Local Moduli Spaces and Kuranishi Maps

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In this note we will describe a formal setting for the general construction of local semi-universal deformations in the context of differential graded (Banach) Lie algebras. The ideas behind it are not quite new and go finally back to Kodaira/Spencer and in particular, to Kuranishi (see [10]). A more general construction of local moduli spaces has been given in our book ([2] vol.1, Chap.II, section 12). For many situations, deformations are described by certain infinite dimensional differential graded Lie algebras which admit splitting operators for its differential. We will essentially stick to this last case (in the framework of Banach spaces) and relate the general construction of loc.cit. to it. The initial intention of this note was to construct a Kuranishi map (which means to describe *explicitly* a semi-universal deformation in terms of the DG-Lie algebra), since it does not seem to exist in sufficient generality in the literature (a related approach is already contained in the paper of Goldman and Millson, [6]). Moreover, we use the opportunity to state and prove certain general facts in local deformation theory which seem to be known (in various versions), unfortunately only to experts and where non experts are not able to find references. We also give some standard criteria for smoothness and universality of semi-universal local deformations, one for formality of DG-Lie algebras and discuss the relation with Massey products. A notion of a generalized Kuranishi space is indicated in (4.6), by taking the whole graded DG-algebra and not only the first homogenous part of it. In the last sections, we treat (very briefly) some applications, for example to deformations of Calabi-Yau manifolds, but in a more abstract way than usually. In the past, I was often asked about the existence of Kuranishi maps for a particular situation. So hopefully this note will be useful for such purposes.

Contents

1 A construction of semi-universal deformations

We first describe the general context where local semi-universal deformations exist. For this, we consider a contravariant functor

$$F : \mathcal{C} \to (sets)$$

and a group-valued contravariant functor

$$G : \mathcal{C} \to (groups)$$

which operates on F. Here $\mathcal{C}$ denotes the category of complexe space germs. To these data one can associate in a natural way a fibered category F/G over $\mathcal{C}$ such that the isomorphy classes in the category $(F/G)(S)$ of objects over the germ S identify with the quotient set $F(S)/G(S)$. We are looking for a semi-universal "deformation" for an element of the set $F(Spec(\mathbf{C}))$. For this purpose we can suppose that it contains only one element. *"Versal"* in a fibered category $p : \mathcal{F} \to \mathcal{C}$ over $\mathcal{C}$ means that there is an object a over a germ R which has the base change lifting property: For a given morphism $i : b \to b'$ in $\mathcal{F}$ such that $p(i)$ is an embedding $S \hookrightarrow S'$ of complex spaces germs and for $\varphi : b \to a$, there is a lifting $\varphi : b' \to a$ of φ, i.e. $\varphi' \circ i = \varphi$. *"Semi-universal"* means that in addition the tangent map of the induced lifting $S' \to R$ of $S \to R$ is unique (or equivalently that the dimension of the tangent space of R is minimal). We define the *tangent module* $D_a(M)$ of an object a over a germ S as the isomorphy classes of extensions $i : a \to b$ with $p(i) : S \hookrightarrow S[M]$, the trivial extension of S by the $\mathcal{O}_S$-module M.

Let us now consider a still quite general situation in the Banach context. Let E and F be Banach spaces and G be a Banach Lie group (in fact a Banach Lie group *germ* would be sufficient and even a certain class of Fréchet spaces and Lie groups can be treated by our method, theses are the so-called $uniformly\,bounded$ spaces, introduced by V.Palamodov in [12] and further developped in the our book [2]). Moreover, we suppose to have operations

$$\omega : G \times E \to E, \qquad \omega : G \times F \to F$$

of G on both spaces such that $0 \in F$ is a fixed point, and an equivariant analytic map

$$q : E \to F$$

which is defined in a neighborhood of $0 \in E$. Then G operates also on the (banach-analytic) space germ

$$Z := q^{-1}(0)$$

and so we may consider Z and G as contravariant $functors$ on the category of complex space germs. Then we have the following theorem (see [2] Chap.II, 12.13).

Theorem (1.1). *Let the tangent maps (in the distinguished points) of q and the orbit map $\beta : G \to E$, $g \mapsto \omega(g, 0)$, be direct (i.e.splitting) linear maps of Banach spaces and, moreover, assume that*

$$dim\,\mathrm{Ker}(Tq)/\mathrm{Im}(T\beta) < \infty.$$

Then there is a semi-universal element in the quotient category Z/G and especially, the functor $S \mapsto \mathrm{Hom}(S, Z)/\mathrm{Hom}(S, G)$ has a semi-universal deformation.

Proof. Let p be a linear projection onto $\mathrm{Im}(Tq)$ then pq is a (local) submersion and $M := (pq)^{-1}(0)$ is a manifold containing Z, having the same tangent space in (0). Let $N \subset M$ be a submanifold such that

$$(+) \qquad TM \;=\; \mathrm{Im}(T\beta) \oplus TN.$$

By assumption, this N will be of finte dimension. Putting

$$R \;=\; Z \cap N$$

we get a finite dimensional subgerm of Z which is the desired semi-universal deformation. In fact, it has the minimal possible tangent space and one can show that the restriction of the operation map onto $G \times R$

$$\omega : \, G \times R \;\to\; Z$$

is a smooth morphism of banach-analytic spaces. This follows essentially from the relativ implicit function theorem, q.e.d.

REMARK (1.2). We note that the above smooth map is an isomorphism if in addition, $T\beta$ is injective. So one obtains a *universal* germ R in this case. But universality is even true under a weaker condition, namely if the kernel of the tangent maps associated to all orbit maps close to the given one β, is a Banach *bundle* over R. In fact, let $H \subset G \times R$ denote the stabilizor of points in R and assume that it is smooth over R. Then the inclusion $H \subset \omega^{-1}(R)$ must be, by the relativ implicit function theorem, an isomorphism over R. This implies that the morphism of functors

$$\mathrm{Hom}(-, R) \;\to\; \mathrm{Hom}(-, Z)/\mathrm{Hom}(-, G)$$

induced by the inclusion $R \hookrightarrow Z$, is an isomorphism.

It is also obvious that openness of *versality* holds in our case, since smoothness is trivially an open condition.

2 Differential graded Banach Lie algebras

We take a graded differential Banach Lie algebra

$$L \;=\; \oplus_{i \in Z} L^i$$

with a differential d which is a dervation of degree 1. We will consider the Maurer-Cartan equation

$$q(s) := \, d(s) \, + \, \frac{1}{2}[s, s] \;=\; 0$$

defining in particular a banach-analytic subspace Z of $E := L^1$. Moreover, we put $F := L^2$. We consider Z always as a space germ in $0 \in E$.

The Banach space L^o is a Banach Lie algebra and posseses an associated *germ* of a Banach Lie group, denoted by G (it can be defined by the Campell-Hausdorff series). Since we are only interested in questions of local nature, we therefore may assume that L^o is the Lie algebra of a Banach Lie group G and that we have an adjoint operation of G on L, inducing the Lie-bracket of L.

REMARK. Not in all applications it is possible to stay completely in the context of Banach Lie groups. For instance for deformations of compact complex spaces, the group G will not be of that kind, due to the fact of being forced to making shrinkings when composing holomorphic maps which are close to the identity on a fixed open set. For deformations of vector bundles and many other situations, the Banach frame can be kept.

3 Kuranishi maps

We want to show here that one can obtain the semi-universal deformation R, just called "local moduli space" for short, as the zero set of an explicite holomorphic map

$$\kappa : H^1 \to H^2$$

between the first and second cohomology groups of the complex (L, d). By our construction, the map κ will *only* depend on the choice of a *splitting* r of this complex, i.e. a linear continuous map $r : L \to L$ of degre -1 such that

$$drd = d.$$

We even can assume that r satisfies

$$r^2 = 0, \qquad rdr = r$$

which is always possible after modifying r eventually (see [2] vol.I, p.67). We note that the condition $drrd = 0$ is weaker, but sufficient for our purpose.

Theorem (3.1). *A model for the local moduli space R is given by the zero set of the composition of the following two mappings*

$$K^{-1} : \mathcal{H}^1 \to N, \qquad h([-,-]) : N \to \mathcal{H}^2$$

where h ("harmonic projection") is defined by the equality

$$id = h + [d, r],$$

$$\mathcal{H} = \mathrm{Fix}(h)$$

and

$$K : L^1 \to L^1$$

$$w \mapsto w + \frac{1}{2} r[w,w].$$

Then the germ $N := V(K - h)$ satisfied property $(+)$ in the proof of theorem (1.1).

REMARK(3.2). The "harmonic" subspace $\mathcal{H}$ of L can be identified via h with the cohomology groups H of the differential graded Lie algebra L and so we finally get the desired map κ. We also note that

$$\mathcal{H} = \mathrm{Ker}([d,r])$$

and, moreover

$$\mathcal{H} = \mathrm{Ker}(d) \cap \mathrm{Ker}(r)$$

if $r^2 = 0$ and $rdr = r$.

PROOF *of theorem (3.1).* We first note that the map K is, by the implicit function theorem, obviously a local isomorphism. We put

$$N := \{w \in L^1 \mid w - h(w) + (1/2)r[w,w] = 0\} = V(K - h).$$

Then N is a smooth with tangent space $\mathcal{H}^1$, and we have

$$Z \cap V(r) \subset N \subset M$$

where $M = V(drq)$ is the above mentioned Banach manifold (see the proof of theorem (1.1)). The second inclusion can be seen by formula (*) beyond, using $drh = 0$, $drrd = 0$. We want to show next that

$$w \in N \Rightarrow hK(w) = K(w).$$

But this is easily verified, since $h^2 = h$ (the above implication is in fact an equivalence by the implicit function theorem). In order to show that the composition $\mathcal{H}^1 \to \mathcal{H}^2$ will define a model for the local moduli space

$R = Z \cap N$, it is sufficient to verify the following two properties for elements $w \in N$

a) $q(w) = 0 \;\Leftrightarrow\; h[w,w] = 0,\; rd[w,w] = 0$

b) $h[w,w] = 0 \;\Rightarrow\; rd[w,w] = 0$

The first one results directly from the identity

$$(*) \qquad dK(w) \;=\; q(w) - \frac{1}{2}(h + rd)[w,w]$$

and $dK = 0$ on N. The second one is more subtle. Using the fact that d is a derivation of degre 1, we get

$$rd[w,w] = 2r[dw,w]$$

and, by $dK(w) = 0$

$$rd[w,w] = -r[dr[w,w],w] = -r[(1-h-rd)[w,w],w].$$

This means that

$$rd[w,w] = -r\{[[w,w],w] - [rd[w,w],w]\} = r[rd[w,w],w]$$

since $[[w,w],w] = 0$ and $h[w,w] = 0$, by our hypothesis. If we now estimate the element $rd[w,w]$ and suppose that $\|w\|$ is sufficiently small, it follows, by the last identity, that $rd[w,w]$ must be zero, q.e.d.

COMMENTARY. In the classical cases, studied by Kodaira/Spencer, Kuranishi and others, the role of the diffential d is played by a $\overline{\partial}$ -operator and the splitting r is constructed via the composition of $\overline{\partial}^*$ with a Green's operator. In the holomorphic context, it is in general very hard to find such splittings. If one deforms local objects, then privileged polycylinders are of great importance for finding splittings. For global situations, one can use techniques from the theory of topological vector spaces, first applied by Palamodov and generalized further in [2].

4 Some general results of deformation theory

4.1 Universality and smoothness of local moduli spaces

We already remarked in (1.2) that we have openness of *versality* in the situation of section 1 and 2 which is due to the fact that the operation map $\omega : G \times R \to Z$ is smooth.

Let us now consider the tautologically defined differential D on the relativ graded Lie algebra L_R, defined by the inclusion $R \hookrightarrow Z$, and its cohomology

$$\mathcal{H}^i(L, D)$$

for $i \in \mathbf{Z}$, considered as a *sheaf* of $\mathcal{O}_R$ -modules. We assume that these are *coherent* for $i = 0$ and/or $i = 1$. Then we have the following criterion for universality, respectively smoothness, of the local moduli space.

Theorem (4.1.1). *Under the above assumptions (in the situation of section 2), the semi-universal deformation space germ R is*
(a) *universal, iff $\mathcal{H}^o(L, D)$ commutes with base change,*
(b) *smooth, iff $\mathcal{H}^1(L, D)$ commutes with base change.*

This theorem is a special case of the following general principle for deformation groupoïds (with an obstruction theory):

Theorem (4.1.2). *The sheaf of (relativ) infinitesimal automorphisms of a semi-universal object commutes with base change , iff the semi-universal deformation is universal.*

The (relativ) tangent sheaf of a semi-universal object commutes with base change, iff the semi-universal deformation is smooth.

PROOF. The first part is already contained in the paper of D.S.Rim, see [13] corollary 2.20, but may also be left to the reader as a nice exercise on deformation groupoïds. For the second assertion, let us denote the tangent modules by $\mathcal{D}_a(\mathcal{M})$ for an object a over the base germ R and a coherent $\mathcal{O}_R$-module $\mathcal{M}$. The "Kodaira-Spencer map" for a is a canonically defined homomorphism $\mathcal{T}_R \to \mathcal{D}_a(\mathcal{O})$ of $\mathcal{O}_R$-modules, where $\mathcal{T}_R$ is the sheaf of holomorphic vector fields on R. By our base change assumption, the natural map

$$\mathcal{D}_a(\mathcal{O}) \to \mathcal{D}_a(\mathcal{O}/m)$$

is surjectiv, where m denotes the ideal sheaf of the distinguished point. But since the last vector space identifies canonically with the tangent space of R at the distinguished point, we get finally $(\dim_{\mathbf{C}} T_o R)$ vector fields which are independent and form a basis of $T_o R$. So R is smooth by a standard argument, c.f. [3] p. 91/92. The converse is a rather formal reasoning. -

4.2 Formality

We give here the following simple criterion for the formality of the DG-Lie algebra L, where formality means that there is a homomorphism of DG-Lie algebras $H(L) \to L$ which is a quasi-isomorphism. Here $H(L)$ has differential 0 and the Lie bracket is induced by that of L.

Theorem (4.2.1). *If there is a splitting r of (L, d) which is a derivation of degre -1, then L is formal.*

PROOF. This is quite obvious from the definitions, since the harmonic subspace is the kernel of $[d, r]$ which is a derivation of degree zero and therefore $\mathrm{Ker}([d, r])$ is a Lie subalgebra of L, isomorphic to $H(L)$, q.e.d.

REMARK. Assume that L is formal and that $H^1(L)$ is finite dimensional. Then the semi-universal deformation space germ is at most a *quadratic* singularity.

It is known that L and $H(L)$ are always isomorphis as so-called L^∞-algebras (for this notion and applications of it, see [8]), if one fixes a splitting of the complex (L, d).

4.3 Kuranishi maps for general deformation groupoïds over Artin rings

It is not difficult to prove that *formal* Kuranishi maps exist in a very general setting, namely for groupoïds or functors with a so-called *obstruction theory* which means that the possibility of extending an object to an infinitesimal thickening of its artinian base is controlled by the vanishing of a certain obstruction element in some vector space (which has to be functorial; compare [1], [4]). Under rather mild assumptions (which are grouppoïd versions of Schlessinger's conditions), there is even a canonical obstruction theory (see [4] p.458). The *formal* semi-universal deformation (in the sense of Schlessinger) can be obtained as the zero set of a *formal* map between two vector spaces

$$\kappa : T \to Ob$$

where T denotes the tangent space of the semi-universal deformation and Ob an obstruction space (which has to be the same for *all* infinitesimal extensions of length one). There does not seem to exist a direct reference for this fact, but κ can be easily constructed inductively, only by using (standar)d definitions in formal deformation groupoïds (as for instance contained in [4]). So the existence of a formal Kuranishi map is valid under rather mild assumptions.

4.4 Relation with Massey products

In the context of graded Lie algebras and deformations, it is possible to describe the power series expansion of a Kuranshi map in terms of Massey products. In the case of deformations of complex spaces, this has been indicated by Palamodov in [12].

We will adopt a more general point of view for the Massey power description which uses the notion of a Hopf algebra and which was inspired by the paper [5]. Let L be a (Banach) DG-Lie algebra and H a (topological) Hopf algebra over $\mathbf{C}$ with coproduct Δ (H can also be graded, if not, it will be concentrated in degree 1). A *homomorphism* from H to L is defined to be a $\mathbf{C}$-linear map (of degree zero)

$$\alpha : H \to L$$

such that the formula

$$d\alpha + \mu \circ (\alpha \hat{\otimes} \alpha) \circ \Delta = 0$$

holds, where $\mu : L \hat{\otimes} L \to L$ is the multiplication map, given by the Lie bracket. We want to associate to a semi-universal deformation R (described in terms of L) a Hopf algebra H and a homomorphism $\alpha : H \to L^1$. For this, we may use the contruction of R given in section 1. We can identify canonically

$$L \hat{\otimes} \mathcal{O}_R \cong \mathrm{Hom}_{\mathbf{C}}(\mathcal{O}'_R, L)$$

where $\mathcal{O}'_R$ is the topological dual of $\mathcal{O}_R$ and $\mathrm{Hom}_{\mathbf{C}}(-,-)$ denotes the functor of *continuous* $\mathbf{C}$-linear maps. Obviously

$$H := \mathcal{O}'_R$$

is a topological (counital, cocommutativ and coassociativ) Hopf algebra, since $\mathcal{O}_R$ is a dual Fréchet nuclear $\mathbf{C}$-algebra. Then the following statement is a consequence of the definitions.

Theorem (4.4.1). *The tautological element in* $\mathrm{Mor}(R, L) \cong L \hat{\otimes} \mathcal{O}_R$, *given by the inclusions* $R \subset Z \subset L^1$, *defines a homomorphism from the Hopf algebra* H *into the DG-Lie algebra* L.

REMARK. If one is only interested in *formal* aspects, then one can replace H by the smaller Hopf subalgebra $\underline{H}$ defined by

$$\underline{H} := \varinjlim \mathcal{O}'_{R_k}$$

where one takes the inductiv limit over all dual spaces of the artinian algebras $\mathcal{O}_{R_k} = \mathcal{O}_R / m^{k+1}$.

4.5 Global deformations

For many situations, one can describe global deformations by the set of differentials in a **Z**-graded (Banach) Lie algebra L, so

$$Y := \{s \in L^1 \mid \frac{1}{2}[s,s] = s^2 = 0\}$$

defines a deformation functor by $\mathrm{Hom}(-, Y)$. We assume that the Lie subalgebra L^o should be the Lie algebra associated to a Banach Lie group G which acts by the adjoint representation on L. Moreover, isomorphic deformations will be described by conjugation under the action of G. The *global* moduli problem can now be stated in the following way

When is the functor sheaf, associated to the functor presheaf on the category of complex spaces

$$S \mapsto \mathrm{Hom}(S, Y)/\mathrm{Hom}(S, G)$$

representable by a complex space ?

In this case we will speak of a *coarse* moduli space (it will be *fine*, if the above presheaf is already a sheaf, see for instance the paper [9]). The *local* moduli spaces can be obtained in the following way: Take an element s_o of Y and put

$$d := [s_o, -].$$

Then d will be a differential on L which is a derivation. Setting

$$Z := q^{-1}(0)$$

where

$$q(s) := \ d(s) + \frac{1}{2}[s,s] \ = \ \frac{1}{2}[s_o + s,\ s_o + s]$$

then Z coincides with Y, locally around s_o. Now the constructions in section 2 and 3 are applicable and we obtain in particular local semi-universal deformations.

4.6 Operations on the Kuranishi space

We sketch here how to obtain an operation on the semi-universal deformation of the automorphism group of the object which is under deformation. This problem has been discussed very generally by D.S.Rim (see [14)] in the language of groupoïds over *artinian* germs. The convergence of his construction is difficult to show, even in concret examples. By our point of view, we express the situation in terms of group operations, but starting with global objects. So let G be a (Banach Lie) group which operates on a (banach-analytic) space Z. We assume that Z is the zero set of a G-equivariant map $q : E \to F$ and that the operation of G on E and F is *linear.* Moreover, let s_o be a distinguished point in Z. The incarnation of the "automorphism group" of s_o is, by definition, the *stabilizor* subgroup G_o of this point. Our aim is to construct an operation on the Kuranishi space $R \subset (Z, s_o)$. Firstly, we observe that the (global) group G_o operates on the germ (Z, s_o), by restricting the G-operation on Z. So we try to find an R which is *invariant* with respect to the G_o-operation. This is possible under the following additionnal conditions (adopting the notations of section one):

(1) there are G_o-equivariant projections onto the subspaces $\mathrm{Ker}(T_{s_o}q) \subset \mathrm{E}$ and $\mathrm{Im}(T_{s_o}q) \subset F$,

(2) there is a G_o-equivariant section of the canonical (G_o-equivariant) projection map $T_{s_o}Z \to T_{s_o}Z/\mathrm{Im}(T\beta)$.

Theorem (4.6.1). *Under these assumptions, there is a G_o-invariant Kuranishi space R. The operation of G_o is induced by that of G on Z.*

PROOF. In fact, the conditions (1)and (2) permit to make the construction of R, given explicitely in section one, G_o-equivariant, q.e.d.

REMARK. If G_o is reductiv, then our conditions are valid. Moreover, in the above argument one can replace G_o by a subgroup.

4.7 Generalized Kuranishi spaces

We replace L^1 by the product $\Pi_i L^i$ which we denote simply also by L and consider the Maurer-Cartan equation on the *whole* space L. This means that the graded structure loses importance. Moreover, this observation will be a guiding idea for a general approach to a "non commutativ" deformation theory. In fact, this theory should become even easier and more natural, because there is only *one* cohomology group. The space Z will here be also the zero set of the Maurer-Cartan equation. Elements of L can be regarded as constant vector fields on the supermanifold L (up to shiftings). The generalized Kuranishi space in this context will be constructed as usual as a transversal slice to the action of a group (germ) G on Z, such that the tangent space of G in the neutral element identifies canonically with whole cohomology $H(L)$. The action of G on L has to be an adjoint one. We remark that this point of view is suggerated by M.Kontsevich and S.Merkulov, see[8],[11]. The generalization to L_∞-algebras seems to be natural.

4.8 Some classes of examples

4.8.1 Vector bundles

Let X be a compact complex space. Then the moduli space of holomorphic vector bundles on X of rank r is governed by the following graded Lie algebra

$$\varinjlim \mathrm{C}^{\cdot}(\mathcal{U}, End_X(\mathcal{O}_X^r))$$

where the (inductiv) limit is taken over all finite Stein coverings $\mathcal{U}$ of X. In order to have Banach Lie algebras, it is necessary to introduce norms and fix *one* finite covering $\mathcal{U}$ which is Stein (and simply connected). If X is a complex manifold, there is another graded Lie algebra which controls deformations, namely

$$A_X^{o,\cdot}(\mathrm{End}(E))$$

where E is a fixed C^∞ or C^ω bundle on the complex manifold X. For the Banach context, one usually takes here completions with respect to Sobolev norms (see [10]). Fixing a holomorphic structure on E, means to prescribe a differential on this graded Lie algebra. We note that both DG-Lie algebras will be isomorphic as $L^\infty-$algebras.

4.8.2 Calabi-Yau manifolds

We take a compact Kähler manifold X with trivial canonical bundle K_X, for example a Calabi-Yau manifold, a complex torus, etc. Let us consider a *local* deformation $f : Y \to S$ of X. Then the relativ dualizing sheaf $K_{Y/S}$ is trivial on S (since $f_* K_{Y/S}$ is locally free and commutes with base change by Hodge theory). By relativ duality (see [7]) and Hodge theory again, one shows that also all $R^i f_* \Lambda^j T_{Y/S}$ are locally free and commute with base change, so that the conditions of theorem (4.1.1) are satisfied. Therefore, taking into account that the DG-Lie algebra

$$A^{o,\cdot}(T_X)$$

controls the deformations of X, we obtain the following result, proved by F.Bogomolov, Z.Ran, G.Tian, .. .

Theorem. *The semi-universal deformation of a compact complex Kähler manifold with $K_X = 0$ is universal and smooth.*

4.8.3 Singularities

We consider a singularity X over the complex numbers and take a Tate resolution A of $\mathcal{O}_X$. Then A is a (topologically) *free* graded (anti)commutativ algebra, concentrated in non-positiv degres, with a differential s which is a derivation of degree 1, such that (A, s) is a resolution of $\mathcal{O}_X$. The topological DG-Lie algebra which controls the deformations of X, will be the following

$$L = \mathrm{Der}^{\bullet}_{\mathbf{C}}(A, A).$$

In order to work within the Banach setting, one first has to introduce suitable truncation ideals F in L and consider the quotient L/F instead of L. This is necessary, since A might not be finitly generated and therefore, taking values on generateurs will not give finite products. Then one has to introduce norms and work over (privileged) polycylinders for the Banach context. Unfortunately, L^o is *not* a Banach Lie algebra for the usual sup-norms. This turns out to be only a technical difficulty which can be overcome by introducing a category of *uniformly bounded Frechet spaces* (the PO-spaces of Palamodov) and the notion of a PO-Lie group. Theorem (3.1) and theorem (4.1.1)(b) are still valid and so there are Kuranshi maps for describing semi-universal deformations under the following finiteness condition on tangent cohomology

$$\mathit{supp}\, \mathcal{T}^1(X, \mathcal{O}_X) \subset \{0\}$$

where $\{0\}$ is the distinguished point of X.

We remark that for deformations of complex spaces a *simplicial* version of the above DG-Lie algebra can be defined which governs again the local deformations.

The following question is rather natural: *Which are the singularities X, having a formal DG-Lie algebra $L = \mathrm{Der}^{\bullet}_{\mathbf{C}}(A, A)$?* - For these singularities, the semi-universal deformation germ (if it exists) is then at most a quadratic singularity.

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