Matrices related to the Pascal triangle

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Résumé. Le but de ce papier est d'étudier les déterminants de quelques familles de matrices reliées au triangle de Pascal.

Abstract[1]*: The aim of this paper is to study determinants of matrices related to the Pascal triangle.*

1 The Pascal triangle

Let P be the infinite symmetric "matrix" with entries $p_{i,j} = \binom{i+j}{i}$ for $0 \leq i, j \in \mathbf{N}$. The matrix P is hence the famous Pascal triangle yielding the binomial coefficients and can be recursively constructed by the rules $p_{0,i} = p_{i,0} = 1$ for $i \geq 0$ and $p_{i,j} = p_{i-1,j} + p_{i,j-1}$ for $1 \leq i, j$.

In this paper we are interested in (sequences of determinants of finite) matrices related to P.

The present section deals with determinants of some minors of the above Pascal triangle P, perhaps slightly perturbed.

Sections 2-6 are devoted to the study of matrices satisfying the Pascal recursion rule $m_{i,j} = m_{i-1,j} + m_{i,j-1}$ for $1 \leq i, j < n$ (with various choices for the first row $m_{0,j}$ and column $m_{i,0}$). Our main result is the experimental observation (Conjecture 3.3 and Remarks 3.4) that given such an infinite matrix whose first row and column satisfy linear recursions (for instance given by the Fibonacci sequence), then the determinants of a suitable sequence of minors seem also to satisfy a linear recursion. We give a proof if all linear recursions are of length at most 2 (Theorem 3.1).

Section 7 is seemingly unrelated since it deals with matrices which are "periodic" along strips parallel to the diagonal. If such a matrix consists only of a finite number of such strips, then an appropriate sequence of determinants satisfies a linear recursion (Theorem 7.1).

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Section 8 is an application of section 7. It deals with matrices which are periodic on the diagonal and off-diagonal coefficients satisfy a different kind of Pascal-like relations.

We come now back to the Pascal triangle P with coefficients $p_{i,j} = \binom{i+j}{i}$. Denote by $P(n)$ the $n \times n$ minor defined by the first n rows and columns of P. One has then the following well-known result.

Proposition 1.1. *(i) The matrix $P(n)$ has determinant* 1 *for all $n \geq 0$.*

First proof. Subtract row number $n-2$ (rows and columns of $P(n)$ are indexed from 0 to $n-1$) from row number $n-1$. Subtract then row number $n-3$ from row number $n-2$ and keep on going until subtracting row number 0 from row number 1. Do the same operations on columns. The result is essentially $P(n-1)$ which implies the result by induction on n. QED

Second proof. The matrix T with coefficients $t_{i,j} = \binom{i}{j}$, $0 \leq i, j < n$ is an inferior triangular matrix of determinant 1. The classical and easy identity

$$\sum_{j=0} \binom{i_1}{j}\binom{i_2}{j} = \sum_{j=0} \binom{i_1}{j}\binom{i_2}{i_2 - j} = \binom{i_1 + i_2}{i_2}$$

implies then $T\, T^t = P(n)$. QED

Proposition 1.1 has the following easy generalization. Let $Q(x,y) \in \mathbf{Q}[x,y]$ be a polynomial in two variables x, y and let $P_Q(n)$ be the matrix with coefficients $p_{i,j} = \binom{i+j}{i} + Q(i,j)$, $0 \leq i, j < n$. Write the polynomial $Q(x,y)$ in the form

$$Q(x,y) = \sum_{s,t=0} c_{s,t} \binom{x}{s}\binom{y}{t}$$

where $\binom{x}{s} = \frac{1}{s!}x(x-1)\cdots(x-s+1)$. Let $C_Q(n)$ be the square matrix of order n with coefficients $c_{i,j}$ for $0 \leq i, j < n$.

Elementary operations on rows and columns similar to those in the first proof above show the following result.

Proposition 1.2. *One has for all n*

$$det(P_Q(n)) = det(C_Q(n) + Id_n)$$

where Id_n denotes the identity matrix of order n.

In particular, the sequence of determinants

$$det(P_Q(0)) = 1, det(P_Q(1)) = 1 + c_{0,0}, det(P_Q(2)), \ldots$$

becomes constant for $n \geq \mu$ where $\mu = min(degree_x(Q), degree_y(Q))$ with $degree_x(Q)$ (respectively $degree_x(Q)$) denoting the degree of Q with respect to x (respectively y).

What about other minors of the infinite matrix P? Denote by $P_{s,t}(n)$ the $n \times n$ minor of P with coefficients $\binom{i+j+s+t}{i+s}$, $0 \leq i, j < n$ and denote by $D_{s,t}(n) = \det(P_{s,t}(n))$ its determinant.

Theorem 1.3. *We have*

$$D_{s,t}(n)=\prod_{k=0}^{s-1}\frac{\binom{n+k+t}{t}}{\binom{k+t}{t}}\quad ,s,t,n\geq 0\ .$$

This Theorem follows for instance from the formulas contained in section 5 of **[GV]** (a beautiful paper studying mainly determinants of finite minors of the matrix T with coefficients $t_{i,j}=\binom{j}{i}$). We give briefly a different proof.

Proof of Theorem 1.3. Proposition 1.1 and elementary row-operations establish the Theorem for $s=0$ and all t. Multiplying the matrix $P_{s,t}(n)$ at the left with the diagonal matrix having entries $s!,(s+1)!,\dots,(s+n-1)!$ and at the right by a diagonal matrix with entries $t!,(t+1)!,\dots,(t+n-1)!$ yields a matrix with entries $(i+s+j+t)!,\ 0\leq i,j<n$ (this matrix has determinant $\prod_{i=0}^{n-1}i!\ (i+s+t)!$ as can fairly easily be checked) which implies the equality

$$(s+n)!\ t!\ D_{s+1,t}(n)=s!\ (t+n)!\ D_{s,t+1}(n)\ .$$

A small computation shows that the formula of Theorem 1.3 for $D_{s,t}(n)$ satisfies the same identity which proves the result. QED

Remarks 1.4. (i) For $s,t\geq 1$, let $\tilde{D}_{s,t}(n)$ be the determinant of the matrix with entries $a_{i,j}=\binom{i+j+s+t}{i+s}-1,\ 0\leq i,j<n$. The function $n\longmapsto \tilde{D}_{s,t}(n),\ n\geq 1$ seems then to be polynomial of degree $st-1$.

(ii) (Cf. Proposition 1.2.) More generally, consider a matrix $A(n)$ having coefficients $a_{i,j}=\binom{i+j+s+t,}{i+s}+Q(i,j)$ with $Q(x,y)\in\mathbf{C}[x,y]$ a polynomial. Then the function

$$n\longmapsto\det(A(n))$$

seems to be polynomial of degree $\leq st$ for n big enough.

(iii) Consider the symmetric matrix G of order k with coefficients $g_{i,j}=\sum_{s=0}^{n+k-1}\binom{s}{i}\binom{s}{j}$ for $0\leq i,j<k$. Theorem 1.3 implies $\det(G)=D_{k,k}(n)$.

Let us now consider the following variation of the Pascal triangle. Recall that a complex matrix of rank 1 and order $n\times n$ has coefficients $\alpha_i\beta_j$ (for $0\leq i,j<n$) where $\alpha=(\alpha_0,\dots,\alpha_{n-1})$ and $\beta=(\beta_0,\dots,\beta_{n-1})$ are two complex sequences, well defined up to $\lambda\alpha,\frac{1}{\lambda}\beta$ for $\lambda\in\mathbf{C}^*$.

Given two infinite sequences $\alpha=(\alpha_0,\alpha_1,\dots)$ and $\beta=(\beta_0,\beta_1,\dots)$ consider the $n\times n$ matrix $A(n)$ with coefficients $a_{i,j}=a_{i-1,j}+a_{i,j-1}+\alpha_i\beta_j$ for $0\leq i,j<n$ (where we use the convention $a_{i,-1}=a_{-1,i}=0$ for all i).

Proposition 1.5. *(i) The coefficient $a_{i,j}$ (for $0\leq i,j<n$) of the matrix $A(n)$ is given by*

$$a_{i,j}=\sum_{s=0}^{i}\sum_{t=0}^{j}\alpha_{i-s}\beta_{j-t}\binom{s+t}{s}\ .$$

(ii) The matrix $A(n)$ has determinant $(\alpha_0\beta_0)^n$.

Proof. Assertion (i) is elementary and left to the reader.

Assertion (ii) obviously holds if $\alpha_0 = 0$ or $\beta_0 = 0$. We can hence suppose $\beta_0 = 1$. Proposition 1.1 and elementary operations on rows establish the result easily for arbitrary α and $\beta = (1, 0, 0, 0, \ldots)$. The case of an arbitrary sequence β with $\beta_0 = 1$ is then reduced to the previous case using elementary operations on columns. QED

Determinants of matrices with coefficients among $\binom{a}{b}^{-1}$ seem also to have interesting properties: Given three integers $s, t, n \geq 0$ let $d_{s,t}(n)$ denote the determinant of the $n \times n$ matrix M with coefficients

$$m_{i,j} = \binom{i+s+j+t}{i+s}^{-1} \quad \text{for } 0 \leq i, j < n \ .$$

Conjecture 1.6. *One has*

$$d_{s,t}(n) = (-1)^{\binom{n}{2}} \frac{1}{\prod_{k=0}^{n-1} \binom{2k+s+t}{k+s} \binom{2k-1+s+t}{k}} \ .$$

Remark 1.7. For $0 \leq k \in \mathbf{N}$ introduce the symmetric matrix $A_k(n)$ of order n with coefficients $a_{i,j} = \frac{1}{(i+j+k)!}$, $0 \leq i, j < n$. A small computation shows then that conjecture 1.6 is equivalent to

$$\det(A_k(n)) = (-1)^{\binom{n}{2}} \prod_{i=0}^{n-1} \frac{i!}{(n+k+i-1)!}$$

(with $k = s + t$).

Another variation on theme of Pascal triangles is given by onsidering the $n \times n$ matrix $A(n)$ with coefficients $a_{i,0} = \rho^i$, $a_{0,i} = \sigma^i$, $0 \leq i < n$ and $a_{i,j} = a_{i-1,j} + a_{i,j-1} + x \ a_{i-1,j-1}$, $1 \leq i, j < n$. Setting $x = 0$, $\rho = \sigma = 1$ we get hence the matrix defined by binomial coefficients considered above. Cf. Problem ?? in Am. Math. Monthly, ??? for the case $\rho = \sigma = 1$.

Conjecture 1.8. *One has*

$$det(A(n)) = (1+x)^{\binom{n-1}{2}} \ (x + \rho + \sigma - \rho\sigma)^{n-1} \ .$$

Let now $A(n)$ be the symplectic (antisymmetric) $n \times n$ matrix defined by $a_{i,i} = 0$, $0 \leq i < n$, $a_{i,0} = -a_{0,i} = \rho^{i-1}$, $1 \leq i < n$, $a_{i,j} = a_{i-1,j} + a_{i,j-1} + x \ a_{i-1,j-1}$, $1 \leq i, j < n$. One seems to have

$$\det(A(2n)) = (1+x)^{2(n-1)^2} \ (\rho + x)^{2n-2} \ .$$

Finally, consider the symplectic $n \times n$ matrix $\tilde{A}(n)$ defined by $\tilde{a}_{i,0} = -\tilde{a}_{0,i} = i$, $0 \leq i < n$, $\tilde{a}_{i,j} = \tilde{a}_{i-1,j} + \tilde{a}_{i,j-1} + x \ \tilde{a}_{i-1,j-1}$, $1 \leq i, j < n$. One seems then to have

$$\det(\tilde{A}(2n)) = (1+x)^{4\binom{n}{2}} \ .$$

Let us close this introduction by mentioning a somewhat trivial generalisation of the Pascal triangle: Given a power series $F(z)=\sum_{k=0}^{\infty} s_k z^k$ consider the $n\times n$ matrix $A(n)$ with coefficient

$$a_{i,j} = \text{coefficient of } z^j \text{ in } (F(z))^{i+1}\ ,\ 0\le i,j<n\ .$$

It is straightforward to check that $F(z)=\frac{1}{1-z}=\sum_{k=0}^{\infty} z^k$ yields the Pascal triangle.

Similarly, considering the matrix $\tilde A(n)$ defined by

$$\tilde a_{i,j} = \text{coefficient of } z^j \text{ in } (F(z))^{i}\ ,\ 0\le i,j<n\ .$$

Theorem 1.9. *Given $F(z)=\sum_{k=0}^{\infty} s_k z^k$ the matrices $A(n)=A_F(n)$ and $\tilde A(n)$ defined as above have determinants*

$$det(A(n)) = s_0^n\ s_1^{\binom{n}{2}} \quad \text{and} \quad det(\tilde A(n)) = s_1^{\binom{n}{2}}\ .$$

Sketch of proof. The formula for $\det(A(n))$ is easily seen to hold if $s_0 s_1=0$ for $F(z)=\sum_{k=0}^{\infty} s_k z^k$. Otherwise, the coefficients $a_{i,j}$ of the $j-$th column are polynomial in i with leading coefficient $\binom{n}{j} s_0^{n-j} s_1^j$. This shows that $\det(A(n))$ depends only on s_0 and s_1. Proposition 1.1 shows that $A(n)$ has determinant 1 if $s_0=s_1=1$. The general case follows now easily.

The proof for $\tilde A(n)$ is similar. QED

2 Generalized Pascal triangles

Let $\alpha=(\alpha_0,\alpha_1,\ldots)$ and $\beta=(\beta_0,\beta_1,\ldots)$ be two sequences starting with a common first term $\gamma_0=\alpha_0=\beta_0$. Define a matrix $P_{\alpha,\beta}(n)$ of order n with coefficients $p_{i,j}$ by setting $p_{i,0}=\alpha_i,\ p_{0,i}=\beta_i$ for $0\le i<n$ and $p_{i,j}=p_{i-1,j}+p_{i,j-1}$ for $1\le i,j<n$.

It is easy to see that the coefficient $p_{i,j}$ of $P_{\alpha,\beta}(n)$ is also given by the formula

$$p_{i,j}=\gamma_0\binom{i+j}{i}+\left(\sum_{s=1}^{i}(\alpha_s-\alpha_{s-1})\binom{i-s+j}{j}\right)+\left(\sum_{t=1}^{j}(\beta_t-\beta_{t-1})\binom{i+j-t}{i}\right)\ .$$

We call the infinite "matrix" $P_{\alpha,\beta}(\infty)$ the *generalized Pascal triangle* associated to α,β.

We will mainly be interested in the sequence of determinants

$$(\det(P_{\alpha,\beta}(1))=\gamma_0, \det(P_{\alpha,\beta}(2))=\gamma_0(\alpha_1+\beta_1)-\alpha_1\beta_1,\ldots,\det(P_{\alpha,\beta}(n)),\ldots)\ .$$

Example 2.1. Take an arbitrary sequence $\alpha=(\alpha_0,\alpha_1,\ldots)$ and let β be the constant sequence $\beta=(\alpha_0,\alpha_0,\alpha_0,\ldots)$. Proposition 1.5 implies

$\det(A_{(\alpha_0,\alpha_1,\ldots),(\alpha_0,\alpha_0,\ldots)}(n)) = \alpha_0^n$ (using perhaps the convention $0^0 = 1$). This yields an easy way of writing down matrices with determinant 1 by choosing a sequence $\alpha = (\alpha_0 = 1, \alpha_1, \ldots)$. The finite sequence $\alpha = (1, -2, 5, 11)$ for instance yields the determinant 1 matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \\ 5 & 4 & 4 & 5 \\ 11 & 15 & 19 & 24 \end{pmatrix} .$$

3 Linear recursions

This section is devoted to general Pascal triangles constructed from sequences satisfying linear recursions. Conjecturally, the sequence of determinants of such matrices satisfies then again a (generally much longer) linear recursion. We prove this in the particular case where the defining sequences are of order at most 2.

Definition. A sequence $\sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots)$ satisfies a *linear recursion of order d* if there exist constants $D_1, D_2, \ldots, D_d$ such that

$$\sigma_n = \sum_{i=1}^{d} D_i \ \sigma_{n-i} \quad \text{for all } n \geq d \ .$$

The polynomial

$$z^d - \sum_{i=1}^{d} D_i \ z^{d-i}$$

is then called the *characteristic polynomial* of the linear recursion.

Let us first consider generalized Pascal triangles defined by linear recursion sequences of order at most 2:

Given $\gamma_0, \alpha_1, \beta_1, A_1, A_2, B_1, B_2$ we set $\alpha_0 = \beta_0 = \gamma_0$ and consider the square matrix $M(n)$ of order n with entries

$$\begin{aligned} m_{i,0} &= \alpha_i, \ 0 \leq i < n \text{ where } \alpha_k = A_1\alpha_{k-1} + A_2\alpha_{k-2}, \ k \geq 2, \\ m_{0,i} &= \beta_i, \ 0 \leq i < n \text{ where } \beta_k = B_1\beta_{k-1} + B_2\beta_{k-2}, \ k \geq 2, \\ m_{i,j} &= m_{i-1,j} + m_{i,j-1}, \ 1 \leq i, j < n \ . \end{aligned}$$

The matrix $M(3)$ for instance is hence given by

$$M(3) = \begin{pmatrix} \gamma_0 & \beta_1 & B_1\beta_1 + B_2\gamma_0 \\ \alpha_1 & \alpha_1 + \beta_1 & \alpha_1 + \beta_1 + B_1\beta_1 + B_2\gamma_0 \\ A_1\alpha_1 + A_2\gamma_0 & \alpha_1 + \beta_1 + A_1\alpha_1 + A_2\gamma_0 & m_{3,3} \end{pmatrix}$$

where $m_{3,3} = 2\alpha_1 + 2\beta_1 + A_1\alpha_1 + B_1\beta_1 + A_2\gamma_0 + B_2\gamma_0$.

We have hence $M(n) = P_{\alpha,\beta}(n)$ where $P_{\alpha,\beta}$ is the generalized Pascal triangle introduced in the previous section.

We set $d(n) = \det(M(n))$ for $n \geq 1$ and introduce the constants

$$D_1 = -(A_1\beta_1 + B_1\alpha_1 - 2(\alpha_1 + \beta_1) + \gamma_0\,(A_1B_2 + A_2B_1 - (A_2 + B_2) + A_2B_2))$$
$$D_2 = -\,(A_2\gamma_0 + \alpha_1 + (1 - A_1 - A_2)\beta_1)\,(B_2\gamma_0 + \beta_1 + (1 - B_1 - B_2)\alpha_1)\ .$$

Theorem 3.1. *The sequence $d(n),\ n \geq 1$ defined as above satisfies the following equalities*

$$\begin{aligned} d(1) &= \gamma_0\ , \\ d(2) &= \gamma_0(\alpha_1 + \beta_1) - \alpha_1\beta_1\ , \\ d(n) &= D_1\ d(n-1) + D_2\ d(n-2)\ \textit{for all}\ n \geq 3\ . \end{aligned}$$

Theorem 3.1 will be proven below.

Example 3.2 (a) The sequence $(\det(P_{\alpha,\beta}(n)))_{n=1,2,\ldots}$ of determinants associated to two geometric sequences

$$\begin{aligned} \alpha &= (1, A, A^2, \ldots, \alpha_k = A^k, \ldots) \\ \beta &= (1, B, B^2, \ldots, \beta_k = B^k, \ldots) \end{aligned}$$

is given by

$$\det(P_{\alpha,\beta}(n)) = (A + B - AB)^{n-1}\ .$$

Let $\alpha = (\alpha_0, \alpha_1, \ldots)$ and $\beta = (\beta_0, \beta_1, \ldots)$ be two sequences satisfying $\alpha_0 = \beta_0 = \gamma_0$ and linear recursions

$$\begin{aligned} \alpha_n &= \textstyle\sum_{i=1}^{a} A_i\alpha_{n-i}\ \text{for}\ n \geq a\ , \\ \beta_n &= \textstyle\sum_{i=1}^{b} B_i\beta_{n-i}\ \text{for}\ n \geq b \end{aligned}$$

of order a and b.

Theorem 3.1 and computations suggest that the following might be true.

Conjecture 3.3. *If two sequences $\alpha = (\alpha_0, \alpha_1, \ldots), \beta = (\beta_0, \beta_1, \ldots)$ satisfy both linear recurrence relations then there exist a natural integer $d \in \mathbf{N}$ and constants $D_1, \ldots, D_d$ (depending on α, β) such that*

$$\det(P_{\alpha,\beta}(n)) = \sum_{i=1}^{d} D_i\ \det(P_{\alpha,\beta}(n-i)) \quad \textit{for all}\ n > d\ .$$

Remarks 3.4. (i) Generically, (ie. for α and β two generic sequences of order a and b such that $\alpha_0 = \beta_0$) the integer d of Conjecture 3.3 seems to be given by $d = \binom{a+b-2}{a-1}$.

(ii) Generically, the coefficient D_i seems to be a homogeneous form (with polynomial coefficients in $A_1, \ldots, A_a, B_1, B_b$) of degree i in $\gamma_0, \alpha_1, \alpha_{a-1}, \beta_1, \ldots \beta_{b-1}$. For non-generic pairs of sequences (try $\beta = -\alpha$ with $\alpha = (0, \alpha_1, \ldots)$ satisfying a linear recursion of order 3) the coefficients D_i may be rational fractions in the variables.

(iii) If $a = b > 1$ and the recursive sequences α, β are generic, then the coefficients $D_0 = -1, D_1, \ldots, D_d$ of the linear recursion in Conjecture 3.3 seem to have the symmetry

$$D_{d-i} = q^{(d-2i)/2} D_i$$

where q is a quadratic form in $\gamma_0, \alpha_i, \beta_i$ factorizing into a product of two linear forms which are symmetric under the exchange of parameters α_i with β_i and A_i with B_i (this corresponds to transposing $P_{\alpha,\beta}$).

Theorem 3.1 shows that the generic quadratic form q_2 working for $a = b = 2$ is given by

$$q_2 = (A_2\gamma_0 + \alpha_1 + (1 - A_1 - A_2)\beta_1)\,(B_2\gamma_0 + \beta_1 + (1 - B_1 - B_2)\alpha_1)\ .$$

The generic quadratic form q_3 working for $a = b = 3$ seems to be

$$\begin{aligned} q_3 \;=\; & (A_3\gamma_0 + \alpha_1 + \alpha_2 + (1 - A_1 - A_2 - A_3)\beta_1) \\ & \quad (B_3\gamma_0 + \beta_1 + \beta_2 + (1 - B_1 - B_2 - B_3)\alpha_1)\ . \end{aligned}$$

Example 3.5. Consider the 3−periodic sequence $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_k = \alpha_{k-3}, \ldots)$. The sequence $d(n) = \det(P_{\alpha,\alpha}(n))$ seems then to satisfy the recursion relation

$$\begin{aligned} d(n) &= D_1\ d(n-1) + D_2\ d(n-2) - (\alpha_0 + \alpha_1 + \alpha_2)D_2\ d(n-3) \\ &\quad -(\alpha_0 + \alpha_1 + \alpha_2)^3 D_1\ d(n-4) + (\alpha_0 + \alpha_1 + \alpha_2)^5\ d(n-5) \end{aligned}$$

where

$$\begin{aligned} D_1 &= 11\alpha_1 + 5\alpha_2 \\ D_2 &= -(3\alpha_0^2 + 37\alpha_1^2 + 3\alpha_2^2 + 15\alpha_0\alpha_1 + 5\alpha_0\alpha_2 + 24\alpha_1\alpha_2)\ . \end{aligned}$$

In the general case

$$\begin{aligned} \alpha &= (\alpha_0 = \gamma_0, \alpha_1, \alpha_2, \ldots, \alpha_k = \alpha_{k-3}, \ldots) \\ \beta &= (\beta_0 = \gamma_0, \beta_1, \beta_2, \ldots, \beta_k = \beta_{k-3}, \ldots) \end{aligned}$$

of two 3− periodic sequences (starting with a common value γ_0) one seems to have

$$d(n) = D_1 d(n-1) + D_2 d(n-2) + D_3 d(n-3) + q D_2 d(n-4) + q^2 D_1 d(n-5) - q^3 d(n-6)$$

where

$$q = (\gamma_0 + \alpha_1 + \alpha_2)(\gamma_0 + \beta_1 + \beta_2)$$

$$D_1 = \gamma_0 + 6(\alpha_1 + \beta_1) + 3(\alpha_2 + \beta_2)$$

$$\begin{aligned} D_2 = -(&3\gamma_0^2 + 12(\alpha_1^2 + \beta_1^2) + 13\gamma_0(\alpha_1 + \beta_1) + 5\gamma_0(\alpha_2 + \beta_2) \\ &+9(\alpha_1\alpha_2 + \beta_1\beta_2) + 11(\alpha_1\beta_2 + \alpha_2\beta_1) + 24\alpha_1\beta_1 + 8\alpha_2\beta_2) \end{aligned}$$

$$
\begin{aligned}
D_3 = {} & 6\gamma_0^3+\gamma_0^2(18(\alpha_1+\beta_1)+8(\alpha_2+\beta_2))+\gamma_0(25(\alpha_1^2+\beta_1^2)+3(\alpha_2^2+\beta_2^2)\\
& +18(\alpha_1\alpha_2+\beta_1\beta_2)+54\alpha_1\beta_1+26(\alpha_1\beta_2+\alpha_2\beta_1)+10\alpha_2\beta_2)\\
& +9(\alpha_1^3+\beta_1^3)+9(\alpha_1^2\alpha_2+\beta_1^2\beta_2)+28(\alpha_1^2\beta_1+\alpha_1\beta_1^2)+22(\alpha_1^2\beta_2+\alpha_2\beta_1^2)\\
& +3(\alpha_1\beta_2^2+\alpha_2^2\beta_1)+3(\alpha_2^2\beta_2+\alpha_2\beta_2^2)+30(\alpha_1\alpha_2\beta_1+\alpha_1\beta_1\beta_2)+24(\alpha_1\alpha_2\beta_2+\alpha_2\beta_1\beta_2)
\end{aligned}
$$

Let us briefly explain how Conjecture 3.3 can be tested on a given pair α,β of linear recurrence sequences.

First Step. Guess d.

Second step. Compute at least $2d+1$ terms of the sequence

$$w_1=\det(P_{\alpha,\beta}(1)), w_2=\det(P_{\alpha,\beta}(2)),\dots\ .$$

Third step. Check that the so-called *Hankel matrix*

$$H_{d+1}(w)=\begin{pmatrix} w_1 & w_2 & w_3 & \dots & w_{d+1}\\ w_2 & w_3 & w_4 & \dots & w_{d+2}\\ \vdots & & & & \\ w_{d+1} & w_{d+2} & w_{d+3} & \dots & w_{2d+1}\end{pmatrix}$$

of order $d+1$ has zero determinant (otherwise try again with a higher value for d) and choose a vector of the form

$$L=(D_d,D_{d-1},D_{d-2},\dots,D_2,D_1,-1)$$

in its kernel. One has then by definition

$$\det(P_{\alpha,\beta}(n))=\sum_{i=1}^{d} D_i\ \det(P_{\alpha,\beta}(n-i))$$

for $d+1\le n\le 2d+1$.

Finally, check (perhaps) the above recursion for a few more values of $n>2d+1$.

3.1 Proof of Theorem 3.1.

The assertions concerning $d(1)$ and $d(2)$ are obvious. One checks (using for instance a symbolic computation program on a computer) that the recursion relation holds for $d(3)$, $d(4)$ and $d(5)$.

Introduce now the lower and upper triangular square matrices

$$T_A=\begin{pmatrix} 1 & 0 & 0 & 0 & \dots\\ -A_1 & 1 & 0 & 0 & \dots\\ -A_2 & -A_1 & 1 & 0 & \dots\\ 0 & -A_2 & -A_1 & 1 & \dots\\ \vdots & & \ddots & & \end{pmatrix}$$

$$T_B = \begin{pmatrix} 1 & -B_1 & -B_2 & 0 & 0 & \dots \\ 0 & 1 & -B_1 & -B_2 & 0 & \dots \\ 0 & 0 & 1 & -B_1 & -B_2 & \dots \\ & \vdots & & \ddots & & \end{pmatrix}$$

of order n and set $\tilde{M} = T_A \; M \; T_B$. The entries $\tilde{m}_{i,j},\ 0 \leq i, j < n$ of $\tilde{M}$ satisfy $\tilde{m}_{i,j} = \tilde{m}_{i-1,j} + \tilde{m}_{i,j-1},\ (i,j) \neq (2,2)$ for $2 \leq i, j < n$. One has

$$\tilde{M} = \begin{pmatrix} \gamma_0 & \beta_1 - B_1\gamma_0 & 0 & 0 & 0 & \dots \\ \alpha_1 - A_1\gamma_0 & \tilde{m}_{1,1} & \tilde{m}_{1,2} & \tilde{m}_{1,3} & \tilde{m}_{1,4} & \dots \\ 0 & \tilde{m}_{2,1} & \tilde{m}_{2,2} & \tilde{m}_{2,3} & \tilde{m}_{2,4} & \dots \\ \vdots & & \vdots & & & \end{pmatrix}$$

where

$$\begin{aligned} &\tilde{m}_{1,1} = \alpha_1 + \beta_1 - A_1\beta_1 - B_1\alpha_1 + A_1B_1\gamma_0 \\ &\tilde{m}_{1,2} = \tilde{m}_{1,3} = \tilde{m}_{1,4} = \ldots = (1 - B_1 - B_2)\alpha_1 + \beta_1 + B_2\gamma_0 \\ &\tilde{m}_{2,1} = \tilde{m}_{3,1} = \tilde{m}_{4,1} = \ldots = (1 - A_1 - A_2)\beta_1 + \alpha_1 + A_2\gamma_0 \\ &\tilde{m}_{2,2} = (2 - B_1)\alpha_1 + (2 - A_1)\beta_1 + (A_2 + B_2 - A_1B_2 - A_2B_1 - A_2B_2)\gamma_0 \end{aligned}$$

Developping the determinant $d(n) = \det(\tilde{M})$ along the second row of $\tilde{M}$ one obtains

$$d(n) = (\gamma_0(\alpha_1 + \beta_1) - \alpha_1\beta_1)\overline{d}(n-2) + \gamma_0 \det(P(n-1))$$

where $\overline{d}(n-2) = \det(\overline{M}(n-2))$ with coefficients $\overline{m}_{i,j} = \tilde{m}_{i+2,j+2}$ for $0 \leq i, j < n-2$ (ie. $\overline{M}(n-2)$ is the principal minor of $\tilde{M}$ obtained by erasing the first two rows and columns of $\tilde{M}$) and where $P(n-1)$ is the square matrix of order $(n-1)$ with entries $p_{0,0} = 0$ and $p_{i,j} = \tilde{m}_{i+1,j+1}$ for $0 \leq i, j < n-1, (i,j) \neq (0,0)$.

The matrix $\overline{M}(m)$ $(m \leq n-2)$ is a generalized Pascal triangle associated to the linear recursion sequences $\overline{\alpha} = (\overline{\alpha}_0, \overline{\alpha}_1, \ldots)$ and $\overline{\beta} = (\overline{\beta}_0, \overline{\beta}_1, \ldots)$ of order 2 defined by

$$\begin{aligned} &\overline{\alpha}_0 = \overline{\beta}_0 = (2 - B_1)\alpha_1 + (2 - A_1)\beta_1 + (A_2 + B_2 - A_1B_2 - A_2B_1 - A_2B_2)\gamma_0, \\ &\overline{\alpha}_1 = (3 - B_1)\alpha_1 + (3 - 2A_1 - A_2)\beta_1 + (2A_2 + B_2 - A_1B_2 - A_2B_1 - A_2B_2)\gamma_0 \\ &\overline{\beta}_1 = (3 - 2B_1 - B_2)\alpha_1 + (3 - A_1)\beta_1 + (A_2 + 2B_2 - A_1B_2 - A_2B_1 - A_2B_2)\gamma_0 \\ &\overline{\alpha}_n = 2\overline{\alpha}_{n-1} - \overline{\alpha}_{n-2} \quad \text{for } n \geq 2 \ , \\ &\overline{\beta}_n = 2\overline{\beta}_{n-1} - \overline{\beta}_{n-2} \quad \text{for } n \geq 2 \ . \end{aligned}$$

Induction on n and a computation (with $\overline{A}_1 = \overline{B}_1 = 2,\ \overline{A}_2 = \overline{B}_2 = -1$) shows the equality

$$\overline{d}(m) = D_1 \; \overline{d}(m-1) + D_2 \; \overline{d}(m-2)$$

for $3 \leq m < n$ where D_1 and D_2 are as in the Theorem.

Introducing the $(n-1)\times(n-1)$ lower triangular square matrix

$$T_P = \begin{pmatrix} 1 & 0 & 0 & \dots \\ -1 & 1 & 0 & \dots \\ 0 & -1 & 1 & \dots \\ & \vdots & & \ddots \end{pmatrix}$$

we get $\tilde{P} = T_P\ P(n-1)\ T_P^t$ with coefficients $\tilde{p}_{i,j},\ 0 \le i, j < n-1$ given by

$$\begin{aligned}
&\tilde{p}_{0,0} = \tilde{p}_{i,0} = \tilde{p}_{0,i} = 0 \text{ for } 2 \le i < n-1, \\
&\tilde{p}_{0,1} = (1 - B_1 - B_2)\alpha_1 + \beta_1 + B_2\gamma_0 \\
&\tilde{p}_{1,0} = (1 - A_1 - A_2)\beta_1 + \alpha_1 + A_2\gamma_0 \\
&\tilde{p}_{1,i} = \tilde{p}_{0,1} \text{ for } 2 \le i < n-1, \\
&\tilde{p}_{i,1} = \tilde{p}_{1,0} \text{ for } 2 \le i < n-1, \\
&\tilde{p}_{2,2} = (2 - B_1)\alpha_1 + (2 - A_1)\beta_1 + (A_2 + B_2 - A_1B_2 - A_2B_1 - A_2B_2)\gamma_0 \\
&\tilde{p}_{i,j} = \tilde{p}_{i-1,j} + \tilde{p}_{i,j-1}\ , 2 \le i, j < n-1,\ (i,j) \ne (2,2)\ .
\end{aligned}$$

Let $\overline{P}(n-3)$ denote the square matrix of order $(n-3)$ with coefficients $\overline{p}_{i,j} = \tilde{p}_{i+2,j+2}\ , 0 \le i, j < n-3$ (ie. $\overline{P}(n-3)$ is obtained by erasing the first two rows and columns of $\tilde{P}(n-1)$). One checks the equality

$$\overline{P}(n-3) = \overline{M}(n-3)$$

where $\overline{M}(n-3)$ is defined as above. This implies the identity

$$\begin{aligned}
d(n) &= (\gamma_0(\alpha_1 + \beta_1) - \alpha_1\beta_1)\overline{d}(n-2) \\
&\quad -\gamma_0((1 - B_1 - B_2)\alpha_1 + \beta_1 + B_2\gamma_0)((1 - A_1 - A_2)\beta_1 + \alpha_1 + A_2\gamma_0)\overline{d}(n-3)\ .
\end{aligned}$$

Using the recursion relation $\overline{d}(m) = D_1\ \overline{d}(m-1) + D_2\ \overline{d}(m-2)$ (which holds by induction for $3 \le m < n$) we can hence express $d(n)$ as a linear function (with polynomial coefficients in $\gamma_0, \alpha_1, \beta_1, A_1, A_2, B_1, B_2$) of $\overline{d}(m-4)$ and $\overline{d}(m-5)$.

Comparison of this with the linear expression in $\overline{d}(m-4)$ and $\overline{d}(m-5)$ obtained similarly from $D_1\ d(n-1) + D_2\ d(n-2)$ finishes the proof.

4 Symmetric matrices

Take an arbitrary sequence $\alpha = (\alpha_0, \alpha_1, \dots)$. The generalized Pascal triangle associated to the pair of identical sequences α, α is the *generalized symmetric Pascal triangle* associated to α and yields symmetric matrices $P_{\alpha,\alpha}(n)$ by considering principal minors consisting of the first n rows and columns of $P_{\alpha,\alpha}$.

The main example is of course the classical Pascal triangle obtained from the constant sequence $\alpha = (1, 1, 1, \dots)$. Other sequences satisfying linear recursions like for instance the Fibonacci sequence

$$(0, 1, 1, 2, 3, 5, 8, \dots)$$

and shifts of it yield also nice examples.

Conjecture 3.3 should of course also hold for symmetric matrices obtained by considering the generalised symmetric Pascal triangle associated to a sequence satisfying a linear recurrence relation.

The generic order $d_s(a)$ (where a denotes the order of the defining linear sequence) of the linear recursion satisfied by $\det(P_{\alpha,\alpha}(n))$ seems however usually to be smaller than in the generic non-symmetric case. Examples yield the following first values

$a =$	1	2	3	4	5	6
$d_s(a) =$	1	2	5	14	41	122

and suggest that perhaps $d_a = (3^{a-1} + 1)/2$.

The coefficients D_i seem still to be polynomial in α_i and A_i.

The symmetry relation has also an analogue (in the generic case) which is moreover somewhat simpler in the sense that it is given by a linear form ρ (in $\alpha_0, \ldots, \alpha_{a-1}$) and we seem to have

$$D_{d_s - i} = \rho^{d_s - 2i}\ D_i$$

(where $D_0 = -1$).

Example 4.1. If a sequence

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_k = A_1\alpha_{k-1} + A_2\alpha_{k-2} + A_3\alpha_{k-3}, \ldots)$$

satisfies a linear recursion relation of order 3, then the sequence $d(n) = \det(P_{\alpha,\alpha}(n))$ (the matrix $P_{\alpha,\alpha}(n)$ has coefficients $p_{0,i} = p_{i,0} = \alpha_i,\ 0 \leq i < n$ and $p_{i,j} = p_{i-1,j} + p_{i,j-1}$ for $1 \leq i, j < n$) of the associated determinants seems to satisfy

$$\begin{aligned}
&d(1) = \alpha_0\ ,\\
&d(2) = 2\alpha_0\alpha_1 - \alpha_1^2\ ,\\
&d(3) = (2\alpha_1 - \alpha_2)\left(\alpha_0(2\alpha_1 + \alpha_2) - 2\alpha_1^2)\right)\ ,\\
&d(n) = D_1\ d(n-1) + D_2\ d(n-2) + \rho D_2\ d(n-3) + \rho^3\ D_1\ d(n-4) - \rho^5\ d(n-5)
\end{aligned}$$

where

$$\begin{aligned}
&\rho = -A_3\alpha_0 + (-2 + 2A_1 + A_2 + A_3)\alpha_1 - \alpha_2\ ,\\
&D_1 = A_3(1 - 2A_1 - 2A_2 - A_3)\alpha_0 + (10 - 10A_1 - A_2 + A_3 + 4A_1^2 + 2A_1A_2)\alpha_1\\
&\qquad + (5 - 4A_1 - 2A_2)\alpha_2\ ,\\
&D_2 = c_{0,0}\alpha_0^2 + c_{1,1}\alpha_1^2 + c_{2,2}\alpha_2^2 + c_{0,1}\alpha_0\alpha_1 + c_{0,2}\alpha_0\alpha_2 + c_{1,2}\alpha_1\alpha_2
\end{aligned}$$

with

$$
\begin{aligned}
c_{0,0} &= -A_3^2(2-2A_1+2A_2+A_3+A_1^2)\ ,\\
c_{1,1} &= -40+80A_1+16A_2+4A_3\\
&\quad -64A_1^2-2A_2^2-A_3^2-28A_1A_2-20A_1A_3-2A_2A_3\\
&\quad +2A_1(2A_1+A_2+A_3)(6A_1+A_2+A_3)-A_1^2(2A_1+A_2+A_3)^2\ ,\\
c_{2,2} &= -10+12A_1+6A_2+8A_3-(2A_1+A_2+A_3)^2\ ,\\
c_{0,1} &= -A_3(16-28A_1+16A_1^2-2A_2^2-A_3^2+2A_1A_3-3A_2A_3\\
&\quad -2A_1^2(2A_1+A_2+A_3))\ ,\\
c_{0,2} &= -A_3(8-10A_1-3A_3+2A_1(2A_1+A_2+A_3))\ ,\\
c_{1,2} &= 2\left(-20+32A_1+10A_2+9A_3-18A_1^2-A_2^2-A_3^2\right.\\
&\quad \left.-11A_1A_2-12A_1A_3-2A_2A_3+A_1(2A_1+A_2+A_3)^2\right)
\end{aligned}
$$

We conclude this section by mentionning the following more exotic example:

Example 4.2. (Central binomial coefficients) Consider the sequence

$$\alpha=(1,2,6,20,70,252,924,3432,12870,48620,184756,\dots,\alpha_k=\binom{2k}{k},\dots)$$

of central binomial coefficients. For $1\le n\le 36$ the values of $\det(P_{\alpha,\alpha}(n))$ are zero except if $n\equiv 1,3 \pmod 6$ and for $n\equiv 1,3 \pmod 6$ the values of $\det(P_{\alpha,\alpha}(n))$ have the following intriguing factorisations:

n :	$\det(P_{\alpha,\alpha}(n))$	n :	$\det(P_{\alpha,\alpha}(n))$
1 :	1	3 :	-2^2
7 :	-2^6	9 :	$2^8\ 3^2$
13 :	$2^{16}\ 3^6$	15 :	$-2^{26}\ 3^6$
19 :	$-2^{30}\ 3^2\ 103^4$	21 :	$2^{24}\ 103^4\ 4229^2$
25 :	$2^{24}\ 3^4\ 31^4\ 431^4\ 4229^2$	27 :	$-2^{26}\ 3^6\ 31^4\ 59^2\ 431^4\ 11701^2$
31 :	$-2^{30}\ 3^{10}\ 59^2\ 11701^2\ p^4$	33 :	$2^{32}\ 3^{12}\ 11^2\ 2017^2\ 28349^2\ p^4$

where $p=4893589$.

The matrix $P_{\alpha,\alpha}(n)$ seems to have rank n if $n\equiv 1,3 \pmod 6$, rank $n-1$ if $n\equiv 0 \pmod 2$ and rank $n-2$ if $n\equiv 5 \pmod 6$.

5 Symplectic matrices

Given an arbitrary sequence $\alpha=(\alpha_0,\alpha_1,\dots)$ with $\alpha_0=0$, the matrices $P_{\alpha,-\alpha}(n)$ are symplectic (antilinear).

Determinants of integral symplectic matrices are squares of integers and are zero in odd dimensions. We restrict hence ourself to even dimensions and consider sometimes also the (positive) square-roots of the determinants. Even if Conjecture 3.3 holds there is of course no reason that the square roots of the determinants satisfy a linear recursion.

The conjectural recurrence relation for symplectic matrices has the form

$$\det(P_{\alpha,-\alpha}(2n)) = \sum_{i=1}^{d(\alpha)} D_i \det(P_{\alpha,-\alpha}(2n-2i)) \ .$$

However the coefficients $D_1, \ldots, D_{d(\alpha)}$ seem no longer to be polynomial but rational for generic α. Moreover, the nice symmetry properties of the coefficients D_i present in the other cases seem to have disappeared too.

Proposition 5.1. *(i) The symplectic matrices $P_{\alpha,-\alpha}(2n)$ associated to the sequence $\alpha = (0,1,1,1,1,\ldots)$ have determinant 1 for every natural integer n.*

(ii) The symplectic matrices $P_{\alpha,-\alpha}(2n)$ associated to the sequence $\alpha = (0,1,2,3,4,5,\ldots)$ have determinant 1 for every natural integer n.

Both assertions follow of course from Theorem 3.1. We will however reprove them independently.

Proof. Consider the generalized Pascal triangle

$$P = P_{(1,1,1,\ldots,1,\ldots),(1,-1,-1,\ldots,-1,\ldots)}(\infty)$$

$$= \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & -1 & -1 & \ldots \\ 1 & 0 & -1 & -2 & -3 & -4 & -5 & \ldots \\ 1 & 1 & 0 & -2 & -5 & -9 & -14 & \ldots \\ 1 & 2 & 2 & 0 & -5 & -14 & -28 & \ldots \\ \vdots & & & & & & & \end{pmatrix} .$$

The matrices $P(m)$ given by retaining only the first m rows and columns of $P(\infty)$ are all of determinant 1 (compare the transposed matrix $P(m)^t$ with Example 2.1).

Expanding the determinant along the first row one gets

$$1 = \det(P(m))$$

$$= \det(P_{(0,1,1,1,\ldots),-(0,1,1,\ldots)}(m)) + \det(P_{(0,1,2,3,4,\ldots),-(0,1,2,3,\ldots)}(m-1)) \ .$$

The fact that symplectic matrices of odd order have zero determinant proves now assertion (i) for even m and assertion (ii) for odd m. QED

Remark 5.2. The coefficients $p_{i,j}$ of the infinite symplectic matrix

$$P_{(0,1,1,1,\ldots),-(0,1,1,1,\ldots)}(\infty)$$

have many interesting properties: One can for instance easily check that

$$p_{i,j} = \binom{i+j-1}{j} - \binom{i+j-1}{j-1}$$

(with the correct definition for $\binom{k}{-1}$ given by $\binom{-1}{-1} = \binom{-1}{0} = 1$ and $\binom{k}{-1} = 0$ for $k = 0,1,2,3,\ldots$). These numbers are closely related to the so-called Temperley-Lieb algebras (see for instance **[GHJ]**).

There are other matrices constructed using the numbers $\binom{i+j-1}{j}-\binom{i+j-1}{j-1}$ whose determinants seem to have interesting properties: Let $A(n)$ and $B(n)$ be the $n\times n$ matrices with entries

$$a_{i,j}=\binom{2(i+j)}{i}-\binom{2(i+j)}{i-1} \text{ and } b_{i,j}=\binom{2(i+j)+1}{i}-\binom{2(i+j)+1}{i-1}$$

for $0\leq i,j<n$. One seems then to have

$$\det(A(n))=\det(B(n))=2^{\binom{n}{2}}\ .$$

Similarly, considering $\tilde{A}(n)$ and $\tilde{B}(n)$ defined by

$$\tilde{a}_{i,j}=\binom{2(i+j+2)}{i+1}-\binom{2(i+j+2)}{i} \text{ and } \tilde{b}_{i,j}=\binom{2(i+j)+5}{i+1}-\binom{2(i+j)+5}{i}$$

for $0\leq i,j<n$, ie. $\tilde{A}(n)$ (respectively $\tilde{B}(n)$) is the minor obtained by erasing the first row and column of $A(n+1)$ (respectively $B(n+1)$) one seems to have

$$\det(\tilde{A}(n))=\frac{(2n+1)!}{n!\ n!}\ \frac{2^{\binom{n-1}{2}}}{2} \qquad \det(\tilde{B}(n))=(n+1)\ 2^{\binom{n+1}{2}}\ .$$

Principal minors of $P_{(0,1,1,1,\dots),-(0,1,1,1,\dots)}(\infty)$ consisting of $2n$ consecutive rows and columns and starting at rows and columns of index $k=0,1,2,\dots$ seem also to have interesting properties as suggested by computations.

Conjecture 5.3. *Denote by $T_k(2n)$ the $2n\times 2n$ symplectic matrix with coefficients $t_{i,j}=\binom{2k+i+j-1}{k+j}-\binom{2k+i+j-1}{k+j-1}$ for $0\leq i,j<2n$. One has*

$$\sqrt{det(T_k(2n))}=\prod_{t=1}^{k-1}\frac{\binom{2n+2t}{t}}{\binom{2t}{t}}\ ,n=0,1,2,\dots\ .$$

The first polynomials

$$D_k(n)=\prod_{t=1}^{k-1}\frac{\binom{2n+2t}{t}}{\binom{2t}{t}}\quad (=\sqrt{\det(T_k(2n))}\quad ?)$$

are given by

$$\begin{aligned}
D_0(n) &= 1\\
D_1(n) &= 1\\
D_2(n) &= (n+1)\\
D_3(n) &= (2n+3)(n+1)(n+2)/6\\
D_4(n) &= (2n+5)(2n+3)(n+3)(n+2)^2(n+1)/180
\end{aligned}$$

The sequences $(D_k(n))_{k=0,1,2,\ldots}$ (for fixed n) seem also to be of interest since they have appeared elsewhere. They start as follows:

$$\begin{array}{rcl}(D_0(0),D_1(0),D_2(0),\ldots) & = & (1,1,1,\ldots)\\ (D_0(1),D_1(1),D_2(1),\ldots) & = & (1,1,2,5,14,\ldots)\text{ (Catalan numbers)}\\ (D_0(2),D_1(2),D_2(2),\ldots) & = & (1,1,3,14,84,\ldots)\text{ (cf }\mathbf{A}005700\text{ in }[\mathbf{IS}])\\ (D_0(3),D_1(3),D_2(3),\ldots) & = & (1,1,4,30,330,\ldots)\text{ (cf }\mathbf{A}006149\text{ in }[\mathbf{IS}])\\ (D_0(4),D_1(4),D_2(4),\ldots) & = & (1,1,5,55,1001,\ldots)\text{ (cf }\mathbf{A}006150\text{ in }[\mathbf{IS}])\\ (D_0(5),D_1(5),D_2(5),\ldots) & = & (1,1,6,91,2548,\ldots)\text{ (cf }\mathbf{A}006151\text{ in }[\mathbf{IS}])\end{array}$$

Geometric sequences provide other nice special cases of Theorem 3.1.

Example 5.4. (i) The sequence $\alpha = (0,1,A,A^2,A^3,\ldots)$ (for $A>0$) yields $\det(P_{\alpha,-\alpha}(2n)) = A^{2(n-1)}$.

(ii) The slightly more general example $\alpha = (0,1,A+B,\ldots,\alpha_k = \frac{A^k-B^k}{A-B},\ldots)$ yields $\det(P_{\alpha,-\alpha}(2n)) = (A-AB+B)^{2(n-1)}$.

Finally, we would like to mention the following exotic example.

Example 5.5. The sequences

$$\alpha_C = (0,1,1,2,5,14,42,\ldots)$$
$$\alpha_B = (0,1,2,6,20,70,\ldots)$$

related to Catalan numbers and central binomial coefficients yield the sequences $r_C(n) = \sqrt{\det(P_{\alpha_C,-\alpha_C}(2n))}$ and $r_B(n) = \sqrt{\det(P_{\alpha_B,-\alpha_B}(2n))}$:

$n =$	1	2	3	4	5	6	7
$r_C(n) =$	1	2	6	31	286	4600	130664
$r_B(n) =$	1	$2\cdot 2$	$6\cdot 2^2$	$31\cdot 2^3$	$286\cdot 2^4$	$4600\cdot 2^5$	$130664\cdot 2^6$

$n =$	8	9	10	...
$r_C(n) =$	6619840	591478944	93683332808	...
$r_B(n) =$	$6619840\cdot 2^7$	$591478944\cdot 2^8$	$93683332808\cdot 2^9$	...

suggesting the conjecture $r_B(n) = 2^{n-1}r_C(n)$ for $n \geq 1$.

5.1 The even symplectic construction and the even symplectic unimodular tree

Given an arbitrary sequence $\beta = (\beta_0,\beta_1,\ldots)$ we consider the sequence $\alpha = (0,\beta_0,0,\beta_1,0,\beta_2,\ldots)$ defined by $\alpha_{2n} = 0$ and $\alpha_{2n+1} = \beta_n$. We call this way of constructing a symplectic matrix $P_{\alpha,-\alpha}(2n)$ out of a sequence $\beta = (\beta_0,\beta_1,\ldots)$ the *even symplectic construction (of Pascal triangles)*.

Example 5.1.1. The symplectic matrix of order 6 associated to the the sequence $\beta = (1,1,-1,\ldots)$ by the even symplectic construction is the following determinant 1 matrix

$$\begin{pmatrix} 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -2 & -2 \\ 1 & 1 & 1 & 0 & -2 & -4 \\ 0 & 1 & 2 & 2 & 0 & -4 \\ -1 & 0 & 2 & 4 & 4 & 0 \end{pmatrix} .$$

By elementary operations on rows and columns it is easy to check the identity

$$\begin{aligned} &\det(P_{(0,\beta_0,0,\beta_1,0,\beta_2,\dots),-(0,\beta_0,0,\beta_1,\dots)}(2n)) \\ &\quad = \det(P_{(0,\beta_0,\beta_0,\beta_1,\beta_1,\beta_2,\beta_2,\dots),-(0,\beta_0,\beta_0,\beta_1,\beta_1,\dots)}(2n)) \end{aligned}$$

for all n and $\beta = (\beta_0, \beta_1, \dots)$.

The main feature of the even symplectic construction is perhaps given by the following result.

Theorem 5.1.2. *(i) Let $(\beta_0, \beta_1, \dots, \beta_{n-1})$ be a sequence of integers such that*

$$det(P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1}),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1})}(2n)) = 1 .$$

Then there exists a unique even integer $\tilde{\beta}_n$ such that

$$\begin{aligned} &det(P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde{\beta}_n+1),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde{\beta}_n+1)}(2n+2)) = 1 \\ &det(P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde{\beta}_n),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde{\beta}_n)}(2n+2)) = 0 \\ &det(P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde{\beta}_n-1),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,\tilde{\beta}_n-1)}(2n+2)) = 1 . \end{aligned}$$

(ii) If $\beta = (\beta_0, \beta_1, \beta_2, \dots)$ and $\beta' = (\beta'_0, \beta'_1, \beta'_2, \dots)$ are two infinite sequences of integers satisfying the assumption of assertion (i) above for all n, then there exists a unique integer m such that $\beta_i = \beta'_i$ for $i < m$ and $\beta_m = \tilde{\beta}_m + \epsilon$, $\beta'_m = \tilde{\beta}_m - \epsilon$ with $\tilde{\beta}_m$ as in assertion (i) above and $\epsilon \in \{\pm 1\}$.

Proof. The determinant of the symplectic matrix

$$P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,x),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0,x)}(2n+2)$$

is of the form $D(x) = (ax + b)^2$ for some suitable integers a and b (which are well defined up to multiplication by -1).

It is easy to see that it is enough to show that $a = \pm 1$ in order to prove the Theorem (the integer $\tilde{\beta}_m$ equals then $-ab$ and is even by a consideration (mod 2)). This is of course equivalent to showing that the polynomial $D(x)$ has degree 2 and leading term 1.

Consider now the symplectic matrix M of order $2n+2$ defined as follows: The entries of M except the last row and column are given by the odd-order (and hence degenerate) symplectic matrix

$$P_{(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0),-(0,\beta_0,0,\beta_1,\dots,0,\beta_{n-1},0)}(2n+1) .$$

The last row (which determines by antisymetry the last column) of M is given by

$$(1,1,1,\ldots,1,1,0)\ .$$

It is obvious to check that $\det(M)$ is the coefficient of x^2 in the polynomial $D(x)$ introduced above.

Subtract now row number $2n-1$ from row number $2n$ of M (with rows and columns of M indexed from 0 to $2n+1$), subtract then row number $2n-2$ from row number $2n-1$ etc. until subtracting row number 0 from row number 1. Do the same operations on columns thus producing a symplectic matrix $\tilde{M}$ which is equivalent to M and whose last row is given by $(1,0,0,\ldots,0,0)$. The determinant of M equals hence the determinant of the minor of $\tilde{M}$ obtained by deleting the first and last rows and columns in $\tilde{M}$. This minor is given by

$$P_{(0,\beta_0,0,\beta_1,\ldots,0,\beta_{n-1}),-(0,\beta_0,0,\beta_1,\ldots,0,\beta_{n-1})}(2n)$$

thus showing that $\det(M)=1=a^2$. QED

The set of sequences

$$\{\alpha=(0,\beta_0,0,\beta_1,0,\beta_2,\ldots)\mid \det(P_{\alpha,-\alpha}(2n)=1,\ n=1,2,3,\ldots\}$$

associated to unimodular symplectic matrices $P_{\alpha,-\alpha}(2n)$ consists hence of integral sequences and has the structure of a tree. We call this tree the *even symplectic unimodular tree.*

The beginning of this tree is shown below and is to be understood as follows:

Column i displays the integer $\tilde{\beta}_i$ of the Theorem. Indices indicate if $\beta_i=\tilde{\beta}_i+1$ or $\tilde{\beta}_i-1$. Hence the row

$$0_{+1}\quad 0_{+1}\quad 0_{-1}\quad -8_{+1}\quad 68_{+1}\quad 434748_{\pm}$$

corresponds for instance to the sequence

$$(1,1,-1,-7,69)$$

implying $\tilde{\beta}_5=434748$ (the sequence $(1,1,-1,-7,69)$ can hence be extended either to $(1,1,-1,-7,69,434749)$ or to $(1,1,-1,-7,69,434747)$).

We have only displayed sequences starting with 1 since sequences starting with -1 are obtained by a global sign change.

Table 5.1.3. (Part of the even symplectic unimodular tree).

0_{+1}	0_{+1}	0_{+1}	0_{+1}	0_{+1}	$0_{\pm}$
0_{+1}	0_{+1}	0_{+1}	0_{+1}	0_{-1}	$-100_{\pm}$
0_{+1}	0_{+1}	0_{+1}	0_{-1}	-42_{+1}	$32658_{\pm}$
0_{+1}	0_{+1}	0_{+1}	0_{-1}	-42_{-1}	$-39754_{\pm}$
0_{+1}	0_{+1}	0_{-1}	-8_{+1}	68_{+1}	$434748_{\pm}$
0_{+1}	0_{+1}	0_{-1}	-8_{+1}	68_{-1}	$-400344_{\pm}$
0_{+1}	0_{+1}	0_{-1}	-8_{-1}	-254_{+1}	$12922350_{\pm}$
0_{+1}	0_{+1}	0_{-1}	-8_{-1}	-254_{-1}	$-13258926_{\pm}$
0_{+1}	0_{-1}	2_{+1}	0_{+1}	240_{+1}	$13257990_{\pm}$
0_{+1}	0_{-1}	2_{+1}	0_{+1}	240_{-1}	$-12923278_{\pm}$
0_{+1}	0_{-1}	2_{+1}	0_{-1}	-74_{+1}	$400664_{\pm}$
0_{+1}	0_{-1}	2_{+1}	0_{-1}	-74_{-1}	$-434420_{\pm}$
0_{+1}	0_{-1}	2_{-1}	0_{+1}	36_{+1}	$39594_{\pm}$
0_{+1}	0_{-1}	2_{-1}	0_{+1}	36_{-1}	$-32810_{\pm}$
0_{+1}	0_{-1}	2_{-1}	0_{-1}	2_{+1}	$92_{\pm}$
0_{+1}	0_{-1}	2_{-1}	0_{-1}	2_{-1}	$0_{\pm}$

6 A sympletric tree?

The construction of a generalized Pascal triangle $P_{\alpha,\beta}(\infty)$ needs two sequences $\alpha = (\alpha_0, \alpha_1, \ldots)$ and $\beta = (\beta_0, \beta_1, \ldots)$. Starting with only one sequence $\alpha = (\alpha_0, \alpha_1, \ldots)$ and considering $P_{\alpha,\alpha}(\infty)$ we get generalized symmetric Pascal triangles and considering $P_{\alpha,-\alpha}(\infty)$ we get generalized symplectic Pascal triangles. Since the sequence $\tilde{\alpha} = (\tilde{\alpha}_0 = \alpha_0, \tilde{\alpha}_1 = -\alpha_1, \tilde{\alpha}_2 = \alpha_2, \ldots, \tilde{\alpha}_i = (-1)^i \alpha_i, \ldots)$ is half-way between α and $-\alpha$, we call the generalized Pascal triangle $P_{\alpha,\tilde{\alpha}}(\infty)$ the *generalized sympletric Pascal triangle.*

The two sequences

$$\alpha = (0, 1, 1, 2, 3, 5, 8, 13, \ldots) \qquad \text{Fibonacci}$$
$$\alpha = (0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, \ldots) \qquad 6 - \text{periodic}$$

and the associated sequences $\tilde{\alpha}$ satisfy all linear recursions of order 2. Theorem 3.1 and a computation of the first few values show that both sequences $\det(P_{\alpha,\tilde{\alpha}}(n))$ equal $0, 1, 2, 2^2, 2^3, \ldots, 2^{n-2}, \ldots$. All the following finite sequences yield matrices $P_{\alpha\tilde{\alpha}}(n)$ with determinants $0, 1, 2, 4, 8, 16, 32, 64$ (for

$n = 1, 2, 3, \ldots$) too:

0	1	1	2	5	13	34	85 ± 4
0	1	1	2	5	13	28	79 ± 4
0	1	1	2	5	9	20	77 ± 38
0	1	1	2	5	9	2	-193 ± 110
0	1	1	2	3	5	10	19 ± 6
0	1	1	2	3	5	8	12 ± 1
0	1	1	2	3	3	10	-3 ± 6
0	1	1	2	3	3	-2	9 ± 30
0	1	1	0	1	3	4	-31 ± 38
0	1	1	0	1	3	-14	167 ± 110
0	1	1	0	1	-1	2	1 ± 4
0	1	1	0	1	-1	-4	-17 ± 4
0	1	1	0	-1	1	10	33 ± 6
0	1	1	0	-1	1	-2	-27 ± 30
0	1	1	0	-1	-1	2	3 ± 6
0	1	1	0	-1	-1	0	2 ± 1

Problem 6.1. Has the set of all infinite integral sequences $\alpha = (0, 1, 1, \alpha_3, \ldots)$ such that $\det(P_{\alpha,\tilde{\alpha}}(n)) = (0, 1, 2, 4, \ldots, 2^{n-2}, \ldots)_{n=1,2,\ldots}$ the structure of a tree (ie. can every finite such sequence of length at least 3 be extended by one next term in exactly two ways)?

7 Periodic matrices

In this section we are interested in matrices coming from a sort of "periodic convolution with compact support on $\mathbf{N}$".

We say that an infinite matrix A with coefficients $a_{i,j},\ 0 \leq i, j$ is $(s,t)-$*bounded* $(s, t \in \mathbf{N})$ if $a_{i,j} = 0$ for $(j - i) \notin [-s, t]$.

We call a matrix with coefficients $a_{i,j},\ 0 \leq i, j$ $p-$*periodic* if $a_{i,j} = a_{i-p,j-p}$ for $i, j \geq p$.

An infinite matrix P with coefficients $p_{i,j},\ 0 \leq i, j$ is a *finite perturbation* if it has only a finite number of non-zero coefficients.

As before, given an infinite matrix M with coefficients $m_{i,j},\ 0 \leq i, j$ we denote by $M(n)$ the matrix with coefficients $m_{i,j},\ 0 \leq i, j < n$ obtained by erasing all but the first n rows and columns of M.

Theorem 7.1. *Let $A = \tilde{A} + P$ be a matrix where $\tilde{A}$ is a $p-$periodic $(s,t)-$bounded matrix and where P is a finite perturbation. Then there exist constants $N, d \leq \binom{s+t}{s}, C_1, \ldots, C_d$ such that*

$$det(A(n)) = \sum_{i=1}^{d} C_i \ det(A(n - ip))$$

for $n > N$.

We will prove the theorem for $p = 1$, $s = t = 2$ and then describe the necessary modifications in the general case.

Proof in the case $p = 1$, $s = t = 2$. Suppose n huge. The matrix $A(n)$ has then the form

$$A(n) = \begin{pmatrix} \ddots & & & & \vdots \\ & c & d & e & 0 \\ & b & c & d & e \\ & a & b & c & d \\ \dots & 0 & a & b & c \end{pmatrix} .$$

Developping the determinant possibly several times along the last row one gets only matrices of the following six types

$$T_1 = \begin{pmatrix} \ddots & d & e & 0 \\ & c & d & e \\ & b & c & d \\ & a & b & c \end{pmatrix} \quad T_2 = \begin{pmatrix} \ddots & d & e & 0 \\ & c & d & 0 \\ & b & c & e \\ & a & b & d \end{pmatrix} \quad T_3 = \begin{pmatrix} \ddots & d & 0 & 0 \\ & c & e & 0 \\ & b & d & e \\ & a & c & d \end{pmatrix}$$

$$T_4 = \begin{pmatrix} \ddots & d & e & 0 \\ & c & d & 0 \\ & b & c & 0 \\ & a & b & e \end{pmatrix} \quad T_5 = \begin{pmatrix} \ddots & d & 0 & 0 \\ & c & e & 0 \\ & b & d & 0 \\ & a & c & e \end{pmatrix} \quad T_6 = \begin{pmatrix} \ddots & d & 0 & 0 \\ & c & 0 & 0 \\ & b & e & 0 \\ & a & d & e \end{pmatrix}$$

and writing $t_i(m) = \det(T_i(m))$ we have the identity

$$\begin{pmatrix} t_1(m) \\ t_2(m) \\ t_3(m) \\ t_4(m) \\ t_5(m) \\ t_6(m) \end{pmatrix} = \begin{pmatrix} c & -b & a & 0 & 0 & 0 \\ d & 0 & 0 & -b & a & 0 \\ 0 & d & 0 & -c & 0 & a \\ e & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 \end{pmatrix} \begin{pmatrix} t_1(m-1) \\ t_2(m-1) \\ t_3(m-1) \\ t_4(m-1) \\ t_5(m-1) \\ t_6(m-1) \end{pmatrix}$$

for m huge enough. Writing R the above 6×6 matrix relating $t_i(m)$ to $t_j(m-1)$ we have $t(n) = R^{n-N} t(N)$ for $n \geq N$ huge enough and for $t(m)$ the vector with coordinates $t_1(m), \dots, t_6(m)$. Choosing a basis of a Jordan normal form of R and expressing the vector $t(N)$ with respect to this basis shows now that the determinants $t_i(n)$ (and hence $\det(A(n)) = t_1(n)$) satisfy for $n > N$ a linear recursion with characteristic polynomial dividing

$$\det(z \ \mathrm{Id}_6 - R) .$$

Proof of the general case. Let us first suppose $p = 1$. There are then $\binom{s+t}{s}$ (count the possibilities for the highest non-zero entry in the last s columns) different possible types T_i obtained by developping the determinant $\det(A(n))$ for huge n several times along the last row and one gets hence a square matrix R of order $\binom{s+t}{s}$ expressing the determinants $\det(T_i(n))$ linearly in $\det(T_j(n-1))$ for n huge enough. This shows that the determinants $\det(T_i(n))$ satisfy for n huge enough a linear recursion with characteristic polynomial dividing the characteristic polynomial of the square matrix R.

If $p > 1$, develop the determinant of $\det(A(n))$ a multiple of p times along the last row and proceed as above. One gets in this way matrices $R_0, \ldots, R_{p-1}$ according to $n \pmod p$ with identical characteristic polynomials yielding recursion relations between $\det(A(n))$ and $\det(A(n-ip))$. QED

8 The diagonal construction

Let $\gamma = (\gamma_0, \gamma_1, \gamma_2, \ldots,)$ be a sequence and let u_1, u_2, l_1, l_2 be four constants. The *diagonal-construction* is the (infinite) matrix $D_\gamma^{(u_1,u_2,l_1,l_2)}$ with entries

$$\begin{array}{ll} d_{i,i} = \gamma_i & 0 \le i \\ d_{i,j} = u_1 d_{i,j-1} + u_2 d_{i+1,j} & 0 \le i < j \\ d_{i,j} = l_1 d_{i-1,j} + l_2 d_{i,j+1} & 0 \le j < i \end{array}$$

and we denote by $D(n) = D_\gamma^{(u_1,u_2,l_1,l_2)}(n)$ the $n \times n$ principal minor with coefficients $d_{i,j},\ 0 \le i, j < n$ obtained by considering the first n rows and columns of $D_\gamma^{(u_1,u_2,l_1,l_2)}$.

The cases where $u_1 u_2 l_1 l_2 = 0$ are degenerate. For instance, in the case $u_2 = 0$ one sees easily that the matrix $D_\gamma^{(u_1,u_2,l_1,l_2)}(n)$ has determinant

$$\gamma_0 \prod_{j=1}^{n-1} (\gamma_j - u_1(l_1\gamma_{j-1} + l_2\gamma_j)) \ .$$

The other cases are similar.

The following result shows that we loose almost nothing by assuming $u_1 = u_2 = 1$.

Proposition 8.1. *For λ, μ two invertible constants we have*

$$D_{\tilde\gamma}^{(\lambda u_1, \mu u_2, \mu^{-1} l_1, \lambda^{-1} l_2)}(n) = \left(\frac{\lambda}{\mu}\right)^{\binom{n}{2}} D_\gamma^{(u_1,u_2,l_1,l_2)}(n)$$

where

$$\tilde\gamma = (\gamma_0, \frac{\lambda}{\mu}\gamma_1, \frac{\lambda^2}{\mu^2}\gamma_2, \ldots, \tilde\gamma_k = \frac{\lambda^k}{\mu^k}\gamma_k, \ldots) \ .$$

Proof. Check that the coefficients $\tilde d_{i,j}$ of $D_{\tilde\gamma}^{(\lambda u_1, \mu u_2, \mu^{-1} l_1, \lambda^{-1} l_2)}(n)$ are given by $\tilde d_{i,j} = \mu^{-i}\lambda^j\ d_{i,j}$ where $d_{i,j}$ are the coefficients of $D_\gamma^{(u_1,u_2,l_1,l_2)}(n)$. This implies the result easily. QED

Proposition 8.2. *For $n \geq 1$ the sequence*

$$d(n) = det(D_\gamma^{(u_1,u_2,l_1,l_2)}(n))$$

associated to the geometric sequence $\gamma = (1, x, x^2, x^3, \ldots)$ is given by

$$d(n) = \left(-u_1 l_1 + (1 - u_1 l_2 - u_2 l_1)x - u_2 l_2 x^2\right)^{n-1} x^{\binom{n-1}{2}} .$$

A nice special case is given by $u_1 = u_2 = l_1 = l_2 = 1$. The associated matrix $D_\gamma(4) = D_\gamma^{(1,1,1,1)}(4)$ for example is then given by

$$\begin{pmatrix} 1 & 1+x & 1+2x+x^2 & 1+3x+3x^2+x^3 \\ 1+x & x & x+x^2 & x+2x^2+x^3 \\ 1+2x+x^2 & x+x^2 & x^2 & x^2+x^3 \\ 1+3x+3x^2+x^3 & x+2x^2+x^3 & x^2+x^3 & x^3 \end{pmatrix}$$

and the reader can readily check that the coefficient $d_{i,j}$ of $D_\gamma(n)$ is given by

$$d_{i,j} = x^{\min(i,j)}(1+x)^{|i-j|} .$$

Proposition 8.2 shows that the determinant $\det(D_\gamma(n))$ is given by

$$\det(D_{(1,x,x^2,x^3,\ldots)}(n)) = \left(-1 - x - x^2\right)^{n-1} x^{\binom{n-1}{2}}$$

for $n \geq 1$.

Setting $x = 1$ in this special case $u_1 = u_2 = l_1 = l_2 = 1$, we get a matrix M with entries $m_{i,j} = 2^{|i-j|}$ for $0 \leq i, j < n$. Its determinant is $(-3)^{n-1}$. It is easy to show that the matrix M_a of order n with entries $m_{i,j} = a^{|i-j|}$ for $0 \leq i, j < n$ has determinant $(1 - a^2)^{n-1}$.

A more involved computation shows that the determinant of the $n \times n$ matrix N_a with coefficients $n_{i,j} = a^{(i-j)^2}$ $(0 \leq i, j < n)$ is given by

$$\prod_{j=1}^{n-1} \left(1 - a^{2(n-j)}\right)^j$$

(cf. Problem ?? in Am. Math. Monthly).

A similar example is the special case $-u_1 = u_2 = -l_1 = l_2 = 1$ which yields for instance the matrix $D_\gamma(4) = D_\gamma^{(-1,1,-1,1)}(4)$ given by

$$\begin{pmatrix} 1 & -1+x & 1-2x+x^2 & -1+3x-3x^2+x^3 \\ -1+x & x & -x+x^2 & x-2x^2+x^3 \\ 1-2x+x^2 & -x+x^2 & x^2 & -x^2+x^3 \\ -1+3x-3x^2+x^3 & x-2x^2+x^3 & -x^2+x^3 & x^3 \end{pmatrix}$$

and the reader can readily check that the coefficient $f_{i,j}$ of $D_\gamma(n)$ is given by

$$d_{i,j} = x^{\min(i,j)}(x-1)^{|i-j|} .$$

The determinant $\det(D_{(1,x,x^2,x^3,\dots)}(n))$ is given by

$$\det(D_{(1,x,x^2,x^3,\dots)}(n)) = \left(-x^2+3x-1\right)^{n-1} x^{\binom{n-1}{2}}$$

for $n \geq 1$.

Proof of Proposition 8.2. By continuity and Proposition 8.1 it is enough to prove the formula in the case $u_1 = u_2 = 1$.

This implies $d_{i,j} = x^i(1+x)^{(j-i)}$ for $i \leq j$.

Subtracting $(1+x)$ times column number $(n-2)$ from column number $(n-1)$ (which is the last one) etc. until subtracting $(1+x)$ times column number 0 from column number 1 transforms the matrix $D(n)$ into a lower triangular matrix with diagonal entries

$$1, x-(1+x)(l_1+l_2x), x\left(x-(1+x)(l_1+l_2x)\right), \dots, x^{n-2}(-l_1+(1-l_1-l_2)x-l_2x^2) .$$

Theorem 8.3. *Let $\gamma = (\gamma_0, \dots, \gamma_{p-1}, \gamma_0, \dots, \gamma_{p-1}, \dots)$ be a $p-$periodic sequence and let*

$$d(n) = \det(D_\gamma^{(u_1,u_2,l_1,l_2)}(n))$$

be the determinants of the associated matrices (for fixed (u_1, u_2, l_1, l_2)).

Then there exist an integer d and constants $C_1, \dots C_d$ such that

$$d(n) = \sum_{i=1}^{d} C_i \ d(n-ip)$$

for all n huge enough.

Remark 8.4. Generically, the coefficients C_i seem to display the symmetry

$$C_{d-i} = \rho^{(d-2i)/2} \ C_i$$

(with $C_0 = -1$) for some constant ρ which seems to be polynomial in $\gamma_0, \dots, \gamma_{p-1}, u_1, u_2, l_1, l_2$.

Proof of Theorem 8.3. For $k \geq p$ add to the $k-$th row a linear combination (with coefficients depending only on (u_1, u_2, l_1, l_2)) of rows $k-1, k-2, \dots, k-p$ such that $d_{k,k-i} = 0$ for $i \geq p$. Do the analogous operation on columns and apply Theorem 7.1 to the resulting matrices. QED

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