ON THE LENGTH OF GENERALIZED FRACTIONS

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Abstracts.[1] Let M be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m})$ with $\dim M = d$. Let $(x_1, \cdots, x_d)$ be a system of parameters of M and $(n_1, \ldots, n_d)$ a set of positive integers. Consider the length of generalized fraction $1/(x_1^{n_1}, \cdots, x_d^{n_d}, 1)$ as a function in $n_1, \cdots, n_d$. R. Y. Sharp and M. A. Hamieh [Sh-H] asked that if this function is a polynomial for $n_1, \cdots, n_d$ large enough. In this paper, we will give counterexamples to this question. We also study conditions for the system of parameters $\underline{x}$, in order to show that the length of generalized fraction $1/(x_1^{n_1}, \cdots, x_d^{n_d}, 1)$ is not a polynomial for $n_1, \ldots, n_d$ large enough.

1. Introduction

In this paper we always assume that $(R, \mathfrak{m})$ is a noetherian local ring and M is a finitely generated $R-$module with $\dim M = d$. Sharp and Zakeri [Sh-Z1] gave a procedure for constructing so-called modules of generalized fractions which generalizes the usual theory of localization of modules. The theory of generalized fraction has a wide range of application in commutative algebra. Especially, the top local cohomology modules $H^d_{\mathfrak{m}}(M)$ may be viewed as a module of generalized fractions of M with respect to a certain triangular subset of R^{d+1}, and this is used to study Hochster's Monomial Conjecture.

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Let us recall briefly the main ingredients in the construction of modules of generalized fractions. Let k be a positive integer, denote by $D_k(R)$ the set of all $k \times k$ lower triangular matrices with entries in R; we use ^{T} to denote matrix transpose. A *triangular subset* of R^k is a non-empty subset U in R^k such that (i) whenever $(u_1, \dots, u_k) \in U$, then $(u_1^{n_1}, \dots, u_k^{n_k}) \in U$ for all positive integer $n_1, \dots, n_k$, and (ii) whenever $(u_1, \dots, u_k)$ and $(v_1, \dots, v_k) \in U$, then there exist $(w_1, \dots, w_k) \in U$ and $H, H' \in D_k(R)$ such that

$$H[u_1, \dots, u_k]^T = [w_1, \dots, w_k]^T = H'[v_1, \dots, v_k]^T.$$

Given such a U, Sharp and Zakeri constructed an $R-$module $U^{-k}M$ and they call it *the module of generalized fractions* of M with respect to U. Especially, the set

$$U(M)_{d+1} = \{(y_1, \dots, y_d, 1) \in R: \text{ there exists } j \text{ with } 0 \leq j \leq d \text{ such that } (y_1, \dots, y_j) \text{ form a subset of a s.o.p of } M \text{ and } y_{j+1} = \dots = y_d = 1\}$$

is a triangular subset of R^{d+1}. Let $\underline{x} = (x_1, \dots, x_d)$ be a s.o.p of M and $\underline{n} = (n_1, \dots, n_d)$ a set of positive integers. We denote by $M(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))$ the submodule $\{m/(x_1^{n_1}, \dots, x_d^{n_d}, 1) : m \in M\}$ of $U(M)_{d+1}^{-d-1}M$. This submodule is annihilated by $\operatorname{Ann} M + (x_1^{n_1}, \dots, x_d^{n_d})R$. Therefore $\ell(M(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))) < \infty$. Let

$$q_{\underline{x};M}(\underline{n}) = \ell(M(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))).$$

$q_{\underline{x};M}(\underline{n})$ is called *the length of the generalized fraction* $1/(x_1^{n_1}, \dots, x_d^{n_d}, 1)$. Sharp and Hamieh naturally asked the following question [Sh-H, Question 1.2]:

Open question: Does exist a polynomial $F(\underline{X})$ in d variables $X_1, \dots, X_d$ with rational coefficients such that $q_{\underline{x};R}(\underline{n}) = F(n_1, \dots, n_d)$ for all $n_1, \dots, n_d$ large enough?

They have proved in that paper that the answer is positive when $\dim R \leq 2$ or R is generalized Cohen-Macaulay.

In this paper we give counterexamples to this question in the case where R has any dimension $d \geq 3$ (Theorem 1.1). We also study conditions for a s.o.p $\underline{x}$ of module M, in order to show that $q_{\underline{x};M}(\underline{n})$ is not a polynomial for large $\underline{n}$ (Theorem 1.2).

Set

$$J_{\underline{x};M}(\underline{n}) = n_1 \dots n_d e(\underline{x}; M) - q_{\underline{x};M}(\underline{n})$$

In general we have $J_{\underline{x};M}(\underline{n}) \geq 0$ (see [C-M1]). Especially, the least degree of all polynomials in $\underline{n}$ bounding above the function $J_{\underline{x};M}(\underline{n})$ does not depend on the choice of $\underline{x}$. This invariant is denoted by $pf(M)$. If $d \geq 3$ then $pf(M) \leq d-2$ (see [C-M2]).

Theorem 1.1. *Let $d \geq 3$ and $0 < v \leq d-2$ be integers. Let $S = K[x_1, \dots, x_d]$, the polynomial ring in variables $x_1, \dots, x_d$ over a field K. Let $\mathfrak{m} = (x_1, \dots, x_d)S$ and $R = S_{\mathfrak{m}}$, the localization of S with respect to $\mathfrak{m}$. Let $M = (x_1, \dots, x_{d-v})R$ and $R \propto M$ the idealization of M. Then we have*

$$q_{(x_1+x_d,0),(x_2,0)\dots,(x_d,0);R\propto M}(\underline{n}) = 2n_1n_2\dots n_d - n_{d-v+1}\dots n_{d-1}.min\{n_1, n_d\},$$

for all integers $n_1, \dots, n_d \geq 1$. In particular, $q_{(x_1+x_d,0),(x_2,0),\dots,(x_d,0);R\propto M}(\underline{n})$ is not a polynomial for $\underline{n}$ large enough. Moreover, $pf(R \propto M) = v$.

Set

$$I_{\underline{x};M}(\underline{n}) = \ell(M/(x_1^{n_1}, \dots, x_d^{n_d})M) - n_1 \dots n_d e(\underline{x}; M).$$

It was proved in [C2] that the least degree of all polynomials in $\underline{n}$ bounding above the function $I_{\underline{x};M}(\underline{n})$ does not depend on the choice of $\underline{x}$. It is called *polynomial type* of M and is denoted by $p(M)$.

If we stipulate the degree of the polynomial 0 is $-\infty$ then M is Cohen-Macaulay if and only if $p(M) = -\infty$ and M is generalized Cohen-Macaulay if and only if $p(M) \leq 0$. In more general cases, the invariant $p(M)$ makes an important role to study structure of modules (see [C1,2] and [C-M1,2].

Theorem 1.2. *(i). If $p(M) \leq 2$ and $pf(M) > 0$ then there exists a s.o.p $\underline{x}$ of M such that $q_{\underline{x};M}(\underline{n})$ is not a polynomial for $\underline{n}$ large enough.*

(ii). Suppose that R admits dualizing complexes. If $p(M) = 3$ and $pf(M) > 0$ then there exists a s.o.p $\underline{x}$ of M such that $q_{\underline{x};M}(\underline{n})$ is not a polynomial for $\underline{n}$ large enough.

2. Proof of Theorem 1.1

Let $d \geq 3$ and $0 < v \leq d-2$ be integers. Firstly we need the following lemmas.

Lemma 2.1. *([C-K, Lemma 2.3]). Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated $R-$module with $\dim M = d$. Let $\underline{x} = (x_1, \ldots, x_d)$ be a s.o.p of M. Set*

$$Q(\underline{x}; M) = \bigcup_{t>0} (x_1^{t+1}, \cdots, x_d^{t+1})M :_M x_1^t \ldots x_d^t.$$

Then we have

$$M/Q(\underline{x}; M) \cong M(1/(x_1, \ldots, x_d, 1)).$$

Lemma 2.2. *Let $S = K[x_1, \ldots, x_d]$ be the polynomial ring in variables $x_1, \ldots, x_d$ over a field K and $n_1, \ldots, n_d$ positive integers. For any integer $t \geq \dfrac{n_d}{n_1}$ we have*

$$(x_1^{n_1t+n_1}, \ldots, x_d^{n_dt+n_d})(x_1 - x_d, x_2, \ldots, x_{d-v})S :_S x_1^{n_1t} \ldots x_d^{n_dt} = (x_1^{n_1}, \ldots, x_d^{n_d})S.$$

Proof. Set

$$\mathfrak{a} = (x_1^{n_1t+n_1}, \ldots, x_d^{n_dt+n_d})(x_1 - x_d, x_2, \ldots, x_{d-v})S :_S x_1^{n_1t} \ldots x_d^{n_dt}.$$

Since $(x_1^{n_1t+n_1}, \ldots, x_d^{n_dt+n_d})S$ is a monomial ideal, we have

$$\mathfrak{a} \subseteq (x_1^{n_1t+n_1}, \ldots, x_d^{n_dt+n_d})S :_S x_1^{n_1t} \ldots x_d^{n_dt} = (x_1^{n_1}, \ldots, x_d^{n_d})S.$$

Conversely, since $d - v \geq 2$, we can easily check that $x_i^{n_i} \in \mathfrak{a}$, for all $i \neq 2$. Let

$$\mathfrak{b} = (x_1^{n_1t+n_1}, \ldots, x_d^{n_dt+n_d})(x_1 - x_d, x_2, \ldots, x_{d-v})S.$$

We have

$$x_2^{n_2t+n_2} x_1^{n_1t} x_3^{n_3t} \ldots x_d^{n_dt} = x_2^{n_2t+n_2}(x_1 - x_d) x_1^{n_1t-1} x_3^{n_3t} \ldots x_d^{n_dt} + x_2^{n_2t+n_2} x_1^{n_1t-1} x_3^{n_3t} \ldots x_d^{n_dt+1}.$$

So, $x_2^{n_2t+n_2} x_1^{n_1t} x_3^{n_3t} \ldots x_d^{n_dt} \in \mathfrak{b}$ if and only if $x_2^{n_2t+n_2} x_1^{n_1t-1} x_3^{n_3t} \ldots x_d^{n_dt+1} \in \mathfrak{b}$. It follows that, for any integer t such that $n_1t \geq n_d$, after n_1t steps we get

$$x_2^{n_2t+n_2} x_1^{n_1t} x_3^{n_3t} \ldots x_d^{n_dt} \in \mathfrak{b}$$

since $x_2^{n_2t+n_2} x_3^{n_3t} \ldots x_{d-1}^{n_{d-1}t} x_d^{n_dt+n_1t} \in \mathfrak{b}$. Therefore $x_2^{n_2} \in \mathfrak{a}$. □

Lemma 2.3. *Let $S = K[x_1, \dots, x_d]$ be the polynomial ring in variables $x_1, \dots, x_d$ over a field K and $n_1, \dots, n_d$ positive integers. For any integer $t \geq \dfrac{n_d}{n_1}$ we have*

$$((x_1 + x_d)^{n_1 t + n_1}, x_2^{n_2 t + n_2}, \dots, x_d^{n_d t + n_d})(x_1, \dots, x_{d-v})S :_S (x_1 + x_d)^{n_1 t} x_2^{n_2 t} \dots x_d^{n_d t}$$
$$= ((x_1 + x_d)^{n_1}, x_2^{n_2}, \dots, x_d^{n_d})S.$$

Proof. Let $\mathfrak{a}$ be the ideal

$$((x_1 + x_d)^{n_1 t + n_1}, x_2^{n_2 t + n_2}, \dots, x_d^{n_d t + n_d})(x_1, \dots, x_{d-v})S :_S (x_1 + x_d)^{n_1 t} x_2^{n_2 t} \dots x_d^{n_d t}.$$

Clearly $(x_1 + x_d)^{n_1}, x_3^{n_3}, \dots, x_d^{n_d} \in \mathfrak{a}$. We need to prove $x_2^{n_2} \in \mathfrak{a}$. Note that there exist a polynomial f such that

$$x_2^{n_2 t + n_2}(x_1 + x_d)^{n_1 t} x_3^{n_3 t} \dots x_d^{n_d t} = x_1 x_2^{n_2 t + n_2} x_3^{n_3 t} \dots x_d^{n_d t} f + x_2^{n_2 t + n_2} x_3^{n_3 t} \dots x_d^{n_d t + n_1 t}.$$

Therefore, for any integer t such that $n_1 t \geq n_d$ we have $x_2^{n_2} \in \mathfrak{a}$ since both elements $x_1 x_2^{n_2 t + n_2} x_3^{n_3 t} \dots x_d^{n_d t} f$ and $x_2^{n_2 t + n_2} x_3^{n_3 t} \dots x_d^{n_d t + n_1 t}$ belong to

$$((x_1 + x_d)^{n_1 t + n_1}, x_2^{n_2 t + n_2}, \dots, x_d^{n_d t + n_d})(x_1, \dots, x_{d-v}).$$

Conversely, let $f(x_1, x_2 \dots, x_d)$ be an arbitrary polynomial in $\mathfrak{a}$. By replacing $x_1 = x_1 - x_d; x_2 = x_2; \dots; x_d = x_d$ we have $f(x_1 - x_d, x_2, \dots, x_d)$ belongs to

$$(x_1^{n_1 t + n_1}, x_2^{n_2 t + n_2}, \dots, x_d^{n_d t + n_d})(x_1 - x_d, x_2, \dots, x_{d-v})S :_S x_1^{n_1 t} x_2^{n_2 t} \dots x_d^{n_d t}.$$

Therefore $f(x_1 - x_d, x_2, \dots, x_d) \in (x_1^{n_1}, x_2^{n_2}, \dots, x_d^{n_d})S$ by Lemma 2.2. Now by replacing $x_1 = x_1 + x_d; x_2 = x_2; \dots; x_d = x_d$, we have

$$f(x_1, x_2, \dots, x_d) \in ((x_1 + x_d)^{n_1}, x_2^{n_2}, \dots, x_d^{n_d})S.$$

□

Lemma 2.4. *Let $S = K[x_1, \dots, x_d]$ be the polynomial ring in variables $x_1, \dots, x_d$ over a field K. Let $\mathfrak{m} = (x_1, \dots, x_d)S$ and $R = S_{\mathfrak{m}}$, the localization of S with respect to $\mathfrak{m}$. Let $M = (x_1, \dots, x_{d-v})R$. Then we have*

$$q_{x_1 + x_d, x_2, \dots, x_d; M}(\underline{n}) = n_1 \dots n_d - n_{d-v+1} \dots n_{d-1}.min\{n_1, n_d\},$$

for all integers $n_1, \dots, n_d \geq 1$. In particular, $q_{x_1 + x_d, x_2, \dots, x_d; M}(\underline{n})$ is not a polynomial for $\underline{n}$ large enough. Moreover, $pf(M) = v$.

Proof. By the flatness of the natural homomorphism $S \longrightarrow S_{\mathfrak{m}}$, we get by [Mat, (3.H)] and by Lemma 2.3 that

$$Q((x_1 + x_d)^{n_1}, x_2^{n_2}, \dots, x_d^{n_d}; M) = (x_1, \dots, x_{d-v})S_{\mathfrak{m}} \cap ((x_1 + x_d)^{n_1}, x_2^{n_2}, \dots, x_d^{n_d})S_{\mathfrak{m}}.$$

It follows from this relation and Lemma 2.1 that

$$\begin{aligned} &q_{x_1 + x_d, x_2, \dots, x_d; M}(\underline{n}) \\ &= \ell\big((x_1, \dots, x_{d-v})S_{\mathfrak{m}} / (x_1, \dots, x_{d-v})S_{\mathfrak{m}} \cap ((x_1 + x_d)^{n_1}, x_2^{n_2}, \dots, x_d^{n_d})S_{\mathfrak{m}}\big) \\ &= \ell\big((x_1, \dots, x_{d-v}, x_{d-v+1}^{n_{d-v+1}}, \dots, x_d^{\min\{n_1, n_d\}})S_{\mathfrak{m}} / ((x_1 + x_d)^{n_1}, x_2^{n_2}, \dots, x_d^{n_d})S_{\mathfrak{m}}\big). \end{aligned}$$

And since $S_{\mathfrak{m}}$ is Cohen-Macaulay, we get

$$
\begin{aligned}
& q_{x_1+x_d,x_2,\dots,x_d;M}(\underline{n}) = \ell(S_{\mathfrak{m}}/((x_1+x_d)^{n_1}, x_2^{n_2},\dots,x_d^{n_d})S_{\mathfrak{m}}) \\
& - \ell(S_{\mathfrak{m}}/(x_1,\dots,x_{d-v},x_{d-v+1}^{n_{d-v+1}},\dots,x_d^{\min\{n_1,n_d\}})S_{\mathfrak{m}}) \\
& = e((x_1+x_d)^{n_1}, x_2^{n_2},\dots,x_d^{n_d};S_{\mathfrak{m}}) - e(x_1,\dots,x_{d-v},x_{d-v+1}^{n_{d-v+1}},\dots,x_d^{\min\{n_1,n_d\}};S_{\mathfrak{m}}) \\
& = n_1\dots n_d - n_{d-v+1}\dots n_{d-1}.\min\{n_1,n_d\},
\end{aligned}
$$

it finishes the proof. □

Now we need the concept of principle of idealizations, which was introduced by Nagata [Na, p.2]. Let $(R,\mathfrak{m})$ be a Noetherian local ring and M a finitely generated $R-$module. We make the Cartesian product $R\times M$ into a commutative ring with respect to componentwise addition and multiplication defined by $(r,m).(r',m') = (rr', rm' + r'm)$. We call this the *idealization* of M (over R) and denote it by $R\propto M$. The idealization $R\propto M$ is Noetherian local ring with identity $(1,0)$, its maximal ideal is $\mathfrak{m}\times M$ and its Krull dimension is $\dim R$. There is a canonical projection $\rho : R\propto M \longrightarrow R$ defined by $\rho((r,m)) = r$ and a canonical inclusion $\sigma : R \longrightarrow R\propto M$ defined by $\sigma(r) = (r,0)$. These maps are local homomorphisms and we can regard any $R-$module (resp. $R\propto M-$module) as an $R\propto M-$module (resp. $R-$module) by ρ (resp. σ). Note that the structure of $R-$modules induced by the composition $\rho\sigma$ coincides with the original one.

Remark 2.5. Let $\mathfrak{c}$ be an ideal of $R\propto M$. Then $\mathfrak{c}$ is $\mathfrak{m}\times M-$ primary if and only if $\rho(\mathfrak{c})$ is $\mathfrak{m}-$primary. In particular if $\underline{x} = (x_1,\cdots,x_d)$ is a s.o.p of R then $\underline{(x,0)} = ((x_1,0),\cdots,(x_d,0))$ is a s.o.p of $R\propto M$.

Lemma 2.6. *Let* $\dim M = \dim R = d$. *Let* $\underline{x} = (x_1,\cdots,x_d)$ *be a s.o.p of* R. *Let* $Q(\underline{x};R), Q(\underline{x};M)$ *and* $Q(\underline{(x,0)};R\propto M)$ *be defined as in Lemma 2.1. Then we have*

$$\ell\big(R\propto M/Q(\underline{(x,0)};R\propto M)\big) = \ell(R/Q(\underline{x};R)) + \ell(M/Q(\underline{x};M)).$$

Proof. We have $(x_1,0)^t\dots(x_d,0)^t(r,m) = (x_1^t\dots x_d^t r, x_1^t\dots x_d^t m)$, for any element (r,m) belongs to $R\propto M$ and any integer $t>0$. Moreover,

$$((x_1,0)^{t+1},\cdots,(x_d,0)^{t+1}))R\propto M = (x_1^{t+1},\cdots,x_d^{t+1})R\times(x_1^{t+1},\cdots,x_d^{t+1})M,$$

for any integer $t>0$. It follows that $Q(\underline{(x,0)};R\propto M) = Q(\underline{x};R)\times Q(\underline{x};M)$. Therefore we have the exact sequence of $R\propto M-$modules

$$0\longrightarrow M/Q(\underline{x};M)\overset{\epsilon'}{\longrightarrow} R\propto M/Q(\underline{(x,0)};R\propto M)\overset{\rho'}{\longrightarrow} R/Q(\underline{x};R)\longrightarrow 0,$$

where ϵ' (resp. ρ') is deduced from the canonical inclusion $\epsilon : M\longrightarrow R\propto M$ with $\epsilon(m) = (0,m)$ for all $m\in M$ (resp. the projection ρ). These imply the result. □

Proof of Theorem 1.1. Since R is Cohen-Macaulay, $(x_1+x_d)^{n_1}, x_2^{n_2},\dots,x_d^{n_d}$ is a regular $R-$sequence. It follows that

$$Q((x_1+x_d)^{n_1}, x_2^{n_2},\dots,x_d^{n_d};R) = ((x_1+x_d)^{n_1}, x_2^{n_2},\dots,x_d^{n_d})R.$$

Therefore, by Lemma 2.1 we have

$$q_{x_1+x_d,x_2,\dots,x_d;R}(n_1,\dots,n_d)=n_1\dots n_d.$$

Now the results follows easily by Lemmas 2.1, 2.4 and 2.6. □

3. Proof of Theorem 1.2

Firstly, we recall some basic notions and properties of Artinian modules. Following [R] and [Kir], the *Noetherian dimension* of an Artinian $R-$module A, denoted by $\text{N-dim}_R A$, is defined inductively as follows: when $A=0$, put $\text{N-dim}_R A=-1$. Then by induction, for an integer $d\geq 0$, we put $\text{N-dim}_R A=d$ if $\text{N-dim}_R A<d$ is false and for every ascending sequence $A_0\subseteq A_1\subseteq\dots$ of submodules of A, there exists n_0 such that $\text{N-dim}_R(A_n/A_{n+1})<d$ for all $n>n_0$.

The theory of *secondary representation* of Artinian modules is a useful tool in this section. Here we review some facts of this theory from [Mac] and [Sh-H]: Any Artinian $R-$module A has a *minimal secondary representation* $A=A_1+\dots+A_n$ of $\mathfrak{p}_i-$secondary submodules A_i. The set $\{\mathfrak{p}_1,\mathfrak{p}_2,\dots,\mathfrak{p}_n\}$ is independent of the choice of minimal representation of A and is denoted by $\operatorname{Att}_R A$. Note that $\text{N-dim}_R A=0$ if and only if $\operatorname{Att} A=\{\mathfrak{m}\}$ and if and only if $\ell(A)<\infty$. Moreover, for any exact sequence of Artinian $R-$module $0\longrightarrow A'\longrightarrow A\longrightarrow A"\longrightarrow 0$, we have

$$\text{N-dim}_R A=\max\{\text{N-dim}_R A',\text{N-dim}_R A"\}\ \text{and}\ \operatorname{Att} A''\subseteq\operatorname{Att} A\subseteq\operatorname{Att} A'\cup\operatorname{Att} A".$$

From now on, we denote by $s(A)$ the least integer s such that $\mathfrak{m}^sA=\mathfrak{m}^nA$ for all $n\geq s$ and by $Rl(A)$ the length of $A/\mathfrak{m}^{s(A)}A$. It should be noticed that if $x\in\mathfrak{m}$ and $x\notin\mathfrak{p}$ for all $\mathfrak{p}\in\operatorname{Att} A\setminus\{\mathfrak{m}\}$ then $x^nA=\mathfrak{m}^{s(A)}A$ for all $n\geq s(A)$.

Lemma 3.1. *([C-H-M]) (i).* $p(M)=\max\limits_{i=0,1,\dots,d-1}\{\text{N-dim}_R H^i_{\mathfrak{m}}(M)\}$.
(ii). Let $p(M)>0$. *Set*

$$T(M)=\big(\operatorname{Ass} M\bigcup_{i=1}^{d-1}\operatorname{Att}(H^i_{\mathfrak{m}}(M)\big)\setminus\{\mathfrak{m}\}.$$

Let $x\in\mathfrak{m}$ *such that* $x\notin\mathfrak{p}$ *for all* $\mathfrak{p}\in T(M)$. *Then we have* $p(M/xM)=p(M)-1$.

Note that $pf(M)\leq p(M)\leq d-1$. Moreover, if $pf(M)\leq 0$ then $q_{\underline{x};M}(\underline{n})$ is always a polynomial for $\underline{n}$ large enough, for any s.o.p $\underline{x}$ of M. In the case $p(M)=1=pf(M)$ and R admits dualizing complexes, it was proved by [C-M1, Theorem 4.5] that there exists a s.o.p $\underline{x}$ of M such that $q_{\underline{x};M}(\underline{n})$ is a polynomial for $\underline{n}$ large enough. Howeiver, Theorem 1.2 shows that this is not the case for any s.o.p of M.

Lemma 3.2. *Let* $p(M)=1=pf(M)$. *Then there exists a s.o.p* $\underline{x}$ *of* M *such that* $q_{\underline{x};M}(\underline{n})$ *is not a polynomial for* $\underline{n}$ *large enough.*

Proof. Let $T(M)$ as in Lemma 3.2, (ii). Let $(x_1,y_2,\dots,y_d)$ be a s.o.p of M such that $x_1\notin\mathfrak{p}$ for all $\mathfrak{p}\in T(M)$. Since $(x_1,y_2,\dots,y_d)\not\subseteq\mathfrak{p}$ for all $\mathfrak{p}\in T(M)$, we can choose by [K, Theorem 124] an element $a\in(x_1,y_3,\dots,y_d)$ such that $y_2+a\notin\mathfrak{p}$ for all $\mathfrak{p}\in T(M)$. Set $x_2=y_2+a$. Set $x_i=y_i$ for $i\geq 3$. It is easily seen that $\underline{x}=(x_1,\dots,x_d)$ is a s.o.p of M. We will show that $q_{\underline{x};M}(\underline{n})$ is not a polynomial for

$\underline{n}$ large enough. Without loss of generality, we may assume that depth $M > 0$. Let $\overline{M} = M/x_1^{n_1}M$. By [Sh-H, Proposition 2.2] we have the exact sequence:

$$0 \longrightarrow H_{\mathfrak{m}}^{d-1}(M)/x_1^{n_1}H_{\mathfrak{m}}^{d-1}(M) \longrightarrow U(\overline{M})_d^{-d}\overline{M} \stackrel{\Psi_{d+1}}{\longrightarrow} U(M)_{d+1}^{-d-1}M,$$

where Ψ_{d+1} is defined by $\Psi_{d+1}(\overline{m}/(u_2,\dots,u_d,1)) = m/(x_1^{n_1},u_2,\dots,u_d,1)$, for all $\overline{m} \in \overline{M}$ and $(u_2,\dots,u_d,1) \in U(\overline{M})_d$. Let $s = s(H_{\mathfrak{m}}^{d-1}(M))$. It should be noticed that $\mathrm{Ker}(\Psi_{d+1}) = H_{\mathfrak{m}}^{d-1}(M)/\mathfrak{m}^s H_{\mathfrak{m}}^{d-1}(M)$ is of finite length. Therefore it is generated by finitely many elements, say $f_1,\dots,f_l$. On the other hand, it follows by [Sh-Z2] that

$$U(\overline{M})_d^{-d}\overline{M} = \bigcup_{n_2,\dots,n_d \geq 0} \overline{M}(1/(x_2^{n_2},\dots,x_d^{n_d},1)).$$

Moreover, we have $\overline{M}(1/(x_2^{m_2},\dots,x_d^{m_d},1)) \subseteq \overline{M}(1/(x_2^{n_2},\dots,x_d^{n_d},1))$ if $n_i \geq m_i$ for $i = 2,\dots,d$. Therefore, given $n_1 \geq s$, there exists some integer $r(n_1)$ (depends on n_1) such that $f_1,\dots,f_l \in \overline{M}(1/(x_2^{n_2},\dots,x_d^{n_d},1))$ for all $n_2,\dots,n_d \geq r(n_1)$. So, the above exact sequence implies the following exact sequence

$$0 \longrightarrow \mathrm{Ker}(\Psi_{d+1}) \longrightarrow \overline{M}(1/(x_2^{n_2},\dots,x_d^{n_d},1)) \stackrel{\Psi_{d+1}}{\longrightarrow} M(1/(x_1^{n_1},\dots,x_d^{n_d},1)) \longrightarrow 0,$$

for all $n_1 \geq s$, and all $n_2,\dots,n_d \geq r(n_1)$. Therefore we have

$$(1) \qquad q_{\underline{x};M}(\underline{n}) = q_{x_2,\dots,x_d;\overline{M}}(n_2,\dots,n_d) - Rl(H_{\mathfrak{m}}^{d-1}(M)).$$

Note that $e(x_2,\dots,x_d;\overline{M}) = n_1 e(\underline{x};M)$ and $\overline{M}$ is a generalized Cohen-Macaulay by Lemma 3.1. Therefore for given $n_1 \geq s$, there exists by [Sh-H, Theorem 3.7] an integer $s(n_1)$ (depends on only n_1) such that

$$(2) \quad q_{x_2,\dots,x_d;\overline{M}}(n_2,\dots,n_d) = n_1\dots n_d e(\underline{x};M) - \sum_{i=1}^{d-2}\binom{d-2}{i-1}\ell(H_{\mathfrak{m}}^i(M/x_1^{n_1}M)),$$

for all $n_2,\dots,n_d \geq s(n_1)$. Now, assume that there exists a polynomial $f(\underline{X})$ of degree 1 in d variables $X_1,\dots,X_d$ such that $q_{\underline{x};M}(\underline{n}) = n_1\dots n_d e(\underline{x};M) - f(\underline{n})$ for $\underline{n}$ large enough. Then by (1) and (2), for given $n_1 \geq s$ and for all $n_2,\dots,n_d \geq \max\{r(n_1), s(n_1)\}$, $f(\underline{n})$ depends on only n_1. Therefore all variables $X_2,\dots,X_d$ can not appear in any term of $f(\underline{X})$. By repeating the above procees for x_2, all variables $X_1, X_3,\dots,X_d$ can not appear in any term of $f(\underline{X})$. Therefore $f(\underline{X})$ must be a constant. This give a contradiction because the degree of $f(\underline{X})$ is 1. □

Example 3.3. Let R and M be as in Lemma 2.4 with $d = 3$ and $v = 1$. Let $T(M)$ as in Lemma 3.1. Then $T(M) = \{0, (x_1,x_2)R\}$. Let (g_1,g_2,g_3) is a s.o.p of M. If there exist two elements $g_i, g_j, i \neq j, i,j = 1,2,3$ such that $g_i, g_j \in (x_1,x_2)R$ then $q_{g_1,g_2,g_3;M}(n_1,n_2,n_3)$ is a polynomial for n_1,n_2,n_3 large enough by [C-M1, Theorem 4.5]. In other cases, there exist two elements $g_i, g_j, i \neq j, i,j = 1,2,3$ such that $g_i, g_j \notin (x_1,x_2)R$. Therefore by Lemma 3.2, $q_{x_1,x_2,x_3;M}(n_1,n_2,n_3)$ is not a polynomial for n_1,n_2,n_3 large enough. In particular, for all $n_1,n_2,n_3 \geq 1$, we have

$$q_{x_1,x_2,x_3;M}(n_1,n_2,n_3) = q_{x_1,x_1+x_2,x_3;M}(n_1,n_2,n_3) = n_1n_2n_3 - n_3,$$

$$q_{x_1+x_3,x_2,x_3;M}(n_1,n_2,n_3) = q_{x_1+x_3,x_2,x_2+x_3;M}(n_1,n_2,n_3) = n_1n_2n_3 - \min\{n_1,n_3\}.$$

Lemma 3.4. *Let $p(M) = 2$ and $pf(M) > 0$. Then there exists a s.o.p $\underline{x}$ of M such that $q_{\underline{x};M}(\underline{n})$ is not a polynomial for $\underline{n}$ large enough.*

Proof. Let $T(M)$ be as in Lemma 3.1 and $(x_1, y_2 \dots , y_d)$ be a s.o.p of M such that $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in T(M)$. Let

$$T(x_1; M) = \big(\bigcup_{n_1 \geq 1} \mathrm{Ass}(M/x_1^{n_1}M) \bigcup_{i=1}^{d-2} \bigcup_{n_1 \geq 1} \mathrm{Att}(H^i_{\mathfrak{m}}(M/x_1^{n_1}M)\big) \setminus \{\mathfrak{m}\}.$$

We have by [B] that $\bigcup_{n_1 \geq 1} \mathrm{Ass}(M/x_1^{n_1}M)$ is a finite set. Moreover, since $0 :_M x_1^{n_1}$ is of finite length, we get from the exact sequences

$$0 \longrightarrow 0 :_M x_1^{n_1} \longrightarrow M \longrightarrow M/0 :_M x_1^{n_1} \longrightarrow 0$$

$$0 \longrightarrow M/0 :_M x_1^{n_1} \xrightarrow{x_1^{n_1}} M \longrightarrow M/x_1^{n_1}M \longrightarrow 0$$

the following exact sequences for $i = 1, \dots , d-1$,

$$0 \longrightarrow H^i_{\mathfrak{m}}(M)/x_1^{n_1}H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(M/x_1^{n_1}M) \longrightarrow 0 :_{H^{i+1}_{\mathfrak{m}}(M)} x_1^{n_1} \longrightarrow 0.$$

Note that $\ell(H^i_{\mathfrak{m}}(M)/x_1^{n_1}H^i_{\mathfrak{m}}(M)) < \infty$ and $\bigcup_{n_1 \geq 1} \mathrm{Att}(0 :_{H^{i+1}_{\mathfrak{m}}(M)} x_1^{n_1})$ is a finite set by [Sh]. Therefore, $T(x_1; M)$ is a finite set. Because $(x_1, y_2, \dots , y_d) \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in T(M) \cup T(x_1; M)$, we can choose an element $a \in (x_1, y_3, \dots , y_d)$ such that $y_2 + a \notin \mathfrak{p}$ for all $\mathfrak{p} \in T(M) \cup T(x_1; M)$. Set $x_2 = y_2 + a$. Let

$$T(x_2; M) = \big(\bigcup_{n_2 \geq 1} \mathrm{Ass}(M/x_2^{n_2}M) \bigcup_{i=1}^{d-2} \bigcup_{n_2 \geq 1} \mathrm{Att}(H^i_{\mathfrak{m}}(M/x_2^{n_2}M)\big) \setminus \{\mathfrak{m}\}.$$

By similar reasons, $T(x_2; M)$ is a finite set. Therefore we can choose an element $b \in (x_1, x_2, y_4, \dots , y_d)$ such that $y_3 + b \notin \mathfrak{p}$ for all $\mathfrak{p} \in T(x_1; M) \cup T(x_2; M)$. Set $x_3 = y_3 + b$. Let $\underline{x} = (x_1, \dots , x_d)$ with $x_i = y_i$ for all $i \geq 4$. Then $\underline{x}$ is a s.o.p of M. Let $s = s(H^{d-1}_{\mathfrak{m}}(M))$. Similarly to the proof of Lemma 3.2, for given $n_1 \geq s$, there exists an integer $r(n_1)$ such that for all $n_2, \dots , n_d \geq r(n_1)$, we have

$$q_{\underline{x};M}(\underline{n}) = q_{x_2,\dots,x_d;M/x_1^{n_1}M}(n_2, \dots , n_d) - Rl(H^{d-1}_{\mathfrak{m}}(M)). \tag{3}$$

Note that $e(x_2, \dots , x_d; M/x_1^{n_1}M) = n_1 \dots n_d e(\underline{x}; M)$ and $p(M/x_1^{n_1}M) = 1$ by the choice of x_1. Let $s(n_1) = s(H^{d-2}_{\mathfrak{m}}(M/x_1^{n_1}M))$. Similarly to the proof of Lemma 3.2, for given $n_1 \geq s$ and $n_2 \geq \max\{r(n_1), s(n_1)\}$, there exists an integer $r'(n_1, n_2)$ such that

$$\begin{aligned} q_{x_2,\dots,x_d;M/x_1^{n_1}M}(n_2, \dots , n_d) &= n_1 \dots n_d e(\underline{x}; M) - Rl(H^{d-2}_{\mathfrak{m}}(M/x_1^{n_1}M)) \\ &- \sum_{i=1}^{d-3} \binom{d-3}{i-1} \ell(H^i_{\mathfrak{m}}(M/(x_1^{n_1}, x_2^{n_2})M)), \end{aligned} \tag{4}$$

for all $n_3, \dots , n_d \geq r'(n_1, n_2)$. Now, assume that there exists a polynomial $f(\underline{X})$ in variables $X_1, \dots , X_d$ such that $q_{\underline{x};M}(\underline{n}) = n_1 \dots n_d e(\underline{x}; M) - f(\underline{n})$ for $\underline{n}$ large enough. Then by (3) and (4), all variables $X_3, \dots , X_d$ can not appear in any terms of $f(\underline{X})$. Since $x_3 \notin \mathfrak{p}$ for all $\mathfrak{p} \in T(x_1; M)$, we can repeat the above procees for two elements x_1, x_3. It follows that all variables $X_2, X_4, \dots , X_d$ can not appear in any terms of $f(\underline{X})$. Since $x_2 \notin \mathfrak{p}$ for all $\mathfrak{p} \in T(M)$ and $x_3 \notin \mathfrak{p}$ for all $\mathfrak{p} \in T(x_2; M)$, we can also repeat the above procees for two elements x_2, x_3. Therefore all variables $X_1, X_4, \dots , X_d$ can not appear in any terms of $f(\underline{X})$. These follow that $f(\underline{X})$ must be a constant. It give a contradiction because degree of $f(\underline{X})$ is positive. □

Lemma 3.5. *Let $p(M) = 3$ and $pf(M) > 0$. If R admits dualizing complexes then there exists a s.o.p $\underline{x}$ of M such that $q_{\underline{x};M}(\underline{n})$ is not a polynomial for $\underline{n}$ large enough.*

Proof. Let $\mathfrak{a}(M) = \mathfrak{a}_0(M)\dots\mathfrak{a}_{d-1}(M)$, where $\mathfrak{a}_i(M) = \operatorname{Ann} H^i_{\mathfrak{m}}(M)$, $i = 0,\dots,d-1$. Then we have by [C, Theorem 1.1] that $p(M) = 3 = \dim R/\mathfrak{a}(M)$. Therefore, similarly to the proof of Lemma 3.4, we can choose a s.o.p $(x_1, x_2, x_3, y_4\dots, y_d)$ of M such that $(y_4,\dots,y_d)R \subseteq \mathfrak{a}(M)$, $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in T(M)$, $x_2 \notin \mathfrak{p}$ for all $\mathfrak{p} \in T(M) \cup T(x_1;M)$ and $x_3 \notin \mathfrak{p}$ for all $\mathfrak{p} \in T(x_1;M) \cup T(x_2;M)$, where $T(M), T(x_1;M), T(x_2;M)$ defined similarly to the proof of Lemma 3.4. Let $x_4 = y_4 + x_1 + x_2 + x_3$ and $x_i = y_i$ for $i \geq 5$. Let $\underline{x} = (x_1,\dots,x_d)$. Then $\underline{x}$ is a s.o.p of M. Let $s = s(H^{d-1}_{\mathfrak{m}}(M))$. Similarly to the proof of Lemma 3.2, for given $n_1 \geq s$, there exist $r(n_1)$ such that

$$q_{\underline{x};M}(\underline{n}) = q_{x_2,\dots,x_d;M/x_1^{n_1}M}(n_2,\dots,n_d) - Rl(H^{d-1}_{\mathfrak{m}}(M)), \tag{5}$$

for all $n_2,\dots,n_d \geq r(n_1)$. Let $s(n_1) = s(H^{d-2}_{\mathfrak{m}}(M/x_1^{n_1}M))$. Similarly to the proof of Lemma 3.2, for given $n_1 \geq s$ and $n_2 \geq \max\{r(n_1), s(n_1)\}$, there exists an integer $r(n_1, n_2)$ such that

$$\begin{aligned} q_{x_2,\dots,x_d;M/x_1^{n_1}M}(n_2,\dots,n_d) &= q_{x_3,\dots,x_d;M/(x_1^{n_1},x_2^{n_2})M}(n_3,\dots,n_d) \\ &\quad - Rl(H^{d-2}_{\mathfrak{m}}(M/x_1^{n_1}M)), \end{aligned} \tag{6}$$

for all $n_3,\dots,n_d \geq r(n_1,n_2)$. Let

$$T(M/(x_1^{n_1},x_2^{n_2})M) = \big(\operatorname{Ass}(M/(x_1^{n_1},x_2^{n_2})M) \bigcup_{i=1}^{d-3} \operatorname{Att}(H^i_{\mathfrak{m}}(M/(x_1^{n_1},x_2^{n_2})M)\big) \setminus \{\mathfrak{m}\}.$$

Since $(x_3, y_4, y_5,\dots,y_d)$ is a s.o.p of $M/(x_1^{n_1},x_2^{n_2})M$ and

$$(y_4, y_5,\dots,y_d)R \subseteq \mathfrak{a}(M) \subseteq \operatorname{Rad}(\mathfrak{a}(M/(x_1^{n_1},x_2^{n_2})M)),$$

we can easily check that $x_3 \notin \mathfrak{p}$ for all $\mathfrak{p} \in T(M/(x_1^{n_1},x_2^{n_2})M)$ and for all $n_1, n_2 \geq 1$. Let $s(n_1,n_2) = s(H^{d-3}_{\mathfrak{m}}(M/(x_1^{n_1},x_2^{n_2})M))$. Note that $p(M/(x_1^{n_1},x_2^{n_2})M) = 1$ and

$$e(x_3,\dots,x_d; M/(x_1^{n_1},x_2^{n_2}M) = n_1\dots n_d e(\underline{x};M).$$

So, similarly to the proof of Lemma 3.2, for given $n_1 \geq s, n_2 \geq \max\{r(n_1), s(n_1)\}$ and $n_3 \geq \max\{r(n_1,n_2), s(n_1,n_2)\}$, there exists an integer $r''(n_1,n_2,n_3)$ such that

$$\begin{aligned} q_{x_3,\dots,x_d;M/(x_1^{n_1},x_2^{n_2})M}(n_3,\dots,n_d) &= n_1\dots n_d e(\underline{x};M) - Rl\big(H^{d-3}_{\mathfrak{m}}(M/(x_1^{n_1},x_2^{n_2})M)\big) \\ &\quad - \sum_{i=1}^{d-4}\binom{d-4}{i-1}\ell(H^i_{\mathfrak{m}}(M/(x_1^{n_1},x_2^{n_2},x_3^{n_3})M)), \end{aligned} \tag{7}$$

for all $n_4,\dots,n_d \geq r''(n_1,n_2,n_3)$. Now, assume that there is a polynomial $f(\underline{X})$ in variables $X_1,\dots,X_d$ such that $q_{\underline{x};M}(\underline{n}) = n_1\dots n_d e(\underline{x};M) - f(\underline{n})$ for $\underline{n}$ large enough. Then by (5), (6) and (7), all variables $X_4,\dots,X_d$ can not appear in any terms of $f(\underline{X})$. Define similarly the set $T(M/(x_1^{n_1},x_3^{n_3})M)$. Since $(x_4, y_4, y_5,\dots,y_d)$ is a s.o.p of $M/(x_1^{n_1},x_3^{n_3})M$ and

$$(y_4, y_5,\dots,y_d)R \subseteq \mathfrak{a}(M) \subseteq \operatorname{Rad}(\mathfrak{a}(M/(x_1^{n_1},x_3^{n_3})M)),$$

we can check that $x_4 \notin \mathfrak{p}$ for all $\mathfrak{p} \in T(M/(x_1^{n_1}, x_3^{n_3})M)$ and for all $n_1, n_3 \geq 1$. So, we can repeat the above procees for three elements x_1, x_3, x_4 and we get that all variables $X_2, X_5, \dots, X_d$ can not appear in any terms of $f(\underline{X})$. By the same reasons, we can repeat the above procees for x_1, x_2, x_4 and x_2, x_3, x_4 and we get that all variables $X_3, X_5, \dots, X_d$ and all variables $X_1, X_5, \dots, X_d$ can not appear in any terms of $f(\underline{X})$. Therefore $f(\underline{X})$ must be a constant. It gives a contradiction because $pf(M) > 0$. □

Proof of Theorem 1.2. Follows by Lemmas 3.2, 3.4 and 3.5. □

Remark 3.6. All our attempts to obtain an extension of Theorem 1.2 which applies to the case where $p(M) > 3$ have failed. The difficulty is that for a subset s.o.p $x_1, \dots, x_u$ of M with $d-2 \geq u \geq 2$, we do not know when the sets

$$\bigcup_{n_1,\dots,n_u>0} \operatorname{Ass}(M/(x_1^{n_1}, \dots, x_u^{n_u})M) \text{ and } \bigcup_{n_1,\dots,n_u>0} \operatorname{Att}\left(0 :_{H^i_{\mathfrak{m}}(M)} (x_1^{n_1}, \dots, x_u^{n_u})R\right)$$

are finite sets. It was proved in [B-R-Sh] that there are saveral special cases in which the set $\bigcup_{n_1,\dots,n_u>0} \operatorname{Ass}(M/(x_1^{n_1}, \dots, x_u^{n_u})M)$ is a finite. However, in general, this problem is still open.

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