THE VIRTUAL POINCARÉ POLYNOMIALS OF HOMOGENEOUS SPACES

MICHEL BRION AND EMMANUEL PEYRE

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Abstract. We factor the virtual Poincaré polynomial of every homogeneous space G/H, where G is a complex connected linear algebraic group and H is an algebraic subgroup, as $t^{2u}(t^2-1)^r Q_{G/H}(t^2)$ for a polynomial $Q_{G/H}$ with non-negative integer coefficients. Moreover, we show that $Q_{G/H}(t^2)$ divides the virtual Poincaré polynomial of every regular embedding of G/H, if H is connected.

Introduction and statement of the results

One associates to every complex algebraic variety X (possibly singular, or reducible) its *virtual Poincaré polynomial* $P_X(t)$, uniquely determined by the following properties:
(i) (additivity) $P_X(t) = P_Y(t) + P_{X-Y}(t)$ for every closed subvariety Y.
(ii) If X is smooth and complete, then $P_X(t) = \sum_m \dim H^m(X)\ t^m$ is the usual Poincaré polynomial.
Then $P_X(t) = P_Y(t)\ P_F(t)$ for every fibration $F \to X \to Y$ which is locally trivial for the Zariski topology.

Specifically, we have

$$P_X(t) = \sum_{j,m} (-1)^{j+m} \dim \mathrm{gr}_W^m(H_c^j(X))\ t^m,$$

where $gr_W^m(H_c^j(X))$ denotes the m-th subquotient of the weight filtration on the j-th cohomology group of X with compact supports and complex coefficients (see [15] 4.5 and [11]). More generally, the mixed Hodge structure on $H_c^*(X)$ yields a polynomial $E_X(s,t)$ in two variables, satisfying the same properties of additivity and multiplicativity, and such that $P_X(t) = E_X(-t,-t)$ (see [9] and [2] §3 for more details).

In this paper, we investigate the E-polynomials of homogeneous spaces under linear algebraic groups, and of their *regular* embeddings in the sense of [4]. It turns out that these polynomials behave much better than the usual Poincaré polynomials; the latter are generally unknown for homogeneous spaces. To state our main results, we introduce the following notation.

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Let G be a complex connected linear algebraic group and let H be a closed subgroup. Let r_H (resp. u_H) be the rank (resp. the dimension of a maximal unipotent subgroup) of H, and define similarly r_G, u_G. Choose maximal reductive subgroups $H^{\mathrm{red}} \subseteq H$, $G^{\mathrm{red}} \subseteq G$ such that $H^{\mathrm{red}} \subseteq G^{\mathrm{red}}$, and maximal tori $T_H \subseteq H^{\mathrm{red}}$, $T_G = T \subseteq G^{\mathrm{red}}$ such that $T_H \subseteq T$; let W_H, $W_G = W$ be the corresponding Weyl groups. The Lie algebras of $G, H, \ldots$ will be denoted $\mathfrak{g}, \mathfrak{h}, \ldots$

The group W_H acts on the Lie algebra $\mathfrak{t}_H$ and on its ring of polynomial functions, $\mathbb{C}[\mathfrak{t}_H] = R(T_H)$. The invariant subring $\mathbb{C}[\mathfrak{t}_H]^{W_H} = R(H)$ is a finitely generated, graded algebra over $\mathbb{C}$, isomorphic to $\mathbb{C}[\mathfrak{h}^{\mathrm{red}}]^{H^{\mathrm{red}}}$. Its Hilbert series $\sum_{m=0}^{\infty} \dim R(H)_m \; t^m$ is the expansion of a rational function of t, denoted $F_H(t)$.

Since G is connected, $R(G)$ is a polynomial ring, and there exists a graded subspace $\mathcal{H}$ of $R(T)$ such that the multiplication map induces an isomorphism of $R(G) \otimes \mathcal{H}$ onto $R(T)$. Moreover, $\mathcal{H}$ is isomorphic to the cohomology space of the flag variety $\mathcal{F}(G)$, with complex coefficients. This isomorphism doubles degrees, and the Hodge structure on $H^*(\mathcal{F}(G))$ is pure. Therefore, the Poincaré polynomial $P_{\mathcal{F}(G)}$ is even, and we have

$$E_{\mathcal{F}(G)}(s,t) = P_{\mathcal{F}(G)}((st)^{1/2}) \text{ and } \frac{1}{(1-t)^{r_G}} = F_T(t) = F_G(t) P_{\mathcal{F}(G)}(t^{1/2}).$$

Moreover, we have

$$P_{\mathcal{F}(G)}(q^{1/2}) = |\mathcal{F}(G)(\mathbb{F}_q)|$$

for every finite field $\mathbb{F}_q$ with q elements. Here $|\mathcal{F}(G)(\mathbb{F}_q)|$ denotes the number of points over $\mathbb{F}_q$ of $\mathcal{F}(G)$ regarded as the flag variety of the split $\mathbb{Z}$-form of G^{red}.

Our first main result generalizes this to an arbitrary homogeneous space G/H, with some twists. Notice that both G and its closed subgroup H are defined over a finitely generated subring of $\mathbb{C}$, so that $(G/H)(\mathbb{F}_q)$ makes sense for a large power q of a large prime number.

Theorem 1. *(a) With preceding notation, the virtual Poincaré polynomial $P_{G/H}$ is even, and we have*

$$E_{G/H}(s,t) = P_{G/H}((st)^{1/2}) \text{ and } F_H(t) = F_G(t)\, t^{\dim(G/H)} P_{G/H}(t^{-1/2}).$$

Moreover, we have for all large q:

$$|(G/H)(\mathbb{F}_q)| = P_{G/H}(q^{1/2}).$$

(b) There exists a polynomial $Q_{G/H}$ with non-negative integer coefficients, such that

$$P_{G/H}(t^{1/2}) = t^{u_G - u_H}(t-1)^{r_G - r_H} Q_{G/H}(t).$$

Moreover, $Q_{G/H}(t) = Q_{G^{\mathrm{red}}/H^{\mathrm{red}}}(t)$. The degree of $Q_{G/H}$ equals $\dim \mathcal{F}(G) - \dim \mathcal{F}(H^0)$, with leading coefficient 1, and $Q_{G/H}(1)$ equals $\frac{|W_G|}{|W_H|}$.

(c) If H is connected, then

$$Q_{G/H}(t) = \frac{P_{\mathcal{F}(G)}(t^{1/2})}{P_{\mathcal{F}(H)}(t^{1/2})} = t^{\dim \mathcal{F}(G) - \dim \mathcal{F}(H)} Q_{G/H}(t^{-1}).$$

In particular, $Q_{G/H}(0) = 1$.

It follows that $u_G - u_H$, $r_G - r_H$ and $Q_{G/H}$ depend only on the complex algebraic variety G/H (in fact, $r_G - r_H$ is a topological invariant, see [1] 4.3).

Theorem 1 is proved in Section 1 by arguments of equivariant cohomology ; it would be interesting to deduce it from a deeper motivic result. Notice that (a) can be deduced from the fibration

$$G/H \to BH \to BG,$$

where BH (resp. BG) denotes the classifying space of H (resp. G); then the cohomology ring of BH is isomorphic to $R(H)$ with degrees doubled, so that the Poincaré series of BH is $F_H(t^2)$. If moreover H is connected, then

$$P_{G/H}(t^{1/2}) = \frac{P_G(t^{1/2})}{P_H(t^{1/2})} = t^{u_G - u_H}(t-1)^{r_G - r_H} \frac{P_{\mathcal{F}(G)}(t^{1/2})}{P_{\mathcal{F}(H)}(t^{1/2})},$$

as follows from [11] Theorem 6.1 (ii); and a similar relation holds for $|(G/H)(\mathbb{F}_q)|$, by Lang's theorem.

So the main point of Theorem 1 is (b), especially the non-negativity of coefficients of $Q_{G/H}$. We deduce it (together with (a) and (c)) from a geometric construction that may be of independent interest. In loose words, we obtain a locally trivial fibration (for the Zariski topology)

$$S \to G/H \to Z$$

where S is a torus of dimension $r_G - r_H$, and Z is an algebraic variety satisfying Poincaré duality and whose cohomology is purely algebraic (see Lemmas 1 and 2 for a precise statement). Thus, $E_{G/H}(s,t) = (1-st)^{r_G - r_H} E_Z(s,t)$, and $E_Z(s,t)$ is the value at $(st)^{1/2}$ of the Poincaré polynomial of $H_c^*(Z)$. In the case where G and H have the same rank, it follows that $P_{G/H}(t)$ is the Poincaré polynomial of $H_c^*(G/H)$.

As a consequence of Theorem 1, the Poincaré polynomial of the flag variety of a semi-simple group is divisible by the Poincaré polynomial of the flag variety of every semi-simple subgroup, and the quotient has non-negative coefficients. Our second main result generalizes this to closed subvarieties of flag varieties. Here the cohomology $H^*(X)$ is replaced by $IH^*(X;\mathcal{L})$, the middle intersection cohomology of X with coefficients in a local system $\mathcal{L}$ on a dense open nonsingular subvariety (see [3] and also the survey [5]).

Proposition 1. *With preceding notation, let X be a closed H-invariant subvariety of the flag variety $\mathcal{F}(G)$ and let $\mathcal{L}$ be a local system on a dense open nonsingular subvariety of X. If H is connected and if $\mathcal{L}$ is semi-simple and H-equivariant, then the intersection cohomology Poincaré polynomial*

$$IP_{X,\mathcal{L}}(t) = \sum_m \dim IH^m(X;\mathcal{L})\, t^m$$

is divisible by $P_{\mathcal{F}(H)}(t)$, and the quotient has non-negative integer coefficients.

This is proved at the end of Section 2 by adapting part of the proof of Theorem 1.

Next we turn to the E-polynomials of *regular embeddings*. Recall from [4] that a regular embedding of the homogeneous space G/H is a smooth complex algebraic variety X endowed with an algebraic action of G, such that:
(i) X contains an open orbit isomorphic to G/H.
(ii) The complement of this open orbit is a union of smooth irreducible divisors (the *boundary divisors*), with normal crossings.
(iii) Every orbit closure is a partial intersection of the boundary divisors, and its normal bundle contains an open orbit.

Recall also that those homogeneous spaces under a connected reductive group G which admit a complete regular embedding are exactly the *spherical* homogeneous spaces, i.e., those where a Borel subgroup of G acts with an open orbit.

Since every regular embedding X contains only finitely many orbits, we have

$$E_X(s,t) = P_X((st)^{1/2})$$

by Theorem 1 and additivity. Therefore, it suffices to consider the virtual Poincaré polynomial P_X. Our third main result yields a factorization of that polynomial:

Theorem 2. *Let X be a regular embedding of G/H, where H is connected. Then, for every orbit G/H' in X, the polynomial $Q_{G/H}(t)$ divides $Q_{G/H'}(t)$, and the quotient has non-negative integer coefficients.*

As a consequence, there exists a polynomial $R_X(t)$ with integer coefficients, such that

$$P_X(t^{1/2}) = Q_{G/H}(t)R_X(t).$$

If moreover X is complete, then the coefficients of $R_X(t)$ are non-negative.

The assumption that H is connected cannot be suppressed, as shown by an example at the end of Section 2. This section is devoted to the proof of Theorem 2. Again, the main point is the non-negativity of coefficients of $R_X(t)$; for this, we show that the equivariant cohomology ring of X is a free module of finite rank over a polynomial subring generated by $R(H)$ and indeterminates of degree 2. It would be interesting to obtain a topological interpretation of the polynomial $R_X(t)$. However, the factorization $P_X(t^{1/2}) = Q_{G/H}(t)R_X(t)$ does not originate in a fibration with total space X, as shown by the following simple example.

Consider the complex projective space $X = \mathbb{P}^{2m+1}$ of odd dimension, where the projective special orthogonal group $G = SO(2m+2)/\{\pm 1\}$ acts linearly. Then X consists of 2 orbits: the quadric Q^{2m}, and its complement with isotropy group $H \cong O(2m+1)/\{\pm 1\} \cong SO(2m+1)$, a connected subgroup; one checks that X is a regular completion of G/H. We have

$$P_{G/H}(t^{1/2}) = P_{\mathbb{P}^{2m+1}}(t^{1/2}) - P_{Q^{2m}}(t^{1/2}) = t^m(t^{m+1}-1),$$

so that $Q_{G/H}(t) = t^m + t^{m-1} + \cdots + 1$ and that $R_X(t) = t^{m+1}+1$. How to explain the factorization

$$P_{\mathbb{P}^{2m+1}}(t^{1/2}) = t^{2m+1} + t^{2m} + \cdots + 1 = (t^m + t^{m-1} + \cdots + 1)(t^{m+1}+1)$$

in topological terms ?

Notice that the complex projective space $\mathbb{P}^{2m}$ of even dimension is a regular completion of the homogeneous space $SO(2m+1)/O(2m)$ (where $O(2m)$ is not connected) by the quadric $\mathbb{Q}^{2m-1}$; this yields $Q_{SO(2m+1)/O(2m)}(t) = 1$.

These are examples of complete symmetric varieties. In fact, the Poincaré polynomials of all such varieties were determined by De Concini and Springer (see [10]) who deduced the virtual Poincaré polynomials of adjoint symmetric spaces. Their results were the starting point for the present work, as the factorizations of Theorems 1 and 2 can be seen on examples of [10].

For instance, by Theorem 2, the virtual Poincaré polynomial of any regular embeding X of a connected reductive group G (viewed as a homogeneous space under the action of $G \times G$ by left and right multiplication) is divisible by $Q_G(t^2) = P_{\mathcal{F}(G)}(t^2)$. When G is semi-simple adjoint and X is its canonical completion, this agrees with the closed formula for $P_X(t)$ given in [10] p. 96.

1. Proof of Theorem 1 and of Proposition 1

In what follows, we use [14] as a general reference for mixed Hodge structures, and [20] for algebraic groups.

Proof of Theorem 1.

We begin with an easy reduction to the case where both groups G and H are reductive. Let $R_u(H)$ be the unipotent radical of H. This unipotent group is isomorphic, as an algebraic variety, to some $\mathbb{C}^u$. Since H is the semi-direct product of $R_u(H)$ with H^{red}, we have $u = u_H - u_{H^{\mathrm{red}}}$. The quotient map $G \to G/H$ factors through

$$p : G/H^{\mathrm{red}} \to G/H,$$

a fibration with fiber $R_u(H) \cong \mathbb{C}^u$. Thus, the pullback map $H^*(G/H^{\mathrm{red}}) \to H^*(G/H)$ is an isomorphism of mixed Hodge structures. By Poincaré duality, it follows that

$$E_{G/H^{\mathrm{red}}}(s,t) = (st)^u \, E_{G/H}(s,t).$$

We now show that

$$|(G/H^{\mathrm{red}})(\mathbb{F}_q)| = q^u \, |(G/H)(\mathbb{F}_q)|$$

for q such that H^{red} is defined over $\mathbb{F}_q$ and that H is the semidirect product of $R_u(H)$ with H^{red} over $\overline{\mathbb{F}_q}$. This follows from Grothendieck's trace formula; as an alternative proof using elementary arguments of Galois descent, we check that

$$\pi : (G/H^{\mathrm{red}})(\mathbb{F}_q) \to (G/H)(\mathbb{F}_q)$$

is surjective with all fibers of order q^u. We denote Fr_q the Frobenius endomorphism of $G(\overline{\mathbb{F}_q})$, with fixed point subgroup $G(\mathbb{F}_q)$.

Let $x \in G(\overline{\mathbb{F}_q})$ such that $xH \in (G/H)(\mathbb{F}_q)$. Then $x^{-1}\,\mathrm{Fr}_q(x) \in H(\overline{\mathbb{F}_q})$. Thus, we can write $x^{-1}\,\mathrm{Fr}_q(x) = yz$ where $y \in R_u(H)(\overline{\mathbb{F}_q})$ and $z \in H^{\mathrm{red}}(\overline{\mathbb{F}_q})$. Since $R_u(H)(\overline{\mathbb{F}_q})$ is connected and invariant under $\mathrm{Int}(z)\circ\mathrm{Fr}_q$, there exists $h \in R_u(H)(\overline{\mathbb{F}_q})$ such that $y = hz\,\mathrm{Fr}_q(h^{-1})z^{-1}$. Thus, $x^{-1}\,\mathrm{Fr}_q(x) = hz\,\mathrm{Fr}_q(h^{-1})$. Replacing x by xh, we may assume that $x^{-1}\,\mathrm{Fr}_q(x) \in H^{\mathrm{red}}(\overline{\mathbb{F}_q})$. This proves the surjectivity of π.

Let now $x, y \in G(\overline{\mathbb{F}_q})$ such that $xH^{\mathrm{red}}, yH^{\mathrm{red}} \in (G/H^{\mathrm{red}})(\mathbb{F}_q)$ and that $y \in xH$. We may assume that $y = xz$ where $z \in R_u(H)(\overline{\mathbb{F}_q})$. Then $H^{\mathrm{red}}(\overline{\mathbb{F}_q})$ contains $x^{-1}\operatorname{Fr}_q(x)$ and $y^{-1}\operatorname{Fr}_q(y) = z^{-1}x^{-1}\operatorname{Fr}_q(x)\operatorname{Fr}_q(z)$. Since $H^{\mathrm{red}}(\overline{\mathbb{F}_q})$ normalizes $R_u(H)(\overline{\mathbb{F}_q})$ and their intersection is trivial, it follows that $z^{-1}x^{-1}\operatorname{Fr}_q(x)\operatorname{Fr}_q(z)\operatorname{Fr}_q(x^{-1})x = 1$. Therefore, $xzx^{-1} \in (xR_u(H)x^{-1})(\mathbb{F}_q)$, and $xR_u(H)x^{-1}$ is a Fr_q-stable connected unipotent group of dimension u. So every fiber of π has order q^u.

Therefore, if Theorem 1 holds for G/H^{red}, then it holds for G/H, and

$$Q_{G/H^{\mathrm{red}}}(t) = Q_{G/H}(t).$$

So we may assume that $H = H^{\mathrm{red}}$. Then, using the fibration

$$G/H \to G/R_u(G)H \cong G^{\mathrm{red}}/H^{\mathrm{red}}$$

with fiber $R_u(G)$, one reduces similarly to the case where $G = G^{\mathrm{red}}$.

We assume from now on that G and H are reductive; as a consequence, G/H is affine.

Lemma 1. *The following conditions are equivalent for a subtorus S of T, with Lie algebra $\mathfrak{s} \subseteq \mathfrak{t}$:*
(i) All isotropy subgroups of S acting on G/H are finite, and S is maximal for this property.
(ii) $\mathfrak{s} \oplus w\mathfrak{t}_H = \mathfrak{t}$ for all $w \in W$.

As a consequence, there exist subtori S satisfying (i), and all of them have dimension $r_G - r_H$. Moreover, the double coset space $S\backslash G/H$ is an affine algebraic variety, with at worst quotient singularities by finite abelian groups.

Proof. Let $g \in G$, then the finiteness of the isotropy group of gH in S is equivalent to: $\mathfrak{s} \cap \operatorname{Ad}(g)\mathfrak{h} = 0$. As there are only finitely many isotropy groups for a torus action on an algebraic variety, the finiteness of all isotropy groups for the S-action on G/H is equivalent to: $\mathfrak{s} \cap \operatorname{Ad}(G)\mathfrak{h} = 0$. Since

$$\mathfrak{s} \cap \operatorname{Ad}(G)\mathfrak{h} = \mathfrak{s} \cap (\mathfrak{t} \cap \operatorname{Ad}(G)\mathfrak{t}_H) = \mathfrak{s} \cap W\mathfrak{t}_H,$$

this amounts to: $\mathfrak{s} \cap w\mathfrak{t}_H = \{0\}$ for all $w \in W$.

Now $\mathfrak{t}$ has a W-invariant rational structure, defined by the lattice of differentials at 1 of one-parameter subgroups of T; the rational subspaces are exactly the Lie algebras of subtori. Moreover, any rational subspace $\mathfrak{s}$ intersecting trivially all subspaces $w\mathfrak{t}_H$ is contained in a rational complement to all these subspaces. This proves equivalence of conditions (i) and (ii), and the assertion on existence of subtori S and their dimension. For any such subtorus S, all orbits in the affine variety G/H are closed, and the isotropy groups are finite abelian groups. This implies the latter assertion. □

Remark. Lemma 1 extends to arbitrary homogeneous spaces G/H, except for the assertion that $S\backslash G/H$ is an affine algebraic variety. In fact, the quotient space $S\backslash G/H$ may well be non-separated if G/H is not affine. For example, let $G = \mathrm{SL}(2)$ and let H be its standard unipotent subgroup. The diagonal torus $D \cong \mathbb{C}^*$ of G acts on $G/H \cong \mathbb{C}^2 - \{0\}$ by $t \cdot (x, y) = (tx, t^{-1}y)$. All isotropy groups are trivial, but the quotient space is a classical example of a non-separated scheme : the affine line with its origin doubled.

Next choose a subtorus S of T satisfying the conditions of Lemma 1 and let

$$Z = S\backslash G/H$$

with quotient map $f : G/H \to Z$. Then there exists a decomposition of Z into finitely many disjoint, locally closed subvarieties Z_j ($j \in J$), together with finite subgroups F_j ($j \in J$) of S, such that every $f^{-1}(Z_j)$ is equivariantly isomorphic to $S/F_j \times Z_j$. Since S/F_j is a torus of dimension $r_G - r_H$, we have $E_{S/F_j}(s,t) = (st-1)^{r_G - r_H}$, whence

$$E_{G/H}(s,t) = (st-1)^{r_G - r_H} E_Z(s,t).$$

Likewise, we have for all large q:

$$|(G/H)(\mathbb{F}_q)| = (q-1)^{r_G - r_H} |Z(\mathbb{F}_q)|.$$

Since Z has at worst finite quotient singularities, it satisfies Poincaré duality over $\mathbb{C}$. As a consequence, each closed algebraic subvariety of codimension (say) r in Z has a cohomology class in $H^{2r}(Z)$. This yields the (degree doubling) cycle map

$$\mathrm{cl} : A^*(Z) \to H^*(Z),$$

where the left hand side is the Chow group of Z, graded by codimension (see [16] Chapter 19).

Lemma 2. *With preceding notation,* cl *is an isomorphism over* $\mathbb{C}$. *Moreover, the graded ring* $H^*(Z)$ *is isomorphic to* $R(S) \otimes_{R(G)} R(H)$, *and the usual Poincaré polynomial of* Z *equals*

$$\frac{F_S(t^2)F_H(t^2)}{F_G(t^2)} = \frac{F_H(t^2)}{(1-t^2)^{r_G - r_H} F_G(t^2)}.$$

Proof. We use equivariant cohomology, see e.g. [17]. Consider the action of T on G/H, then the equivariant cohomology ring $H_T^*(G/H)$ is clearly isomorphic to $H_H^*(G/T)$. Since $H_G^*(G/T) = H^*(BT) = R(T)$ is a free module of rank $|W|$ over $H_G^*(pt) = H^*(BG) = R(G)$, the Eilenberg-Moore spectral sequence (see [17] III.2) yields an isomorphism

$$H_H^*(G/T) \cong H^*(BH) \otimes_{H^*(BG)} H_G^*(G/T),$$

that is,

$$H_T^*(G/H) \cong R(T) \otimes_{R(G)} R(H).$$

This is a commutative, positively graded algebra, finite and free of rank $|W|$ over its subring $R(H)$. The latter is a Cohen-Macaulay ring of dimension r_H. Thus, the ring $H_T^*(G/H)$ is Cohen-Macaulay of dimension r_H as well, with Poincaré series

$$\frac{F_T(t^2)F_H(t^2)}{F_G(t^2)} = \frac{F_H(t^2)}{(1-t^2)^{r_G} F_G(t^2)}.$$

Since the subtorus S of T acts on G/H with finite isotropy groups, we have

$$H_T^*(G/H) \cong H_{T/S}^*(S\backslash G/H) \cong H_{T/S}^*(Z).$$

This is a finitely generated module over $H_{T/S}^*(pt) = R(T/S)$. But T/S is a torus of dimension r_H, so that $R(T/S)$ is a polynomial ring in r_H variables of degree 2. Since

$H_T^*(G/H)$ is Cohen-Macaulay of dimension r_H and finite over $R(T/S)$, it is a free module over that ring, by the Auslander-Buchsbaum formula (see [13] 19.3).

We claim that the canonical map

$$\mathbb{C} \otimes_{R(T/S)} H^*_{T/S}(Z) \to H^*(Z)$$

is an isomorphism. This follows from the Eilenberg-Moore spectral sequence again; we provide an alternative argument which adapts to other settings. We may assume that $S \neq T$. Choose then a primitive character χ of T restricting trivially to S. Consider the T-action on $\mathbb{C}$ by multiplication with χ, the corresponding diagonal action on $Z \times \mathbb{C}$, and the closed T-invariant subset $Z \times \{0\}$, with complement $Z \times \mathbb{C}^* \cong Z \times T/T'$. The long exact sequence of equivariant cohomology for the pair $(Z \times \mathbb{C}, Z \times \{0\})$, together with the Thom isomorphism, yields exact sequences

$$\cdots \to H^{m-2}_{T/S}(Z) \to H^m_{T/S}(Z) \to H^m_{T'/S}(Z) \to \cdots$$

for all m, where the left map is multiplication by the image of χ in $H^2_{T/S}(pt)$. Since this multiplication is injective, it follows that the canonical map

$$H^*_{T/S}(Z)/\chi H^*_{T/S}(Z) \to H^*_{T'/S}(Z)$$

is an isomorphism. Now induction on $\dim(T/S)$ completes the proof of the claim.

By that claim, we have

$$H^*(Z) \cong \mathbb{C} \otimes_{R(T/S)} R(T) \otimes_{R(G)} R(H).$$

But $\mathbb{C} \otimes_{R(T/S)} R(T) \cong R(S)$; thus, we obtain $H^*(Z) \cong R(S) \otimes_{R(G)} R(H)$. Moreover, $H^*(Z)$ is the quotient of $H^*_{T/S}(Z)$ by a regular sequence consisting of r_H homogeneous elements of degree 2. Therefore, the usual Poincaré polynomial of Z equals

$$(1-t^2)^{r_H} \frac{F_T(t^2)F_H(t^2)}{F_G(t^2)} = \frac{F_H(t^2)}{(1-t^2)^{r_G - r_H} F_G(t^2)}.$$

It remains to compare cohomology of Z with its Chow group. For this, we use equivariant intersection theory, see [12] and also [6]. The equivariant Chow group with complex coefficients (graded by codimension) $A_T^*(G/H)$ is again isomorphic to $R(T) \otimes_{R(G)} R(H)$, by [6] Corollary 12. Moreover, for any scheme X with an action of T, the canonical map

$$R(S) \otimes_{R(T)} A_T^*(X) \to A_S^*(X)$$

is an isomorphism (to see this, choose χ as above; then we obtain exact sequences

$$A_T^{m-1}(X \times \{0\}) \to A_T^m(X \times \mathbb{C}) \to A_T^m(X \times \mathbb{C}^*) \to 0,$$

that is,

$$A_T^{m-1}(X) \to A_T^m(X) \to A_{T'}^m(X) \to 0$$

where the left map is multiplication by the image of χ in $A_T^1(pt)$). As a consequence, the map

$$R(S) \otimes_{R(G)} R(H) \to A_S^*(G/H)$$

is an isomorphism; it follows that the cycle map

$$\mathrm{cl} : A_S^*(G/H) \to H_S^*(G/H) = H^*(Z)$$

defined in [12] 2.8, is an isomorphism as well. Finally, $A^*_S(G/H) \cong A^*(S\backslash G/H) = A^*(Z)$ by [12] Proposition 4 and Theorem 4. □

Remark. By Lemma 2, the Betti numbers of $Z = S\backslash G/H$ are independent of the choice of S. But the algebra structure of $H^*(Z)$ may depend on S, as shown by the example where $H = \mathrm{SL}(2)\times\mathrm{SL}(2)$ is embedded diagonally in $H \times H = G$. Furthermore, there may exist no subtorus S acting on G/H with finite constant isotropy groups; this happens, for instance, if $G = \mathrm{SL}(3)$ and $H = \mathrm{SO}(3)$.

As a final preparation for the proof of Theorem 1, we need the following easy result of invariant theory.

Lemma 3. *We have*

$$\lim_{t\to 1}(1-t)^{r_H} F_H(t) = \frac{1}{|W_H|}.$$

Moroever, the degree of the rational function $F_H(t)$ is at most $-\dim\mathcal{F}(H^0)$, with equality if H is connected.

Proof. The former assertion is a (well-known) consequence of Molien's formula for the invariant ring $R(H) = \mathbb{C}[\mathfrak{t}_H]^{W_H}$:

$$F_H(t) = \frac{1}{|W_H|}\sum_{w\in W_H}\frac{1}{\det_{\mathfrak{t}_H}(1-tw^{-1})}.$$

For the latter assertion, recall that $R(H^0)$ is a graded polynomial ring with homogeneous generators of degrees $d_1 \le \cdots \le d_r$, where $r = r_H$. Thus, the degree of $F_{H^0}(t)$ is $-d_1 - \cdots - d_r = -\dim\mathcal{F}(H^0)$. Moreover, denoting Γ the finite group H/H^0, we have an exact sequence

$$1 \to W_{H^0} \to W_H \to \Gamma \to 1.$$

Thus, Γ acts on $R(H^0)$ with invariant subring $R(H)$. Since $R(H^0)$ is a graded polynomial ring, it contains a graded Γ-stable subspace V such that the map $\mathrm{Sym}(V) \to R(H^0)$ is an isomorphism. It follows that V decomposes as a direct sum of homogeneous components V_m ; the increasing sequence of their degrees (with multiplicities given by the dimensions of the V_m) is the same as $(d_1,\ldots,d_r)$. Now

$$F_H(t) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{1}{\prod_m \det_{V_m}(1-t^m\gamma^{-1})}$$

is a sum of rational functions of the same degree, equal to $-d_1-\cdots-d_r = -\dim\mathcal{F}(H^0)$. □

We can now complete the proof of Theorem 1. By Lemma 2, the cohomology of Z vanishes in all odd degrees, and every space $H^{2m}(Z)$ is generated by algebraic classes. Thus, the Hodge structure on that space is pure of type (m,m), and the same holds for the dual space $H^{2m}_c(Z)$. In other words,

$$E_Z(s,t) = \sum_m \dim H^{2m}_c(Z)\ (st)^m.$$

Using Poincaré duality and Lemma 2, it follows that

$$E_Z(s,t) = (st)^{\dim(Z)} \frac{F_H((st)^{-1})}{(1-(st)^{-1})^{r_G - r_H} F_G((st)^{-1})},$$

so that

$$E_{G/H}(s,t) = \frac{(st)^{\dim(G/H)} F_H((st)^{-1})}{F_G((st)^{-1})}.$$

On the other hand, we have

$$|Z(\mathbb{F}_q)| = \sum_m \dim H_c^{2m}(Z)\; q^m.$$

For, by Grothendieck's trace formula [19], one has

$$|Z(\mathbb{F}_q)| = \sum_m (-1)^m \operatorname{Tr}(\mathrm{Fr}_q, H_c^m(Z_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l))$$

the equality, for large q, then follows from the proper base change theorem and the fact that the cycle class map is an isomorphism. (alternatively, one may show directly that

$$|(G/H)(\mathbb{F}_q)| = \frac{q^{\dim(G/H)} F_H(q^{-1})}{F_G(q^{-1})},$$

by arguments of Galois descent). This implies (a). Taking degrees in the equality of rational functions

$$F_H(t) = F_G(t) t^{\dim(G/H)}\; P_{G/H}(t^{-1/2})$$

and using Lemma 3, we obtain that $P_{G/H}(t^{1/2})$ is divisible by

$$t^{\dim(G/H) - \dim \mathcal{F}(G) + \dim \mathcal{F}(H)} = t^{u_G - u_H}.$$

Thus, we can write $P_{G/H}(t^{1/2}) = t^{u_G-u_H}(t-1)^{r_G-r_H} Q_{G/H}(t)$ for a polynomial $Q_{G/H}(t)$ with integer coefficients. Since $P_Z(t) = t^{u_G-u_H} Q_{G/H}(t)$, these coefficients are non-negative. Moreover, Lemma 3 implies that $Q_{G/H}(1) = \frac{|W_G|}{|W_H|}$.

For any irreducible variety X, the degree of $P_X(t)$ is $2\dim(X)$, with leading coefficient 1. It follows that the degree of $Q_{G/H}(t)$ is $\dim \mathcal{F}(G) - \dim \mathcal{F}(H^0)$, with leading coefficient 1. This completes the proof of (b). Finally, (c) follows from (a), (b) and Poincaré duality for $\mathcal{F}(G)$ and $\mathcal{F}(H)$.

Proof of Proposition 1.

We may assume that both G and H are semi-simple. We adapt the arguments of the proof of Lemma 2 to equivariant intersection cohomology; for the latter, see [8] and [18].

Choose a Borel subgroup B of G containing T. This identifies $\mathcal{F}(G)$ with G/B and hence $H_H^*(\mathcal{F}(G))$ with

$$H_H^*(G/B) \cong H_H^*(G/T) \cong R(H) \otimes_{R(G)} R(T),$$

compatibly with the isomorphism $H_H^*(pt) \cong R(H)$. Consider now the H-equivariant intersection cohomology $IH_H^*(X;\mathcal{L})$ with complex coefficients. This is a module over the

H-equivariant cohomology ring $H^*_H(X)$ and hence over $H^*_H(\mathcal{F}(G))$. Moreover, we have an isomorphism of $H^*_H(pt)$-modules

$$IH^*_H(X;\mathcal{L}) \cong H^*_H(pt) \otimes_{\mathbb{C}} IH^*(X;\mathcal{L})$$

(this is proved in [18] Proposition 13, in the case where H is a one-dimensional torus and $\mathcal{L}$ is constant; the general case is similar, see [7] (1.5.2) for details).

In particular, the $H^*_H(pt)$-module $IH^*_H(X;\mathcal{L})$ is finitely generated and free. Thus, $IH^*_H(X;\mathcal{L})$ is a finitely generated Cohen-Macaulay module of dimension r_H over $H^*_H(\mathcal{F}(G))$. Since the latter ring is a finitely generated free module over $R(T/S)$, a polynomial subring in r_H variables, it follows that $IH^*_H(X;\mathcal{L})$ is finitely generated and free over $R(T/S)$ as well. Thus, we have an isomorphism of $R(T/S)$-modules

$$IH^*_H(X;\mathcal{L}) \cong R(T/S) \otimes_{\mathbb{C}} V$$

where V is a finite dimensional graded vector space. Let $IQ_{X,\mathcal{L}}(t) = \sum_m \dim(V_m)\, t^m$, then the preceding isomorphisms imply the equality

$$F_H(t)\, IP_{X,\mathcal{L}}(t) = \frac{1}{(1-t^2)^{r_H}}\, IQ_{X,\mathcal{L}}(t)$$

and hence $IP_{X,\mathcal{L}}(t) = P_{\mathcal{F}(H)}(t)\, IQ_{X,\mathcal{L}}(t)$.

Remark. We formulate a topological interpretation of the quotient polynomial $IQ_{X,\mathcal{L}}(t)$. Let $\pi_B : G \to G/B$ be the quotient map, then $\pi_B^{-1}(X)$ is a closed subvariety of G, invariant under left multiplication by H and right multiplication by B. Next let $\pi_H : G \to H\backslash G$ be the quotient map and let $Y = \pi_H(\pi_B^{-1}(X))$. This is a closed B-invariant subvariety of the affine algebraic variety $H\backslash G$. Moreover, the H-equivariant semi-simple local system $\mathcal{L}$ on a dense open nonsingular subvariety of X corresponds to a B-equivariant semi-simple local system $\mathcal{L}_Y$ on a dense open nonsingular subvariety of Y, since π_B and π_H are principal fibrations.

Now one may show like in the proof of Lemma 2 that

$$IQ_{X,\mathcal{L}}(t) = \sum_m \dim IH^m_S(Y;\mathcal{L}_Y)\, t^m,$$

where S is any subtorus of T satisfying the conditions of Lemma 1. In particular, if $\mathcal{L}$ is the constant local system, then $IQ_{X,\mathcal{L}}(t)$ is the intersection cohomology Poincaré polynomial of Y/S, so that the intersection cohomology Betti numbers of Y/S are independent of the choice of S.

2. Proof of Theorem 2

Let Y be an orbit in X. Replacing X by the union of all orbits whose closure contains Y (an open G-invariant subset of X), we may assume that Y is closed in X. Then Y is the transversal intersection of boundary divisors, say $X_1, \ldots, X_r$. Choose $x \in Y$ and denote by H' its isotropy subgroup. Then H' acts on the normal space to Y at x; this action is diagonalizable and given by r linearly independent characters, see [4]. This defines a surjective group homomorphism $H' \to (\mathbb{C}^*)^r$, whence an exact sequence

$$1 \to K \to H' \to (\mathbb{C}^*)^r \to 1$$

where K is the kernel of the H'-action on the normal space. Let K^{red} be a maximal reductive subgroup of K.

We claim that K^{red} is contained in a conjugate of H. To check this, consider the linear action on K^{red} on the tangent space T_xX and choose a K^{red}-invariant complement N to the K^{red}-invariant subspace T_xY; by construction, K^{red} fixes N pointwise. Then we can choose a K^{red}-invariant subvariety Z of X, such that Z is smooth at x and that $T_xZ = N$. Therefore, K^{red} fixes pointwise a neighborhood of x in Z, and this neighborhood meets the open orbit G/H.

Thus, we may assume that K^{red} is contained in H. Since H is connected, we can apply [11] Theorem 6.1 (ii) to the fibration $G/K^{\mathrm{red}} \to G/H$ with fiber H/K^{red}, to obtain

$$P_{G/K^{\mathrm{red}}}(t) = P_{G/H}(t)P_{H/K^{\mathrm{red}}}(t).$$

Together with Theorem 1, it follows that

$$Q_{G/K}(t) = Q_{G/H}(t)Q_{H/K^{\mathrm{red}}}(t).$$

On the other hand, the action of $H'/K \cong (\mathbb{C}^*)^r$ on G/K by right multiplication defines a principal $(\mathbb{C}^*)^r$-bundle $G/K \to G/H'$. All such bundles are locally trivial, whence $P_{G/K}(t) = (t^2-1)^r P_{G/H'}(t)$, and

$$Q_{G/K}(t) = Q_{G/H'}(t).$$

So, $Q_{G/H}(t)$ divides $Q_{G/H'}(t)$ and the quotient has non-negative coefficients.

By additivity, it follows that $Q_{G/H}(t)$ divides $P_X(t^{1/2})$; the quotient is an even polynomial, $R_X(t)$. Since $Q_{G/H}(0) = 1$, the coefficients of $R_X(t)$ are integers. However, their non-negativity for complete X is not an obvious fact, because of the factor $t^{u_G - u_{H'}}(t-1)^{r_G - r_{H'}}$ in each $P_{G/H'}(t^{1/2})$. For this reason, we shall present an alternative proof of the existence of $R_X(t)$, which will also yield this non-negativity property.

We begin by relating the virtual Poincaré polynomial $P_X(t)$ to equivariant cohomology of X. If V is a $\mathbb{Z}$-graded complex vector space such that every homogeneous component V_m is finite dimensional, let $F_V(t) = \sum_{m=-\infty}^{\infty} \dim(V_m)\, t^m$ be its Poincaré series. If X is a variety where G acts algebraically, then $H^*_G(X)$ is a finitely generated, graded module over $H^*(BG) = R(G)$. As a consequence, the series $F_{H^*_G(X)}(t)$ is the expansion of a rational function, for which we use the same notation.

Lemma 4. *For every regular embedding X, the rational function $F_{H^*_G(X)}(t)$ is even, and*

$$F_{H^*_G(X)}(t^{1/2}) = F_G(t)\, t^{\dim(X)} P_X(t^{-1/2}).$$

Proof. In the case where $X = G/H$ is a unique orbit, we have $H^*_G(X) \cong H^*(BH) \cong R(H)$, whence $F_{H^*_G(X)}(t) = F_H(t^2)$. So the assertion follows from Theorem 1.

In the general case, choose a closed orbit Y in X, of codimension r, with complement U. The inclusion map $i : Y \to X$ defines a Gysin morphism

$$i_* : H^*_G(Y) \to H^*_G(X),$$

of degree $2r$. By [4], this map and the restriction map $H^*_G(X) \to H^*_G(U)$ fit into a short exact sequence

$$0 \to H^*_G(Y) \to H^*_G(X) \to H^*_G(U) \to 0.$$

It follows that

$$F_{H^*_G(X)}(t) = t^{2r} F_{H^*_G(Y)}(t) + F_{H^*_G(U)}(t).$$

Since $P_X = P_Y + P_U$, our assertion follows by induction. □

Remark. Lemma 4 admits a simpler formulation in terms of equivariant Borel-Moore homology $H^G_*(X)$, as defined in [12]. Indeed, by Poincaré duality, the rational function $F_{H^G_*(X)}(t)$ is even, and

$$F_{H^G_*(X)}(t^{1/2}) = F_G(t^{-1}) P_X(t^{1/2}).$$

In fact this holds, more generally, for every variety X where G acts with finitely many orbits.

Next let $X_1, \ldots, X_n$ be the boundary divisors of the regular embedding X, and let $z_1, \ldots, z_n \in H^2_G(X)$ be their equivariant cohomology classes. In the ring $H^*_G(X)$, consider the ideal I_X of $H^*_G(X)$ generated by $z_1, \ldots, z_n$, and the ideal J_X, kernel of the restriction map

$$\rho : H^*_G(X) \to H^*_G(G/H) \cong R(H).$$

Clearly, I_X is contained in J_X, and the latter ideal is prime. Moreover, ρ is surjective by [4], so that we have an exact sequence

$$0 \to J_X \to H^*_G(X) \to R(H) \to 0.$$

Examples show that I_X may differ from J_X; but these ideals are closely related, as shown by the following result.

Lemma 5. *We have $J_X^{2^N} \subseteq I_X$, where N denotes the number of G-orbits in X.*

Proof. We argue by induction on N. If $N = 1$, then $X = G/H$ so that both I_X and J_X are trivial. In the general case, we use the notation of the proof of Lemma 4. The (surjective) restriction map $H^*_G(X) \to H^*_G(U)$ sends I_X (resp. J_X) onto I_U (resp. J_U).

Let $\alpha \in J_X$. Since $J_U^{2^{N-1}} \subseteq I_U$ by the induction assumption, we may assume that

$$\alpha^{2^{N-1}} = i_* \beta$$

for some $\beta \in H^*_G(Y)$. Now we have in $H^*_G(X)$:

$$\alpha^{2^N} = (i_*\beta) \cup (i_*\beta) = i_*(\beta \cup i^* i_* \beta) = i_*(\beta^2 \cup i^* i_* 1) = (i_*\beta^2) \cup (i_* 1),$$

by the projection formula. Moreover, i_*1 is the equivariant cohomology class of Y in X. Since Y is a transversal intersection of r boundary divisors, say $X_1, \ldots, X_r$, we have $i_*1 = z_1 \cdots z_r \in I_X$, and $\alpha^{2^N} \in I_X$ as well. □

Since H is connected, $R(H)$ is a graded polynomial ring, so that we can choose a graded subalgebra R of $H^*_G(X)$ that restricts isomorphically to $H^*_G(G/H) \cong R(H)$ via ρ.

Lemma 6. *$H^*_G(X)$ is finite over its subring generated by R and $z_1, \ldots, z_n$.*

Proof. Since the algebra $H^*_G(X)$ is positively graded, it suffices to prove that the quotient

$$H^*_G(X)/(z_1, \ldots, z_n) = H^*_G(X)/I_X$$

is a finitely generated R-module. By Lemma 5, $H^*_G(X)/I_X$ is a quotient of $H^*_G(X)/J_X^m$ for some positive integer m. Consider the finite filtration of $H^*_G(X)/J_X^m$ by the powers of the image of J_X, and notice that all the subquotients $J_X^p H^*_G(X)/J_X^{p+1} H^*_G(X)$ are finite modules over $H^*_G(X)/J_X = R(H)$. Since the latter is isomorphic to R, the assertion follows. □

We now need the following variant of the Noether normalization theorem.

Lemma 7. *Let A be a finitely generated, positively graded algebra over an infinite field k. Let $y_1, \ldots, y_m$ be homogeneous, algebraically independent elements of A and let $z_1, \ldots, z_n$ be homogeneous elements of degree 1, such that A is finite over its subalgebra generated by $y_1, \ldots, y_m, z_1, \ldots, z_n$. Then there exist a non-negative integer n' and homogeneous elements $y'_1, \ldots, y'_m, z'_1, \ldots, z'_{n'}$ of A such that:*
(i) $y'_i - y_i \in k[z_1, \ldots, z_n]$ for $1 \leq i \leq m$.
(ii) $z'_1, \ldots, z'_{n'}$ are linear combinations of $z_1, \ldots, z_n$.
(iii) $y'_1, \ldots, y'_m, z'_1, \ldots, z'_{n'}$ are algebraically independent, and A is finite over the subring that they generate.

Proof. The argument is similar to that of the classical Noether normalization theorem, see [13] 13.1; we present it for completeness. We argue by induction on n, the case where $n = 0$ being trivial. In the general case, we may assume that $y_1, \ldots, y_m, z_1, \ldots, z_n$ are algebraically dependent, and we choose a polynomial relation

$$P(y_1, \ldots, y_m, z_1, \ldots, z_n) = 0.$$

We may assume that this relation is homogeneous and involves z_n. Let $d_1, \ldots, d_m$ be the degrees of $y_1, \ldots, y_m$. Define $y'_1, \ldots, y'_m, z'_1, \ldots, z'_{n-1}$ by

$$y_i = y'_i + a_i z_n^{d_i}, \ z_j = z'_j + b_j z_n$$

where $a_1, \ldots, a_m, b_1, \ldots, b_{n-1}$ are in k. Then

$$P(y'_1 + a_1 z_n^{d_1}, \ldots, y'_m + a_m z_n^{d_m}, z'_1 + b_1 z_n, \ldots, z'_{n-1} + b_{n-1} z_n, z_n) = 0.$$

Regarding the right-hand side as a polynomial in z_n, the coefficient of the leading term equals $P(a_1, \ldots, a_m, b_1, \ldots, b_{n-1}, 1)$. Since k is infinite and by our assumptions on P, we may choose $a_1, \ldots, a_m, b_1, \ldots, b_{n-1}$ so that this coefficient is non-zero. Then z_n is integral over the subring A' of A generated by $y'_1, \ldots, y'_m$ and $z'_1, \ldots, z'_{n-1}$. We conclude by the induction assumption for A'. □

We can now show that $Q_{G/H}(t)$ divides $P_X(t^{1/2})$. Apply Lemma 7 to the algebra $H^*_G(X)$ and to homogeneous, algebraically independent generators of its polynomial subalgebra R; then we obtain another polynomial subalgebra R' (restricting isomorphically to $R(H)$) and linear combinations $z_1',\ldots,z_{n'}'$ of $z_1,\ldots,,z_n$, such that $H_G^*(X)$ is finite over its polynomial subring $R'[z_1',\ldots,z_{n'}']$. Let $f(t)$ be the associated Hilbert polynomial, then

$$F_{H_G^*(X)}(t^{1/2})=\frac{F_H(t)f(t)}{(1-t)^{n'}}.$$

Moreeover, $f(1)$ is the rank of the $R'[z_1',\ldots,z_{n'}']$-module $H_G^*(X)$, a positive integer. On the other hand, we have by Lemma 4:

$$F_{H_G^*(X)}(t^{1/2})=F_G(t)\,t^{\dim(G/H)}P_X(t^{-1/2})$$

and, by Theorem 1:

$$F_H(t)=F_G(t)\,t^{\dim(G/H)-u_G+u_H}(t^{-1}-1)^{r_G-r_H}Q_{G/H}(t^{-1}).$$

This yields

$$P_X(t^{1/2})=t^{n'+u_G-u_H}(t-1)^{r_G-r_H-n'}Q_{G/H}(t)f(t^{-1}).$$

Since $f(1)Q_{G/H}(1)\neq 0$, we must have $r_G-r_H-n'\geq 0$; and since $Q_{G/H}(0)=1$, the Laurent polynomial $t^{n'+u_G-u_H}(t-1)^{r_G-r_H-n'}f(t^{-1})$ must be a polynomial. Thus, $Q_{G/H}(t)$ divides $P_X(t^{1/2})$.

If moreover X is complete, then the $R(G)$-module $H_G^*(X)$ is free by [4]. Thus, the ring $H_G^*(X)$ is Cohen-Macaulay of dimension r_G. Since this ring is finite over $R'[z_1',\ldots,z_{n'}']$, a polynomial subring, $H_G^*(X)$ is a free module over that subring, and we have $r_G=r_H+n'$. Therefore, the Hilbert polynomial $f(t)$ has non-negative coefficients, so that the same holds for the polynomial

$$t^{n'+u_G-u_H}f(t^{-1})=\frac{P_X(t^{1/2})}{Q_{G/H}(t)}.$$

Example. We show that Theorem 2 does not extend to all homogeneous spaces G/H. Let $G=\mathrm{SL}(2)\times\mathrm{SL}(2)$ with maximal torus $T=D\times D$, where D denotes the diagonal torus of SL(2). Let $n=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, then the element (n,n) of G normalizes T. Let H be the subgroup of G generated by T and by (n,n). The homogeneous space G/H is spherical, and we have $T_H=T$. Denoting by x, y the obvious coordinates on $\mathfrak{t}$, one obtains $R(G)=\mathbb{C}[x^2,y^2]$ and $R(H)=\mathbb{C}[x^2,xy,y^2]$, whence

$$F_G(t)=\frac{1}{(1-t^2)^2},\ F_H(t)=\frac{1+t^2}{(1-t^2)^2},\ P_{G/H}(t^{1/2})=t^4+t^2 \text{ and } Q_{G/H}(t)=1+t^2.$$

We now construct a regular completion X of G/H, such that $P_X(t^{1/2})$ is not divisible by $Q_{G/H}(t)$. Consider the variety $Y=\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ where G acts by $(g_1,g_2)(a,b,c,d)=(g_1a,g_1b,g_2c,g_2d)$. Then Y is a regular embedding of G/T. Moreover, the right action of (n,n) on G/T extends to the involution σ of Y, defined by $\sigma(a,b,c,d)=(b,a,d,c)$. The fixed point subset Y^σ is the closed G-orbit, $\mathrm{diag}(\mathbb{P}^1)\times\mathrm{diag}(\mathbb{P}^1)$. Since the actions of G and σ commute, G acts on the quotient Y/σ. The latter is singular along the image Z of Y^σ; the normal space to Y/σ at every point of Z is isomorphic to the quotient of $\mathbb{C}^2$ by the involution $(s,t) \mapsto (-s,-t)$. Thus, blowing up Z along Y/σ yields a smooth projective embedding X of G/H.

One may check that X is regular and that $P_X(t^{1/2}) = t^4 + 3t^3 + 6t^2 + 3t + 1$, which is prime to $Q_{G/H}(t) = t^2 + 1$. One may also check that $Q_{G/H'}(t)$ equals $t+1$ or $(t+1)^2$ for the other orbits; thus, $Q_{G/H}(t)$ is prime to all other $Q_{G/H'}(t)$.

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Université de Grenoble I, Département de Mathématiques, Institut Fourier, UMR 5582 du CNRS, 38402 Saint-Martin d'Hères Cedex, France

E-mail address: Michel.Brion@ujf-grenoble.fr, Emmanuel.Peyre@ujf-grenoble.fr