QUANTUM STOPPING TIMES
AND QUASI-LEFT CONTINUITY

by Stéphane ATTAL & Agnès COQUIO

ABSTRACT. — We study the general properties of quantum stopping times on Hilbert spaces equipped with a filtration. We define and investigate notions such as the spaces of anterior events, the spaces of strictly anterior events, the property $S < T$ for two stopping times, predictable stopping times,... . When applied to the particular case of the symmetric Fock space Φ over $L^2(\mathbb{R}^+)$ (the natural space for quantum stochastic calculus) we show an intrinsic quasi-left continuity property shared by all the probabilistic interpretations of Φ. We finally apply these definitions to the case of a non commutative stochastic base $(\mathcal{U}_\infty, \Phi, (\mathcal{U}_t)_{t \geq 0}, (M_t)_{t \geq 0})$ where $\mathcal{U}_\infty$ is a von Neumann algebra, generated by an increasing family of sub-von Neumann algebras $(\mathcal{U}_t)_{t \geq 0}$, Φ is a normal, faithfull state and $(M_t)_{t \geq 0}$ is a family of conditional expectations on $(\mathcal{U}_t)_{t \geq 0}$. We show, in this context, that the fermionic Fock space over $L^2(\mathbb{R}^+)$, the quasi-free boson and fermion spaces are also quasi-left continuous.

I. Introduction

The fundamental importance of stopping times in the classical theory of stochastic processes does not need to be demonstrated any more. That is a reason why the difficulties to define a serious and efficient theory of stopping times in the framework of quantum processes can be felt as an obstacle to important developments.

The theory of quantum processes and quantum noises has had an impressive development since the last 25 years and has found many deep applications in quantum physics. For example, the quantum statistical description of the dilation of the dynamic of a quantum open system, with the help of quantum noises, is one of the most remarkable application of quantum probability theory ([H-P]).

The theory of quantum statistical mechanics is now having a very quick development and follows in parallel the tracks of the older (classical) statistical mechanics theory: Gibbs states become K.M.S. states, generators of Feller processes become Lindblad generators of quantum dynamical semigroups,.... The important open problems in quantum statistical mechanics are those of return to equilibrium, recurrence, existence of invariant states, spectral gaps,.... It is well-known that the most remarkable answers to the corresponding problems in the classical theory were obtained with the help of Markov processes and stopping time theory.

But why are stopping times so difficult to handle in quantum theory? The first obstacle is physical and philosophical. It is very delicate (and sometime taboo) to associate an observable with the time when something happens in a quantum system. The main reason is that in order to observe such a time one should be continuously monitoring the system. This is of course very delicate in quantum mechanics as the system is definitely affected by the observation. The brutal continuous observation of a quantum system leads to surprising consequences such as freezing the system in the initial state (quantum Zeno effect). But on the other hand it is also true that the theory of continuous observation of quantum systems has made impressive progresses recently, both theoretically and experimentally (non-demolition measurement).

It is even more remarkable that, forgetting the physical constraints, it is very easy to mathematically associate an observable to the time when some even occurs in a quantum system. For example, when studying the quantum stochastic differential equations describing the dynamics of some quantum open systems (such as in quantum optics for example), one can exhibit abelian subalgebras of observables which are invariant under the dynamic. This thus gives rise to commutative processes which can be realised (diagonalised) on some probability space. As a consequence, any classical stopping time associated to this process (exit times, hitting times,...) gives rise to a quantum stopping time when pulled back in the general setup.

The theory of quantum stopping times has been initiated by R. L. Hudson ([Hud]) in the framework of Fock space. The basic idea is to say that a classical stopping time is a positive random variable (with value $+\infty$ admitted) which satisfies some adaptedness property with respect to a given filtration of σ-fields. Thus a quantum stopping time is a quantum random variable (a self-adjoint operator on a spectral measure) which is positive, which admits the value $+\infty$ and which satisfies some adaptedness property on the Fock space. This theory has been developed by several authors in the same framework: [App], [P-S], [A-S], [Att], but also in the framework of filtered families of σ-finite and finite von Neumann algebras: [BN1], [BN2], [B-L],....

In this article, a stopping time is an increasing family of projections on a filtered

2

Hilbert space, adapted to an increasing family of algebras. This approach thus covers all the preceding cases.

In the five first parts we study stopping times on filtered Hilbert spaces $(\mathcal{H}, (\mathcal{H}_t)_{t \geq 0})$, without really mentioning algebras and we define for a stopping time T, the spaces $\mathcal{H}_T$, $\mathcal{H}_{T-}$, the property $S < T$ for two stopping times and the notion of previsible stopping time.

In the last two parts, we apply the preceding definitions and results to the cases of the symmetric Fock space, the antisymmetric Fock space over $L^2(\mathbb{R}^+)$ and the quasi-free representations of the CCR and CAR. We prove that in all these cases the quasi-left continuity property is verified.

II. Quantum stopping times

A *filtered Hilbert space* is a complex separable Hilbert space $\mathcal{H}$ together with a family of orthogonal projections $(E_t)_{t \in \mathbb{R}^+}$ with range $(\mathcal{H}_t)_{t \in \mathbb{R}^+}$ satisfying

   i) $\mathcal{H}_0 \simeq \mathbb{C}$ and $s - \lim\limits_{t \to +\infty} E_t = I$ (*i.e.* $\bigwedge\limits_{t \in \mathbb{R}^+} \mathcal{H}_t = \mathcal{H}$).

   ii) $E_s E_t = E_t E_s = E_s$ for all $s \leq t$ (*i.e.* $\mathcal{H}_s \subset \mathcal{H}_t$ for $s \leq t$).

   iii) $s - \lim\limits_{\substack{u \to t \\ u > t}} E_u = E_t$ (*i.e.* $\bigcap\limits_{u > t} \mathcal{H}_u = \mathcal{H}_t$).

We write $\mathcal{H}_{t-} = \bigvee\limits_{s < t} \mathcal{H}_s$ and E_{t-} is the orthogonal projection onto $\mathcal{H}_{t-}$, $t \in \mathbb{R}^+$ (with the convention $\mathcal{H}_{0-} = \mathcal{H}_0$).

An operator X on $\mathcal{H}$ is said to be *adapted at time t* if

   i) $E_u(\mathrm{Dom}\, X) \subset \mathrm{Dom}\, X$, for all $u \geq t$,

and

   ii) $E_u X = X E_u$ on $\mathrm{Dom}\, T$, for all $u \geq t$.

A *stopping time* (or *quantum stopping time*) T on a filtered Hilbert space $(\mathcal{H}, (E_t)_{t \in \mathbb{R}^+})$ is a (right-continuous) spectral measure on $\mathbb{R}^+ \cup \{+\infty\}$ with values in the set of orthogonal projectors on $\mathcal{H}$, such that, for all $t \in \mathbb{R}^+$, the operator $T([0, t])$ is adapted at time t.

In the following we adopt probabilistic-like notations: for every Borel subset $E \subset \mathbb{R}^+ \cup \{+\infty\}$ we write $\mathbb{1}_{T \in E}$ instead of $T(E)$. In the same way $\mathbb{1}_{T \leq t}$ means $T([0, t])$, $\mathbb{1}_{T = t}$ means $T(\{t\})$, ….

3

Note that in particular

$$\mathbb{1}_{T \leq t} = s - \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \mathbb{1}_{T \leq t + \varepsilon} \,,$$

and

$$\mathbb{1}_{T < t} = s - \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \mathbb{1}_{T \leq t - \varepsilon} \,.$$

In particular, $\mathbb{1}_{T < t}$ is also adapted at time t, for all $t \in \mathbb{R}^+$.

Let us see briefly how this definition connects to the classical one. If T is a classical stopping time on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, P)$, then taking $\mathcal{H} = L^2(\Omega, \mathcal{F}, P)$, $\mathcal{H}_t = L^2(\Omega, \mathcal{F}_t, P)$ (if we assume that the filtration verifies the "usual conditions") and $E_t = \mathbb{E}[\cdot / \mathcal{F}_t]$ makes up a filtered Hilbert space. The operators $\mathcal{M}_{\mathbb{1}_{T \leq t}}$ of multiplication by $\mathbb{1}_{(T \leq t)}$ on $\mathcal{H}$ then define a quantum stopping time on $\mathcal{H}$.

Conversely, if T is a quantum stopping time on a filtered Hilbert space $(\mathcal{H}, (E_t)_{t \in \mathbb{R}^+})$ then the operators $\mathbb{1}_{T \leq t}$, $t \in \mathbb{R}^+ \cup \{+\infty\}$, are two by two commuting. Thus they simultaneously diagonalise on a probability space $(\Omega, \mathcal{F}, P)$ to give rise to operators of multiplication by indicator functions of the form $\mathbb{1}_{(\tau \leq t)}$ for some random variable τ valued in $\mathbb{R}^+ \cup \{+\infty\}$. Taking $\mathcal{F}_t$ to be the σ-field generated by the image of $\mathcal{H}_t$ into $L^2(\Omega, \mathcal{F}, P)$, make τ being a classical stopping time.

Thus when considering one quantum stopping time (on a commuting family of quantum stopping times) leads to a theory which is exactly equivalent to the classical one. Of course the difference appears when considering several non-commuting stopping times on $\mathcal{H}$. Each of them can be individually interpreted classically, but not together. They come from different probabilistic context and they are put together in the same context, exactly like observables in quantum mechanics.

A point $t \in \mathbb{R}^+$ is a *continuity point* for a quantum stopping time T if $\mathbb{1}_{T = t} = 0$. Note that as $\mathcal{H}$ is separable, then any stopping time T admits an at most countable set of points which are not of continuity for T. Also note that if t is a continuity point for T then the map $s \mapsto \mathbb{1}_{T \leq s}$ is strongly continuous at t.

A stopping time T is *discrete* if there exists a finite set $E = \{0 \leq t_1 < t_2 < \cdots < t_n \leq +\infty\}$ in $\mathbb{R}^+ \cup \{+\infty\}$ such that $\mathbb{1}_{T \in E} = I$.

A sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ is said to *converge* to a stopping time T if $s - \lim_{n \to +\infty} \mathbb{1}_{T_n \leq t} = \mathbb{1}_{T \leq t}$ for all continuity point t of T.

A stopping time T is *finite* if $\mathbb{1}_{T = +\infty} = 0$.

Two stopping times S, T satisfy $S \leq T$ if $\mathbb{1}_{S \leq t} \geq \mathbb{1}_{T \geq t}$ for all $t \in \mathbb{R}^+$ (in the sense of comparison of projectors). In particular

$$\mathbb{1}_{T \leq t} = \mathbb{1}_{S \leq t} \mathbb{1}_{T \leq t} = \mathbb{1}_{T \leq t} \mathbb{1}_{S \leq t}$$

4

for all $t \in \mathbb{R}^+$.

Let S and T be two stopping times on $\mathcal{H}$. We define the stopping times $S \vee T$ and $S \wedge T$ on $\mathcal{H}$ by, for all $t \in \mathbb{R}^+$,

$$\mathbb{1}_{S \vee T \leq t} = \quad \text{orthogonal projection onto} \quad \mathrm{Ran}\,\mathbb{1}_{S \leq t} \cap \mathrm{Ran}\,\mathbb{1}_{T \leq t}$$

$$\mathbb{1}_{S \wedge T \leq t} = \quad \text{orthogonal projection onto} \quad \mathrm{Ran}\,\mathbb{1}_{S \leq t} \vee \mathrm{Ran}\,\mathbb{1}_{T \leq t}\,.$$

PROPOSITION 1. — *For any two stopping times S, T on $\mathcal{H}$ we have*

i) $S \wedge T \leq S$ and $S \wedge T \leq T$.

ii) $S \vee T \geq S$ and $S \vee T \geq T$.

iii) *If U is any stopping time satisfying $U \leq S$ and $U \leq T$ then one has $U \leq S \wedge T$.*

iv) *If U is any stopping time satisfying $U \geq S$ and $U \geq T$ then one has $U \geq S \vee T$.*

Proof.

i) We have $\mathrm{Ran}\,\mathbb{1}_{S \wedge T \leq t} \supset \mathrm{Ran}\,\mathbb{1}_{S \leq t} \cup \mathrm{Ran}\,\mathbb{1}_{T \leq t}$. One concludes easily. The proof of *ii)* is identical.

iii) If $U \leq S$ and $U \leq T$ then $\mathrm{Ran}\,\mathbb{1}_{U \leq t} \supset \mathrm{Ran}\,\mathbb{1}_{S \leq t} \cup \mathrm{Ran}\,\mathbb{1}_{T \leq t}$. Thus $\mathrm{Ran}\,\mathbb{1}_{U \leq t} \supset \mathrm{Ran}\,\mathbb{1}_{S \leq t} \vee \mathrm{Ran}\,\mathbb{1}_{T \leq t} = \mathrm{Ran}\,\mathbb{1}_{S \wedge T \leq t}$. One concludes easily. The proof of *iv)* is identical. ∎

Note that the stopping time T given by

$$\mathbb{1}_{T \leq s} = \begin{cases} 0 & \text{for } s < t \\ I & \text{for } s \geq t \end{cases}$$

is noting but the deterministic time $T = tI$, denoted t simply.

Finally, for a stopping time T, by a *sequence of refining T-partitions of* $\mathbb{R}^+$ we mean a sequence $(E_n)_{n \in \mathbb{N}}$ of finite subsets $E_n = \{0 = t_0^n < t_1^n < \cdots < t_k^n < +\infty\}$ of $\mathbb{R}^+$ such that:

i) all the t_j^n are continuity points for T, $n \in \mathbb{N}$, $j \geq 1$;

ii) $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$;

iii) the diameter $\delta_n = \sup\{t_{i+1}^n - t_i^n \,;\, i \in \mathbb{N}\}$ of E_n tends to 0 when n tends to $+\infty$;

iv) $\sup E_n$ tends to $+\infty$ as n tends to $+\infty$.

Note that, for any stopping time T such a sequence always exists.

PROPOSITION 2. — *For every stopping time T there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of discrete stopping times such that $T_0 \geq T_1 \geq \cdots \geq T$ and $(T_n)_{n \in \mathbb{N}}$ converges to T.*

5

Proof. — Let $E = \{0 = t_0 < t_1 < t_2 < \cdots < t_n < +\infty\}$ be a partition of $\mathbb{R}^+$. Define the spectral measure T_E by

$$\begin{cases} T_E(\{t_i\}) = T([t_{i-1}, t_i[) & \text{for } i = 1, \ldots, n, \\ T_E(\{+\infty\}) = T([t_n, +\infty]). \end{cases}$$

Then T_E is clearly a quantum stopping time.

Now let $(E_n)_{n \in \mathbb{N}}$ be a sequence of refining T-partitions of $\mathbb{R}^+$ and put $T_n = T_{E_n}$ for all $n \in \mathbb{N}$. Let us first check that $T_n \geq T_{n+1} \geq T$ for all $n \in \mathbb{N}$. For all $t \in \mathbb{R}^+$, let $t_{i_0}^{(n)}$ be given by $t_{i_0}^{(n)} = \max\{t_i \in E_n ; t_i \leq t\}$. As $E_n \subset E_{n+1}$ we have $t_{i_0}^{(n)} \leq t_{i_0}^{(n+1)} \leq t$ and

$$\mathbb{1}_{T_n \leq t} = \mathbb{1}_{T \in [0, t_{i_0}^{(n)}[},$$

$$\mathbb{1}_{T_{n+1} \leq t} = \mathbb{1}_{T \in [0, t_{i_0}^{(n+1)}[}.$$

This proves $T_n \geq T_{n+1} \geq T$.

Let us now prove the convergence. Let $f \in \mathcal{H}$ and $t \in \mathbb{R}^+$. We have

$$\|\mathbb{1}_{T_n \leq t} f - \mathbb{1}_{T \leq t} f\|^2 = \|\mathbb{1}_{T < t_{i_0}^{(n)}} f - \mathbb{1}_{T \leq t} f\|^2$$

$$= \|\mathbb{1}_{T \in [t_{i_0}^{(n)}, t]} f\|^2,$$

which converges to $\|\mathbb{1}_{T=t} f\|^2$ when n tends to $+\infty$. Thus it converges to 0 if t is a continuity point for T. ∎

III. The space $\mathcal{H}_T$

Let $(\mathcal{H}, (\mathcal{H}_t)_{t \in \mathbb{R}^+})$ be a filtered Hilbert space and T be a stopping time on $\mathcal{H}$. The space $\mathcal{H}_t$ classically interprets as the space of events occurring before time t (see the discussion in section II about the connections with classical theory). Thus mimicking the classical definition of $\mathcal{F}_T$, the σ-field of events anterior to T, that is

$$\mathcal{F}_T = \{A \in \mathcal{F} ; (T \leq t) \cap A \in \mathcal{F}_t \text{ for all } t \in \mathbb{R}^+\},$$

we define in our quantum context: the *space of events anterior to a stopping time T* is the space

$$\mathcal{H}_T = \{f \in \mathcal{H} ; \mathbb{1}_{T \leq t} f \in \mathcal{H}_t \text{ for all } t \in \mathbb{R}^+\}.$$

We denote by E_T the orthogonal projection onto $\mathcal{H}_T$ which is clearly a closed subspace of $\mathcal{H}$.

PROPOSITION 3.

6

1) If $S \leq T$ then $\mathcal{H}_S \subset \mathcal{H}_T$.

2) If $(T_n)_{n \in \mathbb{N}}$ is a decreasing sequence of stopping times converging to T then

$$\mathcal{H}_T = \bigcap_{n \in \mathbb{N}} \mathcal{H}_{T_n}$$

and $(E_{T_n})_{n \in \mathbb{N}}$ decreases and converges to E_T.

3) $\mathcal{H}_T = \{ f \in \mathcal{H} ; \ \mathbb{1}_{T < t} f \in \mathcal{H}_t \text{ for all } t \in \mathbb{R}^+ \}$.

Proof.

1) If $f \in \mathcal{H}_S$ then $\mathbb{1}_{T \leq t} f = \mathbb{1}_{T \leq t} \mathbb{1}_{S \leq t} f$. By hypothesis $\mathbb{1}_{S \leq t} f$ belongs to $\mathcal{H}_t$ and thus so does $\mathbb{1}_{T \leq t} \mathbb{1}_{S \leq t} f$ (by adaptedness). Thus $f \in \mathcal{H}_T$.

2) We have $T \leq T_n$ and thus $\mathcal{H}_T \subset \bigcap_{n \in \mathbb{N}} \mathcal{H}_{T_n}$. Now if f belongs to $\bigcap_{n \in \mathbb{N}} \mathcal{H}_{T_n}$ and t is a continuity point for T, then $\mathbb{1}_{T \leq t} f = \lim_{n \to \infty} \mathbb{1}_{T_n \leq t} f$ and thus $\mathbb{1}_{T \leq t} f$ belongs to $\mathcal{H}_t$. Now, if t is not a continuity point for T we can find a sequence $(t_n)_{n \in \mathbb{N}}$ of continuity points for T which decreases and converges to t. Thus $\mathbb{1}_{T \leq t_n} f$ belongs to $\mathcal{H}_{t_n}$ and $\mathbb{1}_{T \leq t} f = \lim_{n \to \infty} \mathbb{1}_{T \leq t_n} f$ belongs to $\bigcap_{n \in \mathbb{N}} \mathcal{H}_{t_n} = \mathcal{H}_t$.

3) As $\mathbb{1}_{T < t} = \mathbb{1}_{T < t} \mathbb{1}_{T \leq t}$ and $\mathbb{1}_{T < t}$ is adapted at time t, we clearly have $\mathcal{H}_T \subset \{ f \in \mathcal{H} ; \ \mathbb{1}_{T < t} f \in \mathcal{H}_t \text{ for all } t \in \mathbb{R}^+ \}$. Now, if $\mathbb{1}_{T < t} f \in \mathcal{H}_t$ for all t, then $\mathbb{1}_{T \leq t} f = \lim_{n \to +\infty} \mathbb{1}_{T < t + \frac{1}{n}} f$ which belongs to $\bigcap_{n \in \mathbb{N}} \mathcal{H}_{t + \frac{1}{n}} = \mathcal{H}_t$. ∎

THEOREM 4. — *Let T be a discrete stopping time with $\sum_{i=1}^{n} \mathbb{1}_{T = t_i} = I$. We then have*

$$E_T = \sum_{i=1}^{n} \mathbb{1}_{T = t_i} E_{t_i} = \sum_{i=1}^{n} E_{t_i} \mathbb{1}_{T = t_i} ,$$

with the convention $E_\infty = I$.

Proof. — If T is a discrete stopping time with values $\{ 0 \leq t_1 < \cdots < t_n \leq +\infty \}$ then the operator

$$P = \sum_{i=1}^{n} \mathbb{1}_{T = t_i} E_{t_i} \qquad (\text{with } E_\infty = I)$$

is easily seen to be an orthogonal projector. If f belongs to Ran P, if $t \in \mathbb{R}^+$ then

$$\mathbb{1}_{T \leq t} f = \sum_{\substack{i=1 \\ t_i \leq t}}^{n} \mathbb{1}_{T = t_i} E_{t_i} f = E_t \sum_{\substack{i=1 \\ t_i \leq t}}^{n} \mathbb{1}_{T = t_i} E_{t_i} f$$

$$= E_t \mathbb{1}_{T \leq t} f$$

Thus f belongs to $\mathcal{H}_T$. That is, $\mathrm{Ran}\, P \subset \mathcal{H}_T$. Now if f belongs to $\mathcal{H}_T$ then

$$
\begin{aligned}
Pf &= \sum_{i=1}^{n} \mathbb{1}_{T=t_i} E_{t_i} f = \sum_{i=1}^{n} E_{t_i} \mathbb{1}_{T=t_i} f \\
&= \sum_{i=1}^{n} \mathbb{1}_{T=t_i} f \quad (\text{for } \mathbb{1}_{T=t_i} f \in \mathcal{H}_{t_i} \text{ by hypothesis}) \\
&= f .
\end{aligned}
$$

Thus $\mathrm{Ran}\, P = \mathcal{H}_T$ and $P = E_T$. ∎

THEOREM 5. — *Let T be a quantum stopping time on $\mathcal{H}$. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of discrete stopping times converging to T, such as in Proposition 2, then the sequence $(E_{T_n})_{n\in\mathbb{N}}$ strongly converges to E_T. In other words*

$$
E_T = s-\lim_{n\to+\infty} \sum_{i=1}^{N_n} \mathbb{1}_{T\in[t_{i-1},t_i[} E_{t_i} + \mathbb{1}_{T\geq t_{N_n}}
$$

where the diameter of the partition $\{0 = t_0 < t_1 < \cdots < t_{N_n}\}$ tends to 0 and t_{N_n} tends to $+\infty$.

Proof. — This is now an easy consequence of Proposition 2, Proposition 3, part 2) and Theorem 4. ∎

PROPOSITION 6.

1) If S and T are two discrete stopping times then

$$E_{S \vee T} = E_S \vee E_T .$$

2) If S and T are two stopping times then

$$E_{S \wedge T} = E_S \wedge E_T .$$

Proof.

1) We always have $\mathcal{H}_S \vee \mathcal{H}_T \subset \mathcal{H}_{S \vee T}$. Now let S and T be discrete and $\mathcal{S}$ be the union of their support. Suppose $\mathcal{S} \cap \mathbb{R}^+ = \{0 \leq t_1 < \cdots < t_N\}$. Let $f \in \mathcal{H}_S^\perp \cap \mathcal{H}_T^\perp$. We have $E_S f = 0$ but

$$E_S f = \mathbb{1}_{S=+\infty} f + \sum_{i=0}^{N} \mathbb{1}_{S=t_i} E_{t_i} f$$

$$= \mathbb{1}_{S=+\infty} f + \mathbb{1}_{S \leq t_N} E_{t_N} - \sum_{i=0}^{N-1} \mathbb{1}_{S \leq t_i} (E_{t_{i+1}} - E_{t_i}) f$$

$$= f - \sum_{i=0}^{N} \mathbb{1}_{S \leq t_i} (E_{t_{i+1}} - E_{t_i}) f$$

with the convention $t_{N+1} = +\infty$.

In the same way

$$E_{S \vee T} f = f - \sum_{i=0}^{N} \mathbb{1}_{S \vee T \leq t_i} (E_{t_{i+1}} - E_{t_i}) f .$$

But

$$\mathbb{1}_{S \vee T \leq t_i} = s - \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(\mathbb{1}_{S \leq t_i} \mathbb{1}_{T \leq t_i} \right)^k$$

thus

$$E_{S \vee T} f = f - \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{i=0}^{N} \left(\mathbb{1}_{S \leq t_i} \mathbb{1}_{T \leq t_i} \right)^k (E_{t_{i+1}} - E_{t_i}) f.$$

Note that

$$\sum_{i=0}^{N} \mathbb{1}_{S \leq t_i} (E_{t_{i+1}} - E_{t_i}) \sum_{i=0}^{N} \mathbb{1}_{T \leq t_j} (E_{t_{j+1}} - E_{t_j}) = \sum_{i=0}^{N} \mathbb{1}_{S \leq t_i} \mathbb{1}_{T \leq t_i} (E_{t_{i+1}} - E_{t_i}).$$

In the same way

$$\sum_{i=0}^{N} \left(\mathbb{1}_{S \leq t_i} \mathbb{1}_{T \leq t_i} \right)^k (E_{t_{i+1}} - E_{t_i}) = \left[\sum_{i=0}^{N} \mathbb{1}_{S \leq t_i} (E_{t_{i+1}} - E_{t_i}) \sum_{j=0}^{N} \mathbb{1}_{T \leq t_j} (E_{t_{j+1}} - E_{t_j}) \right]^k$$

$$= \left[(I - E_S)(I - E_T) \right]^k .$$

9

This gives

$$E_{S \vee T} f = f - \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left[(I - E_S)(I - E_T) \right]^k f$$
$$= f - (E_S^\perp \vee E_T^\perp) f$$
$$= 0 \,.$$

This proves 1).

2) We always have $\mathcal{H}_S \cap \mathcal{H}_T \supset \mathcal{H}_{S \wedge T}$. Now let f belong to $\mathcal{H}_S \cap \mathcal{H}_T$. For all $t \in \mathbb{R}^+$ we have

$$\mathbb{1}_{S \wedge T \leq t} f = f - \mathbb{1}_{S \wedge T > t} f$$
$$= f - \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} (\mathbb{1}_{S > t} \mathbb{1}_{T > t})^k f \,.$$

But

$$(\mathbb{1}_{S > t} \mathbb{1}_{T > t})^k = \left[(I - \mathbb{1}_{S \leq t})(I - \mathbb{1}_{T \leq t}) \right]^k$$
$$= I - L_t$$

where L_t satisfies $L_t f \in \mathcal{H}_t$.

Thus

$$\frac{1}{n} \sum_{k=0}^{n-1} (\mathbb{1}_{S > t} \mathbb{1}_{T > t})^k f = f - g$$

with $g \in \mathcal{H}_t$. This shows $\mathbb{1}_{S \wedge T \leq t} f \in \mathcal{H}_t$ and proves 1). ∎

IV. The space $\mathcal{H}_{T-}$

As our definition of $\mathcal{H}_T$ in our setup seems to fit very well, we pursue the analogy with classical probability theory and define the *space of event strictly anterior* to a stopping time T as the space $\mathcal{H}_{T-}$ which is the closure of the subspace of $\mathcal{H}$ generated by $\mathcal{H}_0$ and $\{ \mathbb{1}_{T > t} f \; ; \; f \in \mathcal{H}_t, \; t \in \mathbb{R}^+ \}$.

PROPOSITION 7. — *For every stopping time T we have $\mathcal{H}_{T-} \subset \mathcal{H}_T$.*

Proof. — First of all, if $f \in \mathcal{H}_0$ then $\mathbb{1}_{T \leq t} f = \mathbb{1}_{T \leq t} E_0 f = \mathbb{1}_{T \leq t} E_t f = E_t \mathbb{1}_{T \leq t} f$ and thus $\mathbb{1}_{T \leq t} f$ belongs to $\mathcal{H}_t$, that is $\mathcal{H}_0 \subset \mathcal{H}_T$.

Now if $f \in \mathcal{H}_t$ let us consider $g = \mathbb{1}_{T>t}f$. We then have

$$\mathbb{1}_{T \le s}g = \begin{cases} 0 & \text{if } s \le t \\ \mathbb{1}_{T \in]t,s]}f & \text{if } s > t. \end{cases}$$

In particular $\mathbb{1}_{T \le s}g$ belongs to $\mathcal{H}_s$ for all s. Thus g belongs to $\mathcal{H}_T$.

We have proved that all the generators of $\mathcal{H}_{T-}$ are elements of $\mathcal{H}_T$, which is closed. Thus $\mathcal{H}_{T-} \subset \mathcal{H}_T$. ∎

PROPOSITION 8. — *If the filtration is continuous (i.e. $\mathcal{H}_{t-} = \mathcal{H}_t$ for all t) then $\mathcal{H}_{T-} = \mathcal{H}_T$ for every discrete stopping time T.*

Proof. — Let $g \in (\mathcal{H}_{T-})^\perp$. We thus have, for all $t \in \mathbb{R}^+$, all $f \in \mathcal{H}$,

$$\langle \mathbb{1}_{T>t}E_t f, g \rangle = 0, \quad \text{i.e.} \quad \langle f, \mathbb{1}_{T>t}E_t g \rangle = 0.$$

This means $\mathbb{1}_{T>t}E_t g = 0$ for all $t \in \mathbb{R}^+$.

Now suppose that T is discrete with $\sum_{i=1}^{n} \mathbb{1}_{T=t_i} = I$. If $t < t_1$ then $\mathbb{1}_{T>t} = I$ and $E_t g = 0$ thus by the continuity hypothesis $E_{t_1} g = 0$.

If $t_i \le t < t_{i+1}$ then $E_t g = \mathbb{1}_{T \le t}E_t g = \sum_{j=0}^{i} \mathbb{1}_{T=t_j}E_t g$. Now let t tend to t_{i+1}, this gives $E_{t_{i+1}} g = \sum_{j=0}^{i} \mathbb{1}_{T=t_j}E_{t_{i+1}} g$. Thus in particular $\mathbb{1}_{T=t_{i+1}}E_{t_{i+1}} g = 0$.

All together we have proved that $E_T g = \sum_i \mathbb{1}_{T=t_i}E_{t_i} g = 0$ and thus $g \in (\mathcal{H}_T)^\perp$. ∎

PROPOSITION 9. — *If S and T are two stopping times such that $S \le T$, then $\mathcal{H}_{S-} \subset \mathcal{H}_{T-}$.*

Proof. — The space $\mathcal{H}_{S-}$ is generated by $\mathcal{H}_0$ and the $\mathbb{1}_{S>t}E_t f$, $f \in \mathcal{H}$, $t \in \mathbb{R}^+$. But $\mathbb{1}_{S>t} = \mathbb{1}_{S>t}\mathbb{1}_{T>t}$ and thus $\mathbb{1}_{S>t}E_t f = \mathbb{1}_{S>t}\mathbb{1}_{T>t}E_t f = \mathbb{1}_{T>t}E_t \mathbb{1}_{S>t}f \in \mathcal{H}_{T-}$. This proves $\mathcal{H}_{S-} \subset \mathcal{H}_{T-}$. ∎

PROPOSITION 10. — *If $(T_n)_{n \in \mathbb{N}}$ is an increasing sequence of stopping times converging to T then*

$$\mathcal{H}_{T-} = \bigvee_n \mathcal{H}_{T_n-}.$$

Proof. — By Proposition 9, each space $\mathcal{H}_{T_n-}$ is a subspace of $\mathcal{H}_{T-}$, thus $\bigvee_n \mathcal{H}_{T_n-} \subset \mathcal{H}_{T-}$. Conversely, if f belongs to $\mathcal{H}$ and if t is a continuity point for T,

11

then $\mathbb{1}_{T>t}E_t f = \lim\limits_{n\to+\infty}\mathbb{1}_{T_n>t}E_t f$ which belongs to $\bigvee_n \mathcal{H}_{T_n-}$. If t is not a continuity point for T, then choose a decreasing sequence $(t_n)_{n\in\mathbb{N}}$ of continuity points for T, converging to t. We conclude by the identity

$$\mathbb{1}_{T>t}E_t f = \lim\limits_{n\to+\infty}\mathbb{1}_{T>t_n}E_t f\,. \qquad\blacksquare$$

Let T be a finite stopping time (*i.e.* $\mathbb{1}_{T=+\infty} = 0$). Then the spectral integral $\int_{\mathbb{R}^+} t\,\mathbb{1}_{T\in dt}$ defines a self-adjoint operator on $\mathcal{H}$, which we denote by T again.

PROPOSITION 11. — *If T is any finite stopping time then the operator T maps* Dom $T\cap\mathcal{H}_T$ *to $\mathcal{H}_T$.*

If moreover T is bounded, it maps $\mathcal{H}_{T-}$ to $\mathcal{H}_{T-}$.

Proof. — If f belongs to Dom $T\cap\mathcal{H}_T$ then

$$Tf = \lim\sum_i t_i\mathbb{1}_{T\in[t_i,t_{i+1}[}f$$

(where the limits is taken as usual under a sequence of partitions whose diameter tends to 0) and thus, as $\mathbb{1}_{T\le s}\big(\sum_i t_i\mathbb{1}_{T\in[t_i,t_{i+1}[}f\big)$ belongs to $\mathcal{H}_s$, we have $\mathbb{1}_{T\le s}Tf \in \mathcal{H}_s$ and thus $Tf\in\mathcal{H}_T$. This proves the first part.

If T is bounded, we have

$$T\mathbb{1}_{T>t}E_t f = \lim\sum_i t_i\mathbb{1}_{T\in[t_i,t_{i+1}[}\mathbb{1}_{T>t}E_t f$$

where each term in the sum is an element of $\mathcal{H}_{T-}$. Thus $T\mathbb{1}_{T>t}E_t f$ belongs to $\mathcal{H}_{T-}$. This proves $T\mathcal{H}_{T-}\subset\mathcal{H}_{T-}$. $\qquad\blacksquare$

V. Strictly smaller stopping times

We wish to give a correct meaning to the relation $S < T$ for two quantum stopping times, S and T on $\mathcal{H}$. If S is a discrete stopping time with $\sum_{i=1}^n \mathbb{1}_{S=s_i} = I$ and if T is any stopping time, it is then natural to say that $S < T$ if and only if

$$\mathbb{1}_{S=s_i} = \mathbb{1}_{T>s_i}\mathbb{1}_{S=s_i}\ \text{ for all }\ i\,.$$

Note that this in particular implies $S \le T$ and we have $S < T$ if and only if

$$\sum_{i=1}^n \mathbb{1}_{T>s_i}\mathbb{1}_{S=s_i} = I$$

or else if and only if

$$\sum_{i=1}^{n} \mathbb{1}_{S=s_i} \mathbb{1}_{T>s_i} = I .$$

This motivates the general definition.

Two stopping times S and T on $\mathcal{H}$ are said to satisfy $S < T$ if and only if one has that the expression

$$\sum_{i=1}^{N_n} \mathbb{1}_{T>t_i} \mathbb{1}_{S\in[t_{i-1},t_i[}$$

weakly converges to I when $\{t_i, \ i = 1, \ldots, N_n\}$ follows a sequence of refining S-partitions of $\mathbb{R}^+$.

Note that $S < T$ implies $S \leq T$ for if $S < T$ then

$$\mathbb{1}_{T\leq t} \sum_{i=1}^{N_n} \mathbb{1}_{T>t_i} \mathbb{1}_{S\in[t_{i-1},t_i[} \mathbb{1}_{S\leq t} \tag{1}$$

converges weakly to $\mathbb{1}_{T\leq t} \mathbb{1}_{S\leq t}$, but (1) also equals

$$\mathbb{1}_{T\leq t} \sum_{i=1}^{N_n} \mathbb{1}_{T>t_i} \mathbb{1}_{S\in[t_{i-1},t_i[}$$

which weakly converges to $\mathbb{1}_{T\leq t}$.

Note that if S and T commute, that is if for all s and t in $\mathbb{R}^+$, $\mathbb{1}_{S\leq s} \mathbb{1}_{T\leq t} = \mathbb{1}_{T\leq t} \mathbb{1}_{S\leq s}$, then for all sequence of refining partitions of $\mathbb{R}^+$, $R_n = \sum_{i=1}^{N_n} \mathbb{1}_{T>t_i} \mathbb{1}_{S\in[t_{i-1},t_i[}$ converges strongly. In fact, in this case, $(R_n)_{n\geq 0}$ is an increasing sequence of projections.

PROPOSITION 12. — *Let S and T be two stopping times. Then the following assertions are equivalent.*

i) $S < T$.

ii) $\sum_{i=1}^{N_n} \mathbb{1}_{S\in[t_{i-1},t_i[} \mathbb{1}_{T>t_i}$ *converges weakly to I.*

iii) $\sum_{i=1}^{N_n} \mathbb{1}_{T\leq t_i} \mathbb{1}_{S\in[t_{i-1},t_i[}$ *converges weakly to 0 and $\mathbb{1}_{S=+\infty} = 0$.*

iv) $\sum_{i=1}^{N_n} \mathbb{1}_{S\in[t_{i-1},t_i[} \mathbb{1}_{T\leq t_i}$ *converges weakly to 0 and $\mathbb{1}_{S=+\infty} = 0$.*

Proof. — Assumption *i)* implies that

$$\sum_{i=1}^{N_n} \mathbb{1}_{S\in[t_{i+1},t_i[} - \sum_{i=1}^{N_n} \mathbb{1}_{T\leq t_i} \mathbb{1}_{S\in[t_{i-1},t_i[} \text{ converges weakly to } I.$$

13

That is

$$\sum_{i=1}^{N_n} \mathbb{1}_{T \leq t_i} \mathbb{1}_{S \in [t_{i-1}, t_i[} \text{ converges weakly to } - \mathbb{1}_{S=+\infty}$$

thus

$$\sum_{i=1}^{N_n} \mathbb{1}_{T \leq t_i} \mathbb{1}_{S \in [t_{i-1}, t_i[} \mathbb{1}_{S=+\infty} \text{ converges weakly to } - \mathbb{1}_{S=+\infty}$$

thus $\mathbb{1}_{S=+\infty} = 0$. All the others parts of the proof are obvious. ∎

PROPOSITION 13. — *If S and T are two stopping times on $\mathcal{H}$ such that $S < T$ then $\mathcal{H}_S \subset \mathcal{H}_{T-}$.*

Proof. — For all $f \in \mathcal{H}$, the quantity

$$\sum_{i=1}^{N_n} \mathbb{1}_{T > t_i} \mathbb{1}_{S \in [t_{i-1}, t_i[} E_{t_i} f$$

belongs to $\mathcal{H}_{T-}$, but it is also equal to

$$\sum_{i=1}^{N_n} \mathbb{1}_{T > t_i} \mathbb{1}_{S \in [t_{i-1}, t_i[} E_{S_E} f \ ,$$

where $E = \{t_i, \ i = 1, \ldots, N_n\}$, (with the notation S_E of the proof of Proposition 2) which converges weakly to $E_S f$. Thus $E_S f$ belongs to $\mathcal{H}_{T-}$. ∎

PROPOSITION 14.

i) *If $(T_n)_{n \in \mathbb{N}}$ is an increasing sequence of stopping times converging to T and with $T_n < T$ for all n, then $\mathcal{H}_{T-} = \bigvee_{n \in \mathbb{N}} \mathcal{H}_{T_n}$.*

ii) *If $(T_n)_{n \in \mathbb{N}}$ is a decreasing sequence of stopping times converging to T and with $T_n > T$ for all n, then $\mathcal{H}_T = \bigcap_{n \in \mathbb{N}} \mathcal{H}_{T_n-}$.*

Proof.

i) We have $\mathcal{H}_{T_n} \subset \mathcal{H}_{T-}$ for all $n \in \mathbb{N}$, thus $\bigvee_{n \in \mathbb{N}} \mathcal{H}_{T_n} \subset \mathcal{H}_{T-}$. But by Proposition 10 we have $\mathcal{H}_{T-} = \bigvee_n \mathcal{H}_{T_n-} \subset \bigvee \mathcal{H}_{T_n}$.

ii) We have $\mathcal{H}_T \subset \mathcal{H}_{T_n-}$ for all $n \in \mathbb{N}$, thus $\mathcal{H}_T \subset \bigcap_{n \in \mathbb{N}} \mathcal{H}_{T_n-}$. But $\bigcap_{n \in \mathbb{N}} \mathcal{H}_{T_n-}$ is included in $\bigcap_n \mathcal{H}_{T_n}$ which is equal to $\mathcal{H}_T$ by Proposition 3, part. 2). This proves *ii)*. ∎

Let us study a rather pathological example. Consider a filtered Hilbert spaces $(\mathcal{H}, (E_t)_{t \in \mathbb{R}^+})$. Then the spectral measure $(E_t)_{t \in \mathbb{R}^+}$ itself defines a quantum stopping time T by putting

$$\mathbb{1}_{T \leq t} = E_t \ , \ \ t \in \mathbb{R}^+ \ .$$

14

Let us then compute $\mathcal{H}_T$ and $\mathcal{H}_{T-}$. We have
$$\mathcal{H}_T = \big\{ f \in \mathcal{H} \,;\, \mathbb{1}_{T \leq t} f \in \mathcal{H}_t \text{ for all } t \big\}$$
$$= \big\{ f \in \mathcal{H} \,;\, E_t f \in \mathcal{H}_t \text{ for all } t \big\}$$
and thus $\mathcal{H}_T = \mathcal{H}$.

We have $\mathbb{1}_{T > t} E_t f = (I - E_t) E_t f = 0$ for all $t \in \mathbb{R}^+$. Thus $\mathcal{H}_{T-} = \mathcal{H}_0$.

Now let T_n be defined by $\mathbb{1}_{T_n \leq t} = E_{t+1/n}$, $t \in \mathbb{R}^+$. Then T_n is a stopping time again and the sequence $(T_n)_{n \in \mathbb{N}}$ is increasing and converging to T. We obviously have $T_n < T$ for all n. Thus $\mathcal{H}_{T_n} = \mathcal{H}_{T_n-} = \mathcal{H}_0$.

We end this section with two definitions, which follow from the classical corresponding definitions.

A stopping time T is *previsible* if there exists an increasing sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ which converges to T and such that $T_n < T$ for all $n \in \mathbb{N}$.

A filtered Hilbert space $(\mathcal{H}, (E_t)_{t \in \mathbb{R}^+})$ is *quasi-left continuous* if $\mathcal{H}_T = \mathcal{H}_{T-}$ for every previsible stopping time T.

The pathological example above shows that a filtered Hilbert space is never quasi-left continuous. We actually have to enlarge our definitions.

Let $(\mathcal{H}, (E_t)_{t \in \mathbb{R}^+})$ be a filtered Hilbert space. Let $\mathcal{U}$ be a closed subalgebra of $\mathcal{B}(\mathcal{H})$ and $(\mathcal{U}_t)_{t \in \mathbb{R}^+}$ an increasing family of closed subalgebras of $\mathcal{U}$ such that $\bigcup_{t \in \mathbb{R}} \mathcal{U}_t$ generated $\mathcal{U}$, and which satisfies
$$X E_u = E_u X$$
for all $X \in \mathcal{U}_t$, all $u \geq t$.

We define a $(\mathcal{U}_t)_{t \in \mathbb{R}^+}$-stopping time T to be a spectral measure on $\mathbb{R}^+ \cup \{+\infty\}$, valued in $\mathcal{H}$ and such that $\mathbb{1}_{T \leq t}$ belongs to $\mathcal{U}_t$ for all t. *Then note that all what has been proved before remains valid for any $(\mathcal{U}_t)_{t \in \mathbb{R}^+}$-stopping times.*

Examples.

1) If $\mathcal{U}_t = \{ X \in \mathcal{B}(\mathcal{H}) \,;\, \forall u \geq t,\, E_u X = X E_u \}$ then we recover the case studied in the previous sections.

2) If $\mathcal{U}$ is a von Neumann algebra acting on an Hilbert space $\mathcal{H}$ and $(\mathcal{U}_t)_{t \in \mathbb{R}^+}$ is an increasing family of von Neumann subalgebras which generates $\mathcal{U}$. Assume there exists a unit vector $\Omega \in \mathcal{H}$ which is cyclic and separating for $\mathcal{U}$ and a family $(M_t)_{t \in \mathbb{R}^+}$ of normal ω-invariant conditional expectations $M_t : \mathcal{U} \to \mathcal{U}_t$, where $\omega(\cdot) = \langle \Omega, \cdot \Omega \rangle$. We denote by $\mathcal{H}_t$ the closure of $\mathcal{U}_t \Omega$ in $\mathcal{H}$ and by E_t the orthogonal projection onto $\mathcal{H}_t$. We then have
$$E_t(X\Omega) = M_t(X)\Omega$$

15

and thus for all $u \geq t$, all $X \in \mathcal{U}_t$, we have $E_u X = X E_u$ (indeed $E_u X A \Omega = M_u(XA)\Omega = X M_u(A)\Omega = X E_u A \Omega$).

Thus our definitions covers the case of stopping times in von Neumann algebras such as studied in [BW1] or [BSW].

3) Let Φ be the symmetric Fock space on $L^2(\mathbb{R}^+; \mathbb{C})$: $\Phi = \Gamma_s(L^2(\mathbb{R}^+; \mathbb{C}))$. If we define $\Phi_{t]} = \Gamma_s(L^2([0, t]; \mathbb{C}))$ and $\Phi_{[t} = \Gamma_s(L^2([t, +\infty[; \mathbb{C}))$, we then have the well-known "continuous tensor product" property of Fock spaces:

$$\Phi \simeq \Phi_{t]} \otimes \Phi_{[t}$$

and we can consider $\Phi_{t]}$ as a subspace of Φ.

In the framework of quantum stochastic calculus ([H-P]) a bounded operator H on Φ is said to be adapted at time t if it is of the form $H = K \otimes I$ for some $K : \Phi_{t]} \to \Phi_{t]}$.

By considering the algebras $\mathcal{U}_t$ of t-adapted bounded operators, $t \in \mathbb{R}^+$, our set up covers all the theory of quantum stopping times on the Fock space Φ.

We extend our definitions into:

• A $(\mathcal{U}_t)_{t \in \mathbb{R}^+}$-stopping time T is *previsible* if there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of $(\mathcal{U}_t)_{t \in \mathbb{R}^+}$-stopping times such that $(T_n)_{n \in \mathbb{N}}$ converges to T and $T_n < T$ for all $n \in \mathbb{N}$.

• A filtered Hilbert space $(\mathcal{H}, (E_t)_{t \in \mathbb{R}^+})$ together with a family $(\mathcal{U}_t)_{t \in \mathbb{R}^+}$ is *quasi-left continous* if for all previsible $(\mathcal{U}_t)_{t \in \mathbb{R}^+}$-stopping time T we have $\mathcal{H}_{T-} = \mathcal{H}_T$.

VI. The Fock space case

We here just recall few facts about the symmetric Fock space. Details can be found in [Att] or [Mey].

The Fock space Φ is the symmetric Fock space over $L^2(\mathbb{R}^+; \mathbb{C})$: $\Phi = \Gamma_s(L^2(\mathbb{R}^+; \mathbb{C}))$. This space can be advantageously understood as the space $L^2(\mathcal{P})$ where $\mathcal{P}$ is the set of finite subsets of $\mathbb{R}^+$ equipped with the Guichardet symmetric measure. That is, an element f of $\Phi = L^2(\mathcal{P})$ is a function $f : \mathcal{P} \to \mathbb{C}$ such that

$$\|f\|^2 = \int_{\mathcal{P}} |f(\sigma)|^2 \, d\sigma$$
$$= |f(\emptyset)|^2 + \sum_{n=1}^{\infty} \int_{0 < s_1 < \cdots < s_n} |f(\{s_1, \ldots, s_n\})|^2 ds_1 \cdots ds_n < \infty \, .$$

We then have the following properties:

i) If we write $\Phi_{t]}$ (resp. $\Phi_{[t}$) for the subspace of Φ made of those f such that $f(\sigma) = 0$ unless $\sigma \subset [0, t]$ (resp. $\sigma \subset [t, +\infty[)$, then the mapping

$$\Phi_{t]} \otimes \Phi_{[t} \longrightarrow \Phi$$
$$f \otimes g \longmapsto h\,,$$

with $h(\sigma) = f(\sigma \cap [0, t]) g(\sigma \cap [t, +\infty[)$, extends to a unitary operator. We thus identify Φ to $\Phi_{t]} \otimes \Phi_{[t}$ for all $t \in \mathbb{R}^+$.

ii) For all $f \in \Phi$, if we define $D_t f$ by

$$[D_t f](\sigma) = f(\sigma \cup \{t\}) \mathbb{1}_{\sigma \subset [0,t]}$$

we then have that $D_t f$ belongs to Φ for a.a.t and

$$\|f\|^2 = |f(\emptyset)|^2 + \int_0^\infty \|D_t f\|^2 \, dt\,.$$

iii) If $(g_t)_{t \in \mathbb{R}^+}$ is a family of elements of Φ such that

    a) $g_t \in \Phi_{t]}$ for all t

    b) $t \mapsto g_t$ is measurable

    c) $\int_0^\infty \|g_t\|^2 \, dt < \infty$

then $(g_t)_{t \in \mathbb{R}^+}$ is said to be *Ito-integrable*. In this case we write

$$\int_0^\infty g_t \, d\chi_t$$

for the element h of Φ given by

$$h(\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset \\ g_{t_n}(\{t_1, \ldots, t_{n-1}\}) & \text{if } \sigma = \{t_1 < t_2 < \cdots < t_n\}. \end{cases}$$

This element h of Φ is called the *Ito integral* of $(g_t)_{t \in \mathbb{R}^+}$ and we have

$$\|h\|^2 = \int_0^\infty \|g_t\|^2 \, dt\,.$$

If we denote by $\mathbb{1}$ the vacuum of Φ, that is the element of Φ given by

$$\mathbb{1}(\sigma) = \begin{cases} 1 & \text{if } \sigma = \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

and by E_t the orthogonal projection from Φ onto $\Phi_{t]}$, $t \in \mathbb{R}^+ \smallsetminus \{0\}$, we then easily have the following theorem, cf. [Att] for details.

THEOREM 15. — *For every $f \in \Phi$, the family $(D_t f)_{t \in \mathbb{R}^+}$ is Ito integrable and we have*

$$f = f(\emptyset) \mathbb{1} + \int_0^\infty D_t f \, d\chi_t\,. \tag{2}$$

17

For all $t \in \mathbb{R}^+ \smallsetminus \{0\}$ we have

$$E_t f = f(\emptyset) \mathbb{1} + \int_0^t D_s f \; d\chi_s \tag{3}$$

(in the sense $\int_0^\infty \mathbb{1}_{[0,t]}(s) D_s f \; d\chi_s$).

We have the isometry formula

$$\|f\|^2 = |f(\emptyset)|^2 + \int_0^\infty \|D_s f\|^2 \; ds \,. \tag{4}$$

We put $\mathcal{U}$ to be the algebra $\mathcal{B}(\Phi)$ of bounded operators on Φ and, for all $t \in \mathbb{R}^+$, $\mathcal{U}_t$ is the algebra of t-adapted bounded operators on Φ in the sense of Hudson-Parthasarathy, that is the algebra of bounded operators H on Φ of the form

$$H = k \otimes I$$

on $\Phi_{t]} \otimes \Phi_{[t}$, for some bounded operator k on $\Phi_{t]}$. Note that, for all $t \in \mathbb{R}^+$, all $u \geq t$ and all $H \in \mathcal{U}_t$ we have $E_u H = H E_u$.

Clearly, $\left(\Phi, (E_t)_{t \in \mathbb{R}^+}\right)$ is a filtered Hilbert space and from now on we define stopping times on Φ as being affiliated to the family $(\mathcal{U}_t)_{t \in \mathbb{R}^+}$.

In particular $(E_t)_{t \in \mathbb{R}^+}$ is not a stopping time on Φ.

The following theorem is proved in [A-S], Proposition 6.

THEOREM 16. — *Let T be a stopping time on Φ. Then for all $f \in \Phi$ we have*

$$E_T f = f(\emptyset) \mathbb{1} + \int_0^\infty \mathbb{1}_{T>s} D_s f \; d\chi_s \,.$$

COROLLARY 17. — *Let $(T_n)_{n \in \mathbb{N}}$ be any sequence of stopping times on Φ converging to T. Then $(E_{T_n})_{n \in \mathbb{N}}$ converges strongly to E_T.*

Proof. — Indeed, we have by Theorem 16 and Theorem 15 (4),

$$\|E_T f - E_{T_n} f\|^2 = \int_0^\infty \|(\mathbb{1}_{T>s} - \mathbb{1}_{T_n>s}) D_s f\|^2 \; ds$$

which converges to 0. ∎

PROPOSITION 18. — *Let S and T be any two stopping times on Φ. We then have*

$$E_{T \vee S} = E_T \vee E_S \,.$$

Proof. — We know that $E_T \vee E_S \leq E_{T \vee S}$. Let $f \in (\Phi_T \vee \Phi_S)^\perp = \Phi_T^\perp \wedge \Phi_S^\perp$. We have $E_T f = E_S f = 0$ and thus for a.a. $s \in \mathbb{R}^+$

$$\mathbb{1}_{T>s} D_s f = \mathbb{1}_{S>s} D_s f = 0$$

18

and $f(\emptyset) = 0$.

As a consequence, $D_s f$ belongs to $\mathrm{Ran}\,\mathbb{1}_{T \leq s} \cap \mathrm{Ran}\,\mathbb{1}_{S \leq s}$ for a.a. $s \in \mathbb{R}^+$, that is to $\mathrm{Ran}\,\mathbb{1}_{T \vee S \leq s}$. This says that $\mathbb{1}_{T \vee S > s} D_s f = 0$ for a.a. $s \in \mathbb{R}^+$ and finally $E_{S \vee T} f = 0$. $\blacksquare$

The Fock space Φ admits several probabilistic interpretations (cf. [Att] or [Mey]) in terms of the Brownian motion, the compensated Poisson process or the Azema martingales. All these classical martingales have in common that their canonical space and filtration is quasi-left continuous.

The theorem to come proves that this property is actually intrinsic to the Fock space structure, and does not depend on any classical probabilistic interpretation of it.

THEOREM 19. — *The filtered Fock space* $\left(\Phi, (E_t)_{t \in \mathbb{R}^+}, (\mathcal{U}_t)_{t \in \mathbb{R}^+}\right)$ *is quasi-left continuous.*

Proof. — Let T be a previsible stopping time on Φ and $(T_n)_{n \in \mathbb{N}}$ an increasing sequence of stopping times converging to T with $T_n < T$ for all $n \in \mathbb{N}$.

By Proposition 14 we know that $\Phi_{T-} = \bigvee_{n \in \mathbb{N}} \Phi_{T_n}$. But if $f \in \Phi_T$ we have $f = E_T f = \lim_{n \to +\infty} E_{T_n} f$ (by Corollary 17) and thus $f \in \bigvee_{n \in \mathbb{N}} \Phi_{T_n}$. This proves that $\Phi_T \subset \bigvee_{n \in \mathbb{N}} \Phi_{T_n}$. Thus $\Phi_{T-} = \Phi_T$. $\blacksquare$

We are now going to discuss some interesting examples of stopping times on Φ:

1) Projection on chaoses. — For every $n \in \mathbb{N}$, we denote by C_n the space of $f \in \Phi$ such that $f(\sigma) = 0$ unless $\#\sigma = n$. It is a closed subspace of Φ and we have

$$\Phi = \bigoplus_{n \in \mathbb{N}} C_n \,.$$

The space C_n is called the *n-th chaos* of Φ. We denote by Q_n the orthogonal projection from Φ onto $\bigoplus_{i=0}^{n} C_i$, that is

$$[Q_n f](\sigma) = f(\sigma) \mathbb{1}_{\#\sigma \leq n}$$

and by $Q_{n,t}$ the operator

$$[Q_{n,t} f](\sigma) = f(\sigma) \mathbb{1}_{\#(\sigma \cap [0,t]) \leq n} \,.$$

The operator $Q_{n,t}$ is t-adapted and equal to

$$Q_{n|\Phi_{t]}} \otimes I_{|\Phi_{[t}} \,.$$

It is an orthogonal projection also and $Q_{n,t} \leq Q_{n,s}$ if $s \leq t$. We define a stopping time T_n by putting

$$\begin{cases} \mathbb{1}_{T_n > t} = Q_{n,t} \\ \mathbb{1}_{T_n = +\infty} = Q_n \,. \end{cases}$$

19

We clearly have $T_n \leq T_{n+1}$ for all $n \in \mathbb{N}$. Note that for all $s, t \in \mathbb{R}^+$ we have

$$\mathbb{1}_{T_n \leq s} \mathbb{1}_{T_{n+1} \leq t} = \mathbb{1}_{T_{n+1} \leq t} \mathbb{1}_{T_n \leq s} \, .$$

We also have

$$
\begin{aligned}
E_{T_n} f &= f(\emptyset) \mathbb{1} + \int_0^\infty \mathbb{1}_{T_n > t} D_t f \, d\chi_t \\
&= f(\emptyset) \mathbb{1} + \int_0^\infty Q_n D_t f \, d\chi_t \\
&= f(\emptyset) \mathbb{1} + Q_{n+1} \int_0^\infty D_t f \, d\chi_t \\
&= Q_{n+1} f \, .
\end{aligned}
$$

Thus $\Phi_{T_n} = \bigoplus_{i=0}^{n+1} C_i$. But note that $\mathbb{1}_{T_n > t} E_t f = Q_n E_t f$ and thus $\Phi_{T_n -} = \bigoplus_{i=0}^{n} C_i$.

In particular the T_n's are not previsible.

2) Jumping times of the Poisson process. — For this example only we refer to quantum stochastic integration on Φ (cf. [Att] or [Mey] for details) and we consider the reader very familiar with it.

Let $(a_t^+)_{t \in \mathbb{R}^+}$, $(a_t^-)_{t \in \mathbb{R}^+}$ and $(a_t^0)_{t \in \mathbb{R}^+}$ be the usual *creation*, *annihilation* and *conservation* processes on Φ. Let $N_t = a_t^+ + a_t^- + a_t^0 + tI$ be the Poisson process on Φ. We define a family of stopping times $(T_n)_{n \in \mathbb{N}}$ by

$$
\begin{cases}
T_0 = 0 \\
\mathbb{1}_{T_n > t} = I - \int_0^t (\mathbb{1}_{T_n > s} - \mathbb{1}_{T_{n-1} > s}) dN_s, \ n \geq 1 \, .
\end{cases}
$$

Indeed, straightforward applications of the quantum Ito formula show that the family $(\mathbb{1}_{T_n > t})_{t \in \mathbb{R}^+}$ is a decreasing family of projectors, adapted at time t. Thus they define a stopping time T_n.

More straightforward applications of the quantum Ito formula show that, for all $s, t \in \mathbb{R}^+$, all $n, m \in \mathbb{N}$

$$\mathbb{1}_{T_n \leq s} \mathbb{1}_{T_m \leq t} = \mathbb{1}_{T_m \leq t} \mathbb{1}_{T_n \leq s} \, ,$$

and that $T_n \leq T_m$ for $n \leq m$.

One can even be more precise.

PROPOSITION 20. — *For all $t \in \mathbb{R}^+$, the self-adjoint operator N_t admits a spectrum equal to $\mathbb{N}$ and the spectral projection onto the eigenspace associated to $n \in \mathbb{N}$ is*

$$\mathbb{1}_{T_n \leq t} \mathbb{1}_{T_{n+1} > t} \, .$$

Proof. — Consider, for $t \in \mathbb{R}^+$, $n \in \mathbb{N}^*$, the operators

$$X_t^n = \mathbb{1}_{T_n \leq t} \mathbb{1}_{T_{n+1} > t} \, .$$

The family $(X_t^n)_{n \in \mathbb{N}^*}$ is family of two by two orthogonal projections whose sum is equal to I (for $\mathbb{1}_{T_n > t}$ converges strongly to I when n tends to $+\infty$). Thus $(X_t^n)_{n \in \mathbb{N}}$ is a spectral measure.

Furthermore,

$$\begin{aligned}
X_t^n &= \mathbb{1}_{T_{n+1} > t} - \mathbb{1}_{T_n > t} \\
&= \int_0^t \left(-\mathbb{1}_{T_{n+1} > s} + \mathbb{1}_{T_n > s} + \mathbb{1}_{T_n > s} - \mathbb{1}_{T_{n-1} > s} \right) dN_s \\
&= \int_0^t \left(X_s^{n-1} - X_s^n \right) dN_s \, .
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} n X_t^n &= \int_0^t \sum_{n=0}^{\infty} X_s^n \, dN_s \\
&= \int_0^t I \, dN_s \\
&= N_t \, .
\end{aligned}$$

Details are left to the reader. ∎

PROPOSITION 21. — *For all $n \geq 1$, we have $T_n < T_{n+1}$.*

Proof. — Let

$$\begin{aligned}
R &= \sum_{i=0}^{N-1} \mathbb{1}_{T_n \in [t_i, t_{i+1}[} \mathbb{1}_{T_{n+1} \in [t_i, t_{i+1}[} \\
&= \sum_{i=0}^{N-1} \mathbb{1}_{T_{n+1} \in [t_i, t_{i+1}[} \mathbb{1}_{T_n \in [t_i, t_{i+1}[} \, .
\end{aligned}$$

From the identity

$$\langle \varepsilon(f), \mathbb{1}_{T_n > t} \varepsilon(g) \rangle = \sum_{j=0}^{n-1} \frac{(\int_0^t f(s) \, ds)^j}{j!} e^{-\int_0^t f(s) \, ds} \langle \varepsilon(f), \varepsilon(g) \rangle \, ,$$

where $\varepsilon(h)(\sigma) = \prod_{s \in \sigma} h(s)$ for all $\sigma \in \mathcal{P}$, all $h \in L^2(\mathbb{R}^+; \mathbb{C})$, we can prove that R converges strongly to 0 when the partition refines and that $\mathbb{1}_{T_n = +\infty} = 0$. ∎

PROPOSITION 22. — *We have $\Phi_{T_n} = \Phi_{T_n-}$ for all $n \in \mathbb{N}^*$, but T_n is not previsible, whatever is $n \in \mathbb{N}^*$.*

Proof. — Let $f \in \Phi_{T_n-}^{\perp}$, we have $\mathbb{1}_{T_n>t} E_t f = 0$ for all $t \in \mathbb{R}_+$. But also (see [Att]) we have

$$\mathbb{1}_{T_n>t} E_t f = f(\emptyset)\mathbb{1} + \int_0^t \left(\mathbb{1}_{T_{n-1}>s} D_s f - (\mathbb{1}_{T_n>s} - \mathbb{1}_{T_{n-1}>s}) E_s f \, d\chi_s\right)$$

$$- \int_0^t (\mathbb{1}_{T_n>s} - \mathbb{1}_{T_{n-1}>s})(D_s f + E_s f) \, ds \, .$$

Thus, for a.a.t

$$\mathbb{1}_{T_{n-1}>t} D_t f = (\mathbb{1}_{T_n>t} - \mathbb{1}_{T_{n-1}>t}) E_t f$$

and $\left(\mathbb{1}_{T_n>t} - \mathbb{1}_{T_{n-1}>t}\right)(D_t f + E_t f) = 0$.

In particular, $\mathbb{1}_{T_n>t} D_t f = 0$ for a.a.t. and thus $E_{T_n} f = 0$ (Theorem 16). This proves $\Phi_{T_n} = \Phi_{T_n-}$.

It is proved in [A-S] that if, for all $t \in \mathbb{R}^+$

$$x_t = \int_0^t m_s \, d\chi_s + \int_0^t a_s \, ds$$

with, for all $s \in \mathbb{R}^+$, m_s and a_s belong to $\Phi_{s]}$, $\int_0^\infty \|m_s\|^2 \, ds < \infty$ and $\int_0^\infty \|a_s\| \, ds < \infty$, then for every stopping time T, the limit (over refining partitions as usual)

$$x_T = \lim \sum_i \mathbb{1}_{T \in [t_i, t_{i+1}[} x_{t_{i+1}}$$

exists and is equal to

$$x_T = \int_0^\infty \mathbb{1}_{T>s} m_s \, d\chi_s + \int_0^\infty \mathbb{1}_{T>s} a_s \, ds \, .$$

Suppose T_n is previsible and let $(S_p)_{p\in\mathbb{N}}$ be a sequence of stopping times converging to T_n and with $S_p < T_n$ for all p.

We know that

$$\sum_i \mathbb{1}_{S_p \in [t_i, t_{i+1}[} \mathbb{1}_{T_n>t_{i+1}} \mathbb{1}$$

converges to $\mathbb{1}$. But $x_t^{(n)} = \mathbb{1}_{T_n>t} \mathbb{1} = \mathbb{1} - \int_0^t (x_s^{(n)} - x_s^{(n-1)})(d\chi_s + ds)$ thus

$$x_{S_p}^{(n)} = \mathbb{1} - \int_0^{+\infty} \mathbb{1}_{S_p>s}(x_s^{(n)} - x_s^{(n-1)})(d\chi_s + ds) = \mathbb{1} \, .$$

But $\mathbb{1}_{S_p>s}(x_s^{(n)} - x_s^{(n-1)}) = \mathbb{1}_{S_p>s}\mathbb{1} - \mathbb{1}_{S_p>s}\mathbb{1}_{T_{n-1}>s}\mathbb{1}$. This quantity converges to $\mathbb{1}_{T_n>s}\mathbb{1} - \mathbb{1}_{T_{n-1}>s}\mathbb{1}$ when p tends to $+\infty$.

Finally,

$$\mathbb{1} = \mathbb{1} - \int_0^\infty (\mathbb{1}_{T_n>s}\mathbb{1} - \mathbb{1}_{T_{n-1}>s}\mathbb{1})(d\chi_s + ds) \, .$$

As

$$\lim_{t\to+\infty} \mathbb{1}_{T_n>t}\mathbb{1} = 0 = \mathbb{1} - \int_0^{+\infty} (\mathbb{1}_{T_n>s}\mathbb{1} - \mathbb{1}_{T_{n-1}>s}\mathbb{1})(d\chi_s + ds)$$

we have a contradiction. ∎

VII. Applications to (strictly non-Fock) quasifree boson or fermion quantum stochastic theories

In the case of quasifree boson and fermion quantum stochastic theories, we have a family $(\mathcal{U}_t)_{t \in \mathbb{R}^+}$ of von Neumann algebras acting on a Hilbert space $\mathcal{H}$ and such that $\mathcal{U}_s \subset \mathcal{U}_t$ for all $s \leq t$. We put $\mathcal{U}_\infty = \bigvee_{t \in \mathbb{R}^+} \mathcal{U}_t$. We also suppose that there exists a cyclic and separating unit vector Ω for $\mathcal{U}_\infty$ in $\mathcal{H}$, and that there exists a family $(M_t)_{t \in \mathbb{R}^+}$ of normal, ω-invariant, conditional expectations $M_t : \mathcal{U}_\infty \to \mathcal{U}_t$ where ω is the vector state associated to Ω. We denote by $\mathcal{H}_t$ the closure of $\mathcal{U}_t \Omega$ in $\mathcal{H}$ and by E_t the orthogonal projection from $\mathcal{H}$ to $\mathcal{H}_t$. We have $E_t X \Omega = M_t(X)\Omega$ for all $X \in \mathcal{U}_\infty$. Furthermore since $\mathcal{H}_t$ is invariant under $\mathcal{U}_t$, it follows that E_t belongs to $\mathcal{U}_t'$.

This setup includes the Ito-Clifford (fermion) theory and the quasi-free CAR and CCR theories. In the former case, ω is a tracial state.

In all these three cases we have a representation theorem for the elements of $\mathcal{H}$, see [H-L], [BSW], [Lin], [BW1], [B-L]. This representation implies that if $(T_n)_n$ is a sequence of $(\mathcal{U}_t)_{t \in \mathbb{R}^+}$-stopping times converging to T, then E_{T_n} converges strongly to E_T (Corollary 3.4 of [BW1], Theorem 3.7 of [B-L]). Thus, as in Theorem 19 the filtered space $(\mathcal{H}, (E_t)_{t \in \mathbb{R}^+}, (\mathcal{U}_t)_{t \in \mathbb{R}^+})$ is quasi-left continuous.

In fact, in [B-L], [BW1], [BW2] and others, one defines M_T as the strong limit of $\sum_i \mathbb{1}_{T \in [t_{i-1}, t_i[} M_{t_i}$ and one proves that M_T is an orthogonal projection on $L^2(\mathcal{U}_\infty)$. It is easy to see that, for $X \in L^2(\mathcal{U}_\infty)$, we have $M_T(X)\Omega = E_T X \Omega$. Thus, using the isometry between $L^2(\mathcal{U}_\infty)$ and $\mathcal{H}$ (given by $X \mapsto X\Omega$) makes the study of E_T or M_T equivalent.

Actually, in [B-L] and [BW1], M_T is the limit of $\sum_i M_{t_i} \mathbb{1}_{T \in [t_{i-1}, t_i[}$, but if one denotes by $\widetilde{M}_T$ this limit, then clearly $M_T(X) = \widetilde{M}_T(X^*)^*$.

The case of tracial state.

Let us suppose now that ω is tracial. This is for example the case of the Ito-Clifford theory and of the CAR algebra over $L^2(\mathbb{R}^+)$ where ω is the gauge-invariant quasi-free state given by

$$\omega(b^*(f)b(g)) = \frac{1}{2} \int_0^{+\infty} \bar{f}(s)g(s)\, ds\,.$$

PROPOSITION 23. — *Let S, T be two $(\mathcal{U}_t)_{t \geq 0}$-stopping times such that $S \leq T$. Let $(\mathcal{S}_n)_{n \in \mathbb{N}}$ be an increasing family of partitions of $\mathbb{R}^+$ and $R_{\mathcal{S}_n}$ be defined by $R_{\mathcal{S}_n} = \sum_{t_i \in \mathcal{S}_n} \mathbb{1}_{T \in [t_{i-1}, t_i[} \mathbb{1}_{S \in [t_{i-1}, t_i[}$.*

Then the sequence $(R_{\mathcal{S}_n})_{n\in\mathbb{N}}$ is always strongly convergent.

Proof. — We have

$$\omega\Big((R_{\mathcal{S}_n} - R_{\mathcal{S}_m})^*(R_{\mathcal{S}_n} - R_{\mathcal{S}_m})\Big)$$
$$= \langle \Omega, (R_{\mathcal{S}_n} - R_{\mathcal{S}_m})^*(R_{\mathcal{S}_n} - R_{\mathcal{S}_m})\Omega \rangle$$
$$= \omega(R_{\mathcal{S}_n}^* R_{\mathcal{S}_n}) + \omega(R_{\mathcal{S}_m}^* R_{\mathcal{S}_m}) - \langle \Omega, R_{\mathcal{S}_n}^* R_{\mathcal{S}_m}\Omega \rangle - \langle \Omega, (R_{\mathcal{S}_m})^* R_{\mathcal{S}_n}\Omega \rangle \, .$$

We claim that if $\mathcal{S} \subset \mathcal{S}'$ then $\langle R_{\mathcal{S}}\Omega, R_{\mathcal{S}'}\Omega \rangle = \langle \Omega, R_{\mathcal{S}'}\Omega \rangle$. Indeed,

$$R_{\mathcal{S}'} = \sum_{t_i \in \mathcal{S}'} \mathbb{1}_{T\in[t_{i-1},t_i[}\mathbb{1}_{S\in[t_{i-1},t_i[}$$

and

$$R_{\mathcal{S}}^* R_{\mathcal{S}'} = \sum_{s_j\in\mathcal{S}}\sum_{t_i\in\mathcal{S}'} \mathbb{1}_{S\in[s_{j-1},s_j[}\mathbb{1}_{T\in[s_{j-1},s_j[}\mathbb{1}_{T\in[t_{i-1},t_i[}\mathbb{1}_{S\in[t_{i-1},t_i[}$$
$$= \sum_{s_j\in\mathcal{S}}\sum_{t_i;s_{j-1}\le t_{i-1}<t_i<s_j} \mathbb{1}_{S\in[s_{j-1},s_j[}\mathbb{1}_{T\in[t_{i-1},t_i[}\mathbb{1}_{S\in[t_{i-1},t_i[}$$

thus, by traciality

$$\omega(R_{\mathcal{S}}^* R_{\mathcal{S}'}) = \sum_{s_j\in\mathcal{S}}\sum_{t_i;s_{j-1}\le t_{i-1}<t_i<s_j} \omega\big(\mathbb{1}_{T\in[t_{i-1},t_i[}\mathbb{1}_{S\in[t_{i-1},t_i[}\big)$$
$$= \omega(R_{\mathcal{S}'}) \, .$$

This gives

$$\omega\Big((R_{\mathcal{S}_n} - R_{\mathcal{S}_m})^*(R_{\mathcal{S}_n} - R_{\mathcal{S}_m})\Big) = \omega(R_{\mathcal{S}_n}^* R_{\mathcal{S}_n}) - \omega(R_{\mathcal{S}_m}^* R_{\mathcal{S}_n})$$

and $(R_{\mathcal{S}_n})_{n\in\mathbb{N}}$ is converging in $L^2(\mathcal{U}_\infty)$. Let R be the limit. We have $R_{\mathcal{S}_n} \in \mathcal{U}_\infty$ and $\|R_{\mathcal{S}_n}\| \le 1$ for all n. Therefore $R_{\mathcal{S}_n}$ converges to R strongly and R belongs to $\mathcal{U}_\infty$. ∎

In this context we are thus always able to say if two stopping times such that $S \le T$ are such that $S < T$ or not. Indeed, this is the case if and only if $R = 0$ and $\mathbb{1}_{S=+\infty} = 0$.

Remark. — If we want the following property to be satisfied:

$$S < T \text{ and } T \le R \Longrightarrow S < R$$

we need to define $S < T$ by a *strong* convergence of $\sum_i \mathbb{1}_{T>t_i}\mathbb{1}_{S\in[t_{i-1},t_i[}$ to I.

Indeed, in this case

$$\sum_i \mathbb{1}_{R\in[t_{i-1},t_i[}\mathbb{1}_{S\in[t_{i-1},t_i[} = \sum_i \mathbb{1}_{R\in[t_{i-1},t_i[}\mathbb{1}_{T\in[t_{i-1},t_i[}\sum_i \mathbb{1}_{T\in[t_{i-1},t_i[}\mathbb{1}_{S\in[t_{i-1},t_i[} \, .$$

But the second sum in the right hand side converges strongly to 0 and the first one is bounded by 1. This gives the claim.

REFERENCES

[App] D. Applebaum. — *The strong Markov property for fermion Brownian motion*, J. Funct. Anal. **65** (1986), 273–291.

[Att] S. Attal. — *Extensions of quantum stochastic calculus*, Quantum probability summer school, Grenoble, June 15th–July 3rd 1998, Ed. S. Attal and J.M. Lindsay, World Scientific, to appear.

[A-S] S. Attal & K.B. Sinha. — *Stopping semimartingales on Fock space*, Quantum Probability Communications , vol. **X**, World Scientific (1998), 171–186.

[B-L] C. Barnett & T. Lyons. — *Stopping non-commutative processes*, Math. Proc. Camb. Phil. Soc. (1986), 99–151.

[BSW] C. Barnett, R.-F. Streater, I.F. Wilde. — *The Ito-Clifford integral*, Journal of Functional Analysis **48** (1982), 172–212.

[BN1] C. Barnett & I. Wilde. — *Random times and time projections*, Proceedings of the American Mathematical Society, **110**(2), October 1990.

[BN2] C. Barnett & I. Wilde. — *Random times, predictable processes and stochastic integration in finite Von Neumann Algebra*, Proc. London Math. Soc **3** (1993), 355–383.

[Hud] R.L. Hudson. — *The strong Markov property for canonical Wiener process*, Journal of Functional Analysis **34** (1979), 266–281.

[H-P] R.L. Hudson & K.R. Parthasarathy. — *Quantum Ito's formula and stochastic evolutions*, Communications in Mathematical Physics **93** (1984), 301–323.

[Lin] J.M. Lindsay. — *Fermion martingales*, Probability Theory and Related Fields **71** (1986), 307–320.

[Mey] P.-A. Meyer. — *Quantum probability for probabilists*, Springer Lecture Notes in Math., 1538, 1993.

[P-S] K.R. Parthasarathy & K.B. Sinha. — *Stop times in Fock space stochastic calculus*, Probability Theory and Related Fields **75** (1987), 317–349.

$$- \Diamond -$$

Université de Grenoble I
Institut Fourier
UMR 5582 CNRS-UJF
UFR de Mathématiques
B.P. 74
38402 ST MARTIN D'HÈRES Cedex (France)

(6 novembre 2001)