb{1} + \sum_{i\geq 0} d_i f\, X_i \;=\; f(\emptyset)\,\mathbb{1} + I(d_. f)$$

and one has the associated isometry formula:

$$\|f\|^2 = |f(\emptyset)|^2 + \sum_i \|d_i f\|^2.$$

This decomposition is called the previsible representation *of f.*

■

The isometry formula polarizes to the following adjoint relation:

$$\langle I(f_.), g\rangle = \sum_{i\geq 0} \langle f_i, d_i g\rangle$$

for all $g \in T\Phi$ and all Ito-integrable process $(f_.)$ of vectors of $T\Phi$.
We also define (time) summation of vectors, which will appear many times in the sequel, as follows: a process (f_i) of vectors is said to be *summable* if for all $A \in \mathcal{P}$, $\sum_i |f_i(A)| < +\infty$ and $A \mapsto \sum_i f_i(A)$ is square integrable. We then denote by $\sum_i f_i$ the corresponding element of $T\Phi$.

1.3 Previsible operators and conditional expectations

We have defined previsibility of vectors of $T\Phi$. Previsibility of operators can be seen in a purely algebraic way if one does not worry about domain conditions: i-previsibility of an operator h is simply the fact that h acts only on $T\Phi_{i)}$ and leaves $T\Phi_{[i}$ unchanged, that is, for example that

$$hX_A \;\in\; T\Phi_{i)} \text{ for } A < i \text{ and}$$

$$hX_A = (hX_{A_{i)}}) \otimes X_{A_{[i}}$$

Domain considerations make a more technical definition necessary; following Attal and Lindsay in [5], we put the following:

Definition 1.3 *A subspace of $T\Phi$ is said to be i-previsible if it is stable by p_i and all d_j, $j \geq i$.*

An operator h is said to be i-previsible if one of the following equivalent conditions 1. or 2. holds true:

1. - *Dom h is i-previsible*
   - $\left\{ \begin{array}{ll} hp_i = p_i h & \\ hd_j = d_j h & \forall j \geq i \end{array} \right.$ *on Dom h.*

2. - *Dom h is i-previsible*
   - $hf(A) = (hp_i d_{A_{[i}} f)(A_{i)})$ *for all $f \in$ Dom h and for all $A \in \mathcal{P}$.*

In particular, if a X_A is in the domain of a i-previsible operator h, then $d_{A_{[i}} X_A = X_{A_{i)}}$ is, and

$$\begin{aligned} hX_A(M) = p_i d_{M_{[i}}[hX_A](M_{i)}) &= [hp_i d_{M_{[i}} X_A](M_{i)}) \\ &= \mathbb{1}_{M_{[i}=A_{[i}}[hX_{A_{i)}}](M_{i)}) = [(hX_{A_{i)}}) \otimes X_{A_{[i}}](M), \end{aligned}$$

hence $hX_A = (hX_{A_{i)}})X_{A_{[i}}$. One easily checks that the set of i-previsible operators is stable under addition of operators and multiplication by a scalar. It fails to be a ring only for the same reasons that prevent the set of operators on $T\Phi$ to be a ring, for example that 0 times an operator is not the null operator, but a restriction of it.
The preceding definition provides us with a natural way to turn an operator into an i-previsible one. First of all, one has to make the domain i-previsible; let us define the *i-previsible conditional expectation of a subspace V* as

$$\mathbb{D}_i V = \{ f \in T\Phi \ \text{ s.t. } p_i d_A f \in V \quad \forall A \geq i \}.$$

Lemma 1.4 *The conditional expectation of subspaces has the following properties:*

- $\mathbb{D}_i V$ *is j-previsible for all $j \geq i$.*
- $\mathbb{D}_i T\Phi = T\Phi$ *for all i.*
- $\mathbb{D}_i(V \cap V') = \mathbb{D}_i V \cap \mathbb{D}_i V'$.
- *V is i-previsible if and only if $V \subset \mathbb{D}_i V$.*
- *for all $i \leq j$, $\mathbb{D}_i(\mathbb{D}_j V) = \mathbb{D}_j(\mathbb{D}_i V) = \mathbb{D}_i V$.*

■

One then defines the i-previsible conditional expectation of an operator h as the operator $\mathbb{E}_i h$ with domain

$$Dom\, \mathbb{E}_i h = \mathbb{D}_i(Dom\, h),$$

and for $f \in Dom\, \mathbb{E}_i h$, $A \in \mathcal{P}$

$$\mathbb{E}_i h f(A) = hp_i d_{A_{[i}} f(A_{i)}),$$

which defines a element of $T\Phi$, that is, a square-integrable function of A. Indeed,

$$\begin{aligned}\sum_A |h|^2 p_i d_{A_{[i}} f(A_{i)}) &= \sum_A \left|p_i h p_i d_{A_{[i}} f(A_{i)})\right|^2 \\ &= \sum_{A\geq i} \|p_i h p_i d_A f\|^2 \\ &\leq \|p_i h p_i\|^2 \sum_{A\geq i} \|p_i d_A f\|^2\end{aligned}$$

and doing the same manipulations backwards shows that $\sum_{A\geq i} \|p_i d_A f\|^2$ is $\sum_A \left|d_{A_{[i}} f(A_{i]})\right|^2 = \|f\|^2$.

Lemma 1.5 *The conditional expectation of operators has the following properties:*

- $\mathbb{E}_i h$ *is a* j*-previsible operator for all* $j \geq i$
- *an operator* h *is* i*-previsible if and only if* $h \subset \mathbb{E}_i h$
- $\mathbb{E}_i \circ \mathbb{E}_j h = \mathbb{E}_j \circ \mathbb{E}_i h = \mathbb{E}_i h$ *for all* $i \leq j$
- *for all operators* h, k *on* $T\Phi$*, all* $\lambda,\ \mu\ \in \mathbb{C}$*,* $\mathbb{E}_i(\lambda h + \mu k) = \lambda \mathbb{E}_i(h) + \mu \mathbb{E}_i(k)$
- $\mathbb{E}_i h = \mathbb{E}_i(h p_i) = \mathbb{E}_i(p_i h) = \mathbb{E}_i(p_i h p_i)$
- $p_i(\mathbb{E}_i h) = (\mathbb{E}_i h) p_i = p_i h p_i$.

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Remark in strong contrast with the regular Fock space case, if $T\Phi_{i)} \subset Dom\, h$, and in particular as soon as $\{X_M;\ M < i\} \subset Dom\, h$, then $Dom\, \mathbb{E}_i h$ is equal to the whole of $T\Phi$ and, what's more, $\mathbb{E}_i h$ is a bounded operator with norm dominated by $\|p_i h p_i\|$, as we have showm above. This is an example of the power of "stopping the time" and of previsibility in our discrete time framework.

1.4 Quantum stochastic integration on $T\Phi$

The easiest way to define quantum stochastic integrals with respect to the three quantum noises is as follows. For a *maximally previsible process* of operators, that is, a process $(h_i)_{i\geq 0}$ of operators such that for every i, h_i is equal to $\mathbb{E}_i h$, we define the integral $a^\epsilon(h)$ as the operator $\sum_i a_i^\epsilon h_i$, *i.e.*, the operator with domain

$$\{f \in T\Phi \text{ such that } (a_i^\epsilon h_i f)_i \text{ is a summable process}\}$$

and equal to $\sum_i a_i^\epsilon h_i f$ on that domain. For the definition of a summable process, see the end of section 1.2. This definition is meaningful for $\epsilon = +, -, \circ$; we extend it to the case $\epsilon = \times$ with $a_i^\times$ equal to the identity operator, in order to include time integrals in this notation.

This definition requires an explanation: it would seem, indeed, more natural to define integrals as $\sum_i h_i a_i^\epsilon$; but one shows, using the i-previsible maximality of an h_i, that

$$f \in Dom\, h_i \Leftrightarrow a_i^+ f, a_i^- f \text{ and } a_i^\circ f \text{ are in } Dom\, h_i$$

and that, when one of the sides holds, $h_i a_i^\epsilon f = a_i^\epsilon h_i f$. Therefore, the operators $\sum_i h_i a_i^\epsilon$ would be extensions of the operators $\sum_i a_i^\epsilon h_i$, that is, of the integrals we defined. But with such pathological cases as $Dom\, h_i = \{0\}$, where $Dom\, h_i a_i^+ = Ker\, a_i^+$, the domain of the sum $\sum_i h_i a_i^\epsilon$ would contain vectors which are intuitively unwanted in the domain of the integral. Such pathologies will not appear with our definition and, on the domain of the integrals, integrand and integrator do commute.

One can wonder if this definition coincides with natural transcriptions in discrete time of the Fock space integration theories. Indeed, if one remarks that

- the Skorohod integral of a process (f_i), defined as $A \mapsto \sum_{i\in A} f_i(A \setminus i)$ is just the sum $\sum_i a_i^+ f_i$, even from the point of view of summability conditions,
- the gradient operator d_i is just $p_i a_i^-$,

it is straightforward that our definitions are the transcriptions of Attal-Lindsay's maximal definitions of the integrals; we can therefore do the same formal computations as are made in [5], to show that our integrals satisfy Attal-Meyer type equations. We give without proof a formulary of equations satisfied by our integrals, as all proofs are similar tbeir counterparts in continuous time (see [5], [6], [10]).

Hudson-Parthasarathy formulas : for all $u, v \in l^2(\mathbb{N})$ such that $e(u), e(v)$ are in the domain of $a^\epsilon(h)$, one has

$$\langle e(u), a^\epsilon(h)e(v)\rangle = \sum_i \phi(i)\, \langle e(u'_i), h_i^\epsilon e(v'_i)\rangle \tag{1.1}$$

where

$$\phi(i) = \begin{cases} \overline{u(i)} & \text{if } \epsilon = + \\ v(i) & \text{if } \epsilon = - \\ \overline{u(i)}v(i) & \text{if } \epsilon = \circ \\ 1 & \text{if } \epsilon = \times \end{cases}$$

and $u'_i = u$ if $\epsilon = -, \times$, $u'_i = u\mathbb{1}_{\neq i}$, *i.e.* the sequence equal to u, except for the i-th term which is zero, if $\epsilon = +, \circ$.

Attal-Meyer equations : for all f such that all quantities are defined, we have

$$a^\epsilon(h)f = \sum_{i\geq 0} a_i^\epsilon(h) d_i f\, X_i + \sum_{i\geq 0} h_i q_i f\, r_i \tag{1.2}$$

where

$$q_i = \begin{cases} d_i & \text{if } \epsilon = -, \circ \\ p_i & \text{if } \epsilon = +, \times \end{cases}$$

$$r_i = \begin{cases} X_i & \text{if } \epsilon = +, \circ \\ \mathbb{1} & \text{if } \epsilon = -, \times \end{cases}$$

and $a_i^\epsilon(h)$ is the integral of the process $(h_0, h_1, \cdots, h_{i-1}, 0, \cdots)$.

Attal-Lindsay formulas : for all f in the domain of the integral $a^\epsilon(h)$, one has for all $A \in \mathcal{P}$

$$a^\epsilon(h)f(A) = \sum_{i\in A} h_i q_i d_{A_{(i}} f(A_{i)}) \text{ if } \epsilon = +, \circ \tag{1.3}$$

$$a^\epsilon(h)f(A) = \sum_{i\geq 0} h_i q_i d_{A_{[i}} f(A_{i)}) \text{ if } \epsilon = -, \times, \tag{1.4}$$

where q_i is as above.

With our definitions of integrals it is easy to prove the following Ito formula for quantum integrals, which gives an identity between an expression for $a^\epsilon(h)a^\eta(k)$ on a subdomain.

Theorem 1.6 *Let (h_i) and (k_i) be two maximally previsible processes of operators on $T\Phi$. Let ϵ and η belong to $\{+, \circ, -, \times\}$. Then*

$$a^\epsilon(h)a^\eta(k) - a^\epsilon(h_. a^\eta_.(k)) - a^\eta(a^\epsilon_.(h)k_.) - a^{\epsilon\cdot\eta}(h_.k_.),$$

is a restriction of the zero process, where $a^{\epsilon\cdot\eta}$ is given by the following table:

$\urcorner$	$-$	$\circ$	$+$	$\times$
$-$	0	a^-	$a^\times - a^\circ$	a^-
$\circ$	0	a°	a^+	a°
$+$	a°	0	0	a^+
$\times$	a^-	a°	a^+	$a^\times$

■

We turn now to results which are no transcriptions of continuous time theory; despite their extremely simple proofs, these results are crucial for the following section, and may be useful for other applications.

Integral representations of operators In [11], we obtained results giving the representability of operators on $T\Phi$ as kernel operators. The main result was the following:

Theorem 1.7 *Let h be an operator such that all vectors X_M belong to $Dom\, h \cap Dom\, h^*$. Then h is the restriction of a kernel operator $\sum_{A,B,C} k(A,B,C)a^+_A a^\circ_B a^-_C$.*

■

One also has explicit formulas for the scalar coefficients $k(A, B, C)$ which we do not give here. The series $\sum_{A,B,C} k(A,B,C)a^+_A a^\circ_B a^-_C$ is to be seen as a series of operators, that is, an operator with domain the set of all f such that:

- for all $M \in \mathcal{P}$, $\sum_{A,B,C} \left|k(A,B,C)a^+_A a^\circ_B a^-_C f(M)\right| < +\infty$,
- $M \mapsto \sum_{A,B,C} k(A,B,C)a^+_A a^\circ_B a^-_C f(M)$ defines a square-integrable function.

It is therefore straightforward that any operator h defined as a series obtained by grouping the terms of such a kernel operator extends that operator. Therefore any operator satisfying the assumptions of the above theorem is a restriction of

$$\sum_i \sum_{A,B,C<i} \left(k(A+i,B,C)a^+_A a^\circ_B a^-_C a^+_i + k(A,B+i,C)a^+_A a^\circ_B a^-_C a^\circ_i + k(A,B,C+i)a^+_A a^\circ_B a^-_C a^-_i\right)$$

which we can write in the form

$$\sum_i \left(h_i^+ a_i^+ + h_i^\circ a_i^\circ + h_i^- a_i^-\right). \tag{1.5}$$

Such an operator is itself an extension of the integral operator

$$\sum_i h_i^+ a_i^+ + \sum_i h_i^\circ a_i^\circ + \sum_i h_i^- a_i^- \tag{1.6}$$

but it is straightforward that both operators coincide on their common domain. Besides, since any X_M is in the domain of h, it is also in the domain of the integral operator: indeed, since the a_i° and a_i^- vanish on X_M for large enough i, it is equivalent for a X_M to be in the domain of (1.5) or in the domain of (1.6).

Besides, we have kernel decompositions of the integrands h_i^+, h_i°, h_i^-: for example $h_i^+ = \sum_{A,B,C<i} k(A+i, B, C) a_A^+ a_B^\circ a_C^-$. Using the formulas in [11] giving the kernel of an operator we obtain more accessible expressions of the integrands. These results are summarized in the following:

Theorem 1.8 *Let h be an operator on $T\Phi$ such that all vectors X_A belong to $\mathrm{Dom}\, h \cap \mathrm{Dom}\, h^*$. Then if we defined the integral operator*

$$\tilde{h} = \lambda + \sum_{\epsilon=+,\circ,-} \sum_{i\geq 0} h_i^\epsilon a_i^\epsilon$$

where the h_i^ϵ 's are i-previsible and given by:

$$\begin{aligned} h_i^+ \; p_i &= d_i h p_i \\ h_i^- \; p_i &= p_i h a_i^+ p_i \\ h_i^\circ \; p_i &= d_i h a_i^+ p_i - p_i h p_i. \end{aligned} \tag{1.7}$$

then h, $\tilde{h}$ coincide whenever they are both defined, and $\{X_M, M \in \mathcal{P}\} \subset \mathrm{Dom}\, h \cap \mathrm{Dom}\, \tilde{h}$.

■

This theorem is not quite enough if we want to consider previsible processes of operators, that is, sequences $(h_i)_i$ of operators such that h_i is i-previsible, and represent such a process by

$$h_i = \sum_{\epsilon=+,\circ,-,\times} \sum_{j<i} h_j^\epsilon a_j^\epsilon.$$

The presence of an integral with respect to $a^\times$ is unavoidable if we want the h_j^ϵ 's to be independent of i. Minor adaptations of the above result lead to the same representability result for processes with the slighly different formulas:

$$\begin{aligned} h_i^+ \; p_i &= d_i h_{i+1} p_i &&= d_i (h_{i+1} - h_i) p_i \\ h_i^- \; p_i &= p_i h_{i+1} a_i^+ p_i &&= p_i (h_{i+1} - h_i) a_i^+ p_i \\ h_i^\circ \; p_i &= d_i h_{i+1} a_i^+ p_i &&= d_i (h_{i+1} - h_i) a_i^+ p_i \\ h_i^\times \; p_i &= p_i (h_{i+1} - h_i) p_i. \end{aligned} \tag{1.8}$$

It is a particular feature of the toy Fock space as opposed to the Fock space that one can isolate terms in a sum or in an Ito integral using the fundamental Ito calculus operators. In continous time, one could *a priori* obtain nothing but averages of the desired quantities, and would need unaffordable regularity properties to "pass to the limit".

2 Approximations of continuous-time integrals

2.1 A reminder on quantum stochastic calculus

We shall here recall briefly some necessary definitions and results from quantum stochastic calculus on regular Fock space, which will seem very analogous to the discrete time theory developed above. First of all, the Fock space is defined as $\Phi = L^2(\mathcal{P})$, *i.e.* the set of functions on the set $\mathcal{P}$ of finite subsets of $\mathbb{R}_+$ when $\mathcal{P}$ is equipped with the measure such that the empty set is the only atom, of mass one, and the measure is equal to Lebesgue measure of order n on sets of cardinality n. The canonical variable will be denoted by σ, and the infinitesimal volume element by $d\sigma$. We keep the same kind of conventions for the notations on sets as in the discrete case. Informally, the elements of Φ are the functions defined on all increasing simplexes $\Sigma_n = \{t_1 < \cdots < t_n\}$ of $\mathbb{R}_+$ such that

$$\sum_n \int_{\Sigma_n} |f(t_1, \cdots, t_n)|^2 \, dt_1 \cdots dt_n < +\infty. \tag{2.1}$$

It is clear from this *chaotic representation* that Φ is isomorphic to the chaos space of any normal martingale (*e.g.* the brownian motion, the compensated Poisson process, the Azéma martingales, etc.). We shall label Φ_t the analogous set of functions defined on simplexes of $[0, t]$; Φ_t will be canonically included in Φ.

A particular set of elements in Φ is relevant, that is the *exponential domain*: an exponential over a function $u \in L^2(\mathbb{R}_+)$ is defined by

$$\mathcal{E}(u)(\sigma) = \prod_{s \in \sigma} u(s). \tag{2.2}$$

It is an element of Φ, as one can see that $\|\mathcal{E}(u)\|^2 \leq e^{\|u\|^2}$.

Abstract Ito calculus Let us consider for all t the element χ_t of Φ defined as follows:

$$\chi_t(\sigma) = \begin{cases} \mathbb{1}_{s<t} \text{ if } \sigma = \{s\} \\ 0 \text{ otherwise.} \end{cases}$$

The isomorphism from Φ to any chaos space sends χ_t to the brownian motion at time t when onto the chaos space of brownian motion, to the Poisson process at time t when onto the chaos space of Poisson process, etc. One can define an integral of *adapted processes* $(f_t)_{t \geq 0}$ of elements of Φ (that is, such that $f_t \in \Phi_t$ for almost all t), with respect to the curve $(\chi_t)_{t \geq 0}$ (see [4]), denoted

$$I(f) = \int f_t \, d\chi_t$$

and satisfying

$$\|I(f)\|^2 = \int_{\mathbb{R}_+} \|f_t\|^2 dt \tag{2.3}$$

as soon as the latter real-valued integral is finite; the complete construction uses the isometry property (2.3) for step processes. This integral is called the (abstract) *Ito integral.*

There is an alternate construction for this integral:

$$I(f)(\sigma) = f_{\sup \sigma}(\sigma -) \tag{2.4}$$

where $\sup \sigma$ is the largest element in σ and $\sigma- = \sigma \setminus (\sup \sigma)$. The natural conditions for this to be well defined can be seen to be the same as above, namely the square-integrability of the process $(\|f\|_t)_{t\geq 0}$.

Let us define the two fundamental operators of abstract Ito calculus on Φ:

- the *adapted projection* P_t for all t, as the orthogonal projection onto Φ_t. Explicitly, for a $f \in \Phi$,
$$P_t f(\sigma) = \mathbb{1}_{\sigma<t} f(\sigma), \tag{2.5}$$

- the *adapted gradient* by
$$D_t f(\sigma) = \mathbb{1}_{\sigma<t} f(\sigma \cup t). \tag{2.6}$$

Substituting (2.6) in (2.4) yields immediately

$$f = f(\emptyset) + \int D_t f \, d\chi_t \tag{2.7}$$

and

$$\|f\|^2 = |f(\emptyset)|^2 + \int \|D_t f\|^2 dt. \tag{2.8}$$

That is, all elements of Φ have a *previsible representation* (2.7) together with the associated isometry formula (2.8).

Quantum stochastic integrals We shall here define integrals

$$\int_0^\infty H_s da_s^\epsilon$$

with respect to the three quantum noises da^+, da°, da^- and to time, which corresponds to the noise $da^\times$.

The heuristics of the Attal-Meyer quantum stochastic calculus, which we present in a simplified way, derives from the fact that the noises, which will turn out to be differentials of continuous-time fundamental operators, should act just like the fundamental operators of toy Fock space, that is:

- any da_t^ϵ acts only on $\Phi_{[t,t+dt]}$, which from (2.7) can be seen as "generated" by $\mathbb{1}$ and $d\chi_t$ and

- the da_t^ϵ are given by the following table:

$$da_t^+ \mathbb{1} = d\chi_t \text{ and } da_t^+ d\chi_t = 0$$

$$da_t^- \mathbb{1} = 0 \text{ and } da_t^- d\chi_t = dt\mathbb{1}$$

$$da_t^\circ \mathbb{1} = 0 \text{ and } da_t^\circ d\chi_t = d\chi_t$$

$$da_t^\times \mathbb{1} = dt\mathbb{1} \text{ and } da_t^\times d\chi_t = 0.$$

These heuristics allow us to define integrals $\int H_s da_s^\epsilon$ for adapted processes $(H_s)_{s\geq 0}$, that is, processes of operators such that for almost all s, all $f \in Dom\, H_s$,

$$P_s f \in Dom\, H_s,\ D_u f \in Dom\, H_s \text{ for a.a. } u \geq s \text{ and}$$

$$H_s P_s f = P_s H_s f \text{ and } H_s D_u f = D_u H_s f \text{ for a.a. } u \geq s.$$

In that case, a formal computation leads us to give the following definition: an adapted operator process $(T_t)_{t_geq0}$ is said to be the integral process $(\int_0^t H_s da_s^\epsilon)_{t\geq 0}$ if the following equality holds for almost all $t \geq 0$:

$$T_t f = \int_0^\infty T_{t\wedge s} D_s f d\chi_s + \begin{cases} \int_0^t H_s P_s f d\chi_s & \text{if } \epsilon = + \\ \int_0^t H_s D_s f ds & \text{if } \epsilon = - \\ \int_0^t H_s D_s f d\chi_s & \text{if } \epsilon = \circ \\ \int_0^t H_s P_s f ds & \text{if } \epsilon = \times, \end{cases} \tag{2.9}$$

that is, f is in the common domain of the integrals if and only if the right-hand side is well defined and equality holds, an integral $\int_a^b H_s da_s^\epsilon$ is then simply an integral of the process equal to H_s for $s \in [a, b]$ and zero otherwise.

The fundamental operators a_t^ϵ are recovered as the integrals $\int_0^t da_s^\epsilon$ in the above sense.

We give here as a corollary the formulas of Hudson and Parthasarathy, which originally were the cornerstone of the first theory of quantum stochastic integration on Fock space

Hudson-Parthasarathy formula Let us consider a quantum stochastic integral process $(T_t)_{t\geq 0}$ defined on the exponential domain (see (2.2)). Then the following equality holds for all $u, v \in L^2(\mathbb{R}_+)$, almost all $t \in \mathbb{R}_+$:

$$\langle \mathcal{E}(u), T_t \mathcal{E}(v)\rangle = \int_0^t \phi(s) \langle \mathcal{E}(u), H_s \mathcal{E}(v)\rangle\, ds \tag{2.10}$$

where

$$\phi(s) = \begin{cases} \overline{u}(s) & \text{if } \epsilon = + \\ v(s) & \text{if } \epsilon = - \\ \overline{u}(s) v(s) & \text{if } \epsilon = \circ \\ 1 & \text{if } \epsilon = \times. \end{cases}$$

2.2 Explicit formulas for the projections of integrals

We use here the embedding of toy Fock space in regular toy Fock space defined by Attal in [1] and the formulas (1.7) to obtain the projection of an integral operator in discrete time. Let us first recall the definitions of [1]: for any partition $\mathcal{S} = \{0 = t_0 < t_1 < \cdots\}$ of $\mathbb{R}_+$, one defines the following on Φ:

$$X_i = \frac{\chi_{t_{i+1}} - \chi_{t_i}}{\sqrt{t_{i+1} - t_i}}$$

$$a_i^- = \frac{a_{t_{i+1}}^- - a_{t_i}^-}{\sqrt{t_{i+1} - t_i}}$$

$$a_i^+ = P_i^{(1)} \frac{a_{t_{i+1}}^+ - a_{t_i}^+}{\sqrt{t_{i+1} - t_i}}$$

$$a_i^\circ = a_{t_{i+1}}^\circ - a_{t_i}^\circ,$$

where $P_i^{(1)}$ is the projection on the chaoses of order zero and one restricted to the Fock space on $[t_i, t_{i+1}]$, that is, denoting by $P^{(i)}$ the projection on the chaoses of order zero and one, $P_i^{(1)}$ is $Id \otimes P^{(1)} \otimes Id$ in the decomposition $\Phi = \Phi_{t_i]} \otimes \Phi_{[t_i,t_{i+1}]} \otimes \Phi_{[t_{i+1}}$. The space $\mathrm{T}\Phi(\mathcal{S}) \subset \Phi$ is defined as the closed subspace spanned by the vectors $X_A = \prod_{i\in A} X_i$ for $A \in \mathcal{P}$; it is isomorphic to the toy Fock space, and the restrictions of the above operators a_A^ϵ coincide with the operators defined in the previous section. We denote by $\mathbb{E}_\mathcal{S}$ the projection on the subspace $\mathrm{T}\Phi(\mathcal{S})$.

The main relations for our computations are given in the following lemma:

Lemma 2.1 *Let us fix a given partition $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$. One has for all $f \in \Phi$,*

$$p_i \mathbb{E}_\mathcal{S} f = \mathbb{E}_\mathcal{S} P_{t_i} f \quad \text{and} \quad d_i \mathbb{E}_\mathcal{S} f = \frac{1}{\sqrt{t_{i+1} - t_i}} \mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} P_{t_i} D_t f \, dt$$

$$\mathbb{E}_\mathcal{S} \int_0^\infty f_t d\chi_t = \frac{1}{\sqrt{t_{i+1} - t_i}} \sum_{i\geq 0} \mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} P_{t_i} f_t \, dt \, X_i.$$

■

Proof.
We shall prove the first two equalities only, the third one being a consequence of the previsible representation property on toy Fock space and of the second equality.

For all $A = \{i_1 < \cdots < i_n\}$ in $\mathcal{P}$, one has

$$\begin{aligned}
\mathbb{E}_\mathcal{S} P_{t_i} f(A) &= \langle X_A, P_{t_i} f \rangle \\
&= \frac{1}{\sqrt{t_{i_1+1} - t_{i_1}} \cdots \sqrt{t_{i_n+1} - t_{i_n}}} \int_{t_{i_1}}^{t_{i_1+1}} \cdots \int_{t_{i_n}}^{t_{i_n+1}} P_{t_i} f(s_1, \cdots, s_n) ds_1 \cdots ds_n.
\end{aligned}$$

The right-hand side is null if one of the t_{i_j}'s is larger than t_i, that is, if one of the i_j's is larger than i; it is otherwise equal to the same integral without P_{t_i}, *i.e.* to $\mathbb{E}_\mathcal{S} f(A)$. We have proved $p_i \mathbb{E}_\mathcal{S} f(A) = \mathbb{E}_\mathcal{S} P_{t_i} f(A)$.

Since $d_i = p_i a_i^-$, one has:

$$d_i \mathbb{E}_\mathcal{S} f = p_i a_i^- \mathbb{E}_\mathcal{S} f = \frac{1}{\sqrt{t_{i+1} - t_i}} \mathbb{E}_\mathcal{S} P_{t_i} ((a_{t_{i+1}}^- - a_{t_i}^-) f).$$

But

$$(a_{t_{i+1}}^- - a_{t_i}^-) f = \int_{t_i}^\infty (a_{t\wedge t_{i+1}}^- - a_{t_i}^-) D_t f d\chi_t + \int_{t_i}^{t_{i+1}} D_t f dt$$

by the Attal-Meyer formulas, so

$$\begin{aligned}
d_i \mathbb{E}_\mathcal{S} f &= \frac{1}{\sqrt{t_{i+1} - t_i}} p_i \mathbb{E}_\mathcal{S} \left(\int_{t_i}^{t_{i+1}} D_t f dt \right) \\
&= \frac{1}{\sqrt{t_{i+1} - t_i}} \mathbb{E}_\mathcal{S} \left(\int_{t_i}^{t_{i+1}} P_{t_i} D_t f dt \right).
\end{aligned}$$

□

Since we wish to compute approximations $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ of operators on Φ, and therefore to compose H and $\mathbb{E}_{\mathcal{S}}$, and need the operators on $\mathrm{T}\Phi$ to have large enough domains to apply our representation theorems, we need to make the following assumptions on the considered operator H:

$$(\mathbf{HD}) \begin{cases} 1.\ \mathcal{E}(L^2(\mathbb{R}_+)) \text{ is contained in } Dom\, H \cap Dom\, H^* \\ \\ 2.\ Dom\, H \text{ and } Dom\, H^* \text{ are stable by all } \mathbb{E}_{\mathcal{S}}, \end{cases}$$

where all domains are meant in the sense of Attal-Meyer integrals.

In particular, the projections $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ are defined on all finite linear combinations of X_A's. Indeed, by Lemma 3.1

$$\mathbb{E}_{\mathcal{S}} \mathcal{E}(\mathbb{1}) = \mathbb{1}$$
$$\mathbb{E}_{\mathcal{S}} \mathcal{E}(\mathbb{1}_{[t_i,t_{i+1}]}) = (t_{i+1}-t_i)X_i + \mathbb{1}$$
$$\mathbb{E}_{\mathcal{S}} \mathcal{E}(\mathbb{1}_{[t_i,t_{i+1}]\cup[t_j,t_{j+1}]}) = (t_{i+1}-t_i)(t_{j+1}-t_j)X_{i,j} + t_{i+1}-t_i X_i + t_{j+1}-t_j X_j + \mathbb{1}$$

and so on.

In this paragraph, we will reduce our computations to the case where integrals are of the type $H = \int_{t_i}^{t_{i+1}} H_t^\epsilon da^\epsilon$ with $\epsilon \in \{+, \circ, -\}$, and such that their domain and the domain of their adjoint is stable by $\mathbb{E}_{\mathcal{S}}$. By the above remark, the approximations $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ are defined on all vectors X_A of $\mathrm{T}\Phi(\mathcal{S})$. Besides, by Theorem 1.8, $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ has an integral representation which is valid anywhere both $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ and the integral are defined, and in particular on all X_A's. Since the adjoint operators satisfy the same hypotheses, the integral has a densely defined adjoint, hence is closable, and can be extended to the whole of $Dom\, \mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}} = \mathrm{T}\Phi$. Since $P_{t_i} H P_{t_i} = 0$ and $P_{t_{i+1}} H P_{t_{i+1}} = H$, one has $p_i \mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}} p_i = 0$ and $p_{i+1} \mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}} p_{i+1} = \mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$, so the integral is simply of the form

$$\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}} = h_i^+ a_i^+ + h_i^\circ a_i^\circ + h_i^- a_i^-,$$

and we have explicit formulas for the operators h_i^ϵ:

Theorem 2.2 *Let $H = \int H_t^\epsilon da_t^\epsilon$ be a quantum stochastic integral on Φ that satisfies the assumptions ($\boldsymbol{HD}$). Then $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ is a discrete quantum stochastic integral on $\mathrm{T}\Phi(\mathcal{S})$ such that the integrands h_i^+, h_i^-, h_i° are given by:*

- *for $\epsilon = +$,*

$$\begin{aligned} h_i^+ &= \frac{1}{\sqrt{t_{i+1}-t_i}} \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} P_{t_i} H_t^+ dt \\ h_i^- &= 0 \\ h_i^\circ &= \frac{1}{t_{i+1}-t_i} \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} P_{t_i} H_t^+ (a_t^+ - a_{t_i}^+) dt \end{aligned}$$

- *for $\epsilon = -$,*

$$\begin{aligned} h_i^+ &= 0 \\ h_i^- &= \frac{1}{\sqrt{t_{i+1}-t_i}} \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} P_{t_i} H_t^- dt \\ h_i^\circ &= \frac{1}{t_{i+1}-t_i} \mathbb{E}_{\mathcal{S}} \int_{t_i}^{t_{i+1}} P_{t_i} (a_t^- - a_{t_i}^-) H_t^- dt \end{aligned}$$

- *for* $\epsilon = \circ$,

$$\begin{aligned} h_i^+ &= 0 \\ h_i^- &= 0 \\ h_i^\circ &= \frac{1}{t_{i+1}-t_i}\mathbb{E}_\mathcal{S}\int_{t_i}^{t_{i+1}} P_{t_i} H_t^\circ \, dt \end{aligned}$$

where the equalities are over $T\Phi_{i)}$ *(considering, for the right-hand-side,* $T\Phi_{i)}$ *as a subspace of* Φ_{t_i}*) and where all operator integrals are in the strong sense.*

■

Proof.
Let us prove for example the case $\epsilon = +$. Let us consider the action of h_i^+, h_i^-, h_i° on a vector X_A with $A < i$. One has by (1.7) and Lemma 2.1,

$$\begin{aligned} h_i^+ X_A &= d_i \mathbb{E}_\mathcal{S} H X_A \\ &= \frac{1}{\sqrt{t_{i+1}-t_i}}\mathbb{E}_\mathcal{S}\int_{t_i}^{t_{i+1}} P_{t_i} D_t H X_A \, dt \end{aligned}$$

and by the classical Attal-Meyer equations $HX_A = \int_{t_i}^\infty H_{t\wedge t_{i+1}} D_t X_A \, d\chi_t + \int_{t_i}^{t_{i+1}} H_t^+ P_t X_A \, d\chi_t$, but $D_t X_A = 0$ and $P_t X_A = X_A$ for $t \geq t_i$, so $D_t H X_A = H_t^+ X_A$.

$$\begin{aligned} h_i^- X_A &= p_i \mathbb{E}_\mathcal{S} H a_i^+ X_A \\ &= \mathbb{E}_\mathcal{S} P_{t_i} a_i^+ X_A \end{aligned}$$

and $P_{t_i} a_i^+ X_A = 0$.

$$\begin{aligned} h_i^\circ X_A &= d_i \mathbb{E}_\mathcal{S} H a_i^+ X_A - p_i \mathbb{E}_\mathcal{S} H X_A \\ &= \frac{1}{\sqrt{t_{i+1}-t_i}}\mathbb{E}_\mathcal{S}\int_{t_i}^{t_{i+1}} P_{t_i} D_t H a_i^+ X_A - \mathbb{E}_\mathcal{S} P_{t_i} H X_A \end{aligned}$$

the second term is null just like above, and

$$\begin{aligned} H X_{A+i} &= \int_{t_i}^{t_{i+1}} H_t D_t X_{A+i} \, d\chi_t + \int_{t_i}^{t_{i+1}} H_t^+ P_t X_{A+i} \, d\chi_t \\ &= \int_{t_i}^{t_{i+1}} H_t \frac{X_A}{\sqrt{t_{i+1}-t_i}} \, d\chi_t + \int_{t_i}^{t_{i+1}} H_t^+ X_A \frac{\chi_t - \chi_{t_i}}{\sqrt{t_{i+1}-t_i}} \, d\chi_t \end{aligned}$$

but $H_t X_A = \int_{t_i}^t H_s^+ X_A d\chi_s$ and $P_{t_i} H_t X_A = 0$, so $D_t H X_{A+i} = \frac{1}{t_{i+1}-t_i} H_t^+ X_A(\chi_t - \chi_{t_i})$ and the proof is complete. The other two cases are treated exactly in the same way.

□

The surprising fact is of course that the approximation of integrals with respect to a^+ or a^- may contain non-zero integrals with respect to a°; it will be contained in the proof of the next section that, under some analytical conditions, these unwanted terms vanish, at least in a weak sense.

Now let us consider the more general case of processes of integrals $(H_t = \int_0^t H_s^\epsilon da_s^\epsilon)_{t\geq 0}$ and project them considering for a fixed partition $\mathcal{S}$ the discrete-time process $(h_i)_{i\geq 0} = (\mathbb{E}_{\mathcal{S}} \int_0^{t_i} H_s^\epsilon da_s^\epsilon)$. We assume that the domain assumptions **(HD)** hold for each operator H_t of the process; in that case it is straightforward from

$$\mathbb{E}_{\mathcal{S}} H_{t_i} \mathbb{E}_{\mathcal{S}} = \sum_{j<i} \mathbb{E}_{\mathcal{S}} \int_{t_j}^{t_{j+1}} H_s^\epsilon da_s^\epsilon \mathbb{E}_{\mathcal{S}}$$

that for $\epsilon \in +, \circ, -$,

$$h_i = \sum_{j<i} h_j^+ a_j^+ + \sum_{j<i} h_j^- a_j^- + \sum_{j<i} h_j^\circ a_j^\circ,$$

where the h_j^ϵ are as in theorem 2.2.

But for $\epsilon = \times$, if we consider the same discrete-time process and apply the formulas (1.8), one also has an *a priori* surprising feature: the projected process is written

$$h_i = \sum_{j<i} h_j^+ a_j^+ + \sum_{j<i} h_j^- a_j^- + \sum_{j<i} h_j^\circ a_j^\circ + \sum_{j<i} h_j^\times,$$

and the terms $\sum_{j<i} h_j^\epsilon a_j^\epsilon$ can be non-zero; we do not give explicit formulas for them since there is no better way to write the integrands than (1.8). We will show in the next section that, with additional analytic conditions, the $a^\times$ integral alone converges to the continuous-time operator with the mesh size of the partition, hence meaning in particular that the other terms (at least taken as a whole) do vanish.

In that proof we will use the following straightforward proposition, which shows that an alternative way to project integrals with respect to to time can seem more relevant than the one we described.

Proposition 2.3 *Let $H = \int_0^{+\infty} H_t^\times da_t^\times$ be an integral in Φ such that each stopped process $\int_0^t H_s^\times ds$ satisfies the assumptions **(HD)**. Then $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ can be written as the sum $\sum_{i\geq 1} h_i^{'\times}$, where*

$$h_i^{'\times} = \mathbb{E}_{\mathcal{S}} \int_{t_{i-1}}^{t_i} H_t^\times dt\, \mathbb{E}_{\mathcal{S}}$$

is i-previsible for each i.

In particular, for a fixed t belonging to all partitions $\mathcal{S}$,

$$\sum_{i, t_i \leq t} h_i^{'\times} da_i^\times = \mathbb{E}_{\mathcal{S}} \int_0^t H_s^\times ds \mathbb{E}_{\mathcal{S}}.$$

■

We should emphasize here on the fact that the last equality above is not a contradiction to the formulas (1.8): here we allowed the discrete-time to go one step further than in our canonical choice of representations of processes, and shifted all terms. This is just another consequence of the fact that in discrete-time also, there is no unicity of integral representations involving integrals with respect toto time if one considers a single operator; the representation of processes such that the i-th, i previsible operator of the process involves summations until time $i-1$ is unique nevertheless.

3 Convergence of the Ito table

Here we want to prove that the Ito formula for continuous-time quantum stochastic integrals is a limit of the one for discrete-time integrals; to achieve this we actually reprove the quantum Ito formula for regular semimartingales as defined in [2], using nothing but our approximation scheme and the Ito formula on toy Fock space. For this we define below our assumptions on the considered quantum stochastic integrals. The proof will be done in three steps: first, we will show that the unwanted a° integrals that appear when projecting A^+ or A^- integrals vanish, as well as the terms they create when two projections are composed. Then we shall prove that, asymptotically, one can compute the composition of two projections using the *continuous* Ito table. The third step will be dedicated to showing that the remaining discrete-time integrals obtained after composition do converge to the continuous-time integrals we are looking for.

Troughout this section we will make the following assumptions on the operator integrals

$$H = \int_0^\infty H_t^\epsilon da_t^\epsilon :$$

(HS)
1. the integrands H_t^ϵ are bounded operators such that $t \mapsto \|H_t^\epsilon\|$ is square-integrable if $\epsilon = +, -$, integrable if $\epsilon = \times$, essentially bounded if $\epsilon = \circ$
2. H is a bounded operator on Φ.

These assumptions are the ones made for all terms of a regular semimartingale process as defined in [2]. One can regret the apparently too demanding assumption 2., but as it is shown in [3], there is a one-to-one correspondence between the operators that satisfy 1. and 2. and those that satisfy only 1.

With such assumptions, it is straightforward from general stochastic integration theory on Φ that $H = \int_0^\infty H_t^\epsilon da_t^\epsilon$ is the strong limit of the $\int_0^T H_t^\epsilon da_t^\epsilon$ as T goes to infinity, with uniform norm estimates. As a consequence, H is the strong sum of all $\int_{t_i}^{t_{i+1}} H_t^\epsilon da_t^\epsilon$ and $\mathbb{E}_\mathcal{S} H \mathbb{E}_\mathcal{S}$ is the strong sum of the $h_i^+ a_i^+ + h_i^- a_i^- + h_i^\circ a_+^\circ h_i^\times a_i^\times$ computed in the previous section. We consider here the projections we obtain when considering *processes* and not simply operators; this makes no difference for $\epsilon \in \{+, circ, -\}$ but justifies the fact that we consider also continuous-time integrals with respect toto time. Let us fix general notations as we go on: for such an integral $H = \int H_t^\epsilon da_t^\epsilon$ on the Fock space, we will denote by $\widetilde{h}$ its projection on the toy Fock space; it is represented in the form

$$\widetilde{h} = a^+(h_.) + a^-(h_.^-) + a^\circ(h_.^\circ). + a^\times(h_.^\times) \tag{3.1}$$

and the different parts $a_.^\epsilon(h_.^\epsilon)$ will be denoted by $\widetilde{h}^\epsilon$. One shall remark that the above representation holds on all of $T\Phi$; indeed, it is a simple consequence from [11] that it holds on the dense set $\{X_A,\, A \in \mathcal{P}\}$, and $\widetilde{h}$ is a bounded operator.

We need a few lemmas to start the technical proofs:

Lemma 3.1 *Let u belong to $L^2(\mathbb{R}_+)$. The projection $\mathbb{E}_\mathcal{S}\mathcal{E}(u)$ is again an exponential vector in $T\Phi$ over the function $\widetilde{u}(i) = \frac{1}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} u(s)ds$. When seen as a vector of Φ, it is not necessarily an exponential vector, but one has for all $t_i \leq t < t_{i+1}$,*

$$D_t e(\widetilde{u}) = \frac{\widetilde{u}(i)}{\sqrt{t_{i+1}-t_i}} e(\widetilde{u}_{i)})$$

■

Proof.
For all $A = \{i_1, \ldots, i_n\} \in \mathcal{P}$, we have

$$\begin{aligned}
\mathbb{E}_{\mathcal{S}}\mathcal{E}(u)(A) &= \frac{1}{\sqrt{(t_{i_1+1}-t_{i_1})\ldots(t_{i_n+1}-t_{i_n})}} \int_{[t_{i_1},t_{i_1+1}]\times\cdots\times[t_{i_n},t_{i_n+1}]} \mathcal{E}(u)(s_1,\ldots,s_n)ds_1\ldots ds_n \\
&= \prod_{i\in A} \frac{1}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} u(t)dt \\
&= \prod_{i\in A} \widetilde{u}(i).
\end{aligned}$$

□

Lemma 3.2 *For all $s < t$, the operator $(a_t^+ - a_s^+)\, P_s$ is bounded on the exponential domain, with norm $\sqrt{t-s}$.*

■

Proof.
On Φ_s, the operator $(a_t^+ - a_s^+)$ is simply the tensor multiplication by $\chi_t - \chi s$.

□

Lemma 3.3 *Let $\epsilon = +$ or $-$. Then for all $u \in L^2(\mathbb{R}_+)$, we have*

$$\left\|h_i^\circ e(\widetilde{u}_{i)})\right\| \leq \frac{1}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} \|H_t^\epsilon\|\, dt\, \left\|e(\widetilde{u}_{i)})\right\| \tag{3.2}$$

$$\left\|\widetilde{h}_i^\circ e(\widetilde{u}_{i]})\right\| \leq \|u\|_{L^2} \sqrt{\int_0^\infty \|H_t^\epsilon\|^2 dt} \quad \exp\frac{\|u\|_{L^2}^2}{2} \tag{3.3}$$

■

Proof.
The first inequality is trivial from Lemma 3.2. The second one comes from $a_i^\circ e(\widetilde{u}) = \widetilde{u}(i)e(\widetilde{u}_{i)})e(\widetilde{u}_{(i})$ and $\widetilde{h}_i^\circ = \sum_{j<i} h_j^\circ a_j^\circ$.

□

Lemma 3.4 *Let f be an integrable function over $\mathbb{R}_+$. One has*

$$sup_{|t-s|<\delta} \int_s^t |f(u)|\, du \underset{\delta\to 0}{\to} 0$$

■

Proof.
The function $x \mapsto \int_0^x |f(u)|\, du$ has arbitrarily small variations outside of a compact set, and is continuous, hence uniformly continuous, on that compact set.

□

We shall first simplify the case when $\epsilon = \times$, proving the fact that one can choose to consider the alternative description described at the end of the preceding section.

Lemma 3.5 *Let $(H_t = \int_0^{+\infty} H_s^\times ds)_{t\geq 0}$ be an operator satisfying* **(HS)**. *Then for all $t \in [0, +\infty]$, H_t is the strong limit of*

$$\sum_{i|t_i\leq t} h_i^\times .$$

■

Proof.
First let us observe that, by the strong left-continuity of the process, which is obvious from **(HS)**, and Proposition 2.3, the result holds if we consider $h_i^{'\times}$ in place of $h_i^\times$. Now let us prove that

$$\begin{aligned}
&\sum_i (h_i^{'\times} - h_i^\times) \\
&= \sum_i (\mathbb{E}_\mathcal{S} \int_{t_{i-1}}^{t_i} H_s^\times ds \mathbb{E}_\mathcal{S} - \mathbb{E}_\mathcal{S}(p_i \int_{t_i}^{t_{i+1}} H_s^\times ds p_i) \otimes Id\mathbb{E}_\mathcal{S})
\end{aligned}$$

converges strongly to zero on the exponential set. Since thhe considered operators are norm-bounded (with bound $2\int_0^t \|H_s\| \, ds$), convergence on any vector in Φ will follow. We recall that the $\otimes$ decomposition appears because the projection h_i is the action of $\int_{t_i}^{t_{i+1}} H_s^\times ds$ before time t_i.

The above operator, when applied to an exponential vector $\mathcal{E}(u)$, yields

$$\begin{aligned}
&\sum_i (\mathbb{E}_\mathcal{S} \int_{t_{i-1}}^{t_i} H_s^\times ds \, e(\widetilde{u}_{i)}) - p_i \mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} H_s^\times ds \, e(\widetilde{u}_{i)}))e(\widetilde{u}_{[i})\\
&= \sum_i \Big((I - p_i)\mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} H_s^\times ds \, e(\widetilde{u}_{i)}) \cdots \\
&\quad \cdots - \mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} H_s^\times ds \, e(\widetilde{u}_{i]}) + \mathbb{E}_\mathcal{S} \int_{t_{i-1}}^{t_i} H_s^\times ds \, e(\widetilde{u}_{i)}) \cdots \\
&\quad \cdots + \mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} H_s^\times ds (e(\widetilde{u}_{i]}) - e(\widetilde{u}_{i)})) \Big) \, e(\widetilde{u}_{[i})\\
&= \sum_i (p_{i+1} - p_i)\mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} H_s^\times ds \, e(\widetilde{u}_{i)}) e(\widetilde{u}_{[i}) \cdots \\
&\quad \cdots - \sum_i (\mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} H_s^\times ds \, e(\widetilde{u}_{i]}) + \mathbb{E}_\mathcal{S} \int_{t_{i-1}}^{t_i} H_s^\times \, ds e(\widetilde{u}_{i]})) \cdots \\
&\quad \cdots \sum_i (\mathbb{E}_\mathcal{S} \int_{t_i}^{t_{i+1}} H_s^\times ds \, \big(e(\widetilde{u}_{i]}) - e(\widetilde{u}_{i)})\big)).
\end{aligned}$$

The last sum is smaller in norm than $\sum_i |\widetilde{u}(i)| \int_{t_i}^{t_{i+1}} \|H_s^\times\| \, ds \, \|e(\widetilde{u})\|$, hence smaller than the vanishing sequence $\int \|H_s^\times\|\,ds\,\sup_i \|\widetilde{u}(i)\|\,\|\mathcal{E}(u)\|$. The second sum is

$$\sum_i \mathbb{E}_{\mathcal{S}}(\int_{t_i}^{t_{i+1}} H_s^\times ds\, e(\widetilde{u}) - \int_{t_{i-1}}^{t_i} H_s^\times ds\, e(\widetilde{u})),$$

hence appears as the sum of increments of a summable sequence; therefore it is either zero or of the form $\int_{t_i}^t H_s^\times ds\, e(\widetilde{u})$ for the largest i s.t. $t_i \leq t$, in which case it is bounded by $\int_{t_i}^t \|H_s^\times\|\,ds\,\|\mathcal{E}(u)\|$ and therefore vanishes by Lemma 3.4. The first term is equal to

$$\mathbb{E}_{\mathcal{S}} \int_0^t (P_{i+1} - P_i) H_s^\times e(\widetilde{u}_{i)}) e(\widetilde{u}_{[i}) ds$$

where i is actually an $i(s)$, which is defined by $t_{i(s)} \leq s < t_{i(s)+1}$. The integrand is pointwise bounded by $\|H_s^\times\|\,\|\mathcal{E}(u)\|$, which is an integrable function, independently of the partition, and converges pointwise to zero with the mesh size of the partition. Lebesgue's dominated convergence theorem thus applies and the proof is complete.

□

We have shown in particular that the unwanted terms that arise when projecting a process $(\int_0^t H_s^\times ds)_{t\geq 0}$ as a process on toy Fock space converge strongly to zero.

The following proposition constitutes the first of the three announced steps of our demonstration, that is, getting rid of the unwanted a° from the projection, and of the terms they induce after composition:

Proposition 3.6 *Let $\epsilon, \eta \in \{+,-,\circ,\times\}$ and let H, K be two operator integrals satisfying the assumptions (**HS**). Then for all $u, v \in L^2(\mathbb{R}_+)$,*

$$\langle \mathcal{E}(u)\,, \mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}\, \mathbb{E}_{\mathcal{S}} K \mathbb{E}_{\mathcal{S}} \mathcal{E}(v)\rangle - \langle e(\widetilde{u})\,, a^\epsilon(h^\epsilon_\cdot) a^\eta(k^\eta_\cdot) e(\widetilde{v})\rangle$$

tends to zero as the partition's mesh size tends to zero.

■

Proof.
If both ϵ and η are $\circ$, there is nothing to do but recall that $\mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}$ and $\mathbb{E}_{\mathcal{S}} K \mathbb{E}_{\mathcal{S}}$ converge strongly on Φ and are uniformly bounded in norm. If one of ϵ, η is $\times$, the corresponding projection can be immediately replaced by the integral with respect to $a^\times$: if for example $\eta = \times$ (the other case being proved by adjonction), then

$$\langle e(\widetilde{u})\,, \mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}\, \mathbb{E}_{\mathcal{S}} K \mathbb{E}_{\mathcal{S}} e(\widetilde{v})\rangle - \langle e(\widetilde{u})\,, \mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}} a^\eta(k^\eta_\cdot) e(\widetilde{v})\rangle$$

is smaller than $\|H^*\|\,\|\mathcal{E}(u)\|\,\|(\mathbb{E}_{\mathcal{S}} K \mathbb{E}_{\mathcal{S}} - a^\times(k^\times_\cdot))\mathcal{E}(v)\|$ which tends to zero by the previous proposition. Because of this exceptionally strong feature (strong convergence), we will make no difference in the sequel between a projection $\mathbb{E}_{\mathcal{S}} \int K_s^\times ds \mathbb{E}_{\mathcal{S}}$ and the associated $a^\times(k^\times_\cdot)$.

To work out the other cases, let us consider first a term in which no Ito bracket is involved, for example when ϵ and η are both $-$. Then

$$\begin{aligned} \mathbb{E}_{\mathcal{S}} H \mathbb{E}_{\mathcal{S}}\, \mathbb{E}_{\mathcal{S}} K \mathbb{E}_{\mathcal{S}} &= (a^-(h^-) + a^\circ(h^\circ))(a^-(k^-) + a^\circ(k^\circ))\mathbb{E}_{\mathcal{S}} \\ &= (a^-(h^-)a^-(k^-) + \widetilde{h}\widetilde{k}^\circ + \widetilde{h}^\circ \widetilde{k}^+)\mathbb{E}_{\mathcal{S}} \end{aligned}$$

and since $\widetilde{h}^\circ \widetilde{k}^+$ is $\widetilde{h}^\circ(\widetilde{k} - \widetilde{k}^\circ)$, one only has to show that $\widetilde{h}^\circ \widetilde{k}$, $\widetilde{h}\widetilde{k}^\circ$ and $\widetilde{h}^\circ \widetilde{k}^\circ$ tend to zero in our weak sense. Performing the same scheme for other cases, and using adjoint relations shows that all one needs to prove is

$$\widetilde{h}^\circ \widetilde{k} \text{ tends to zero for } \epsilon = -, + \text{ and any } \eta \tag{3.4}$$

and

$$\widetilde{h}^\circ \widetilde{k}^\circ \text{ tends to zero when } \epsilon \text{ and } \eta \text{ are both } + \text{ or } - \tag{3.5}$$

where convergence is meant "asymptotically on the exponential domain" in the same sense as in the proposition.

For the first assumption (3.4), let us observe that

$$\left\langle e(\widetilde{u}) \,, \widetilde{h}^\circ \widetilde{k} e(\widetilde{v}) \right\rangle = \left\langle \widetilde{h^\circ}^* e(\widetilde{u}), \widetilde{k} e(\widetilde{v}) \right\rangle$$

and that, since $\widetilde{k} = \mathbb{E}_\mathcal{S} K \mathbb{E}_\mathcal{S}$ is bounded and $(\widetilde{h}^\circ)^* e(\widetilde{u})$ is uniformly bounded by Lemma 3.3, one can replace $e(\widetilde{v})$ by anything which tends to it in norm with the mesh size $|\mathcal{S}|$. But one can approximate $K\mathcal{E}(v)$ by a linear combination of exponential vectors; let us suppose for simplicity that $K\mathcal{E}(v)$ is approximated by a single vector $\mathcal{E}(w)$. Then, since

$$\begin{aligned}
\left\| \widetilde{k} e(\widetilde{v}) - e(\widetilde{w}) \right\| &= \| \mathbb{E}_\mathcal{S} \mathcal{E}(w) - \mathbb{E}_\mathcal{S} K \mathbb{E}_\mathcal{S} \mathcal{E}(v) \| \\
&\le \| \mathbb{E}_\mathcal{S} \mathcal{E}(w) - \mathbb{E}_\mathcal{S} K \mathcal{E}(v) \| + \| \mathbb{E}_\mathcal{S} K \mathcal{E}(v) - \mathbb{E}_\mathcal{S} K \mathbb{E}_\mathcal{S} \mathcal{E}(v) \| \\
&\le \| K\mathcal{E}(v) - \mathcal{E}(w) \| + \|K\| \, \| \mathcal{E}(v) - \mathbb{E}_\mathcal{S} \mathcal{E}(v) \| \,,
\end{aligned}$$

and both terms on the right-hand side can be made arbitrarily small if the partition is refined enough, one can replace $\widetilde{k} e(\widetilde{v})$ by $e(\widetilde{w})$. Now our assumption reduces to showing that $\left\langle e(\widetilde{u}) \,, \widetilde{h}^\circ e(\widetilde{w}) \right\rangle$ tends to zero with $|\mathcal{S}|$. But it is equal to

$$\begin{aligned}
&\sum_i \overline{\widetilde{u}(i)} \widetilde{v}(i) \left\langle e(\widetilde{u}_{i)}) \,, h_i^\circ e(\widetilde{w}_{i)}) \right\rangle \left\langle e(\widetilde{u}_{(i}), e(\widetilde{v}_{(i}) \right\rangle \\
&= \sum_i \frac{\overline{\widetilde{u}(i)} \widetilde{v}(i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} \left\langle e(\widetilde{u}_{i)}) \,, P_{t_i} (a_t^- - a_{t_i}^-) H_t^- e(\widetilde{w}_{i)}) \right\rangle dt \left\langle e(\widetilde{u}_{(i}), e(\widetilde{v}_{(i}) \right\rangle
\end{aligned}$$

if for example $\epsilon = -$ (the case $\epsilon = +$ is just the same). Using Lemma 3.2, one has

$$\begin{aligned}
\left| \left\langle e(\widetilde{u}) \,, \widetilde{h}^\circ e(\widetilde{w}) \right\rangle \right| &\le \sum_i \frac{|\widetilde{u}(i) \widetilde{w}(i)|}{\sqrt{t_{i+1} - t_i}} \int_{t_i}^{t_{i+1}} \|H_t^-\| dt \, \|e(\widetilde{u}_{i)})\| \, \|e(\widetilde{w}_{i)})\| \, \|e(\widetilde{u}_{(i})\| \, \|e(\widetilde{w}_{(i})\| \\
&\le \sum_i |\widetilde{u}(i) \widetilde{w}(i)| \sqrt{\int_{t_i}^{t_{i+1}} \|H_t^-\|^2 dt} \; \|e(\widetilde{u})\| \, \|e(\widetilde{w})\| \\
&\le \|u\| \, \|w\| \exp \|u\| \exp \|w\| \sqrt{\sup_i \int_{t_i}^{t_{i+1}} \|H_t^-\|^2 dt}
\end{aligned}$$

and the last term converges to zero by Lemma 3.4.

To prove (3.5) let us write

$$\widetilde{h}^\circ \widetilde{k}^\circ = \sum_i h_i^\circ \widetilde{k}_i^\circ a_i^\circ + \sum_i \widetilde{h}_i^\circ k_i^\circ a_i^\circ + \sum_i h_i^\circ k_i^\circ a_i^\circ$$

using the discrete time Ito formula. Hence

$$\left\langle e(\widetilde{u})\,, \widetilde{h}^\circ \widetilde{k}^\circ e(\widetilde{v}) \right\rangle = \sum_i \overline{\widetilde{u}}(i)\widetilde{v}(i) \left\langle e(\widetilde{u}_{i)})\,, (h_i^\circ \widetilde{k}_i^\circ + \widetilde{h}_i^\circ k_i^\circ + h_i^\circ k_i^\circ) e(\widetilde{v}_{i)}) \right\rangle \langle e(\widetilde{u}_{(i)}), e(\widetilde{v}_{(i)}) \rangle$$

and using Lemma 3.3 and the fact that $\widetilde{h}_i^\circ,\ \widetilde{k}_i^\circ$ are bounded with norms $\leq \|H\|,\ \|K\|$, one has a majoration of $\left|\left\langle e(\widetilde{u})\,, \widetilde{h}^\circ \widetilde{k}^\circ e(\widetilde{v}) \right\rangle\right|$ by three terms of the kind

$$\sum_i |\widetilde{u}(i)\widetilde{v}(i)| \sqrt{\int_{t_i}^{t_{i+1}} \|K_t\|^2 dt} \times C_{u,v}$$

and since the series $\sum \widetilde{u}(i)\widetilde{v}(i)$ is convergent, one concludes again using Lemma 3.4.

□

Proposition 3.7 *With the assumptions of proposition 3.6, one has*

$$\langle \mathcal{E}(u)\,, \mathbb{E}_\mathcal{S} H \mathbb{E}_\mathcal{S} K \mathbb{E}_\mathcal{S} \mathcal{E}(v) \rangle - \langle e(\widetilde{u})\,, (a^\epsilon(h^\epsilon_\cdot a^\eta_\cdot(k^\eta)) + a^\eta(a^\epsilon_\cdot(h^\epsilon)k^\eta_\cdot) + a^{\epsilon.\eta}(h.k.))e(\widetilde{v}) \rangle$$

tends to zero as the partition's mesh size tends to zero, where $\epsilon.\eta$ is computed using formally the continuous *Ito formula.*

■

Proof.
All that is left to prove is that

$$\text{for } (\epsilon,\eta) = (+,-),\ \text{one has } \sum_i h_i^- k_i^+ a_i^\circ \underset{|\mathcal{S}|\to 0}{\to} 0 \tag{3.6}$$

and

$$\text{for } (\epsilon,\eta) = (-,+),\ \text{one has } \sum_i h_i^- k_i^+ a_i^\circ \underset{|\mathcal{S}|\to 0}{\to} 0. \tag{3.7}$$

plus the convergence to zero in all cases involving an integral with respect to $\times$.

The proofs of (3.6) (3.7) are the same; let us prove for example (3.6):

$$\left|\left\langle e(\widetilde{u})\,, \sum_i h_i^- k_i^+ a_i^\circ e(\widetilde{v}) \right\rangle\right| \leq \sum_i |\widetilde{u}(i)|\, |\widetilde{v}(i)|\, \left\|h_i^{-*} e(\widetilde{u}_{i)})\right\|\, \left\|k_i^+ e(\widetilde{v}_{i)})\right\|\, \left\|e(\widetilde{u}_{(i)})\right\|\, \left\|e(\widetilde{v}_{(i)})\right\|$$

and since

$$\left\|h_i^{-*} e(\widetilde{u}_{i)})\right\| \leq \sqrt{\int_{t_i}^{t_{i+1}} \left\|H_t^{-*}\right\|^2 dt}\, \left\|e(\widetilde{u}_{i)})\right\|$$

and

$$\left\|k_i^+ e(\widetilde{v}_{i)})\right\| \leq \sqrt{\int_{t_i}^{t_{i+1}} \left\|K_t^+\right\|^2 dt}\, \left\|e(\widetilde{v}_{i)})\right\|$$

one concludes as before.

Now in the case where ϵ, for example, is $\times$, one writes the usual equalities

$$\left\langle e(\widetilde{u})\,, a^{\eta}(h^{\times}k^{\eta})e(\widetilde{v})\right\rangle \;=\; \sum_{i\geq 0}\left\langle e(\widetilde{u}_{i)})\,, h_i^{\times}k_i^{\epsilon}e(\widetilde{v}_{i)})\right\rangle\left\langle e(\widetilde{u}_{[i)}), e(\widetilde{v}_{[i)})\right\rangle \;=$$

$somui \geq 0int_{[t_i,t_{i+1}]^2} \left\langle e(\widetilde{u}_{i)})\,, H_t^{\times}K_s^{\eta}e(\widetilde{v}_{i)})\right\rangle\left\langle e(\widetilde{u}_{[i)}), e(\widetilde{v}_{[i)})\right\rangle$

(keep in mind that $\times.\eta = \eta$ in all cases), and this is dominated by the following quantities:

- $\sum_i \int_{t_i}^{t_{i+1}}\left\|H_t^{\times}\right\| dt \int_{t_i}^{t_{i+1}}\left\|K_t^{\times}\right\| dt$ if $\eta = \times$,
- $\sum_i \sqrt{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}\left\|H_t^{\times}\right\| dt \sqrt{\int_{t_i}^{t_{i+1}}\|K_t^{\eta}\|^2 dt}$ if $\eta = +, -$,
- $\sum_i (t_{i+1}-t_i)\int_{t_i}^{t_{i+1}}\left\|H_t^{\times}\right\| dt\, \|K^{\circ}\|_{\infty}$ if $\eta = \circ$

where all majorations are up to constant factors. In all three cases the summed term in the majorant is a summable one multiplied by a vanishing one.

□

We now apply these results to prove the final result:

Theorem 3.8 (Ito formula in continous time) *Let $H = A^{\epsilon}(H^{\epsilon})$ and $K = A^{\eta}(K^{\eta})$ be two continuous-time integrals satisfying the assumptions (**HS**). Then the following equality holds on the exponential domain:*

$$A^{\epsilon}(H^{\epsilon})A^{\eta}(K^{\eta}) = A^{\epsilon}(H^{\epsilon}_{\cdot}A^{\eta}_{\cdot}(K^{\eta}) + A^{\eta}(A^{\epsilon}_{\cdot}(H^{\epsilon})K^{\eta}_{\cdot}) + A^{\epsilon.\eta}(H^{\epsilon}_{\cdot}K^{\eta}_{\cdot}))$$

where $\epsilon.\eta$ is computed using the continuous *Ito table.*

■

Remark we insist on the fact that this reproves the Ito formula on regular Fock space knowing nothing but its counterpart on the toy Fock space.

Proof.
We will prove that for any $u, v \in L^2(\mathbb{R}_+)$ one has

$$\langle \mathcal{E}(u)\,, A^{\epsilon}(H^{\epsilon})A^{\eta}(K^{\eta})\mathcal{E}(v)\rangle = \langle \mathcal{E}(u)\,, (A^{\epsilon}(H^{\epsilon}_{\cdot}A^{\eta}_{\cdot}(K^{\eta})) + A^{\eta}(A^{\epsilon}_{\cdot}(H^{\epsilon})K^{\eta}_{\cdot}) + A^{\epsilon.\eta}(H^{\epsilon}_{\cdot}K^{\eta}_{\cdot}))\mathcal{E}(v)\rangle\,.$$

By Proposition 3.7, it suffices to show that

$$\begin{aligned}
\left\langle e(\widetilde{u})\,, a^{\epsilon}(h^{\epsilon}\widetilde{k})e(\widetilde{v})\right\rangle &\to \langle \mathcal{E}(u)\,, A^{\epsilon}(H^{\epsilon}K)\mathcal{E}(v)\rangle\\
\left\langle e(\widetilde{u})\,, a^{\epsilon}(\widetilde{h}k^{\eta})e(\widetilde{v})\right\rangle &\to \langle \mathcal{E}(u)\,, A^{\eta}(HK^{\eta})\mathcal{E}(v)\rangle\\
\left\langle e(\widetilde{u})\,, a^{\epsilon.\eta}(h^{\epsilon}k^{\eta})e(\widetilde{v})\right\rangle &\to \langle \mathcal{E}(u)\,, A^{\epsilon.\eta}(H^{\epsilon}K^{\eta})\mathcal{E}(v)\rangle
\end{aligned}$$

but by the previous propositions, one also has that

$$\begin{array}{rcl}\left\langle e(\widetilde{u})\,,a^{\epsilon}(\widetilde{H^{\epsilon}K}^{\epsilon})e(\widetilde{v})\right\rangle & \to & \langle\mathcal{E}(u)\,,A^{\epsilon}(H^{\epsilon}K)\mathcal{E}(v)\rangle \\ \left\langle e(\widetilde{u})\,,a^{\eta}(\widetilde{HK^{\eta}}^{\eta})e(\widetilde{v})\right\rangle & \to & \langle\mathcal{E}(u)\,,A^{\eta}(HK^{\eta})\mathcal{E}(v)\rangle \\ \left\langle e(\widetilde{u})\,,a^{\epsilon.\eta}(\widetilde{H^{\epsilon}K^{\eta}}^{\epsilon.\eta})e(\widetilde{v})\right\rangle & \to & \langle\mathcal{E}(u)\,,A^{\epsilon.\eta}(H^{\epsilon}K^{\eta})\mathcal{E}(v)\rangle\end{array}$$

where $\widetilde{H^{\epsilon}K^{\eta}}^{\epsilon}$ denotes the integrand with respect to a^{ϵ} in the projection of $H^{\epsilon}K^{\eta}$; it is indeed obvious that the considered integrals satisfy our assumptions (**HS**) (in the case where $\epsilon.\eta$ is $\times$, all we use is Proposition 2.3 where we have shifted the discrete-times integrands so that they are adapted and not previsible). It suffices then to prove that

$$\left\langle e(\widetilde{u})\,,a^{\epsilon}(h^{\epsilon}\widetilde{k}-\widetilde{H^{\epsilon}K}^{\epsilon})e(\widetilde{v})\right\rangle \to 0 \tag{3.8}$$

$$\left\langle e(\widetilde{u})\,,a^{\eta}(\widetilde{h}k^{\eta}-\widetilde{HK^{\eta}}^{\eta})e(\widetilde{v})\right\rangle \to 0 \tag{3.9}$$

$$\left\langle e(\widetilde{u})\,,a^{\epsilon.\eta}(h^{\epsilon}k^{\eta}-\widetilde{H^{\epsilon}K^{\eta}}^{\epsilon.\eta})e(\widetilde{v})\right\rangle \to 0 \tag{3.10}$$

(3.8) and (3.9) derive one from another by adjointness. Let us prove different cases one by one.

(3.8) in the case $\epsilon = -$ or $+$: let us take for example $\epsilon = +$.

$$\left\langle e(\widetilde{u})\,,a^{\epsilon}(h^{\epsilon}\widetilde{k}-\widetilde{H^{\epsilon}K}^{\epsilon})e(\widetilde{v})\right\rangle = \sum_i \overline{\widetilde{u}(i)}\left\langle e(\widetilde{u}_{i)})\,,(h_i^{+}\widetilde{k}_i-\widetilde{H^{+}K_i}^{+})e(\widetilde{v}_{i)})\right\rangle \left\langle e(\widetilde{u}_{(i)}),e(\widetilde{v}_{(i)})\right\rangle.$$

The last bracket on the right is uniformly bounded and we will forget it and its avatars in the sequel for the sake of lisibility. The above quantities are equal to

$$\begin{array}{rl} & \displaystyle\sum_i \overline{\widetilde{u}(i)}\left\langle e(\widetilde{u}_{i)})\,,\frac{1}{\sqrt{t_{i+1}-t_i}}\int_{t_i}^{t_{i+1}}P_{t_i}(H_t^{+}\widetilde{k}_i-H_t^{+}K_t)e(\widetilde{v}_{i)})dt\right\rangle \\ = & \displaystyle\sum_i \frac{\overline{\widetilde{u}(i)}}{\sqrt{t_{i+1}-t_i}}\int_{t_i}^{t_{i+1}}\left\langle H_t^{+*}e(\widetilde{u}_{i)}),(\widetilde{k}_i-K_t)e(\widetilde{v}_{i)})\right\rangle dt\end{array}$$

So the norm of the left-hand side is smaller than

$$\begin{array}{rl} & \displaystyle\sum_i \frac{|\widetilde{u}(i)|}{\sqrt{t_{i+1}-t_i}}\int_{t_i}^{t_{i+1}}\|H_t^{+*}\|\left\|(\widetilde{k}_i-K_t)e(\widetilde{v}_{i)})\right\|dt \\ \leq & \displaystyle\sum_i |\widetilde{u}(i)|\sqrt{\int_{t_i}^{t_{i+1}}\|H_t^{+*}\|^2\left\|(\widetilde{k}_i-K_t)e(\widetilde{v}_{i)})\right\|^2dt} \\ \leq & \displaystyle\|\widetilde{u}\|_{l^2}^2\sqrt{\int_0^{\infty}\|H_t^{+*}\|^2\left\|(\widetilde{k}_i-K_t)e(\widetilde{v}_{i)})\right\|^2dt}\end{array} \tag{3.11}$$

by repeated use of the Cauchy-Schwarz formula and convenient erasing of constant terms. The index i in the last line is actually a $i(t)$.

But, since $\widetilde{k}_i = \mathbb{E}_{\mathcal{S}} K_{t_i} \mathbb{E}_{\mathcal{S}}$,

$$\left\| (\widetilde{k}_i - K_t) e(\widetilde{v}_{i)}) \right\| \le \left\| K_{t_i} e(\widetilde{v}_{i)}) \right\| + \left\| K_t e(\widetilde{v}_{i)}) \right\| .$$

If $\eta = +, \circ, -$, then (K_t) is an operator martingale, so, since $t_i \le t$ with $e(\widetilde{v}_{i)}) \in \Phi_{t_i}$,

$$\left\| (k_i - K_t) e(\widetilde{v}_{i)}) \right\| \le 2 \left\| K e(\widetilde{v}_{i)}) \right\| \le 2 \left\| K \right\| \left\| e(\widetilde{v}) \right\| . \tag{3.12}$$

A majoration of the same kind is immediately obtained in the case $\eta = \times$ since $\|K_t\| \le \int \|K_s^{\times}\| \, ds$. One can then apply Lebesgue's dominated convergence theorem to the integral in (3.11). Besides,

$$\left\| (\tilde{k}_i - K_t) e(\widetilde{v}_{i)}) \right\| \le \| (\mathbb{E}_{\mathcal{S}} K_{t_i} - K_{t_i}) e(\widetilde{v}) \| + \| (K_{t_i} - K_t) P_{t_i} \mathbb{E}_{\mathcal{S}} \mathcal{E}(v_{t_i}) \| .$$

And both terms on the right-hand side tend to zero a.e.; the proof of (3.8) with $\epsilon = +$ or $-$ is now complete.

<u>(3.8) in the case $\epsilon = \circ$:</u>
we now consider the quantity

$$\sum_i \overline{\widetilde{u}(i)} \widetilde{v}(i) \left\langle e(\widetilde{u}_{i)}) \, , (h_i^{\circ} \widetilde{k}_i - \widetilde{H^{\circ} K}_i^{\circ}) e(\widetilde{v}_{i)}) \right\rangle$$

where we forget once again the last factor. It is equal to

$$\begin{aligned} &\sum_i \frac{\overline{\widetilde{u}(i)} \widetilde{v}(i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} \left\langle e(\widetilde{u}_{i)}) \, , H_t^{\circ} (\widetilde{k}_i - K_t) e(\widetilde{v}_{i)}) \right\rangle dt \\ = \; &\int_0^{\infty} \frac{\overline{\widetilde{u}(i)} \widetilde{v}(i)}{t_{i+1} - t_i} \left\langle e(\widetilde{u}_{i)}) \, , H_t^{\circ} (\widetilde{k}_i - K_t) e(\widetilde{v}_{i)}) \right\rangle dt. \end{aligned}$$

The bracket is uniformly bounded, and $t \mapsto \frac{\widetilde{u}(i)}{\sqrt{t_{i+1} - t_i}}, \frac{\widetilde{v}(i)}{\sqrt{t_{i+1} - t_i}}$ where $i = i(t)$ tend to u,v in $L^2(\mathbb{R}_+)$ by the martingale convergence theorem. One can therefore consider, instead of the above, the quantity

$$\int_0^{\infty} \overline{u(t)} v(t) \left\langle e(\widetilde{u}_{i)}) \, , H_t^{\circ} (\widetilde{k}_i - K_t) e(\widetilde{v}_{i)}) \right\rangle dt$$

and one can now apply Lebesgue's theorem in the same way as in the previous case.

<u>(3.8) in the case $\epsilon = \times$:</u> we consider

$$\begin{aligned} &\sum_i \left\langle e(\widetilde{u}_{i)}) \, , (h_i^{\times} \widetilde{k}_i - \widetilde{H^{\times} K}_i)^{\times} e(\widetilde{v}_{i)}) \right\rangle \\ = \; &\sum_i \int_{t_{i-1}}^{t_i} \langle H_t^{\times *} e(\widetilde{u}_{i)}), (\mathbb{E}_{\mathcal{S}} K_{t_i} - K_t) e(\widetilde{v}_{i)}) \rangle \, dt. \end{aligned}$$

Its norm is smaller than

$$\sum_i \int_{t_{i-1}}^{t_i} \left\| H_t^\times \right\| dt \, \left\| (\mathbb{E}_\mathcal{S} K_{t_i} - K_t) e(\widetilde{v}_{i)}) \right\|$$

and a majoration was established in (3.12), that is sufficient to conclude here.

Let us now prove (3.10). First let us settle the problem when one of ϵ or η is $\times$; we take for example $\epsilon = \times$. In this case we have to show that

$$\left\langle e(\widetilde{u}) \, , \sum_i h_i^\times k_i^\eta a_i^\eta e(\widetilde{v}) \right\rangle$$

vanishes when the mesh size of the partition tends to zero. But that quantity is smaller in norm than

$$\sum_i \left\| (h_i^\times)^* e(\widetilde{u}) \right\| \, \| k_i^\eta a_i^\eta e(\widetilde{v}) \|$$

which in turn is smaller than a constant times

$$somui \int_{t_i}^{t_i+1} \left\| H_t^{\times *} \right\| dt \, \left\| k_i^\eta a_i^\eta e(\widetilde{v}_{i]}) \right\| .$$

Now

- if $\eta = +$, then $k_i^\eta a_i^\eta e(\widetilde{v}_{i]}) = k_i^\eta e(\widetilde{v} \mathbb{1}_{\neq i}) X_i$, and

$$\left\| k_i^\eta a_i^\eta e(\widetilde{v}_{i]}) \right\| \leq \sqrt{\int_{t_i}^{t_{i+1}} \| K_t^\eta \|^2 dt} \, \| \mathcal{E}(v) \| .$$

- if $\eta = -$, then $k_i^\eta a_i^\eta e(\widetilde{v}_{i]}) = \widetilde{v}(i) k_i^\eta e(\widetilde{v} \mathbb{1}_{\neq i})$ and

$$\left\| k_i^\eta a_i^\eta e(\widetilde{v}_{i]}) \right\| \leq \sqrt{\int_{t_i}^{t_{i+1}} \| K_t^\eta \|^2 dt} \, |\widetilde{v}(i)| \, \| \mathcal{E}(v) \| .$$

- if $\eta = \circ$, then $k_i^\eta a_i^\eta e(\widetilde{v}_{i]}) = \widetilde{v}(i) k_i^\eta e(\widetilde{v} \mathbb{1}_{\neq i}) X_i$ and

$$\left\| k_i^\eta a_i^\eta e(\widetilde{v}_{i]}) \right\| \leq \| K^\circ \|_\infty \, |\widetilde{v}(i)| \, \| \mathcal{E}(v) \| .$$

- if $\eta = \times$, then $k_i^\times a_i^\times e(\widetilde{v}_{i]}) = \left(k_i^\times e(\widetilde{v}_{i)}) \right) (\mathbb{1} + \widetilde{v}(i) X_i)$ and

$$\left\| k_i^\eta a_i^\eta e(\widetilde{v}_{i]}) \right\| \leq \int_{t_i}^{t_{i+1}} \left\| K_t^\times \right\| dt (1 + |\widetilde{u}(i)|).$$

In all cases, (3) is a sum of which the general term is the summable $\int_{t_i}^{t_{i+1}} \left\| H_t^{\times *} \right\|$ times a term that vanishes uniformly in i with the mesh size of the partition.

There are four non trivial cases left: (ϵ, η) equal to $(-,+)$, $(\circ,\circ)$, $(\circ,+)$ and $(-,\circ)$. The last two cases have similar proofs; let us prove them first.

(3.10) in the case $(\epsilon,\eta) = (-,\circ)$ or $(\circ,+)$:
we take for example $(\epsilon,\eta) = (\circ,+)$. What we want to prove is that

$$\left\langle e(\widetilde{u})\,, \sum_i (h_i^\circ k_i^- - \widetilde{H^\circ H^+}^{\,+}) a_i^+ e(\widetilde{v}) \right\rangle \underset{|\mathcal{S}|\to 0}{\to} 0.$$

This quantity is equal to

$$\sum_i \overline{\widetilde{u}(i)} \left\langle e(\widetilde{u}_{i)})\,, (h_i^\circ k_i^+ - \frac{1}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} P_{t_i} H_t^\circ K_t^+ dt) e(\widetilde{v}_{i)}) \right\rangle \langle e(\widetilde{u}_{i)}), e(\widetilde{v}_{i)})\rangle$$
$$= \sum_i \overline{\widetilde{u}(i)} \left\langle e(\widetilde{u}_{i)})\,, \frac{1}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} H_t^\circ (\frac{1}{\sqrt{t_{i+1}-t_i}}\, k_i^+ - K_t^+) dt e(\widetilde{v}_{i)}) \right\rangle \langle e(\widetilde{u}_{i)}), e(\widetilde{v}_{i)})\rangle$$

hence the norm of the left-hand side is, up to a factor term depending only on u, v, smaller than:

$$\sum_i \frac{|\widetilde{u}(i)|}{\sqrt{t_{i+1}-t_i}} \int_{t_i}^{t_{i+1}} \|H_t^\circ\| \left\| (\frac{1}{\sqrt{t_{i+1}-t_i}}\, k_i^+ - K_t^+) e(\widetilde{v}_{i)}) \right\| dt$$
$$\leq \sup \|H_t^\circ\| \sum_i |\widetilde{u}(i)| \sqrt{\int_{t_i}^{t_{i+1}} \left\| (\frac{1}{\sqrt{t_{i+1}-t_i}}\, k_i^+ - K_t^+) e(\widetilde{v}_{i)}) \right\|^2 dt}.$$

Substituting $e(\widetilde{v}_{i)})$ with $e(\widetilde{v}_{i]})$, since $e(\widetilde{v}_{i]}) - e(\widetilde{v}_{i)}) = \widetilde{v}(i) e(\widetilde{v}_{i)}) X_i$, creates an error term which is smaller than

$$\left(\sum_i \int_{t_i}^{t_{i+1}} \left\| (\frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} K_s^+ ds - K_t^+) \widetilde{v}(i) e(\widetilde{v}_{i)}) X_i) \right\|^2 dt \right)^{1/2}$$
$$\leq \left(\sum_i 2 \int_{t_i}^{t_{i+1}} (\frac{1}{(t_{i+1}-t_i)^2} \left(\int_{t_i}^{t_{i+1}} \|K_s\|\, ds \right)^2 + \|K_t\|^2)\, |\widetilde{v}(i)|\, dt \right)^{\frac{1}{2}}$$

up to a constant factor; but that is smaller by the Cauchy-Schwarz inequality than

$$\left(\sum_i \int_{t_i}^{t_{i+1}} (\frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} \|K_s\|^2 ds + \|K_t\|^2)\, |\widetilde{v}(i)|^2\, dt \right)^{\frac{1}{2}}$$
$$= \left(\sum_i |\widetilde{v}(i)|^2 \, (\int_{t_i}^{t_{i+1}} \|K_s\|^2 ds + \int_{t_i}^{t_{i+1}} \|K_t\|^2 dt) \right)^{\frac{1}{2}}$$

up to constant factors again. This tends to zero by Lemma 3.4.

Using the adaptation of operators, one sees easily that, one $e(\widetilde{v}_{i]})$ has been substituted to $e(\widetilde{v}_{i)})$, it can be in turn substituted with $e(\widetilde{v})$; the usual majorations allow one to substitute it then with $\mathcal{E}(v)$. The convergence to zero of

$$\sum_i \int_{t_i}^{t_{i+1}} \left\| \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} K_s^+ \mathcal{E}(v) ds - K_t^+ \mathcal{E}(v) \right\|^2 dt$$

is then a simple consequence of the L^2 martingale convergence theorem.

<u>(3.10) in the case $(\epsilon,\eta)=(-,+)$</u>: what we must show vanishes is

$$\sum_i \left\langle e(\widetilde{u}_{i)})\,,(h_i^- k_i^+ - \widetilde{H^- K^+}^{\times}_i)e(\widetilde{v}_{i)})\right\rangle \langle e(\widetilde{u}_{[i)}),e(\widetilde{v}_{[i)})\rangle\,,$$

thanks to Lemma 3.5.

We show that

$$\sum_i \left\langle e(\widetilde{u}_{i)})\,,(h_i^- k_i^+ - \mathbb{E}_{\mathcal{S}}\int_{t_i}^{t_{i+1}} H_t^- K_t^+ dt)e(\widetilde{v}_{i)})\right\rangle \langle e(\widetilde{u}_{[i)}),e(\widetilde{v}_{[i)})\rangle$$

vanishes. Its norm is easily shown to be smaller than

$$\begin{aligned}
&\sum_i \int_{t_i}^{t_{i+1}} \|H_t^-\| \left\|(\frac{1}{\sqrt{t_{i+1}-t_i}}\,k_i^+ - K_t^+)e(\widetilde{v}_{i)})\right\| dt \\
\leq\ & \sum_i \sqrt{\int_{t_i}^{t_{i+1}} \|H_t^-\|^2 dt}\sqrt{\int_{t_i}^{t_{i+1}} \left\|(\frac{1}{\sqrt{t_{i+1}-t_i}}\,k_i^+ - K_t^+)e(\widetilde{v}_{i)})\right\|^2 dt} \\
\leq\ & \|H^-\|_2 \sqrt{\sum_i \int_{t_i}^{t_{i+1}} \left\|(\frac{1}{\sqrt{t_{i+1}-t_i}}\,k_i^+ - K_t^+)e(\widetilde{v}_{i)})\right\|^2 dt}
\end{aligned}$$

and one concludes using the final result in the proof of the previous case.

<u>(3.10) in the case $(\epsilon,\eta)=(\circ,\circ)$:</u>
<u>the last step to freedom is the</u> proof that

$$\left\langle e(\widetilde{u})\,,\sum_i (h_i^\circ k_i^\circ - \widetilde{H^\circ K^\circ}^{\circ})a_i^\circ e(\widetilde{v})\right\rangle \underset{|\mathcal{S}|\to 0}{\longrightarrow} 0.$$

But it is equal to

$$\sum_i \overline{\widetilde{u}(i)}\widetilde{v}(i)\left\langle e(\widetilde{u}_{i)})\,,(h_i^\circ k_i^\circ - \widetilde{H^\circ K^\circ}^{\circ})e(\widetilde{v}_{i)})\right\rangle$$

up to the usual last factor in the sum. The above line is equal to:

$$\begin{aligned}
&\sum_i \overline{\widetilde{u}(i)}\widetilde{v}(i)\left\langle e(\widetilde{u}_{i)})\,,\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}(H_t^\circ k_i^\circ - H_t^\circ K_t^\circ)e(\widetilde{v}_{i)})dt\right\rangle \\
=\ & \sum_i \frac{\overline{\widetilde{u}(i)}\widetilde{v}(i)}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}\left\langle H_t^{\circ *}e(\widetilde{u}_{i)}),\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}(K_s^\circ - K_t^\circ)e(\widetilde{v}_{i)})ds\right\rangle dt \\
=\ & \int_0^\infty \frac{\overline{\widetilde{u}(i)}\widetilde{v}(i)}{t_{i+1}-t_i}\left\langle H_t^{\circ *}e(\widetilde{u}_{i)}),\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}(K_s^\circ - K_t^\circ)e(\widetilde{v}_{i)})ds\right\rangle dt
\end{aligned}$$

As in the proof of 3.4 we can replace $\frac{\overline{\widetilde{u}(i)}\widetilde{v}(i)}{t_{i+1}-t_i}$ by $\overline{u(t)}v(t)$. The norm of the integrated function is then smaller than

$$|u(t)v(t)|\ 2\sup\|H_\cdot^{\circ *}\|\,\|K_\cdot^\circ\|\,\|\mathcal{E}(u)\|\,\|\mathcal{E}(v)\|$$

which is integrable. By Lebesgue's theorem, the considered quantity tends to zero with the mesh size of the partition.

□

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