The additive group actions on $\mathbb{Q}$-homology planes

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Abstract

In this article, we consider the conjecture that a $\mathbb{Q}$-homology plane with constant Makar-Limanov invariants is isomorphic to either the affine plane $\mathbb{A}^2$ or the complement of a smooth conic on the projective plane $\mathbb{P}^2$. Though the conjecture is not fully solved yet, we can show strong evidences to support the conjecture. Furthermore, it is shown that such a $\mathbb{Q}$-homology plane is a quotient of a hypersurface $xy = p(z)$ by a cyclic group $\mathbb{Z}/m\mathbb{Z}$, where the hypersurface was investigated in [1] by Bandman and Makar-Limanov.

0 Introduction

A $\mathbb{Q}$-homology plane is, by definition, a smooth algebraic surface X defined over the complex field $\mathbb{C}$ such that $H_i(X; \mathbb{Q}) = (0)$ for every $i > 0$ [11]. It is known that X is affine and rational [5]. If there is a nontrivial action of the additive group scheme G_a on X, the orbits will form the fibers of an $\mathbb{A}^1$-fibration $\rho : X \to \mathbb{A}^1$. Hence X has log Kodaira dimension $\overline{\kappa}(X) = -\infty$. Write $A = \Gamma(X, \mathcal{O}_X)$. Then there is a well-known bijective correspondence

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between the set of G_a-actions on X and the set of locally nilpotent derivations on A. The correspondence is given by assigning to a locally nilpotent derivation δ on A an algebra homomorphism $\varphi : A \to A \otimes_{\mathbb{C}} \mathbb{C}[t]$ giving rise to the coaction :

$$\varphi(a) = \sum_{i=0}^{\infty} \frac{1}{n!} \delta^n(a) t^n.$$

The set of invariant elements of A under the given G_a-action is obtained as $\mathrm{Ker}\,\delta$ consisting of elements annihilated by δ. Then $\mathrm{Ker}\,\delta$ is isomorphic to a polynomial ring in one variable and the base curve of the $\mathbb{A}^1$-fibration which is isomorphic to $\mathbb{A}^1$ is obtained as the spectrum of $\mathrm{Ker}\,\delta$.

The Makar-Limanov invariant $\mathrm{ML}\,(X)$ for X is then introduced by Kaliman and Makar-Limanov [6] as the set $\bigcap \mathrm{Ker}\,\delta$, where δ ranges over all possible locally nilpotent derivations of A. Then it is shown that $\mathrm{ML}\,(X)$ for a $\mathbb{Q}$-homology plane X is the coordinate ring A, a polynomial ring in one variable $\mathbb{C}[x]$ or $\mathbb{C}$. We are particularly interested in such $\mathbb{Q}$-homology planes X that the Makar-Limanov invariant $\mathrm{ML}\,(X)$ is equal to $\mathbb{C}$. We shall consider two algebraically independent G_a-actions σ, σ' and define the intertwining number $\iota(\sigma, \sigma')$ associated with these G_a-actions. It is then shown that the intertwining number is actually a multiple of m^2, where $m = |H_1(X; \mathbb{Z})|$. We define a *minimal* pair $\{\sigma, \sigma'\}$ of algebraically independent G_a-actions as such with $\iota(\sigma, \sigma') = m^2$. We show that there are no minimal pairs of G_a-actions if $m \geq 3$.

Recently, Bandman and Makar-Limanov [1] considered a problem of characterizing in terms of the boundary divisors the smooth affine rational surfaces with trivial Makar-Limanov invarinats. They succeeded in obtaining a characterization in the case where the surfaces are embedded into $\mathbb{A}^3$ as hypersurfaces. Furthermore, the hypersurfaces are defined by the equations of the form $xy = p(z)$ with respect to a suitable system of coordinates $\{x, y, z\}$, where $p(z)$ is a polynomial in z such that $p(z) = 0$ has distinct roots.

1 Intertwining number

Let X be a smooth affine surface defined over the ground field $k = \mathbb{C}$. We assume always that $\Gamma(X, \mathcal{O}_X)^* = k^*$. The *Makar-Limanov invariant* $\mathrm{ML}\,(X)$

is defined as the intersection

$$\mathrm{ML}\,(X) = \bigcap_{\delta} \mathrm{Ker}\ \delta,$$

where δ runs over all locally nilpotent derivations δ on the coordinate ring $A = \Gamma(X, \mathcal{O}_X)$, where δ corresponds in a bijective way to an algebraic G_a-action σ on X. We *assume* that X is rational and $A^* = k^*$. Then it is known that Ker $\delta = k[t]$ a polynomial ring in one variable for any locally nilpotent derivation δ.

We begin with the following result.

Lemma 1.1 *We have one of the following three cases.*

(1) $\mathrm{ML}\,(X) = A$ *and there are no nontrivial* G_a*-actions on* X*. In particular,* $\overline{\kappa}(X) \geq 0$.

(2) $\mathrm{ML}\,(X) = k[t]$*, and any two locally nilpotent derivations* δ, δ' *on* A *are conjugate to each other in the sense that* $a\delta = a'\delta'$ *for nonzero elements* $a, a' \in \mathrm{ML}\,(X)$*. The surface* X *has a unique* $\mathbb{A}^1$*-fibration defined by the inclusion* $\mathrm{ML}\,(X) \hookrightarrow A$.

(3) $\mathrm{ML}\,(X) = k$*, and there are two non-conjugate locally nilpotent derivations on* A.

Proof. Our proof consists of several steps.

(I) Suppose that X has a nontrivial G_a-action σ. Let δ be the corresponding locally nilpotent derivation. Let $A_0 = \mathrm{Ker}\ \delta$. Then A_0 is a normal rational algebra of dimension one with $A_0^* = k^*$. Hence $A_0 = k[t]$. The G_a-action σ gives rise to an $\mathbb{A}^1$-fibration. Hence $\overline{\kappa}(X) = -\infty$. Conversely, if $\overline{\kappa}(X) = -\infty$, X has an $\mathbb{A}^1$-fibration $\rho : X \to \mathbb{A}^1 = \mathrm{Spec}\ A_0$.

(II) Suppose that δ and δ' are locally nilpotent derivations on A. Then Ker $\delta = k[t]$ and Ker $\delta' = k[u]$. If t and u are algebraically independent over k, we have $k[t] \cap k[u] = k$. In this case, we say that δ and δ' (or the corresponding G_a-actions σ and σ') are *algebraically independent* over k. Then $ML(X) = k$.

(III) Suppose that u is algebraic over $k(t)$. Then there exists an algebraic equation

$$a_0(t)u^n + a_1(t)u^{n-1} + \cdots + a_{n-1}(t)u + a_n(t) = 0, \qquad (1)$$

where $a_i(t) \in k[t]$, and we may assume that (1) is minimal. Since $\mathrm{Ker}\delta = k[t]$, we have

$$\left\{na_0(t)u^{n-1} + (n-1)a_1(t)u^{n-2} + \cdots + a_{n-1}(t)\right\}\delta(u) = 0. \qquad (2)$$

Since (1) is minimal, $na_(t)u^{n-1} + \cdots + a_{n-1}(t) \neq 0$. This implies that $\delta(u) = 0$. Hence $k[u] \subseteq k[t]$, and t is then algebraic over $k(u)$. By the same reasoning as above, we infer that $k[t] \subseteq k[u]$. So, $k[t] = k[u]$. The $\mathbb{A}^1$-fibrations associated with σ and σ' coincide with the morphism $X \to \mathbb{A}^1$ defined by the inclusion $k[t] = k[u] \hookrightarrow A$. By (1) above, $A[a^{-1}] = k[t, a^{-1}][\xi] = k[u, a^{-1}][\xi]$ for $a \in k[t]$ and an element $\xi \in A$ which is algebraically independent over $k(t)$. Then $a_1\delta = b_1\frac{\partial}{\partial\xi}$ and $a_2\delta' = b_2\frac{\partial}{\partial\xi}$ for $a_1, a_2, b_1, b_2 \in k[t]$. By adjusting the coefficients, we have $a\delta = a'\delta'$ for some nonzero elements $a, a' \in k[t]$. Namely, δ and δ' are conjugate to each other. These observations yields the assertions (2) and (3). Q.E.D.

Remark. Note that there exists an $\mathbb{A}^1$-fibration $\rho : X \to B$ if and only if there exists an algebraic G_a-action on X. In fact, if there exists a nontrivial G_a-action σ, then there is an $\mathbb{A}^1$-fibration as in the above proof of the assertion (1). Suppose that there is an $\mathbb{A}^1$-fibration $\rho : X \to B \cong \mathbb{A}^1$. Write $B = \mathrm{Spec}\, k[t]$ and $X = \mathrm{Spec}\, A$. Then there exists an element $a \in k[t]$ such that $\rho^{-1}(U) \cong U \times \mathbb{A}^1$, where $U = \mathrm{Spec}\, k[t, a^{-1}]$. Hence $A[a^{-1}] = k[t, a^{-1}][\xi]$, where we can take ξ to be an element of A. Consider a derivation $\delta = a^N\frac{\partial}{\partial\xi}$ with $N > 0$. This is a locally nilpotent derivation on $k[t, a^{-1}][\xi]$. Since A is finitely generated over k, it follows that $\delta(A) \subseteq A$ if $N \gg 0$. Then δ defines a G_a-action σ and the associated $\mathbb{A}^1$-fibration consisting of σ-orbits is the given $\mathbb{A}^1$-fibration ρ.

We consider first the case where $ML(X) = k$. In this case, there are two G_a-actions σ, σ' which are algebraically independent over k. Let T, T' be general G_a-orbits with respect to the actions σ, σ', respectively. We have the following result.

Lemma 1.2 *There exists a non-empty open set U of X such that, for $P \in U$ and the σ-orbit T and σ'-orbit T' passing through P, the number*

$$\iota(\sigma, \sigma'; P) = \sum_{Q \in T \cap T'} i(T, T'; Q)$$

is independent of the choice of P. Furthermore, T and T' meet transversally in each point $Q \in T \cap T'$.

Proof. Let $\rho : X \to B \cong \mathbb{A}^1$ and $\rho' : X \to B' \cong \mathbb{A}^1$ be the $\mathbb{A}^1$-fibrations defined by σ and σ', respectively. Then there exists a smooth compactification V of X such that the $\mathbb{A}^1$-fibrations ρ, ρ' are extended to the $\mathbb{P}^1$-fibrations $p : V \to \overline{B}$ and $p' : V \to \overline{B}'$, respectively, where $\overline{B}$ and $\overline{B}'$ are isomorphic to $\mathbb{P}^1$. Let $\overline{T}$ and $\overline{T}'$ be respectively the closures of T and T'. Consider the restriction $p_{\overline{T}'} : \overline{T}' \to \overline{B}$ of p. Since $\overline{T}'$ has only one place outside of X, which must dominate the point of $\overline{T}'$ where $\overline{T}'$ intersects the fiber of p lying over the point at infinity of $\overline{B}$, the restriction $\rho_{T'} : T' \to B$ is a finite morphism. Then $\rho_{T'}$ is unramified over an open set W of B. This means that the intersection of T' and a fiber $\rho^{-1}(Q)$ with $Q \in W$ is transversal and consists of the same number of points.

Similarly, there exists an open set W' of B' such that the intersection of T and a fiber ${\rho'}^{-1}(Q')$ with $Q' \in W'$ is transversal and consists of the same number of points. Now choose an open set U so that $U \subseteq \rho^{-1}(W) \cap {\rho'}^{-1}(W')$. Then, for $P \in U$, the fibers $T := \rho^{-1}(\rho(P))$ and $T' := {\rho'}^{-1}(\rho'(P))$ are respectively the σ-orbit and σ'-orbit passing through P. Hence we have the property for T, T' as required in the statement. Q.E.D.

We call $\iota(\sigma, \sigma'; P)$ the *intertwinig number* of σ and σ', and denote it by $\iota(\sigma, \sigma')$. By the abuse of the notations, we denote it by $(T \cdot T')$ if we choose T, T' as in the above proof and treat it as the intersection number of divisors on a smooth projective surface.

Choose a point $P \in U$ as above and defines a morphism $\Phi_P : \mathbb{A}^2 \to X$ by $\Phi_P(g, g') = \sigma(g)\sigma'(g')P$, where $(g, g') \in \mathbb{A}^2 \cong G_a \times G_a$. Then we have the following result.

Lemma 1.3 *The morphism Φ_P has degree $\iota(\sigma, \sigma')$.*

Proof. For $(g, g') = (0, 0)$, we have $\Phi_P(0, 0) = P$. With the above notations, any point of $T \cap T'$ is written as $\sigma(g_i)(P) = \sigma'(g'_i)(P), 1 \le i \le n$, where $n = |T \cap T'| = \iota(\sigma, \sigma')$. Conversely, $\Phi_P^{-1}(P)$ consists of the (g, g') such that $\sigma(g)\sigma'(g')P = P$, i.e., $\sigma(g^{-1})P = \sigma'(g')P$.

Let Q be a general point of X, say $Q \in U$. Then $\Phi^{-1}(Q)$ consists of the $(g, g') \in \mathbb{A}^2$ such that $\sigma(g)\sigma'(g')P = Q$, i.e., $\sigma(g^{-1})Q = \sigma'(g')P$. Suppose $\sigma(g_1)\sigma'(g'_1)P = \sigma(g)\sigma'(g')P$. Then we have

$$\sigma'(g'_1)P = \sigma(g_1^{-1}g)\sigma'(g')P \in \sigma(G_a)(\sigma'(g')P) \cap \sigma'(G_a)P.$$

This implies that $\Phi_P^{-1}(Q)$ corresponds bijectively to the set of intersection points of the σ-orbit $\sigma(G_a)(\sigma'(g')P)$ and the σ'-orbit $\sigma'(G_a)P$. So, $\Phi_P^{-1}(Q)$ consists of $\iota(\sigma, \sigma')$ points. Q.E.D.

As an immediate consequence of Lemma 1.3, we have:

Corollary 1.4 *With the notations and assumptions, $\pi_1(X)$ is a finite group of order less than or equal to $\iota(\sigma, \sigma')$.*

Let σ, σ' be algebraically independent G_a-actions on X and let δ, δ' be the corresponding locally nilpotent derivations on A. We can interpret the intertwining number $\iota(\sigma, \sigma')$ in terms of δ, δ'. Write $\operatorname{Ker}\delta = k[t]$ and $\operatorname{Ker}\delta' = k[t']$ for two elements t, t' of A which are algebraically independent over k. Then we have:

Lemma 1.5 *With the notations as above, the following equalities hold:*

$$\begin{aligned} \iota(\sigma, \sigma') &= \min\{n \mid \delta^n(t') = 0\} - 1 \\ &= \min\left\{n \mid \delta'^n(t) = 0\right\} - 1. \end{aligned}$$

Proof. By [8], there exist $a \in \operatorname{Ker}\delta$ and $\xi \in A$ such that $A[a^{-1}] = k[t, a^{-1}][\xi]$. Then t' is written as

$$t' = c_0\xi^N + c_1\xi^{N-1} + \cdots + c_N,$$

where $c_i \in k[t, a^{-1}]$ and $c_0 \neq 0$. We may assume, after replacing t' by $t' + \lambda$ with $\lambda \in k$, that $t' = 0$ defines a general σ'-orbit T'. Similarly, we can take $\mu \in k$ so that $c_i(\mu)$ is defined for $0 \leq i \leq N$, $c_0(\mu) \neq 0$ and the curve $t = \mu$ is a general σ-orbit T. Then the intersection number $(T \cdot T')$ is equal to the number of roots of the equation

$$c_0(\mu)\xi^N + c_1(\mu)\xi^{N-1} + \cdots + c_N(\mu) = 0,$$

where each root is counted with multiplicity. Namely $(T \cdot T') = N$. On the other hand, since δ is equivalent to the derivation $\partial/\partial\xi$, it follows that $N = \min\{n \mid \delta^n(t') = 0\} - 1$. So, we have the assertion. Q.E.D.

2 $\mathbb{Q}$-homology planes and the Makar-Limanov invariants

In this section, X denotes a $\mathbb{Q}$-homology plane, that is, a smooth algebraic surface defined over the complex field such that $H_i(X; \mathbb{Q}) = (0)$ for every

$i > 0$. In particular, X is affine and rational [5]. Furthermore, $\pi_1(X) \cong H_1(X : \mathbb{Z}) \cong \mathrm{Pic}\,(X)$. We consider the existence of G_a-actions on X and the structure of X when X has enough G_a-actions.

We recall the following result [11, Th.1.2].

Lemma 2.1 *Let X be a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration $\rho : X \to B$. Then every fiber $\rho^{-1}(P)$ is irreducible and $\rho^{-1}(P)_{\mathrm{red}}$ is isomorphic to $\mathbb{A}^1$. Let $m_1A_1, \ldots, m_nA_n$ exhaust all multiple fibers with $A_i \cong \mathbb{A}^1$. Then $H_1(X; \mathbb{Z}) \cong \prod_{i=1}^n \mathbb{Z}/m_i\mathbb{Z}$.*

We need the following result.

Lemma 2.2 *Let $X = \mathrm{Spec}\ A$ be an affine variety defined over k and let $f : Y \to X$ be an étale finite morphism. Suppose that there exists a G_a-action σ on X. Then σ lifts up uniquely to a G_a-action $\widetilde{\sigma}$ on the variety Y.*

Proof. Let δ be the locally nilpotent derivation associated with σ. Let $A_0 = \mathrm{Ker}\ \delta$. Then $A[a^{-1}] = A_0[a^{-1}][\xi]$ for some element $a \in A_0$, and δ is conjugate to $\partial/\partial\xi$, i.e., $a_0\delta = a_1\frac{\partial}{\partial\xi}$ for nonzero elements $a_0, a_1 \in A_0$. Let $B = \Gamma(Y, \mathcal{O}_Y)$. Then the derivation δ extends *uniquely* to a derivation $\widetilde{\delta}$ on B because $\mathrm{Der}_k(B, B) \cong \mathrm{Der}_k(A, A) \otimes_A B$, which follows from the hypothesis that B is étale over A. On the other hand, δ extends uniquely to a derivation δ on the function field $Q(A)$ and to a derivation on $Q(B)$ which must coincide with the extension of $\widetilde{\delta}$ on $Q(B)$. Since $f : Y \to X$ is étale and finite and since $D(a) \cong \mathrm{Spec}\ A_0[a^{-1}] \times \mathbb{A}^1$, it follows that $f^{-1}(D(a)) \cong \mathrm{Spec}\ B_0 \times \mathbb{A}^1$, where $f\mid_{f^{-1}(D(a))}$ is induced by an étale finite morphism $f_0 : \mathrm{Spec} B_0 \to \mathrm{Spec} A_0[a^{-1}]$ via the fiber product $f = f_0 \times \mathbb{A}^1$. Hence $B[a^{-1}] = B_0[\xi]$. Then the derivation $\widehat{\delta} = \frac{a_1}{a_0}\frac{\partial}{\partial\xi}$ is a derivation on $Q(B)$ which is zero on $Q(B_0)$. Since $\widehat{\delta}$ is clearly an extension of δ on $Q(B)$, the uniqueness of the extension implies that $\widehat{\delta} = \widetilde{\delta}$. In particular, $\widehat{\delta}$ is zero on B_0. This implies that $\widehat{\delta}$ is a locally nilpotent derivation on B, and $\widetilde{\delta}$ defines a G_a-action $\widetilde{\sigma}$ on Y which extends σ on X.

Q.E.D.

The existence of two algebraically independent G_a-actions on a $\mathbb{Q}$-homology plane gives a strong restriction on the structure of X. Namely we have:

Lemma 2.3 *Let X be a $\mathbb{Q}$-homology plane with algebraically independent G_a-actions σ, σ'. Then each of the $\mathbb{A}^1$-fibrations $\rho : X \to B$ and $\rho' : X \to B'$*

associated respectively with σ and σ' has a unique multiple fiber of multiplicity m, where $m = |H_1(X;\mathbb{Z})|$. Furthermore, $\iota(\sigma, \sigma')$ is a multiple of m^2.

Proof. Consider the $\mathbb{A}^1$-fibration $\rho : X \to B$. Let $m_1A_1, \ldots, m_nA_n$ exhaust all multiple fibers of ρ. Then there is a Galois covering $\pi : C \to \overline{B}$ which ramifies over the points $P_1 = \rho(A_1), \ldots, P_n = \rho(A_n)$ and P_∞ with respective multiplicities $m_1, \ldots, m_n$ and m_∞, where $\overline{B}$ is the smooth compactification of B and $\{P_\infty\} = \overline{B} - B$. By [2]and [3], such a covering exists for a suitable choice of $m_\infty > 1$ provided $n \geq 1$. The genus g of C is computed by the Riemann-Hurwitz formula

$$\begin{aligned} 2g - 2 &= -2d + \sum_{i=1}^{n} \frac{d}{m_i}(m_i - 1) + \frac{d}{m_\infty}(m_\infty - 1) \\ &= d\left\{(n-1) - \left(\frac{1}{m_1} + \cdots + \frac{1}{m_n} + \frac{1}{m_\infty}\right)\right\}, \end{aligned}$$

where d is the degree of the morphism π. Hence $g \geq 1$ if and only if

$$n - 1 \geq \frac{1}{m_1} + \cdots + \frac{1}{m_n} + \frac{1}{m_\infty}.$$

Since $m_i \geq 2$ $(1 \leq i \leq n)$ and $m_\infty \geq 2$, it follows that $g = 0$ only if $n - 1 < (n+1)/2$, i.e., $n \leq 2$. If $n = 2$, then $g = 0$ only if

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_\infty} > 1.$$

If $n = 1$, then $g = 0$ always. The above observation implies that we can choose $\{m_1, \ldots, m_n, m_\infty\}$ to make the genus $g > 0$ unless one of the following cases takes place:

(1) $n = 1$

(2) $\{m_1, m_2\} = \{2, 2\}$.

Suppose we can take C to have genus $g \geq 1$. Let $C_0 = C - \pi^{-1}(P_\infty)$. Let Y be the normalization of the fiber product $X \times_B C_0$ and let $f : Y \to X$ be the composite of the normalization morphism and the projection $X \times_B C_0 \to C_0$. Then f is a finite etale morphism. Hence the $\mathbb{A}^1$-fibration ρ lifts up to the $\mathbb{A}^1$-fibration $\widetilde{\rho} : Y \to C_0$. Let T' be a general orbit of the G_a-action σ'. Then $\pi^{-1}(T')$ splits into a disjoint union of the affine lines $\widetilde{T}'_1, \ldots, \widetilde{T}'_d$, where

$d = \deg \pi$. Since T' is transversal to ρ, each of $\widetilde{T}'_1, \ldots, \widetilde{T}'_n$ is transversal to the $\mathbb{A}^1$-fibration $\widetilde{\rho}$. Then $\widetilde{\rho} : \widetilde{T}'_j \to C_0$ is dominant. Since the genus of C is positive by the assumption, this is a contradiction.

In the case (2) above, we have $H_1(X; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. By Lemma 2.1, the $\mathbb{A}^1$-fibration ρ' then has also two multiple fibers of multiplicity two. Let $2A_1, 2A_2$ be the multiple fibers of ρ and let $2A'_1, 2A'_2$ be the multiple fibers of ρ'. Since $\iota(\sigma, \sigma') = (2A_1, 2A'_1) = 4(A_1, A'_1)$, write $\iota(\sigma, \sigma') = 4d$. Consider the restriction $\rho'_1 : A'_1 \to B$ of ρ' onto A'_1. Since A'_1 has only one place point lying over the point $P_\infty := \overline{B} - B$, the Riemann-Hurwitz formula applied to ρ'_1, which has degree $2d$, yields

$$\begin{aligned} -2 &= -4d + (2d-1) + \{\text{contributions from ramifying points over } B\} \\ &\geq -4d + (2d-1) + d + d, \end{aligned}$$

which is a contradiction, where we obtain the above inequality by counting the ramifications at the intersection points of A'_1 with A_1 and A_2. This implies that the case (2) does not occur.

In the case (1), let mA_1 (resp. mA'_1) be a unique multiple fiber of ρ (resp. ρ'), where $m = m_1$. Then $\iota(\sigma, \sigma') = (mA_1, mA'_1) = m^2(A_1, A'_1)$. Hence $\iota(\sigma, \sigma')$ is a multiple of m^2. Q.E.D.

A pair (σ, σ') of two algebraically independent G_a-actions on a $\mathbb{Q}$-homology plane X is *minimal* if $\iota(\sigma, \sigma') = m^2$, where $m = |H_1(X; \mathbb{Z})|$. The following result guarantees the existence of a minimal pair of G_a-actions in the case $m = 2$.

Lemma 2.4 *Let C be a smooth conic on $\mathbb{P}^2$ and let $X = \mathbb{P}^2 - C$. Then the following assertions hold:*

(1) *X is a $\mathbb{Q}$-homology plane with $m = 2$.*

(2) *Let Q be a point on C and let ℓ_Q be the tangent line of C at Q. Let Λ_Q be the linear pencil spanned by C and $2\ell_Q$. Then the pencil Λ_Q defines an $\mathbb{A}^1$-fibration $\rho_Q : X \to \mathbb{A}^1$, and hence the conjugate class of G_a-actions σ_Q on X.*

(3) *If Q, Q' are distinct points on C, then $\sigma_Q, \sigma_{Q'}$ are algebraically independent. Furthermore, $\iota(\sigma_Q, \sigma_{Q'}) = 4$. Hence $(\sigma_Q, \sigma_{Q'})$ is a minimal pair.*

Proof. All the assertions are verified by a straightforward argument. Note that there is an infinite family of mutually algebraically independent G_a-actions on X. Q.E.D.

On the contrary, the following result denies the existence of minimal pairs of G_a-actions in the case $m \geq 3$.

Theorem 2.5 *There are no minimal pairs of two algebraically independent G_a-actions on a $\mathbb{Q}$-homology plane with $m = |H_1(X;\mathbb{Z})| \geq 3$.*

Proof. Suppose that (σ, σ') is a minimal pair of two algebraically independent G_a-actions on a $\mathbb{Q}$-homology plane X with $m \geq 3$. We consider the associated $\mathbb{A}^1$-fibrations $\rho : X \to B$ and $\rho' : X \to B'$. With the previous notations, let mA_1 and mA'_1 be the unique multiple fibers of ρ and ρ', respectively. Since $\iota(\sigma, \sigma') = m^2$ by the hypothesis, we have $(A_1 \cdot A'_1) = 1$. We consider the normalization Y of $X \times_B C_0$, where $C_0 \to B \cong \mathbb{A}^1$ is a finite covering of degree m totally ramifying over the point $P_1 = \rho(A_1)$ and the point at infinity P_∞. Let $\pi : Y \to X$ be a Galois covering with Galois group $G \cong \mathbb{Z}/m\mathbb{Z}$, which is a composite of the normalization morphism $Y \to X \times_B C_0$ and the second projection $X \times_B C_0 \to C_0$. Then $\pi^*(A_1) = E_1 + \cdots + E_m$ and $\pi^*(A'_1) = B_1 + \cdots + B_m$, where the E_i and the B_j are mutually disjoint and isomorphic to $\mathbb{A}^1$. Furthermore, we may assume that $(E_i \cdot B_j) = 1$ if $i = j$ and 0 otherwise. In fact, $Y_i := Y - \bigcup_{j \neq i} E_j$ is isomorphic to the affine plane and Y is obtained by glueing the Y_i $(1 \leq i \leq m)$ along the open set $Y - \bigcup_i E_i$.

Let T' be a general fiber of the $\mathbb{A}^1$-fibration ρ'. Then $\pi^*(T')$ splits into a disjoint sum of $\widetilde{T}'_1, \ldots, \widetilde{T}'_m$ which are isomorphic to $\mathbb{A}^1$. In fact, when T' ranges over the fibers of ρ', the family of curves consisting of the connected components of the $\pi^*(T')$ defines an $\mathbb{A}^1$-fibration $\widetilde{\rho}' : Y \to \widetilde{B}' \cong \mathbb{A}^1$, which is different from the $\mathbb{A}^1$-fibration $\widetilde{\rho} : Y \to C_0$ induced by the lifting of ρ.

Claim. *For $1 \leq i \leq m$, $\widetilde{T}'_i$ meets each of the E_j in one point transversally.*

Indeed, we may consider $\widetilde{T}'_i$ as a general fiber of $\widetilde{\rho}'$. Then it follows that the intersection number $(\widetilde{T}'_i \cdot E_j)$ is independent of the choice of $\widetilde{T}'_i$ and E_j. Since $(\pi^*(T') \cdot \pi^*(A_1)) = m^2$, we obtain $(\widetilde{T}'_i \cdot E_j) = 1$.

Hence each of the curves E_j is considered as a cross-section of the $\mathbb{A}^1$-fibration $\widetilde{\rho}'$. Similarly, each of the B_j is a cross-section of the $\mathbb{A}^1$-fibration $\widetilde{\rho}$. Consider $Y_1 := Y - \bigcup_{i \neq 1} E_i$ which is isomorphic to the affine plane as remarked above. Let $\widetilde{\rho}'_1 : Y_1 \to \widetilde{B}'$ be the fibration induced by the restriction of $\widetilde{\rho}'$ onto Y_1. Then E_1 and B_1 are two affine lines meeting in one point transversally, and we can choose E_1 and B_1 as the coordinate axes. Furthermore, the general fibers of $\widetilde{\rho}'_1$ are generically rational polynomial curves of simple type with m places at infinity (see [10] for the definition and the relevant results). The fibration $\widetilde{\rho}'_1$ has a unique reducible fiber consisting of $B_1 \cong \mathbb{A}^1$ and $B_j^0 \cong \mathbb{A}^1_*$ $(2 \le j \le m)$.

We may choose a system of coordinates $\{x, y\}$ on Y_1 in such a way that B_1 (resp. E_1) is defined by $x = 0$ (resp. $y = 0$). Suppose $m \ge 3$. Set $n = m - 1$. By [10, Th.3.3], the fibration $\widetilde{\rho}'_1$ is defined by a polynomial f which has one of the following forms, where the symbol $\sim$ means that f is given upto constant multiples by the polynomial on the right hand side:

(1)

$$f \sim \left(\prod_{j=1}^{n} (x - d_j)^{\alpha_j} \right) \cdot \left(y \cdot \prod_{j=1}^{n} (x - d_j)^{\varepsilon_j} + P(x) \right) + 1,$$

where $d_1, \ldots, d_n$ are mutually distinct elements in k and $P(x) \in k[x]$, $\alpha_j > 0$ and $\varepsilon_j \ge 0$ for $1 \le j \le n$; $P(d_j) \ne 0$ if $\varepsilon_j > 0$.

(2)

$$f \sim x \cdot \prod_{j=1}^{n} \left(x^{\ell} \left(x^t y + P(x) \right) - d_j \right)^{\alpha_j} + 1,$$

where $\ell > 0, t \ge 0$ and $P(x) \in k[x]$; $\deg P(x) < t$ and $P(0) \ne 0$ if $t > 0$ and $P(x) = 0$ if $t = 0$; the α_j and the d_j are as in the case (1).

(3)

$$f \sim x^{\beta} y^{\alpha_1} \cdot \prod_{j=2}^{n} \left(x^{\ell} y - d_j \right)^{\alpha_j} + 1,$$

where $d_2, \ldots, d_n$ are mutually distinct elements in k^*; $\beta > 0, \ell > 0$ and $\alpha_j > 0$ for $1 \le j \le n$; $\beta - \alpha_1 \ell = \pm 1$.

(4)

$$f \sim x^{\beta} \cdot \left(x^t y + P(x) \right)^{\alpha_1} \cdot \prod_{j=2}^{n} \left(x^{\ell} \left(x^t y + P(x) \right) - d_j \right)^{\alpha_j} + 1,$$

where $t > 0$ and $P(x) \in k[x]$ with $\deg P(x) < t$ and $P(0) \neq 0$; β, ℓ, the α_j and the d_j are as in the case (3).

Note that the unique reducible fiber of $\widetilde{\rho'_1}$ is defined by $f = 1$. Then, in the case (1) for example, $\widetilde{\rho'_1}^{-1}(1)$ consists of $n \geq 2$ components isomorphic to $\mathbb{A}^1$ and one component isomorphic to $\mathbb{A}^1_*$. Hence the case (1) is ruled out. Similarly, the case (3) is ruled out. In the case (2), $\widetilde{\rho'_1}^{-1}(1)$ consists of one component isomorphic to $\mathbb{A}^1$ and n components isomorphic to $\mathbb{A}^1_*$. Meanwhile, a general fiber, say H, of $\widetilde{\rho'_1}$ meets the curve $E_1 = \{y = 0\}$ in the points given by the equation

$$x \cdot \prod_{j=1}^{n} \left(x^{\ell} P(x) - d_j\right)^{\alpha_j} + 1 = 0.$$

Namely, H meets E_1 in not less than two points. So, the case (2) is ruled out. Consider the case (4). The singular fiber $\widetilde{\rho'_1}^{-1}(1)$ consists of one component isomorphic to $\mathbb{A}^1$ and n components isomorphic to $\mathbb{A}^1_*$. The points of intersection where H meets E_1 are given by

$$x^{\beta} \cdot P(x)^{\alpha_1} \prod_{j=2}^{n} \left(x^{\ell} P(x) - d_j\right)^{\alpha_j} + 1 = 0.$$

So, H meets E_1 in not not less than two points. This case is thus ruled out. Hence the case $m \geq 3$ does not occur in our settings. Q.E.D.

Remark. On the affine plane $\mathbb{A}^2$, a minimal pair of the G_a-actions (σ, σ') has $\iota(\sigma, \sigma') = 1$. Hence the general orbits T, T' of σ, σ' respectively meets in one point transversally. Consider, for example, the G_a actions σ, σ' such that the associated $\mathbb{A}^1$-fibrations ρ, ρ' are given respectively by the inclusions $k[y] \hookrightarrow k[x, y]$ and $k[y + P(x)] \hookrightarrow k[x, y]$, where $P(x) \in k[x]$. Then σ corresponds to a locally nilpotent derivation $\partial/\partial x$. Hence the intertwining number $\iota(\sigma, \sigma')$ is equal to $\deg P(x)$. Hence there exist non-minimal pairs of G_a-actions on $\mathbb{A}^2$.

Let X be a $\mathbb{Q}$-homology plane with two algebraically independent G_a-actions σ, σ'. *Suppose that* $|H_1(X; \mathbb{Z})| = m > 1$. Embed X into a smooth projective surface V in such a way that the following conditions are satisfied:

(1) There exists a $\mathbb{P}^1$-fibration $p : V \to \overline{B}$ which restricts to the $\mathbb{A}^1$-fibration $\rho : X \to B$ associated with σ, where $\overline{B}$ is isomorphic to $\mathbb{P}^1$.

(2) The boundary divisor $D := V - X$ is a divisor with simple normal crossings.

(3) The divisor D is written as $D = F_\infty + S + G$, where F_∞ is a smooth fiber of p lying over the point $P_\infty = \overline{B} - B$, S is a cross-section of p and G together with the closure $\overline{A}_0$ of a unique multiple fiber mA_0 of ρ supports a fiber of p lying over the point $P_0 := \rho(A_0)$.

(4) The connected component G contains no (-1) components.

We consider the linear pencil Λ' on V generated by the closures of σ'-orbits. Then we have the following result.

Lemma 2.6 *We may furthermore assume that the following conditions are satisfied:*

(5) *Λ' has a unique base point Q on F_∞, which is different from the point $Q_0 = S \cap F_\infty$.*

(6) *$(S^2) = -1$.*

Proof. Let $\overline{T}'$ be the closure of a general σ'-orbit T'. If $\overline{T}' \cap F_\infty = \emptyset$, then the $\mathbb{A}^1$-fibrations ρ, ρ' associated respectively with σ, σ' coincide with each other, which is impossible. Thence it follows that $\overline{T}' \cap F_\infty \neq \emptyset$. Suppose that Λ' has no base points. Since $\overline{T}'$ has a single one-place point on F_∞, this implies that F_∞ is a cross-section of Λ'. This implies that $\iota(\sigma, \sigma') = 1$, which is impossible because $\iota(\sigma, \sigma')$ is a multiple of m^2 by Lemma 2.3 and $m > 1$ by the hypothesis. So, Λ' has a unique one-place base point Q on F_∞. Suppose that $Q = Q_0$. Then blow up the point Q_0 to obtain an exceptional (-1) curve E and the proper transform E' of F_∞ with $(E'^2) = -1$. Then contract E' to obtain a smooth projective surface V'. We call this process of obtaining V' from V the *elementary transformation* with center Q_0. By this process we have a new compactification $X \hookrightarrow V'$ which satisfies the same conditions (1) $\sim$ (4) as above. By applying the elementary transformations with center Q_0 several times, the proper transform of Λ' will have no base points on the proper transform of S. We may assume that this situation is already realized on the surface V at the beginning.

Then the components of $S+G$ are contained in one and the same member M_0 of Λ'. Since these components are untouched until the base points of Λ' are eliminated, it follows that $(S^2) \leq -1$. Suppose that $(S^2) \leq -2$. Let μ be the multiplicity of $\overline{T}'$ at the point Q. Let $\iota(\sigma, \sigma') = m^2 d$. Suppose $\mu = m^2 d$. Blow up the point Q. Let E be the exceptional curve and let F'_∞ be the proper transform of F_∞. Then E is a component of the member M'_0 of the proper transform of Λ' corresponding to M_0. Otherwise, E is a cross-section and $m^2 d = \mu = 1$, which is impossible. By contracting F'_∞, we obtain a new compactification of X with the same property but with (S^2) increased by 1. Hence we may assume that $m^2 d > \mu$. Then $(S^2) = -1$. For otherwise, the member M_0 of Λ' containing $S + G$ will have no (-1) components when the base points of Λ' are eliminated and the last (-1) curve arising from the elimination process gives rise to a cross-section. This is impossible. Q.E.D.

Lemma 2.6 has the following consequence (cf. [9]).

Theorem 2.7 *With the notations as in Lemma 2.6, the dual graph of G is a linear chain. In particular, if C is a projective plane curve defined by an equation $X_0 X_1^{m-1} = X_2^m$ with $m > 2$, then the surface $X := \mathbb{P}^2 - C$ has a unique G_a-action up to equivalence which is associated with the pencil generated by C and $m\ell_0$, where ℓ_0 is the line $X_1 = 0$.*

Proof. Let $\varphi : \widetilde{V} \to V$ be the shortest sequence of blowing-ups to eliminate the base points of the pencil Λ' and let $\widetilde{\Lambda}'$ be the proper transform of Λ' by φ. Let $\widetilde{M}_0$ be the member of $\widetilde{\Lambda}'$ containing $S + G$, where we denote the proper transforms of S, G by the same symbols. Then S is a unique (-1) curve in $\widetilde{M}_0$ because $m^2 d > \mu$ with the notations in the proof of Lemma 2.6. One can obtain a smooth member by a sequence of blowing-downs which starts with the contraction of S. If the dual graph of G contains a branch point, then there appears in the course of the above sequence of blowing-downs a (-1) component meeting three or more components, one of which might be replaced by the cross-section. Hence the dual graph of G must be a linear chain. The second assertion is a straightforward consequence if one notices that a smooth compactification V of X satisfying the conditions (1) $\sim$ (6) as listed above is obtained by blowing up the point $(1, 0, 0)$ and its infinitely near points and that the dual graph of D is then as given in [9, Figure 1, p.23], where $r = m > 2$ and $n = 1$. Hence the dual graph of the component G is not linear. Q.E.D.

Another consequence of Lemma 2.6 (and also Theorem 2.7) is the following result.

Theorem 2.8 *Let X be a $\mathbb{Q}$-homology plane with $H_1(X; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Sup-*

pose that X has two algebraically independent G_a-actions. Then X is isomorphic to $\mathbb{P}^2 - C$, where C is a smooth conic.

Proof. With the notations in Lemma 2.6, we consider the fiber F_0 which restricts on X a unique multiple fiber $2T_0$. The fiber F_0 is supported by $\overline{T}_0+G$ and $\overline{T}_0$ is a unique (-1) component. By Theorem 2.7, the dual graph of G is a linear chain. Then it is readily verified that G consists of three irreducible components $G_1+G_2+G_3$ which are all (-2) curves. Furthermore, $\overline{T}_0$ meets the component G_2, and we may assume that G_1 meets the cross-section S of the $\mathbb{P}^1$-fibration $p: V \to \overline{B}$. Now contract $S+G_1+G_2+G_3$. Then we obtain a projective plane $\mathbb{P}^2$ and the proper transforms of $F_\infty, \overline{T}_0$ become respectively a smooth conic C and a line tangent to the conic. Hence X is isomorphic to $\mathbb{P}^2 - C$. Q.E.D.

We assume that the conditions (1) $\sim$ (6) are satisfied. By Theorem 2.7, the dual graph of G is a linear chain. The linear pencil Λ' has a base point Q on F_∞ which is different from the point $S \cap F_\infty$. Let $\overline{T}'$ be a general member of Λ'. As in the proof of Lemma 2.6, we may assume that $\mu < m^2 d$, where $m^2 d = i(\overline{T}', F_\infty; Q)$ and $\mu = \text{mult}_Q \overline{T}'$. The pencil contains a member $m\overline{A}'$, where mA' with $A' := \overline{A}' \cap X$ is a unique multiple fiber of the $\mathbb{A}^1$-fibration $\rho' : X \to B'$ which is induced by Λ'. Let $\mu' := \text{mult}_Q \overline{A}'$. Let $\varphi : \widetilde{V} \to V$ be the shortest sequence of blowing-ups which eliminates the base points of Λ' and let $\widetilde{\Lambda}'$ be the proper transform of Λ' by φ. Let E be the last (-1) curve appearing in the process φ and write $\varphi^{-1}(Q) = \Gamma + E + \Delta$, where Γ (resp. Δ) is the connected component of $\varphi^{-1}(Q) - E$ which meets the proper transform $\widetilde{F}_\infty$ (resp. $\widetilde{A}'$) of F_∞ (resp. $\overline{A}'$). Theorem 2.7 applied to the σ'-action implies that the dual graph of Δ is a linear chain.

Lemma 2.9 *The following assertions hold true.*

(1) $m\mu' \geq \mu$.

(2) *Suppose that $m\mu' > \mu$. Then the dual graph of Γ is either an emptyset or a linear chain. Furthermore, $m\mu' - \mu = 1$.*

(3) *Suppose that $m\mu' = \mu$. Then the dual graph of Γ has a branch point.*

Proof. (1) This is clear because the multiplicity $\text{mult}_Q \overline{T}' = \mu$ is the minimum of the multiplicities which the members of Λ' take at the point Q.

(2) Let φ_1 be the first blowing-up in the process φ and let E_1 be the exceptional curve. Then we have

$$\begin{aligned} \varphi_1^*(m\overline{A}') &= m\varphi_1'(\overline{A}') + m\mu' E_1 \\ \varphi_1^*(\overline{T}') &= \varphi_1'(\overline{T}) + \mu E_1. \end{aligned}$$

Hence in the proper transform Λ_1' of Λ' by φ_1, the (-1) curve E_1 belongs to the member containing $\varphi_1'(\overline{A}')$. If the dual graph $\varphi^{-1}(Q) = \Gamma + E + \Delta$ has a branching point, the member $\widetilde{M_0'}$ of $\widetilde{\Lambda}'$ containing $S+G$ has to coincide with the member containing $\varphi'(\overline{A}')$, which is a contradiction. So, the dual graph of Γ is a linear chain. Under the assumption $m\mu' > \mu$, the proper transform of E_1 by $\varphi \cdot \varphi_1^{-1}$ is the end component of Δ. Since $\Delta + \varphi'(\overline{A}')$ is contractible to a smooth fiber of a $\mathbb{P}^1$-fibration, it follows that $m\mu' - \mu = 1$.

(3) With the above notation, E_1 belongs to the member $\widetilde{M_0'}$. Let $\psi : \widehat{V} \to V$ be the oscilating sequence of blowing-ups with the data (md, μ') (cf. [11]) and let E' be the last (-1) curve. Since the proper transforms of E_1 and F_∞ by φ are contained in the member $\widetilde{M_0'}$, all the exceptional curves of ψ are also contained in $\widetilde{M_0'}$. In order to eliminate the base points of Λ', we have therefore to blow up a point on E'. Hence the dual graph of Γ has a branch point which represent the proper transform of E'. Q.E.D.

Lemma 2.10 *The following assertions hold.*

(1) *Suppose $\mu' = 1$ and $m\mu' > \mu$. Then $m = 2$.*

(2) *Suppose $\mu' \le d$ and $m\mu' > \mu$. Then $\mu' = 1$.*

Proof. (1) By Lemma 2.9 and the hypothesis $\mu' = 1$, we have $\mu = m - 1$. Then the curve $\overline{A}'$ touches F_∞ with multiplicity md. Let $\psi : V' \to V$ be a sequence of md blowing-ups with centers Q and its infinitely near points lying on the proper transforms of F_∞. Let $E_1, \ldots, E_{md}$ be the irreducible exceptional curves. Then $\psi'(F_\infty) + E_{md} + \cdots + E_1$ is a linear chain and $\psi'(\overline{A}')$ meets E_{md} transversally. Let M_0' (resp. M_1') be the member of $\psi'(\Lambda')$ containing $\psi'(F_\infty)$ (resp. $\psi'(\overline{A}')$). Then we have

$$\begin{aligned} M_0' &= (m-1)\psi'(F_\infty) + \text{a divisor supported by } \psi'(S) + \psi^*(G)_{\text{red}} \\ M_1' &= m\psi'(\overline{A}') + E_1 + 2E_2 + \cdots + mdE_{md}. \end{aligned}$$

The general member $\psi'(\overline{T}')$ passes the point $Q' := \psi'(F_\infty) \cap E_{md}$ with

$$\begin{aligned} i(\psi'(F_\infty), \psi'(\overline{T}'); Q') &= m^2 d - (m-1)md = md, \\ i(\psi'(\overline{T}'), E_{md}; Q') &= m-1. \end{aligned}$$

Let $\varphi : \widetilde{V} \to V$ be the sequence of blowing-ups as above which eliminates the base points of Λ'. Then the member $\widetilde{M_1}$ of $\varphi'(\Lambda')$ containing $\varphi'(\overline{A}')$ is a degenerate fiber of a $\mathbb{P}^1$-fibration which contains only one (-1) curve $\varphi'(\overline{A}')$. Since the coefficient of $\varphi'(\overline{A}')$ in $\widetilde{M_1}$ is m, it is the largest coefficient among those for the components of $\widetilde{M_1}$. This implies that $md \le m$. Hence $d = 1$. So, the pair (σ, σ') is a minimal pair, and Theorem 2.5 implies that $m = 2$.

(2) Suppose on the contrary that $\mu' \ge 2$. Write

$$md = c_1\mu' + \mu'_1, \quad 0 \le \mu'_1 < \mu'.$$

Then

$$m^2 d = m(c_1\mu' + \mu'_1) = c_1\mu + (c_1 + m\mu'_1).$$

Since $\mu' \le d$, we have $c_1 \ge m$. In the case $c_1 > m$, we abuse the notations to denote by $\psi : V' \to V$ a sequence of c_1 blowing-ups with center Q and its infinitely near points lying on F_∞. It produces the member M'_1 of $\psi'(\Lambda')$ such that

$$M'_1 = m\psi'(\overline{A}') + E_1 + 2E_2 + \cdots + c_1 E_{c_1},$$

which leads to a contradiction as in the proof of the previous assertion. Consider the case $c_1 = m$. Suppose $\mu'_1 > 0$. Then we have

$$\begin{aligned} i(\psi'(F_\infty), \psi'(\overline{A}); Q') &= \mu'_1, \\ i(\psi'(\overline{A}'), E_{c_1}; Q') &= \mu', \end{aligned}$$

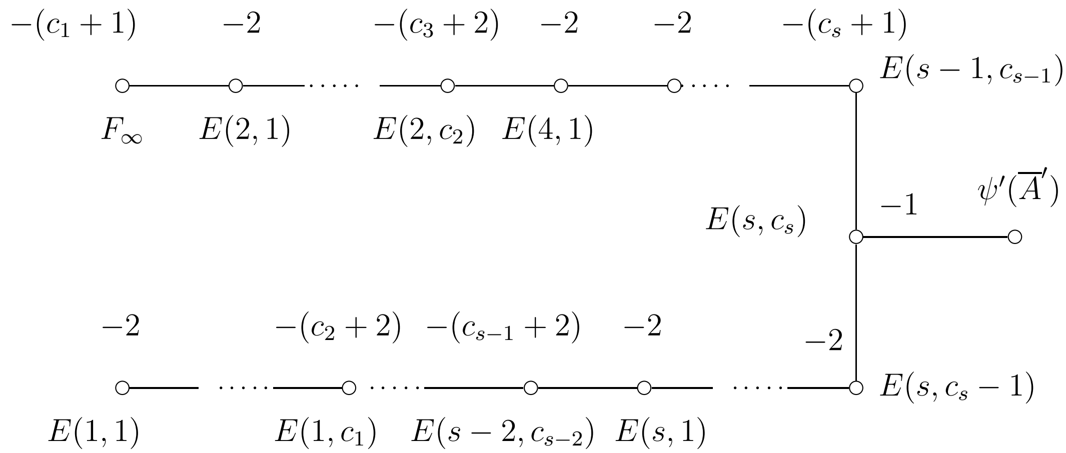
where $Q' = \psi'(F_\infty) \cap E_{c_1}$. Then, after the base points of Λ' are removed by $\varphi : \widetilde{V} \to V$, $\varphi'(\overline{A}')$ does not meet any one of the proper transforms of $E_1, \ldots, E_{c_1}$. This implies that a component of the member $\widetilde{M_1}$ has coefficient greater than m, where $\widetilde{M_1}$ is a member of the proper transform $\varphi'(\Lambda')$ containing $\varphi'(\overline{A}')$. This is a contradiction. So, we must have $\mu'_1 = 0$. Then $c_1 = m$ and $\mu' = d$. Since $\mu' \ge 2$, $\psi'(\overline{A}')$ meets E_m in a single point with multiplicity μ', and this point is untouched in the further process of eliminating the base points of Λ'. This is a contradiction. Q.E.D.

We continue the analysis of the case $m\mu' > \mu$ and keep the same notations as above. In particular, we abuse the notations M_0' and M_1' to denote respectively the members of Λ' such that $\mathrm{Supp}\, M_0' = F_\infty + S + G$ and $M_1' = m\overline{A}'$, while $\overline{T}'$ denotes a general member of Λ'. Let $\varphi : \widetilde{V} \to V$ be the shortest sequence of blowing-ups with centers at the base point Q of Λ' and its infinitely near points such that the proper transform $\widetilde{\Lambda}'$ of Λ' has no base points. We denote by $\widetilde{M_0'}$ and $\widetilde{M_1'}$ the members of $\widetilde{\Lambda}'$ corresponding to M_0' and M_1' respectively. Let $\varphi^{-1}(Q) = \Gamma + E + \Delta$ as before, where $\Gamma \cap \varphi'(F_\infty) \neq \emptyset$ and $\Delta \cap \varphi'(\overline{A}') \neq \emptyset$. *We assume that* $m\mu' > \mu$. Then Γ is a linear chain and $m\mu' - \mu = 1$ by Lemma 2.9.

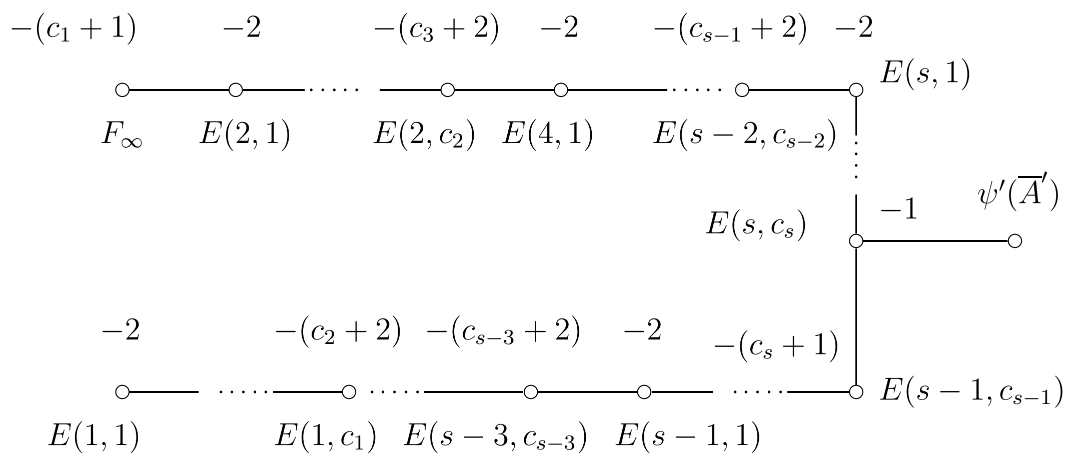
By the Euclidean algorithm with respect to md and μ', we introduce the integers c_i, μ_i' for $1 \le i \le s$ as follows:

$$\begin{array}{rll} md & = c_1\mu' + \mu_2', & 0 < \mu_2' < \mu' \\ \mu_1' & = c_2\mu_2' + \mu_3', & 0 < \mu_3' < \mu_2' \\ & \cdots\cdots & \\ \mu_{s-2}' & = c_{s-1}\mu_{s-1}' + \mu_s', & 0 < \mu_s' < \mu_{s-1}' \\ \mu_{s-1}' & = c_s\mu_s', & c_s \ge 2, \end{array}$$

where we set $\mu_1' = \mu'$. Let $\psi : \widehat{V} \to V$ be an oscilating sequence of blowing-ups with respect the data (md, μ') (cf. [11]). Then we have the following exceptional dual graph of $\psi^{-1}(Q)$. See also [8] for similar dual graphes and relevant explanations.



CASE s is odd



CASE s is even

Lemma 2.11 *The following assertions hold true.*

(1) $\psi'(\overline{A}')$ *meets the component* $E(s,c_s)$ *in one point transversally and does not meet any other components of* $\psi^{-1}(Q)$. *In particular,* $\mu'_s = 1$.

(2) *The components located on the rught side of* $E(s,c_s)$, *i.e.,* $E(1,1),\ldots,$ $E(s,1),\ldots,E(s-1,c_{s-1})$ *if* s *is odd and* $E(1,1),\ldots,E(s-1,c_{s-1})$, $E(s,c_s-1)$ *if* s *is even, are contained in the member* $\widehat{M'_1}$ *of* $\psi'(\Lambda')$ *corresponding to* M'_1 *of* Λ'.

(3) $\psi'(\overline{T}')$ *passes through the point* $E(s, c_s) \cap E(s-1, c_{s-1})$ *if* s *is odd and the point* $E(s, c_s) \cap E(s, c_s - 1)$ *if* s *is even.*

(4) *The components located on the left side of* $E(s, c_s)$ *are contained in the member* $\widehat{M_0'}$ *of* $\psi'(\Lambda')$, *where* $\widehat{M_0'}$ *corresponds to* M_0' *of* Λ'.

Proof. Let $\widehat{M_0'}$ and $\widehat{M_1'}$ be respectively the members of the proper transform $\psi'(\Lambda')$ of Λ' such that $\widehat{M_0'}$ (resp. $\widehat{M_1'}$) contains $\psi'(F_\infty)$ (resp. $\psi'(\overline{A}')$). Since every member of $\psi'(\Lambda')$ is connected, $\widehat{M_1'}$ contains a connected linear chain $\psi'(\overline{A}') + E(s, c_s) + \cdots + E(1, 1)$, which contains the lower half of the whole chain. We note that $\psi'(\overline{A}')$ meets $E(s, c_s)$ in one point with multiplicity μ_s' which is different from the points of $E(s, c_s)$ where $E(s, c_s)$ meets the other components $E(i, j)$'s.

The member $\widehat{M_0'}$ contains some connected part of the linear chain $E(2, 1) + \cdots + E(s-1, c_{s-1})$ if s is odd (resp. $E(2, 1) + \cdots + E(s, c_s - 1)$ if s is even). We claim that $\widehat{M_0'}$ contains all of this linear chain and hence the point $E(s-1, c_{s-1}) \cap E(s, c_s)$ (resp. $E(s, c_s - 1) \cap E(s, c_s)$) is the base point of $\psi'(\Lambda')$ if s is odd (resp. if s is even). Suppose on the contrary that the rightmost component E of $\widehat{M_0'}$ is not $E(s-1, c_{s-1})$ (resp. $E(s, c_s - 1)$) if s is odd (resp. if s is even). Then, from the mid-stage of ψ onward when E was the last (-1) curve, the general member $\overline{T}'$ (or precisely, its proper transform) keeps meeting the component E. Namely, the process φ is branched at this stage and should constitute of the blowing-ups with centers at the intersection point of E and $\overline{T}'$ and its infinitely near points. This implies that the component $\varphi'(\overline{A}')$ in the corresponding member $\widetilde{M_1'}$ of $\varphi'(\Lambda')$ has a singular point or meets two other components in a point. This is a contradiction. Hence our claim is ascertained. Furthermore, the point $Q_1 = E(s-1, c_{s-1}) \cap E(s, c_s)$ if s is odd (resp. $Q_1 = E(s, c_s - 1) \cap E(s, c_s)$ if s is even) is a base point of the pencil $\psi'(\Lambda')$.

Now the process φ is a sequence of blowing-ups with centers Q_1 and its infinitely near points. Let $\psi_1 = \psi^{-1} \cdot \varphi : \widetilde{V} \to \widehat{V}$ be the necessary process of eliminating the base points of $\psi'(\Lambda')$. Since $Q_1 \neq \psi'(\overline{A}') \cap E(s, c_s)$, it follows that $\mu_s' = 1$ because the proper transforms of $\psi'(\overline{A}')$ and $E(s, c_s)$ in $\widetilde{M_1'}$ meet each other transversally. All the assertions of Lemma 2.11 follows from these observations. Q.E.D.

Now let $\psi_1^{-1}(Q_1) = \Gamma_1 + E_1 + \Delta_1$, where E_1 is the last (-1) curve and Γ_1

(resp. Δ_1) is contained in $\widetilde{M}'_0$ (resp. $\widetilde{M}'_1$). Then

$$\Delta_1 + \varphi'(\overline{A}') + \psi'_1(E(s, c_s) + \cdots + E(1, 1))$$

is contracted to a smooth $\mathbb{P}^1$-fiber, and the dual graph of Δ_1 (hence Γ_1) is therefore uniquely determined. In fact, the dual graph of Δ_1 coincides with the dual graph $F_\infty + E(2, 1) + \cdots + E(s-1, c_{s-1})$ if s is odd (resp. $F_\infty + E(2, 1) + \cdots + E(s, c_s - 1)$ if s is even).

We shall determine the multiplicity of $\psi'_1(E(s, c_s))$ as a component of a degenerate $\mathbb{P}^1$-fiber supported by $\Delta_1 + \varphi'(\overline{A}') + \psi'_1(E(s, c_s) + \cdots + E(1, 1))$. For this purpose, identify Δ_1 with $F_\infty + E(2, 1) + \cdots + E(s-1, c_{s-1})$ (resp. $F_\infty + E(2, 1) + \cdots + E(s, c_s - 1)$) if s is odd (resp. if s is even), and let $\mu(i, j)$ be the multiplicity of $E(i, j)$ for $1 \leq i \leq s$ and $1 \leq j \leq c_i$, where $\mu(1, 1) = 1$ and the multiplicity of F_∞ is 1. Then we have the following relations:

$$\begin{array}{lll}
\mu(1, j) & = j, & 1 \leq j \leq c_1 \\
\mu(2, j) & = 1 + j\mu(1, c_1), & 1 \leq j \leq c_2 \\
\mu(3, j) & = \mu(1, c_1) + j\mu(2, c_2), & 1 \leq j \leq c_3 \\
 & \cdots\cdots & \\
\mu(t, j) & = \mu(t-2, c_{t-2}) + j\mu(t-1, c_{t-1}), & 1 \leq j \leq c_t \\
 & \cdots\cdots & \\
\mu(s, j) & = \mu(s-2, c_{s-2}) + j\mu(s-1, c_{s-1}), & 1 \leq j \leq c_s.
\end{array}$$

Thence we have

$$\frac{\mu(s, c_s)}{\mu(s-1, c_{s-1})} = c_s + \cfrac{1}{c_{s-1} + \cfrac{1}{c_{s-2} + \cfrac{1}{\ddots - \cfrac{1}{c_1}}}} = [c_s, c_{s-1}, \ldots, c_1],$$

while $md/\mu' = [c_1, \ldots, c_s]$. Note that $\mu'_s = 1$ implies $\gcd(md, \mu') = 1$. Then it follows that $\mu(s, c_s) = md$. Meanwhile, the multiplicity of $\varphi'(\overline{A}')$ (and hence the one of $\psi'_1(E(s, c_s))$) is m. So, we conclude that $d = 1$ and that the pair (σ, σ') is minimal. Then $m \geq 3$ is impossible by Theorem 2.5. Hence we have the following result.

Theorem 2.12 *Suppose that $m\mu' > \mu$. Then the pair (σ, σ') is minimal, and hence $m = 1$ or 2.*

3 Observations in the case $m\mu' = \mu$

Inheriting the notations in the previous section, we shall explain the elimination process $\varphi : \widetilde{V} \to V$ of the base points of the pencil Λ' in the case $m\mu' = \mu$. Let $\varphi_1 : V_1 \to V$ be the oscilating sequence of blowing-ups with center Q and data (md, μ'). With the observations before Lemma 2.11 taken into account, the proper transform $\varphi_1'(\Lambda')$ has a base point Q_1 on the last exceptional curve $E_1 := E(s, c_s)$, which does not lie on any other components of $\varphi_1^{-1}(Q)$. Note that the following assertions hold:

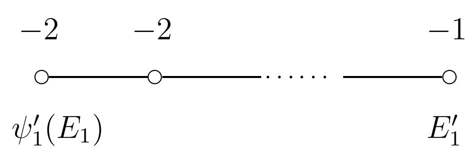
(1) Every component of $\varphi_1^{-1}(Q)$ belongs to the member $M_0'(1)$ of $\varphi_1'(\Lambda')$ which corresponds to the member M_0' of Λ'.

(2) Write $\varphi_1^{-1}(Q) = \Gamma_1 + E_1 + \Delta_1$, where Γ_1 and Δ_1 are the connected components of $\varphi_1^{-1}(Q) - E_1$ such that $\Gamma_1 \cap \varphi_1'(F_\infty) \neq \emptyset$ and $\Delta_1 \cap \varphi_1'(F_\infty) = \emptyset$. Then $\varphi'(G + S + F_\infty) + \Gamma_1$ contracts to a smooth point.

(3) The general member $\varphi_1'(\overline{T}')$ of $\varphi_1'(\Lambda')$ satisfies

$$i(E_1, \varphi_1'(\overline{T}'); Q_1) = \text{mult}_{\,Q_1} \varphi_1'(\overline{T}') = \mu_s = m\mu_s'.$$

Let $\psi_1 : V_1' \to V_1$ be a sequence of blowing-ups such that $\psi^{-1}(Q_1)$ has the dual graph



where the proper transform $\Lambda_1' := (\varphi_1\psi_1)'(\Lambda')$ has a base point Q_1' lying only on the last (-1) curve E_1' and not on the other components, and where

$$m\mu_s' = i(E_1', (\varphi_1\psi_1)'(\overline{T}'); Q_1') > \mu^{(2)} := \text{mult}_{\,Q_1'} (\varphi_1\psi_1)'(\overline{T}').$$

We note that $m(\varphi_1\psi_1)'(\overline{A}')$ is the member of Λ_1' and hence passes through the point Q_1' with

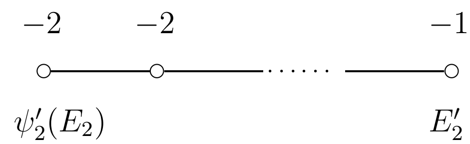
$$\mu_s' = i(E_1', (\varphi_1\psi_1)'(\overline{A}'); Q_1') \geq \mu'^{(2)} := \text{mult}_{\,Q_1'} (\varphi\psi_1)'(\overline{A}').$$

Here $m\mu'^{(2)} \geq \mu^{(2)}$.

Suppose $\mu^{(2)} = m\mu'^{(2)}$. The the next process is similar to the sequence φ_1 above. We let $\varphi_2 : V_2 \to V_1'$ be the oscilating sequence of blowing-ups with center Q_1' and data $(\mu_s', \mu'^{(2)})$. Let E_2 be the last (-1) curve of φ_2. Then the pencil $(\varphi_1\psi_1\varphi_2)'(\Lambda')$ has a base point Q_2 on E_2 not lying on any other components of $\varphi_2^{-1}(Q_1')$. Write $(\psi_1\varphi_2)^{-1}(Q_1) = \Gamma_2 + E_2 + \Delta_2$, where Γ_2 and Δ_2 are the connected components of $(\psi_1\varphi_2)^{-1}(Q_1) - E_2$ such that $\Gamma_2 \cap (\psi_1\varphi_2)'(E_1) \neq \emptyset$.

(4) Then $(\psi_1\varphi_2)'\left(\varphi_1'(G + S + F_\infty) + \Gamma_1 + E_1 + \Delta_1\right) + \Gamma_2$ contracts to a smooth point.

After a possible sequence of blowing-ups $\psi_2 : V_2' \to V_2$ like ψ_1 whose dual graph is a (-2) sequence



the proper transform $\Lambda_2' := (\varphi_2\psi_2)'(\Lambda_1')$ has a base point Q_2' lying only on the last (-1) curve E_2' and not lying on the other components. Furthermore,

$$i(E_2', (\varphi_1\psi_1\varphi_2\psi_2)'(\overline{T}'); Q_2') > \mu^{(3)} = \text{mult}_{Q_2'}((\varphi_1\psi_1\varphi_2\psi_2)'(\overline{T}')).$$

We note that $m(\varphi_1\psi_1\varphi_2\psi_2)'(\overline{A}')$ is the member of Λ_2' and passes through the point Q_2' with

$$i(E_2', (\varphi_1\psi_1\varphi_2\psi_2)'(\overline{A}'); Q_2') \geq \mu'^{(3)} = \text{mult}_{Q_2'}((\varphi_1\psi_1\varphi_2\psi_2)'(\overline{A}')),$$

where $m\mu'^{(3)} \geq \mu^{(3)}$.

After this process repeated several times, we reach to the t-th stage where $m\mu'^{(t)} > \mu^{(t)}$. As in Lemma 2.9, it then follows that $m\mu'^{(t)} - \mu^{(t)} = 1$. As in the proof of Lemma 2.11 and the subsequent arguments, the oscilating sequence of blowing-ups with center Q_{t-1}' and data $(i(E_{t-1}', \widehat{T}'; Q_{t-1}'), \mu^{(t)})$ eliminates the base points of the pencil Λ_{t-1}', where $\widehat{T}'$ is the proper transform of $\overline{T}'$. Hence $V_t = \widetilde{V}$. Let E_t be the last (-1) curve of φ_t and write $(\psi_{t-1}\varphi_t)'(Q_{t-1}') = \Gamma_t + E_t + \Delta_t$ as above, where Γ_t is connected to the proper transform of F_∞. Then we have:

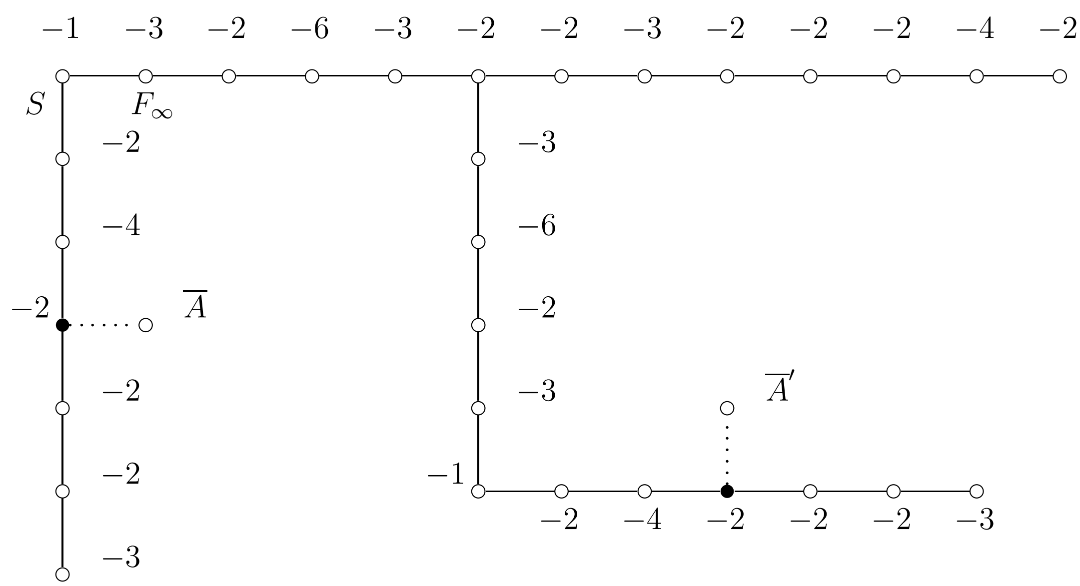
(5) All the components lying on the left side of E_t, i.e., the connected component containing Γ_t and the proper transform of $G + S + F_\infty$ contracts to a smooth $\mathbb{P}^1$-fiber.

(6) Δ_t together with the proper transform of $\overline{A}'$ contracts to a smooth $\mathbb{P}^1$-fiber. In fact, the component of Δ_t where $\overline{A}'$ meets is the proper transform of the (-1) curve which appears as the last exceptional curve of the oscilating sequence of blowing-ups with center Q'_{t-1} and data $(i(E'_{t-1}, \widehat{A}'; Q'_{t-1}), \mu'^{(t)})$, where $\widehat{A}'$ is the proper transform of $\overline{A}'$ on V'_{t-1}.

(7) The same argument as the one leading to Theorem 2.12 shows that $(i(E'_{t-1}, \widehat{A}'; Q'_{t-1}), \mu'^{(t)}) = m$.

We do not know if such a pencil Λ' exists as satisfying all the above conditions. But the following example shows that the dual graph of exceptional curves of $\varphi : \widetilde{V} \to V$ together with the proper transform of $G + S + F_\infty$ is realizable.

EXAMPLE **3.1** *Let* $m = 7, d = 76, \mu' = 31, \mu = m\mu', s = 5, \mu'_s = 7, t = 1, \mu^{(1)} = 27, \mu'^{(1)} = 4$. *The dual graph is as given as follows.*



Although we have this example, we have an impression that the linear pencil Λ' does not exist. Hence we propose the following

Conjecture *Let X be a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration $\rho : X \to B$. Suppose that ρ has a single multiple fiber of multiplicity $m \geq 3$. Then the Makar-Limanov invariant of X is not constant.*

We shall include here a remark made by a doctoral student Adrien Dubouloz of the Université de Grenoble, which gives a relation between the $\mathbb{Q}$-homology planes with trivial Makar-Limanov invariants and the hypersurfaces $xy = p(z)$ in [1].

Theorem 3.2 *Let X be a $\mathbb{Q}$-homology plane with trivial Makar-Limanov invariant and let $\rho : X \to B$ be an $\mathbb{A}^1$-fibration with a unique multiple fiber mA of multiplicity $m > 1$. Let $B' \to B$ be a cyclic Galois covering of order m ramifying totally over the point $P_0 = \rho(A)$ and let Y be the normalization of the fiber product $X \times_B B'$. Then Y is isomorphic to a hypersurface $xy = p(z)$, where $p(z)$ is a polynomial of degree m in z with distinct linear factors. The given $\mathbb{Q}$-homology plane X is regained as the quotient of X with respect to a $\mathbb{Z}/m\mathbb{Z}$-action.*

Proof. We shall give a rough sketch of the proof, leaving the details to a paper by A. Dubouloz. We use the projective embedding $X \hookrightarrow V$ considered in Lemmas 2.6 and 2.7. In particular, the fiber F_0 of $p : V \to \overline{B}$ over the point P_0 is supported by $G + \overline{A}$, where the dual graph is a linear chain and $\overline{A}$ is the closure of A in V. Let G_1 be the irreducible component of G such that $(G_1 \cdot \overline{A}) = 1$. Let $\sigma : \overline{B}' \to \overline{B}$ be a cyclic Galois covering of order m ramifying totally over the points P_0 and $P_\infty = p(F_\infty)$. Let W' be the normalization of V in the function field of V and let $\tau' : W' \to V$ be the normalization morphism. Then the branch locus of τ' contains F_∞ and is contained in the sum $F_\infty + G$. Hence W' has a $\mathbb{P}^1$-fibration $q' : W' \to \overline{B}'$. The singularity of W' are at most cyclic quotient singularities which arise from the intersection points of the branch locus and lie on the fiber ${q'}^{-1}(P'_0)$, where P'_0 is the point of $\overline{B}'$ lying over P_0. Let $\nu : W \to W'$ be the minimal resolution of the singular points of W' and let $\tau = \tau' \cdot \nu : W \to V$. Then there is an induced $\mathbb{P}^1$-fibration $q : W \to \overline{B}'$, which satisfies $\sigma \cdot q = p \cdot \tau$. Remind that the component A splits into a disjoint union of m affine lines $B_1, \ldots, B_m$. This implies that the component G_1 is not contained in the branch locus of τ' and hence τ. Let H_1 be the irreducible component of $q^{-1}(P'_0)$ lying over G_1. Then $\tau \mid_{H_1}: H_1 \to G_1$ is a cyclic covering of order m, and there are m irreducible components $\overline{B}_1, \ldots, \overline{B}_m$ of $q^{-1}(P'_0)$ such that $(H_1 \cdot \overline{B}_i) = 1$ and $\overline{B}_i \cap Y = B_i$ for $1 \leq i \leq m$. Since $\overline{B}_1, \ldots, \overline{B}_m$ are reduced in $q^{-1}(P'_0)$, the multiplicity of H_1 in $q^{-1}(P'_0)$ is accordingly equal to 1. So, we can contract all the components of $q^{-1}(P'_0)$ except for H_1 and $\overline{B}_1, \ldots, \overline{B}_m$. Let $\widetilde{W}$ be the surface thus obtained from W. Then $\widetilde{W}$ has a $\mathbb{P}^1$-fibration $\widetilde{q} : \widetilde{W} \to \overline{B}$ and Y is embedded into $\widetilde{W}$ as an open set, and the boundary divisor $\widetilde{D} := \widetilde{W} - Y$ consists of the cross-section $\widetilde{S}$ of $\widetilde{q}$, the fiber $\widetilde{F}_\infty$ lying above the point at infinity Q_∞, and the fiber $\widetilde{F}_0 = \widetilde{H}_1 + \sum_{i=1}^m \widetilde{B}_i$, where Q_∞ is a unique point lying above P_∞, $\widetilde{S}$ is the inverse image of S and $\widetilde{H}_1, \widetilde{B}_1, \ldots, \widetilde{B}_m$ are respectively the proper transforms of $H_1, \overline{B}_1, \ldots, \overline{B}_m$. Then it is straightforward to see that the canonical divisor K_Y, that is to say, the restriction of $K_{\widetilde{W}}$ onto Y, is trivial. On the other hand, since all the G_a-actions on X lifts up to Y by Lemma 2.2, Y is a smooth affine surface with trivial Makar-Limanov invariant. Hence the Makar-Limanov invariant of Y is trivial by [1, Lemma 4], and Y is isomorphic to a hypersurface $xy = p(z)$ with $\deg p(z) = m$. Q.E.D.

4 Etale endomorphisms of $\mathbb{Q}$-homology planes

In [4], the generalized Jacobian conjecture for $\mathbb{Q}$-homology planes is considered. It is shown that any étale endomorphism of a $\mathbb{Q}$-homology plane X is an automorphism if one of the following conditions is satisfied:

(1) $\overline{\kappa}(X) = 2$ or 1.

(2) $\overline{\kappa}(X) = -\infty$ and X has an $\mathbb{A}^1$-fibration $\rho : X \to B$ with at least two multiple fibers.

In this section, we shall consider the generalized Jacobian conjecture for a $\mathbb{Q}$-homology plane X with an action of the additive group, which has accordingly $\overline{\kappa}(X) = -\infty$. We shall rectify some of the arguments in [4]. We recall the following two lemmas (cf. [4, Lemma 6.1] and [4, 7, Lemma 3.1]).

Lemma 4.1 *Let $\rho : X \to B$ be an $\mathbb{A}^1$-fibration on a $\mathbb{Q}$-homology plane. Suppose that ρ has at least two singular fibers. Let $g : \mathbb{A}^1 \to X$ be a non-constant morphism. Then the image of g is a fiber of ρ.*

Lemma 4.2 *For $i = 1, 2$, let $\rho_i : X_i \to B_i$ be $\mathbb{A}^1$-fibrations on $\mathbb{Q}$-homology planes. Let $\phi : X_1 \to X_2$ and $\beta : B_1 \to B_2$ be dominant morphisms such that $\rho_2 \cdot \phi = \beta \cdot \rho_1$. Let $m\Gamma$ be an irreducible fiber of ρ_2 lying over a point*

$p \in B_2$ with $m \geq 1$ and Γ reduced, and let $q \in B_1$ be a point such that $\beta(q) = p$. Suppose $\rho_1^*(q) = \ell\Delta$, where Δ is reduced and irreducible and ℓ is its multiplicity. Suppose furthermore that ϕ is an étale morphism. If the ramification index of β at q is e then $\ell e = m$. In particular, if $m = 1$ then $\ell = e = 1$.

Applying these lemmas, we shall show the following result.

Lemma 4.3 *Let X be a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration $\rho : X \to B$. Let $m_1A_1, \ldots, m_nA_n$ exhaust all multiple fibers of ρ. Let $\phi : X \to X$ be an étale endomorphism. Then the following assertions hold:*

(1) *If $n \geq 2$, then there exists an endomorphism β of B such that $\rho \cdot \phi = \beta \cdot \rho$.*

(2) *The above endomorphism β is an automorphism provided $n \geq 3$ or $n = 2$ and $\{m_1, m_2\} \neq \{2, 2\}$.*

Proof. The first assertion is an immediate consequence of Lemma 4.1. So, we consider the second assertion. We employ the arguments in [7, Lemmas 3.1 and 3.2]. Note that $\beta : B \to B$ is a finite morphism because B is the affine line. By Lemma 4.1, the set $\{p_1, \ldots, p_n\}$ is mapped to itself by β, where $p_i = \rho(A_i)$. Suppose, furthermore, that the points $q_1, \ldots, q_s$, none of which belongs to $\{p_1, \ldots, p_n\}$, are mapped to $\{p_1, \ldots, p_n\}$. Then, by Lemma 4.2, the ramification index of β at q_j, say e_j, is larger than 1. In fact, if $\beta(q_j) = p_i$ then $e_j = m_i$.

Since β induces an étale finite morphism

$$\beta : B - \{p_1, \ldots, p_n, q_1, \ldots, q_s\} \longrightarrow B - \{p_1, \ldots, p_n\},$$

the comparison of the Euler numbers gives rise to an equality

$$1 - (n + s) = d(1 - n), \tag{1}$$

where $d = \deg \beta$. On the other hand, by summing up the ramification indices, we have an inequality

$$2s + n \leq dn. \tag{2}$$

So, by combining (1) and (2) together, we have an inequality

$$2(d - 1)(n - 1) = 2s \leq (d - 1)n. \tag{3}$$

Suppose $d > 1$. Then $n \leq 2$. Hence, if $n \geq 3$ then $d = 1$ and β is an automorphism. Suppose that $d > 1$ and $n = 2$. Then the equality occurs in (3), and hence the equality occurs in (2). Namely, the ramification index e_j at q_j is two for all j, and $s = d - 1$. Since $d > 1$ implies $s > 0$, we may assume that q_1 is mapped to p_1. Then $m_1 = 2$. Suppose $d \geq 3$. Then $2s = 2(d-1) > d$. Hence one of the q_j is mapped to $p_2, \ldots, p_n$, say p_2. Hence $m_2 = 2$. In this case, after a suitable change of indices, one of the following two cases is possible:

(1) $s = s_1 + s_2 = d - 1$, and $q_1, \ldots, q_{s_1}, p_1$ (or p_2) (resp. $q_{s_1+1}, \ldots, q_s p_2$ (or p_1) are mapped to p_1 (resp. p_2).

(2) $s = s_1 + s_2, d = 2s_1 = 2s_2 + 2$, and $q_1, \ldots, q_{s_1}$ (resp. $q_{s_1+1}, \ldots, q_s, p_1, p_2$) are mapped to p_1 (resp. p_2).

Finally, suppose that $d = n = 2$ and $s = 1$. Then we may assume that $\beta(q_1) = p_1$ and $\beta(p_1) = \beta(p_2) = p_2$. Then $m_2 = 2$ as well by Lemma 4.2. So, if $\{m_1, m_2\} \neq \{2, 2\}$, then $d = 1$ and β is an automorphism. Q.E.D.

As a consequence of Lemma 4.3, we can prove the following result, which rectifies Theorem 6.1 in [4].

Theorem 4.4 *Let X be a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration $\rho : X \to B$. Let $m_1A_1, \ldots, m_nA_n$ exhaust all multiple fibers of ρ. Suppose that either $n \geq 3$ or $n = 2$ and $\{m_1, m_2\} \neq \{2, 2\}$. Then any étale endomorphism $\phi : X \to X$ is an automorphism.*

Proof. By Lemma 4.3, there exists an automorphism β of B such that $\rho \cdot \phi = \beta \cdot \rho$. Since β is an automorphism, Lemma 4.2 implies that β induces a permutation of the finite set $\{p_1, \ldots, p_n\}$. By replacing β by its suitable iteration β^r, we may assume that β induces the identity on $\{p_1, \ldots, p_n\}$. Since $n \geq 2$ and β (or rather an induced automorphism of the smooth compactification $\overline{B}$ of B) fixes the point at infinity p_∞. Hence β is then the identity automorphism.

Let $K = k(B)$ be the function field of B and let X_K be the generic fiber of ρ. Then X_K is isomorphic to the affine line over K, and ϕ induces an étale endomorphism ϕ_K of X_K. Since ϕ_K is then finite, ϕ_K is an automorphism. Hence ϕ is birational. Then Zariski's Main Theorem implies that ϕ is an open immersion. Note that Pic $(X)_{\mathbb{Q}} = 0$ and $\Gamma(\mathcal{O}_X)^* = \mathbb{C}^*$. Suppose that $X \neq \phi(X)$. Then $X - \phi(X)$ has pure codimension one. Since Pic $(X)_{\mathbb{Q}} = 0$, there exists a regular function h on X such that the zero locus $(h)_0$ of h is supported by $X - \phi(X)$. Then $\phi^*(h)$ is a non-constant invertible function on X, which contradicts the property $\Gamma(\mathcal{O}_X)^* = \mathbb{C}^*$. So, ϕ is an automorphism. Q.E.D.

In the case $\{m_1, m_2\} = \{2, 2\}, d = n = 2$ and $s = 1$, there exists the following counter-example to the generalized Jacobian conjecture.

EXAMPLE **4.5** *Let $V_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Let M_0 be a cross-section and let $\ell_0, \ell_1, \ell_\infty$ be distinct three fibers with respect to the second projection $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Let $\varphi : V \to V_0$ be q sequence of blowing-ups with centers at $\ell_0 \cap M_0, \ell_1 \cap M_0$ and their infinitely near points such that $\varphi^*(\ell_0) = \ell'_0 + E_1 + 2E_2 + 2E_3$ and $\varphi^*(\ell_1) = \ell'_1 + F_1 + 2F_2 + 2F_3$, where $({\ell'_0}^2) = ({\ell'_1}^2) = ({E_i}^2) = ({F_i}^2) = -2$ for $i = 1, 2$ and $({E_3}^2) = ({F_3}^2) = -1$. Let*

$$X := V - (\ell_\infty + M'_0 + \ell'_0 + \ell'_1 + E'_1 + F'_1 + E'_2 + F'_2).$$

Hence X has an $\mathbb{A}^1$-fibration $\rho : X \to B$ with two multiple fibers $2E_3 \cap X, 2F_3 \cap X$ of multiplicity 2. Then X has a degree two, non-finite étale endomorphism.

In fact, let $\sigma : B' \to B$ be a degree two covering ramifying over the point at infinity p_∞ and p_0, where $p_0 = \rho(E_3 \cap X)$. Let $\widetilde{X}$ be the normalization of $X \times_B B'$, let $\tau : \widetilde{X} \to X$ be the composite of the normalization morphism and the first projection $X \times_B B' \to X$ and let $\widetilde{\rho} : \widetilde{X} \to B'$ be the $\mathbb{A}^1$-fibration induced naturally by ρ. Then $\widetilde{\rho}^*(q_0)$ is a disjoint sum $G_1 + G_2$ of two affine lines and $\tau : \widetilde{X} \to X$ is a finite étale morphism, where q_0 is a point of B' lying over p_0. Then $\widetilde{X} - G_1 \cong \widetilde{X} - G_2 \cong X$, and $\tau|_{\widetilde{X}-G_1}$ and $\tau|_{\widetilde{X}-G_2}$ induce a non-finite étale endomorphism of X.

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