Algebras with finitely generated invariant subalgebras

Ivan Arzhantsev *

Prépublication de l'Institut Fourier n^0 551 (2001)
http://www-fourier.ujf-grenoble.fr/prepublications.html

Abstract

We classify all finitely generated integral algebras with a rational action of a reductive group such that any invariant subalgebra is finitely generated. Some results on affine embeddings of homogeneous spaces are also given.

1. Introduction. Let $\mathcal{A}$ be an associative commutative finitely generated integral algebra with unit over an algebraically closed field K and let G be a connected reductive algebraic group over K acting rationally on $\mathcal{A}$. The latter condition means that there is a homomorphism $G \to Aut(\mathcal{A})$ such that the orbit Ga of any element $a \in \mathcal{A}$ is contained in a finite-dimensional subspace in $\mathcal{A}$ where G acts rationally. We say in this case that $\mathcal{A}$ is a *G-algebra*. Let us introduce three special types of G-algebras.

Type C. Here $\mathcal{A}$ is a finitely generated domain of Krull dimension $\operatorname{Kdim} \mathcal{A} = 1$ (i.e. the transcendence degree of the quotient field $Q\mathcal{A}$ equals

*The work was supported by CRDF grant RM1-2088, by RFBR grant 01-01-00756, and by CNRS.

[0]**Keywords:** algebraic groups, rational G-algebras, quasi-affine homogeneous spaces, affine embeddings.

AMS Classification: 13A50, 13E15, 14L30, 14M17, 14R20.

one) with any (for example, trivial) G-action. Such algebras may be considered as the algebras of regular functions on irreducible affine curves.

Type HV. Let H be a closed subgroup of G and

$$\mathcal{A}(H) = K[G]^H = K[G/H] = \{f \in K[G] \mid f(gh) = f(g) \text{ for any } g \in G,\ h \in H\}.$$

The left G-action $(l(g')f)(g) := f(g'^{-1}g)$ is rational.

Further we follow notation of the book [Gr]. Let $B = TU$ be a Borel subgroup of G with the unipotent radical U and a maximal torus T. Here T normalizes U and there is a G-equivariant T-action on $K[G]^U$ defined by right translation $(r(t)f)(g) := f(gt)$. For a character $\omega \in X(T)$ consider the G-invariant subspace

$$E(\omega^*) = \{f \in K[G]^U \mid r(t)f = \omega(t)f \text{ for all } t \in T\ \}.$$

The G-module $E(\omega^*)$ is $\{0\}$ unless ω is dominant. Denote by $X^+(T)$ the set of dominant weights. For every $\omega \in X^+(T)$ $E(\omega^*)$ contains the simple submodule of G having highest weight denoted by ω^* and in zero characteristic $E(\omega^*)$ is simple. The map $\omega \to \omega^*$ is an involution on $X^+(T)$. Since each element in $K[G]^U$ is a direct sum of T-weight vectors (where T acts by right translation), we see that $K[G]^U$ is the direct sum of the $E(\omega)$, $\omega \in X^+(T)$. From the definition, it is obvious that if $\omega,\ \omega' \in X^+(T)$, then $E(\omega)E(\omega') \subseteq E(\omega + \omega')$.

Consider the G-algebra

$$\mathcal{A}(\lambda) = \oplus_{m \geq 0} E(m\lambda) \subset K[G]^U,$$

where λ is a dominant weight. (More geometrically, the algebra $\mathcal{A}(\lambda)$ may be realized as

$$\mathcal{A}(\lambda) = \oplus_{m \geq 0} H^0(G/B, L_{m\lambda^*}),$$

where $L_{m\lambda^*} = G *_B K(-m\lambda^*)$ is the G-line bundle on the flag variety G/B corresponding to the character $m\lambda^*$.)

We say that a G-algebra $\mathcal{A}$ is an algebra of type HV if it is G-isomorphic to an invariant subalgebra of $\mathcal{A}(\lambda)$ for some $\lambda \in X^+(T)$. Any G-algebra of type HV is finitely generated, see Lemma 2 below.

Suppose for a moment that $\operatorname{char} K = 0$. The algebra $\mathcal{A}(\lambda)$ may be considered as the algebra of regular functions on the closure of a highest weight vector in the simple G-module with highest weight λ^*. Clearly, any invariant subalgebra in $\mathcal{A}(\lambda)$ has the form

$$\mathcal{A}(P, \lambda) = \oplus_{p \in P} E(p\lambda),$$

where P is a subsemigroup in the additive semigroup Z_+ of non-negative integers, cf. [PV1].

Example 1. Let G be $SL_n(K)$ and $\omega_1, \ldots, \omega_{n-1}$ be its fundamental weights. The natural linear action $G : K^n$ induces an action on regular functions

$$G : \mathcal{A} = K[x_1, \ldots, x_n], \quad (g * f)(v) := f(g^{-1}v).$$

The homogeneous polynomials $K[x_1, \ldots, x_n]_m$ of degree m form an (irreducible) isotypic component corresponding to the weight $m\omega_{n-1}$. Hence $\mathcal{A} = \mathcal{A}(\omega_{n-1})$ and any invariant subalgebra in $\mathcal{A}$ is composed of homogeneous components indexed by the elements of a subsemigroup $P \subseteq Z_+$.

In positive characteristic the situation is more complicated.

Example 2. Suppose that $\mathrm{char}\, K = 2$, $G = SL_2(K)$ and G acts on $\mathcal{A} = K[x_1, x_2]$ as in Example 1. Then the invariant subalgebras $K[x_1^2, x_2^2]$, or $K[x_1^2, x_2^2, x_1^3x_2, x_1x_2^3]$, are not of the form $\mathcal{A}(P, \lambda)$.

The author does not know a "constructive" description of G-algebras of type HV in the case $\mathrm{char}\, K > 0$.

Type N. Let H be a reductive subgroup of G. The algebra $\mathcal{A}(H)$ is finitely generated. We say that a G-algebra $\mathcal{A}$ is of type N if there exists a reductive subgroup $H \subset G$ such that the index of H in its normalizer $N_G(H)$ is finite and $\mathcal{A}$ is G-isomorphic to an invariant subalgebra of $\mathcal{A}(H)$. Any G-algebra of type N is finitely generated (Lemma 1).

Example 3. Suppose that $\mathrm{char}\, K \neq 2$. Let $G = SL_n$ and $H = SO_n$. The group G acts on the space of symmetric $n \times n$-matrices by $(g, s) \to g^T sg$. The stabilizer of the identity matrix E is the subgroup H and the orbit GE is the set X of symmetric matrices with determinant 1. This yields that the algebra $\mathcal{A} = K[X]$ with the G-action $(g * f)(s) := f((g^{-1})^T sg^{-1})$ is an algebra of type N.

In zero characteristic a G-algebra $\mathcal{A}$ is a G-algebra of type N if and only if

(*) *$\mathcal{A}$ contains no proper G-invariant ideals and the group of G-equivariant automorphisms of $\mathcal{A}$ is finite*

(see Remark 2 in Section 4). Moreover, any G-algebra of type N is G-isomorphic to $\mathcal{A}(H')$ for a reductive subgroup $H' \subset G$ such that the group $N_G(H')/H'$ is finite. In positive characteristic this is no longer the case.

It is natural to expect that property (*) characterizes G-algebras of type N in arbitrary characteristic. At this moment the author has only a proof that for any G-algebra of type N property (*) holds (Section 4).

Now we are able to formulate the main result.

Theorem 1. *Let $\mathcal{A}$ be an associative commutative finitely generated integral K-algebra with a rational G-action. Then any G-invariant subalgebra of $\mathcal{A}$ is finitely generated if and only if $\mathcal{A}$ is an algebra of one of the types C, HV or N.*

The proof of Theorem 1 is given in Section 4. Now we begin with some auxiliary results.

2. Non-finitely generated subalgebras. Let X be an irreducible affine algebraic variety and Y be a proper closed irreducible subvariety. Consider the subalgebra

$$\mathcal{A}(X,Y) = \{f \in K[X] \mid f(y_1) = f(y_2) \text{ for any } y_1, y_2 \in Y\} \subset \mathcal{A} = K[X].$$

Proposition 1. *The subalgebra $\mathcal{A}(X,Y)$ is finitely generated if and only if Y is a point.*

Proof. If Y is a point, then $\mathcal{A}(X,Y) = K[X]$. Suppose that Y has positive dimension. Consider the ideal $\mathcal{I} = \mathcal{I}(Y) = \{f \in K[X] \mid f(y) = 0 \text{ for any } y \in Y\}$. Then $\mathcal{A}/\mathcal{I}$ is an infinite-dimensional vector space. By the Nakayama lemma, we can find $i \in \mathcal{I}$ such that in the local ring of Y the element i is not in $\mathcal{I}^2$. For any $a \in k|X] \setminus \mathcal{I}$ the element $ia \in \mathcal{I} \setminus \mathcal{I}^2$. Hence the space $\mathcal{I}/\mathcal{I}^2$ has infinite dimension.

On the other hand, suppose that $f_1, \ldots, f_n$ are generators of $\mathcal{A}(X,Y)$. Subtracting constants, one may suppose that all f_i are in $\mathcal{I}$. Then $\dim \mathcal{A}(X,Y)/\mathcal{I}^2 \leq n+1$, a contradiction. $\diamondsuit$

Proposition 2. *Let $\mathcal{A}$ be a finitely generated domain. Then any subalgebra in $\mathcal{A}$ is finitely generated if and only if* $\operatorname{Kdim} \mathcal{A} \leq 1$.

Proof. If $\mathrm{Kdim}\,\mathcal{A} \geq 2$, then the statement follows from the previous proposition. The case $\mathrm{Kdim}\,\mathcal{A} = 0$ is obvious. It remains to prove that if $\mathrm{Kdim}\,\mathcal{A} = 1$, then any subalgebra is finitely generated. By taking the integral closure, one may suppose that $\mathcal{A}$ is the algebra of regular functions on a smooth affine curve C_1. Let C be the smooth projective curve such that $C_1 \cong C \setminus \{P_1, \ldots, P_k\}$. The elements of $\mathcal{A}$ are the rational functions on C that may have poles only at points P_i. Let $\mathcal{B}$ be a subalgebra in $\mathcal{A}$. By induction on k, we may suppose that the subalgebra $\mathcal{B}' \subset \mathcal{B}$ consisting of functions regular at P_1 is finitely generated, say $\mathcal{B}' = K[s_1, \ldots, s_m]$. (Functions that are regular at any point P_i are constants.) Let $v(f)$ be the order of the zero/pole of $f \in \mathcal{B}$ at P_1. The set $V = \{v(f) \mid f \in \mathcal{B}\}$ is an additive subsemigroup of integers. Such a subsemigroup is finitely generated. Let $f_1, \ldots, f_n$ be elements of $\mathcal{B}$ such that $v(f_i)$ generate V. Then for any $f \in \mathcal{B}$ there exists a polynomial $P(y_1, \ldots, y_n)$ with $v(f - P(f_1, \ldots, f_n)) \geq 0$, thus $f - P(f_1, \ldots, f_n) \in \mathcal{B}'$. This shows that $\mathcal{B}$ is generated by $f_1, \ldots, f_n, s_1, \ldots, s_m$. $\diamondsuit$

3. Affine embeddings. To go further we need some definitions.

Definition 1. Let H be a closed subgroup of G. We say that an affine variety X with a regular G-action is *an affine embedding* of the homogeneous space G/H if there exists a point $x \in X$ such that the orbit Gx is dense in X and the orbit map $G \to Gx$ defines an isomorphism between G/H and Gx. We denote this as $G/H \hookrightarrow X$. An embedding is *trivial* if $X = Gx$.

Note that a homogeneous space G/H admits an affine embedding if and only if G/H is quasi-affine (as an algebraic variety), see [PV2, Th.1.6]. In this situation, the subgroup H is said to be *observable* in G. For a group-theoretic description of observable subgroups see [Su] ($\mathrm{char}\,K = 0$) and [Gr, Th.7.4] ($\mathrm{char}\,K$ is arbitrary). It is known that G/H is affine if and only if H is reductive [Rich, Th.A], [Gr, Th.7.2]. In particular, any reductive subgroup is observable.

Definition 2. A homogeneous space is said to be *affinely closed* if it admits only the trivial affine embedding. (In this case G/H is affine.)

The following theorem is a reformulation of a theorem due to D. Luna [Lu2].

Theorem 2. *A homogeneous space G/H is affinely closed if and only if H is reductive and the group $N_G(H)/H$ is finite.*

Remark. Moreover, it is proved in [Lu2] that if G acts on an affine variety X and the stabilizer of a point $x \in X$ contains a reductive subgroup H such that the group $N_G(H)/H$ is finite, then the orbit Gx is closed in X. This implies that if $H \subseteq H' \subseteq G$ and H' is observable, then H' is reductive and G/H' is affinely closed.

Theorem 2 is proved in [Lu2] under the assumption $\operatorname{char} K = 0$. We shall give a characteristic-free proof of Theorem 2 and of the above remark in Section 5 in terms of Kempf's adapted one-parameter subgroups [Kem].

4. Proof of Theorem 1. Let $\mathcal{A}$ be a G-algebra with $\operatorname{Kdim} \mathcal{A} \geq 2$ such that any invariant subalgebra in $\mathcal{A}$ is finitely generated. Consider the corresponding affine variety $X = \operatorname{Spec} \mathcal{A}$. The action $G : \mathcal{A}$ induces a regular algebraic action $G : X$.

Suppose that there exists a proper irreducible closed invariant subvariety $Y \subset X$ of positive dimension. Then $\mathcal{A}(X, Y)$ is an invariant subalgebra that is not finitely generated. In particular, this is the case if G acts on X without a dense orbit. Hence we may suppose that either

(i) the action $G : X$ is transitive or
(ii) X consists of an open orbit $\mathcal{O}$ and a G-fixed point o.

In case (i), fix a point $x \in X$ and denote by H the (reduced) stabilizer of x in G. The orbit map $g \to gx$ defines a bijective purely inseparable and finite morphism $\pi : G/H \to X$ [Hum, 4.3, 4.6]. Note that $K(G/H)^{p^s} \subseteq \pi^* K(X)$ for some $s \geq 0$, and $K[G/H]^{p^s} \subseteq \pi^* K[X]$, where $p = \operatorname{char} K$ if $\operatorname{char} K > 0$ and $p = 1$ otherwise. If G/H is not affinely closed, then there is a nontrivial affine embedding $G/H \hookrightarrow X'$. The algebra $\mathcal{B} := K[X'] \cap \pi^* K(X)$ is finite over $K[X']^{p^s}$. Hence $\mathcal{B}$ is finitely generated, and $\tilde{X} := \operatorname{Spec} \mathcal{B}$ contains X as an open subset.

$$\begin{array}{ccc} G/H & \hookrightarrow & X' \\ \downarrow \pi & & \downarrow \pi' \\ X & \hookrightarrow & \tilde{X} \end{array}$$

On the other hand, the morphism $\pi' : X' \to \tilde{X}$ defined by the inclusion $\mathcal{B} \subset K[X']$ is finite. This shows that $X \neq \tilde{X}$. The complement in $\tilde{X}$ of the open affine subset X is a union of irreducible divisors. Let Y be one of these divisors. The algebra $\mathcal{A}(\tilde{X}, Y)$ is a non-finitely generated invariant subalgebra in $K[\tilde{X}]$ and the inclusion $X \subset \tilde{X}$ defines an embedding $K[\tilde{X}] \subset$

$K[X] = \mathcal{A}$. We conclude that G/H should be affinely closed. In this case $\mathcal{A}$ is of type N by Theorem 2.

Lemma 1. *If $X = G/H$ is affinely closed, then any invariant subalgebra in $\mathcal{A}(H)$ is finitely generated.*

Proof. Suppose that there exists an invariant subalgebra $\mathcal{B} \subset \mathcal{A}(H)$ that is not finitely generated. Let $f_1, f_2, \ldots$ be a system of generators of $\mathcal{B}$. Consider the finitely generated subalgebras $\mathcal{B}_i = K[< Gf_1, \ldots, Gf_i >]$. Infinitely many of them are pairwise different. For the corresponding varieties $X_i := \operatorname{Spec} \mathcal{B}_i$ one has natural dominant G-morphisms

$$X_1 \longleftarrow X_2 \longleftarrow X_3 \longleftarrow \ldots$$

We claim that the action $G : X_i$ is transitive for any i. In fact, let $\mathcal{O}_i$ be the open orbit in X_i and $\pi_i : G/H^i \to \mathcal{O}_i$ be the bijective purely inseparable finite morphism from a quasi-affine homogeneous space G/H^i. Consider the integral closure $\mathcal{C}$ of $\pi_i^* K[X_i]$ in $K(G/H^i)$. The algebra $\mathcal{C}$ is finitely generated and, if X_i contains more than one orbit, then the G-variety $\operatorname{Spec} \mathcal{C}$ defines a nontrivial affine embedding of G/H^i. But $H \subseteq H^i$, and the remark after Theorem 2 concludes the arguments.

One may consider any X_i as a homogeneous variety G/H_i, where H_i is a (non-reduced) group subscheme of G containing H^i as its reduced part. The infinite sequence of group subschemes

$$H_1 \supset H_2 \supset H_3 \supset \ldots$$

leads to a contradiction. $\diamondsuit$

Remarks. 1) In the case $K = \mathbb{C}$, Lemma 1 follows also from [Lat]. In fact, the paper [Lat] was the starting point for the present note.

2) A G-algebra $\mathcal{A}$ contains no proper invariant ideals if and only if the action $G : X = \operatorname{Spec} \mathcal{A}$ is transitive. The group of equivariant automorphisms of the homogeneous space G/H (and of the algebra $\mathcal{A}(H)$) is isomorphic to the group $N_G(H)/H$. Suppose that $\operatorname{char} K = 0$, H is reductive and the group $N_G(H)/H$ is finite. As is obvious from what has been said any invariant subalgebra in $\mathcal{A}(H)$ has the form $\mathcal{A}(H')$, where $H \subseteq H' \subseteq G$, H' is reductive and $N_G(H')/H'$ is finite. Hence G-algebras of type N are characterized by property (*) (see Introduction).

Now consider case (ii). We are going to prove that here $\mathcal{A} = K[X]$ is an algebra of type HV following the proof of [Br, Lemme 1.2] (in the case char $K = 0$ see also [Po, Th.4], [Akh, Th.1]). Without loss of generality it can be assumed that X is contained as a closed G-invariant subvariety in a finite-dimensional G-module V with o at the origin. Let $\mathbb{P}(V \oplus K)$ be the projective space of $V \oplus K$, where G acts trivially on K. Denote by $\overline{X}$ the closure of X in $\mathbb{P}(V \oplus K)$, then $\overline{X}$ intersects the hyperplane at infinity $\mathbb{P}(V)$. This shows that U contains at least two fixed points in $\overline{X}$. But the set of points fixed by a connected unipotent group on a connected complete variety is connected [Hor, Th.4.1]. This proves that for the open orbit $\mathcal{O} \subset X$ one has $\mathcal{O}^U \neq \emptyset$. Let v be a U-fixed vector in $\mathcal{O}$. The vector v has the form $v = \sum v_i$, where $tv_i = \chi_i(t)v_i$ with $\chi_i \in X^+(T)$ for any i and any $t \in T$. Find a one-parameter subgroup $\theta : K^* \to T$ such that

(1) $< \theta, \chi_i > \geq 0$ for any i;

(2) there exists a non-zero χ_k (denote it by λ^*) such that $< \theta, \chi_i >= 0$ if and only if χ_i is a multiple of λ^*.

Then $v_1 = \lim_{t \to 0} \theta(t)v = \sum v_j$, where the corresponding χ_j are multiples of λ^*. By assumption on X, $v_1 \in X$. Let H be the stabilizer of v_1 in G. The bijective morphism $G/H \to \mathcal{O}$ defines an inclusion $K[\mathcal{O}] \subseteq K[G/H]$. Moreover, the subgroup H contains U and $K[G/H] = \oplus_\omega E(\omega)$, where $\omega^* \mid T_1 = 1$ for $T_1 = H \cap T$ [Gr, p.98]. This shows that $K[G/H] \subseteq \mathcal{A}(\lambda)$ and $\mathcal{A} = K[X] \subseteq K[\mathcal{O}]$ is a G-algebra of type HV.

Lemma 2. *Any invariant subalgebra of the algebra $\mathcal{A}(\lambda)$ is finitely generated.*

Proof. Let $\mathcal{B}$ be an invariant subalgebra of $\mathcal{A}(\lambda)$. It is known that $\mathcal{B}$ is finitely generated if and only if the algebra $\mathcal{B}^U$ of U-invariants is finitely generated [Gr, Th.16.2]. But Kdim $\mathcal{A}(\lambda)^U = 1$, and, by Proposition 2, $\mathcal{B}^U \subseteq \mathcal{A}(\lambda)^U$ is finitely generated. $\diamondsuit$

The proof of Theorem 1 is completed. $\diamondsuit$

5. Some results on affine embeddings. The next proposition is a modification of a construction due to G. Kempf [Kem].

Proposition 3. *Let G/H be a quasi-affine non affinely closed homogeneous space. Then G/H admits an affine embedding with a G-fixed point.*

Proof. Let $G/H \hookrightarrow X$ be a nontrivial embedding and $Y \subset X$ be a proper closed irreducible invariant subvariety. Denote by $f_1, \ldots, f_k$ generators of $K[X]$ and by $g_1, \ldots, g_s$ generators of the ideal $\mathcal{I}(Y)$. One may suppose that the f_i form a basis of $< Gf_1, \dots, Gf_k >$ and the g_i form a basis of $< Gg_1, \dots, Gg_s >$. Consider the G-equivariant morphism

$$\psi : \ X \to K^{s(k+1)},$$

$$x \to (g_1(x), .., g_s(x), g_1(x)f_1(x), .., g_s(x)f_1(s), .., g_1(x)f_k(x), .., g_s(x)f_k(x)).$$

Let Z be the closure of $\psi(X)$. It is clear that Z is birationally isomorphic to X and is an affine embedding of G/H. But $\psi(Y) = \{0\}$ is a G-fixed point on Z. $\diamondsuit$

Proof of Theorem 2. Suppose that for a reductive subgroup H the group $N_G(H)/H$ is infinite. Since $N_G(H)/H$ is reductive there exists a one-parameter subgroup $\theta : K^* \to N_G(H)$ such that $H \cap \theta(K^*)$ is finite. Consider the subgroup $H_1 = \theta(K^*)H$. The homogeneous fiber space $G *_{H_1} K$, where H acts on K trivially and $H_1/(H \cap \theta(K^*))$ acts on K by dilation, is a two-orbit embedding of G/H.

Conversely, suppose that X is an affine G-variety and $x \in X$ is an H-fixed point. As in the proof of Theorem 1, one may reduce the general case to the case where the orbit Gx is isomorphic to a quasi-affine homogeneous space G/H_1. We need to prove that if G/H_1 is a quasi-affine homogeneous space that is not affinely closed and H is a reductive subgroup contained in H_1, then $N_G(H)/H$ is infinite. By Proposition 3, there exists an affine embedding $G/H_1 \hookrightarrow X$ with a G-fixed point o. Fix a point x_0 in the open orbit on X. Let $\gamma : K^* \to G$ be an adapted (to x_0) one-parameter subgroup. Consider the parabolic subgroup

$$P(\gamma) = \{ g \in G \mid \lim_{t \to 0} \gamma(t) g \gamma(t)^{-1} \text{ exists in } G \ \}.$$

Then $P(\gamma) = L(\gamma)U(\gamma)$, where $L(\gamma)$ is a Levi subgroup that is the centralizer of $\gamma(K^*)$ in G, and $U(\gamma)$ is the unipotent radical of $P(\gamma)$. By [Kir], [Nes] (see also [PV2, Th.5.5]), the stabilizer G_{x_0} is contained in $P(\gamma)$. There is an element $u \in U(\gamma)$ such that $H' = uHu^{-1} \subset L(\gamma)$.

We claim that $\gamma(K^*)$ is not contained in H'. In fact, assume the converse: $\gamma(t)ux_0 = ux_0$ for any $t \in K^*$. Denote $\gamma(t)u\gamma(t)^{-1}$ by u_t. Then $u_t\gamma(t)x_0 = ux_0$, so that $\gamma(t)x_0 \in U(\gamma)x_0$. By assumption, $\lim_{t \to 0} \gamma(t)x_0 = o \notin Gx_0$. On the other hand, the orbit $U(\gamma)x_0$ is contained in Gx_0 and is closed as an orbit of a unipotent group on an affine variety [PV2, p.151]. (The proof of the latter statement is based only on the Lie-Kolchin theorem, which holds in arbitrary characteristic [Hum, 17.5].) This contradiction shows that $\gamma(K^*)$ is not contained in H' and $\gamma(K^*)$ centralizes H'. Hence the group $N_G(H')/H'$ (and $N_G(H)/H$) is infinite. $\diamondsuit$

Let us recall that a subgroup $Q \subset G$ is said to be *quasi-parabolic* if Q is the stabilizer of a highest weight vector v in some finite-dimensional irreducible G-module, say V_{λ^*}. If P_{λ^*} is the parabolic subgroup fixing the line $< v >$, then $Q = Q_{\lambda^*} = \{g \in P_{\lambda^*} \mid \lambda^*(g) = 1\}$.

Proposition 4. *A homogeneous space G/H admits an affine embedding $G/H \hookrightarrow X$ such that $X = G/H \cup \{o\}$, where o is a G-fixed point if and only if H is a quasi-parabolic subgroup of G.*

Proof. If H is quasi-parabolic, then consider $X' = \overline{Gv} \subset V_{\lambda^*}$. Let $\pi : G/H \to Gv$ be the orbit map and $\mathcal{B}$ be the integral closure of $\pi^*K[X']$ in $K[G/H]$. We have obtained the desired embedding $G/H \hookrightarrow \operatorname{Spec}\mathcal{B}$.

Conversely, as was shown in the proof of Theorem 1, the subgroup H (up to conjugation) is the stabilizer of a sum of highest weight vectors with proportional weights. This shows that H is a quasi-parabolic subgroup. $\diamondsuit$

Remarks. 1) Proposition 4 was proved by V. L. Popov [Po, Th.4 and Cor.5] in zero characteristic. For a description of complete embeddings with an isolated fixed point over the field $\mathbb{C}$ see [Akh, Th.2].

2) The assumption that G is reductive is not essential in Proposition 4, see [Po, Th.3].

Proposition 5. *Let H be an observable subgroup of G.*

(1) *If either G/H is affinely closed or H is a quasi-parabolic subgroup of G, then G/H admits only one normal affine embedding (up to G-isomorphisms);*

(2) *if $G = K^*$ and H is finite, then there exist only two normal affine embeddings, namely K^*/H and K/H;*

(3) *in all other cases there exists an infinite sequence*

$$X_1 \xleftarrow{\phi_1} X_2 \xleftarrow{\phi_2} X_3 \xleftarrow{\phi_3} \dots$$

of pairwise nonisomorphic normal affine embeddings X_i of G/H and equivariant dominant morphisms ϕ_i.

Proof. (1) The statement is obvious for affinely closed spaces. If H is quasi-parabolic, then consider the subalegra $\mathcal{B}$ in $\mathcal{A} = K[G/H]$ corresponding to a normal affine embedding of G/H. We claim that $\mathcal{A}^U = \mathcal{B}^U$. Indeed, $\mathcal{A}^U \cong K[x]$ is isomorphic to the polynomial algebra in one variable and $\mathcal{B}^U$

is a graded integrally closed subalgebra. Hence $\mathcal{B}^U = K[x^d]$. But $Q\mathcal{A} = Q\mathcal{B}$ implies $Q\mathcal{A}^U = Q\mathcal{B}^U$ and $d = 1$.

Any element of $\mathcal{A}$ is contained in $Q\mathcal{B}$. On the other hand, the algebra $\mathcal{A}$ is integral over $G\mathcal{A}^U$ [Gr, Th.14.3] and $G\mathcal{A}^U = G\mathcal{B}^U \subseteq \mathcal{B}$. But $\mathcal{B}$ is integrally closed and finally $\mathcal{A} = \mathcal{B}$.

(2) is obvious.

(3) In this case $K[G/H]$ contains a non-finitely generated subalgebra of type $\mathcal{A}(X,Y)$. One may suppose that X is normal. Then $\mathcal{A}(X,Y)$ is an integrally closed subalgebra in $K[G/H]$. Fix an element $g \in \mathcal{I}(Y)$ and generators $f_1, \dots, f_n$ of $K[X]$. Extend the sequence $g_0 = g, g_1 = gf_1, \dots, g_n = gf_n$ to an (infinite) generating set $g_0, g_1, \dots, g_n, g_{n+1} \dots$ of $\mathcal{A}(X,Y)$. Let $\mathcal{A}_k$ be the integral closure of $K[< Gg_0, \dots, Gg_{n+k} >]$ in $\mathcal{A}(X,Y)$. The varieties $X_i = \operatorname{Spec} \mathcal{A}_i$ are birationally isomorphic to X and $G/H \hookrightarrow X_i$. Infinitely many of X_i are pairwise nonisomorphic. Renumbering, one may suppose that all X_i are pairwise nonisomorphic. The chain

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \dots$$

corresponds to the desired chain

$$X_1 \xleftarrow{\phi_1} X_2 \xleftarrow{\phi_2} X_3 \xleftarrow{\phi_3} \dots \qquad \diamondsuit$$

6. The canonical embedding. Let us recall that an observable subgroup H of G is said to be *a Grosshans subgroup* if the algebra $K[G/H]$ is finitely generated. The famous Nagata counter-example to Hilbert's fourteenth problem provides an example of a unipotent subgroup in SL_{32}, which is not a Grosshans subgroup, see [Gr].

Definition 3. Let H be a Grosshans subgroup of G. Let us call $G/H \hookrightarrow X = \operatorname{Spec} K[G/H]$ *the canonical embedding* of G/H and denote it as $CE(G/H)$.

It is well-known that the codimension of the complement of the open orbit in $CE(G/H)$ is ≥ 2 and $CE(G/H)$ is the only normal affine embedding of G/H with this property [Gr, Th.4.2]. If H is reductive, then $CE(G/H)$ is the trivial embedding. For non-reductive subgroups $CE(G/H)$ is an interesting object canonically associated to the pair (G,H).

In the rest of this section, we shall assume that $\operatorname{char} K = 0$. Fix some notation. There exists a canonical decomposition $K[G/H] = K \oplus K[G/H]_G$, where the first term corresponds to the constant functions and $K[G/H]_G$ is the sum of all nontrivial irreducible submodules in $K[G/H]$.

Proposition 6. *The following conditions are equivalent:*
(1) *$CE(G/H)$ contains a G-fixed point;*
(2) *any affine embedding of G/H contains a G-fixed point;*
(3) *$K[G/H]_G$ is an ideal in $K[G/H]$;*
(4) *H is not contained in a proper reductive subgroup of G.*

Proof. (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (2). If $G/H \hookrightarrow X$, then $K[X] \subseteq K[G/H]$ and the image of a G-fixed point in $CE(G/H)$ under the corresponding morphism $CE(G/H) \to X$ is a G-fixed point in X.

To prove (1) $\Rightarrow$ (3) we note that $K[G/H]_G$ is the only candidate for a maximal G-invariant ideal in $K[G/H]$.

(1) $\Rightarrow$ (4). Suppose that $H \subseteq L \subset G$, L is reductive. Then $K[G/L] \subseteq K[G/H]$ and $CE(G/H) \to G/L$. Hence G/L contains a G-fixed point, a contradiction.

(4) $\Rightarrow$ (1). Suppose that the closed G-orbit in $CE(G/H)$ is isomorphic to G/L. By the slice theorem [Lu1] H is contained in a subgroup conjugated to L. $\diamondsuit$

Let G be a connected semisimple group and $P \subset G$ be a parabolic subgroup containing no simple component of G. Denote by U_P the unipotent radical of P.

Proposition 7. *The homogeneous space G/U_P satisfies conditions (1)-(4) of Proposition 6.*

Proof. It is known that U_P is a Grosshans subgroup of G [Gr, Th.16.4]. We shall check that $K[G/U_P]_G$ is an ideal in $K[G/U_P]$. For this it is sufficient to find a nonnegative grading on $K[G/U_P]$ with $K[G/U_P]_G$ as the positive part.

Let $B = TU$ be a Borel subgroup in G with $B \subseteq P$ and let $P = LU_P$, where L is a Levi subgroup such that $T \subseteq L$ and $U = (U \cap L)U_P$. Denote by $T_L \subset T$ the center of L. Then $T_L = \{t \in T \mid \alpha_i(t) = 1 \ \ \forall\ i\ \}$, where $\{\alpha_i\}$ is the set of simple roots corresponding to P. Let $\pi : X(T) \to X(T_L)$ be the restriction homomorphism of the groups of characters, and $X^+(T) \subset X(T)$ be the set of dominant weights (with respect to B). It is easy to check that the restriction of π to $X^+(T)$ is injective and $\pi(X^+(T))$ generates a strictly convex cone in $X(T_L) \otimes \mathbb{Q}$. Fix a one-parameter subgroup $\theta : K^* \to T_L$ so that $< \theta, \chi >$ is positive for any $\chi \in \pi(X^+(T))$.

Note that L acts on $K[G/U_P]$ as $(l * f)(gU_P) := f(glU_P)$ and this action commutes with the G-action. The L-module $K[G/U_P]_G$ contains no trivial L-submodules because of $K[G/U_P]^L = K[G/P] = K$. On any nontrivial irreducible L-submodule T_L acts by multiplication by $\chi(t)$, $t \in T_L$, for some non-zero $\chi \in \pi(X^+(T))$. The restriction of the T_L-action to $\theta(K^*)$ defines the desired grading. $\diamondsuit$

Definition 4. Let H be a Grosshans subgroup of G. We say that a reductive subgroup L is *a reductive hull* of H if L is a minimal (with respect to inclusions) reductive subgroup of G containing H.

It follows from the proof of Proposition 6 that the closed orbit in $CE(G/H)$ is isomorphic to G/L. Therefore for any two reductive hulls L_1 and L_2 of H there is an element $g \in G$ such that $L_2 = g^{-1}L_1g$. In fact, a reductive hull is not unique.

Example 4. Let $G = SL_n$, $L = SO_n$, and H be a maximal unipotent subgroup of L. It is clear that L is a reductive hull of H. One has $H \subset U$ for some maximal unipotent subgroup U in G. There exists a subgroup H_1 such that $H \subset H_1 \subseteq U$, $\dim H_1 = \dim H + 1$ and H is a normal subgroup of H_1. Consider an element $h_1 \in H_1 \setminus H$. Then $h_1^{-1}Lh_1$ is another reductive hull of H.

7. Problems. In this section we collect some problems that follow naturally from the discussion above.

Problem 1. *Suppose that* $\operatorname{char} K > 0$, *a G-algebra $\mathcal{A}$ contains no proper G-invariant ideals, and the group of equivariant automorphisms of $\mathcal{A}$ is finite. Is it true that $\mathcal{A}$ is a G-algebra of type N ?*

Problem 2. *Let G be a linear algebraic group. Characterize all finitely generated integral G-algebras $\mathcal{A}$ such that any invariant subalgebra in $\mathcal{A}$ is finitely generated.*

This class of algebras seems to be much wider than in the reductive case.

Proposition 8. *Let G be a reductive group, S be a unipotent group, $H \subset G$ be a reductive subgroup with a finite index in $N_G(H)$, and $F \subset S$ be any closed subgroup. Then any $G \times S$-invariant subalgebra in $\mathcal{A} = K[(G \times S)/(H \times F)]$ is finitely generated.*

Proof. Fix the notation: $\mathcal{A}_1 = K[G/H]$, $\mathcal{A}_2 = K[S/F]$, $\mathcal{B}$ is a subalgebra in $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. It is clear that $\mathcal{A}^S = \mathcal{A}_1 \otimes K = \mathcal{A}_1$.

It is sufficient to prove that $\mathcal{B}$ contains no proper invariant ideals. (After this we complete the proof following the proof of Lemma 1.)

Let $\mathcal{I} \subset \mathcal{B}$ be an invariant ideal. By the Lie-Kolchin theorem, $\mathcal{I}^S \neq 0$. Hence $\mathcal{I}^S$ is a non-zero ideal in $\mathcal{B} \cap \mathcal{A}_1$. But any invariant subalgebra in $\mathcal{A}_1$ contains no proper invariant ideals. We have $\mathcal{I}^S = \mathcal{B} \cap \mathcal{A}_1$ and $\mathcal{I}^S$ contains constants, thus $\mathcal{I} = \mathcal{B}$. $\diamondsuit$

This proof shows that $(G \times S)/(H \times F)$ is an affinely closed homogeneous space.

Problem 3. *Characterize all affinely closed homogeneous spaces of a linear algebraic group G.*

The last problem concerns canonical embeddings. Let us recall that the modality of a G-variety X is the maximal number of parameters in a continuous family of G-orbits on X, or, more formally,

$$\mathrm{mod}_G(X) = \max_{Y \subseteq X} \ \mathrm{tr.deg} K(Y)^G,$$

where Y runs through all closed irreducible invariant subvarieties in X.

Problem 4. *Let H be a Grosshans subgroup of a reductive group G. Find the modality of $CE(G/H)$.*

One may suppose that a reductive hull of H is G. Indeed, if a reductive hull of H is L, then, by the slice theorem, $CE(G/H) = G *_L CE(L/H)$ and $\mathrm{mod}_G(CE(G/H)) = \mathrm{mod}_L(CE(L/H))$.

Example 5. Let $G = SL_n$ and H be the unipotent radical of the maximal parabolic subgroup in G corresponding to the first $(n-2)$ simple roots. It is clear that $CE(G/H) \cong K^n \times \ldots \times K^n$ ($(n-1)$ copies) with the diagonal G-action. This space is covered by finitely many locally closed G-invariant subset $S_{i_1,\ldots,i_k}$, where $S_{i_1,\ldots,i_k}$ is the set of $(n \times (n-1))$-matrices of rank k with linearly independent columns $i_1 \ldots, i_k$. An orbit in $S_{i_1,\ldots,i_k}$ depends on $k(n-1-k)$ parameters, which are the coefficients of linear expressions of the remaining $(n-1-k)$ columns by the columns $i_1, \ldots, i_k$. The maximal number of parameters is

$$\mathrm{mod}_G(CE(G/H)) = s^2 \ \text{ for } \ n = 2s+1$$

and

$$\mathrm{mod}_G(CE(G/H)) = s^2 - s \quad \text{for} \quad n = 2s.$$

8. Acknowledgements. The author is grateful to Michel Brion for numerous discussions and suggestions, and to A. J. Parameswaran for a useful remark.

This paper was written during the staying at Institut Fourier (St. Martin d'Hères, France). The author wishes to thank this institution and especially Michel Brion for invitation and hospitality, and Independent Moscow University for support.

References

[Akh] *D. N. Akhiezer*, Dense orbits with two ends, Izv. Akad. Nauk SSSR, Ser. Mat. **41:2** (1977), 308-324 (in Russian); English trans.: Math. USSR-Izv. **11:2** (1977), 293-307.

[Br] *M. Brion*, Groupe de Picard et nombres caractéristiques des variétés sphériques, Duke Math. J. **58:2** (1989), 397-424.

[Gr] *F. D. Grosshans*, Algebraic Homogeneous Spaces and Invariant Theory, LNM 1673, Springer-Verlag Berlin, 1997.

[Hor] *G. Horroks*, Fixed point schemes of additive group actions, Topology **8** (1969), 233-242.

[Hum] *J. E. Humphreys*, Linear Algebraic Groups, Grad. Texts in Math. **21**, Springer-Verlag, New-York, 1975.

[Kem] *G. Kempf*, Instability in invariant theory, Ann. of Math. **108:2** (1978), 299-316.

[Kir] *F. Kirwan*, Cohomology of quotients in symplectic and algebraic geometry, Math. Notes. Princeton Univ. Press **31** (1984).

[Lat] *I. A. Latypov*, Homogeneous spaces of compact connected Lie groups which admit nontrivial invariant algebras, Journal of Lie Theory **9** (1999), 355-360.

[Lu1] *D. Luna*, Slices étales, Bull. Soc. Math. France, Paris, Memoire **33** (1973), 81-105.

[Lu2] *D. Luna*, Adhérences d'orbite et invariants, Invent. Math. **29** (1975), 231-238.

[Nes] *L. Ness*, A stratification of the null cone via the moment map, Amer. J. of Math. **106** (1984), 1281-1329.

[Po] *V. L. Popov*, Classification of three-dimensional affine algebraic varieties that are quasihomogeneous with respect to an algebraic group, Izv. Akad. Nauk SSSR, Ser. Mat. **39:3** (1975), 566-609 (in Russian); English trans.: Math. USSR-Izv. **9** (1975), 535-576.

[PV1] *V. L. Popov and E. B. Vinberg*, A certain class of quasihomogeneous affine algebraic varieties, Izv. Akad. Nauk SSSR, Ser. Mat. **36** (1972), 749-764 (in Russian); English trans.: Math. USSR-Izv. **6** (1972), 743-758.

[PV2] *V. L. Popov and E. B. Vinberg*, Invariant Theory, Itogy Nauki i Tekhniki, Sovr. Problemy Mat. Fund. Napravlenia, vol. 55, VINITI, Moscow, 1989, pp. 137-309 (in Russian); English trans.: Algebraic Geometry IV, Encyclopaedia of Math. Sciences, vol. 55, Springer-Verlag Berlin, 1994, pp. 123-278.

[Rich] *R. W. Richardson*, Affine coset spaces of reductive algebraic groups, Bull. London Math. Soc. **9** (1977), 38-41.

[Su] *A. A. Sukhanov*, Description of the observable subgroups of linear algebraic groups, Mat. Sbornik **137:1** (1988), 90-102 (in Russian); English trans.: Math. USSR-Sb. **65:1** (1990), 97-108.

Author's address: Russia, 119899, Moscow,
Vorobievy Gory, Moscow State University,
Department of Mathematics and Mechanics, Chair of Higher Algebra
arjantse@mccme.ru

Current address: Institut Fourier,
BP 74, 38402 St.Martin D'Hères, Cedex, France
arjantse@mozart.ujf-grenoble.fr