POSITIVITY IN THE GROTHENDIECK GROUP OF COMPLEX FLAG VARIETIES

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ABSTRACT. We prove a conjecture of A. S. Buch concerning the structure constants of the Grothendieck ring of a flag variety with respect to its basis of Schubert structure sheaves. For this, we show that the coefficients in this basis of the structure sheaf of any subvariety with rational singularities, have alternating signs. Equivalently, the class of the dualizing sheaf of such a subvariety is a nonnegative combination of classes of dualizing sheaves of Schubert varieties.

INTRODUCTION

Consider a complex flag variety X (see the end of this introduction for detailed notation and conventions). The Chow ring $A^*(X)$ has an additive basis consisting of the classes $[X_w]$ of Schubert subvarieties. It is well known that the structure constants of $A^*(X)$ with respect to this basis are positive, i.e.,

$$[X_u] \cdot [X_v] = \sum_w a_{u,v}^w [X_w]$$

for nonnegative integers $a_{u,v}^w$. A generalization to the Grothendieck ring $K(X)$ of vector bundles (or of coherent sheaves) on X was recently formulated by A. S. Buch, see [2]. The classes $[\mathcal{O}_{X_w}]$ of structure sheaves of Schubert subvarieties form an additive basis of $K(X)$; this defines integers $c_{u,v}^w$ such that

$$[\mathcal{O}_{X_u}] \cdot [\mathcal{O}_{X_v}] = \sum_w c_{u,v}^w [\mathcal{O}_{X_w}].$$

Setting $N(u,v;w) = \text{codim}(X_w) - \text{codim}(X_u) - \text{codim}(X_v)$, one easily shows that $c_{u,v}^w = a_{u,v}^w$ if $N(u,v;w) = 0$, whereas $c_{u,v}^w = 0$ if $N(u,v;w) < 0$.

In the case where X is a Grassmannian, Buch obtained a combinatorial description of the structure constants $c_{u,v}^w$ which implies that they have alternating signs:

$$(-1)^{N(u,v;w)} \, c_{u,v}^w \geq 0$$

for all u, v, w. And he conjectured that the latter result holds for all flag varieties ([2] Conjecture 9.2).

This conjecture is proved in the present paper. In fact it is a direct consequence of the following result, which answers a question of W. Graham.

Theorem 1. *Let Y be a closed subvariety of a complex flag variety X; write*

$$[\mathcal{O}_Y] = \sum_w c_Y^w \, [\mathcal{O}_{X_w}]$$

in $K(X)$. If Y has rational singularities (e.g., if Y is nonsingular), then the coefficients c_Y^w satisfy

$$(-1)^{\operatorname{codim}(X_w) - \operatorname{codim}(Y)} \, c_Y^w \geq 0.$$

Using the duality involution of $K(X)$, this may be reformulated in a more appealing way: *the class of the dualizing sheaf ω_Y of any subvariety Y having rational singularities, is a nonnegative combination of classes of dualizing sheaves of Schubert varieties* (the latter classes form another natural basis of $K(X)$).

To deduce Buch's conjecture from Theorem 1, one shows that $[\mathcal{O}_{X_u}] \cdot [\mathcal{O}_{X_v}] = [\mathcal{O}_Y]$, where Y is the intersection of X_u with a general translate of X_v; furthermore, since Schubert varieties have rational singularities, the same holds for Y (see Lemma 2).

We now sketch a proof of Theorem 1 in the simplest case, where X is a projective space and Y is nonsingular. The Schubert varieties in $X = \mathbb{P}^n$ are a flag of linear subspaces $\mathbb{P}^m$, where $0 \leq m \leq n$. Writing

$$[\mathcal{O}_Y] = \sum_{m=0}^{n} c_Y^m \, [\mathcal{O}_{\mathbb{P}^m}]$$

and noticing that the Euler characteristic $\chi(\mathcal{O}_{\mathbb{P}^m}(-1))$ vanishes for $m > 0$, we see that $c_Y^0 = \chi(\mathcal{O}_Y(-1))$. By the Kodaira vanishing theorem, $H^i(\mathcal{O}_Y(-1)) = 0$ for $i < \dim(Y)$, so that

$$(-1)^{\dim(Y)} \, c_Y^0 \geq 0.$$

More generally, one obtains

$$c_Y^m = \chi(\mathcal{O}_{Y \cap \mathbb{P}^{n-m}}(-1))$$

for any linear subspace $\mathbb{P}^{n-m}$ in general position with respect to Y; then $Y \cap \mathbb{P}^{n-m}$ is nonsingular, and the same argument yields

$$(-1)^{\dim(Y) - m} \, c_Y^m \geq 0.$$

This argument adapts to subvarieties Y having rational singularities, by Grauert-Riemenschneider's generalization of the Kodaira vanishing theorem. It also shows that the statement of Theorem 1 does not hold for all closed subvarieties of $\mathbb{P}^n$. For instance, one may check that the projective cone Y over a nonsingular rational curve of degree d in $\mathbb{P}^{n-1}$ satisfies $c_Y^0 \leq n - d$. This yields examples of (singular) projective surfaces Y with arbitrarily negative c_Y^0.

However, Theorem 1 may be extended to all closed subvarieties of complex flag varieties, by replacing $[\mathcal{O}_Y]$ with $\sum_{i=0}^{\dim(Y)} (-1)^i [R^i \varphi_* \mathcal{O}_{\tilde{Y}}]$ where $\varphi : \tilde{Y} \to Y$ is a desingularization (see the Remark at the end of Section 2). On the other hand, Theorem 1 does not extend to positive characteristics: given any prime number p, there exists a nonsingular subvariety Y of projective space in characteristic p, such that $c_Y^0 = \chi(\mathcal{O}_Y(-1))$ is negative. Explicit examples where Y is homogeneous are constructed in [11] and [12].

This article is organized as follows. Section 1 gathers preliminary results concerning products of classes of structure sheaves and dualizing sheaves in the Grothendieck ring of flag varieties. In Section 2, we generalize the decomposition

$$[\mathcal{O}_Y] = \sum_{m=0}^{n} \chi(\mathcal{O}_{Y \cap \mathbb{P}^{n-m}}(-1)) [\mathcal{O}_{\mathbb{P}^m}]$$

to any Cohen-Macaulay subvariety Y of a flag variety X. For this, we construct a degeneration in $X \times X$ of the diagonal of Y, which is interesting in its own right (Theorem 2). In Section 3, we obtain an analogue of the vanishing theorem

$$H^i(\mathcal{O}_{Y \cap \mathbb{P}^{n-m}}(-1)) = 0 \text{ for } i < \dim(Y) - m,$$

where Y has rational singularities ; this completes the proof of Theorem 1.

In the final Section 4, we adapt these arguments to prove another positivity result: the class of the restriction of any globally generated line bundle to any Schubert variety, has nonnegative coefficients in the basis of classes of Schubert structure sheaves. That result was first obtained by W. Fulton and A. Lascoux for the variety of complete flags, using the combinatorics of Grothendieck polynomials (see [4]). It was generalized to all flag varieties by H. Pittie and A. Ram (see [16]), O. Mathieu (see [15]) and also by P. Littelmann and C. S. Seshadri (see [13]); all of them used the Littelmann path method and/or representation theory. Our geometric approach expresses the coefficients as the dimensions of certain spaces of sections of the line bundle (Theorem 4).

Notation and conventions. The ground field is the field $\mathbb{C}$ of complex numbers. An equidimensional reduced scheme of finite type over $\mathbb{C}$ will be called a *variety*; with this convention, varieties need not be irreducible.

A *desingularization* of a variety Y is a nonsingular variety $\tilde{Y}$ together with a proper birational morphism $\varphi : \tilde{Y} \to Y$. The singularities of Y are *rational* if Y is normal and if there exists a desingularization $\varphi : \tilde{Y} \to Y$ such that $R^i \varphi_*(\mathcal{O}_{\tilde{Y}}) = 0$ for all $i \geq 1$. Equivalently, Y is Cohen-Macaulay and the natural map $\varphi_* \omega_{\tilde{Y}} \to \omega_Y$ is an isomorphism, where $\omega_{\tilde{Y}}$ (resp. ω_Y) denotes the dualizing sheaf of $\tilde{Y}$ (resp. Y). If Y has rational singularities, then the preceding conditions hold for any desingularization (for these results, see e.g. [8] p. 50). Furthermore, since Y is normal, ω_Y is the double dual of the sheaf $\wedge^{\dim(Y)} \Omega^1_Y$, that is, the direct image of the sheaf of differential forms of top degree on the nonsingular locus Y^{reg}.

We next turn to notation concerning flag varieties. Let G be a simply connected semisimple algebraic group. Choose opposite Borel subgroups B and B^-, with common torus T; let $\mathcal{X}(T)$ be the group of characters of T, also called weights. In the root system Φ of (G, T), we have the subset Φ^+ of positive roots (that is, of roots of (B, T)), and the subset $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots. Let $\omega_1, \ldots, \omega_r$ be the corresponding fundamental weights; they form a basis of $\mathcal{X}(T)$. Let $\rho = \omega_1 + \cdots + \omega_r$, this equals the half sum of positive roots.

We also have the Weyl group W of (G, T), generated by the simple reflections $s_1, \ldots, s_r$ corresponding to the simple roots. This defines the length function ℓ and the Bruhat order $\leq$ on W. Let w_o be the longest element of W, then $B^- = w_o B w_o$.

Let P be a parabolic subgroup of G containing B and let W_P be the Weyl group of (P, T), a parabolic subgroup of W; let $w_{o,P}$ be the longest element of W_P. Each right W_P-coset in W contains a unique element of minimal length; this defines the subset W^P of minimal representatives of the quotient W/W_P. This subset is invariant under the map $w \mapsto w_o w w_{o,P}$; the induced bijection of W^P reverses the Bruhat order. Notice that $W^P = W$ if and only if $P = B$.

The homogeneous variety $X = G/P$ is called a flag variety, the full flag variety being G/B. For any weight λ regarded as a character of B, let $\mathcal{L}_{G/B}(\lambda)$ be the corresponding G-linearized invertible sheaf on G/B. The assignment $\lambda \mapsto \mathcal{L}_{G/B}(\lambda)$ yields an isomorphism from $\mathcal{X}(T)$ to the Picard group of G/B; the dominant weights correspond to the globally generated invertible sheaves. The dualizing sheaf of G/B is $\mathcal{L}_{G/B}(-2\rho)$. Via pullback under the natural map $G/B \to G/P$, the Picard group of G/P identifies with the subgroup of $\mathcal{X}(T)$ consisting of restrictions of characters of P.

The T-fixed points in G/P are the $e_w = wP/P$ ($w \in W/W_P$); we index them by W^P. The B-orbit $C_w = Be_w$ is a Bruhat cell, isomorphic to affine space of dimension $\ell(w)$; its closure in X is the Schubert variety X_w. The complement $X_w - C_w$ is the boundary ∂X_w; it has pure codimension 1 in X_w.

We shall also need the opposite Bruhat cell $C_w^- = B^- e_w$ of codimension $\ell(w)$ in X, the opposite Schubert variety $X^w = \overline{C_w^-}$, and its boundary ∂X^w. Then

$$X^w = w_o X_{w_o w w_{o,P}}.$$

For any v and w in W^P, we have $v \leq w \Leftrightarrow X_v \subseteq X_w \Leftrightarrow X^w \subseteq X^v$. Equivalently,

$$\partial X_w = \bigcup_{v \in W^P, v < w} X_v \text{ and } \partial X^w = \bigcup_{v \in W^P, v > w} X^v.$$

By [17] Proposition 2 and Theorem 4, all Schubert varieties have rational singularities; in particular, they are normal and Cohen-Macaulay. If in addition X is the full flag variety, then we have

$$\omega_{X_w} = \mathcal{L}_{X_w}(-\rho)(-\partial X_w).$$

1. Preliminaries on Grothendieck groups

For an arbitrary nonsingular variety X, let $K(X)$ be the Grothendieck group of the category of coherent sheaves on X ; the class in $K(X)$ of a coherent sheaf $\mathcal{F}$ will be denoted $[\mathcal{F}]$. Recall that $K(X)$ is isomorphic to the Grothendieck group of vector bundles over X. The tensor product of vector bundles defines a product $\cdot$ on $K(X)$ making it a commutative ring with unit ; the duality of vector bundles defines an involutive automorphism $*$ of that ring. For any coherent sheaves $\mathcal{F}, \mathcal{G}$ on X, we have

$$[\mathcal{F}] \cdot [\mathcal{G}] = \sum_{i=0}^{\dim(X)} (-1)^i \left[Tor_i^X(\mathcal{F}, \mathcal{G})\right] \text{ and } [\mathcal{F}]^* = \sum_{i=0}^{\dim(X)} (-1)^i \left[Ext_X^i(\mathcal{F}, \mathcal{O}_X)\right].$$

If in addition X is complete, then the Euler characteristic

$$\mathcal{F} \mapsto \chi(\mathcal{F}) = \sum_{i=0}^{\dim(X)} (-1)^i \, h^i(\mathcal{F})$$

yields the pushforward map $\chi : K(X) \to \mathbb{Z}$.

We associate to any closed subscheme Y of X the class $[\mathcal{O}_Y]$ of its structure sheaf. If in addition Y is Cohen-Macaulay and equidimensional, its dualizing sheaf

$$\omega_Y = Ext_X^{\text{codim}(Y)}(\mathcal{O}_Y, \omega_X)$$

yields another class $[\omega_Y]$, and we have

$$[\mathcal{O}_Y]^* = (-1)^{\text{codim}(Y)}[\omega_Y] \cdot [\omega_X]^*.$$

The product of two such classes associated with subschemes "in general position" is determined by the following result, a variant of a lemma of Fulton and Pragacz ([5], p. 108).

Lemma 1. *Let Y and Z be closed subschemes of a nonsingular variety X. If Y and Z are equidimensional, Cohen-Macaulay and intersect properly in X, then their scheme-theoretic intersection $Y \cap Z$ is equidimensional and Cohen-Macaulay as well. Furthermore,*

$$\omega_{Y \cap Z} = \omega_Y \otimes_{\mathcal{O}_X} \omega_Z \otimes_{\mathcal{O}_X} \omega_X^{-1},$$

and $Tor_i^X(\mathcal{O}_Y, \mathcal{O}_Z) = 0 = Tor_i^X(\omega_Y, \omega_Z)$ for all $i \geq 1$. As a consequence,

$$[\mathcal{O}_Y] \cdot [\mathcal{O}_Z] = [\mathcal{O}_{Y \cap Z}] \text{ and } [\omega_Y] \cdot [\omega_Z] = [\omega_{Y \cap Z}] \cdot [\omega_X].$$

Proof. Notice that $Y \times Z$ intersects properly the diagonal, $\mathrm{diag}(X)$ in $X \times X$, along $\mathrm{diag}(Y \cap Z)$. Thus, a sequence of local equations of $\mathrm{diag}(X)$ in $X \times X$ at any point of $\mathrm{diag}(Y \cap Z)$ restrict to a regular sequence in the local ring of $Y \times Z$ at that point. Since $Y \times Z$ is Cohen-Macaulay, it follows that the same holds for $\mathrm{diag}(Y \cap Z)$, and that $Tor_i^{X \times X}(\mathcal{O}_{Y \times Z}, \mathcal{O}_{\mathrm{diag}(X)}) = 0$ for any $i \geq 1$.

Let c (resp. d) denote the codimension of Y (resp. Z) in X. Then we may choose a locally free resolution $\mathcal{L}$ (resp. $\mathcal{M}$) of the sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_Y$ (resp. $\mathcal{O}_Z$) of length c (resp. d). Now $\mathcal{L} \otimes_{\mathbb{C}} \mathcal{M}$ is a locally free resolution of the sheaf of $\mathcal{O}_{X \times X}$-modules $\mathcal{O}_{Y \times Z}$. By the preceding step, it follows that

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M} \cong (\mathcal{L} \otimes_{\mathbb{C}} \mathcal{M}) \otimes_{\mathcal{O}_{X \times X}} \mathcal{O}_{\mathrm{diag}(X)}$$

is a locally free resolution of $\mathcal{O}_{Y \cap Z}$. Thus, $Tor_i^X(\mathcal{O}_Y, \mathcal{O}_Z) = 0$ for all $i \geq 1$.

Since Y is Cohen-Macaulay, the dual complex $\mathcal{L}^*$ is a locally free resolution of the sheaf

$$Ext_X^c(\mathcal{O}_Y, \mathcal{O}_X) = \omega_Y \otimes_{\mathcal{O}_X} \omega_X^{-1}.$$

Likewise, $\mathcal{M}^*$ (resp. $(\mathcal{L} \otimes \mathcal{M})^* = \mathcal{L}^* \otimes \mathcal{M}^*$) is a locally free resolution of $\omega_Z \otimes_{\mathcal{O}_X} \omega_X^{-1}$ (resp. of $\omega_{Y \cap Z} \otimes_{\mathcal{O}_X} \omega_X^{-1}$). It follows that

$$\omega_Y \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} \omega_Z \otimes_{\mathcal{O}_X} \omega_X^{-1} \cong \omega_{Y \cap Z} \otimes_{\mathcal{O}_X} \omega_X^{-1}$$

and that $Tor_i^X(\omega_Y, \omega_Z) = 0$ for all $i \geq 1$. $\square$

We next turn to the case where X is a flag variety ; we shall obtain variants of Kleiman's transversality theorem, see [9]. In what follows, a statement holds "for general $g \in G$" if it holds for all g in a nonempty open subset of G.

Lemma 2. *Let Y be a closed subvariety of $X = G/P$ and let $w \in W^P$. Then the translate gX_w intersects properly Y, for general $g \in G$.*

If in addition Y is Cohen-Macaulay, then $Y \cap gX_w$ is Cohen-Macaulay as well, for general g. As a consequence,

$$[\mathcal{O}_Y] \cdot [\mathcal{O}_{X_w}] = [\mathcal{O}_{Y \cap g \cdot X_w}] \text{ and } [\omega_Y] \cdot [\omega_{X_w}] = [\omega_{Y \cap g \cdot X_w}] \cdot [\omega_X].$$

If in addition Y is normal, then $Y \cap gX_w$ is normal as well.

Finally, if Y has rational singularities, then $Y \cap gX_w$ has rational singularities as well.

Proof. The first assertion follows from Kleiman's transversality theorem ; we recall the proof, since we shall repeatedly use its ingredients. Let

$$i : Y \to X$$

be the inclusion and let

$$m : G \times X_w \to X, \ (g, x) \mapsto gx$$

be the "multiplication" map. Notice that G acts on $G \times X_w$ via left multiplication on G, and that m is a G-equivariant morphism to G/P. Thus, m is a locally trivial fibration with fiber $m^{-1}(P/P)$. The latter is isomorphic to $\overline{Pw^{-1}B}$, the pullback in G of a Schubert variety in G/B. Therefore, the fiber of m has rational singularities.

Now consider the cartesian product

$$Z = Y \times_X (G \times X_w)$$

with projections ι to $G \times X_w$, and μ to Y. Let $p : G \times X_w \to G$ be the projection and let $\pi = p \circ \iota$ as displayed in the following commutative diagram:

$$
\begin{array}{ccccc}
G & \xleftarrow{\ \pi\ } & Z & \xrightarrow{\ \mu\ } & Y \\
{\scriptstyle id}\downarrow & & {\scriptstyle \iota}\downarrow & & {\scriptstyle i}\downarrow \\
G & \xleftarrow{\ p\ } & G \times X_w & \xrightarrow{\ m\ } & X
\end{array}
$$

By definition, the square on the right is cartesian, so that μ is also a locally trivial fibration with fiber $\overline{Pw^{-1}B}$. As a consequence, Z is a variety, and we have

$$\dim(Z) = \dim(Y) + \dim(G \times X_w) - \dim(X).$$

Furthermore, the morphism π is proper, and its fiber at each $g \in G$ identifies with $Y \cap gX_w$. Thus, for general g, the intersection $Y \cap gX_w$ is either empty or equidimensional of dimension $\dim(Y) + \dim(X_w) - \dim(X)$. This proves the first assertion.

If in addition Y is Cohen-Macaulay, then by Lemma 1, the intersection $Y \cap gX_w$ is Cohen-Macaulay whenever it is proper. If in addition Y is normal, then $Y \cap gX_w$ is nonsingular in codimension 1, for general g (since $Y^{\mathrm{reg}} \cap gX_w^{\mathrm{reg}}$ is nonsingular). Therefore, $Y \cap gX_w$ is normal, by Serre's criterion.

Assume now that Y has rational singularities ; then Z has rational singularities as well. Now the following easy result completes the proof of Lemma 2.

Lemma 3. *Let Z and S be varieties and let $\pi : Z \to S$ be a morphism. If Z has rational singularities, then the same holds for the general fibers of π.*

Proof. We may reduce to the situation where Z is affine, S is nonsingular and π is flat with connected fibers. Let F be a general fiber of π ; then F is Cohen-Macaulay, since Z is.

Choose a desingularization $\varphi : \tilde{Z} \to Z$ and let $\tilde{\pi} = \pi \circ \varphi$. Then $\tilde{F} = \varphi^{-1}(F)$ is a general fiber of $\tilde{\pi}$, and hence is connected. By generic smoothness, $\tilde{F}$ is nonsingular, so that φ restricts to a desingularization $\psi : \tilde{F} \to F$. Since Z has rational singularities, the map $\varphi_* \omega_{\tilde{Z}} \to \omega_Z$ is an isomorphism ; since ω_F is the restriction of $\omega_Z \otimes \pi^* \omega_S^{-1}$ to F, and similarly for $\omega_{\tilde{F}}$, it follows that the map $\psi_* \omega_{\tilde{F}} \to \omega_F$ is an isomorphism as well. $\square$

$\square$

Consider, for example, $Y = X^v$ where $v \in W^P$. Since Y (resp. X_w) is invariant under B^- (resp. B) and the product $B^- B$ is open in G, we see that $X^v \cap X_w$ is the intersection of X^v with a general translate of X_w, and hence has rational singularities.

In fact $X^v \cap X_w$ is the closure of $C^v \cap C_w$, and hence is irreducible of dimension $\ell(w) - \ell(v)$ (or empty), see [18] Theorem 3.7. Considering T-fixed points, one sees that $X^v \cap X_w$ is nonempty if and only if $v \leq w$.

If in addition X is the full flag variety, then we obtain

$$\omega_{X^v \cap X_w} = \mathcal{O}_{X^v \cap X_w}(-(X_w \cap \partial X^v) - (X^v \cap \partial X_w)).$$

Likewise, $X_v \cap w_o X_w = X_v \cap X^{w_o w w_{o,P}}$ is the intersection of X_v with a general translate of X_w.

As a final preliminary result, we study the boundaries of Schubert varieties:

Lemma 4. *For every $w \in W^P$, the boundary ∂X_w is Cohen-Macaulay. Moroever, for every Cohen-Macaulay subvariety Y of X, the intersection $Y \cap g X_w$ is Cohen-Macaulay for general $g \in G$. As a consequence,*

$$Ext^i_{Y \cap g X_w}(\mathcal{O}_{Y \cap g X_w}(-Y \cap g \partial X_w), \omega_{Y \cap g X_w}) = 0 \text{ for any } i \geq 1.$$

If in addition Y is normal, then

$$Hom(\mathcal{O}_{Y \cap g X_w}(-Y \cap g \partial X_w), \omega_{Y \cap g X_w}) = \omega_{Y \cap g X_w}(Y \cap g \partial X_w).$$

Proof. If $X = G/B$ then $\mathcal{O}_{X_w}(-\partial X_w)$ is locally isomorphic to ω_{X_w}, so that the ideal sheaf of ∂X_w in X_w is Cohen-Macaulay. Since X_w is also Cohen-Macaulay and ∂X_w has pure codimension 1, it follows easily that ∂X_w is Cohen-Macaulay as well.

In the general case where $X = G/P$, we set (in this proof) $X' = G/B$ and denote the Schubert varieties in X' by X'_w. For any $w \in W^P$, notice that the natural map $X' \to X$ restricts to a proper surjective morphism

$$\eta : X'_w \to X_w$$

which maps isomorphically the Bruhat cell $\hat{C}_w$ to C_w. Thus, $\eta^{-1}(\partial X_w) = \partial X'_w$ (as sets). By [17] Theorem 2 and Proposition 3, we have

$$\eta_* \mathcal{O}_{X'_w} = \mathcal{O}_{X_w} \text{ and } R^i \eta_*(\mathcal{O}_{X'_w}) = 0 \text{ for } i \geq 1.$$

It follows that $\eta_* \mathcal{O}_{X'_w}(-\partial X'_w) = \mathcal{O}_{X_w}(-\partial X_w)$. We claim that

$$R^i \eta_* (\mathcal{O}_{X'_w}(-\partial X'_w)) = 0 \text{ for } i \geq 1.$$

To see this, choose a reduced decomposition of w and let

$$\psi : \tilde{X}_w \to X'_w$$

be the corresponding standard desingularization (see [17]). Then B acts in $\tilde{X}_w$, and ψ is B-equivariant. Furthermore, $\tilde{X}_w$ contains a dense B-orbit, mapped isomorphically to $\hat{C}_w$ by ψ; the complement $\partial \tilde{X}_w$ of this orbit is a union of nonsingular irreducible divisors $\tilde{X}_1, \ldots, \tilde{X}_\ell$ intersecting transversally. By [17] Proposition 2, we have

$$\omega_{\tilde{X}_w} = \mathcal{O}_{\tilde{X}_w}(-\partial \tilde{X}_w) \otimes \psi^* \mathcal{L}_{X'_w}(-\rho).$$

Let

$$\tilde{\eta} = \eta \circ \psi : \tilde{X}_w \to X_w,$$

this is a desingularization of X_w. Since $\mathcal{L}_{X'_w}(\rho)$ is ample, we have

$$R^i \tilde{\eta}_* (\omega_{\tilde{X}_w} \otimes \psi^* \mathcal{L}_{X'_w}(\rho)) = 0$$

for $i \geq 1$, by the Grauert-Riemenschneider vanishing theorem (see [7]) ; furthermore,

$$R^i \psi_* (\omega_{\tilde{X}_w} \otimes \psi^* \mathcal{L}_{X'_w}(\rho)) = R^i \psi_* (\omega_{\tilde{X}_w}) \otimes \mathcal{L}_{X'_w}(\rho)$$

vanishes for $i \geq 1$ as well, and

$$\psi_* (\omega_{\tilde{X}_w} \otimes \psi^* \mathcal{L}_{X'_w}(\rho)) = \psi_* \omega_{\tilde{X}_w} \otimes \mathcal{L}_{X'_w}(\rho) = \omega_{X'_w} \otimes \mathcal{L}_{X'_w}(\rho) = \mathcal{O}_{X'_w}(-\partial X'_w).$$

Now the Leray spectral sequence for $\tilde{\eta} = \eta \circ \psi$ implies the claimed vanishing.

Using that claim and duality for the morphism $X'_w \to X$, we obtain

$$Ext^i_X(\mathcal{O}_{X_w}(-\partial X_w), \omega_X) = R^{i-\mathrm{codim}(X_w)} \eta_* (\omega_{X'_w}(\partial X'_w)) = R^{i-\mathrm{codim}(X_w)} \eta_* (\mathcal{L}_{X'_w}(-\rho)).$$

Thus, to prove that the sheaf $\mathcal{O}_{X_w}(-\partial X_w)$ is Cohen-Macaulay, it suffices to check the vanishing of $R^i \eta_* (\mathcal{L}_{X'_w}(-\rho))$ for $i \geq 1$. We deduce this from the Kawamata-Viehweg vanishing theorem (see e.g. [3]) as follows.

Notice that $\partial \tilde{X}_w$ is the support of a very ample divisor of $\tilde{X}_w$ (to see this, consider a B-linearized very ample invertible sheaf $\mathcal{M}$ on $\tilde{X}_w$ and a B-semi-invariant section σ of $\mathcal{M}$ that vanishes on $\partial \tilde{X}_w$; then the zero set of σ is exactly $\partial \tilde{X}_w$). Thus, we may choose positive integers $b_1, \ldots, b_\ell$ such that the divisor $b_1 \tilde{X}_1 + \cdots + b_\ell \tilde{X}_\ell$ is very ample. Choose also a positive integer $N > \max(b_1, \ldots, b_\ell)$ and let $a_1 = N - b_1, \ldots, a_\ell = N - b_\ell$. Finally, let $\mathcal{L} = \mathcal{O}_{\tilde{X}_w}(\tilde{X}_w)$ and $D = a_1 \tilde{X}_1 + \cdots + a_\ell \tilde{X}_\ell$. Then the invertible sheaf

$$\mathcal{L}^N(-D) = \mathcal{O}_{\tilde{X}_w}(b_1 \tilde{X}_1 + \cdots + b_\ell \tilde{X}_\ell)$$

is very ample. By [3] Corollary 6.11, it follows that $R^i \tilde{\eta}_* (\omega_{\tilde{X}_w}(\tilde{X}_w)) = 0$ for $i \geq 1$, that is, $R^i \tilde{\eta}_* (\psi^* \mathcal{L}_{X'_w}(-\rho)) = 0$. As above, the Leray spectral sequence for $\tilde{\eta} = \eta \circ \psi$

yields the vanishing of $R^i \eta_*(\mathcal{L}_{X'_w}(-\rho))$ for $i \geq 1$; thus, $\mathcal{O}_{X_w}(-\partial X_w)$, and hence ∂X_w, is Cohen-Macaulay.

By Lemma 2, the same holds for $Y \cap g\partial X_w$ for general g. Since $Y \cap g\partial X_w$ has pure codimension 1 in $Y \cap gX_w$, and the latter is Cohen-Macaulay, the ideal sheaf $\mathcal{O}_{Y \cap gX_w}(-Y \cap g\partial X_w)$ is Cohen-Macaulay as well. This implies the vanishing of $Ext^i_{Y \cap gX_w}(\mathcal{O}_{Y \cap gX_w}(-Y \cap g\partial X_w, \omega_{Y \cap gX_w})$ for $i \geq 1$. If in addition Y is normal, then so is $Y \cap gX_w$ by Lemma 2. Thus, the sheaf $\omega_{Y \cap gX_w}$ is reflexive of rank 1 ; this implies the latter assertion. $\square$

2. A DEGENERATION OF THE DIAGONAL

Let Y be a closed subvariety of a flag variety X. Consider the diagonal, $\mathrm{diag}(Y) \subseteq X \times X$. We shall construct a flat degeneration of $\mathrm{diag}(Y)$ to a union of products of subvarieties, by taking limits under all "positive" one-parameter subgroups.

Specifically, let T act linearly in affine space $\mathbb{A}^r$ with weights $-\alpha_1, \ldots, -\alpha_r$. Then the T-orbit of the point $(1, \ldots, 1)$ is the complement of the union of all coordinate hyperplanes, and the isotropy group of that point is the center $Z(G)$. Thus, the orbit $T \cdot (1, \ldots, 1)$ is isomorphic to the adjoint torus $T_{\mathrm{ad}} = T/Z(G)$.

Let $\mathcal{Y}$ be the closure in $X \times X \times \mathbb{A}^r$ of the subset

$$\{(ty, y, \alpha_1(t^{-1}), \ldots, \alpha_r(t^{-1})) \mid y \in Y, \ t \in T\}$$

and let

$$\pi_Y : \mathcal{Y} \to \mathbb{A}^r, \ p_Y : \mathcal{Y} \to X \times X$$

be the projections. Clearly, the morphism π_Y is proper and its fibers identify via p_Y with closed subschemes of $X \times X$. This yields an isomorphism

$$\pi_Y^{-1}(1, \ldots, 1) \cong \mathrm{diag}(Y).$$

Notice that T acts in $X \times X \times \mathbb{A}^r$ by

$$t \cdot (x_1, x_2, t_1, \ldots, t_r) = (tx_1, x_2, \alpha_1(t^{-1})t_1, \ldots, \alpha_r(t^{-1})t_r)$$

and leaves $\mathcal{Y}$ invariant ; furthermore, π_Y is equivariant. Thus, π_Y is surjective, and restricts to a trivial fibration over the orbit $T \cdot (1, \ldots, 1)$ with fiber $\mathrm{diag}(Y)$. Some less obvious properties of π_Y are summarized in the following statement.

Theorem 2. *Let Y be a Cohen-Macaulay subvariety of X, such that the intersection $Y \cap X^w$ is proper and reduced for all $w \in W^P$. Then, with preceding notation, $\mathcal{Y}$ is Cohen-Macaulay ; furthermore, π_Y is flat with reduced fibers, and*

$$\pi_Y^{-1}(0, \ldots, 0) = \bigcup_{w \in W^P} X_w \times (Y \cap X^w).$$

Proof. We begin with the case where $Y = X$; we then set $\mathcal{Y} = \mathcal{X}$. We show how to obtain $\pi_X : \mathcal{X} \to \mathbb{A}^r$ by base change from a degeneration of $\operatorname{diag}(X)$ constructed in [1] §7.

Let $G_{\mathrm{ad}} = G/Z(G)$ be the adjoint group of G and let $\overline{G}_{\mathrm{ad}}$ be its wonderful completion ; this is a nonsingular projective variety where $G \times G$ acts with a dense orbit isomorphic to $(G \times G)/(Z(G) \times Z(G)) \operatorname{diag}(G) \cong G_{\mathrm{ad}}$, and a unique closed orbit isomorphic to $(G \times G)/(B \times B^-)$. Let $P_{\mathrm{ad}} = P/Z(G)$ and let $\overline{P}$ be its closure in $\overline{G}_{\mathrm{ad}}$. Since $\overline{P}_{\mathrm{ad}}$ is invariant under the action of $P \times P$, we may form the associated fiber bundle

$$p : G \times G \times^{P \times P} \overline{P}_{\mathrm{ad}} \to G/P \times G/P = X \times X.$$

On the other hand, the map $G \times G \times \overline{P}_{\mathrm{ad}} \to \overline{G}_{\mathrm{ad}}$, $(x, y, z) \mapsto (x, y)z$ factors through a map

$$\pi : G \times G \times^{P \times P} \overline{P}_{\mathrm{ad}} \to \overline{G}_{\mathrm{ad}},$$

which is clearly surjective and $G \times G$-equivariant. Furthermore, the product map

$$p \times \pi : G \times G \times^{P \times P} \overline{P}_{\mathrm{ad}} \to X \times X \times \overline{G}_{\mathrm{ad}}$$

is a closed immersion, with image the "incidence variety"

$$\{(xP, yP, z) \mid (x, y) \in G \times G, z \in (x, y)\overline{P}_{\mathrm{ad}}\}.$$

By [1] §7, π is flat, with reduced Cohen-Macaulay fibers ; these identify with closed subschemes of $X \times X$ via p. The fiber at the identity element of G_{ad} (resp. at the unique $(B \times B^-)$-fixed point z) identifies with $\operatorname{diag}(X)$ (resp. $\cup_{w \in W^P} X_w \times X^w$). Furthermore, the closure $\overline{T}_{\mathrm{ad}}$ of the torus T_{ad} in $\overline{G}_{\mathrm{ad}}$ is a nonsingular $T \times T$-equivariant completion of that torus, containing z as a fixed point.

Let $\overline{T}_{\mathrm{ad},z}$ be the unique $T \times T$-invariant open affine neighborhood of z in $\overline{T}_{\mathrm{ad}}$. Then $\overline{T}_{\mathrm{ad},z}$ is equivariantly isomorphic to affine r-space where $T \times T$ acts linearly with weights $(-\alpha_1, \alpha_1), \ldots, (-\alpha_r, \alpha_r)$. Thus, for the action of T by left multiplication, $\overline{T}_{\mathrm{ad},z}$ is isomorphic to $\mathbb{A}^r$.

We claim that the subvariety $\pi^{-1}(\overline{T}_{\mathrm{ad},z})$ of $X \times X \times \mathbb{A}^r$ equals $\mathcal{X}$. To see this, note that

$$\pi^{-1}(T) = \{(xP, yP, z) \mid (x, y) \in G \times G, z \in T_{\mathrm{ad}} \cap (x, y)\overline{P}_{\mathrm{ad}}\}$$
$$= \{(zyP, yP, z) \mid y \in G, z \in T_{\mathrm{ad}}\} = \{(t\xi, \xi, t \cdot (1, \ldots, 1)) \mid \xi \in X, \ t \in T\}$$

since $T_{\mathrm{ad}} \cap (x, y)\overline{P}_{\mathrm{ad}} = T_{\mathrm{ad}} \cap xP_{\mathrm{ad}}y^{-1}$. It follows that $\pi^{-1}(\overline{T}_{\mathrm{ad},z})$ contains $\mathcal{X}$ as an irreducible component. Furthermore, π restricts to a flat morphism from the complement $\pi^{-1}(\overline{T}_{\mathrm{ad},z}) - \mathcal{X}$, to $\overline{T}_{\mathrm{ad},z}$. If this complement is not empty, then its image meets the open subset T_{ad} of $\overline{T}_{\mathrm{ad},z}$, a contradiction. This proves the claim, and hence all assertions of Theorem 2 in the case where $Y = X$.

In the general case, we consider

$$\mathcal{Y}' = \mathcal{X} \cap (X \times Y \times \mathbb{A}^r)$$

(scheme-theoretical intersection in $X \times X \times \mathbb{A}^r$), with projection $\pi_Y' : \mathcal{Y}' \to \mathbb{A}^r$. Notice that $\mathcal{Y}$ is contained in $\mathcal{Y}'$, and that

$$\mathcal{Y} \cap \pi_Y^{-1}(T \cdot (1, \ldots, 1)) = \mathcal{Y}' \cap \pi_Y'^{-1}(T \cdot (1, \ldots, 1)).$$

Thus, $\mathcal{Y}$ is an irreducible component of $\mathcal{Y}'$; the latter is invariant under the action of T in $X \times X \times \mathbb{A}^r$.

We claim that $\mathcal{X}$ intersects properly $X \times Y \times \mathbb{A}^r$ in $X \times X \times \mathbb{A}^r$, that is, every irreducible component C of $\mathcal{Y}'$ has dimension equal to

$$\dim(\mathcal{X}) + \dim(X \times Y \times \mathbb{A}^r) - \dim(X \times X \times \mathbb{A}^r) = \dim(Y) + r.$$

In fact, it suffices to check that $\dim(C) \leq \dim(Y) + r$. Since C is T-invariant and T acts attractively in $\mathbb{A}^r$ with fixed point $(0, \ldots, 0)$, it suffices in turn to show that $\dim(C \cap \pi_Y'^{-1}(0, \ldots, 0)) \leq \dim(Y)$. But

$$C \cap \pi_Y'^{-1}(0, \ldots, 0) \subseteq \mathcal{X} \cap (X \times Y \times \{(0, \ldots, 0)\}$$

$$\cong (\bigcup_{w \in W^P} X_w \times X^w) \cap (X \times Y) = \bigcup_{w \in W^P} X_w \times (Y \cap X^w).$$

And the latter is equidimensional of dimension $\dim(Y)$, since Y intersects properly all X^w. This proves our claim.

Since $\mathcal{X}$ and $X \times Y \times \mathbb{A}^r$ are Cohen-Macaulay subvarieties of $X \times X \times \mathbb{A}^r$ intersecting properly, then $\mathcal{Y}'$ is equidimensional and Cohen-Macaulay, by Lemma 1. Furthermore, the morphism $\pi_Y' : \mathcal{Y}' \to \mathbb{A}^r$ is equidimensional by the proof of the preceding claim ; therefore, π_Y' is flat. As in the first step of the proof, it follows that $\mathcal{Y}'$ equals $\mathcal{Y}$ as sets. Furthermore,

$$\pi_Y'^{-1}(T \cdot (1, \ldots, 1)) = \mathcal{X} \cap (X \times Y \times T \cdot (1, \ldots, 1))$$

is clearly reduced, so that $\mathcal{Y}'$ is generically reduced. Since it is Cohen-Macaulay, it is reduced, and $\mathcal{Y}'$ equals $\mathcal{Y}$ as subschemes.

Likewise, the fiber $\pi_Y^{-1}(0, \ldots, 0)$ equals $\cup_{w \in W^P} X_w \times (Y \cap X^w)$ as sets. Furthermore, this fiber is generically reduced (since each $Y \cap X^w$) is), and Cohen-Macaulay (since π_Y is flat and $\mathcal{Y}$ is Cohen-Macaulay). Thus, $\pi_Y^{-1}(0, \ldots, 0)$ is reduced. By semicontinuity, it follows that all fibers are reduced. $\square$

For Y satisfying the assumptions of Theorem 2, let

$$Y_0 = \pi_Y^{-1}(0, \ldots, 0) = \bigcup_{w \in W^P} X_w \times (Y \cap X^w)$$

regarded as a subvariety of $X \times X$.

Corollary 1. *(i) With preceding notation and assumptions, the structure sheaf $\mathcal{O}_{Y_0}$ has an ascending filtration with associated graded*

$$\bigoplus_{w \in W^P} \mathcal{O}_{X_w} \otimes_{\mathbb{C}} \mathcal{O}_{Y \cap X^w}(-Y \cap \partial X^w).$$

It has also a descending filtration with associated graded

$$\bigoplus_{w \in W^P} \mathcal{O}_{X_w}(-\partial X_w) \otimes_{\mathbb{C}} \mathcal{O}_{Y \cap X^w}.$$

(ii) If in addition Y is normal, then the dualizing sheaf ω_{Y_0} has a descending filtration with associated graded

$$\bigoplus_{w \in W^P} \omega_{X_w} \otimes_{\mathbb{C}} \omega_{Y \cap X^w}(Y \cap \partial X^w).$$

It has also an ascending filtration with associated graded

$$\bigoplus_{w \in W^P} \omega_{X_w}(\partial X_w) \otimes_{\mathbb{C}} \omega_{Y \cap X^w}.$$

Proof. (i) We adapt the argument of [1] Theorem 11 to this setting. We may index the finite poset $W^P = \{w_1, \ldots, w_N\}$ so that $i \leq j$ whenever $w_i \leq w_j$. Let

$$Z_i = X_{w_i} \times (Y \cap X^{w_i}), \ \ Z_{\geq i} = \cup_{j \geq i} Z_j \text{ and } Z_{>i} = \cup_{j > i} Z_j$$

for $1 \leq i \leq N$. Then $Z_{\geq 1} = Y_0$ and $Z_{\geq N} = X_{w_N} \times (Y \cap X^{w_N})$. We claim that

$$Z_i \cap Z_{>i} = X_{w_i} \times (Y \cap \partial X^{w_i}).$$

To see this, we may assume that $Y = X$. Then $Z_i \cap Z_{>i}$ is a union of products $X_u \times X^v$ for certain u, v in W^P. We must have $u \leq w_i \leq v$ (since $X_u \times X^v \subseteq Z_i$) and $w_i \neq v$ (since $X_u \times X^v \subseteq Z_{>i}$). Thus, $Z_i \cap Z_{>i}$ is contained in $X_{w_i} \times \partial X^{w_i}$. Conversely, if $X^v \subseteq \partial X^{w_i}$, then $v = w_j$ for some j such that $w_j > w_i$, whence $j > i$; this yields the opposite inclusion. The claim is proved.

Now consider the exact sequence

$$0 \to \mathcal{I}_i \to \mathcal{O}_{Z_{\geq i}} \to \mathcal{O}_{Z_{>i}} \to 0$$

where $\mathcal{I}_i$ denotes the ideal sheaf of $Z_{>i}$ in $Z_{\geq i}$. Then $\mathcal{I}_i$ identifies with the ideal sheaf of $Z_i \cap Z_{>i}$ in Z_i; by the claim, this is the ideal sheaf of $X_{w_i} \times (Y \cap \partial X^{w_i})$ in $X_{w_i} \times (Y \cap X^{w_i})$. This yields the ascending filtration of $\mathcal{O}_{Y_0}$.

With obvious notation, we obtain likewise

$$Z_i \cap Z_{<i} = \partial X_{w_i} \times (Y \cap X^{w_i})$$

which yields the descending filtration.

(ii) By Lemma 4, the sheaf $\mathcal{I}_i$ is Cohen-Macaulay of depth $\dim(Y)$. Now a descending induction on i shows that each $Z_{\geq i}$ is a Cohen-Macaulay variety of dimension $\dim(Y)$. Furthermore, we obtain exact sequences

$$0 \to Ext_{X \times X}^{\dim(X \times X) - \dim(Y)}(\mathcal{O}_{Z_{>i}}, \omega_{X \times X}) \to Ext_{X \times X}^{\dim(X \times X) - \dim(Y)}(\mathcal{O}_{Z_{\geq i}}, \omega_{X \times X}) \to$$
$$\to Ext_{X \times X}^{\dim(X \times X) - \dim(Y)}(\mathcal{I}_i, \omega_{X \times X}) \to 0,$$

that is,

$$0 \to \omega_{Z_{>i}} \to \omega_{Z_{\geq i}} \to \omega_{X_{w_i}} \otimes_{\mathbb{C}} Ext_X^{\operatorname{codim}(Y \cap X^{w_i})}(\mathcal{O}_{Y \cap X^{w_i}}(-Y \cap \partial X^{w_i}), \omega_X) \to 0.$$

Since $Y \cap X^{w_i}$ and $Y \cap \partial X^{w_i}$ are Cohen-Macaulay by Lemma 4, we obtain

$$Ext_X^{\operatorname{codim}(Y \cap X^{w_i})}(\mathcal{O}_{Y \cap X^{w_i}}(-Y \cap \partial X^{w_i}), \omega_X) \cong Hom_X(\mathcal{O}_{Y \cap X^{w_i}}(-Y \cap \partial X^{w_i}), \omega_{Y \cap X^{w_i}}).$$

The latter is isomorphic to $\omega_{Y \cap X^{w_i}}(Y \cap \partial X^{w_i})$, by Lemma 4 again. This yields a descending filtration of $\omega_{Z_{\geq 1}} = \omega_{Y_0}$, with associated graded as claimed. The ascending filtration is obtained by replacing $Z_{\geq i}$ with $Z_{\leq i}$. $\square$

Next we derive from Corollary 1 several formulae for decomposing $[\mathcal{O}_Y]$ and $[\omega_Y]$ in the Grothendieck group $K(X)$. Recall that this group is freely generated by the classes $[\mathcal{O}_{X_w}]$ where $w \in W^P$ (see [10] §4). Another natural basis of $K(X)$ consists of the classes $[\mathcal{O}_{X_w}(-\partial X_w)] = [\mathcal{O}_{X_w}] - [\mathcal{O}_{\partial X_w}]$. Using the duality involution $*$, we obtain two additional bases: the $[\omega_{X_w}]$, and the $[\omega_{X_w}(\partial X_w)]$.

Corollary 2. *For any Cohen-Macaulay closed subvariety Y of X, we have in $K(X)$:*

$$[\mathcal{O}_Y] = \sum_{w \in W^P} \chi(\mathcal{O}_{Y \cap gX^w}(-Y \cap g\partial X^w))[\mathcal{O}_{X_w}] = \sum_{w \in W^P} \chi(\mathcal{O}_{Y \cap gX^w})[\mathcal{O}_{X_w}(-\partial X_w)]$$

for general $g \in G$. If in addition Y is normal, then

$$[\omega_Y] = \sum_{w \in W^P} \chi(\omega_{Y \cap gX^w}(Y \cap g\partial X^w))[\omega_{X_w}] = \sum_{w \in W^P} \chi(\omega_{Y \cap gX^w})[\omega_{X_w}(\partial X_w)].$$

Proof. Since $\operatorname{diag}(Y)$ and Y_0 are two fibers of the flat family $\mathcal{Y}$ over the affine space $\mathbb{A}^r$, we have $[\mathcal{O}_{\operatorname{diag}(Y)}] = [\mathcal{O}_{Y_0}]$ in $K(X \times X)$. By Corollary 1 (i), it follows that

$$[\mathcal{O}_{\operatorname{diag}(Y)}] = \sum_{w \in W^P} [\mathcal{O}_{X_w} \otimes_{\mathbb{C}} \mathcal{O}_{Y \cap gX^w}(-Y \cap g\partial X^w)] = \sum_{w \in W^P} [\mathcal{O}_{X_w}(-g\partial X_w) \otimes_{\mathbb{C}} \mathcal{O}_{Y \cap gX^w}].$$

Now let

$$p_1 : X \times X \to X$$

be the projection to the first factor, and denote

$$p_{1*} : K(X \times X) \to K(X)$$

the corresponding pushforward map. Then $p_{1*}[\mathcal{O}_{\operatorname{diag}(Y)}] = [\mathcal{O}_Y]$, whereas

$$p_{1*}([\mathcal{F} \otimes_{\mathbb{C}} \mathcal{G}]) = \chi(\mathcal{G})[\mathcal{F}]$$

for all coherent sheaves $\mathcal{F}$, $\mathcal{G}$ on X. This yields our formulae for $\mathcal{O}_Y$.

To obtain the formulae for ω_Y, notice that the sheaf $\omega_{\mathcal{Y}}$ is flat over $\mathbb{A}^r$, since $\mathcal{Y}$ is Cohen-Macaulay and flat over $\mathbb{A}^r$. Furthermore, the restriction to $\omega_{\mathcal{Y}}$ to the fiber at $(1,\ldots,1)$ (resp. $(0,\ldots,0)$) is isomorphic to $\omega_{\mathrm{diag}(Y)}$ (resp. ω_{Y_0}). Thus, $\omega_{\mathrm{diag}(Y)} = \omega_{Y_0}$ in $K(X \times X)$. Now both formulae follow from Corollary 1 (ii) by the preceding argument.

Alternatively, these formulae may be derived from those for $[\mathcal{O}_Y]$ by applying the involution $*$ and duality in $Y \cap gX^w$. For, using Lemma 4, we obtain isomorphisms

$$H^i(\mathcal{O}_{Y \cap gX^w}(-Y \cap g\partial X^w))^* = H^{\dim(Y \cap gX^w)-i}(\omega_{Y \cap gX^w}(Y \cap g\partial X^w)),$$

whence

$$\chi(\mathcal{O}_{Y \cap gX^w}(-Y \cap g\partial X^w)) = (-1)^{\dim(Y \cap gX^w)}\chi(\omega_{Y \cap gX^w}(Y \cap g\partial X^w)).$$

$\square$

3. A VANISHING THEOREM

Consider a normal Cohen-Macaulay subvariety Y of a flag variety X. In Section 2, we constructed a flat family in $X \times X$ with general fibers certain translates of $\mathrm{diag}(Y)$. Furthermore, the special fiber Y_0 is Cohen-Macaulay, and its canonical sheaf has a filtration with layers the $\omega_{X_w} \otimes_{\mathbb{C}} \omega_{Y \cap gX^w}(Y \cap g\partial X^w)$, where g is a general element of G.

Let $p_1 : X \times X \to X$ be the first projection. If all higher cohomology groups of the sheaves $\omega_{Y \cap gX^w}(Y \cap g\partial X^w)$ vanish, then $R^i p_{1*}(\omega_{Y_0}) = 0$ for $i \geq 1$. Furthermore, the sheaf $p_{1*}\omega_{Y_0}$ has a filtration with layers the ω_{X_w} of respective multiplicities $h^0(\omega_{Y \cap gX^w}(Y \cap g\partial X^w))$. Thus, the following equalities hold in $K(X)$:

$$p_{1*}[\omega_{Y_0}] = \sum_{w \in W^P} h^0(\omega_{Y \cap gX^w}(Y \cap g\partial X^w))\,[\omega_{X_w}],$$

$$p_{1*}[\mathcal{O}_{Y_0}] = \sum_{w \in W^P} (-1)^{\mathrm{codim}(X_w)-\mathrm{codim}(Y)} h^0(\omega_{Y \cap gX^w}(Y \cap g\partial X^w))\,[\mathcal{O}_{X_w}].$$

On the other hand, since p_1 maps isomorphically $\mathrm{diag}(Y)$ to Y, we obtain

$$p_{1*}[\mathcal{O}_{Y_0}] = [\mathcal{O}_Y] \text{ and } p_{1*}[\omega_{Y_0}] = [\omega_Y].$$

Together with the preceding equalities, this yields a sharper version of Theorem 1.

By our next result, this vanishing condition holds in the case where Y has rational singularities. In fact we shall prove a slightly more general vanishing theorem, in view of further applications in the next section.

Theorem 3. *Consider a flag variety $X = G/P$, a closed subvariety Y and an invertible sheaf $\mathcal{L} = \mathcal{L}_X(\lambda)$. Assume that Y has rational singularities, and that λ is dominant. Then we have for general $g \in G$:*

$$H^i(\mathcal{L}_{Y \cap gX^w}(-\lambda)(-Y \cap g\partial X^w)) = 0$$

for all $w \in W^P$ and $i < \dim(Y \cap gX^w) = \mathrm{codim}(X_w) - \mathrm{codim}(Y)$. Equivalently,

$$H^i(\mathcal{L}_{Y \cap gX^w}(\lambda) \otimes \omega_{Y \cap gX^w}(Y \cap g\partial X^w)) = 0$$

for all $w \in W^P$ and $i \geq 1$.

Proof. We first consider the case where X is the full flag variety ; furthermore, we replace X^w by X_w for simplicity.

Recall that each intersection $Y \cap gX_w$ is the fiber $\pi^{-1}(g)$, with notation displayed by the commutative diagram

$$
\begin{array}{ccccc}
G & \xleftarrow{\ \pi\ } & Z & \xrightarrow{\ \mu\ } & Y \\
{\scriptstyle id}\downarrow & & {\scriptstyle \iota}\downarrow & & {\scriptstyle i}\downarrow \\
G & \xleftarrow{\ p\ } & G \times X_w & \xrightarrow{\ m\ } & X
\end{array}
$$

where the square on the right is cartesian. Recall also that Z has rational singularities. Let

$$\partial Z = Y \times_X (G \times \partial X_w)$$

a subvariety of codimension 1 in $Z = Y \times_X (G \times X_w)$. For general $g \in G$, we have

$$(\mu^*\mathcal{L}_Y(\lambda) \otimes \omega_Z(\partial Z))|_{\pi^{-1}(g)} \cong \mathcal{L}_{Y \cap gX_w}(\lambda) \otimes \omega_{Y \cap gX_w}(Y \cap g\partial X_w).$$

Thus, our statement (ii) is a consequence of the following assertion:

$$R^i\pi_*(\mu^*\mathcal{L}_Y(\lambda) \otimes \omega_Z(\partial Z)) = 0 \text{ for } i \geq 1. \tag{1}$$

We shall deduce (1) from the Kawamata-Viehweg vanishing theorem, like in the proof of Lemma 4. Since that theorem concerns nonsingular varieties, we first construct a desingularization of Z.

Let $\varphi : \tilde{Y} \to Y$ be a desingularization. On the other hand, let $\psi : \tilde{X}_w \to X_w$ be a standard desingularization as in the proof of Lemma 4. Composing $id \times \psi : G \times \tilde{X}_w \to G \times X_w$ with the multiplication map $m : G \times X_w \to X$ defines $\tilde{m} : G \times \tilde{X}_w \to X$. Define likewise $\tilde{i} = i \circ \varphi$, $\tilde{p} = p \circ (id \times \psi)$ and consider the commutative diagram

$$
\begin{array}{ccccc}
G & \xleftarrow{\ \tilde{\pi}\ } & \tilde{Z} & \xrightarrow{\ \tilde{\mu}\ } & \tilde{Y} \\
{\scriptstyle id}\downarrow & & {\scriptstyle \tilde{\iota}}\downarrow & & {\scriptstyle \tilde{i}}\downarrow \\
G & \xleftarrow{\ \tilde{p}\ } & G \times \tilde{X}_w & \xrightarrow{\ \tilde{m}\ } & X
\end{array}
$$

where the square on the right is cartesian. Since $\tilde{m}$ is a G-equivariant morphism from the nonsingular variety $G \times \tilde{X}_w$ to $X = G/B$, it is a locally trivial fibration with nonsingular fiber. Thus, the same holds for $\tilde{\mu}$, so that $\tilde{Z}$ is nonsingular as well. The map $id \times \tilde{f} \times \varphi : G \times \tilde{X}_w \times \tilde{Y} \to G \times X_w \times Y$ is a desingularization ; it restricts to a proper morphism

$$f : \tilde{Z} \to Z$$

which is clearly birational. Thus, f is a desingularization of Z. The subset

$$\partial \tilde{Z} = \tilde{Y} \times_X (G \times \partial \tilde{X}_w)$$

is a union of nonsingular irreducible divisors intersecting transversally in $\tilde{Z}$; clearly, $f(\partial \tilde{Z}) = \partial Z$.

We claim that

$$\omega_{\tilde{Z}}(\partial \tilde{Z}) = \tilde{\mu}^*(\omega_{\tilde{Y}} \otimes \varphi^* \mathcal{L}_Y(\rho)).$$

To verify this, notice that

$$\omega_{\tilde{Z}} = \tilde{\mu}^* \omega_{\tilde{Y}} \otimes \omega_{\tilde{Z}/\tilde{Y}},$$

since μ is a locally trivial fibration. Furthermore,

$$\omega_{\tilde{Z}/\tilde{Y}} = \tilde{\iota}^* \omega_{(G \times \tilde{X}_w)/X} = \tilde{\iota}^*(\omega_{G \times \tilde{X}_w} \otimes \tilde{m}^* \omega_X^{-1}) = \tilde{\iota}^*(\mathcal{O}_{G \times \tilde{X}_w}(-G \times \partial \tilde{X}_w) \otimes \tilde{m}^* \mathcal{L}_X(\rho)),$$

since $\omega_G = \mathcal{O}_G$ and $\omega_{\tilde{X}_w} = \mathcal{O}_{\tilde{X}_w}(-\partial \tilde{X}_w) \otimes \tilde{m}^* \mathcal{L}_X(-\rho)$. Therefore,

$$\omega_{\tilde{Z}} = \tilde{\mu}^* \omega_{\tilde{Y}} \otimes \mathcal{O}_{\tilde{Z}}(-\partial \tilde{Z}) \otimes \tilde{\iota}^* \tilde{m}^* \mathcal{L}_X(\rho)$$

which implies the claim.

We next obtain the analogue of (1) for $\tilde{Z}$, that is,

$$R^i \tilde{\pi}_*(\tilde{\mu}^* \varphi^* \mathcal{L}_Y(\lambda) \otimes \omega_{\tilde{Z}}(\partial \tilde{Z})) = 0 \text{ for } i \geq 1. \tag{2}$$

This follows from a relative version of the Kawamata-Viehweg vanishing theorem. Specifically, recall that $\partial \tilde{X}_w$ is the support of a very ample divisor $b_1 \tilde{X}_1 + \cdots + b_\ell \tilde{X}_\ell$ of $\tilde{X}_w$, with the notation of the proof of Lemma 4. Let $\tilde{Z}_i = \tilde{Y} \times_X (G \times \tilde{X}_i)$ for $i = 1, \ldots, \ell$; then the nonempty $\tilde{Z}_i$ are the irreducible components of $\partial \tilde{Z}$. Define positive integers $N, a_1, \ldots, a_\ell$ as in the proof of Lemma 4 and let $\mathcal{M} = (\tilde{\mu}^* \varphi^* \mathcal{L}_Y(\lambda))(\partial \tilde{Z})$ and $D = a_1 \tilde{Z}_1 + \cdots + a_\ell \tilde{Z}_\ell$. Then the invertible sheaf

$$\mathcal{M}^N(-D) = (\tilde{\mu}^* \tilde{\iota}^* \mathcal{L}_X(N\lambda))(b_1 \tilde{Z}_1 + \cdots + b_\ell \tilde{Z}_\ell) = \tilde{\iota}^*(\tilde{m}^* \mathcal{L}_X(N\lambda)(G \times (b_1 \tilde{X}_1 + \cdots + b_\ell \tilde{X}_\ell)))$$

is the pullback under $\tilde{\iota}$ of a very ample invertible sheaf on $G \times \tilde{X}_w$. Since $\tilde{\iota}$ is generically injective, it follows that $\mathcal{M}^N(-D)$ is $\tilde{\pi}$-numerically effective and $\tilde{\pi}$-big. Therefore, $R^i \tilde{\pi}_*(\mathcal{M} \otimes \omega_{\tilde{Z}}) = 0$ for $i \geq 1$, by [3] Corollary 6.11. This proves (2).

Likewise, $\mathcal{M}^N(-D)$ is f-numerically effective, so that

$$R^i f_*(\tilde{\mu}^* \varphi^* \mathcal{L}_Y(\lambda) \otimes \omega_{\tilde{Z}}(\partial \tilde{Z})) = 0 \text{ for } i \geq 1. \tag{3}$$

Finally, we claim that

$$f_*(\tilde{\mu}^*\varphi^*\mathcal{L}_Y(\lambda) \otimes \omega_{\tilde{Z}}(\partial\tilde{Z})) = \mu^*\mathcal{L}_Y(\lambda) \otimes \omega_Z(\partial Z). \tag{4}$$

Together with (2), (3) and the Leray spectral sequence for $\tilde{\pi} = \pi \circ f$, this will imply assertion (1).

To check (4), we factor f into $\varphi' \circ f'$, with notation displayed in the commutative diagram

$$
\begin{array}{ccccc}
\tilde{Z} & \xrightarrow{f'} & Z' & \xrightarrow{\varphi'} & Z \\
{\scriptstyle\tilde{\mu}}\downarrow & & {\scriptstyle\mu'}\downarrow & & {\scriptstyle\mu}\downarrow \\
\tilde{Y} & \xrightarrow{id} & \tilde{Y} & \xrightarrow{\varphi} & Y
\end{array}
$$

where the square on the right is cartesian. Notice that Z' has rational singularities, and that f' is a desingularization ; furthermore, we obtain

$$\omega_{Z'}(\partial Z') = {\mu'}^*(\omega_{\tilde{Y}} \otimes \varphi^*\mathcal{L}_Y(\rho)),$$

by the preceding arguments for determining $\omega_{\tilde{Z}}(\partial\tilde{Z})$, applied to the regular locus of Z'. Thus,

$$f'_*(\tilde{\mu}^*\varphi^*\mathcal{L}_Y(\lambda) \otimes \omega_{\tilde{Z}}(\partial\tilde{Z})) = f'_*{f'}^*{\mu'}^*(\omega_{\tilde{Y}} \otimes \varphi^*\mathcal{L}_Y(\lambda+\rho)) = {\mu'}^*(\omega_{\tilde{Y}} \otimes \varphi^*\mathcal{L}_Y(\lambda+\rho)).$$

It follows that

$$f_*(\tilde{\mu}^*\varphi^*\mathcal{L}_Y(\lambda) \otimes \omega_{\tilde{Z}}(\partial\tilde{Z})) = \varphi'_*{\mu'}^*(\omega_{\tilde{Y}} \otimes \varphi^*\mathcal{L}_Y(\lambda+\rho)) = \mu^*\varphi_*(\omega_{\tilde{Y}} \otimes \varphi^*\mathcal{L}_Y(\lambda+\rho)),$$

where the latter equality holds since μ is flat. By the projection formula and rationality of singularities of Y, this yields

$$f_*(\tilde{\mu}^*\varphi^*\mathcal{L}_Y(\lambda) \otimes \omega_{\tilde{Z}}(\partial\tilde{Z})) = \mu^*(\omega_Y \otimes \mathcal{L}_Y(\lambda+\rho)).$$

And one may check as above that the latter equals $\mu^*\mathcal{L}_Y(\lambda)\otimes\omega_Z(\partial Z)$. This completes the proof of (4) and hence of Theorem 3, in the case where $X = G/B$.

Finally, in the case where $X = G/P$, one argues by reducing to G/B as in the proof of Lemma 4 ; we skip the details. $\square$

Remark. Consider an arbitrary closed subvariety Y of X and a desingularization $\varphi : \tilde{Y} \to Y$. Then the sheaves $R^i\varphi_*\mathcal{O}_{\tilde{Y}}$ are independent of the choice of φ, so that the same holds for the class

$$\varphi_*[\mathcal{O}_{\tilde{Y}}] = \sum_{i=0}^{\dim(Y)} (-1)^i [R^i\varphi_*\mathcal{O}_{\tilde{Y}}]$$

in $K(X)$. Thus, we may define integers b_Y^w by

$$\varphi_*[\mathcal{O}_{\tilde{Y}}] = \sum_{w \in W^P} b_Y^w [\mathcal{O}_{X_w}].$$

Then one may adapt the arguments of Sections 1 and 2 to obtain

$$b_Y^w = \chi(\mathcal{O}_{\tilde{Y} \times_X gX^w}(-\tilde{Y} \times_X g\partial X^w))$$

for general $g \in G$. And the proof of Theorem 3 actually shows that

$$H^i(\mathcal{O}_{\tilde{Y} \times_X gX^w}(-\tilde{Y} \times_X g\partial X^w)) = 0 \ \text{ for } \ i < \mathrm{codim}(X_w) - \mathrm{codim}(Y).$$

As a consequence,

$$(-1)^{\mathrm{codim}(X_w) - \mathrm{codim}(Y)} \, b_Y^w \geq 0$$

for all $w \in W^P$.

This admits a simpler formulation in terms of the sheaf $\varphi_* \omega_{\tilde{Y}}$. The latter is also independent of the choice of φ, and is called the sheaf of absolutely regular differential forms on Y ; we denote it by $\tilde{\omega}_Y$. Furthermore, $R^i \varphi_* \omega_{\tilde{Y}} = 0$ for $i \geq 1$ (see [7]). Using duality for the morphism $\tilde{Y} \to X$, it follows that

$$[\tilde{\omega}_Y] = \sum_{w \in W^P} (-1)^{\mathrm{codim}(X_w) - \mathrm{codim}(Y)} \, b_Y^w \, [\omega_{X_w}].$$

In other words, the class of $\tilde{\omega}_Y$ is a nonnegative combination of classes of dualizing sheaves of Schubert varieties. This generalizes Theorem 1 to all closed subvarieties of flag varieties.

4. Restricting homogeneous line bundles to Schubert subvarieties

In this section, X is the full flag variety. Then $K(X)$ is generated as an additive group by classes of invertible sheaves, see [14]. This raises the question of describing the product of such classes with classes of structure sheaves of Schubert varieties. For any weight λ and for any v in W, we have in $K(X)$:

$$[\mathcal{L}_X(\lambda)] \cdot [\mathcal{O}_{X_v}] = [\mathcal{L}_{X_v}(\lambda)] = \sum_{w \in W} c_v^w(\lambda) \, [\mathcal{O}_{X_w}]$$

for uniquely defined integer coefficients $c_w^v(\lambda)$. Our next result expresses these coefficients in geometric terms.

Theorem 4. *For any weight λ and for any $v \in W$, we have*

$$c_v^w(\lambda) = \chi(\mathcal{L}_{X_v \cap X^w}(\lambda)(-X_v \cap \partial X^w)).$$

As a consequence, we have the duality formula

$$c_v^w(-\lambda) = c_{w_o w}^{w_o v}(-w_o \lambda).$$

If in addition λ is dominant, then

$$c_v^w(\lambda) = h^0(\mathcal{L}_{X_v \cap X^w}(\lambda)(-X_v \cap \partial X^w)).$$

Proof. We apply Theorem 2 and Corollary 1 to the normal, Cohen-Macaulay variety X_v. This yields

$$[\mathcal{O}_{\mathrm{diag}(X_v)}] = \sum_{w \in W} [\mathcal{O}_{X_w}] \times [\mathcal{O}_{X_v \cap X^w}(-X_v \cap \partial X^w)]$$

in $K(X \times X)$. Multiplying both sides by $p_2^*[\mathcal{L}_X[\lambda)]$ (where $p_1, p_2 : X \times X \to X$ are the projections) and then applying p_{1*}, we obtain

$$p_{1*}([\mathcal{O}_{\mathrm{diag}(X_v)}] \cdot p_2^*[\mathcal{L}_X(\lambda)]) = \sum_{w \in W} \chi(\mathcal{L}_{X_v \cap X^w}(\lambda)(-X_v \cap \partial X^w)) [\mathcal{O}_{X_w}]$$

for any weight λ. But since $p_1 : \mathrm{diag}(X_v) \to X_v$ is an isomorphism, we have

$$p_{1*}([\mathcal{O}_{\mathrm{diag}(X_v)}] \cdot p_2^*[\mathcal{L}_X(\lambda)]) = p_{1*}[\mathrm{diag}(\mathcal{L}_{X_v}(\lambda))] = [\mathcal{L}_{X_v}(\lambda)].$$

This proves our first formula.

Recalling that

$$\omega_{X_v \cap X^w} = \mathcal{O}_{X_v \cap X^w}\big(-(X_v \cap \partial X^w) - (\partial X_v \cap X^w)\big)$$

as seen in Section 1, we obtain

$$\chi(\mathcal{L}_{X_v \cap X^w}(\lambda)(-X_v \cap \partial X^w)) = \chi(\mathcal{L}_{X_v \cap X^w}(\lambda) \otimes \omega_{X_v \cap X^w}(\partial X_v \cap X^w)).$$

By Lemma 4 and duality in the variety $X_v \cap X^w$ of dimension $\ell(v) - \ell(w)$, it follows that

$$c_v^w(\lambda) = (-1)^{\ell(v) - \ell(w)} \chi(\mathcal{L}_{X_v \cap X^w}(-\lambda)(-\partial X_v \cap X^w)).$$

Since $X_v = w_o X^{w_o v}$ and $X^w = w_o X_{w_o w}$, this implies our second formula.

If in addition λ is dominant, then $H^i(\mathcal{L}_{X_v \cap X^w}(\lambda) \otimes \omega_{X_v \cap X^w}(\partial X_v \cap X^w)) = 0$ for every $i \geq 1$, as follows from Theorem 3. This yields our third formula. □

As a consequence, $c_v^w(\lambda) = 0$ unless $w \leq v$, and $c_v^v(\lambda) = 1$; furthermore, $c_v^w(\lambda) \geq 0$ if λ is dominant.

The definition of the coefficients $c_v^w(\lambda)$ implies that

$$c_v^w(\lambda + \mu) = \sum_{x \in W, \, w \leq x \leq v} c_v^x(\lambda) \, c_x^w(\mu)$$

for all weights λ and μ. Together with the second formula in Theorem 4, this shows that the $c_v^w(\lambda)$ may be expressed in terms of the $c_v^w(\omega_i)$, where the ω_i are the fundamental weights. The latter are related to certain structure constants $c_{u,v}^w$ as follows.

Lemma 5. *With preceding notation, we have for $v \neq w$:*

$$c_v^w(-\omega_i) = -c_{w_o s_i, v}^w \;\; and \;\; c_v^w(\omega_i) = (-1)^{\ell(v) - \ell(w) - 1} \, c_{s_i w_o, w_o v}^{w_o w}.$$

Proof. The invertible sheaf $\mathcal{L}_X(\omega_i)$ has a section with zero subscheme the Schubert variety $X_{w_o s_i}$. This yields an exact sequence

$$0 \to \mathcal{L}_X(-\omega_i) \to \mathcal{O}_X \to \mathcal{O}_{X_{w_o s_i}} \to 0,$$

whence $[\mathcal{L}_X(-\omega_i)] = [\mathcal{O}_X] - [\mathcal{O}_{X_{w_o s_i}}]$ in $K(X)$. Multiplying this equality by $[\mathcal{O}_{X_v}]$ yields

$$[\mathcal{L}_{X_v}(-\omega_i)] = [\mathcal{O}_{X_v}] - \sum_{w \in W} c^w_{w_o s_i, v} [\mathcal{O}_{X_w}]$$

which implies our first formula. The second formula follows by duality.   □

Notice that all results of this section extend to the setting of T-equivariant K-theory (see [10]). For the intersections $X_v \cap X^w$ are invariant under the action of T on X, and the constructions of Section 2 are equivariant with respect to this action. This yields a geometric proof for the positivity of $[\mathcal{L}_{X_v}(\lambda)]$ in $K^T(X)$, due to Pittie and Ram (see [16] Corollary, p. 106) and independently to Mathieu [15]; see also [13]. And this raises the question of a positivity result for the structure constants of $K^T(X)$; see [6] for such a result in the setting of T-equivariant cohomology.

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