Geometric and Categorical Nonabelian Duality in Complex Geometry

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In many domains of mathematics there is a natural notion of duality or of dual objects, such as the dual vector space, the dual complex, the dual abelian group etc. Sometimes this notion is less obvious, as for instance in the case of finite nonabelian groups. This paper makes a contribution to the problem of constructing dual objects for *abstract* complex varieties (for embedded projective varieties, there is a classical notion of a dual variety). A good duality theory should satifsy contravariancy for the duality functor and biduality, i.e. the original space should be recovered from its double dual. This is not always true as one can see by rather natural examples. Our approach intends to consider the dual space as a moduli space of certain geometric objects (as vector bundles for instance) on the original variety or as mappings into a classifying space which plays the role of a dualizing object. It seems to be a general phenomenon that the dual carries more algebraic structure than the original space. In the case of moduli spaces of vector bundles, these structures are induced by the tensor product and direct sum of bundles. By adopting a categorical point of view, the dual should consist of a *category* of geometric objects on the original space, more precisely a *tannakian* category. We show, for instance, how to recover a projective variety from its category of vector bundles by a spectrum contruction, similar to the affine or Stein case. More generally, one can define the spectrum of a tannakian category and a dualizing category. It is evident that one loses a priorily information by passing to isomorphy classes, so that biduality, using moduli spaces, could not be expected in general. This is the case for the complex projective line which cannot be recovered from its dual (consisting of moduli of vector bundles) by biduality.

For some restricted classes of varieties, one wants the dual to stay in the same class (thinking of abelian varities, K3 surfaces, Calabi-Yau manifolds). Here the "restricted" dual should be a component of the general one which does not involve the particular character of the original variety. I do not know if mirror symmetry will fit into this framework, but it served as one motivation for this work.

In most of our constructions, moduli spaces of vector bundles play an essential role. This is not so by accident, since various classical results, properly interpreted, are just biduality statements (as for abelian varities or compact Riemann surfaces of genus > 1 by a theorem of M.S.Narasimhan and S.Ramanan, see [16]).

The spaces considered in this paper are still "abelian", or better "classical", so we do not touch the quantum context (approaches for noncommutative geometry, for example that in the book [17], is a different story, although there seem to be commun points).

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Contents

1 Review of some duality results

1.1 The Stein case

In the paper [11], I established a certain nonabelian biduality result on a Stein complex space X. We define a dual of X by means of a complex Lie group G. The evaluation biduality map is

$$\begin{aligned} \chi :\ X \ &\longrightarrow \ \mathrm{Hom}_e(\mathrm{Mor}(X,G),G) \\ x &\mapsto (f \mapsto f(x)) \end{aligned}$$

$\mathrm{Mor}(X,G)$ denotes the topological group of *holomorphic* maps from X to G and $\mathrm{Hom}_e(-,-)$ is the set of continuous group homomorphisms which are right and left equivariant with respect to the actions of G on both sides.

In general, χ need not to be surjective as one can see in the case of the additive group $G := (\mathbf{C},+)$. Even if we take G to be $\mathrm{Gl}_n(\mathbf{C})$ or $\mathrm{Sl}_n(\mathbf{C})$, this is not always true. From the main result of [11], one can deduce the following theorem:

Theorem (1.1.1). *Let X be a Stein complexe space and $X^\vee$ the connected component of the neutral element of the topological group* $\mathrm{Mor}(X,\mathrm{Sl}_n(\mathbf{C}))$ *for $n \geq 2$.*

Then the biduality map

$$\chi :\ X \ \longrightarrow \ \mathrm{Hom}_e(X^\vee,\mathrm{Sl}_n(\mathbf{C}))$$

is bijective.

We expect that χ is a *homeomorphism,* as it is the case for the classical situation of the spectrum of the topological $\mathbf{C}$-algebra $\Gamma(X, \mathcal{O}_X) = \mathrm{Mor}(X, \mathbf{C})$. But we do not intend to treat this particular aspect here. Obvously, the dual carries a much richer structure than the original space. This structure has to be taken into account for defining the bidual space. In the Stein situation, the dual is an infinite dimensional object.

REMARK (1.1.2). One can pose similar questions in the affine algebraic situation and, moreover, try to establish a result of GAGA type for holomorphic mappings into algebraic groups. Unfortunately, this last property turns out to be wrong for many groups. The natural map $\mathrm{Mor}(X, \mathrm{Gl}_2(\mathbf{C})) \to \mathrm{Mor}(X^{\mathrm{an}}, \mathrm{Gl}_2(\mathbf{C}))$ does not have dense image in general (take for instance $X = \mathrm{A}^1_{\mathbf{C}}$). It should be noted that the proof of the main theorem on biduality in [11] does not apply to the affine algebraic case.

1.2 Tannaka duality

This kind of duality is very instructive in order to understand better what a dual should be and how one can establish a biduality property.

Let G be a compact topological group. Then the dual of G is usually considered to be the *category*

$$\mathrm{Rep}(G)$$

of all complex finite dimensional representations of G. This category has a very rich structure. It is a tensor category (in such a category, one has finite direct sums, tensor products, dual objects and a neutral object, satisfying the usual compatability relations). The next question is: *What is a representation of* $\mathrm{Rep}(G)$ *?* - The answer is quite easy: It should be a functor

$$\phi : \ \mathrm{Rep}(G) \longrightarrow (\mathrm{endvect})$$

with values in the category of endomorphisms of complex vector spaces such that ϕ commutes with the forgetful functor on both sides which takes values in the category of complex vector spaces and, moreover, ϕ respects all the above mentioned operations as sums, tensor products etc. Such a functor will be called a *tensor functor.* Then Tannaka duality can be stated in the following way:

Theorem (1.2.1) *For a compact topological group, the natural map*

$$\begin{aligned} G &\longrightarrow \mathrm{Rep}(\mathrm{Rep}(G)) \\ g &\mapsto \phi_g \end{aligned}$$

where ϕ_g is the evaluation in g, is bijective.

For a sketch of the proof, see [10]. The above map will be a homeomorphism of topological groups if one endows the right hand side with the *weak* topology. The main point in the proof of theorem (1.2.1) is the surjectivity which can be reduced to the following property:

Let $\phi : \mathrm{Rep}(G) \to (\mathrm{endvect})$ *be a representation of* $\mathrm{Rep}(G)$*. Then one has*

$$\phi(\rho) \in \mathrm{Im}(\rho)$$

for every representation $\rho : G \to \mathrm{Gl}(V)$*.*

The surjectivity is deduced from this by rather easy and formal arguments whereas the assertion itself is shown by contradiction. If we want to interpret Tannaka duality analogously to the biduality in the Stein situation, we would have to consider the category (endvect) as a "dualizing" category and a representation ρ as a functor into a fiber category of the forgetful functor (endvect) $\to$ (vect). Here the group G has the structure of a category with objects as the set of elements of G. But there is another natural way to consider G as a category: The category which consists only of one object e with $\mathrm{Mor}(e, e) = G$. In this case, a representation is a functor from G to the category

$$\mathcal{T}_o := (\mathrm{vect})$$

of (finite dimensional) complex vector spaces and we have

$$\mathrm{Rep}(G) = \mathrm{Hom}(G, \mathcal{T}_o)$$

The bidual will now be

$$\mathrm{Hom}_{\mathcal{T}_o,\otimes}(\mathrm{Rep}(G),\ \mathcal{T}_o),$$

the category of all functors from $\mathrm{Rep}(G)$ to $\mathcal{T}_o$ which are compatible with the tensor category structures, but also the left and right operations of $\mathcal{T}_o$ on $\mathrm{Rep}(G)$, given by $(V, \rho) \mapsto (id_V \otimes \rho)$, $(\rho, V) \mapsto (\rho \otimes id_V)$, and which are the identity on the subcategory $\mathcal{T}_o$ of $\mathrm{Rep}(G)$ of trivial representations of G. This new category has only one object (a representation is mapped to its underlying vector space). The set of endomorphisms of this object is a group G'' and the biduality map is the natural group homomorphism $G \to G''$ which is bijective by Tannaka duality.

It should be mentioned that one can also consider the representation ring $R(G)$ of G as a candidate for a dual of G. Unfortunately, it is not always possible to recover G from its representation ring. So, in general, one loses information by passing to isomorphy classes.

Another viewpoint of Tannaka duality, which is slightly different from our one, is that by fixing a *fiber functor*, as considered by A.Grothendieck. The forgetful functor

$$\omega : \text{Rep}(G) \longrightarrow (\text{vect})$$

is such a fiber functor and G is identified with the group of automorphisms of ω.

For higher order Tannaka duality, see the recent work of B.Toen [22].

1.3 Abstract Spanier-Whitehead duality

In stable homotopy theory, there is a nice duality theory, namely that of Spanier and Whitehead (see for instance [1] and [23]). It concerns (spectra of) CW-complexes. One has a duality statement expressed in a natural bijection of stable classes of homotopy of maps

$$[W \wedge X, S] \cong [W, X^*]$$

where S is the sphere spectrum and X^* the Spanier-Whitehead dual of X. Under finiteness conditions on X, biduality holds.

We observe here that the sphere spectrum plays the role of a dualizing object in the stable homotopy category. It is selfdual and a neutral object too. A.Dold and D.Puppe gave a nice abstract presentation and axiomatisation of this duality in their paper [9]. We reproduce here those parts of it which are of interest for our purpose.

Let $\mathcal{C}$ be a monoidal category with multiplication functor $\otimes$ and neutral object I. This means that $\otimes$ is a bi-functor $(A, B) \mapsto A \otimes B$ of $\mathcal{C}$ into itself and we have given natural isomorphisms

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

$$I \otimes A \cong A \cong A \otimes I$$

$$\gamma = \gamma_{AB} : A \otimes B \longrightarrow B \otimes A$$

which satisfy the obvious compatibility relations. It should be noted that γ_{AA} need not be the identity.

We shall define two notions of a dual object (a weak and a strong one). An object B of $\mathcal{C}$ is called a *weak dual* of A, if it represents the functor

$$X \longmapsto \text{Mor}_{\mathcal{C}}(X \otimes A, I)$$

so there is a natural bijection

$$\mathrm{Mor}_{\mathcal{C}}(X \otimes A, I) \cong \mathrm{Mor}_{\mathcal{C}}(X, B). \tag{1}$$

In the case $X = B$, we obtain by this identification the *evaluation* morphism

$$\epsilon = \epsilon_A : B \otimes A \longrightarrow I$$

applied to id_B.

Let $\mathcal{C}_d$ be the full subcategory of those objects of $\mathcal{C}$, possessing a dual. Then we can define a contravariant functor

$$D : \mathcal{C}_d \longrightarrow \mathcal{C}$$

which associates to each object A of $\mathcal{C}_d$ a dual DA. The object I is obviously dual to itself , so we may assume $DI = I$. We also have a biduality morphism $\delta_A : A \to DDA$ (in the case where A and DA belong to $\mathcal{C}_d$) which is obtained by (1), applying to the composition

$$A \otimes DA \stackrel{\gamma}{\longrightarrow} DA \otimes A \stackrel{\epsilon}{\longrightarrow} I$$

If δ_A is an isomorphism, we call A *reflexiv*e. Let A, B, $A \otimes B$ be objects of $\mathcal{C}_d$. Then we define

$$\mu = \mu_{AB} : DA \otimes DB \longrightarrow D(A \otimes B)$$

by

$$DA \otimes DB \otimes B \otimes A \stackrel{id \otimes \epsilon_B \otimes id}{\longrightarrow} DA \otimes I \otimes A \stackrel{\epsilon_A}{\to} I.$$

Defintion (1.3.3). An object A of $\mathcal{C}$ is called *strongly dualizable* if it is reflexiv and $\mu_{A,DA}$, or equivalently, the composition

$$DA \otimes A \stackrel{id \otimes \delta}{\longrightarrow} DA \otimes DDA \stackrel{\mu}{\longrightarrow} D(A \otimes DA)$$

is an isomorphism. The latter means that $DA \otimes A$ is canonically selfdual.

If A is strongly dualizable, there is a coevaluation

$$\eta = \eta_A : I \longrightarrow A \otimes DA$$

which is define to be the composition

$$I = DI \xrightarrow{D\epsilon} D(DA \otimes A) \xrightarrow{\mu^{-1}} DA \otimes DDA \longrightarrow$$

$$\xrightarrow{id \otimes \delta^{-1}} DA \otimes A \xrightarrow{\gamma} A \otimes DA$$

The proof of the following theorem is sketched in [9]:

Theorem (1.3.4) *Let A and B be objects of a monoidal category $\mathcal{C}$ and let $\epsilon : B \otimes A \to I$ be a morphism. Then the following assertions are equivalent:*

(a) *B is a strong dual of A with evaluation ϵ,*

(b) *there exists $\eta : I \to A \otimes B$ such that the following compositions are the identity morphisms of A and B respectively*

$$A = I \otimes A \xrightarrow{\eta \otimes id_A} A \otimes B \otimes A \xrightarrow{id_A \otimes \epsilon} A \otimes I = A,$$

$$B = B \otimes I \underset{id_B \otimes \eta}{\longrightarrow} B \otimes A \otimes B \underset{\epsilon \otimes id_B}{\longrightarrow} I \otimes B = B,$$

(c) *the map*

$$\varphi_{XY} : \mathrm{Mor}_{\mathcal{C}}(X, Y \otimes B) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X \otimes A, Y)$$

which sends $f : X \to Y \otimes B$ into the composition $(id_Y \otimes \epsilon) \circ (f \otimes id_A)$, is bijective for all objects X, Y of $\mathcal{C}$.

Furthermore, if one of these properties (and hence all of them) is satisfied, then the morphism η in (b) *is necessarily the coevaluation and the bijection φ_{IA} of* (c) *sends it into id_A.*

Suppose now that X and Y are weakly dualizable objects in $\mathcal{C}$. Then we have a natural bijection

$$\mathrm{Mor}(X, DY) \cong \mathrm{Mor}(Y, DX) \tag{2}$$

which is deduced from the following commutativ diagram

$$\begin{array}{ccc}
\mathrm{Mor}(X, DY) & \longleftarrow & \mathrm{Mor}(X \otimes Y, I) \\
\downarrow & & \downarrow \\
\mathrm{Mor}(Y, DX) & \longleftarrow & \mathrm{Mor}(Y \otimes X, I)
\end{array}$$

where the horizontal arrows are bijections. We also note that for reflexive objects X, Y, we can identify

$$\begin{aligned} \mathrm{Mor}(X,Y) &\to \mathrm{Mor}(DY,DX) \\ f &\mapsto Df\,. \end{aligned}$$

It is possible to strengthen the notion of a weakly dualizable object, being less strong than a strongly dualizable one. This is achieved by the viewpoint of adjoint functors. Let A be an object of $\mathcal{C}$ and consider the functor

$$X \longmapsto F_A(X) := X \otimes A$$

The existence of a right adjoint functor G_A to F_A is expressed as an isomorphism of bifunctors

$$\mathrm{Mor}(F_A(X),Y) \cong \mathrm{Mor}(X, G_A(Y)) \tag{3}$$

This property is a little weaker than (1.3.4)(c) but stronger than (2). Obviously, G_A is contravariant in A and $G_A(Y)$ should be considered as the dual of A with "values" in Y. From (3) one deduces directly evaluation and coevaluation morphisms.

The formula (3) admits another interpretation, in the case where every object has a weak dual. Since the functor D is contravariant, it can be regarded for example, as a *covariant* functor

$$D : \mathcal{C} \longrightarrow \mathcal{C}^\circ$$

from $\mathcal{C}$ to its dual category $\mathcal{C}^\circ$. Then (3) says that D is *selfdual.* We will come back later to this point of view.

1.4 Duality for compact Riemann surfaces

Let X be a compact Riemann surface of genus $g \geq 2$ and L be a line bundle of degree d on X. The moduli space

$$M_{r,L}$$

of stable vector bundles on X of degree r and determinant L is known to be a smooth projective variety. In the case where r and d are coprime, there is a tautological universal bundle W on the product space $X \times M_{r,L}$. We denote by $ad(W)$ the sheaf of traceless endomorphisms of W and by $ad_x(W)$ those of $W_x := W \mid \{x\} \times M_{r,L}$.

M.S.Narasimhan and S.Ramanan showed in their paper [16] (see also [18]) that the infinitesimal deformations of $M_{r,L}$ and of X coincide (with maybe some exceptions for low values of g). It is also shown that the Kodaira-Spencer map, obtained by considering X as a parameter space of deformations of $M_{r,L}$, is bijective

$$T_xX \longrightarrow \mathrm{H}^1(M_{r,L}, ad_x(W)) \tag{4}$$

for every $x \in X$. Moreover, we have

$$\mathrm{H}^i(M_{r,L}, ad_x(W)) = 0, \qquad for\, all\, i \neq 1. \tag{5}$$

Especially, the family W is a family of *simple* bundles on $M_{r,L}$. So we get a holomorphic map

$$\chi_{_{r,L}} : X \longrightarrow \mathrm{Sim}_r(M_{r,L})$$

from X into the moduli space of simple bundles of rank r on $M_{r,L}$ (see the paper [12] for moduli spaces of simple bundles; one could replace *simple* by *semi-stable*, c.f. [3]). A point x of X is mapped to the isomorphy class of W_x. The image of $\chi_{_{r,L}}$ is, by construction, contained in the fiber of the determinant map

$$det : \mathrm{Sim}(M_{r,L}) \longrightarrow \mathrm{Pic}(M_{r,L})$$

The bijectivity of (4) is equivalent to the property that

$$\chi_{_{r,L}} : X \longrightarrow det^{-1}([\det(W_{x_o})])$$

is locally (in X) biholomorphic where x_o is a fixed point in X. In fact, it is already globally biholomorphic, see [16]. Moreover, the local deformations of $M_{r,L}$ correspond exactly to those of X which will be seen later. By our viewpoint, these facts should be considered as a *biduality* statement (the last one as a *relative* biduality).

In section (4.2), we will indicate another approach to the theorem of Narasimhan/Ramanan which is more analytic and seems to have the advantage to apply in much more genral situations.

Let us now consider the following triangle where $M = M_{r,L}$

$$\begin{array}{ccc} & X \times M & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & M \end{array}$$

where p and q denote the canical projections. Then we have natural identifications

$$R^1 q_* ad(W) \cong \mathcal{T}_M$$

$$R^1 p_* ad(W) \cong \mathcal{T}_X$$

and

$$R^i q_* ad(W) \cong 0 \quad \forall i \neq 1$$

$$R^j p_* ad(W) \cong 0 \quad \forall j \neq 1.$$

We note that X and M have in general different dimensions, but the above properties are symetric in X and M. By Leray's spectral sequence (applied to p and q), we deduce from theses identities that

$$\mathrm{H}^j(X, \mathcal{T}_X) \cong \mathrm{H}^{j+1}(X \times M, ad(W)),$$

$$\mathrm{H}^j(M, \mathcal{T}_M) \cong \mathrm{H}^{j+1}(X \times M, ad(W))$$

for all i and j. So we see that in particular, the infinitesimal deformations of M and X coincide. But we can deduce more, namely that even the local deformations must be the same (by the vanishing of both $\mathrm{H}^2(..)$ cohomology groups).

The Fourier-Mukai transforms for the above triangle

$$\Psi : \mathrm{D}(X) \longrightarrow \mathrm{D}(M)$$

$$\mathcal{F} \mapsto Rq_*(W \otimes \mathrm{L}p^* \mathcal{F}),$$

$$\Phi : \mathrm{D}(M) \longrightarrow \mathrm{D}(X)$$

$$\mathcal{G} \mapsto Rp_*(W \otimes \mathrm{L}q^* G)$$

between the derived categories of complexes with bounded coherent cohomology are *not* fully faithful, in general: This can be seen by applying the criterion of Bondal/Orlov in [5] and the above identities. The orthogonality property of point objects with respect to the $\mathrm{Ext}^\bullet(-,-)$-scalar product (see (3.2) is not preserved here.

2 Dualizing objects and categories

2.1 Duality functors

We want to introduce the notion of a dualizing functor. The first question which one is supposed to answer, is if a dual should be a co- or contravariant construction. It seems to be more natural to assume the *contravariancy* although there are some non trivial examples of covariant duality functors, such as base change by complex conjugation for complex spaces, i. e. the functor $X \mapsto \overline{X}$. So let us consider now a contravariant functor

$$F : \mathcal{C} \longrightarrow \mathcal{C}$$

on some category $\mathcal{C}$. We call F a *(weakly) dualizing functor* if there is a natural isomorphism of bifunctors

$$\omega_{XY} : \operatorname{Mor}(X, FY) \longrightarrow \operatorname{Mor}(Y, FX) \tag{6}$$

in X and Y. Let

$$\chi : id \longrightarrow F^2 \tag{7}$$

be the induced biduality morphism. We call F *dualizing* if χ is an isomorphism of functors. We may write ω in the following form

$$\omega_{XY}(\alpha) = F(\alpha) \circ \chi_Y$$

and

$$\chi_Y = \omega_{FY,Y}(id_{FY}) \,.$$

We note that

$$\omega_{YX}(\beta) = \omega_{XY}^{-1}(\beta) = F(\beta) \circ \chi_X$$

so that we obtain for $\beta = id_{FX}$ and $Y = FX$

$$\begin{aligned} id_{FX} &= \omega_{X,FX} \circ \omega_{X,FX}^{-1}(id_{FX}) \\ &= F(\omega_{X,FX}^{-1}(id_{FX})) \circ \chi_{FX} \\ &= F(\omega_{FX,X}(id_{FX})) \circ \chi_{FX} \\ &= F(\chi_X) \circ \chi_{FX} \,. \end{aligned}$$

In particular, the biduality morphism for FX is a split monomorphism. More generally, we note that the identity (6) can be interpreted as the *autoadjointness* of the funtor F, if we consider it as a covariant functor from $\mathcal{C}$ to its dual category $\mathcal{C}^\circ$. This has already been remarked at the end of section (1.3). So we have a pair of adjoint functors (F, F). It defines in particular a monad in the sense of Eilenberg and MacLane, see [14], Chap.6. The functors $(F^{2k})_{k\in\mathbf{N}}$ form a cosimplicial system with transition maps $F^k(\chi_{_{F^l X}}),\ k+l \equiv 0\, mod\, 2$.

It is of interest to generalize the above construction in the following direction: Assume that a contravariant "dualizing" functor $F : \mathcal{C} \to \mathcal{C}'$ is given, where $\mathcal{C}'$ is some bigger category which contains $\mathcal{C}$ as a full subcategory, so there is an inclusion functor $h : \mathcal{C} \to \mathcal{C}'$. The fact that F should be dualizing, can now be expressed by a functorial isomorphism

$$\mathrm{Mor}_{\mathcal{C}'}(hX, FY) \cong \mathrm{Mor}_{\mathcal{C}'}(hY, FX)\,.$$

The former situation was that where $h = id_{\mathcal{C}}$.

Next, we define duality between two *contravariant* functors

$$F:\,\mathcal{C} \longrightarrow \mathcal{C}', \qquad G:\,\mathcal{C}' \longrightarrow \mathcal{C}\,.$$

We say that F and G is a pair of *dual* functors if they are adjoint, i. e. there is an isomorphism of bifunctors

$$\mathrm{Mor}_{\mathcal{C}}(X, GA) \cong \mathrm{Mor}_{\mathcal{C}'}(A, FX)$$

where $X \in \mathrm{obj}(\mathcal{C})$ and $A \in \mathrm{obj}(\mathcal{C}')$, and if the adjunction morphism

$$\begin{aligned} id_{\mathcal{C}} &\longrightarrow G \circ F, \\ id_{\mathcal{C}'} &\longrightarrow F \circ G \end{aligned}$$

are both isomorphisms. So biduality is considered as the isomorphy of the adjunction morphisms of *contravariant* adjoint functors.

2.2 Dualizing categories

As we could see in the case of Tannaka duality (1.2), there are some evident candidates for dualizing categories, as for example

$$\mathcal{T}_o = (\mathrm{vect}) \quad \text{or} \quad \mathcal{T}_1 := (\mathrm{endvect}).$$

These are tensor categories in an obvious manner. We will work mainly with $\mathcal{T}_o$ as a dualizing category. Let now $\mathcal{C}$ be any category and

$$\mathcal{C}^{\vee} := \mathrm{Hom}(\mathcal{C}, \mathcal{T}_o)$$

the category of covariant functors from $\mathcal{C}$ to $\mathcal{T}_o$. This new category carries also a structure of a tensor category, induced by that of $\mathcal{T}_o$. Moreover, we have natural left and right operations of $\mathcal{T}_o$ on $\mathcal{C}^\vee$. The bidual of $\mathcal{C}$ with respect to $\mathcal{T}_o$, as a category, should now be defined as

$$\mathcal{C}^{\vee\vee} := \mathrm{Hom}_{\mathcal{T}_o,\otimes}(\mathcal{C}^\vee, \mathcal{T}_o)$$

the category of those tensor functors from $\mathcal{C}^\vee$ to $\mathcal{T}_o$ which are compatible with the operations of $\mathcal{T}_o$ on both categories and which are the identity on the constant functors. There is an evalution functor

$$\chi : \mathcal{C} \longrightarrow \mathcal{C}^{\vee\vee}.$$

We call $\mathcal{C}$ *reflexive* (or more precisely *$\mathcal{T}_o$-reflexiv*) if this is an equivalence of categories. Since we did not use the particular definiton of $\mathcal{T}_o$, it could in principle be replaced by any other tensor category. The fully faithfulness of χ is equivalent to a Yoneda type lemma.

Let us define the notion of a $\mathcal{T}_o$-category. An additive or abelian category $\mathcal{M}$ is called a *$\mathcal{T}_o$-category* if there are given two compatible operations of $\mathcal{T}_o$

$$\mathcal{T}_o \times \mathcal{M} \longrightarrow \mathcal{M}, \qquad \mathcal{M} \times \mathcal{T}_o \longrightarrow \mathcal{M}$$

on both sides which satisfy the usual associativity, commutativity and normalisation constraints. An example for this is $\mathcal{T}_o$ itself (with the tensor product "$\otimes$" as left and right operations). This notion can be considered as a categorical generalisation of the notion of a module over a commutative ring. The dual of $\mathcal{M}$ is the category

$$\mathcal{M}^\vee := \mathrm{Hom}_{\mathcal{T}_o}(\mathcal{M}, \mathcal{T}_o)$$

of additiv functors from $\mathcal{M}$ to $\mathcal{T}_o$ which are compatible with the $\mathcal{T}_o$ operations. We obtain again a $\mathcal{T}_o$-category in this way.

These definitons show that the notion of a tensor category can be regarded as a generalisation of that of a commutative ring and $\mathcal{M}$ as a $\mathcal{T}_o$-module.

2.3 Dualizing spaces

We are looking for certain geometric spaces which could serve as dualizing objects for complex projective varieties. So the category $\mathcal{C}$ will be that of all projective not necessarily smooth varieties. We consider an index category I which is monoidal with multiplication "$\cdot$". This muliplication is supposed to be associative and commutative . Moreover, there should exist a neutral element e. Standard examples are $(\mathbf{N}, \cdot)$ and $((\mathrm{vect}), \otimes)$.

Let us consider an I-object P in the category $\mathcal{C}$ or, more generally, in the category of complex spaces.. We have "multiplication" maps

$$P_i \times P_j \longrightarrow P_{ij}$$

which are associativ and commutativ (up to canonical isomorphism) and $P_e = \mathrm{Spec}(\mathbf{C})$ is the neutral object. A *P-(right)space* will be an I-space M in $\mathcal{C}$ with a compatible system of maps

$$M_i \times P_j \longrightarrow M_{ij}$$

over $(i,j) \mapsto ij$. For $j = e$, this is the identity of M_i. If X is compact, then the system

$$M_i := \mathrm{Mor}(X, P_i), \qquad i \in I$$

is in a natural way a P-space of complex spaces which we denote by $X^\vee$. Let $M = (M_i)_{i \in I}$ and $N = (N_i)_{i \in I}$ be two P-spaces. Then

$$\mathrm{Hom}_P(M, N)$$

will be the set of all I-morphisms from M to N which are compatible with the multiplication by P. Every complex space X is in a trivial way a constant P-space which is also denoted by X. So in particular we put

$$X^\vee = \mathrm{Hom}_P(X, P)$$

and $X^\vee$ has a multiplicative structure too. We obtain a biduality map

$$\chi_X : X \longrightarrow \mathrm{Hom}_{P,\cdot}(X^\vee, P)$$

from X into the *multiplicative* P-morphisms from $X^\vee$ to P. This map is induced by the evaluation map in points of X. The mapping χ_X will be injective if the following two conditions are verified

1) *$P_i \to P_j$ is injective for all $i \to j$,*

2) *for any two points x and y in X, there exists $i \in I$ and $f : X \to P_i$ which separates x and y.*

The question of surjectivity of χ_X is much more subtle. We can show the following result

Theorem (2.3.1). *Suppose that the following conditions on P are satisfied*

a) *all $P_i \to P_j$ are injective,*

b) *the multiplication maps $P_i \times P_j \to P_{ij}$ are injective for all i, j,*

c) *for any i, any proper closed subspace $Y \subset P_i$ and any point $p \in P_i \setminus Y$, there exist $j, j' \in I$ and embeddings $\kappa : P_i \to P_{j'}$ and $\lambda : P_j \to P_{j'}$, obtained from compositions of multiplication maps in P, such that*

$$\kappa(Y) \subset \mathrm{Im}(\lambda), \qquad \kappa(p) \notin \mathrm{Im}(\lambda).$$

Then, if for any two point x and y in X, there exists $i \in I$ and $f : X \to P_i$ which separates x and y, the evaluation map χ_X is bijective.

Corollary (2.3.2) *Let I be the category of finite dimensional non zero complex vector spaces with linear injections as morphisms and with the tensor product as multiplication. Then the system of projective spaces*

$$(\mathbf{P}(V))_{V \in I}$$

is dualizing in the category of complex projective varieties (here $\mathbf{P}(V)$ denotes the set of lines in V). The multiplication morphism

$$\mathbf{P}(V) \times \mathbf{P}(W) \longrightarrow \mathbf{P}(V \otimes W)$$

is the Segre embedding.

PROOF of (2.3.2). The first two properties in the theorem are trivial. For the third, one uses for κ a d-uple embedding which maps Y into a hyperplane of some $P_{j'}$ and p outside of it. If $\lambda : P_j \to P_{j'}$ is the linear embedding corresponding to this hyperplan, then conditon (c) is valid, q.e.d.

REMARK (2.3.3). Instead of projective spaces one can also use the system of grassmanians

$$(\mathbf{Grass}_\bullet(V))_{V \in I}$$

for constructing dualizing spaces.

PROOF of (2.3.1). The injectivity of the biduality map is already a consequence of the first two conditions. In order to prove the surjectivity, we assume for the moment that the following condition (*) is verified

(*) *For every $\phi \in \mathrm{Hom}_{P,\cdot}(X^\vee, P)$, we have $\phi(f) \in \mathrm{Im}(f)$ for all $f \in X^\vee$.*

We will show later that this condition is satisfied. Now we have

$$\phi(f) = f(x_f)$$

for every f and some element $x_f \in X$. It is clearly sufficent to show that x_f is independent of f and so ϕ is an evaluation map. Let us fix an injection $g : X \hookrightarrow P_j$ of X which exists under the assumptions of the theorem. We want to show that $x_f = x_g$. For this, we use the following identiies

$$\begin{aligned} \phi(fg) &= \phi(f)\phi(g) \\ &= f(x_f)g(x_g) \end{aligned}$$

$$\begin{aligned} \phi(fg) &= (fg)(x_{fg}) \\ &= f(x_{fg})g(x_{fg}). \end{aligned}$$

By property (b), we conclude that

$$f(x_f) = f(x_{fg}), \qquad g(x_g) = g(x_{fg}) .$$

Since g is injective, we have $x_g = x_{fg}$ and therefore $f(x_f) = f(x_g)$. So x_f is independent of f.

Now we are going to prove the above mentioned condition (*) by contradiction. Suppose that there is $\phi \in \mathrm{Hom}_{P_\cdot}(X^\vee, P)$ and $f : X \to P_i$ such that $\phi(f) \notin \mathrm{Im}(f)$. We put $p := \phi(f)$, so $p \notin Y := \mathrm{Im}(f)$. By (c), there is a commutative diagram

$$\begin{array}{ccccc} X & \stackrel{f}{\longrightarrow} & P_i & \stackrel{\kappa}{\longrightarrow} & P_{j'} \\ & \searrow & \cup & & \uparrow\lambda \\ & f_1 & Y & \underset{\kappa_1}{\longrightarrow} & P_j \end{array}$$

such that $\kappa(Y) \subset \mathrm{Im}(\lambda)$, but $\kappa(p) \notin \mathrm{Im}(\lambda)$. Since κ and λ commute with ϕ, we get

$$\begin{aligned} \phi(\kappa \circ f) &= \kappa(\phi(f)) = \kappa(p), \\ \phi(\lambda \circ \kappa_1 \circ f_1) &= \lambda(\phi(\kappa_1 \circ f_1)) \in \mathrm{Im}(\lambda) \end{aligned}$$

which is a contradiction, q.e.d.

3 The categorial approach

3.1 The tensor category of vector bundles

Let X be a fixed projective variety over $\mathrm{Spec}(\mathbf{C})$ (or any algebraically closed field). We want to consider the tensor category $\mathcal{T}_X := \mathrm{Vect}(X)$ of algebraic vector bundles on X as a dual of the space X. A vector bundle is thought to be a "linear representation" of X. This point of view is indicated by Tannaka duality since a representation of a group G can be considered as a (trivial) vector bundle on G with a G-action. It can be made rigorous if we take an open covering $\mathcal{U}_0$ of X with intersections $\mathcal{U}_1$, so that there is an exact sequence

$$\mathcal{U}_1 \;\rightrightarrows\; \mathcal{U}_o \;\to\; X .$$

The system $(\mathcal{U}_0, \mathcal{U}_1)$ is a groupoïd and a vector bundle which is trivialisable on $\mathcal{U}_0$, can be regarded as a *(linear) representation* of this groupoïd.

We note that $\mathcal{T}_o = \mathcal{T}_{\mathrm{Spec}(\mathbf{C})}$ and $\mathcal{T}_o$ operates on both sides of $\mathcal{T}_X$. Moreover, $\mathcal{T}_o$ is a subcategory of $\mathcal{T}_X$. We define the bidual of X as the set

$$X^{\vee\vee} := \mathrm{Hom}_{\otimes,e}(\mathcal{T}_X, \mathcal{T}_o)$$

of *right exact* tensor functors which are compatible with the $\mathcal{T}_o$ operations and which are the identity on $\mathcal{T}_o \subset \mathcal{T}_X$. We can easily define a biduality map in the following way: We associate to each point $x \in X$ the base change by the inclusion $\mathrm{Spec}(\mathbf{C}) \hookrightarrow X$ and take the restriction of vector bundles to x. We can show the following result

Theorem (3.1.1) *The biduality map*

$$\chi_{_X} : X \longrightarrow X^{\vee\vee}$$

is bijective.

Before proving this, we want to make some general remarks on tensor categories, especially ideals in those. This is important, since we will regard the closed points of X as *maximal* ideals in $\mathcal{T}_X$. For this, let $\mathcal{T}$ be any tensor category over $\mathbf{C}$. An *ideal* $\mathcal{I} \subset \mathcal{T}$ is an additive subcategory, containing all objects $\mathcal{T}$, and which is stable by left and right composition with any morphisms of $\mathcal{T}$. More precisely, this means that

$$\mathcal{I}(a,b) \subset \mathrm{Hom}_{\mathcal{T}}(a,b)$$

is a subvectorspace for any two objects a, b of $\mathcal{T}$ and

$$\mathrm{Hom}(a_1, a) \circ \mathcal{I}(a,b) \circ \mathrm{Hom}(b, b_1) \subset \mathcal{I}(a_1, b_1)$$

for all objects a, b, a_1, b_1 in $\mathcal{T}$. Moreover, we demand that $\mathcal{I}$ should be stable by forming direct sums und by tensor products with arbitrary morphism in $\mathcal{T}$.

Let $\phi : \mathcal{T} \to \mathcal{T}'$ be a morphism of tensor categories. Putting

$$\mathcal{I}(a,b) := \mathrm{Ker}(\mathrm{Hom}_{\mathcal{T}}(a,b) \to \mathrm{Hom}_{\mathcal{T}'}(\phi(a), \phi(b)),$$

for any two objects a, b. we obtain an ideal in $\mathcal{T}$.

Let $\mathcal{T}$ contain $\mathcal{T}_o$ as a subcategory. We define the *spectrum* of $\mathcal{T}$ (or more precisely, the $\mathcal{T}_o$*-spectrum* of $\mathcal{T}$) by

$$\mathrm{Spec}(\mathcal{T}) := \mathrm{Hom}_{\mathcal{T}_o,\otimes}(\mathcal{T}, \mathcal{T}_o)$$

i. e. the set of all right exact tensor functors from $\mathcal{T}$ to $\mathcal{T}_o$ which are the identity on $\mathcal{T}_o \subset \mathcal{T}$. Theorem (3.1.1) tells us that we can identify a (complex) projective variety with the spectrum of its category of vector bundles.

Suppose that $\mathcal{I} \subset \mathcal{T}$ is an ideal. Then the variety of $\mathcal{I}$ is by definition the subset

$$\mathrm{V}(\mathcal{I}) \subset \mathrm{Spec}(\mathcal{T})$$

of those $\mathcal{T}_o$-tensor functors $\phi : \mathcal{T} \to \mathcal{T}_o$ which satisfy $\mathcal{I} \subset \mathrm{Ker}(\phi)$. Conversely, let M be any subset of $\mathrm{Spec}(\mathcal{T})$. We put

$$\mathcal{I}(M) := \bigcap_{\phi \in M} \mathrm{Ker}(\phi)$$

and consider it as the *ideal* defined by M.

We note that there are standard operations on ideals, like

$$\mathcal{I} \cap \mathcal{J}, \quad \mathcal{I} + \mathcal{J}, \quad \mathcal{I} \cdot \mathcal{J}.$$

Only the last product deserves a formal defintion: $(\mathcal{I} \cdot \mathcal{J})(a, b)$ as a subvectorspace of $\mathrm{Hom}(a, b)$, is generated by all compositions of the form

$$a \stackrel{f}{\to} c \stackrel{g}{\to} b$$

with $f \in \mathcal{I}(a, c)$ and $g \in \mathcal{J}(c, b)$. It should be marked that the identity $\mathcal{I} \cdot \mathcal{J} = \mathcal{J} \cdot \mathcal{I}$ does not hold a priorily. But we have the following rules for $\mathcal{I}$ and V

$$\begin{array}{rcl}
\mathrm{V}(\mathcal{I} + \mathcal{J}) & = & \mathrm{V}(\mathcal{I}) \cap \mathrm{V}(\mathcal{J}) \\
\mathrm{V}(\mathcal{I} \cdot J) & \supset & \mathrm{V}(\mathcal{I} \cap J) \supset \mathrm{V}(\mathcal{I}) \cup \mathrm{V}(\mathcal{J}) \\
\mathrm{V}(\mathcal{I}(M)) & \supset & M \\
\mathcal{I}(M \cup N) & = & \mathcal{I}(M) \cap \mathcal{I}(N) \\
\mathcal{I}(M \cap N) & \supset & \mathcal{I}(M) + \mathcal{I}(N) \\
\mathcal{I}(\mathrm{V}(\mathcal{K})) & \supset & \mathcal{K}.
\end{array}$$

In the case $\mathcal{T} = \mathcal{T}_X$, one can show that in addition

$$\mathrm{V}(\mathcal{I}(Y)) = Y$$

for any algebraic subset $Y \subset X$. For these, we also have

$$\mathrm{V}(\mathcal{I}(Y)) \cup \mathrm{V}(\mathcal{I}(Z)) = \mathrm{V}(\mathcal{I}(Y) \cap \mathcal{I}(Z)).$$

PROOF of theorem (3.1.1). Let x_1,x_2 be two points of X. Then there is an ample line bundle L on X and a section s of L, such that $s(x_1) = 0$, but $s(x_2) \neq 0$. We regard s as a homomorphism $s : \mathcal{O}_X \to L$. Then we obtain

$$\chi_X(x_1)(s) \neq \chi_X(x_2)(s)$$

and so χ_X is injective.

The surjectivity is more involved. Let us fix a tensor functor $\phi : \mathcal{T}_X \to \mathcal{T}_o$ which is right exact and which is the identity on $\mathcal{T}_o$. We want to show that ϕ is the evaluation in some point x. For this, let L be an ample line bundle on X. Now

$$\mathbf{C} = \phi(\mathcal{O}) = \phi(L)^\vee \otimes \phi(L)$$

$\dim_{\mathbf{C}} \varphi(L)$ must be 1 which is also true for any other line bundle.

LEMMA. *If L is generated by global sections, there exists $s : \mathcal{O} \to L$ such that $\phi(s) \neq 0$.*

For this, let $s_1, \ldots, s_k$ generate L. So we have a surjection $\mathcal{O}^k \to L$. Since ϕ is right exact, we obtain that $\phi(\mathcal{O})^k \to \phi(L)$ is surjective. Therefore at least one of the $\phi(s_i)$ must be non zero, q.e.d.

We consider first a vector bundle E of rank r on X and the following map induced by ϕ

$$\varinjlim_k \operatorname{Hom}(E, E \otimes L^k) \longrightarrow \varinjlim_k \operatorname{Hom}(\phi(E), \phi(E) \otimes \phi(L)^k)$$

which is a homomorphism of $\mathbf{C}$-algebras. Let $U := X \setminus \mathrm{V}(s)$ be the complement of the zero locus of s. Then U is affine and

$$\operatorname{Hom}(E \mid U, E \mid U) \cong \varinjlim_k \operatorname{Hom}(E, E \otimes L^k)$$

which gives us finally a homomorphism of $\mathbf{C}$-algebras, if E is trivial of rank r

$$\operatorname{Hom}(E \mid U, E \mid \mathrm{U}) \longrightarrow \operatorname{Hom}(\mathbf{C}^r, \mathbf{C}^r).$$

This is necessarily the evaluation map in a point $x \in U$, because it is true for $r = 1$ by Hilbert's Nullstellensatz and for arbitrary r, since ϕ commutes with finite direct sums. It remains to show that ϕ itself is the evaluation in x. First we note that

$$\dim_{\mathbf{C}}(\phi(E)) = \operatorname{rank}(E) = r$$

for any vector bundle E. This is due to the fact that ϕ is exact which implies that it commutes also with exterior products, especially the determinant. As a consequence, the functor ϕ is determined on objects.

Now let E and F be arbitrary vector bundles on X and $k \in \mathbf{N}$ be sufficiently large such that $E \otimes L^k$ and $F \otimes L^k$ are generated by global sections. In order to show that ϕ is the evaluation in x on $\mathrm{Hom}(E, F)$, it is sufficient to treat the case $E = F$ (as ϕ commutes with $\oplus$). The natural map

$$\mathrm{Hom}(\mathcal{O}, E \otimes L^k) \otimes \mathcal{O} \longrightarrow E \otimes L^k$$

is then an epimorphism. Now since $\mathrm{Hom}(\mathcal{O}, E \otimes L^k) \otimes \mathcal{O}$ is a trivial vector bundle, we deduce that ϕ is the evaluation in x on $\mathrm{End}(E \otimes L^k) \cong \mathrm{End}(E)$ and so we are done, q.e.d.

REMARK (3.1.2) It is very natural to ask the question when the spectrum of a given tensor $\mathcal{T}_o$-category carries the structure of a complex projective variety. One would have to axiomize the notion of an ample line bundle in this context (which is, by the way, quite obvious how to do).

Theorem (3.1.1) is also true in the affine and the Stein case. We will treat the last case, since the affine situation is similar and in fact easier. Let X be a Stein complex space and denote by

$$X^{\vee\vee} := \mathrm{Hom}_{\mathcal{T}_o,\otimes}(\mathcal{T}_X, \mathcal{T}_o)$$

the set of *continuous* tensor functors which are $\mathcal{T}_o$-equivariant and the identity on $\mathcal{T}_o$ (right exactness is automatically valid here). The term "continuous" means for the Fréchet topology on vectorspaces $\mathrm{Hom}_{\mathbf{C}}(E, F)$ for any two holomorphic vector bundles E, F on X.

Theorem (3.1.2). *The evaluation map $\chi_X : X \to X^{\vee\vee}$ is bijective.*

PROOF. The injectivity is again trivial. For proving the surjectivity, let $\phi : \mathcal{T}_X \to \mathcal{T}_o$ be a continuous tensor functor. By considering trivial vector bundles, we can immediately extract a continuous character of $\Gamma(X, \mathcal{O}_X)$ which is necessarily given by evaluation in a point x of X. It remains to show that ϕ itself is the evaluation in x. If we want to apply the argument of the projective situation, we have to use the fact that ϕ can be extended to trivial Fréchet bundles of the form $E \widehat{\otimes} \mathcal{O}_X$, where E is a complex Fréchet space. But this is possible by the continuity of ϕ. The rest of the argument remains valid, since the canonical map $\Gamma(X, \mathcal{E}) \widehat{\otimes} \mathcal{O}_X \to \mathcal{E}$ has dense image for every coherent sheaf $\mathcal{E}$, q.e.d.

3.2 The derived category

Instead of working with the category of vector bundles, one can also take into consideration the derived category $\mathrm{D}(X)$ of *right bounded* complexes. We assume here X to be an algebraic variety. In the case where X is a point, this category will be simply denote by D_o. These categories are triangulated categories and they posses a product, the derived tensor product of complexes. Moreover, D_o is in a natural way a subcategory of $\mathrm{D}(X)$. Using only the triangulated structure of $\mathrm{D}(X)$, it is in general not possible to recover X from it (see [4]). On the other hand, if one takes into account the product structure then one can show the following (certainly known) biduality statement

Theorem (3.2.1). *The natural map*

$$X \longrightarrow \mathrm{Hom}_{\mathrm{D}_o,\otimes}(\mathrm{D}(X), \mathrm{D}_o)$$

which maps a point x to Li_x^* is bijective. Here, the right handside denotes the set of all exact functors from $\mathrm{D}(X)$ to D_o which are multiplicative and the identity on D_o.

PROOF. The injectivity is again obvious. For the surjectivity, let us take a tensor functor $\phi : \mathrm{D}(X) \to \mathrm{D}_o$ which is D_o-linear. Clearly, $\Phi(\mathcal{O}_X) = \mathbf{C}$. Next, we want to show that there is at least an affine W such that $\Phi(j_{W!}\mathcal{O}_W) \neq 0$, where j_W denotes the open embedding of W into X. But this is obvious, since $\mathcal{O}_X$ posseses a (left) resolution by direct sums of sheaves of the form at $j_{W!}\mathcal{O}_W$ and Φ is exact. It follows that $\Phi(j_{W!}\mathcal{O}_W) \cong \mathbf{C}$, since Φ is multiplicative. In this way, we get a homomorphism of $\mathbf{C}$-algebras $\Phi_W : \mathrm{End}(j_{W!}\mathcal{O}_W) \to \mathbf{C}$. Now $\mathrm{End}(j_{W!}\mathcal{O}_W) = \mathrm{End}(\mathcal{O}_W)$ and by Hilbert's Nullstellensatz, Φ_W must by the evaluation in some point x. It remains to show that $\Phi \cong \mathrm{L}i_x^*$where i_x denotes the embedding of the point x. But this is evident, since one has resolutions by sums of sheaves of typ $j_{W!}\mathcal{O}_W$ which shows the surjectivity. -

There is another observation to mention which concerns the Fourier-Mukai transform

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$$

given by some coherent sheaf on the product $X \times Y$ as explained at the end of section (1.4), where X and Y are smooth and proper over $\mathrm{Spec}(\mathbf{C})$. If Φ is an equivalence of (triangulated) categories, we can transport the tensor multiplication in $\mathrm{D}(Y)$ to $\mathrm{D}(X)$ and obtain a new product on $\mathrm{D}(X)$. The spectrum in the above sense with respect to this new product will recover Y.

Let us consider the case where the canonical bundle is trivial (which is in fact almost a necessary condition, see [6]). The pairing

$$<,>: \mathrm{D}(X) \times \mathrm{D}(X) \longrightarrow \mathrm{D}_o$$

$$(\mathcal{F}, \mathcal{G}) \mapsto \mathrm{RHom}(\mathcal{F}, \mathcal{G})$$

serves as a "scalar product" in $\mathrm{D}(X)$. It is easy to show that the structure sheaves of points in X form an orthonormal system for this product. If Φ is an equivalence of categories, this orthogonality property (of structure sheaves of points) is preserved. So in order to reconstruct Y out of $\mathrm{D}(X)$, one has to consider orthogonal families of complexes (complete in an certain sense).

4 Moduli spaces as duals

4.1 Moduli spaces of vector bundles and coherent sheaves

We adopt here the point of view that such moduli spaces should serve as duals for algebraic varieties. More precisely, we want to consider moduli spaces of (semi) stable coherent torsion free sheaves on a fixed projective algebraic variety X (with a given ample line bundle). This moduli space will be denote by M or M_r, if we fix in addition the rank equal to r. It is also useful to consider moduli of *simple* coherent sheaves or vector bundles. These exist by the paper [12] or [2] in the algebraic case. In order to define a biduality map, we encounter several problems. The first one is that of the absence of a universal family, in general. This problem could be overcome by modifying the deformation functor. One way out is that of rigidifying the situation, by introducing an additional point a of X and fixing an isomorphism $\varphi : \mathbf{C}^r \to E_a$ for a simple coherent sheaf E of rank r on X. This new moduli space is fibered over the original one, a technique which is quite well known for the Picard functor, but it applies also here. Another way out was proposed by S.Mukai in the appendix of his paper [15]. He introduced the notion of a "quasi-universal" family of vector bundles which means that one allows more general morphisms in the deformation groupoïd (taking into account inverse images of sheaves or bundles on the parameter spaces). In the algebraic setting, such quasi-universal families exist over each component of the moduli space. Another problem for defining a good biduality map is that the bundles obtained on the moduli space by points of X and restricting a universal family, should remain stable or simple. This is not a priorily clear. But a solution can also be found by considering the Harder-Narasimhan filtration for a non semi-stable sheaf and identifying such a sheaf with its associated graded object for this filtration.

In order to see more clearly the essential features, we may regard the moduli space M as a quotient space

$$M = \mathrm{Mor}(X, P)/G$$

where P is a system of Grassmanians (or classifying spaces), parametrised by a category of vector spaces (as in (2.3.3)) and G is a system of reductive or semi-simple affine algebraic groups, operating on P (and therefore also on $\mathrm{Mor}(X, P)$). In order to be more precise, one has to take of course the action of G on the subspace of semi-stable points $\mathrm{Mor(X,P)}_{ss} \subset \mathrm{Mor}(X, P)$. We note that P allows internal operations $\otimes, \oplus, {}^{\vee}$, which are compatible with the G-action. Evidently, there is a tautological bundle on $X \times P$, the universal quotient bundle. Moreover, under rather mild assumptions on the system P, the natural map

$$X \longrightarrow \mathrm{Hom}_{P,\cdot}(\mathrm{Mor}(X, P), P)$$

is bijective (see (2.3.)). By definition of the index category, every element on the right, is automatically G-equivariant (from both sides). We want to work mainly with this operation and therefore change the context in the following way:

Let $P = (P_{\mathrm{V}})_{\mathrm{V}}$ be a system of projective varieties, parametrized by all finite dimensional non zero $\mathbf{C}$-vector spaces with multiplication morphisms $\cdot : P_{\mathrm{V}} \times P_{\mathrm{W}} \to P_{\mathrm{V}\otimes\mathrm{W}}$ (which are commutative and associative up to canonical isomorphisms). Moreover, we fix a system $G = (G_{\mathrm{V}})_V$ of algebraic groups also with multiplication maps as above and such that G operates transitively on P in a compatible way (as a (V)-system). The associated biduality map

$$X \longrightarrow \mathrm{Hom}_{G,\cdot}(\mathrm{Mor}(X, P), P)$$

from X to the G-equivariant and multiplicative homomorphisms will be bijective under the analogous conditions of theorem (2.3.1) which are verified in the case of Grassmanians or projective spaces and for $G_{\mathrm{V}} = \mathrm{Sl(V)}$ with the usual multiplication and operation maps. We will consider the system of quotients

$$\mathrm{Mor}(X, P)/G$$

which is parametrized by the V's. In order to guarante the existence, one would have again to take only the semi-stable points in$\mathrm{Mor}(X, P)$. The biduality map which we are interested in, is

$$X \longrightarrow \mathrm{Mor}(\mathrm{Mor}(X, P)/G,\ P)/G.$$

We introduce he following condition

(S) *Assume that the components of* $M = \mathrm{Mor}(X, P)/G$ *are compact (or at least concave in the analytic context) and that the quotient map* $\mathrm{Mor}(X, P) \to M$ *has a global section which is compatible with the operations* $\otimes, \oplus, ^\vee$. *Moreover, let* G *operate on* $\mathrm{Mor}(X, P)$ *without fixed points.*

Then we can show

Theorem (4.1.1). *The biduality map* $X \to \mathrm{Mor}(M, P)/G$ *is surjectiv if condition* (S) *is satisfied (here* $\mathrm{Mor}(M, P)$ *are those families of* (V)-*morphisms which commute with* $\otimes, \oplus, ^\vee$*).*

PROOF. Let us take a homomorphism $\phi : M \to P$. By condition (S), we can regard M as a subspace of $\mathrm{Mor}(X, P)$, invariant by the three standard operations. Using the operation of G, we can extend ϕ to a homomorphism $\widetilde{\phi} : \mathrm{Mor}(X, P) \to P$. In this way, we get an inclusion

$$\mathrm{Mor}(M, P) \longrightarrow \mathrm{Hom}_{G,\cdot}(\mathrm{Mor}(X, P), P)$$

which is bijective, but which depends on a chosen section of $\mathrm{Mor}(X, P) \to M$. Passing to the quotient by the operation of G, we get the assertion of the theorem, q.e.d.

REMARK (4.1.2). One cannot expect more than the surjectivity (as one sees for $X = \mathbf{P}^1_{\mathbf{C}}$).

Instead of passing to the quotient, we could replace the G-space $\mathrm{Mor}(X, P)$ by its bar construction. We call this the "derived" approach. It has been investigated in [7],[8] in a different context. This approach seems to be more natural than the one, used above.

4.2 About the formal algebro-geometric structure of moduli spaces

Let X be a projective algebraic variety with an ample line bundle L. We consider the moduli space M_r of stable torsion free sheaves of rank r on X and $\overline{M}_r$ that of semi-stable torsion free sheaves of rank r (for a nice construction of these spaces, see [19]). We put $\partial M_r = \overline{M}_r \setminus M_r$. Then there are natural maps

$$\begin{aligned} \oplus : \overline{M}_r \times \overline{M}_s &\longrightarrow \overline{M}_{r+s} \\ \otimes : \overline{M}_r \times \overline{M}_s &\longrightarrow \overline{M}_{rs} \end{aligned}$$

$$^\vee : \overline{M}_r \longrightarrow \overline{M}_r$$

induced by the corresponding operations on stable bundles. These maps satisfy the usual identities of commutativity, associativity and distributivity. So we have in particular an $\mathbf{N}$-space $\mathcal{M}_\bullet$ (in fact more than this, since there is the operation "dual sheaf").

By viewing torsion free sheaves on X as "singular" representations of X, we are lead to define the notion of a representation of such a space $\mathcal{M}_\bullet$. We encounter of course the same problems already mentioned in section 4.1 (the absence of universal families for example). In any case, one will have to take into consideration the structure of the boundary (where one takes filtrations with stable quotients). If we denote by M the disjoint union of all the M_r, the addition operation gives us a natural map for the symmetric product

$$\mathrm{S}_{>1}(M) \longrightarrow \partial M$$

into the boundary which is in fact bijective. One would like to associate to the system $(\overline{M}_r)_r$ of locally algebraic varieties a limit space, denote by $\mathcal{M}$. It should be connected.

Instead of sheaves, one could also consider the cycle space $\mathcal{C}^\bullet(X)$ of X which does, unfortunately, not completely fit into this picture, because of the failure of a good intersection theory. Nevertheless, the formal algebraic structure is very similar. The *prime* cycles of codimension r play the role of M_r. It seems that, for our purposes, the cycle space is not the good object, that one is finally looking for (see also remark (4.4.4)). Only geometric intuition keeps us attached to these moduli spaces which do not admit standard algebraic operations.

There is another example of a moduli space with a mulitplicative structure, namely that of stable pointed curves of variable genus $\overline{\mathcal{M}}_{\bullet,\bullet} = (\overline{\mathcal{M}}_{g,n})_{g,n}$.

4.3 Duality via nonabelian group cohomology

Moduli spaces are often constructed as quotients of an affine variety by the action of some group. So vector bundles on such moduli spaces can be described by nonabelian (analytic) group cohomology cocycles. Group representations are special cocycles. We intend to make this more precise in this section, but which still remains to be completed in details.

We start with a general context. Let G and A be groups such that G operates on A by group homomorphisms. We consider the natural map of nonabelian group cohomology, induced by the inclusion $A^G \to A$

$$\kappa : \mathrm{H}^1(G, A^G) \longrightarrow \mathrm{H}^1(G, A). \tag{8}$$

Moreover, one has maps

$$\begin{aligned} \iota : \mathrm{Hom}(G, A^G) &\longrightarrow \mathrm{Z}^1(G, A), \\ \bar{\iota} : \mathrm{Hom}(G, A^G)/\sim &\longrightarrow \mathrm{H}^1(G, A) \end{aligned}$$

where $'\sim'$ means conjugation by elements of A^G. We can trivially identify $\mathrm{Hom}(G, A^G)$ with $\mathrm{Z}^1(G, A^G)$, so $\bar{\iota}$ coincides with the map (8). In geometric examples and under conditions on the group G, the image of $\bar{\iota}$ is a distinguished "component" of $\mathrm{H}^1(G, A)$. Let us be more precise and take an operation of G on an "affine space" Z such that the quotient $M = Z/G$ exists (say in the category of complex spaces). We consider vector bundles (of rank r) on M which become trivial after pulling back on Z. The isomorphy classes of such vector bundles can be described by the cohomology set

$$\mathrm{H}^1(G,\ \mathrm{Mor}(Z, \mathrm{Gl}_r(\mathbf{C}))).$$

So we take here $A = \mathrm{Mor}(Z, \mathrm{Gl}_r(\mathbf{C}))$.

Let us fix a compact complex manifold X with a (finite) open covering $\mathcal{U} = (U_i)_{i\in I}$. Vector bundles (of rank r) which are trivialized over $\mathcal{U}$ are classified by the Čech-cocycle set

$$Z^1 := \mathrm{Z}^1(\mathcal{U}, \mathrm{Gl}_r(\mathcal{O}_X))$$

and there is an operation on it by conjugation with elements of the group $G := \mathrm{C}^o(\mathcal{U}, \mathrm{Gl}_r(\mathcal{O}_X))$. We take an open G-invariant subset $Z \subset Z^1$, such that the quotient M by this operation exist. There is a tautological G - bundle over the space $X \times Z$, obtained by gluing together the pieces $U_i \times Z \times \mathbf{C}^r$. It is known that this bundle does not always descend to the quotient M. Let us assume now that M is *compact*. Then we know that

$$A^G = \mathrm{Gl}_r(\mathbf{C}).$$

We have the following commutative diagram

$$\begin{array}{ccc} \coprod_i U_i & \hookrightarrow & \mathrm{H}^1(G, A^G) \\ \downarrow & & \downarrow \kappa \\ X & \longrightarrow & \mathrm{H}^1(G, A) \end{array}$$

where the horizontal arrows are given by evaluation in points. The upper one is injective (by Schur's lemma), but also locally surjective (by a deformation rigidity argument for an evaluation homomorphism). We are interested in the local sujectivity of the lower horizontal arrow. For this, we specify the above situation: Let Z consist only of cocycles, definig *simple* bundles. Moreover, we fix a point a of X. The group G can be replaced by the normal subgroup G_a, consisting of those matrices, giving the unit matrix in a. Then the quotient $N := Z/G_a$ is a $\mathrm{PGL}_r(\mathbf{C})$-fiber bundle over M and $G/G_a \cong \mathrm{Gl}_r(\mathbf{C})$. We expect the following fundamental property to hold

(4.3.1) *If there are only constant holomorphic functions on N, the evaluation map*

$$X \to \mathrm{H}^1(G, \mathrm{Mor}(Z, \mathrm{Gl}_r(\mathbf{C})))$$

is locally surjective .

We indicate a possible proof: By applying the implicit function theorem and the above diagram, we are reduced to the consideration of the tangent map of κ in (the image of) a. Defining $V := \mathrm{Mor}(Z, \mathrm{M}_{r\times r}(\mathbf{C}))$, then one has to show the surjectivity of the comparison map

$$\mathrm{H}^1(G_a, V^{G_a})^{G/G_a} \to \mathrm{H}^1(G_a, V)^{G/G_a}.$$

For this argument , we consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(G, (V^G)_\chi) & \longrightarrow & \mathrm{H}^1(G, (V)_\chi) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(G_a, V^G)^{G/G_a} & \longrightarrow & \mathrm{H}^1(G_a, V)^{G/G_a} \\ \downarrow & & \updownarrow \\ \mathrm{H}^1(G_a, V^{G_a})^{G/G_a} & \longrightarrow & \mathrm{H}^1(G_a, V)^{G/G_a} \end{array}$$

Here χ denotes the twist by the evaluation in a (this twist becomes trivial after restricting to G_a). According to our assumption, $V^G = V^{G_a}$. Since the vertical arrows are bijective (by the five term sequence in Serre's spectral sequence for group cohomology, the vanishing of G/G_a-cohomology and the assumption on N), we are reduced to G_a case.

There are two observations which should imply the above surjectitvity in our case (for the G'-invariant part): We filter the group G_a by the descending sequence of normal subgroups G_i, defined by being the unit matrix in a up to the order at least i in the point a, so $G_1 = G_a$. Note also that each G_i is topologically generated as a group by G_1. We would like to replace G_1 by some G_1/G_i, in order to work with ordinary algebraic groups. We intend to show that the canonical map

$$\mathrm{H}^1(G_1, V) \longrightarrow \lim_{\to i} \mathrm{H}^1(G_i, V)$$

is *zero* (by the five term sequence), for the G/G_a-invariant part at least. This property should follow, after introducing norms, from a Schwarz type lemma for cocyles vanishing in a, and the possibility of shrinking the covering $\mathcal{U}$ (without changing the cohomology !). The second observation is an elementary induction argument for the G_1/G_i-version of the above linear surjectivity (using again the five term sequence for group cohomology): For this, let G be an algebraic group (of the above type) which operates on V such that V^G is trivial. Then the first group cohomology of G with values in V vanishes (note that all G_i/G_{i+1} are direct sums of the complex *additiv* group).

A more general approach to duality in this context should be possible by using the nonabelian cohomology theory of C.Simpson ([20], [21]). We will treat the derived (or simplicial) version of the above method in a second paper.

4.4 Duality via Hilbert schemes

Let X be a projective algebraic variety. As a dual of X, we will take here the Hilbert scheme $\mathrm{H}(X)$ of X. A point in $\mathrm{H}(X)$ will be denote by $[I]$ where $I \subset \mathcal{O}_X$ is a coherent ideal sheaf. The components of $\mathrm{H}(X)$ are again projective and the *set* $\mathrm{H}(X)$ carries the structure of an abelian semi-ring (with unity) by the mappings

$$\begin{array}{rcl} + : \mathrm{H(X)} \times \mathrm{H}(X) & \longrightarrow & \mathrm{H}(X) \\ [I], [J] & \mapsto & [I+J] \\ \cdot : \mathrm{H(X)} \times \mathrm{H}(X) & \longrightarrow & \mathrm{H}(X) \\ [I], [J] & \mapsto & [IJ]. \end{array}$$

We remark that these maps are in general *not* morphisms of locally algebraic varieties. They cannot be defined on the level of the Hilbert functor (since the flatness of families is in general not preserved by these operations). This is a severe disadvantage and it seems to be rather evident that the Hilbert scheme is not the right object for a good functorial intersection theory. One would certainly have to take into account also resolutions of ideal sheaves and drop the flatness condition for families. Since the derived version of this theory will be the subject of another paper, we will here confine ourselves with the usual classical Hilbert scheme $\mathrm{H}(X)$.

For defining a double dual, we have to attribute a sense to the symbol $\mathrm{H}(\mathrm{H}(X))$. A point in this space has to be a closed algebraic subspace of $\mathrm{H}(X)$ which is not necessarily compact. We have to take all of them as we shall see. This new space is not a scheme, but at a projective limit of schemes.

There is a natural biduality map

$$X \longrightarrow \mathrm{H}(\mathrm{H}(X))$$

which associates to a point $x \in X$, the set of all ideal sheaves $I \subset \mathcal{O}_X$ such that $x \in \mathrm{V}(I)$. Indeed, this is a closed subspace of $\mathrm{H}(X)$, namely the fiber over x of the incidence variety

$$W \subset X \times \mathrm{H}(X)$$

given by the condition $y \in \mathrm{V}(I)$ for $\{y\} \times [I] \in X \times \mathrm{H}(X)$. The biduality map is now $x \mapsto W_x$. Obviously, W_x is not compact in general. It is an ideal in the semi-ring $\mathrm{H}(X)$, which is maximal.

Let us now take an ideal $A \subset \mathrm{H}(X)$ and a subset $M \subset X$. Following the usual pattern, we put

$$\begin{aligned} \mathrm{V}(A) &:= \bigcap_{[I] \in A} \mathrm{V}(I) \\ \mathrm{Id}(M) &:= \{[I] \in \mathrm{H}(X) \mid \mathrm{V}(I) \supset M\}. \end{aligned}$$

Then $\mathrm{V}(A)$ is a closed subspace of X and $\mathrm{Id}(M)$ an ideal of $\mathrm{H}(X)$. In our example $A = W_x$, we have $\mathrm{V}(W_x) = \{x\}$. Setting

$$\widetilde{M} := \mathrm{V}(\mathrm{Id}(M))$$

we obtain the smallest algebaic subset of X, containing M. We state some properties which follow easily from the definitions

$$\begin{aligned} A = \mathrm{H}(X) &\Leftrightarrow [\mathcal{O}_X] \in A \\ \mathrm{V}(\mathrm{H}(X)) &= \emptyset \\ A \subset B &\Rightarrow \mathrm{V}(\mathrm{A}) \supset \mathrm{V}(B) \\ \mathrm{V}(A+B) &= \mathrm{V}(\mathrm{A}) \cap \mathrm{V}(B) \\ \mathrm{V}(AB) &= \mathrm{V}(A \cap B) = \mathrm{V}(\mathrm{A}) \cup \mathrm{V}(B) \\ A &\subset \mathrm{Id}(\mathrm{V}(A)). \end{aligned}$$

Defintion (4.4.1). An ideal $A \subset \mathrm{H}(X)$ is called *satured* if it satisfies the condition

$$J \subset I, \quad [I] \in A \Rightarrow [J] \in A.$$

Examples for such ideal are W_x and $\mathrm{Id}(M)$.

Theorem (4.4.2) (Nullstellensatz). *Let $A \subset \mathrm{H}(X)$ be a satured ideal, then $\sqrt{A} = \mathrm{Id}(\mathrm{V}(A))$.*

Proof. One inclusion is clear ("$\subset$"). For the other one, we note that

$$\begin{aligned}\mathrm{Id}(\mathrm{V}(A)) &= \{I \mid \mathrm{V}(I) \supset \cap_{[K]\in A}\mathrm{V}(K)\} \\ &= \{I \mid \mathrm{V}(I) \supset \mathrm{V}(\Sigma_{[K]\in A}K)\} \\ &= \{I \mid \exists\, k \in \mathbf{N} :\ I^k \subset \Sigma_{[K]\in A}K\} \\ &\subset \sqrt{A}\end{aligned}$$

since A is satured, q.e.d.

Theorem (4.4.3). *The biduality map above gives a bijection between X and the maximal ideals of* $\mathrm{H}(X)$.

Proof. The injectivity is obvious. Let now A be a maximal ideal of $\mathrm{H}(X)$ and put $M := \mathrm{V}(A)$. If M is empty, then $[\mathcal{O}_X] \in A$. If M contains more than one point, then A cannot be maximal, so $M = \{x\}$. Now we have $A \subset \mathrm{Id}(x) = W_x$ and finally $A = \mathrm{Id}(x)$, q.e.d.

Remark (4.4.4). Since the operation on Hilbert schemes, induced by intersecting subspaces, is *not* an algebraic map, it seems to be more natural to consider another, bigger moduli space, giving the Hilbert scheme as a quotient. Indeed, one should replace an ideal sheaf by a resolution, work with complexes and, moreover, drop the condition of flatness in families. This approach will be treated in a forthcoming paper.

5 Addenda

5.1 Semi-simple coherent sheaves

Let X be a connected reduced compact complex space and E a torsionfree coherent sheaf on X.

Definition (5.1.1). We call E *semi-simple* if it satisfies the following condition: For any endomorphism φ of E, kernel and image of φ are direct summands of E.

There is the following algebraic characterization of semi-simple sheaves:

Proposition (5.1.2). *The sheaf E is semi-simple if and only if the endomorphism algebra* $\mathrm{End}_X(E)$ *is a semi-simple algebra.*

Proof. The property in definition (5.1.1) can be reformulated by the existence of another endomorphism ψ such that $\varphi\psi\varphi = \varphi$ (we call ψ a *splitting* of φ). In this case, $\varphi \neq 0$ cannot belong to the radical of $\mathrm{A} := \mathrm{End}_X(E)$, since it is not strongly nilpotent, so E semi-simple implies A semi-simple. If now A is semi-simple, then one applies the structure theorem for such algebras which says that here A is a direct product of matrix algebras over the field $\mathbf{C}$. But then the existence of a splitting ψ for any φ is obvious, q.e.d.

Proposition (5.1.3). *Suppose that X is irreducible. Then every torsion-free sheaf E has a (essentially unique) filtration $(E_\bullet)$ such that each $\mathrm{gr}_i(E)$ is simple.*

PROOF. We apply induction on the rank r of E. The case $r = 1$ is clear. Suppose now $r \geq 2$ and that E itself is not simple, so there is some non constant endomorphism φ of E. Putting $I := \mathrm{Ker}(\varphi)$, we see that rank $(I)$and rank (E/I) are both strictly smaller than r (using the determinant of φ). By induction hypothesis, we get the desired filtration. The unicity follows from a standard argument. -

Corollary (5.1.4). *Any torsionfree coherent sheaf E can be deformed to a direct sum of simple sheaves.*

For the proof, one uses the general fact that a filtered object can always be deformed to its associated graded object as special fibre (with parameter space $\mathrm{Spec}(\mathbf{C}[T])$).

REMARK (5.1.5). It should be noted that, in general, there are semi-stable bundles which are not semi-simple. Let C be, for example, an elliptic curve. A nontrivial cohomology class in $\mathrm{H}^1(C, \mathcal{O}_C)$ gives a non trivial extension of the trivial bundle by itself. This bundles is always semi-stable but not semi-simple.

PROBLEM (5.16). *Is there a moduli space of semi-simple sheaves on X ?*

5.2 Moduli of vector bundles on manifolds with trivial canonical bundle and duality

We indicate briefly how to associate to a compact complex manifolds X with trivial K_X another complex manifold of the same dimension and with trivial canonical bundle. For this, we use moduli spaces of very special vector bundles. The idea is rather simple: We consider only those torsionfree sheaves $\mathcal{E}$ for which the Ext-algebra is naturally an exterior algebra. More precisely, this means that

$$\mathrm{Ext}^\bullet_X(\mathcal{E}, \mathcal{E}) \cong \Lambda^\bullet \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E})\,. \tag{9}$$

So in particular $\mathcal{E}$ is a simple sheaf. If it is locally free, we deduce by Serre duality that

$$\mathrm{Ext}^n_X(\mathcal{E},\mathcal{E}) \cong \mathrm{Ext}^0_X(\mathcal{E},\mathcal{E}) \cong \mathbf{C}$$

where n is the dimension of X, and so necessarily

$$\mathrm{dim}\mathrm{Ext}^1_X(\mathcal{E},\mathcal{E}) = \mathrm{dim}X\ .$$

This moduli space of bundles of that type is smooth, since the condition (9) is open for families. The anti-canonical bundle of this moduli space is induced by the assignement

$$\mathcal{E} \mapsto \Lambda^n \mathrm{Ext}^1_X(\mathcal{E},\mathcal{E})$$

which identifies canonically with $\mathrm{Ext}^n_X(\mathcal{E},\mathcal{E}) \cong \mathbf{C}$, so it is trivial. It seems to be unknow if this moduli space is in general non empty or has compact components (or if it is simply connected for a Calabi-Yau manifold X; but in this last case, it is known that at least their derived categories are equivalent, see [6],[13]).

Condition (9) is even necessary if one wants the Fourier-Mukai transform to be fully faithful between the derived category of the considered moduli space of simple sheaves and that of X ([5], Theorem (1.1)), assuming the existence of a universal sheaf on the product.

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