A NEW METHOD OF CONSTRUCTING p-ADIC L-FUNCTIONS ASSOCIATED WITH MODULAR FORMS

by Alexei PANCHISHKIN

Prépublication de l'Institut Fourier n^{o} 553 (2001)

http://www-fourier.ujf-grenoble.fr/prepublications.html

To Yuri Ivanovich Manin for his sixty-fifth birthday

Let p be a prime number. The purpose of this paper is to give a new method of constructing p-adic measures associated with modular forms using distributions with values in spaces of modular forms (holomorphic or nearly holomorphic). At the same time we present a conseptual explanation for the formulas of Yu.I.Manin [Man73, Man76, MazSwD] giving p-adic distributions attached to modular forms (in both the ordinary case (Theorems 3 and 5) and the supersingular case, see [Vi76] (Theorems 4 and 6)). For this purpose we use the canonical projector onto the primary (characteristic) subspace associated to a non-zero eigenvalue α of the Atkin-Lehner operator U. This operator acts similar to the trace operator lowering the level of modular forms. On the other hand, U is invertible on its finite dimensional α-primary subspace so that one can glue its action on forms of various levels. In this way one obtains the desired distributions with values in a finite dimensional vector space starting from naturally defined distributions with values in spaces of modular forms (like Eisenstein distributions, theta distributions etc.).

Next, in order to obtain from them numerically valued distributions interpolating L-values attached to modular forms one applies a suitable linear form coming from the Rankin-Selberg method. The inversion of U on the α-primary subspace corresponds to the factor α^{-n} in the formulas for p-adic distributions by Manin-Mazur-Swinnerton-Dyer, see Proposition 11.1 d).

The paper is an extended version of the talk "A new way of constructing p-adic L functions" on July 4, 2001 in the Oberseminar of the Mathematical Institute in Heidelberg (and of talks in Luminy on September 26, 2001, and in Caen on October 5, 2001); it develops

Key-words : Modular forms, p-adic L-functions, special values.
Math. classification math. : 11F33, 11F67, 11F30.

some techniques of [PaIAS] found during a visit of the author to the Institute for Advanced Study in Princeton in 1999-2000. Let these institutions be thanked for their kind hospitality. It is a great pleasure for me to thank S.Böcherer, E.Freitag, R.Weissauer and G.Robert for valuable discussions and observations.

Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ denote the completion of an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers. Fix a positive integer N, a Dirichlet character $\psi \bmod N$ and consider the commutative profinite group $Y = Y_{N,p} = \varprojlim_m (\mathbb{Z}/Np^m\mathbb{Z})^*$ and its group $X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^\times)$ of (continouos) p-adic characters des (this is a $\mathbb{C}_p$-analytic Lie group analogous to $\mathrm{Hom}_{cont}(\mathbb{R}_+^\times, \mathbb{C}^\times) \cong \mathbb{C}$ (by $s \mapsto (y \mapsto y^s)$. The group X is isomorphic to a finite union of discs $U = \{z \in \mathbb{C}_p \mid |z|_p < 1\}$.

A p-adic L-function $L : X \to \mathbb{C}_p$ is a certain meromorphis function on X coming from a p-adic measure on Y.

1. Traditional method of constructing these functions is from the special critical values of complex L-functions (which are often algebraic, after a suitable normalisation). Let us fix an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ in order to consider algebraic numbers as p-adic numbers.

Example. — The Riemann zeta function.

$$\zeta(s) = \prod_{l\ primes} (1 - l^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s}\ (\mathrm{Re}(s) > 1),\ \zeta(1-k) = -\frac{B_k}{k},$$

where B_k are the Bernoulli numbers given by

$$e^{Bt} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!} = \frac{t\, e^t}{e^t - 1}.$$

Put

$$\zeta_{(p)}^{(c)}(-k) = (1 - p^k)(1 - c^{k+1})\zeta(-k)$$

THEOREM 1 (Kummer). — *For any polynomial $h(x) = \sum_{i=0}^n \alpha_i x^i \in \mathbb{Z}_p[x]$ over $\mathbb{Z}_p$ such that $x \in \mathbb{Z}_p \Longrightarrow h(x) \in p^m\mathbb{Z}_p$ one has*

$$\sum_{i=0}^{n} \alpha_i\, \zeta_{(p)}^{(c)}(-i) \in p^m\mathbb{Z}_p \tag{1.1}$$

This property expresses the fact that the numbers $\zeta_{(p)}^{(c)}(-k)$ depend continuously on k in the p-adic sense:

COROLLARY. — *Let $k_1, k_2 \in \mathbf{N}^*$, $k_1 \equiv k_2 \pmod{(p-1)p^{m-1}}$, then*

$$\zeta_{(p)}^{(c)}(-k_1) \equiv \zeta_{(p)}^{(c)}(-k_2) \pmod{p^m}$$

Indeed, it suffices to take $h(x) = x^{k_1} - x^{k_2}$.

Proof of Theorem 1 is implied by the well-known formula:

$$S_k(N) = \sum_{n=1}^{N-1} n^k = \frac{1}{k+1}[B_{k+1}(N) - B_{k+1}] \tag{1.2}$$

in which $B_k(x) = (x+B)^k = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$ is the Bernoulli polynomial.

Definition 1.1.

a) Let A be a normed topological ring containing $\mathbb{Z}_p$ as a closed subring and let h be a positive integer. Consider the A-submodule $\mathcal{C}^h(Y, A)$ of all locally polynomial functions of degre $< h$ on Y (of the variable $y_p : Y \to \mathbb{Z}_p^*$, the canonical projection); in particular, the A-submodule $\mathcal{C}^1(Y, A)$ consists of locally constant functions on Y with values in A. If $A = \mathbb{C}_p$ then

$$\forall h > 1 \ \mathcal{C}^h(Y, A) \subset \mathcal{C}^{\text{loc}-\text{an}}(Y, A) = \{\varphi : Y \to A \mid \varphi \text{ locally analytic}\} \subset \mathcal{C}(Y, A),$$

where $\mathcal{C}(Y, A) = \{\varphi : Y \to A \mid \varphi \text{ continuous}\}$.

b) A distribution Φ on Y with values in a normed A-module V is an A-linear map $\Phi : \mathcal{C}^1(Y, A) \to V, \varphi \mapsto \int_Y \varphi \, d\mu$.

c) A measure $\Phi : \mathcal{C}^1(Y, A) \to V$ is a bounded distribution: $|\Phi(\varphi)|_p < C|\varphi|_p$ where C does not depend on ϕ.

d) Let $h \in \mathbf{N}^*$. An h-admissible measure on Y with values in V is an A-linear map $\tilde{\Phi} : \mathcal{C}^h(Y, A) \to V$ with the following growth condition: for all $t = 0, 1, \ldots, h-1$,

$$\left| \int_{a+(p^m)} (y_p - a_p)^t d\tilde{\Phi} \right|_p = o\left(p^{m(h-t)}\right).$$

for $m \to \infty$.

If $A = \mathbb{C}_p$ then according to [AV] and [Vi76], such a map $\tilde{\Phi}$ can be uniquely extended to the A-module $\mathcal{C}^{\text{loc}-\text{an}}(Y, A)$ of locally analytic functions on Y of the parameter $y_p : Y \to \mathbb{Z}_p^\times$.

Theorem 2 (Mazur). — *There exists a unique (bounded) measure $\mu^{(c)}$ on $\mathbb{Z}_p^\times$ with values in $\mathbb{Z}_p$ such that*

$$\int_{\mathbb{Z}_p^\times} x^k d\mu^{(c)} = \zeta_{(p)}^{(c)}(-k), \ \ k \geq 0$$

Remark. — Theorem 1 is equivalent to Theorem 2 (by integration of h against $\mu^{(c)}$).

In the present paper we construct p-adic distributions on Y with values in $\mathbb{C}_p$ starting from distributions with values in spaces of modular forms.

The p-adic L-function of Kubota-Leopoldt is the meromorphic function $L_p : X \to \mathbb{C}_p^\times$ given by

$$L_p(x) := \frac{\int_Y x d\mu^{(c)}}{1 - x(c)c}, \ x \in X \tag{1.3}$$

(with a single simple pole at $x = y_p^{-1}$), and the function (1.3) is independente of a choice of c: for all Dirichlet characters χ mod p^m, $\chi : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}^\times \hookrightarrow \mathbb{C}_p^\times$ one has

$$L_p(\chi y_p^k) = (1 - \chi(p)p^k)L(-k, \chi) \in i_p(\mathbb{Q}^{\mathrm{ab}}).$$

In general every distribution on Y with values in $\mathbb{Z}_p$ defines a p-adic L-function (the non-Archimedian Mellin transform of μ):

$$L_\mu : X \to \mathbb{C}_p, \ \mu(x) = \int_Y x(y)\, d\mu.$$

If $\mathcal{D}(s, \chi) = \sum_{n\geq 1} \chi(n)c_n n^{-s}$ is an arithmeticaly defined complex L-function twisted with a Dirichlet character χ with the property $\mathcal{D}^*(-k, \chi) \in \overline{\mathbb{Q}}$ for an infinite set of couples k, χ (an with a normalization $\mathcal{D}^*(s, \chi)$ obtained by multiplying $\mathcal{D}(s, \chi)$ with certain elementary factors), one constructs usually the corresponding p-adic L-function $L = L_{\mu_{\mathcal{D}}}$ starting from the algebraic special values $\mathcal{D}^*(-k, \chi)$ in such a way that

$$L_{\mu_{\mathcal{D}}}(\chi y_p^k) = \int_Y \chi y_p^k \, d\mu_{\mathcal{D}} = i_p(\mathcal{D}^*(-k, \chi)),$$

and the existence of such a measure is equivalent to generalized Kummer congruences for the special values $\mathcal{D}^*(-k, \chi)$. Formulas for these values could be quite complicated and one uses various methods in order to obtain such congruences (like the formulas of type (1.2) in the proof of the Theorem 1). For modular forms one uses geometric tools like modular symboles, continuous fractions, the Rankin-Selberg method etc., (voir [Man73], [Ra52], [Man-Pa], [PLNM]).

We propose a new method which produces a family of p-adic measures starting from a distribution Φ on Y with values in a suitable vector space $\mathcal{M} = \bigcup_{m\geq 0} \mathcal{M}(Np^m)$ of modular forms; this family of p-adic measures $\mu_{\alpha,\Phi,f}$ is parametrized by non-zero eigenvalues $\alpha \in \mathbb{C}_p \neq 0$, of the operator U of Atkin-Lehner on $\mathcal{M}$, and by a primitive cusp eigenform f with an associated eigenvalue $\alpha \neq 0$ (on an easy modification f_0 of f as an eigenfunction). One says that a primitive cusp eigenform $f = \sum_{n\geq 1} a(n, f)e(nz) \in \mathcal{S}_k(\Gamma_0(N), \psi) \subset \mathcal{M}$ (where $(e(nz) = \exp(2\pi i n z))$) is associated to an eigenvalue α if there exists a cusp form $f_0 = f_{0,\alpha} = \sum_{n\geq 1} a(n, f_0)e(nz)$ such that $f_0 \mid U = \alpha f_0$ and $f_0 \mid T(\ell) = a(\ell)f_0$ for all prime numbers $\ell \nmid Np$)

In the ordinary case $|\alpha|_p = 1$ a construction of such measures could be obtained from Hida's idempotent $e = \lim_{r\to\infty} U(p)^{r!}$ (see Hida [Hi93]) acting on p-adic modular forms; the image of e is contained in a subspace $\mathcal{M}^{\mathrm{ord}} \subset \mathcal{M}$ of finite dimension ("the ordinary part of $\mathcal{M}$") which is known to be generated by certain classical modular forms. One obtains $\mu_{\alpha,\Phi,f} = \ell_f(e\Phi)$ for a suitable linear form $\ell_f \in \mathcal{M}^{\mathrm{ord}\,*}$. In a more general case when $\alpha \neq 0$ one could imitate this method using instead of $\mathcal{M}^{\mathrm{ord}}$ the primary (characteristic) subspace $\mathcal{M}^{\alpha} \subset \mathcal{M}$ of U (which is also of finite dimension).

2. Distributions with values in modular forms. — Let A be an algebraic extension K of $\mathbb{Q}_p$ or its ring of integers $\mathcal{O}_K$. Let us fix an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and let $\mathcal{M}_k(\Gamma_1(N); A)$, $\mathcal{M}_k(\Gamma_0(N)\psi; A)$ be the submodules of $A[[q]]$ generated by the q-expansions $f = \sum_{n\geq 0} a_n(f)q^n \in \mathcal{M}_k(\Gamma_1(Np^m), \overline{\mathbb{Q}})$ of classical modular forms with algebraic Fourier coefficients $a_n(f) \in \overline{\mathbb{Q}}$ in $i_p^{-1}(A)$. One puts $\mathcal{M} = \cup_{m\geq 0}\mathcal{M}(Np^m)$, where $\mathcal{M}(Np^m) = \mathcal{M}_k(\Gamma_1(Np^m); A)$, and $\mathcal{S} = \cup_{m\geq 0}\mathcal{S}(Np^m)$ the A-submodule of cusp forms.

Examples of distributions with values in $\mathcal{M}$.

Let $\Phi : \mathcal{C}^1(Y, \mathbb{C}_p) \to \mathcal{M}$ be a distribution on Y with values in $\mathcal{M}$.

a) *Eisenstein distributions.* For a complex number $s \in \mathbb{C}$ and $a, b \bmod N$ put (by analytic continuation):

$$E_{\ell,N}(z, s; a, b) = \sum (cz + d)^{-\ell}|cz + d|^{-2s} \quad (0 \neq (c, d) \equiv (a, b) \bmod N)\,.$$

Starting from this series , one obtains the following Eisenstein distributions: put $s = -r$, $0 \leq r \leq \ell - 1$,

$$\begin{aligned} E_{r,\ell,N}(a, b) &:= \frac{N^{\ell-2r-1}\Gamma(\ell - r)}{(-2\pi i)^{\ell-2r}(-4\pi y)^r} \sum_{a \bmod N} e(-ax/N)E_{\ell,N}(Nz, -r; x, b) \\ &= \varepsilon_{r,\ell,N}(a, b) + (4\pi y)^{-r} \sum_{0<dd', d\equiv a, d'\equiv b \bmod N} \operatorname{sgn} d\cdot \\ &\quad \cdot d^{\ell-2r-1}W(4\pi dd'y, \ell - r, -r)\cdot e(dd'z) \in \mathcal{M}_{r',\ell}(N), \\ &\qquad\qquad \text{where } r' = \max(r, \ell - r - 1)\,, \end{aligned}$$

$$W(y, \ell-r, -r) = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\frac{\Gamma(\ell - r)}{\Gamma(\ell - r - j)}y^{r-j}\,, \;\; \zeta(s; a, N) = \sum_{\substack{n\geq 1\\ n\equiv a (\bmod N)}} n^{-s}\,,$$

$$\begin{aligned} \varepsilon_{r,\ell,N}(a, b) = &\left.\frac{(-4\pi y)^s\Gamma(\ell + s)}{\Gamma(\ell + 2s)}\delta\left(\frac{b}{N}\right)\zeta(1 - \ell - 2s; a, N)\right|_{s=-r} \\ &+ \frac{\Gamma(\ell + 2s - 1)}{(4\pi y)^{\ell+s-1}\Gamma(s)}\delta\left(\frac{a}{N}\right)\Big[\zeta(\ell + 2s - 1; b, N) \\ &\left.+ (-1)^{\ell+2s}\zeta(\ell + 2s - 1; -b, N)\Big]\right|_{s=-r} \end{aligned}$$

These series are nearly holomorphic modular forms (see Section 7) in spaces $\mathcal{M}_{r',\ell}(N^2)$, where $r' = \max(r, \ell - r - 1)$ but in certain cases they are holomorphic, e.g. $\ell \geq 3$, $r = 0$

where $r = l - 1$) and these series produce distributions on $Y \times Y$ with values in $\mathcal{M}$:

$$E_{r,\ell}((a + (Np^m) \times (b + (Np^m)) := E_{r,\ell,Np^m}(a,b) \in \mathcal{M}_l(N^2p^{2m}).$$

b) *Partial modular forms.* For any $f = \sum_{n\geq 0} a_n(f)q^n \in \mathcal{M}_k(\Gamma_1(Np^m)$ one puts

$$\Phi_f(a + (Np^m)) := \sum_{\substack{n\geq 0 \\ n\equiv a(\mod p^m)}} a_n(f)q^n \in \mathcal{M}_k(Np^{2m})$$

c) *Partial theta series* (also with a spherical polynomial), see [Hi85].

Remarks. i) For any Dirichlet character χ mod p^m viewed as a function on Y with values in $i_p(\mathbb{Q}^{\mathrm{ab}})$, the integral

$$\int_Y \chi(y)\, d\Phi_f = \Phi_f(\chi) = \sum_{n\geq 0} a_n(f)q^n \in \mathcal{M}_k(Np^{2m}) \in \mathcal{M}_k(N^2p^{2m})$$

coincides then with the twisted modular form f_χ.

ii) The distributions a), b), c) are bounded (after a regularisation of the constant term in a)) with respect to the p-adic norm on $\mathcal{M} = \cup_{p^m}\mathcal{M}_k(\Gamma_1(Np^m), A) \subset A[[q]]$ given by $|g|_p = \sup_n |a(n,g)|_p$ for $g = \sum_{n\geq 0} a(n,g)g^n \in \mathcal{M}_k(\Gamma_1(N), A)$).

iii) Starting from distributions a), b), c) one can construct many other distributions, for example, using the operation of convolution on Y (as in [Hi85], where the case of the convolution of a theta distribution with an Eisenstein distribution was considered).

However we need distributions μ with scalar values (in $\mathbb{Z}_p$ or in $\mathbb{C}_p$) which we construct starting from distributions Φ with values in $\mathcal{M}$. This will be done in two steps:

The first step is the passage from $\mathcal{M}$ to a certain finite-dimensional part $\mathcal{M}^\circ \subset \mathcal{M}$; one uses a suitable projector $\pi : \mathcal{M} \to \mathcal{M}^\circ$ such that one can keep track of denominators when the level of modular forms grows.

The second step is to apply a suitable linear form to the distribution $\pi(\Phi)$ in order to obtain the special values of the L-functions as certain p-adic integrals against the measure $\pi(\Phi)$.

3. First step: projectors on finite dimensional subspaces. — The first idea would be to use *the trace operator*

$$Tr_N^{Np^m} f = \sum_{\gamma\in\Gamma_0(Np^m)\backslash\Gamma_0(N)} f\,|_k\, \gamma.$$

One obtains after a normalisation a projector $\pi(f) = [\Gamma_0(N) : \Gamma_0(Np^m)]^{-1}Tr_N^{Np^m} f$ which is well defined but which introduces inacceptable denominators.

The second idea is to use the operator $U = U_p$ of Atkin-Lehner which acts on $\mathcal{M}$ and on $\mathcal{S}$ by $g \mid U = \sum_{n\geq 0} a(pn,g)q^n$, where $g = \sum_{n\geq 0} a(n,g)q^n \in \mathcal{M} \subset A[[q]]$, $a(n,g) \in A$.

Let $\alpha \in \mathbb{C}_p$ be a non-zero eigenvalue of $U = U_p$ on $\mathcal{M}$, associated to a primitive cusp eigenform $f = \sum_{n\geq 1} a(n,f)e(nz) \in \mathcal{S}_k(\Gamma_0(N),\psi) \subset \mathcal{M}$.

In the ordinary case $|\alpha|_p = 1$ there exists a p-adic construction of a projector π given by Hida's idempotent $e = \lim_{r\to\infty} U(p)^{r!}$ acting on p-adic modular forms, whoses image is in the finite dimensional subspace $\mathcal{M}^{\mathrm{ord}} \subset \mathcal{M}$ ("the ordinary part of $\mathcal{M}$"). Then one gets $\mu_{\alpha,\Phi,f} = \ell_f(e\Phi)$ with a suitable p-adic linear form $\ell_f \in \mathcal{M}^{\mathrm{ord}\,*}$.

4. A new construction. — It provides a rather simple method which attaches to a distribution Φ on Y with values in a suitable vector space

$$\mathcal{M} = \bigcup_{m\geq 0} \mathcal{M}(Np^m)$$

of modular forms, a family $\mu_{\alpha,\Phi,f}$ of p-adic measures on Y parametrized by non-zero eigenvalues α associated with primitive cusp eigenforms f. This construction does not use any p-adic limit procedure and in fact it uses only standard linear algebra considerations in the finite dimensional primary (characteristic) subspace of the eigenvalue α.

DEFINITION 4.1.

a) For an $\alpha \in A$ put $\mathcal{M}^{(\alpha)} = \mathrm{Ker}(U-\alpha I)$ the A-submodule of $\mathcal{M}$ of eigenfunctions of the A-linear operator U (of the eigenvalue α).

b) Put $\mathcal{M}^{\alpha} = \cup_{n\geq 1} \mathrm{Ker}(U - \alpha I)^n$ the α-primary (characteristic) A-submodule of $\mathcal{M}$.

c) Put $\mathcal{M}^{\alpha}(Np^m) = \mathcal{M}^{\alpha} \cap \mathcal{M}(Np^m)$, $\mathcal{M}^{(\alpha)}(Np^m) = \mathcal{M}^{(\alpha)} \cap \mathcal{M}(Np^m)$.

PROPOSITION 4.2. — *Let $A = \overline{\mathbb{Q}}_p$. Define $N_0 = Np$, then $U^m(\mathcal{M}(N_0p^m)) \subset \mathcal{M}(N_0)$.*

Proof follows from a known formula [Se73],

$$U^m = p^{m(k/2-1)} W_{N_0p^m} \mathrm{Tr}_{N_0}^{N_0p^m} W_{N_0},$$

where $g|_k W_N(z) = (\sqrt{N}z)^{-k}g(-1/Nz) : \mathcal{M}(N) \to \mathcal{M}(N)$ the main involution of level N (over the complex numbers).

PROPOSITION 4.3. — *Let $A = \overline{\mathbb{Q}}_p$ and let α be a non-zero element of A; hence*

a) $(U^{\alpha})^m : \mathcal{M}^{\alpha}(N_0p^m) \xrightarrow{\sim} \mathcal{M}^{\alpha}(N_0p^m)$ is an invertible $\overline{\mathbb{Q}}_p$-linear operator.

b) The $\overline{\mathbb{Q}}_p$-vector subspace $\mathcal{M}^{\alpha}(N_0p^m) = \mathcal{M}^{\alpha}(N_0)$ is independent of m.

c) Let $\pi_{\alpha,m} : \mathcal{M}(N_0p^m) \to \mathcal{M}^\alpha(N_0p^m)$ the canonical projector onto the α-primary subspace of U (of the kernel $\operatorname{Ker}\pi_{\alpha,m} = \bigcap_{n\geq 1}\operatorname{Im}(U-\alpha I)^n = \oplus_{\beta\neq\alpha}\mathcal{M}^\beta(N_0p^m)$), then the following diagram is commutative

$$\begin{array}{ccccc} & \mathcal{M}(N_0p^m) & \xrightarrow[\pi_{\alpha,m}]{} & \mathcal{M}^\alpha(N_0p^m) & \\ U^m & \downarrow & & \downarrow\wr & (U^\alpha)^m \\ & \mathcal{M}(N_0) & \xrightarrow[\pi_{\alpha,0}]{} & \mathcal{M}^\alpha(N_0) & \end{array}$$

Proof. Due to the reduction theory of endomorphisms in a finite dimensional subspace over a field K, the projector $\pi_{\alpha,m}$ onto the α-primary subspace $\bigcup_{n\geq 1}\operatorname{Ker}(U-\alpha I)^n$ has the kernel $\bigcap_{n\geq 1}\operatorname{Im}(U-\alpha I)^n$ and it can be expressed as a polynomial of U with coefficients in K, hence $\pi_{\alpha,m}$ commutes with U. On the other hand, the restriction of $\pi_{\alpha,m}$ on $\mathcal{M}(N_0)$ coincides with $\pi_{\alpha,0} : \mathcal{M}(N_0) \to \mathcal{M}^\alpha(N_0)$ because its image is

$$\bigcup_{n\geq 1}\operatorname{Ker}(U-\alpha I)^n \cap \mathcal{M}(N_0) = \bigcup_{n\geq 1}\operatorname{Ker}(U\mid_{\mathcal{M}(N_0)} -\alpha I)^n,$$

and the kernel is

$$\bigcap_{n\geq 1}\operatorname{Im}(U-\alpha I)^n \cap \mathcal{M}(N_0) = \bigcap_{n\geq 1}\operatorname{Im}(U\mid_{\mathcal{M}(N_0)} -\alpha I)^n.$$

5. Distributions with values in p-adic modular forms. — Let

$$g = \sum_{n\geq 0} a(n,g)g^n \in \mathcal{M}_k(\Gamma_1(N),A)$$

then $|g|_p = \sup_n |a(n,g)|_p$ is a well-defined p-adic norm on

$$\mathcal{M} = \cup_{m\geq 0}\mathcal{M}_k(\Gamma_1(Np^m),A) \subset A[[q]].$$

Let us denote by $\overline{\mathcal{M}}$ the completion of $\mathcal{M}$ in $A[[q]]$ with respect to this norm. Let V be a normed A-module.

Definition 5.1. — *Let $\alpha \neq 0$ be a non-zero eigenvalue of the operator U on the A-module $\mathcal{M}$. The α-primary part Φ^α of a distribution on Y with values in $\mathcal{M}$ is given by $\int_Y \varphi\Phi^\alpha := (U^\alpha)^{-m}\pi_{\alpha,0}\Big(\big(\int_Y \varphi\, d\Phi\big) \mid U^m\Big) \in \mathcal{M}^\alpha$ for all $\varphi \in \mathcal{C}^1(Y,A)$ and for all p^m sufficiently large so that $\int_Y \varphi\, d\Phi$ is a finite linear combination in $\mathcal{M}(N_0p^m)$).*

Put $\Phi(a+(Np^m)) = \int_Y \chi_{a+(Np^m)}d\Phi$ where $\chi_{a+(Np^m)}$ denotes the characteristic function of an open subset $a+(Np^m) \subset Y$; hence there exists $m' \in \mathbf{N}$ such that

$$\Phi(a+(Np^m)) = \Phi(\chi_{(a+(Np^m))}) \in \mathcal{M}(Np^{m'+1}),$$

and the α-primary part Φ^α of Φ is defined by

$$\Phi^\alpha(a+(Np^m)) = (U^\alpha)^{-m'}\left[\pi_{\alpha,0}(\Phi(a+(Np^m)) \mid U^{m'}\right]. \tag{5.1}$$

6. Main theorems. — Let Φ be a bounded distribution with values in $\mathcal{M}$ and α an eigenvalue of U on $\mathcal{M}$.

THEOREM 3. — *If* $|\alpha|_p = 1$ *then* Φ^α *is a bounded distribution on* Y *with values in* $\mathcal{M}^\alpha$ *(an* A*-module of finite rank).*

THEOREM 4. — *Suppose that for all* $m \in \mathbf{N}^*$ *and for* $t = 0, 1, \ldots, h\varkappa - 1$

$$\int_{a+(N_0p^m)} y_p^t \, d\Phi \in \mathcal{M}(N_0p^{\varkappa m}) \ \ (\text{with } h = [\mathrm{ord}_p \, \alpha] + 1) \tag{6.1}$$

for a suitable non-negative integer $\varkappa$ *(the condition of the modularity of a suitable level). Then there exists an* $h\varkappa$*-admissible distribution* $\tilde{\Phi}^\alpha$ *on* Y *with values in* $\mathcal{M}^\alpha$ *such that for all* m' *sufficiently large (with* $m' \geq \varkappa m$*) and for all* $t = 0, 1, \ldots, h\varkappa - 1$ *one has*

$$\int_{a+(N_0p^m)} y_p^t d\tilde{\Phi}^\alpha = (U^\alpha)^{-m'} \pi_{\alpha,0} \left(\left(\int_{a+(N_0p^m)} y_p^t d\Phi \right) \mid U^{m'} \right).$$

Remark. — If $A = \overline{\mathbb{Q}}_p$ then the condition of the theorem 4 is equivalent to $\int_Y \chi y_p^t d\Phi \in \mathcal{M}(N_0 p^{m\varkappa})$ for all Dirichlet characters $\chi \bmod N_0p^m$ (with values in A) because

$$\int_{a+(N_0p^m)} y^t \, d\Phi = \frac{1}{\varphi(N_0p^m)} \sum_{\chi \bmod N_0p^m} \chi^{-1}(a) \int_Y \chi y_p^t \, d\Phi.$$

Proof of Theorem 3. It suffices to show that for a constant $C > 0$ and for all the open subsets of type $a + (Np^m) \subset Y$ one has $|\Phi^\alpha(a + (Np^m))|_p \leq C$. By our assumption there exists $m' \in \mathbf{N}$ such that

$$\Phi(a + (Np^m)) = \Phi(\chi_{(a+(Np^m))}) \in \mathcal{M}(Np^{m'+1}),$$

then the α-primary part Φ^α of Φ is given by (5.1):

$$\Phi^\alpha(a + (Np^m)) = (U^\alpha)^{-m'} \left[\pi_{\alpha,0}(\Phi(a + (Np^m)) \mid U^{m'} \right].$$

On the α-primary subspace $\mathcal{M}^\alpha \subset \mathcal{M}$ one has $U^\alpha) = \alpha I + Z$ for a nilpotent p-integral operator Z: for all $g = \sum_{n\geq 0} a(n,g)q^n \in \mathcal{M}$, $g \mid U = \sum_{n\geq 0} a(pn,g)q^n$, $|g \mid U|_p \leq |g|_p$ and $|g \mid Z|_p \leq |g|_p$.

Next all the functions

$$\Phi^\alpha(a + (Np^m)) = \alpha^{-m}(\alpha(U^\alpha)^{-1})^m \left[\pi_{\alpha,0}(\Phi(a + (Np^m)) \mid U^m \right]$$

are bounded because $|\alpha^{-1}|_p = 1$ and

$$(\alpha(U^\alpha)^{-1})^m = (\alpha^{-1}U^\alpha)^{-m} = (I + \alpha^{-1}Z)^{-m} = \sum_{j=0}^{n-1} \binom{-m}{j} \alpha^{-j} Z^j. \tag{6.2}$$

Proof of Theorem 4. Let $\mathrm{ord}_p \alpha > 0$ and let $h = [\mathrm{ord}_p \alpha] + 1$. Hence one has to bound

$$\int_{a+(N_0 p^m)} (y_p - a_p)^t \, d\tilde{\Phi}^\alpha = \alpha^{-m\varkappa}(\alpha(U^\alpha)^{-1})^{m\varkappa} \left[\pi_{\alpha,0} \Big(\int_{a+(N_0 p^m)} (y_p - a_p)^t \, d\Phi \Big) \mid U^\alpha \right].$$

The norms of the operators $(\alpha(U^\alpha)^{-1})^m = \sum_{j=0}^{n-1} \binom{-m}{j} \alpha^{-j} Z^j$ are uniformly bounded by $C_1 > 0$ as $n = \dim \mathcal{M}^\alpha$ does not depend on m. Hence for all $t = 0, 1, \ldots, h\varkappa - 1$ one has

$$\begin{aligned} \Big| \int_{a+(N_0 p^m)} (y_p - a_p)^t \, d\tilde{\Phi}^\alpha \Big|_p &\le C_1 \cdot |\alpha|_p^{-m\varkappa} \cdot \mathrm{Max}_{y \in a+(N_0 p^m)} |y_p - a_p|^t \cdot |\Phi|_p \\ &\le C_1 \cdot C_2 |p^m|^{t - \mathrm{ord}_p \alpha \cdot \varkappa} = o(p^{m(h\varkappa - t)}), \quad h > \mathrm{ord}_p \alpha \end{aligned}$$

as $m \to \infty$ (because $|\Phi|_p \le C_2$ and $|\alpha|_p^{-m} = p^{m \, \mathrm{ord}_p \alpha}$).

7. Nearly holomorphic p-adic modular forms of type $r \ge 0$. — Let us specialize us now to the case when A is either an algebraic extension K of $\mathbb{Q}_p$ or the ring $\mathcal{O}_K$ of integers of K. Fix an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \widehat{\overline{\mathbb{Q}}}_p = \mathbb{C}_p$. Let r be a non-negative integer and q, ω two variabls (over the complex numbers $q = e(z) = e^{2\pi i z}$, $\omega = 4\pi y = 4\pi \, \mathrm{Im}(z)$, $z \in \mathbb{C}$). In the ring $A[[q]][\omega^{-1}]$ let us consider the $A[[q]]$-submodules $P_r(A) = \Big\{ g = \sum_{j=0}^r \omega^{-j} g_j$ with $g_j = \sum_{n \ge 0} a(j, n, g) q^n \in A[[q]] \Big\}$. Consider also for any positive integer a the complex vector spaces of nearly holomorphic functions (see [Hi85]) $Q_{r,a} = \Big\{ \sum_{j=0}^r \mathrm{Im}(4\pi z)^{-j} g_j(z)$ with $g_j = \sum_{n \ge 0} a(j, n, g) e(nz/a) \Big\}$ and put $Q_r = \cup_{a \ge 1} Q_{r,a}$.

DEFINITION 7.1.

a) The A-module $\mathcal{M}_{r,k}(\Gamma_1(N), A) \subset A[[q_1]][\omega^{-1}]$ of modular forms of type $r \ge 0$ and of weight $k \ge 1$ for $\Gamma_1(N)$ is generated by the series $g \in \mathcal{M}_{r,k}$ with algebraic coefficients $a(j, n, g) \in i_p(\overline{\mathbb{Q}})$ such that the correspondent complex series (denoted also g)

$$g = i_p^{-1}(g) = \sum_{j=0}^{r} \mathrm{Im}(4\pi z)^{-j} \sum_{n \ge 0} i_p^{-1}(a(j, n, g)) e(nz) \in Q_{r,1}$$

satisfy to two conditions: $\forall \gamma \in \Gamma_1(N)$, $g|_k \gamma = g$ and $\forall \gamma \in \mathrm{SL}_2(\mathbb{Z})$, $g|_k \gamma \in Q_r$.

b) Put

$$\mathcal{M}_{r,k} = \mathcal{M}_{r,k}(N, p) = \cup_{m \ge 0} \mathcal{M}_{r,k}(N p^m),$$

where $\mathcal{M}_{r,k}(N p^m) = \mathcal{M}_{r,k}(\Gamma_1(N p^m), A)$.

8. Example: real analytic Eisenstein series of weight $\ell > 0$ (see [Ka76])**. —** For $s \in \mathbb{C}$ and $a, b \bmod N$ define (by analytic continuation):

$$E_{\ell,N}(z, s; a, b) = \sum (cz + d)^{-\ell} |cz + d|^{-2s} \quad (0 \ne (c, d) \equiv (a, b) \bmod N).$$

Starting from this series one obtains the following Eisenstein distribution: put $s = -r$, $0 \le r \le \ell - 1$,

$$E_{r,\ell,N}(a,b) := \frac{N^{\ell-2r-1}\Gamma(\ell-r)}{(-2\pi i)^{\ell-2r}(-4\pi y)^r} \sum_{a \bmod N} e(-ax/N) E_{\ell,N}(Nz,-r;x,b)$$
$$= \varepsilon_{r,\ell,N}(a,b) + (4\pi y)^{-r} \sum_{\substack{0<dd', d\equiv a \\ d'\equiv b \bmod N}} \operatorname{sgn} d \cdot d^{\ell-2r-1} W(4\pi dd'y, \ell-r,-r) \tag{8.1}$$
$$\cdot\, e(dd'z) \in \mathcal{M}_{r',\ell}(N^2),$$

where $r' = \max(r, \ell - r - 1)$,

$$W(y,\ell-r,-r) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \frac{\Gamma(\ell-r)}{\Gamma(\ell-r-j)} y^{r-j}, \quad \zeta(s;a,N) = \sum_{\substack{n\ge 1 \\ n\equiv a (\bmod N)}} n^{-s},$$

$$\varepsilon_{r,\ell,N}(a,b) = \frac{(-4\pi y)^s \Gamma(\ell+s)}{\Gamma(\ell+2s)} \delta\left(\frac{b}{N}\right) \zeta(1-\ell-2s;a,N)\Big|_{s=-r}$$
$$+ \frac{\Gamma(\ell+2s-1)}{(4\pi y)^{\ell+s-1}\Gamma(s)} \delta\left(\frac{a}{N}\right) \Big[\zeta(\ell+2s-1;b,N)$$
$$+ (-1)^{\ell+2s} \zeta(\ell+2s-1;-b,N)\Big]\Big|_{s=-r}$$

9. Distributions with values in nearly holomorphic p-adic modular forms of type $r \ge 0$. — Let $g = \sum_{j=0}^{r} \omega^{-j} \sum_{n\ge 0} a(j,n,g) q^n \in \mathcal{M}_{r,k} \subset A[[q]][\omega^{-1}]$. Put $|g|_p = \sup_{n,j} |a(j,n,g)|_p$. One has a p-adic norm on $\mathcal{M}_{r,k}$.

Let α be an eigenvalue of U on $\mathcal{M}_{r,k}$,

$$g|U^m = p^{m(k/2-1)} \sum_{u \bmod p^m} g\Big| \begin{pmatrix} 1 & u \\ 0 & p^m \end{pmatrix}$$
$$= p^{-m} \sum_{u \bmod p^m} g\left(\frac{z+u}{p^m}\right),$$
$$\operatorname{Im}\left(\frac{z+u}{p^m}\right) = \frac{\operatorname{Im} z}{p^m}$$
$$\Longrightarrow g|U^m = \sum_{j=0}^{k} \omega^{-j} p^{mj} \sum_{n\ge 0} a(j, p^m n, g) q^n$$

(un opérateur entier sur $\overline{\mathcal{M}}_{r,k}$).

Definition 9.1.

a) $\mathcal{M}_{r,k}^{(\alpha)} = \operatorname{Ker}(U - \alpha I)$,

b) $\mathcal{M}_{r,k}^{\alpha} = \bigcup_{n\ge 1} \operatorname{Ker}(U - \alpha I)^n$, *c)* $\mathcal{M}^{\alpha}_{r,k}(Np^m) = \mathcal{M}_{r,k}(Np^m) \cap \mathcal{M}^{\alpha}_{r,k}$.

PROPOSITION 9.2.

a) $\mathcal{M}^{\alpha}_{r,k}(N_0p^m) = \mathcal{M}^{\alpha}_{r,k}(N_0)$ *is of finite rank over* A.

b) The following diagram is commutative (for $A = \overline{\mathbb{Q}}_p$*)*

$$\begin{array}{ccccc} & \mathcal{M}_{r,k}(N_0p^m) & \underset{\pi_{\alpha,m}}{\longrightarrow} & \mathcal{M}^{\alpha}_{r,k}(N_0p^m) & \\ U^m & \downarrow & & \downarrow\wr & (U^{\alpha})^m \\ & \mathcal{M}_{r,k}(N_0) & \underset{\pi_{\alpha,0}}{\longrightarrow} & \mathcal{M}^{\alpha}_{r,k}(N_0) & \end{array}$$

($\pi_{\alpha,m}$ *is the projector onto the* α*-primary subspace with the kernel* $\bigcap_{n\geq 1}\mathrm{Im}(U-\alpha I)^n \cap \mathcal{M}(N_0)$ *(equal to the direct sum of all the other primary subspaces;* $\pi_{\alpha,m} = R_{\alpha,m}(U)$ *with a suitable polynomial* $R_{\alpha,m} \in A[x]$ *see Proposition 4.3).*

DEFINITION 9.3. — *Let* $\alpha \in A$ *be a non-zero eigenvalue of the operator* U *on* $\mathcal{M}_{r,k}$ *and* $\Phi : \mathcal{C}^1(Y,A) \to \mathcal{M}_{r,k}$ *a distribution. The* α*-primary part* $\Phi^{\alpha} : \mathcal{C}^1(Y,A) \to \mathcal{M}^{\alpha}_{r,k}$ *of* Φ *is given by*

$$\Phi^{\alpha}(\varphi) = U^{-m}\big[\pi_{\alpha,0}U^m(\Phi(\varphi)\big] \in \mathcal{M}^{\alpha}_{r,k}$$

for all m *sufficiently large (avec* $\Phi(\varphi) \in \mathcal{M}_{r,k}(N_0p^m)$*).*

The definition is independent of the choice of m (assumed sufficiently large) by Prop. 9.2, b).

THEOREM 5. — *Let* $|\alpha|_p = 1$ *and* Φ *a bounded distribution. Then the distribution* Φ^{α} *is also bounded.*

Proof. It is identical to that of Theorem 3.

10. ***h*-admissible distributions..** — Let $\Phi_j : \mathcal{C}^1(Y,A) \to \mathcal{M}_{r,k}$ be a family of distributions (non necessarily bounded, $j = 0,1,\ldots,r^*$, $r^* \geq 1$). For any open subset $a + (Np^m) \subset Y$ put $\Phi_j(a+(Np^m)) = \Phi_j(\chi_{a+(Np^m)})$.

THEOREM 6 . — *Let* $0 < |\alpha|_p < 1$ *and* $h = [\mathrm{ord}_p\,\alpha] + 1$*. Suppose that there exists* $\varkappa \in \mathbf{N}^*$ *such that for all* $j = 0,1,\ldots,h-1$ *and for all* $m \geq 1$ *one has* $\Phi_j(a+(N_0p^m)) \in \mathcal{M}_{r,k}(N_0p^{m\varkappa})$*. Suppose next that for* $t = 0,1,\ldots,h-1$ *and for all* $a+(Np^m) \subset Y$ *one has*

$$|U^{\varkappa m}\sum_{j=0}^{t}\binom{r}{j}(-a_p)^{t-j}\Phi_j(a+(Np^m))|_p \leq C|p^m|_p^t \qquad (10.1)$$

with a suitable constant $C > 0$*. Then there exists an* $h\varkappa$*-admissible measure* $\tilde{\Phi}^{\alpha} : \mathcal{C}^{h\varkappa}(Y,A) \to \mathcal{M}^{\alpha}_{r,k} = \mathcal{M}_{r,k}$ *such that* $\int_{a+(N_0p^m)} y_p^j d\tilde{\Phi}^{\alpha} = \Phi^{\alpha}_j(a+(N_0p^m))$ *(for* $j = 0,1,\ldots,h\varkappa-1$*).*

Proof. It suffices to verify the condition of growth (1.1 d)) for $\Phi_j^\alpha(a+(N_0p^m)) \in \mathcal{M}_{r,k}(N_0p^{m\varkappa})$. One has $U = \alpha I + Z$, $Z^n = 0$ on $\mathcal{M}_{r,k}^\alpha(N_0)$, $n = rk_A$, $\mathcal{M}_{r,k}^\alpha(N_0p^m; A) = \mathcal{M}_{r,k}^\alpha(N_0p^m; A)$.

On the other hand by the conditions of the theorem we have

$$\int_{a+(N_0p^m)} (y_p - a_p)^t d\tilde{\Phi}^\alpha = \sum_{j=0}^t \binom{t}{j} (-a_p)^{t-j} \Phi_j^\alpha(a+(N_0p^m))$$

$$= \alpha^{-m\varkappa}\alpha^{m\varkappa}(U^\alpha)^{-m\varkappa}\left[\pi_{\alpha,0}U^{m\varkappa}\left(\sum_{j=0}^t \binom{t}{j}(-a_p)^{t-j}\Phi_j(a+(N_0p^m))\right)\right].$$

The operators $(\alpha^{-1}(U^\alpha)^{-m\varkappa} = \sum_{i=0}^{n-1}\binom{-m\varkappa}{i}(\alpha^{-1}Z)^i$ are uniformly bounded by a constant $C_1 > 0$ hence the condition (10.1) implies

$$\left|\int_{a+(N_0p^m)} (y_p - a_p)^t d\tilde{\Phi}^\alpha\right|_p \leq C \cdot C_1 |\alpha|_p^{-m\varkappa} |p^m|_p^t = o(p^{m(h\varkappa - t)})$$

when $m \to \infty$ because $|\alpha|_p = |p|^{\mathrm{ord}_p \alpha}$, $\mathrm{ord}_p \alpha < h$, $|p^m|_p^{-\varkappa\, \mathrm{ord}_p \alpha} = o(p^{m\varkappa h})$.

11. The second step: application of a suitable linear form. — Let $\alpha \in \overline{\mathbb{Q}} \subset \mathbb{C}$ be a non-zero eigenvalue of U on $\mathcal{M}_{r,k}(\mathbb{C})$ associated with a primitive cusp eigenform $f \in \mathcal{S}_k(\Gamma_0(N), \psi)$ and let $f_0 = f_{0,\alpha}$ be a corresponding eigenfunction $(f_0|U = \alpha f_0)$, let us define $f^0 = f_0^\rho | W_{N_0}$, $f_0^\rho = \sum_{n\geq 1} \overline{a_n(f_0)} q^n$, $W_{N_0} = \begin{pmatrix} 0 & -1 \\ N_0 & 0 \end{pmatrix}$.

Proposition 11.1.

a) $U^* = W_{N_0}^{-1} U W_{N_0}$ *in the hermitian vector space* $\mathcal{S}_{r,k}(\Gamma_1(N_0), \mathbb{C})$, *the adjoint operator with respect to the Petersson scalar product.*

b) One has $f^0 | U^* = \bar{\alpha} f^0$, *and for all "good" prime numbers* $l \nmid Np$ *one has* $T_l f^0 = a_l(f) f^0$.

c) The linear form $\Phi \mapsto \langle f^0, \Phi\rangle$ *on* $\mathcal{M}_{r,k}(\Gamma_1(N_0), \mathbb{C})$ *vanishes on* Ker $\pi_{\alpha,0}$, *where* $\pi_{\alpha,0} : \mathcal{M}_{r,k}(\Gamma_1(N_0), \mathbb{C}) \to \mathcal{M}_{r,k}^\alpha(\Gamma_1(N_0), \mathbb{C})$ *(the projector onto the* α*-primary subspace with the kernel* Ker $\pi_{\alpha,0} = \mathrm{Im}(U - \alpha I)^n$*) hence*

$$\langle f^0, \Phi\rangle = \langle f^0, \pi_{\alpha,0}(\Phi)\rangle.$$

d) If $\Phi \in \mathcal{M}(Np^{m+1}, \overline{\mathbb{Q}}) = \mathcal{M}(N_0p^m, \overline{\mathbb{Q}})$ *et* $\alpha \neq 0$, *one has*

$$\langle f^0, \Phi^\alpha\rangle = \alpha^{-m}\langle f^0, \Phi \mid U^m\rangle$$

where

$$\Phi^\alpha = (U^\alpha)^{-m}\pi_{\alpha,0}(\Phi \mid U^m) \in \mathcal{M}^\alpha(Np)$$

is the α*-primary part of* Φ.

e) One puts

$$\mathcal{L}_{f,\alpha}(\Phi) = \frac{\langle f^0, \alpha^{-m}\Phi | U^m\rangle_{N_0}}{\langle f^0, f_0\rangle_{N_0}},$$

hence $\mathcal{L}_{f,\alpha} : \mathcal{M}(Np^{m+1}; \overline{\mathbb{Q}}) \to \overline{\mathbb{Q}}$ *(the linear form* $\mathcal{L}_{f,\alpha}$ *sur* $\mathcal{M}_{r,k}(\Gamma_1(Np), \mathbb{C})$ *is defined over* $\overline{\mathbb{Q}}$*) and there exists a unique A-linear form* $\ell_{f,\alpha} \in \mathcal{M}^{\alpha}_{r,k}(N_0)^*$ *such that*

$$i_p^{-1}(\ell_{\alpha,f}(\Phi^{\alpha})) = \frac{\langle f^0, \alpha^{-m} U^m(\Phi)\rangle_{N_0}}{\langle f^0, f_0\rangle_{N_0}}$$

($\forall\Phi$ *with coefficients in* $i_p^{-1}(A)$*).*

Proof of Proposition 11.1 a) See [Miy], Th. 4.5.5.

b) By definition, $f^0|U^* = f_0^{\rho} \mid W_{Np}W_{Np}^{-1}UW_{Np} = \bar{\alpha} f_0^{\rho} \mid W_{Np} = \bar{\alpha} f^0$.

c) For any function

$$\Psi = (U - \alpha I)^n g \in \operatorname{Ker} \pi_{\alpha,0} = \operatorname{Im}(U - \alpha I)^n$$

one has

$$\langle f^0, \Psi\rangle = \langle f^0, (U - \alpha I)^n g\rangle = \langle (U^* - \bar{\alpha} I) f^0, (U - \alpha I)^{n-1} g\rangle = 0$$

hence for $\Psi = \Phi - \pi_{\alpha,0}(\Phi)$ we get

$$\langle f^0, \Phi\rangle = \langle f^0, \pi_{\alpha,0}(\Phi) + (\Phi - \Phi^{\alpha})\rangle = \langle f^0, \pi_{\alpha,0}(\Phi)\rangle + \langle f^0, \Psi\rangle = \langle f^0, \pi_{\alpha,0}(\Phi)\rangle$$

d) Let us use directly the equality $(U^*)^m f^0 = \bar{\alpha}^m f^0$ of b):

$$\alpha^m \cdot \langle f^0, \Phi^{\alpha}\rangle = \langle (U^*)^m f^0, U^{-m}\pi_{\alpha,0}(\Phi \mid U^m)\rangle = \langle f^0, \pi_{\alpha,0}(\Phi \mid U^m)\rangle = \langle f^0, \Phi \mid U^m\rangle$$

by c) because $\Phi \mid U^m \in \mathcal{M}(Np)$.

e) Note that $\mathcal{L}_{f,\alpha}(f_0) = 1$, $f_0 \in \mathcal{M}(Np; \overline{\mathbb{Q}})$; consider the complex vector space

$$\operatorname{Ker} \mathcal{L}_{f,\alpha} = \langle f^0\rangle^{\perp} = \{g \in \mathcal{M}(Np; \mathbb{C}) \mid \langle f^0, g\rangle = 0\}$$

which admits a $\overline{\mathbb{Q}}$-rational basis because it is stable by the action of all "good" Hecke operators T_l ($l \nmid Np$):

$$\langle f^0, g\rangle = 0 \Longrightarrow \langle f^0, T_l g\rangle = \langle T_l^* f^0, g\rangle = 0$$

and one obtains such a basis by the diagonalisation of the action of all the T_l (a commutative family of normal operators) and e) follows.

12. Relations to the L-functions: convolutions of the Eisenstein distributions. Let ξ mod N be an auxiliary Dirichlet character $\xi : Y \to A^*$, $Y \xrightarrow{y_p} \mathbb{Z}_p^*$, $Y = \varprojlim Y_{Np^m}$, $Y_{Np^m} = (\mathbb{Z}/Np^m\mathbb{Z})^*$. Consider two Eisenstein distributions

$$E_{0,\ell,Np^m}(\xi, b) = \sum_{a \in Y_{Np^m}} \xi(a) E_{0,\ell}(a, b; Np^m) \in \begin{cases} \mathcal{M}_{0,\ell} & \text{si } \xi \neq 1 \\ \mathcal{M}_{1,\ell} & \text{si } \xi = 1, \ell = 1, 2 \end{cases}$$

$$E_{r,\ell,Np^m}(a) = \sum_{b \in Y_{Np^m}} E_{r,\ell,Np^m}(a, b) \in \mathcal{M}_{r',\ell},\ r' = \max(r, \ell - r - 1)\,.$$

PROPOSITION 12.1. — *Let $\chi, \psi : Y \to A^\times$ be two Dirichlet characters* mod Np^m,

a) Let $f \in \mathcal{S}_k(N, \psi)$, $k \geq 2$, $\Phi_j(y)_{Np^m} = \sum_{y \in Y_{Np^m}} \psi\overline{\xi}(a) E_{0,k-1-j}(\xi, ya) E_{j,1-j}(a)$ a twisted convolution, $j = 0, \ldots, k-2$. Hence

$$\Phi_j(\chi) = E_{0,k-1-j}(\xi, \chi) E_{j,1-j}(\psi\overline{\xi\chi}).$$

b) The special values of the function $L_f(s, \chi) = \sum_{n \geq 1} \chi(n) a_n(f) n^{-s}$ satisfy the following

$$\langle f^0, \Phi_j(\chi) \rangle_{N_0} = \frac{L_f(k-1, \overline{\xi}) L_f(1+j, \overline{\chi})}{(-2\pi i)^{j+k}} \cdot t,$$

(where $t \in \overline{\mathbb{Q}}^*$ is an explicit elementary factor) which produces an h-admissible measure (compare with [Vi76]).

13. Application to triple products. — Consider the vector space

$$\mathcal{M} := \bigcup_{m \geq 0} \mathcal{M}_k(\Gamma_1(Np^m)^{\otimes 3}$$

and let $L(f \otimes g \otimes h, s)$ be the triple L-function attached to $f \otimes g \otimes h \in \mathcal{S}_k(\Gamma_1(N)^{\otimes 3}$ associated with an *ordinary* eigenvalue $\alpha\beta\gamma$, hence

$$f_0 \otimes g_0 \otimes h_0 \in \mathcal{S}_k(\Gamma_1(Np)^{\otimes 3}$$

is an eigenfunction of U on $\mathcal{M}$. Let use the restriction on the diagonal $\Phi = E_k^3(z_1, z_2, z_3) \in \mathcal{M}$ of the Siegel-Eisenstein distribution (see [PIsr]) viewed as a formal Fourier series. One obtains a distribution on Y^3 with values in $\mathcal{M}$.

Put

$$l_{f \otimes g \otimes h, \alpha\beta\gamma}(\Phi) := i_p \left(\frac{\langle f_0 \otimes g_0 \otimes h_0, \Phi^{\alpha\beta\gamma} \rangle}{\langle f_0, f_0 \rangle \langle g_0, g_0 \rangle \langle h_0, h_0 \rangle} \right).$$

THEOREM 7 (a work in progress with Siegfried Böcherer). — *The distribution $l_{f \otimes g \otimes h, \alpha\beta\gamma}(\Phi)$ on Y^3 with values in $\mathcal{M}$ is bounded and the integrals $l_{f \otimes g \otimes \otimes h, \alpha\beta\gamma}(\Phi)(\chi_1 \otimes \chi_2 \otimes \chi_3)$ on the characters $\chi_1 \otimes \chi_2 \otimes \chi_3$ coincide with the special values $L^*(f_{\chi_1} \otimes g_{\chi_2} \otimes h_{\chi_3}, s_0)$, where the normalisation of L^* involves Gauss sums, Petersson scalar products , powers of π, $\alpha\beta\gamma$.*

Proof. The existence of $l_{f \otimes g \otimes h, \alpha\beta\gamma}(\Phi)$ follows from the existence of Φ using Theorem 3, and the equality is implied by the integral formula of Garrett-Harris [GaHa], see also [LBP], [PTr].

References

[AV] Amice, Y. ,Vélu, J. . — *Distributions p-adiques associées aux séries de Hecke*, Astérisque **24–25** (1975), 119–131.

[Co] Coates, J.. — *On p-adic L-functions*, Séminaire Bourbaki, 40eme annee, 1987–88, n° 701, Astérisque **Novembre, !988** (1989), 177-178.

[Co-PeRi] Coates, J; Perrin-Riou, B.. — *On p-adic L-functions attached to motives over* $\mathbf{Q}$, Advanced Studies in Pure Math. **17** (1989), 23-54.

[Colm98] Colmez, P.. — *Fonctions L p-adiques*, Séminaire Bourbaki, 51 ème année, 1998–99, n° 851, Novembre 1998, 1998.

[Hi85] Hida, Haruzo. — *A p-adic measure attached to the zeta functions associated with two elliptic cusp forms, I*, Invent. Math. **79** (1985), 545-613.

[Hi93] Hida, Haruzo. — *Elementary theory of L-functions and Eisenstein series*, Cambridge Univ. Press, 386 p, 1993.

[GaHa] Garrett, P.B.; Harris, M.. — *Special values of triple product L-functions*, Am. J. Math. **115, No.1** (1993), 161-240.

[Ka76] Katz, N.M.. — *p-adic interpolation of real analytic Eisenstein series*, Ann. of Math. **104** (1976), 459-571.

[LBP] Le Bras, Yann-Henri; Panchishkin A.A.. — *Sur les produits triples Λ-adiques*, Communications in Algebra **29, No. 9** (2001), 3727-3740.

[Man73] Manin, Yu.I.. — *Periods of cusp forms and p-adic Hecke series*, Mat. Sbornik **92** (1973), 378-401.

[Man76] Manin, Yu.I.. — *Non-Archimedean integration and Jacquet-Langlands p-adic L-functions*, Russ. Math. Surveys **31, N 1** (1976), 5-57.

[Man-Pa] Manin, Yu.I., Panchishkin, A.A.. — *Convolutions of Hecke series and their values at integral points*, Mat. Sbornik **104** (1977), 617–651 (in Russian).

[Maz-SD] Mazur, B., Swinnerton–Dyer, H.P.F.. — *Arithmetic of Weil curves*, Invent. Math. **25** (1974), 1–61.

[MTT] Mazur, B.; Tate, J.; Teitelbaum, J.. — *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. **84** (1986), 1-48 .

[Miy] Miyake, Toshitsune. — *Modular forms. Transl. from the Japanese by Yoshitaka Maeda*, Berlin etc.: Springer-Verlag. viii, 335 p, 1989.

[PLNM] Panchishkin, A.A.. — *Non–Archimedean L-functions of Siegel and Hilbert modular forms*, Lecture Notes in Math., **1471**, Springer–Verlag, 166 p, 1991.

[PTr] Panchishkin, A.A. . — *Produits triples des formes modulaires et leur interpolation p–adique par la méthode d'Amice–Vélu*, Manuscript de l'exposé au Colloque à la mémoire d'Yvette Amice **mars 1994** (1994), 1-27.

[PIsr] Panchishkin, A.A.. — *On the Siegel–Eisenstein measure*, Israel Journal of Mathematics **Vol. 120, Part B** (2000), 467-509.

[PIAS] Panchishkin, A.A.. — *On p-adic integration in spaces of modular forms and its applications*, Preprint of IAS **January 2000** (2000), 1-31.

[Ra52] Rankin, R.A.. — *The scalar product of modular forms*, Proc. London math. Soc **2 (3)** (1952), 198-217.

[Se73] Serre, J.-P. . — *Formes modulaires et fonctions zêta p-adiques*, Lecture Notes in Math **350** (1973), 191-286.

[Vi76] Višik, M. M.. — *Non-archimedean measures connected with Dirichlet series*, Math. USSR Sb. **28** (1976), 216-228.

Université de Grenoble I
Institut Fourier
UMR 5582 CNRS-UJF
UFR de Mathématiques
B.P. 74
38402 ST MARTIN D'HÈRES Cedex (France)