Regularity of $\mathcal{D}$-modules associated to a symmetric pair

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Abstract

The invariant eigendistributions on a reductive Lie algebra are solutions of a holonomic $\mathcal{D}$-module which has been proved to be regular by Kashiwara-Hotta. We solve here a conjecture of Sekiguchi saying that in the more general case of symmetric pairs, the corresponding module are still regular.

Introduction

Let G be a semi-simple Lie group. An irreducible representation of G has a character which is an *invariant eigendistribution*, that is a distribution on G which is invariant under the adjoint action of G and which is an eigenvalue of every biinvariant differential operator on G. A celebrated theorem of Harish-Chandra [2] says that all invariant eigendistributions are locally integrable functions on G.

After transfer to the Lie algebra $\mathfrak{g}$ of G by the inverse of the exponential map, an invariant eigendistribution is a solution of a $\mathcal{D}_\mathfrak{g}$-module $\mathcal{M}_\lambda^F$ for some $\lambda \in \mathfrak{g}^*$. Kashiwara and Hotta studied in [4] these $\mathcal{D}_\mathfrak{g}$-modules $\mathcal{M}_\lambda^F$, in particular they proved that they are holonomic and, using a modified version of the result of Harish-Chandra, proved that they are regular holonomic. This shows in particular that any hyperfunction solution of a module $\mathcal{M}_\lambda^F$ is a distribution, hence that any invariant eigenhyperfunction is a distribution.

In [15], Sekiguchi extended the definition of the modules $\mathcal{M}_\lambda^F$ to a symmetric pair. A symmetric pair is a decomposition of a reductive Lie algebra into a direct sum of an even and an odd part, and the group associated to the even part has an action on the odd part (see section 2.1 for the details). In the diagonal case where even and odd part are identical,

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it is the action of a group on its Lie algebra. Sekiguchi defined a subclass of symmetric pairs ("nice pairs"), for which he proved a kind of Harish-Chandra theorem, that is that there is no hyperfunction solution of a module $\mathcal{M}_\lambda^F$ supported by a hypersurface. He also conjectured that these modules are regular holonomic.

In [11] and [12], Levasseur and Stafford give new proofs of the Harish-Chandra theorem in the original case (the "diagonal" case) and in the Sekiguchi case ("nice pairs"). In [1], we show that the Harish-Chandra theorem (in the diagonal case) may be deduced from results on the roots of the b-functions associated to $\mathcal{M}_\lambda^F$ and we find the regularity of $\mathcal{M}_\lambda^F$ as an easy consequence of this result.

The aim of this paper is to prove Sekiguchi's conjecture, that is the regularity of $\mathcal{M}_\lambda^F$, in the general case of symmetric pairs. Our proof do not use Harish-Chandra's theorem or its generalization, so we do not need to ask here the pairs to be "nice".

In the first section of the paper we study the regularity of holonomic $\mathcal{D}$-modules. In the definition of Kashiwara-Kawaï[6], a holonomic $\mathcal{D}$-module is regular if it is microlocally regular along each irreducible component of its characteristic variety. We had proven in [9], that the microlocal regularity may be connected to some microcharacteristic variety. We show here that an analogous result is still true if homogeneity is replaced by some quasi-homogeneity.

In the second section, we prove Sekiguchi's conjecture in theorem 2.2.1. First by standard arguments, we show that outside of the nilpotent cone, the result may be proved by reduction to a Lie algebra of lower dimension. Then on the nilpotent cone we use the results of the first section to show that the module is microlocally regular along the conormals to the nilpotent orbits.

1 Bifiltrations of $\mathcal{D}$-modules

1.1 V-filtration and microcharacteristic varieties

In this section, we recall briefly the definitions of the V-filtration and microcharacteristic varieties. Details may be found in [10] (see also [5],[8],[13]).

Let X be a complex manifold, $\mathcal{O}_X$ be the sheaf of holomorphic functions on X and $\mathcal{D}_X$ be the sheaf of differential operators with coefficients in $\mathcal{O}_X$. Let Y be a submanifold of X. The ideal $\mathcal{I}_Y$ of holomorphic functions vanishing on Y defines a filtration of the sheaf $\mathcal{O}_X|_Y$ of functions on X defined on a neighborhood of Y by $F_Y^k\mathcal{O}_X = \mathcal{I}_Y^k$. The associate graduate, $gr_Y\mathcal{O}_X = \bigoplus \mathcal{I}_Y^k/\mathcal{I}_Y^{k+1}$ is isomorphic to the sheaf $\lambda_*\mathcal{O}_{[T_YX]}$ where $\lambda : T_YX \to Y$ is the normal bundle to Y in X and $\mathcal{O}_{[T_YX]}$ the sheaf of holomorphic functions on T_YX which are polynomial in the fibers of λ. For f a function of $\mathcal{O}_X|_Y$ we will denote by $\sigma_Y(f)$ its image in $gr_Y\mathcal{O}_X$.

If $\mathcal{I}$ is the ideal of definition of an analytic subvariety Z of X, then $\sigma_Y(\mathcal{I}) = \{\,\sigma_Y(f) \mid f \in \mathcal{I}\,\}$ is an ideal of $\mathcal{O}_{[T_YX]}$ which defines the tangent cone to Z along Y [17].

In local coordinates (x,t) such that $Y = \{t = 0\}$, $\mathcal{I}_Y^k$ is, for $k \geq 0$, the sheaf of functions

$$f(x,t) = \sum_{|\alpha|=k} f_\alpha(x,t)t^\alpha$$

and if k is maximal with $f \in \mathcal{I}_Y^k$, we have $\sigma_Y(f)(x,\tilde{t}) = \sum_{|\alpha|=k} f_\alpha(x,0)\tilde{t}^\alpha$.

Consider now the conormal bundle to Y denoted by $\Lambda = T^*_Y X$ as a submanifold of T^*X. If f is a function on T^*X, $\sigma_\Lambda(f)$ is a function on the normal bundle $T_\Lambda(T^*X)$. The hamiltonian isomorphism $TT^*X \simeq T^*T^*X$ associated to the symplectic structure of T^*X identifies $T_\Lambda(T^*X)$ with the the cotangent bundle $T^*\Lambda$ and thus considered $\sigma_\Lambda(f)$ may be considered as a function on $T^*\Lambda$.

The sheaf $\mathcal{D}_X$ is provided with the filtration by the usual order of operators denoted by $(\mathcal{D}_{X,m})_{m\geq 0}$ and that we will call the "usual filtration". The graduate associated to this filtration is $gr\mathcal{D}_X \simeq \pi_*\mathcal{O}_{[T^*X]}$ where $\pi : T^*X \to X$ is the cotangent bundle and $\mathcal{O}_{[T^*X]}$ is the sheaf of holomorphic functions polynomial in the fibers of π. We have also $gr^m\mathcal{D}_X \simeq \pi_*\mathcal{O}_{[T^*X]}[m]$ where $\mathcal{O}_{[T^*X]}[m]$ is the sheaf of holomorphic functions polynomial homogeneous of degree m in the fibers of π. If P is a differential operator of $\mathcal{D}_X|_Y$, its principal symbol is a function $\sigma(P)$ on T^*X defined in a neighborhood of $\Lambda = T^*_Y X$ and $\sigma_\Lambda(\sigma(P))$ is a function on $T^*\Lambda$ (denoted by $\sigma_\Lambda\{1\}(P)$ in the notations of [10]).

The sheaf $\mathcal{D}_X|Y$ of differential operators on a neighborhood of Y is also provided with the the V-filtration of Kashiwara [5] :

$$V_k\mathcal{D}_X = \{P \in \mathcal{D}_X/\forall j \in \mathbb{Z},\quad P\mathcal{I}_Y^j \subset \mathcal{I}_Y^{j-k}\}$$

where $\mathcal{I}_Y^j = \mathcal{O}_X$ if $j \leq 0$.

In local coordinates (x,t), the operators x_i and $D_{x_i} := \frac{\partial}{\partial x_i}$ have order 0 for the V-filtration while the operators t_i have order -1 and $D_{t_i} := \frac{\partial}{\partial t_i}$ order $+1$.

Remark that the V-filtration induces a filtration on $gr\mathcal{D}_X \simeq \pi_*\mathcal{O}_{[T^*X]}$ which is nothing but the filtration F_Λ associated the conormal bundle $\Lambda = T^*_Y X$. In coordinates, $\Lambda = \{\,(x,t,\xi,\tau) \in T^*X \mid t = 0, \xi = 0\,\}$, a function of $\mathcal{O}_{[T^*X]}[m] \cap \mathcal{I}_\Lambda^{m-k}$ is a function $f(x,t,\xi,\tau)$ which is polynomial homogeneous of degree m in (ξ,τ) and vanishes at order at least $m-k$ on $\{t = 0, \xi = 0\}$.

The two filtrations of $\mathcal{D}_X$ define a bifiltration $F_{kj}\mathcal{D}_X = \mathcal{D}_{X,j} \cap V_k\mathcal{D}_X$. The associated bigraduate is defined by $gr_F\mathcal{D}_X = \oplus gr_F^{kj}\mathcal{D}_X$ with

$$gr_F^{kj}\mathcal{D}_X = F_{kj}\mathcal{D}_X \,/(F_{k-1,j}\mathcal{D}_X + F_{k,j-1}\mathcal{D}_X)$$

and is isomorphic to $gr_\Lambda gr\mathcal{D}_X$ that is to the sheaf $\pi_*\mathcal{O}_{[T^*\Lambda]}$ of holomorphic functions on $T^*\Lambda$ polynomial in the fibers of $\pi : T^*\Lambda \to Y$. The image of a differential operator P in this bigraduate will be denoted by $\sigma_{\Lambda(\infty,1)}(P)$ and may be defined as follows:

If the order of P for the V-filtration is equal to the order of its principal symbol $\sigma(P)$ for the induced V-filtration then $\sigma_{\Lambda(\infty,1)}(P) = \sigma_\Lambda(\sigma(P))$ and if the order of $\sigma(P)$ is strictly lower then $\sigma_{\Lambda(\infty,1)}(P) = 0$.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. A good filtration of $\mathcal{M}$ is a filtration which is locally finitely generated that is locally of the form :

$$\mathcal{M}_m = \sum_{j=1,\ldots,N} \mathcal{D}_{X,m+m_j} u_j$$

where $u_1,\ldots,u_N$ are (local) sections of $\mathcal{M}$ and $m_1,\ldots,m_N$ integers.

It is well known that if $(\mathcal{M}_m)$ is a good filtration of $\mathcal{M}$, the associated graduate $gr\mathcal{M}$ is a coherent $gr\mathcal{D}_X$-module and defines the characteristic variety of $\mathcal{M}$ which is a subvariety of T^*X. This subvariety is involutive for the canonical symplectic structure of T^*X and a $\mathcal{D}_X$-module is said to be holonomic if its characteristic variety is lagrangian that is of minimal dimension.

In the same way, a good bifiltration of $\mathcal{M}$ is a bifiltration which is locally finitely generated. Then the associated bigraduate is a coherent $gr_F\mathcal{D}_X$-module which defines a subvariety $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ of $T^*\Lambda$. It is a homogeneous involutive subvariety of $T^*\Lambda$ but it is not necessarily lagrangian even if $\mathcal{M}$ is holonomic.

If $\mathcal{I}$ is a coherent ideal of $\mathcal{D}_X$ then:

$$Ch(\mathcal{M}) = \{\, \xi \in T^*X \mid \forall P \in \mathcal{I}, \sigma(P)(\xi) = 0 \,\}$$
$$Ch_{\Lambda(\infty,1)}(\mathcal{M}) = \{\, \zeta \in T^*\Lambda \mid \forall P \in \mathcal{I}, \sigma_{\Lambda(\infty,1)}(P)(\zeta) = 0 \,\}$$

Regular holonomic $\mathcal{D}_X$-modules have been defined by Kashiwara and Kawaï in [6, Definition 1.1.16.]. A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is regular if it has regular singularities along the smooth part of each irreducible component of its characteristic variety. It is proved in [6] that the property of regular singularities is generic, that is it suffices to prove it on a dense open subset of Λ, in particular we may assume that Λ is the conormal bundle to a smooth subvariety of X. The definition of regular singularities along a smooth lagrangian variety is given in [6, Definition 1.1.11.] but in this paper, we will use the following characterization which we proved in [9, Theorem 3.1.7.]:

Proposition 1.1.1. *A coherent $\mathcal{D}_X$-module has regular singularities along a lagrangian manifold Λ if and only if $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ is contained in the zero section of $T^*\Lambda$.*

1.2 Weighted V-filtration

The V-filtration is associated to the Euler vector field of the normal bundle T_YX which in coordinates is equal to $\sum \tilde{t}_i D_{\tilde{t}_j}$. We want to define a new filtration associated to a vector field $\sum m_i \tilde{t}_i D_{\tilde{t}_j}$. As this is not invariant under coordinate transform, we have first to give an invariant definition.

Let us consider the fiber bundle $p : T_YX \to Y$. The sheaf $\mathcal{D}_{[T_YX/Y]}$ of relative differential operators is the subsheaf of the sheaf $\mathcal{D}_{T_YX}$ of differential operators on T_YX commuting with all functions of $p^{-1}\mathcal{O}_Y$. A differential operator P on T_YX is homogeneous of degree 0 if for any function f homogeneous of degree k in the fibers of p, Pf is homogeneous of degree k.

In particular, a vector field $\widetilde{\eta}$ on T_YX which is a relative differential operator homogeneous of degree 0 defines a morphism from the set of homogeneous functions of degree 1 into itself which commutes with the action of $p^{-1}\mathcal{O}_Y$, that is a section of

$$\mathcal{H}\mathrm{om}_{p^{-1}\mathcal{O}_Y}\left(\mathcal{O}_{T_YX}[1], \mathcal{O}_{T_YX}[1]\right).$$

Let (x,t) be coordinates of X such that $Y = \{\,(x,t) \in X \mid t = 0\,\}$. Let $(x,\tilde{t})$ be the corresponding coordinates of T_YX. Then $\widetilde{\eta}$ is written as :

$$\widetilde{\eta} = \sum a_{ij}(x) \tilde{t}_i D_{\tilde{t}_j}$$

and the matrix $A = (a_{ij}(x))$ is the matrix of the associated endomorphism of $\mathcal{O}_{T_YX}[1]$ which is a locally free $p^{-1}\mathcal{O}_Y$-module of rank $d = codim_X Y$. Its conjugation class is thus independent of the choice of coordinates (x,t). When the morphism is the identity, $\widetilde{\eta}$ is by definition the Euler vector field of T_YX.

Definition 1.2.1. A vector field $\widetilde{\eta}$ on T_YX is *definite positive* if it is a relative differential operator homogeneous of degree 0 whose eigenvalues are strictly positive rational numbers and which is locally diagonalizable as an endomorphism of $\mathcal{O}_{T_YX}[1]$.

A structure of *local fiber bundle of X over Y* is an analytic isomorphism between a neighborhood of Y in X and a neighborhood of Y in T_YX. For example a local system of coordinates defines such an isomorphism.

Definition 1.2.2. A vector field η on X is *definite positive with respect to Y* if:
(i) η is of degree 0 for the V-filtration associated to Y and the image $\sigma_Y(\eta)$ of η in $gr_V^0\mathcal{D}_X$ is *definite positive* as a vector field on T_YX.
(ii) There is a structure of local fiber bundle of X over Y which identifies η and $\sigma_Y(\eta)$.

It is proved in [10, proposition 5.2.2] that if $\sigma_Y(\eta)$ is the Euler vector field of T_YX the condition (ii) is always satisfied and the local fiber bundle structure of X over Y is unique for a given η, but this is not true in general.

We will now assume that X is provided with such a vector field η. Let $\beta = a/b$ the rational number with minimum positive integers a and b such that the eigenvalues of $\beta^{-1}\eta$ are positive relatively prime integers. Let $\mathcal{D}_X[k]$ be the sheaf of differential operators Q satisfying the equation $[Q,\eta] = \beta k Q$ and let $V_k^\eta\mathcal{D}_X$ be the sheaf of differential operators Q which are equal to a series $Q = \sum_{l\leq k} Q_l$ with Q_l in $\mathcal{D}_X[l]$ for each $l \in \mathbb{Z}$.

By definition of a definite positive vector field, we may find local coordinates (x,t) such that $\eta = \sum m_i t_i D_{t_i}$ and we may assume that the m_i are relatively prime integers after multiplication of η by β^{-1}. In this situation, the operators x_j and D_{x_j} have order 0 while the operators t_i have order $-m_i$ and D_{t_i} order $+m_i$. This shows in particular that any monomial $x^\alpha t^\beta D_x^\gamma D_t^\delta$ is in some $\mathcal{D}_X[k]$ and thus that $\mathcal{D}_X$ is the union of all $V_k^\eta\mathcal{D}_X$. This defines a filtration V^η of the sheaf of rings $\mathcal{D}_X$.

The principal symbol of $[Q,\eta]$ is the Poisson bracket $\{\sigma(P),\sigma(\eta)\}$ which is equal to $H_\eta(\sigma(P))$ where H_η is a vector field on T^*X, the Hamiltonian of η. The V^η-filtration on $\mathcal{D}_X$ induces a filtration on the graduate of $\mathcal{D}_X$ that is on $\mathcal{O}_{[T^*X]}$. A function f of $\mathcal{O}_{[T^*X]}$ will be in $V_k^\eta\mathcal{O}_{[T^*X]}$ if it is a series of functions f_l for $l \geq k$ with $H_\eta f = -lf$. In this case we set $\sigma_k^\eta(f) = f_k$.

We are now in a situation analog to that of section 1.1 with two filtrations on $\mathcal{D}_X$, the usual filtration and the V^η-filtration. The sheaf $\mathcal{D}_X$ is thus provided with a bifiltration by $F_{kj}^\eta\mathcal{D}_X = \mathcal{D}_{X,j} \cap V_k^\eta\mathcal{D}_X$ and this defines a symbol $\sigma^\eta{}_{(\infty,1)}(P)$ which is a function on T^*X. By definition, $\sigma^\eta{}_{(\infty,1)}(P)$ is equal to $\sigma_k^\eta(\sigma(P))$ where k is the order of P for the V^η-filtration. This symbol is thus equal to 0 if the order of $\sigma(P)$ is strictly less than k.

If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, we define a good bifiltration and a microcharacteristic variety $Ch^\eta{}_{(\infty,1)}(\mathcal{M})$. If $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ we will have:

$$Ch^\eta{}_{(\infty,1)}(\mathcal{M}) = \{\, \zeta \in T^*X \mid \forall P \in \mathcal{I}, \quad \sigma^\eta{}_{(\infty,1)}(P)(\zeta) = 0 \,\}$$

The difference with the previous situation is the local identification of T_YX with X which defines isomorphisms $T^*T_Y^*X \simeq T^*T_YX \simeq T^*X$ and make $\sigma^\eta(\sigma(P))$ a function on T^*X. Especially, if $\widetilde{\eta}$ is the Euler vector field of T_YX and η a vector field on X with $\sigma_V(\eta) = \widetilde{\eta}$, the definitions of this section coincide with the definitions of the previous one except for this identification.

1.3 Direct image of V-filtration

Let $\varphi : Y \to X$ be a morphism of complex analytic manifolds. A vector field u on Y is said to be *tangent to the fibers of* φ if $u(f \circ \varphi) = 0$ for all f in $\mathcal{O}_X$. A differential operator P is said to be *invariant under* φ if there exists a $\mathbb{C}$-endomorphism A of $\mathcal{O}_X$ such that $P(f \circ \varphi) = A(f) \circ \varphi$ for all f in $\mathcal{O}_X$. If we assume from now that φ has a dense range in X, A is uniquely determined by P and is a differential operator on X. We will denote by $A = \varphi_*(P)$ the image of P in $\mathcal{D}_X$ under this ring homomorphism.

Let Z be a submanifold of Y and T a submanifold of X. Let η be a vector field on Y invariant under φ. We assume that η is definite positive with respect to Z and that $\eta' = \varphi_*(\eta)$ is definite positive with respect to T. We also multiply η by an integer so that its eigenvalues and those of η' are integers.

Example 1.3.1. Let Y be a complex vector space and $\varphi : Y \to X = \mathbb{C}^d$ given by $\varphi = (\varphi_1, \dots, \varphi_d)$ where $\varphi_1, \dots, \varphi_d$ are holomorphic functions on Y homogeneous of degree $m_1, \dots, m_d$. Let $Z = \{0\}$ and η be the Euler vector field of Y, so that the V^η-filtration is the V-filtration along $\{0\}$. Then $\eta' = \varphi_*(\eta)$ is equal to $\sum m_i t_i D_{t_i}$ on X and is definite positive with respect to $\{0\}$. Remark that we do not assume that φ is defined in a neighborhood of Z.

In the general case, we can choose local coordinates (y,t) on X so that $\eta' = \sum m_j t_j D_{t_j}$, then the map φ is given by $y_i = \varphi_i(x)$ and $t_j = \psi_j(x)$ where the functions $\varphi_i(x)$ is homogeneous of degee 0 for η while the function $\psi_j(x)$ is homogeneous of degee m_j for η.

The sheaf $\mathcal{D}_{Y\to X} = \mathcal{O}_Y \otimes_{\varphi^{-1}\mathcal{O}_X} \varphi^{-1}\mathcal{D}_X$ is a $(\mathcal{D}_Y, \varphi^{-1}\mathcal{D}_X)$-bimodule with a canonical section $1 \otimes 1$ denoted by $1_{Y\to X}$. If we choose coordinates $(x_1, \dots, x_n)$ of X and coordinates $(y_1, \dots, y_p)$ of Y and if $\varphi = (\varphi_1, \dots, \varphi_p)$, then the sections of $\mathcal{D}_{Y\to X}$ are represented by finite sums $\sum f_\alpha(y) \otimes D_x^\alpha$ and the left action of $\mathcal{D}_Y$ is given by

$$D_{y_i}\left(\sum_\alpha f_\alpha(y) \otimes D_x^\alpha\right) = \sum_\alpha \frac{\partial f_\alpha}{\partial y_i}(y) \otimes D_x^\alpha + \sum_{\alpha,j} f_\alpha(y) \frac{\partial \varphi_j}{\partial y_i}(y) \otimes D_{x_j} D_x^\alpha$$

If $\mathcal{N}$ is a coherent $\mathcal{D}_X$-module, its inverse image under φ is the $\mathcal{D}_Y$-module $\varphi^*\mathcal{N} = \mathcal{D}_{Y\to X} \otimes_{\varphi^{-1}\mathcal{D}_X} \varphi^{-1}\mathcal{N}$. In general, $\varphi^*\mathcal{N}$ is not coherent but if $\mathcal{N}$ is holonomic, $\varphi^*\mathcal{N}$ is holonomic (hence coherent).

Let $\mathcal{D}_{Y\to X}[k]$ be the set of sections satisfying $\eta.u - u.\varphi_*\eta = -\beta\beta' k u$ where β (resp. β') is the g.c.d. of the eigenvalues of η (resp. $\varphi_*\eta$). (We may assume that $\beta = 1$ or $\beta' = 1$ but not both in general). We define $V_k\mathcal{D}_{Y\to X}$ as the subsheaf of $\mathcal{D}_{Y\to X}$ of the sections which may be written as series $\sum_{l\geq k} u_l$ with u_l in $\mathcal{D}_{Y\to X}[l]$. Remark that $1_{Y\to X}$ satisfies $\eta.1_{Y\to X} = 1_{Y\to X}.\varphi_*\eta$ hence is of order 0.

If $\mathcal{N}$ is a coherent $\mathcal{D}_X$-module provided with a $V^{\eta'}$-filtration we define a filtration on its inverse image by:

$$V_k^{\eta}\varphi^*\mathcal{N} = \sum_{k=\beta' i+\beta j} V_i\mathcal{D}_{Y\to X} \otimes_{\varphi^{-1}V_0^{\eta'}\mathcal{D}_X} \varphi^{-1}V_j^{\eta'}\mathcal{N}$$

The sheaf $\mathcal{D}_{Y\to X}$ is also provided with a filtration $(\mathcal{D}_{Y\to X})_j$ induced by the usual filtration of $\mathcal{D}_X$ hence of a bifiltration $F^{\eta}\mathcal{D}_{Y\to X}$. If $\mathcal{N}$ is bi-filtrated, we define in the same way a bifiltration on $\varphi_*\mathcal{N}$.

Proposition 1.3.2. *Let $\mathcal{I}$ be an ideal of $\mathcal{D}_Y$ which is generated by all the vector fields tangent to the fibers of φ and by a finite set $(P_1, \dots, P_l)$ of differential operators invariant under φ. Let $\mathcal{J}$ be the ideal of $\mathcal{D}_X$ generated by $(\varphi_*(P_1), \dots, \varphi_*(P_l))$. Let $\mathcal{M} = \mathcal{D}_Y/\mathcal{I}$ and $\mathcal{N} = \mathcal{D}_X/\mathcal{J}$ and put on $\mathcal{M}$ and $\mathcal{N}$ the bifiltrations induced by $F^{\eta}\mathcal{D}_Y$ and $F^{\eta'}\mathcal{D}_X$*

Then, there exists a canonical morphism of $\mathcal{D}_Y$-modules $\mathcal{M} \to \varphi^*\mathcal{N}$ which is a morphism of bi-filtrated $F^{\eta}\mathcal{D}_Y$-modules and an isomorphism at the points where φ is a submersion.

Proof. There is a canonical morphism $\mathcal{D}_Y \to \mathcal{D}_{Y\to X}$ given by $P \mapsto P.1_{Y\to X}$. The vector fields tangent to the fibers cancel $\mathcal{D}_{Y\to X}$ and a differential operator invariant under φ satisfy $P.1_{Y\to X} = 1_{Y\to X}.\varphi_*(P)$ hence this morphism defines a morphism $\mathcal{M} \to \varphi_*\mathcal{N}$ which is a morphism of left $V^{\eta}\mathcal{D}_Y$-modules by the definitions.

In a neighborhood of a point where φ is a submersion, we may choose local coordinates $(x_1, \dots, x_p, y_1, \dots, y_{n-p})$ such that $\varphi(x, y) = x$. Then $\mathcal{D}_{Y\to X}$ is the sheaf of operators $P(x, y, D_x)$, the vector fields tangent to the fibers are generated by $D_{y_1}, \dots, D_{y_{n-p}}$ and the differential operators invariant under φ are of the form $P(x, D_x)$ modulo (D_{y_i}), so $\mathcal{M} \to \varphi_*\mathcal{N}$ is an isomorphism. □

Let $S = \varphi^{-1}(T)$ and x be a point of S where φ is a submersion. In a neighborhood of x, Y is isomorphic to $X \times S$ and if we fix such an isomorphism, η' which is a vector field on X may be considered as a vector field on Y, definite positive relatively to S. Remark that η' differ from η by a vector field tangent to φ. Then proposition 1.3.2 gives:

Corollary 1.3.3. *In a neighborhood of x, the microcharacteristic variety $Ch^{\eta}{}_{(\infty,1)}(\mathcal{M})$ is equal to $Ch^{\eta'}{}_{(\infty,1)}(\mathcal{M})$.*

1.4 Weighted V-filtration and regularity

Definition 1.4.1. Let Z be a submanifold of X and η be a vector field which is definite positive with respect to Z. A coherent $\mathcal{D}_X$-module has *η-weighted regular singularities* along the lagrangian manifold $\Lambda = T^*_Z X$ if there is a dense open subset Ω of Λ such that $Ch^{\eta}{}_{(\infty,1)}(\mathcal{M}) \subset \Lambda$ in a neighborhood of Ω.

If $\sigma_Z(\eta)$ is the Euler vector field of $T_Z X$, proposition 1.1.1 shows that this definition coincide with the definition of Kashiwara-Kawaï.

Let $X = \mathbb{C}^n$ with coordinates $(x_1, \dots, x_{n-p}, t_1, \dots, t_p)$ and $Z = \{t = 0\}$, let $Y = \mathbb{C}^n$ with coordinates $(x_1, \dots, x_{n-p}, y_1, \dots, y_p)$ and $Z' = \{y = 0\}$. Let $m_1, \dots, m_p$ be strictly positive integers , we define the map $\varphi : Y \to X$ by $\varphi(x,y) = (x, y_1^{m_1}, \dots, y_p^{m_p})$ and the vector field $\eta = \sum_{i=1\dots p} m_i t_i D_{t_i}$.

Lemma 1.4.2. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module with η-weighted regular singularities along $T^*_Z X$, then $\varphi^*\mathcal{M}$ is a holonomic $\mathcal{D}_Y$-module with regular singularities along $T^*_{Z'}Y$.*

Proof. We may assume that $\mathcal{M}$ is equal to $\mathcal{D}_X/\mathcal{I}$ for some coherent ideal $\mathcal{I}$ of $\mathcal{M}$. The inverse image of $\mathcal{M}$ by φ is, by definition:

$$\varphi^*\mathcal{M} = \mathcal{D}_{Y\to X} \otimes_{\varphi^{-1}\mathcal{D}_X} \varphi^{-1}\mathcal{M} = \mathcal{D}_{Y\to X}/\mathcal{D}_{Y\to X}\mathcal{I}$$

The sections of $\mathcal{D}_{Y\to X}$ are represented by $P(x,y,D_x,D_t) = \sum a_{\alpha\beta}(x,y) D_x^\alpha D_t^\beta$ and we define the filtration $V^\eta\mathcal{D}_{Y\to X}$ in the same way than in the previous section. For this filtration $x^\gamma y^\delta D_x^\alpha D_t^\beta$ is of order $< m, \beta > -|\delta|$. We also define the usual filtration on $\mathcal{D}_{Y\to X}$, that is the filtration by the order in (D_x, D_t). In this way, $\mathcal{D}_{Y\to X}$ is provided with a bifiltration $F^\eta\mathcal{D}_{Y\to X}$ which is compatible with the bifiltration $F^\eta\mathcal{D}_X$, that is an operator P of $F^\eta_{kl}\mathcal{D}_X$ sends $F^\eta_{ij}\mathcal{D}_{Y\to X}$ into $F^\eta_{i+k,j+l}\mathcal{D}_{Y\to X}$.

Let $\mathcal{D}_{Y\to X}[N]$ be the sub-$\mathcal{D}_Y$-module of $\mathcal{D}_{Y\to X}$ generated by D_t^β for $|\beta| \leq N$. If $\mathcal{M}$ is holonomic, $\varphi^*\mathcal{M}$ is holonomic hence coherent. The images of the morphisms $\mathcal{D}_{Y\to X}[N] \to \varphi^*\mathcal{M}$ make an increasing sequence of coherent submodules of $\varphi^*\mathcal{M}$ which is therefore stationary, so there exists some N_0 such that $\mathcal{D}_{Y\to X}[N] \to \varphi^*\mathcal{M}$ is surjective for all $N \geq N_0$. The bifiltration induced by $F^\eta\mathcal{D}_{Y\to X}$ on $\mathcal{D}_{Y\to X}[N]$ is a good $F\mathcal{D}_Y$-filtration which induces a good filtration on $\varphi^*\mathcal{M}$ if $N \geq N_0$, we will denote it by $F[N]\varphi^*\mathcal{M}$.

The associate graduate is denoted by $gr[N]\varphi^*\mathcal{M}$ and, as $F[N]$ is a good bifiltration, the analytic cycle of T^*Y associated to $gr[N]\varphi^*\mathcal{M}$ is independent of N [10, Prop 3.2.3.]. For $N \geq N_0$, the canonical morphism $gr[N_0]\varphi^*\mathcal{M} \to gr[N]\varphi^*\mathcal{M}$ induces an isomorphism on the associated cycles hence $gr[N_0]\varphi^*\mathcal{M}$ and $gr[N]\varphi^*\mathcal{M}$ have the same support and the kernel and cokernel of the morphism have a support of dimension strictly lower.

An operator P of $F^\eta_{kl}\mathcal{D}_X$ sends $F^\eta_{ij}\mathcal{D}_{Y\to X}[N_0]$ into $F^\eta_{i+k,j+l}\mathcal{D}_{Y\to X}[N_0 + l]$. If P annihilates a section u of $F_{ij}[N_0]\varphi^*\mathcal{M}$, its class in $gr_{kl}\mathcal{D}_X$ that is the function $\sigma^{\eta}{}_{(\infty,1)}(P)$ annihilates the image of u in $gr[N+l]\varphi^*\mathcal{M}$. Let ζ be a point of $\Lambda = T^*_Z X$ such that $Ch^{\eta}{}_{(\infty,1)}(\mathcal{M}) \subset T^*_Z X$ in a neighborhood of ζ. By the hypothesis, there is a dense open subset Ω of such points in Λ. There is a differential operator P which annihilates u and such that $\sigma^{\eta}{}_{(\infty,1)}(P) = t_1^M \mu$ where μ is a function invertible at ζ. Hence there exists some l such that the image of u in $gr[N+l]\varphi^*\mathcal{M}$ is annihilated by $t_1^M = y_1^{Mm_1}$ hence is supported by $y_1 = 0$. As $gr[N_0]\varphi^*\mathcal{M}$ is finitely generated, there exists some $N_1 \geq N_0$ such that the image of $gr[N_0]\varphi^*\mathcal{M}$ in $gr[N_1]\varphi^*\mathcal{M}$ is contained in $y_1 = 0$.

We can do the same for the other equations of $T^*_{Z'}Y$ and show that there exists some $N_2 \geq N_0$ such that the image of $gr[N_0]\varphi^*\mathcal{M}$ in $gr[N_2]\varphi^*\mathcal{M}$ is contained in $T^*_{Z'}Y$. This shows that $gr[N_0]\varphi^*\mathcal{M}$ is supported by the union of $T^*_{Z'}Y$ and of a set W of dimension strictly lower than the dimension of $T^*_{Z'}Y$. But we know that this support is involutive hence all its component have a dimension at least that dimension, so $gr[N_0]\varphi^*\mathcal{M}$ is supported in $T^*_{Z'}Y$ in a neighborhood of $\varphi^{-1}(\zeta)$. By definition $gr[N_0]\varphi^*\mathcal{M}$ is equal to $Ch_{T^*_{Z'}Y}{}_{(\infty,1)}(\varphi^*\mathcal{M})$, hence $\varphi^*\mathcal{M}$ has regular singularities along $T^*_{Z'}Y$. $\square$

Theorem 1.4.3. *Let X be a complex manifold, $\pi : T^*X \to X$ the projection, Z a submanifold of X and η a vector field on X which is definite positive with respect to Z. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. We assume that:*

1. *$\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module on $X - Z$,*
2. *$\mathcal{M}$ has η-weighted regular singularities along $T^*_Z X$,*
3. *The dimension of $Ch(\mathcal{M}) \cap T^*_Z X$ is equal to the dimension of X.*

Then $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module.

Proof. We fix local coordinates $(x_1, \dots, x_{n-p}, t_1, \dots, t_p)$ of X so that $Z = \{t = 0\}$ and $\eta = \sum_{i=1 \dots p} m_i t_i D_{t_i}$. We define a map $\varphi : Y \to X$ by $\varphi(x, y) = (x, y_1^{m_1}, \dots, y_p^{m_p})$ where Y is a neighborhood of 0 in $\mathbb{C}^n$. If Z' is the set $\{y = 0\}$, lemma 1.4.2 shows that $\varphi^*\mathcal{M}$ has regular singularities along $T^*_{Z'}Y$.

The third condition means that the characteristic variety of $\mathcal{M}$ has no irreducible component contained in $\pi^{-1}(Z)$ except $T^*_Z X$. The same is true for $\varphi^*\mathcal{M}$ on Z'. This may be proved as in lemma 1.4.2 but with the usual filtration replacing the bifiltration. This may also be proved easily with the definition of the characteristic variety in terms of microdifferential operators.

By hypothesis, $\mathcal{M}$ is regular on $X - Z$ hence by [6, Cor 5.4.8.] $\varphi^*\mathcal{M}$ is regular holonomic on $Y - Z'$. So, $\varphi^*\mathcal{M}$ has regular singularities along each irreducible component of its characteristic variety, hence by definition, it is a regular holonomic $\mathcal{D}_Y$-module.

Then by [6, theorem 6.2.1.], the direct image $\varphi_*\varphi^*\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module. By definition

$$\varphi_*\varphi^*\mathcal{M} = \mathbb{R}\varphi_*(\mathcal{D}_{X\leftarrow Y} \otimes^{\mathbb{L}}_{\mathcal{D}_Y} \mathcal{D}_{Y\to X} \otimes^{\mathbb{L}}_{\varphi^{-1}\mathcal{D}_Y} \varphi^{-1}\mathcal{M})$$

and the morphism $\mathcal{D}_X \to \mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ is injective hence$\mathcal{M}$ is a submodule of $\varphi_*\varphi^*\mathcal{M}$ hence a regular holonomic $\mathcal{D}_X$-module. □

The following corollary is the generalization of the definition of regular holonomic $\mathcal{D}$-modules and of proposition 1.1.1. It is proved from the previous theorem by descending induction on the dimension of the strata.

Corollary 1.4.4. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Assume that there is a stratification $X = \bigcup X_\alpha$ such that $Ch(\mathcal{M}) \subset \bigcup T^*_{X_\alpha} X$ and for each α there is a vector field η_α positive definite along X_α such that $\mathcal{M}$ has η_α-weighted regular singularities along $T^*_{X_\alpha} X$.*

Then $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module.

2 Symmetric pairs

2.1 Definitions

Let us briefly recall what is a symmetric pair. For the details we refer to [15] and [12]. Let G be a connected complex reductive algebraic group with Lie algebra $\mathfrak{g}$. Fix a non-degenerate, G-invariant symmetric bilinear form κ on the reductive Lie algebra $\mathfrak{g}$ such that

κ is the Killing form on the semi-simple Lie algebra $[\mathfrak{g},\mathfrak{g}]$. Fix an involutive automorphism ϑ of $\mathfrak{g}$ preserving κ and set $\mathfrak{k} = Ker(\vartheta - I)$, $\mathfrak{p} = Ker(\vartheta + I)$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the pair $(\mathfrak{g},\mathfrak{k})$ or $(\mathfrak{g},\vartheta)$ is called a symmetric pair. Recall that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to κ and that $\mathfrak{k}$ is a reductive Lie subalgebra of $\mathfrak{g}$. Denote by K the connected reductive subgroup of G with Lie algebra $\mathfrak{k}$. The group K acts on $\mathfrak{p}$ via the adjoint action.

Let $\mathfrak{p}^*$ be the dual of $\mathfrak{p}$, $\mathcal{O}(\mathfrak{p}) = S(\mathfrak{p}^*)$ the ring of regular functions on $\mathfrak{p}$ ($S(\mathfrak{p}^*)$ is the symmetric algebra), $\mathcal{O}(\mathfrak{p}^*) = S(\mathfrak{p})$ the ring of regular functions on $\mathfrak{p}^*$ and $\mathcal{D}(\mathfrak{p})$ the ring of differential operators on $\mathfrak{p}$ with coefficients in $\mathcal{O}(\mathfrak{p})$. The ring of functions $\mathcal{O}(\mathfrak{p})$ is naturally embedded in $\mathcal{D}(\mathfrak{p})$ and we embed $\mathcal{O}(\mathfrak{p}^*) = S(\mathfrak{p})$ in $\mathcal{D}(\mathfrak{p})$ as differential operators with constant coefficients. That is we associate to an element u of the vector space $\mathfrak{g}$ the derivation in the direction of u

$$D_u(f)(x) = \frac{d}{dt} f(x+tu)|_{t=0}$$

and we extend to the symmetric algebra $S(\mathfrak{p})$. Remark that this embedding is compatible with the filtration by the degree in $S(\mathfrak{p})$ and the filtration by the order in $\mathcal{D}(\mathfrak{p})$.

Notice that K has an induced action on $S(\mathfrak{p})$, $S(\mathfrak{p}^*)$ and $\mathcal{D}(\mathfrak{p})$ and we have natural embeddings of the invariant subrings $S(\mathfrak{p})^K \subset \mathcal{D}(\mathfrak{p})^K$ and $S(\mathfrak{p}^*)^K \subset \mathcal{D}(\mathfrak{p})^K$. The ring $S(\mathfrak{p})^K$ is equal to the ring of polynomials $\mathbb{C}[p_1,\dots,p_r]$ for some $p_1,\dots,p_r$ in $S(\mathfrak{p})^K$ and in the same way $S(\mathfrak{p}^*)^K$ is equal to a ring of polynomials $\mathbb{C}[q_1,\dots,q_r]$ [7].

The differential of the action of K on $\mathfrak{p}$ induces a Lie algebra homomorphism $\tau : \mathfrak{k} \to$ Der $S(\mathfrak{p}^*)$ hence an embedding $\tau : \mathfrak{k} \to \mathcal{D}(\mathfrak{p})$ defined by

$$(\tau(a).f)(v) = \frac{d}{dt} f(e^{-ta}.v)|_{t=0}, \quad \text{for } a \in \mathfrak{k},\ f \in \mathcal{O}(\mathfrak{p}),\ v \in \mathfrak{p}$$

As a section of the tangent bundle, $\tau(A)$ is the map $\mathfrak{p} \to T\mathfrak{p} = \mathfrak{p} \times \mathfrak{p}$ given by $\tau(A)(X) = (X, [X,A])$.

We denote by $\mathbf{N}(\mathfrak{p})$ the nilpotent cone of $\mathfrak{p}$, that is the set of nilpotent elements of $\mathfrak{g}$ which lie in $\mathfrak{p}$, it is also the subvariety of $\mathfrak{p}$ defined by the set of K-invariant functions $S(\mathfrak{p}^*)^K$. In the same way we consider the nilpotent cone $\mathbf{N}(\mathfrak{p}^*)$ which is the subvariety of $\mathfrak{p}^*$ defined by $S(\mathfrak{p})^K$. An important result is that the nilpotent cone $\mathbf{N}(\mathfrak{p})$ is a finite union of K-orbits [7, theorem 2].

The cotangent bundle $T^*\mathfrak{p}$ is equal to $\mathfrak{p} \times \mathfrak{p}^*$. The non-degenerate form κ on $\mathfrak{g}$ defines a non-degenerate symmetric bilinear form on $\mathfrak{p}$ and an isomorphism $\mathfrak{p} \simeq \mathfrak{p}^*$. We identify $T^*\mathfrak{p} = \mathfrak{p} \times \mathfrak{p}^* \simeq \mathfrak{p} \times \mathfrak{p}$. Let $\mathcal{C}(\mathfrak{p}) = \{\, (x,y) \in \mathfrak{p} \times \mathfrak{p} \mid [x,y] = 0 \,\}$, then the dimension of $(\mathfrak{p} \times \mathbf{N}(\mathfrak{p})) \cap \mathcal{C}(\mathfrak{p})$ is equal to the dimension of $\mathfrak{p}$ [12, lemma 2.2.].

The characteristic variety of $\mathcal{D}_\mathfrak{p}/\mathcal{D}_\mathfrak{p}\tau(\mathfrak{k})$ is equal to $\mathcal{C}(\mathfrak{p})$. Let F be an ideal of finite codimension of $S(\mathfrak{p})^K$, its graduate is a power of $S(\mathfrak{p})^K$ hence the characteristic variety of the $\mathcal{D}_\mathfrak{p}$-module $\mathcal{D}_\mathfrak{p}/\mathcal{D}_\mathfrak{p}F$ is $\mathfrak{p} \times \mathbf{N}(\mathfrak{p})$. Finally, if $\mathcal{I}$ be the left ideal of $\mathcal{D}_\mathfrak{p}$ generated by F and $\tau(\mathfrak{k})$, the characteristic variety of $\mathcal{M}_F = \mathcal{D}_\mathfrak{p}/\mathcal{I}$ is contained in $(\mathfrak{p} \times \mathbf{N}(\mathfrak{p})) \cap \mathcal{C}(\mathfrak{p})$ hence $\mathcal{M}_F$ is a holonomic $\mathcal{D}_\mathfrak{p}$-module.

As a special case, we have the diagonal case where $G = G_1 \times G_1$ with $\vartheta(x,y) = (y,x)$ for some reductive group G_1. Thus $(\mathfrak{g},\mathfrak{k}) = (\mathfrak{g}_1 \oplus \mathfrak{g}_1, \mathfrak{g}_1)$ and $K = G_1$ with its adjoint action on $\mathfrak{p} = \mathfrak{g}_1$. Let $\lambda \in \mathfrak{p}^*$ and $F_\lambda = \{\, P - P(\lambda) \mid P \in S(\mathfrak{p})^K \,\}$, then the corresponding module $\mathcal{M}_\lambda^F = \mathcal{D}_\mathfrak{p}/\mathcal{D}_\mathfrak{p}\tau(\mathfrak{k}) + \mathcal{D}_\mathfrak{p}F_\lambda$ is the module of Kashiwara-Hotta [4].

2.2 The conjecture of Sekiguchi

Theorem 2.2.1. *Let F be an ideal of finite codimension of $S(\mathfrak{p})^K$ and $\mathcal{M}_F = \mathcal{D}_\mathfrak{p}/\mathcal{I}$ where $\mathcal{I}$ is the left ideal of $\mathcal{D}_\mathfrak{p}$ generated by F and $\tau(\mathfrak{k})$.*

Then $\mathcal{M}_F$ is a regular holonomic $\mathcal{D}_\mathfrak{p}$-module.

The proof of this theorem will be made in several steps. First we will reduce to the semi-simple case (lemma 2.2.3), then prove by induction on the dimension of the Lie algebra, that the result is true outside of the nilpotent cone (lemma 2.2.4) and the key point of the proof is the case of a nilpotent orbit (lemma 2.2.6).

Lemma 2.2.2. *Let Y be a complex manifold and $X = Y\times\mathbb{C}$. Let $P(t, D_t)$ be a differential operator on $\mathbb{C}$ with principal symbol independent of t and $\mathcal{I}$ be a coherent ideal of $\mathcal{D}_X$ which contains P.*

Let $\mathcal{M}_Y$ be the inverse image of $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ on Y by the immersion $Y \to X$, then $\mathcal{M}$ is isomorphic to the inverse image of $\mathcal{M}_Y$ by the projection $q : X \to Y$, that is

$$\mathcal{M} = \mathcal{D}_{X\to Y} \otimes_{q^{-1}\mathcal{D}_Y} q^{-1}\mathcal{M}_Y = \mathcal{M}_Y \widehat{\otimes} \mathcal{O}_\mathbb{C}$$

In particular, $\mathcal{M}$ is regular holonomic if and only if $\mathcal{M}_Y$ is regular holonomic.

Proof. This lemma is a (very) special case of [14, theorem 5.3.1. ch II]. The first step is to prove that $\mathcal{D}_\mathbb{C}/\mathcal{D}_\mathbb{C}P$ is isomorphic to $(\mathcal{D}_\mathbb{C}/\mathcal{D}_\mathbb{C}D_t)^N$. The proof is the same than that of [14, theorem 5.2.1. ch II], but as there is only one variable, the proof is very simple and use only functions instead of differential operators of infinite order. Then we can follow the proof of [14] but with finite order operators instead of infinite order operators.

Remark that if P were a differential operator in several variables, for example, $P = D_t^2 + D_x$, this result would be true only with the sheaf $\mathcal{D}_X^\infty$ of differential operators with infinite order.

As $X = Y \times \mathbb{C}$, the inverse image of $\mathcal{M}_Y$ by q is isomorphic to the external product of $\mathcal{D}$-modules $\mathcal{M}_Y\widehat{\otimes}\mathcal{O}_\mathbb{C}$. □

Assume that $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, the action of K on $\mathfrak{p}_0$ being trivial. Then $S(\mathfrak{p})^K = S(\mathfrak{p}_0) \otimes S(\mathfrak{p}_1)^K$, this defines a morphism $\delta : S(\mathfrak{p})^K \to S(\mathfrak{p}_1)^K$ by restriction and $F_1 = \delta(F)$ is an ideal of finite codimension of $S(\mathfrak{p}_1)^K$. Let $\mathcal{M}_{F_1} = \mathcal{D}_{\mathfrak{p}_1}/\mathcal{I}_1$ where $\mathcal{I}_1$ is the ideal of $\mathcal{D}_{\mathfrak{p}_1}$ generated by $\tau_{\mathfrak{p}_1}(\mathfrak{k})$ and F_1.

Lemma 2.2.3. *(1) The module $\mathcal{M}_F$ is isomorphic to $\mathcal{O}_{\mathfrak{p}_0}\widehat{\otimes}(\mathcal{M}_F)_{\mathfrak{p}_1}$ where $(\mathcal{M}_F)_{\mathfrak{p}_1}$ is the restriction of $\mathcal{M}_F$ to $\mathfrak{p}_1$.*

(2) $(\mathcal{M}_F)_{\mathfrak{p}_1}$ (hence $\mathcal{M}_F$) is regular if $\mathcal{M}_{F_1}$ is regular.

Proof. By induction on the dimension of $\mathfrak{p}_0$, we may assume that $\mathfrak{p}_0 = \mathbb{C}$ and choose linear coordinates (x, t) of $\mathfrak{p}$ such that $\mathfrak{p}_0 = \{\, (x, t) \in \mathfrak{p} \mid x = 0 \,\}$. The action of K is trivial on $\mathfrak{p}_0$ hence $S(\mathfrak{p})^K$ contains $S(\mathfrak{p}_0)$ and as F is finite codimensional in $S(\mathfrak{p})^K$ it contains a polynomial in D_t. Lemma 2.2.2 shows the first part of the lemma.

We assume now that $\mathcal{M}_{F_1}$ is regular. Recall that $(\mathcal{M}_F)_{\mathfrak{p}_1} = \mathcal{M}_F/t\mathcal{M}_F$ is a holonomic $\mathcal{D}_{\mathfrak{p}_1}$-module generated by the classes of $1, \dots, D_t^{m-1}$. Let $\mathcal{M}'$ be the submodule of $(\mathcal{M}_F)_{\mathfrak{p}_1}$ generated by the class u of D_t^{m-1}. The vector fields of $\tau(\mathfrak{k})$ are independent of (t, D_t)

hence u is annihilated by $\tau(\mathfrak{k})$. If P is an element of F, as an operator of $\mathcal{D}_\mathfrak{p}$ it is equal to $\delta(P) + AD_t$ hence $\delta(P)$ annihilates u. So u is annihilated by $\tau(\mathfrak{k})$ and by F_1 and $\mathcal{M}'$ is a quotient of $\mathcal{M}_{F_1}$. So $\mathcal{M}'$ is regular.

Consider now $\mathcal{M}''$ which is the submodule of $\mathcal{M}$ generated by the classes D_t^{m-1} and D_t^{m-2}. The quotient $\mathcal{M}''/\mathcal{M}'$ is generated by the class v of D_t^{m-2} which is annihilated by $\tau(\mathfrak{k})$ and by F_1, so it is regular. We have an exact sequence $0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M}'' \longrightarrow \mathcal{M}''/\mathcal{M}' \longrightarrow 0$ where two terms are regular hence $\mathcal{M}''$ is regular. Continuing the same argument, we get that $(\mathcal{M}_F)_{\mathfrak{p}_1}$ is regular. $\square$

Let b be a semisimple element of $\mathfrak{p}$. Then $\mathfrak{p} = \mathfrak{p}^b \oplus [\mathfrak{k}, b]$ and $\mathfrak{g}^b = \mathfrak{k}^b \oplus \mathfrak{p}^b$ defines a symmetric pair. Let δ be the restriction map $\delta : S(\mathfrak{p})^K \to S(\mathfrak{p}^b)^{K^b}$, this map is injective and if F is an ideal of finite codimension of $S(\mathfrak{p})^K$ then $\delta(F)$ is an ideal of finite codimension of $S(\mathfrak{p}^b)^{K^b}$ [3, lemma 19]. Let $\mathcal{I}_b$ be the left ideal of $\mathcal{D}_{\mathfrak{p}^b}$ generated by $\delta(F)$ and $\tau(\mathfrak{k}^b)$ and $\mathcal{M}_b = \mathcal{D}_{\mathfrak{p}^b}/\mathcal{I}_b$.

Lemma 2.2.4. *In a neighborhood of b, $\mathcal{M}_F$ is isomorphic to the external product of the holomorphic functions on the orbit of b by a quotient of $\mathcal{M}_b$. In particular, $\mathcal{M}_F$ is regular if $\mathcal{M}_b$ is regular.*

Proof. Let V be a linear subspace of $\mathfrak{k}$ such that $\mathfrak{k} = V \oplus \mathfrak{k}^b$. The map $f : V \times \mathfrak{p}^b \to \mathfrak{p}$ given by $f(y, Z) = exp(y).Z$ is a local isomorphism. If $(x_1, \ldots, x_{n-r})$ are linear coordinates of V and $(t_1, \ldots, t_r)$ are linear coordinates of $\mathfrak{p}^b$, the map f defines local coordinates $(x_1, \ldots, x_{n-r}, t_1, \ldots, t_r)$ of $\mathfrak{p}$ in a neighborhood of b. Lemma 3.7 of [15] shows that in these coordinates, the orbit Kb is $\{\, (x,t) \mid t = 0 \,\}$, $\mathfrak{p}^b = \{\, (x,t) \mid x = 0 \,\}$ and the differential operators $D_{x_1}, \ldots, D_{x_{n-r}}$ belong to $\tau(\mathfrak{k})$. Hence $\mathcal{M}$ is the product of $\mathcal{O}_{Kb}$ by a module $\mathcal{N}$.

If Z is an element of $\mathfrak{k}^b$, $\tau_\mathfrak{p}(Z)$ is by definition the vector field on $\mathfrak{p}$ with value $[Z, A]$ at a point A of $\mathfrak{p}$. The value of $\tau_{\mathfrak{p}^b}(Z)$ at a point A of $\mathfrak{p}^b$ is the projection of $[Z, A]$ on $\mathfrak{p}^b$, hence $\tau_{\mathfrak{p}^b}(\mathfrak{k}^b)$ is equal to $\tau_\mathfrak{p}(\mathfrak{k})$ modulo $D_{x_1}, \ldots, D_{x_{n-r}}$. On the other hand, let $P \in F$, as the coordinates $(t_1, \ldots, t_r)$ are linear coordinates of $\mathfrak{p}^b$, the value of P on a function of t is the restriction of P to $S(\mathfrak{p}^b)^{K^b}$. Hence $\mathcal{N}$ is a quotient of $\mathcal{M}_b$. $\square$

Lemma 2.2.5. *Let Λ be the conormal to 0 in $\mathfrak{p}$. The microcharacteristic variety $Ch_{\Lambda(\infty,1)}(\mathcal{M}_F)$ is contained in $(\mathfrak{p} \times \mathbf{N}(\mathfrak{p})) \cap \mathcal{C}(\mathfrak{p})$.*

Proof. Let E be the Euler vector field of the vector space $\mathfrak{p}$. It is clear on the definition, that the vector fields of $\tau(\mathfrak{k})$ preserve the homogeneity of functions hence that they commute with E. So they are homogeneous of degree 0 for the V-filtration at 0. On the other hand, they are homogeneous of degree 1 for the usual filtration as any vector field. So if $u \in \tau(\mathfrak{k})$, $\sigma_{\Lambda(\infty,1)}(u) = \sigma(u)$.

On differential operators with constant coefficients, the V-filtration at $\{0\}$ and the usual filtration coincide, hence we have also $\sigma^E{}_{(\infty,1)}(P) = \sigma(P)$ for these operators.

So, $Ch_{\Lambda(\infty,1)}(\mathcal{M}_F)$ is contained in the set of points where the symbols of the operators of $\tau(\mathfrak{k})$ and of F vanish that is in $(\mathfrak{p} \times \mathbf{N}(\mathfrak{p})) \cap \mathcal{C}(\mathfrak{p})$. $\square$

Lemma 2.2.6. *For each nilpotent orbit S of $\mathbf{N}(\mathfrak{p})$, there is a vector field η which is positive definite with respect to S and such that $\mathcal{M}_F$ has η-weighted regular singularities along $T^*_S\mathfrak{p}$.*

Proof. Let S be one of these orbits, r the codimension of S and $X \in S$. As in [12, §3] (see also [16, Part I, §5.6]) we can choose a normal $\mathfrak{sl}_2$-triple (H, X, Y) in $\mathfrak{p}$ which generates a Lie algebra isomorphic to $\mathfrak{sl}_2$ and acting on $\mathfrak{p}$ by the adjoint representation. Then $\mathfrak{p}$ splits into a direct sum of irreducible submodules of dimensions $\lambda_i + 1$ for $i = 1 \dots r$. Moreover $\mathfrak{p} = \mathfrak{p}^Y \oplus [X, \mathfrak{k}]$, $\dim \mathfrak{p}^Y = r$ and we can select a basis $(Y_1, \dots, Y_r)$ of $\mathfrak{p}^Y$ such that $[H, Y_i] = -\lambda_i Y_i$. Let V be a linear subspace of $\mathfrak{k}$ such that $\mathfrak{k} = V \oplus \mathfrak{k}^X$. If $(b_1, \dots, b_{n-r})$ is a basis of V, the map $F : \mathbb{C}^n \to \mathfrak{p}$ given by

$$F(x_1, \dots, x_{n-r}, t_1, \dots, t_r) = \exp(x_1 b_1) \dots \exp(x_{n-r} b_{n-r}).(X + \sum t_i Y_i)$$

is a local isomorphism hence defines local coordinates (x, t) of $\mathfrak{p}$ in a neighborhood of X. In these coordinates, $S = \{\, (x, t) \mid t = 0 \,\}$, $\mathfrak{p}^Y = \{\, (x, t) \mid x = 0 \,\}$, and the differential operators $D_{x_1}, \dots, D_{x_{n-r}}$ are in the ideal generated by $\tau(\mathfrak{k})$ [15, lemma 3.7].

Let E be the Euler vector field of the vector space $\mathfrak{p}$. A standard calculation [16, Part I,§5.6] shows that $E(t_i)|_{x=0} = m_i t_i$ with $m_i = \frac{1}{2}\lambda_i + 1$ and if $b_{n-r} = H$ we have $E(t_i) = m_i t_i$ [1]. Hence E is equal to $\eta + w$ where $\eta = \sum_{j=1}^{r} m_j t_j D_{t_j}$ and w is a vector field which vanishes on functions independent of t, that is $w = \sum a_i(x, t) D_{x_i}$. By definition, η is positive definite with respect to S.

Define a map $\varphi : \mathfrak{p} \to V = \mathbb{C}^r$ by $\varphi(x, t) = t$. Let $\eta' = \sum m_j t_j D_{t_j}$ on V. The functions $t_1, \dots, t_r$ satisfy $E(t_i) = \eta'(t_i) = m_i t_i$ hence they are homogeneous and the map φ is defined in a conic neighborhood of X. This also shows that E is invariant under φ and that $\eta' = \varphi_*(E)$.

The module $\mathcal{M}_F$ is equal to $\mathcal{D}_{\mathfrak{p}}/\mathcal{I}$ where $\mathcal{I}$ is a coherent ideal of $\mathcal{D}_{\mathfrak{p}}$ which contains the derivations $D_{x_1}, \dots, D_{x_{n-r}}$ hence $\mathcal{I}$ is generated by $D_{x_1}, \dots, D_{x_{n-r}}$ and a finite set of differential operators $Q_1(t, D_t), \dots, Q_N(t, D_t)$ depending only of (t, D_t). (This result is standard and also a special case of lemma 2.2.2).

The module $\mathcal{M}_F$ satisfies the hypothesis of corollary 1.3.3 hence $Ch^{E}{}_{(\infty,1)}(\mathcal{M}_F)$ is equal to $Ch^{\eta}{}_{(\infty,1)}(\mathcal{M}_F)$ and by lemma 2.2.5 it is contained in $(\mathfrak{p} \times \mathbf{N}(\mathfrak{p})) \cap \mathcal{C}(\mathfrak{p})$.

Assume now that $T^*_S\mathfrak{p}$ is an irreducible component of the characteristic variety $Ch(\mathcal{M}_F)$ and let x^* be a generic point of $T^*_S\mathfrak{p}$, that is a point which does not belong to other irreducible components of $Ch(\mathcal{M}_F)$. We have $T^*_S\mathfrak{p} \subset Ch(\mathcal{M}_F) \subset (\mathfrak{p} \times \mathbf{N}(\mathfrak{p})) \cap \mathcal{C}(\mathfrak{p})$ and as they have the same dimension, they are equal generically. So $Ch^{\eta}{}_{(\infty,1)}(\mathcal{M}_F) = T^*_S\mathfrak{p}$ generically on $T^*_S\mathfrak{p}$ and we are done. □

Proof of theorem 2.2.1. We will argue by induction on the dimension of $\mathfrak{g}$ and first, we reduce to the semi-simple case. Set $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{g}_1$, $\mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{g}_1$, $\mathfrak{z}$ the center of $\mathfrak{g}$ and $\mathfrak{p}_0 = \mathfrak{z} \cap \mathfrak{p}$. We have $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ and by lemma 2.2.3, it suffices to prove the theorem for $\mathfrak{p}_1$. As $\mathfrak{z} \cap \mathfrak{k}$ acts trivially we may assume that $\mathfrak{g}$ is semisimple.

Let x be a non-nilpotent element of $\mathfrak{p}$. It decomposes as $x = b + n$ where b is non zero and semisimple, n is nilpotent and $[b, n] = 0$. As $\mathfrak{g}$ is semisimple, $\mathfrak{p}^b$ is of dimension strictly less than $\mathfrak{p}$, hence we may assume by the induction hypothesis that the theorem is true for $\mathfrak{p}^b$. Lemma 2.2.4 shows that $\mathcal{M}_F$ is regular in a neighborhood of b. As $\mathcal{M}_F$ is constant on the orbits, it is regular on the orbits whose closure contains b, in particular at x.

We proved that $\mathcal{M}_F$ is regular outside of the nilpotent cone. As the nilpotent cone is a finite union of orbits, we will now argue by descending induction on the dimension of these orbits. So let x be a nilpotent point of $\mathfrak{p}$, Kx its orbit and assume that $\mathcal{M}_F$ is regular on $\mathfrak{p} - Kx$ in a neighborhood of x. Lemma 2.2.6 shows that $\mathcal{M}_F$ has η-weighted regular singularities along $T^*_{Kx}\mathfrak{p}$ hence theorem 1.4.3 shows that $\mathcal{M}_F$ is regular at x. $\square$

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