ON THE VANISHING OF HIGHER SYZYGIES OF CURVES

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Abstract

The present paper is related to a conjecture made by Green and Lazarsfeld concerning 1-linear syzygies of curves. Given a complex projective smooth curve, we prove that the least integer q for which the property (M_q) fails for a line bundle L on X does not depend on L as soon as its degree becomes sufficiently large. Consequently, this number is an invariant of the curve, and the statement of Green-Lazarsfeld's conjecture is equivalent to saying that this invariant coincides to the gonality of the curve. We verify the conjecture for plane curves, curve lying on a Hirzebruch surface, and for generic curves having the genus sufficiently large compared to the gonality.

0. Introduction, main results

A main challenge in the theory of syzygies is to interpret the information carried by the graded Betti numbers of a smooth projective variety. Notably, the attempt to understand the way in which the distribution of zeroes in a Betti table interacts with the geometry of the variety has led to a considerable amount of work, motivated by the conjectures that Green, Green-Lazarsfeld, and others had formulated (see, for example, [Gr1], [GL1], [EL], [La2]).

One of the most significant conjectures made by Green and Lazarsfeld (cf. [GL1] 3.7; see also [Gr3] 3.5 and [La2] 2.3), nowadays known as the *gonality conjecture*, predicts that one could read off the gonality of a smooth complex projective curve from the minimal resolution of any line bundle of sufficiently large degree. In order to give a precise statement, the authors introduced the vanishing property (M_k) (see [GL1], [Gr3]), which is the following (we use the notation of Section 1).

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Definition. (Green-Lazarsfeld) If X is a smooth complex projective curve of genus g, L a line bundle on X, and $k \geq 0$ an integer, one says that (X, L) *satisfies the property* (M_k) (or, simply, L *satisfies* (M_k)) if $K_{p,1}(X, L) = 0$, for all $p \geq h^0L - k - 1$.

It is well-known (see, for example, [GL1], [GL2] or [Sch1]) that if X carries a g^1_k, then no line bundle of sufficiently large degree can satisfy (M_k). The gonality conjecture states a converse of this fact: if $\deg(L) >> 2g$, and (M_k) fails for L, then X carries a g^1_q, with $q \leq k$. Green has shown this conjecture holds for $k = 1, 2$ (cf. [Gr1]); Ehbauer proved it in the case $k = 3$ (cf. [Ehb]). Thus hyperelliptic and trigonal curves are characterized by syzygies.

The purpose of the present work is to prove that, under certain conditions, the property (M_k) is preserved when we add an effective divisor to a line bundle. The first result is:

Theorem 1. *Let X be a smooth complex projective curve of genus $g \geq 1$, L_0 be a nonspecial globally generated line bundle on X, and $k \geq 0$ be an integer such that the pair (X, L_0) satisfies the property (M_k). Then, for any effective divisor D on X, the pair $(X, L_0 + D)$ also satisfies the property (M_k). In particular, for any line bundle L with $\deg(L) \geq \deg(L_0) + g$, the pair (X, L) satisfies the property (M_k) as well.*

There are a number of immediate consequences of Theorem 1. For instance, if X carries a g^1_k, and $L_0 \in \mathrm{Pic}(X)$ is nonspecial, and globally generated, then (X, L_0) cannot satisfy (M_k). Further, the least k for which the property (M_k) fails for a nonspecial, globally generated line bundle cannot decrease when adding effective divisors, and cannot pass over the gonality of X, and thus it must be constant when the degree of the line bundle grows large enough. The challenge of the gonality conjecture is to show that this constant always equals the gonality of X.

In view of Theorem 1, verifying the gonality conjecture for a given k-gonal X of genus g reduces to finding a single nonspecial, globally generated line bundle (for example, a line bundle whose degree is sufficiently large compared to $2g$), with property (M_{k-1}). Concretely, we get the following criterion for testing the gonality conjecture:

Corollary 2. *Let X be a smooth complex projective curve of genus $g \geq 1$, which carries a g^1_k, and L_0 be a nonspecial globally generated line bundle on X satisfying the property (M_{k-1}). Then X is k-gonal, and the gonality conjecture is valid for X.*

By semicontinuity of minimal resolutions, and irreducibility of the moduli space $\mathcal{M}_{g,k}$ of k-gonal curves of genus g (cf. [F]), it turns out that the gonality conjecture would also be valid for a generic curve in $\mathcal{M}_{g,k}$, once it was verified for a particular k-gonal curve of genus g. It seems thus very reasonable to predict that for any positive integers k, and g, such that $\mathcal{M}_{g,k} \neq \emptyset$, a generic k-gonal curve of genus g verifies the gonality conjecture (see below).

One could address now the question of what happens if we drop the nonspeciality condition in Theorem 1. A partial answer is given by:

Theorem 3. *Let X be a smooth complex projective curve of genus $g \geq 1$, and $k \geq 0$ be an integer such that the pair (X, K_X) satisfies the property (M_k). Then, for any effective divisor D on X, the pair $(X, K_X + D)$ satisfies the property (M_k).*

Unfortunately, the result just stated is weaker than it might look like at a first sight - it does not show that Green's generic canonical conjecture implies gonality conjecture for a generic curve, as one could think of. The following better version of it would do the job instead.

Conjecture. *Let X be a smooth complex projective curve of Clifford dimension one, and p an integer such that $K_{p,1}(X, K_X) = 0$. Then there exist a positive integer d, and an effective divisor D on X, of degree d, such that $K_{p+d-1,1}(X, K_X + D) = 0$.*

This conjecture is obviously false if we drop the condition that the Clifford dimension be equal to one, as seen by analyzing the case of smooth plane curves. In exchange, it holds for other curves which are not plane curves, such as curves lying on Hirzebruch surfaces (see Sections 6 and 8), and, more generally, it is true for curves which verify both the gonality conjecture, and Green's canonical conjecture (trigonal curves, for instance).

The main ingredient we use to prove Theorem 1, and Theorem 3, is projection of syzygies, concept which was introduced by Ehbauer [Ehb], in a coordinate-based manner. In Section 2, we propose a more abstract view on the subject, in the spirit of [Gr2] 1.b.1, and we think of projections of syzygies as being *corestrictions* of the fiber-restrictions of a certain morphism between vector bundles. We investigate some properties of the projection morphisms, and we show, among other things, that, under some conditions which are almost always satisfied, any nonzero syzygy survives by projection from a generic point (see (2.8)).

The third Section deals with projection of syzygies of varieties. We recall here Ehbauer's approach, and examine the case of syzygies of curves.

In Section 4 we complete the proofs of Theorem 1, and Theorem 3, and further, in the final part of the paper, we analyze some very concrete cases.

We verify first the gonality conjecture for smooth plane curves, and for smooth curves lying on a Hirzebruch surface, in which cases we can make use of the geometry of the ambient surface to produce suitable line bundles. Additionally, we recover the description of the minimal pencils in these cases, which has already been known before (for plane curve, see, for example [ACGH], for curves on a Hirzebruch surface we refer to [Ma1]).

In the fifth Section, we test the gonality conjecture for nodal curves on $\mathbf{P}^1 \times \mathbf{P}^1$, case which eventually shows the following (compare with the main result of [Sch2]):

Theorem 4. *For any integer $k \geq 3$, the gonality conjecture is valid for a generic k-gonal curve of genus $g > (k-1)(k-2)$.*

We conclude this paper by applying the vanishing result (6.2) proved here to show that Green's canonical conjecture holds for smooth curves on Hirzebruch surfaces (compare with [Lo]). As a general philosphy, we expect that smart choices of line bundles on a surface, whose restrictions satisfy the vanishing property required in the gonality conjecture for some curves lying on that surface, be used to prove Green's canonical conjecture for such curves.

1. Some notation, preliminaries

For many of the theoretical facts included in this section we refer to the papers [Gr1], [Gr3], [La1], without further mention. Throughout this section, V is a finite-dimensional complex vector space, and SV denotes the symmetric algebra of V.

(1.1) If $B = \bigoplus_{q \in \mathbf{Z}} B_q$ is a graded SV-module, then there is the *Koszul complex*

$$... \longrightarrow B_{q-1} \otimes \bigwedge^{p+1} V \overset{d_{p+1,q-1}}{\longrightarrow} B_q \otimes \bigwedge^{p} V \overset{d_{p,q}}{\longrightarrow} B_{q+1} \otimes \bigwedge^{p-1} V \longrightarrow ...$$

where

$$d_{p,q}(b \otimes (v_1 \wedge ... \wedge v_p)) = \sum_{j=1}^{p} (-1)^{j-1} (v_j b) \otimes (v_1 \wedge ... \wedge \widehat{v_j} \wedge ... \wedge v_p),$$

for any element $b \in B_q$ and any linearly independent vectors $v_1, ..., v_p \in V$. One denotes

$$K_{p,q}(B,V) = \text{Ker } d_{p,q} / \text{Im } d_{p+1,q-1};$$

they are called the *Koszul cohomology groups of B.*

(1.2) It is well-known (see, for example, [Gr3]) that all $K_{p,q}(SV,V)$ vanish except for $K_{0,0}(SV,V)$ which is isomorphic to $\mathbf{C}$.

(1.3) To any short exact sequence of graded SV-modules, $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ one associates a long exact sequence, for any p (cf. [Gr1], 1.d.4):

$$... \to K_{p+1,0}(C,V) \to K_{p,1}(A,V) \to K_{p,1}(B,V) \to K_{p,1}(C,V) \to K_{p-1,2}(A,V) \to ...$$

In particular, if $C_0 = 0$, then the map $K_{p,1}(A,V) \to K_{p,1}(B,V)$ is injective, and if C_0, and C_1 both vanish, then $K_{p,1}(A,V) \cong K_{p,1}(B,V)$.

(1.4) If all the graded pieces B_q of B are finitely generated complex vector spaces, and $B_q = 0$ for all $q < 0$, then B has a *minimal free resolution* of type

$$... \to \bigoplus_{q \geq 1} SV(-q) \otimes M_{1,q}(B,V) \to \bigoplus_{q \geq 0} SV(-q) \otimes M_{0,q}(B,V) \to B \to 0,$$

where $M_{p,q}(B,V)$ are finitely generated complex vector spaces. They are related to the Koszul cohomology groups of B via the isomorphisms of vector spaces: $K_{p,q}(B,V) \cong M_{p,p+q}(B,V)$, for all integers p and q.

(1.5) In the real life, we often meet graded SV-modules which satisfy the following property:

$$B_q = 0 \text{ for all } q < 0, \text{ and } K_{p,0}(B,V) = 0 \text{ for all } p \geq 1.$$

For example, these conditions are automatically fulfilled if either $B_q = 0$ for all $q \leq 0$ or B is the homogeneous coordinate ring of a nondegenerate subvariety of $\mathbf{P}$. In the latter case, the graded pieces of B are moreover finitely generated vector spaces, and thus (1.4) also applies.

(1.6) If X is an irreducible complex projective variety, $L \in \mathrm{Pic}(X)$ is a line bundle, $\mathcal{F}$ is a coherent sheaf, and $V \subset H^0(X,L)$, take $B = \bigoplus_{q\in\mathbf{Z}} H^0(X, \mathcal{F} \otimes qL)$ (here, qL is the q-th tensor power of L in $\mathrm{Pic}(X)$), and denote $K_{p,q}(X,\mathcal{F},L,V) = K_{p,q}(B,V)$. If we choose $L = \mathcal{O}_X(D)$, for a divisor D on X, then we replace L by D, and write $K_{p,q}(X,\mathcal{F},D,V)$; similarly for an invertible $\mathcal{F}$. If $V = H^0(X,L)$, we drop V and write $K_{p,q}(X,\mathcal{F},L)$, if $\mathcal{F} \cong \mathcal{O}_X$, we suppress it, and write $K_{p,q}(X,L,V)$; the notation $K_{p,q}(X,L)$ corresponds to the choice $V = H^0(X,L)$ and $\mathcal{F} \cong \mathcal{O}_X$.

(1.7) If L is a globally generated line bundle over the smooth irreducible variety X, and $V = H^0(X,L)$, by denoting X' the image of X in $\mathbf{P}V^*$, through the morphism given by the *complete* linear system $|L|$, and $S_{X'} = \mathrm{Im}(SV \to \bigoplus H^0(X,qL))$ its homogeneous coordinate ring, then, in virtue of (1.3), we have natural isomorphisms $K_{p,1}(S_{X'},V) \cong K_{p,1}(X,L)$, for all integers p.

(1.8) If X is a smooth irreducible projective variety with $h^1\mathcal{O}_X = 0$, $L \in \mathrm{Pic}(X)$ a line bundle, and $Y \in |L|$ is irreducible, then $K_{p,1}(X,L) \cong K_{p,1}(Y,L|_Y)$ for any integer p. The proof is very similar to 3.b.7 of [Gr1], and we sketch it next.

Let $V = H^0(X,L)$ as usual. From the exact sequence

$$0 \to H^0\mathcal{O}_X \to V \to H^0(Y,L|_Y) \to 0,$$

we have $\wedge^p V \cong \wedge^p H^0(Y,L|_Y) \oplus \wedge^{p-1} H^0(Y,L|_Y)$, for any p. The long exact sequence associated to the exact sequence of graded SV-modules $0 \to B' \to B \to A \to 0$, where $B' = \bigoplus H^0(X,(q-1)L)$, $B = \bigoplus H^0(X,qL)$, and $A = B/B'$, gives isomorphisms, for any p, $K_{p,1}(A,V) \cong K_{p,1}(X,L) \oplus K_{p-1,1}(X,L)$. Using the above-mentioned decomposition of $\wedge^p V$, and (1.3), we see that $K_{p,1}(A,V) \cong K_{p,1}(Y,L|_Y) \oplus K_{p-1,1}(Y,L|_Y)$, and an induction on p closes the proof of the claim.

2. Projections of syzygies at large

We consider, as in the previous section, $r \geq 1$ an integer, V an $(r+1)$-dimensional complex vector space, and B a graded SV-module.

(2.1) We denote by $\mathbf{P} = \mathbf{P}V^* = \mathrm{Proj}(SV)$ the projective space of one-dimensional quotients of V. A point $x \in \mathbf{P}$ corresponds to a short exact sequence of vector spaces:

$$0 \longrightarrow W_x \longrightarrow V \stackrel{\mathrm{ev}_x}{\longrightarrow} L_x \longrightarrow 0.$$

The one-dimensional vector space L_x is canonically isomorphic to the fiber over x of the line bundle $\mathcal{O}_{\mathbf{P}}(1)$ of hyperplane sections in $\mathbf{P}$, and the surjection $V \xrightarrow{\mathrm{ev}_x} L_x$ is the evaluation of the elements of $V = H^0\mathcal{O}_{\mathbf{P}}(1)$ in the point x. The above-mentioned short exact sequence induces, for any $p \geq 0$, a short exact sequence:

$$0 \longrightarrow \bigwedge^{p+1} W_x \longrightarrow \bigwedge^{p+1} V \longrightarrow L_x \otimes \bigwedge^{p} W_x \longrightarrow 0;$$

the surjection $\bigwedge^{p+1} V \longrightarrow L_x \otimes \bigwedge^p W_x$ can be thought of as being a *projection morphism* for multivectors. In coordinates, for a basis $\{v_1, ..., v_r\}$ of W_x, a vector $v_0 \notin W_x$, and multi-indices $0 \leq i_1 < ... < i_{p+1} \leq r$, the multivector $v_{i_1} \wedge ... \wedge v_{i_{p+1}}$ is projected to $[v_{i_1}] \otimes (v_{i_2} \wedge ... \wedge v_{i_{p+1}})$, where $[v_{i_1}]$ denotes the class of v_{i_1} modulo W_x.

(2.2) By twisting the exact sequences of (2.1) with the graded pieces of B, we get a short exact sequence of complexes, for any integer l:

$$0 \longrightarrow B_* \otimes \bigwedge^{l+1-*} W_x \longrightarrow B_* \otimes \bigwedge^{l+1-*} V \longrightarrow L_x \otimes B_* \otimes \bigwedge^{l-*} W_x \longrightarrow 0,$$

and further, by setting $l = p + q$, a long exact sequence (compare with [Gr2] 1.b.1):

$$... \to K_{p+1,q}(B, W_x) \xrightarrow{\eta_x} K_{p+1,q}(B, V) \xrightarrow{\pi_x} L_x \otimes K_{p,q}(B, W_x) \xrightarrow{\mu_x} K_{p,q+1}(B, W_x) \to ...$$

The map $K_{p+1,q}(B, V) \xrightarrow{\pi_x} L_x \otimes K_{p,q}(B, W_x)$ is called *projection of syzygies centered in* x, and the elements of its image, which coincides to the kernel of the (noncanonical) connecting map

$$L_x \otimes K_{p,q}(B, W_x) \xrightarrow{\mu_x} K_{p,q+1}(B, W_x),$$

are called *projected syzygies.*

(2.3) The composed map $\partial_x := (\eta_x \otimes \mathrm{id}_{L_x}) \circ \pi_x : K_{p+1,q}(B, V) \longrightarrow L_x \otimes K_{p,q}(B, V)$ is in fact the fiber-restriction over x of a natural morphism of sheaves:

$$\mathcal{O}_{\mathbf{P}} \otimes K_{p+1,q}(B, V) \xrightarrow{\partial} \mathcal{O}_{\mathbf{P}}(1) \otimes K_{p,q}(B, V).$$

In order to see this, let us recall first that there is an exact complex of vector bundles on $\mathbf{P}$, derived from the projection morphisms:

$$0 \to \mathcal{O}_{\mathbf{P}}(-r-1) \otimes_{\mathbf{C}} \bigwedge^{r+1} V \to \mathcal{O}_{\mathbf{P}}(-r) \otimes_{\mathbf{C}} \bigwedge^{r} V \to ... \to \mathcal{O}_{\mathbf{P}}(-1) \otimes_{\mathbf{C}} V \to \mathcal{O}_{\mathbf{P}} \to 0.$$

For any integer n, and for any point $x \in \mathbf{P}$, the restriction to the fiber over x, $\bigwedge^{n+1} V \to L_x \otimes \bigwedge^n V$ of the morphism of vector bundles $\mathcal{O}_{\mathbf{P}} \otimes \bigwedge^{n+1} V \to \mathcal{O}_{\mathbf{P}}(1) \otimes \bigwedge^n V$, is obtained by composing the inclusion $L_x \otimes \bigwedge^n W_x \hookrightarrow L_x \otimes \bigwedge^n V$, and the surjection $\bigwedge^{n+1} V \to L_x \otimes \bigwedge^n W_x$. Alternatively, it is also obtained as the composed map of the evaluation $V \otimes \bigwedge^n V \xrightarrow{\mathrm{ev}_x \otimes \mathrm{id}} L_x \otimes \bigwedge^n V$, and the Koszul differential $\bigwedge^{n+1} V \to V \otimes \bigwedge^n V$. For any integer $l \in \mathbf{Z}$, the exact complex above, and the

Koszul complex of B, canonically induce a double complex of vector bundles over $\mathbf{P}$ with general term

$$\mathcal{K}^{s,q} = \mathcal{O}_{\mathbf{P}}(s) \otimes B_q \otimes \bigwedge^{l-s-q} V,$$

Since the rows of this double complex are exact, we get a spectral sequence abutting to zero, with general term

$$\mathcal{E}_1^{s,q} = \mathcal{O}_{\mathbf{P}}(s) \otimes K_{l-s-q,q}(B,V).$$

The differential $\mathcal{O}_{\mathbf{P}} \otimes K_{p+1,q}(B,V) \xrightarrow{\partial} \mathcal{O}_{\mathbf{P}}(1) \otimes K_{p,q}(B,V)$, obtained at the first level of the above spectral sequence by setting $s=0$ and $p=l-q-1$, is the morphism we were looking for.

(2.4) Projection of syzygies is functorial, that is, for any morphism $B \to C$ of graded SV-modules, and any integers p, q, the induced morphisms between syzygies $K_{p+1,q}(B,V) \to K_{p+1,q}(C,V)$, and $L_x \otimes K_{p,q}(B,W_x) \to L_x \otimes K_{p,q}(C,W_x)$ commute with the corresponding projection morphisms $K_{p+1,q}(B,V) \to L_x \otimes K_{p,q}(B,W_x)$, and, respectively, $K_{p+1,q}(C,V) \to L_x \otimes K_{p,q}(C,W_x)$. Furthermore, projection of syzygies is compatible with the connecting morphisms arising from the long cohomology sequence of a short exact sequence of graded SV-modules. More precisely, if $0 \to A \to B \to C \to 0$ is an exact sequence of graded SV-modules, and p, and q, are two integers, then the connecting morphisms $K_{p+1,q}(C,V) \to K_{p,q+1}(A,V)$, and $L_x \otimes K_{p,q}(C,W_x) \to L_x \otimes K_{p-1,q+1}(A,W_x)$ (cf. [Gr1], 1.d.4), commute with the projections $K_{p+1,q}(C,V) \to L_x \otimes K_{p,q}(C,W_x)$, and, respectively, $K_{p,q+1}(A,V) \to L_x \otimes K_{p-1,q+1}(A,W_x)$.

From now on, we will require B to satisfy the property (1.5). In this case, we are able to prove some properties of the projection of syzygies for $q=1$, as follows.

(2.5) For any $p \geq 1$, the map $K_{p+1,1}(B,V) \xrightarrow{H^0(\partial)} V \otimes K_{p,1}(B,V)$ is injective.

To prove this claim, we apply first H^0 to the double complex $\mathcal{K}^{s,q}$. The new double complex we obtain in this way, with general term $H^0\mathcal{K}^{s,q}$, gives a spectral sequence such that $E_1^{s,q} \cong H^0\mathcal{E}_1^{s,q}$, and $E_\infty^{s,q} = 0$ for any pair (s,q), except for $(s,q) = (0,l)$ when $E_\infty^{0,l} = B_l$. Moreover, we see that $H^0(\partial)$ coincides to the differential $E_1^{0,1} \to E_1^{1,1}$ of this new spectral sequence. Because of the assumptions we have made on B, we get $E_2^{0,1} = E_3^{0,1} = ... = E_\infty^{0,1} = 0$. In particular, Ker $H^0(\partial) = 0$, which proves the claim.

In addition, (2.5) goes to show that vanishing of $K_{p,1}(B,V)$ automatically implies vanishing of $K_{p+1,1}(B,V)$.

(2.6) For any $p \geq 0$, the map $K_{p+1,1}(B,W_x) \xrightarrow{\eta_x} K_{p+1,1}(B,V)$ is injective.

Indeed, the long exact sequence of (2.2) applied for $l=p$, shows that $K_{p+1,0}(B,W_x) = 0$ for any $p \geq 0$, and thus B satisfies the property (1.5) as a graded SW_x-module as well. In this case, setting $l = p+1$ in (2.2), the map η_x is injective, as claimed, and the long exact sequence becomes:

$$0 \to K_{p+1,1}(B, W_x) \stackrel{\eta_x}{\to} K_{p+1,1}(B, V) \stackrel{\pi_x}{\to} L_x \otimes K_{p,1}(B, W_x) \stackrel{\mu_x}{\to} K_{p,2}(B, W_x) \to ...$$

(2.7) As an immediate consequence of (2.6), replacing $(p+1)$ by p, we see that, for all $p \geq 1$ the projection of syzygies is actually the corestriction of the map $K_{p+1,1}(B, V) \xrightarrow{\partial_x} L_x \otimes K_{p,1}(B, V)$, obtained by restricting the morphism ∂ to the fibers over the point x. Moreover, the map $K_{p+1,1}(B, V) \xrightarrow{\partial_x} L_x \otimes K_{p,1}(B, V)$ can be seen in two different ways: either as the composed map $(\mathrm{id}_{L_x} \otimes \eta_x) \circ \pi_x$, as before, or, alternatively, as the composition between the evaluation map $V \otimes K_{p,1}(B, V) \longrightarrow L_x \otimes K_{p,1}(B, V)$ and $H^0(\partial)$ (see also (2.3)).

(2.8) Any nonzero element $\alpha \in K_{p+1,1}(B, V)$ survives when we project it from a point x outside a projective subspace of $\mathbf{P}$.

In fact, for any nonzero element $\alpha \in K_{p+1,1}(B, V)$, writing $H^0(\partial)(\alpha) = \sum v_i \otimes \alpha_i$, with $\alpha_i \in K_{p,1}(B, V)$ linearly independent and $v_i \in V$, $\partial_x(\alpha) \neq 0$ as long as the point x does not belong to the projective subspace of $\mathbf{P}$, $\{y \in \mathbf{P},\ v_i \in W_y \text{ for all } i\}$. This subspace is not the whole $\mathbf{P}$, as $H^0(\partial)(\alpha) \neq 0$.

3. Projections of syzygies of projective varieties

In this Section, we move towards a more geometric context, and apply the general theory of Section 2 to syzygies of projective varieties.

First of all, we recover the projections of syzygies as introduced by Ehbauer. For this, consider $X \subset \mathbf{P}$ a nondegenerate irreducible variety, $x \in \mathbf{P}$ a point, and $Y \subset \mathbf{P}W_x^*$ the image of X by the projection centered in x, $\mathbf{P}\backslash\{x\} \longrightarrow \mathbf{P}W_x^*$. Denote by $I_X \subset SV$, $I_Y \subset SW_x$ the homogeneous ideals, and $S_X = SV/I_X$, $S_Y = SW_x/I_Y$ the homogeneous coordinate rings of X, and Y respectively. It is clear that, via the embedding $SW_x \subset SV$, $I_Y = SW_x \cap I_X$, and thus we have a natural embedding $S_Y \hookrightarrow S_X$. It induces, for any integer p, a natural injective map $K_{p,1}(S_Y, W_x) \hookrightarrow K_{p,1}(S_X, W_x)$. Moreover, $L_x \otimes K_{p,1}(S_Y, W_x)$ contains all the projected syzygies, as shown in [Ehb]:

(3.1) Lemma. (Ehbauer) *For any* $p \geq 1$, $\mathrm{Im}(K_{p+1,1}(S_X, V) \xrightarrow{\pi_x} L_x \otimes K_{p,1}(S_X, W_x)) \subset L_x \otimes K_{p,1}(S_Y, W_x)$.

Proof. (cf. [Ehb] Section 6) We begin with the exact sequence of graded SV-modules

$$0 \longrightarrow I_X \longrightarrow SV \longrightarrow S_X \longrightarrow 0;$$

since the image of X in $\mathbf{P}$ is nondegenerate, the long exact sequence gives rise to a canonical isomorphisms, for any $p \geq 1$,

$$K_{p+1,1}(S_X, V) \cong K_{p,2}(I_X, V) \cong \mathrm{Ker}\left((I_X)_2 \otimes \bigwedge^p V \to (I_X)_3 \otimes \bigwedge^{p-1} V\right);$$

similarly for Y. The statement of Lemma reduces then to prove that the projection of an element of $K_{p,2}(I_X, V)$ belongs to $L_x \otimes K_{p-1,2}(I_Y, W_x)$.

Choose next homogeneous coordinates $X_0, ..., X_r$ on $\mathbf{P}$, such that $x = [1 : 0 : ... : 0]$, and all the coordinate points belong to X. An element in $K_{p+1,1}(S_X, V)$ can be seen then as a collection of quadrics vanishing on X, $(Q_{i_1...i_p})_{0 \le i_1 < ... < i_p \le r}$, satisfying the equations:

$$\sum_{k \notin \{k_1,...,k_{p-1}\}} (-1)^{\#\{k_i < k\}} Q_{k_1...k...k_{p-1}} X_k = 0,$$

for all $0 \le k_1 < ... < k_{p-1} \le r$. Since the coordinate points belong to X, and $Q_{i_1...i_p}$'s vanish on X it follows that any such a quadric contains no terms of type X_i^2; the equations above tell moreover that $Q_{i_1...i_p}$ has only terms of type $X_j X_k$ with $j \neq k$, and $j, k \notin \{i_1, ..., i_p\}$. In particular, for any $1 \le i_2 < ... < i_p \le r$, the quadric $Q_{0i_2...i_p}$ does not depend on X_0.

Projection of syzygies simply means removing all the $Q_{i_1...i_p}$'s for $i_1 \neq 0$, and renaming $\widetilde{Q}_{j_1...j_{p-1}} = Q_{0j_1...j_{p-1}}$, for all $1 \le j_1 < ... < j_{p-1} \le r$. We see that $(\widetilde{Q}_{j_1...j_{p-1}})_{1 \le j_1 < ... < j_{p-1} \le r}$ belongs to $L_x \otimes (I_Y)_2 \otimes \bigwedge^{p-1} W_x$, and the proof of Lemma is over.

Let us see next what happens in the particular case of curves, which is the most interesting for our purposes. So, let X be a smooth complex projective curve of genus g, L a line bundle on X, and $x \in X$ a point, and set $L_0 = L - x$, $V = H^0(X, L)$, and $W_x = H^0(X, L_0)$. Using (1.3) for the graded SW_x-modules $B' = \bigoplus H^0(X, qL_0)$, and $B = \bigoplus H^0(X, qL)$, one can easily check that, for any integer p, the natural map $K_{p,1}(X, L - x) \longrightarrow K_{p,1}(X, L, W_x)$ is injective. If L is globally generated, then (1.7) applies, so, if X' denotes the image of X in $\mathbf{P}V^*$, then $K_{p+1,1}(S_{X'}, V) \cong K_{p+1,1}(X, L)$, for all p. We project now our objects from the point x. If the image of X' is $Y \subset \mathbf{P}W_x^*$, the projected syzygies all live in $K_{p,1}(S_Y, W_x)$ (cf. (2.10)). Now, the homogeneous ideal of Y is

$$I_Y = \mathrm{Ker}(SW_x \to \bigoplus H^0(X, qL)),$$

and thus, since the map $SW_x \to \bigoplus H^0(X, qL)$ factors through $\bigoplus H^0(X, qL_0)$, the homogeneous coordinate ring of Y is

$$S_Y = \mathrm{Im}(SW_x \to \bigoplus H^0(X, qL_0)).$$

In particular, by means of (1.3), it follows $K_{p,1}(S_Y, W_x) \cong K_{p,1}(X, L - x)$. We have proved:

(3.2) Proposition. *If L is a globally generated line bundle over the smooth complex curve X, and $x \in X$, then, for all $p \ge 1$, we have*

$$\mathrm{Im}(K_{p+1,1}(X, L) \xrightarrow{\pi_x} L_x \otimes K_{p,1}(X, L, W_x)) \subset L_x \otimes K_{p,1}(X, L - x).$$

Using the same notation, we get, via (2.8), and (3.2):

(3.3) Corollary. *If L is globally generated, and $K_{p+1,1}(X,L)\neq 0$, for an integer $p\geq 1$, then $K_{p,1}(X,L-x)\neq 0$ for a generic point $x\in X$.*

Proof. Observe that, by means of (2.8), any nonzero element in $K_{p+1,1}(X,L)\neq 0$ survives in $K_{p,1}(X,L,W_x)$ by projecting from a generic point $x\in X$, as the image of X in $\mathbf{P}H^0(X,L)^*$ is nondegenerate.

4. Proofs of main results

Theorem 1, and Theorem 3 will follow as easy consequences of two key Lemmas which we show next.

(4.1) Lemma. *The property (M_k) is open in families of nonspecial line bundles on X.*

Proof. Due to the universal property of the Jacobian variety, it suffices to prove the statement for the Poincaré bundle. Let $d\geq g$ be an integer. We will use the following notation: $\mathcal{P}\to X\times \mathrm{Pic}_d(X)$ the Poincaré universal bundle, $p_1: X\times \mathrm{Pic}_d(X)\to X$, and $p_2: X\times \mathrm{Pic}_d(X)\to \mathrm{Pic}_d(X)$ the canonical projections,

$$N_d := \{\xi\in \mathrm{Pic}_d(X),\ h^1(X,\mathcal{P}_\xi)=0\},$$

and, for any positive integer p,

$$Z_{p,d} := \{\xi\in N_d,\ K_{p,1}(X,\mathcal{P}_\xi)\neq 0\}.$$

We aim to prove that $Z_{p,d}$ is closed in N_d.

For a positive integer q, set $E_q := p_{2,*}(\mathcal{P}^{\otimes q})|_{N_d}$; for $q=1$ we use the notation E instead of E_1. By Riemann-Roch, and Grauert, the sheaves E_q are all vector bundles on N_d, and the fiber over a point $\xi\in N_d$, which corresponds to a degree–d line bundle $\mathcal{P}_\xi$ over X, is canonically isomorphic to $H^0(X,q\mathcal{P}_\xi)$. For any q, the natural multiplication maps, defined over open sets U in N_d,

$$H^0(p_2^{-1}(U),\mathcal{P})\otimes H^0(p_2^{-1}(U),\mathcal{P}^{\otimes(q-1)})\longrightarrow H^0(p_2^{-1}(U),\mathcal{P}^{\otimes q})$$

induce multiplication morphisms of sheaves:

$$E\otimes E_{q-1}\longrightarrow E_q.$$

By standard arguments (cf. [Gr1]), mixing the morphisms above with the duals of the natural morphisms

$$E^*\otimes \bigwedge^p E^*\longrightarrow \bigwedge^{p+1} E^*,$$

we obtain further morphisms, for any p:

$$E_{q-1}\otimes \bigwedge^{p+1} E\longrightarrow E_q\otimes \bigwedge^{p} E.$$

Obviously, on each fiber over a point $\xi \in N_d$, these morphisms induce the Koszul complexes of the curve X with respect to the line bundle $\mathcal{P}_\xi$. In particular, we see that, for any p, the set $Z_{p,d}$ is closed, as being the locus where the morphism $E \otimes \bigwedge^{p+1} E \longrightarrow E_2 \otimes \bigwedge^p E$ drops rank.

(4.2) Lemma. *If L_0 is a nonspecial line bundle on X of degree $d \geq g$, which satisfies (M_k), and x_0 is a point of X such that $L = L_0 + x_0$ is globally generated, then L satisfies (M_k).*

Proof. We make use of the Abel-Jacobi embedding $X \hookrightarrow \mathrm{Pic}_d(X)$, given by $x \mapsto L - x$, and intersect the two sets of previous Lemma with X. Then $X \cap N_d$ is open in X, and, since it contains the point x_0, which corresponds to L_0, it is moreover nonempty. Besides, for any positive integer p, the set

$$X \cap Z_{p,d} = \{x \in X \cap N_d,\ K_{p,1}(X, L - x) \neq 0\}$$

is closed in $X \cap N_d$.

Let $p \geq 1$ an integer such that $K_{p+1,1}(X, L) \neq 0$. It follows directly from (3.3) that $K_{p,1}(X, L - x) \neq 0$ for a generic point $x \in X$, hence $X \cap Z_{p,d}$ contains a nonempty open set of $X \cap N_d$. In this case it must be the whole $X \cap N_d$, and, since $x_0 \in X \cap N_d$, we see that $K_{p,1}(X, L_0) \neq 0$. From the hypothesis, it follows that $p+1 < h^0 L_0 - k$. This inequality implies, via Riemann-Roch applied for L_0, and L, that $p + 1 < h^0 L - k - 1$, which eventually shows that (X, L) satisfies (M_k).

Proof of Theorem 1. It is well-known that, for any effective divisor, D, $L_0 + D$ is nonspecial and globally generated, as soon as L_0 itself is nonspecial and globally generated. This remark allows us to proceed by induction on the degree of D, and to reduce to the simpler case of a divisor of degree 1, which is a particular case of (4.2).

For the last part of the statement, observe that $H^0(X, L - L_0) \neq 0$, and thus L itself is of type $L_0 + D$.

In a similar manner, and making use of Theorem 1, we can prove the following:

(4.3) Corollary. *Let X be a smooth complex projective curve of genus $g \geq 1$, L_0 be a nonspecial globally generated line bundle on X and $k \geq 0$ be an integer such that the pair (X, L_0) satisfies the property (M_k). Then, for any effective divisor D, any integer $0 \leq \delta < \deg(D)$, and $x_1, ..., x_\delta$ generic points of X, the pair $(X, L_0 + D - x_1 - ... - x_\delta)$ also satisfies the property (M_k).*

Proof. Using the Abel-Jacobi map defined on the δ-fold symmetric power of X, $X_\delta \to \mathrm{Pic}_\delta(X)$, $(x_1, ..., x_\delta) \mapsto L_0 + D - x_1 - ... - x_\delta$, and (4.1), we see that the set of those $(x_1, ..., x_\delta)$, for which $L_0 + D - x_1 - ... - x_\delta$ is nonspecial, globally generated and satisfies (M_k), is open. Since D is effective and $0 \leq \delta < \deg(D)$, there exists a set of points of X, $\{x_1, ..., x_\delta\}$, such that $D - x_1 - ... - x_\delta$ is effective, hence $L_0 + D - x_1 - ... - x_\delta$ is nonspecial, globally generated and satisfies the property (M_k). Therefore, the above-mentioned open set is nonempty.

Proof of Theorem 3. Firstly, let us remark that if x is a point of X, and $L_0 = K_X + x$, then L_0 is nonspecial, and $H^0(X, K_X) \cong H^0(X, L_0)$. In particular, for any p, $K_{p,1}(X, K_X) \cong K_{p,1}(X, L_0)$, and thus L_0 satisfies (M_k). Then one can apply (4.2) for L_0.

5. Syzygies of plane curves

One of the main results of [Lo] shows that Green's canonical conjecture is true for a plane curve. The key point of his proof was the knowledge of the syzygies of the projective plane, and of their relations, derived from some long cohomology sequence, to the syzygies of the curve. As for the gonality conjecture, is seems natural to try to follow a similar strategy, and to verify it first for some suitable line bundles coming from the plane; Corollary 2 would show then that the gonality conjecture is valid for such a curve. It is the aim of this Section to prove that this idea works indeed.

We begin by pointing out the following useful fact:

(5.1) Lemma. *For any integers $k \geq 2$, and $p \geq 1$, $K_{p,1}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k)) = 0$ if and only if $p \geq N(k) := h^0\mathcal{O}_{\mathbf{P}^2}(k) - k$.*

Proof. Let $r = h^0\mathcal{O}_{\mathbf{P}^2}(k) - 1$. Green's duality ([Gr1], Corollary 2.c.10) in this case translates into

$$K_{p,1}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k)) \cong K_{r-p-2,2}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-3), \mathcal{O}_{\mathbf{P}^2}(k))^*.$$

For $p \geq N(k)$, we have $r - p - 2 \leq -3 + k$, so we can apply Theorem 2.2 of [Gr2], or Theorem 4.1 of [Gr3] to get the vanishing of $K_{p,1}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k))$. The fact that this bound is sharp easily follows from [GL1], by decomposing $\mathcal{O}_{\mathbf{P}^2}(k) = \mathcal{O}_{\mathbf{P}^2}(k-1) \otimes \mathcal{O}_{\mathbf{P}^2}(1)$.

Let $X \subset \mathbf{P}^2$ be a smooth plane curve of degree $k + 1 = d \geq 3$. We prove:

(5.2) Proposition. *The pair $(X, \mathcal{O}_X(k))$ satisfies the property (M_{k-1}).*

Proof. We start with the exact sequences, for any integer q:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-d + qk) \longrightarrow \mathcal{O}_{\mathbf{P}^2}(qk) \longrightarrow \mathcal{O}_X(qk) \longrightarrow 0,$$

denote $V = H^0\mathcal{O}_{\mathbf{P}^2}(k) = H^0\mathcal{O}_X(k)$, and consider the exact sequence of graded SV-modules:

$$0 \longrightarrow \bigoplus_{q \geq 0} H^0\mathcal{O}_{\mathbf{P}^2}(-d + qk) \longrightarrow \bigoplus_{q \geq 0} H^0\mathcal{O}_{\mathbf{P}^2}(qk) \longrightarrow \bigoplus_{q \geq 0} H^0\mathcal{O}_X(qk) \longrightarrow 0,$$

which leads us to the long cohomology sequence (cf. [Gr1], Corollary 1.d.4):

$$\ldots \to K_{p,1}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-d), \mathcal{O}_{\mathbf{P}^2}(k)) \to K_{p,1}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k)) \to K_{p,1}(X, \mathcal{O}_X(k)) \to$$

$$\to K_{p-1,2}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-d), \mathcal{O}_{\mathbf{P}^2}(k)) \to \ldots$$

By means of Green's Vanishing Theorem (cf. [Gr1], Theorem 3.a.1), we see that $K_{p,1}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-d), \mathcal{O}_{\mathbf{P}^2}(k)) = 0$ as soon as $p \geq h^0\mathcal{O}_{\mathbf{P}^2}(-d+k) = h^0\mathcal{O}_{\mathbf{P}^2}(-1) = 0$. The same for $K_{p-1,2}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-d), \mathcal{O}_{\mathbf{P}^2}(k))$: it vanishes for all $p \geq h^0\mathcal{O}_{\mathbf{P}^2}(-d+2k)+1 = N(k)$; in particular, for all $p \geq N(k)$, we have isomorphisms $K_{p,1}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k)) \cong K_{p,1}(X, \mathcal{O}_X(k))$. At this point, (5.2) is a direct consequence of (5.1).

In this special case, Corollary 2 applied to $\mathcal{O}_X(k)$ translates into the following (the first part of the statement has been known for a long time: see, for example, [ACGH] p.56)

(5.3) Corollary. *The curve X is k-gonal, a pencil of minimal degree being obtained by projecting from a point of the curve, and the gonality conjecture is valid for X.*

The property (M_{k-1}) is fulfilled for any line bundle $\mathcal{O}_X(n)$, with $n \geq k$, and it fails for $\mathcal{O}_X(k-2) = K_X$. We can inquire about $\mathcal{O}_X(k-1)$, and address the question of whether it satisfies (M_{k-1}) or not. The answer is NO: since the canonical maps $K_{p,1}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k-1)) \to K_{p,1}(X, \mathcal{O}_X(k-1))$ are injective, for all positive p (Green's vanishing and the long exact sequence, as above), the second part of (5.1) applied for $(k-1)$ shows that the property (M_{k-1}) fails for $(X, \mathcal{O}_X(k-1))$. It is natural then to ask about intermediate line bundles (see also (6.4) below):

(5.4) Problem. Find the least δ such that, for any colinear points $x_1, \ldots, x_\delta \in X$, the property (M_{k-1}) fails for the line bundle $\mathcal{O}_X(k) \otimes \mathcal{O}_X(-x_1 - \ldots - x_\delta)$.

6. Curves on a Hirzebruch surface

Let Σ_e be the Hirzebruch surface of invariant e, and denote by C_0 the minimal section of Σ_e, and by f a fiber of the ruling. For any integers a, and b, we consider the line bundle on Σ_e, $H_{a,b} = (a-1)C_0 + (b-1)f$, to whom we attach the integer $N(a,b) := h^0\mathcal{O}_{\Sigma_e}(H_{a,b}) - a$. We choose two integers $k \geq 2$, and $m \geq \max\{ke, k+e\}$, and a smooth curve $X \in |H_{k+1,m+1}|$; its genus is computed by the formula $g = (k-1)(m-1-ke/2)$.

We will show that the pair $(X, H_{k,m}|_X)$ satisfies the property (M_{k-1}). For this, we need two preliminary Lemmas.

(6.1) Lemma. *We have a natural isomorphism $K_{p,1}(\Sigma_e, H_{k,m}) \cong K_{p,1}(X, H_{k,m}|_X)$, for any $p \geq N(k,m)$.*

Proof. As in the case of plane curves, we use the exact sequence, for any integer q:

$$0 \longrightarrow \mathcal{O}_{\Sigma_e}(-X + qH_{k,m}) \longrightarrow \mathcal{O}_{\Sigma_e}(qH_{k,m}) \longrightarrow \mathcal{O}_{\Sigma_e}(qH_{k,m})|_X \longrightarrow 0.$$

Denote by $V = H^0\mathcal{O}_{\Sigma_e}(H_{k,m}) = H^0(X, \mathcal{O}_{\Sigma_e}(H_{k,m})|_X)$, and consider the graded SV-modules: $B' = \bigoplus_{q\geq 0} H^0\mathcal{O}_{\Sigma_e}(-X+qH_{k,m})$, $B = \bigoplus_{q\geq 0} H^0\mathcal{O}_{\Sigma_e}(qH_{k,m})$, and $A = B/B'$.We have then an exact sequence of graded SV-modules:

$$0 \longrightarrow B' \longrightarrow B \longrightarrow A \longrightarrow 0,$$

leading us to the long exact sequence:

$$... \to K_{p,1}(\Sigma_e, -X, H_{k,m}) \to K_{p,1}(\Sigma_e, H_{k,m}) \to K_{p,1}(A, V) \to$$

$$\to K_{p-1,2}(\Sigma_e, -X, H_{k,m}) \to ...$$

Since $A_0 \cong \mathbf{C}$, $A_1 \cong H^0(X, \mathcal{O}_{\Sigma_e}(H_{k,m})|_X)$, and $A_2 \subset H^0(X, \mathcal{O}_{\Sigma_e}(2H_{k,m})|_X)$, we also have $K_{p,1}(A, V) \cong K_{p,1}(X, \mathcal{O}_{\Sigma_e}(H_{k,m})|_X)$ (cf. (1.3)).

We apply again Green's Vanishing Theorem ([Gr1], Theorem 3.a.1), which implies that $K_{p,1}(\Sigma_e, -X, H_{k,m}) = 0$ as soon as $p \geq h^0\mathcal{O}_{\Sigma_e}(-X + H_{k,m}) = 0$, and $K_{p-1,2}(\Sigma_e, -X, H_{k,m}) = 0$ for all $p \geq h^0\mathcal{O}_{\Sigma_e}(-X+2H_{k,m})+1$. Since $-X+2H_{k,m} = H_{k-1,m-1}$, we have $N(k,m) \geq h^0\mathcal{O}_{\Sigma_e}(-X + 2H_{k,m}) + 1$. In particular, for all $p \geq N(k,m)$, we have isomorphisms $K_{p,1}(\Sigma_e, H_{k,m}) \cong K_{p,1}(X, H_{k,m}|_X)$, as stated.

(6.2) Lemma. *For all $p \geq N(k,m)$, we have $K_{p,1}(\Sigma_e, H_{k,m}) = 0$.*

Proof. It suffices to prove vanishing for $p = N(k,m)$ only, so we stick to this case.

We consider Y a smooth curve in the linear system $|H_{k,m}|$ on Σ_e. In this case, $h^0(Y, H_{k,m}|_Y) = h^0(\Sigma_e, H_{k,m}) - 1$, and the surjection $V \to H^0(Y, H_{k,m}|_Y)$ corresponds to the hyperplane of $\mathbf{P}V^*$ which cuts out the curve Y on the image of Σ_e. Since we work on a rational surface, (1.8) applies, and we get an isomorphism $K_{N(k,m),1}(\Sigma_e, H_{k,m}) \cong K_{N(k,m),1}(Y, H_{k,m}|_Y)$.

We prove the Lemma by induction on k; the first step is $k = 2$. In this case, $N(2,m) = h^0(Y, H_{2,m}|_Y) - 1$; by the general theory of syzygies (apply, for example, Theorem 3.c.1 (1) of [Gr1]), we know that $K_{h^0(Y,H_{2,m}|_Y)-1,1}(Y, H_{2,m}|_Y) = 0$.

The induction step: for $k \geq 3$ we know $K_{N(k-1,m-1),1}(\Sigma_e, H_{k-1,m-1}) = 0$, and we wish to prove that $K_{N(k,m),1}(\Sigma_e, H_{k,m}) = 0$, i.e. $K_{N(k,m),1}(Y, H_{k,m}|_Y) = 0$.

Lemma (6.1) applied for Y yields an isomorphism $K_{N(k-1,m-1),1}(Y, H_{k-1,m-1}|_Y) \cong K_{N(k-1,m-1),1}(\Sigma_e, H_{k-1,m-1})$, which shows that $K_{N(k-1,m-1),1}(Y, H_{k-1,m-1}|_Y) = 0$. The induction step is completed by means of the following facts: $H_{k-1,m-1}.Y \geq 2g(Y) + 1$, $|(H_{k,m} - H_{k-1,m-1})|_Y| \neq \emptyset$, so there exists $D \in |(H_{k,m} - H_{k-1,m-1})|_Y|$, $N(k,m) - N(k-1, m-1) = (H_{k,m} - H_{k-1,m-1}).H_{k,m} = \deg(D)$, which altogether permit us to apply Theorem 1 for Y, and the line bundles $L_0 = H_{k-1,m-1}|_Y$, $L = H_{k,m}|_Y = L_0 + D$.

The obvious inequality $H_{k,m}.X \geq 2g+1$, together with the existence of a g^1_k given by the ruling, show that the gonality conjecture is verified for X (the fact that X is k-gonal was previously proved in [Ma1], by using completely different methods):

(6.3) Theorem. *Let $e \geq 0$, $k \geq 2$, and $m \geq \max\{ke, k+e\}$ be three integers, and $X \equiv kC_0 + mf$ be a smooth curve on the Hirzebruch surface Σ_e. Then X is k-gonal,* *a pencil of minimal degree being given by the ruling, and for any* $L \in \mathrm{Pic}(X)$ *with* $h^0(X, L \otimes \mathcal{O}_{\Sigma_e}(-H_{k,m})|_X) \neq 0$ *the property* (M_{k-1}) *holds for* (X, L).

In the particular case $e = 1$, $m = k+1$, the curve X is the strict transform of a smooth plane curve X_0 of degree $k+1$, which passes through the center x_0 of the blowup. Via the natural isomorphism between X and X_0, the line bundle $H_{k,k+1}|_X$ corresponds to $\mathcal{O}_{X_0}(k) \otimes \mathcal{O}_{X_0}(-x_0)$. This shows that (6.3) does a bit better than (5.2).

7. The gonality conjecture for nodal curves

The aim of this Section is to prove that the gonality conjecture is valid for a nodal curve on $\mathbf{P}^1 \times \mathbf{P}^1$ whose singular points are in general position, fact which makes the proof of Theorem 4 be straightforward; this idea was inspired by the work [Sch2]. We freely use the notation of the previous Sections.

We consider three integers $k \geq 3$, $m \geq k$, $0 \leq \delta \leq k-2$, $Y \in |H_{k,m}|$ a smooth curve of genus $(k-2)(m-2)$ on $\mathbf{P}^1 \times \mathbf{P}^1$, $L_0 = H_{k-1,m-1}|_Y$, and $L = H_{k,m}|_Y$. Then (6.3) applies for Y, hence the pair (Y, L_0) satisfies (M_{k-2}). In this case, for a set of general points $\{x_1, ..., x_\delta\} \subset Y$ the line bundle $L - x_1 - ... - x_\delta$ satisfies the property (M_{k-2}) (cf. (4.3)). Without loss of generality, this set can be chosen such that the points $\{x_1, ..., x_\delta\}$ are in general position in $\mathbf{P}^1 \times \mathbf{P}^1$, in the sense that any two of them are not colinear. We consider furthermore $\Sigma \xrightarrow{\sigma} \mathbf{P}^1 \times \mathbf{P}^1$ the blowup of the points $\{x_1, ..., x_\delta\}$, E the exceptional divisor, $H = \sigma^*(k-1, m-1) - E$, and $\widetilde{Y} \in |H|$ the strict transform of the curve Y.

Proposition 1 of [Sch2] ensures the existence of a smooth connected curve in the linear system $|\sigma^*(k, m) - 2E|$; let us denote it by X. The curve X is k-gonal (see below), its genus equals $g = (k-1)(m-1) - \delta$, and its projection on $\mathbf{P}^1 \times \mathbf{P}^1$ is an irreducible curve of type (k, m), with assigned ordinary nodes at $\{x_1, ..., x_\delta\}$.

(7.1) Proposition. *The pair* $(X, H|_X)$ *satisfies the property* (M_{k-1}).

Proof. It is easy to see that $h^i\mathcal{O}_\Sigma(H - X) = 0$ for all i. Then we denote $V = H^0\mathcal{O}_\Sigma(H) \cong H^0(X, \mathcal{O}_\Sigma(H)|_X)$, and compute $\dim(V) - k = k(m-1) - \delta$. In a similar way as in the Proof of (6.1), we can check that $\dim(V) - k \geq h^0\mathcal{O}_\Sigma(2H - X) + 1$, thence $K_{p,1}(\Sigma, H) \cong K_{p,1}(X, H|_X)$ for all $p \geq \dim(V) - k$. Moreover, (1.8) applied to this case shows that $K_{p,1}(\Sigma, H) \cong K_{p,1}(\widetilde{Y}, H|_{\widetilde{Y}})$. Now, the restriction of σ to $\widetilde{Y}$ gives a natural isomorphism to Y; the line bundle $H|_{\widetilde{Y}}$ on $\widetilde{Y}$ corresponds to the line bundle $L - x_1 - ... - x_\delta$ on Y. Besides, Riemann-Roch implies $h^0(Y, L - x_1 - ... - x_\delta) - (k-1) = \dim(V) - k$. Since $L - x_1 - ... - x_\delta$ satisfies (M_{k-2}), i.e. $K_{p,1}(Y, L - x_1 - ... - x_\delta) = 0$ for $p \geq h^0(Y, L - x_1 - ... - x_\delta) - (k-1)$, we are done.

Consequently, as $H.X \geq 2g + 1$, and X carries a g^1_k, the gonality conjecture is verified for X, and thus Theorem 4 is true.

8. Application: Smooth curves on Hirzebruch surfaces satisfy Green's canonical conjecture

Green's canonical onjecture starts with a corollary of the Green-lazarsfeld nonvanishing result (cf. [GL1])

(8.1) Theorem. *Let X be a smooth complex projective curve of genus g endowed with a g^r_d, with $r \geq 1$ and $d \leq g-1$. Then $K_{g-(d-2r+2),1}(X, K_X) \neq 0$.*

In [Gr1], Green conjectured the converse of this fundamental fact were also true. More precisely, if one denotes by c the Clifford index of the curve X (we refer to [Ma2] for a precise definition) all the $K_{p,1}(X, K_X)$ vanish for $p \geq g-c-1$. In other words, the line bundle K_X satisfies the property M_c.

We show next that smooth curves on Hirzebruch surfaces satisfy Green's canonical conjecture (we use the same notation as in Section 6).

(8.2) Theorem. *Let Σ_e be the Hirzebruch surface of invariant e, and X be a smooth curve on Σ_e, numerically equvalent to $kC_0 + mf$, with $k \geq 3$ and $m \geq max\{ke+1, k+1, k+2e\}$. Then the Clifford dimension of X equals one, and Green's canonical conjecture is valid for X.*

Proof. We start with the exact sequence

$$0 \longrightarrow \mathcal{O}_{\Sigma_e}(-X) \longrightarrow \mathcal{O}_{\Sigma_e} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

and twist it by multiples of $H = K_{\Sigma_e} + X$. We obtain an exact sequence, for any q:

$$0 \longrightarrow \mathcal{O}_{\Sigma_e}(qH - X) \longrightarrow \mathcal{O}_{\Sigma_e}(qH) \longrightarrow K_X \longrightarrow 0.$$

Observe that $H^0\mathcal{O}_{\Sigma_e}(H-X) = H^1\mathcal{O}_{\Sigma_e}(H-X) = 0$, and thus $H^0\mathcal{O}_{\Sigma_e}(H) \cong H^0(X, K_X)$. Obviously, their dimension equals the genus of X. We get next a long exact sequence (see (1.3), and [Gr1] 1.d.4):

$$\dots \to K_{p,1}(\Sigma_e, -X, H) \to K_{p,1}(\Sigma_e, H) \to K_{p,1}(X, K_X) \to K_{p-1,2}(\Sigma_e, -X, H) \to \dots$$

By means of Green's vanishing theorem ([Gr1] 3.a.1), $K_{p-1,2}(\Sigma_e, -X, H)$ equals zero for $p \geq h^0\mathcal{O}_{\Sigma_e}(2H-X)+1$. From the inequality $m \geq max\{k+2e, ke+1\}$, we see that Lemma (6.2) applies, therefore $K_{p,1}(\Sigma_e, H) = 0$ for all $p \geq h^0\mathcal{O}_{\Sigma_e}(H) - (k-1) = g-k+1$. To finish the proof, we remark that, on the one hand, $h^0\mathcal{O}_{\Sigma_e}(2H-X) \leq g-k$, which implies the vanishing of $K_{p,1}(X, K_X)$ for all $p \geq g-k+1$, and, on the other hand, the curve X naturally carries a g^1_k.

(8.3) The role of the vanishing result (6.2) is similiar to the one of Green's vanishing [Gr2] 2.2 for the case of plane curves, studied in [Lo].

(8.4) By following a similar strategy, one can easily prove that Green's canonical conjecture is true for certain nodal curves on $\mathbf{P}^1 \times \mathbf{P}^1$. To do that, we simply apply (7.1), and addapt the proof of (8.1) to the new framework. This also provides a new proof of the main result of [Sch1], which says that Green's canonical conjecture is true for generic curves whose genera are sufficiently large compared to the gonality.

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