SINGULAR LAGRANGIAN MANIFOLDS and SEMI-CLASSICAL ANALYSIS

Yves Colin de Verdière [a, b]

May 2, 2001

[a] Institut Universitaire de France
[b] Institut Fourier, Unité mixte de recherche CNRS-UJF 5582
BP 74, 38402-Saint Martin d'Hères Cedex (France)
yves.colin-de-verdiere@ujf-grenoble.fr
http://www-fourier.ujf-grenoble.fr/~ycolver/

Prépublication de l'Institut Fourier n[0] 534 (2001)
http://www-fourier.ujf-grenoble.fr/prepublications.html

Abstract

Lagrangian submanifolds of symplectic manifolds are very central objects in classical mechanics and microlocal analysis. These manifolds are frequently singular (integrable systems, bifurcations, reduction). There has been a lot of works on singular Lagrangian manifolds initiated by Arnold, Givental and others. The goal of our paper is to extend the classical and semi-classical normal forms of completely integrable systems near non degenerate (Morse-Bott) singularities to more singular systems. It turns out that there is a nicely working way to do that, leading to normal forms and universal unfoldings. We obtain this way natural Ansatz's extending the WKB-Maslov Ansatz. We give more details on the simplest non Morse example, the cusp, which corresponds to a saddle-node bifurcation[1].

Keywords: singular Lagrangian manifolds, integrable Hamiltonian systems, bifurcations, Bohr-Sommerfeld rules, WKB, semi-classics, normal forms, versal deformations.
Mathematics Subject Classification 2000: 32S05, 32S40, 34E20, 35P20, 53D12, 70H06.

[1]**Acknowlegments:** the papers of Eric Delabaere and Frédéric Pham as well as discussions with them were important sources of inspiration. Emmanuel Ferrand provided me a basic list of references on singular Lagrangian manifolds. I thank also Bernard Malgrange for remarks on preliminary versions, discussions and "theorem 1" and San Vũ Ngọc for carefully reading a preliminary version.

Introduction

In the papers [11], [12], [13], [10], [34] and [15], we studied semi-classical completely integrable Hamiltonian systems whose singularities are of Morse-Bott type using normal forms of Birkhoff type. In the nice paper [29] which was an important source of inspiration for us, Frédéric Pham showed the universality of solutions of semi-classical Schrödinger equations with polynomial potentials. Our goal is to extend this analysis allowing (more general) canonical transformations in order to study for example

- the saddle-node bifurcation
- the Birkhoff normal form in case of $k : 1$ resonances with $k \geq 3$ in the spirit of [15]
- the bifurcation of periodic orbits of a Hamiltonian system where the Poincaré map of a periodic orbit admits an eigenvalue which is a cubic root of 1
- the adiabatic limit or the Born-Oppenheimer approximation with crossings of more than 2 eigenvalues.

This way, we propose a general setting inspired by "Thom's catastrophe theory" (see [3]) and present a sketchy study of the *saddle-node bifurcation* (the cusp) $\xi^2 + x^3 = 0$.

A more algebraic (co-homological approach) is presented in [37].

The subject is really the study of the *singularities of Lagrangian manifolds*, of their *deformations* (or *bifurcations*) and of the associated *semi-classical Ansatz's*. Building up classical and semi-classical normal forms leads to study model problems depending on a finite number of parameters among whose the simplest were already described in the litterature: cubic oscillators (see [9], [8], [19]), quartic oscillators (see [30], and polynomial potentials (see [39])). A remarquable fact is that we can use the same methods for the classical and the semi-classical bifurcations and in particular the codimension of the singularities are the same. Of course, the study of the classical Hamiltonian dynamic in a 2D phase space is trivial, but this is no more the case for the semi-classical case which we reduce to special functions.

The reader should take care of the fact that caustic singularities is a different problem for which Lagrangian manifolds usually are smooth. We strongly use canonical transformations which eliminate the problem of caustics.

The main idea is to forget the equations of the manifolds and to focus on the *ideal of functions* which vanish on it. The same idea turned out to be very important in algebraic geometry. On the quantum side, we do the same change of point of view: we consider *left ideals of pseudo-differential operators.* We can do that because any solution of $\hat{P}u = 0$ satisfies also $\hat{B}\hat{P}u = 0$ for any operator $\hat{B}$. It appears that usual singularities, at least for 1 degree of freedom, do admit normal forms and their deformations have a universal model depending of a finite number of parameters. The solutions of this model are the ad'hoc special functions: the smooth case corresponds that way to the BKW-Maslov Ansatz, the Morse-Bott case corresponds to Lagrangian intersections (hyperbolic case) or to coherent states (elliptic case)... An important part of the *programme* is the study of these special functions.

In the case of the cusp $\xi^2 + x^3 = 0$, it is enough to study the differential equation (cubic Schrödinger equation):

$$-u'' + (x^3 + Ax + B)u = 0 \ .$$

We give the general definitions for any dimension and we restrict after section 2 to the case of a 2 dimensional phase space.

The main non trivial result is theorem 6 which is an holomorphic versal deformation result for all quasi-homogeneous isolated singularities of curves.

The semi-classical results follow then from the techniques already developped in [13].

Contents

1 Singular Lagrangian manifolds

1.1 Definitions

There are several possible definitions of germs of singular Lagrangian manifolds. The most appropriate context is the real analytic one. We will denote by $(Z^{2d}, \omega; z_0)$ a germ of non-singular real analytic symplectic manifold of dimension $2d$ which, by Darboux theorem, can be identified with $(T^\star R^d, \sum d\xi_i \wedge dx_i; 0)$. $\mathcal{E}$ will denote the algebra of germs of real valued analytic functions (or smooth functions).

Definition 1 *1. A (germ of)* singular Lagrangian manifold *L in Z^{2d} is a germ of real analytic variety (ie complex variety invariant by complex conjugation) of dimension d which is Lagrangian near all smooth points. We will denote by $\mathcal{L}$ or $\mathcal{L}_L$ the ideal of $\mathcal{E}$ of functions vanishing on L. If $F_j,\ j = 1, \cdots, n$ is a system of generators of $\mathcal{L}$ we will denote $\mathcal{L} =< F_1, \cdots, F_n >$. This ideal is* involutive *meaning that $\{\mathcal{L}, \mathcal{L}\} \subset \mathcal{L}$.*

2. A (germ of)singular Lagrangian manifold L is a complete intersection *if the ideal $\mathcal{L}_L$ is generated by d functions.*

3. A (germ of) singular Lagrangian manifold L is a singular leaf of a Lagrangian foliation *if $\mathcal{L}_L$ admits a set of generators $F_j,\ j = 1, \cdots, d$ such that $\{F_j, F_k\} = 0$ for all j, k.*

In the first case we will speak about a (germ of) singular Lagrangian manifold, *in the second of a* (germ of) singular Lagrangian manifold which is a complete intersection *and in the third of a* (germ of) singular leaf of a completely integrable system.

We can ask the

Question 1 *Are cases 2 and 3 really distinct: does every singular Lagrangian manifold which is a complete intersection admits Poisson commuting generators $F_j,\ j = 1, \cdots, d$?*

1.2 Examples

Example 1.1 *Let $d = 1$ and $f : Z^2 \to \mathbb{R}$ a proper map. If 0 is a critical value of f, the curve $\{f = 0\}$ is a Lagrangian singular manifold with respect to all possible definitions. If f is a Morse function the level sets $L_E = f^{-1}(E)$ are smooth except for a discrete set of energies.*

Example 1.2 *Let us start with an anharmonic oscillator with only one resonance, like $H = |z_1|^2 + |z_2|^2 + \sum_{j=3}^d \omega_j |z_j|^2 + O(|z|^3)$ where $(1, \omega_3, \cdots, \omega_d)$ are linearly independent over the rationals. We get an integrable system using the truncated Birkhoff normal form. The Hamiltonians are $F_1 = |z_1|^2 + |z_2|^2,\ F_2 = |z_3|^2,\ F_{d-1} = |z_d|^2, F_N = K$ where $K = O(|z|^3)$ is a polynomial. Reducing by the action of T^{d-1} given by the $d-1$ first hamiltonians we get a projective line depending of $d-1$ parameters; K can be seen as a function on this projective line depending of $d-1$ parameters and hence the Lagrangian foliation admits generically all singularities of codimension $\leq d$ of functions of 2 variables (this example was described to me by Marc Joyeux, see section 4.2).*

Example 1.3 *The normal bundle L of the cusp $9x^2 - 8y^3 = 0$ is a singular Lagrangian manifold parametrized by*

$$m(u, v) = (u^3/3, u^2/2; v, -uv) \ .$$

The ideal $\mathcal{L}$ of functions vanishing on L is minimally generated by:

$$F_1 = 9x^2 - 8y^3,\ F_2 = 3x\xi + 2y\eta,\ F_3 = \eta^2 - 2y\xi^2, F_4 = 3x\eta + 4y^2\xi\ ,$$

(as computed by Marcelo Morales) hence it is not a complete intersection.

Example 1.4 *The (open) swallowtail S (see [1]) can be defined as the subset S of the set Z of polynomials*

$$P = x^5 + ax^3 + bx^2 + cx + d$$

admitting a zero of order at least 3. We can write $P = (x-u)^3(x^2+3ux+v)$ which give a parametrization of S. There exists a natural symplectic structure on Z for which S is Lagrangian. This manifold is obtained in a generic way in the following problem: if $X \subset \mathbb{R}^3$ is a surface and V a vector field on X whose integral curves are geodesics, the set of affine lines generated by the vectors $V(m)$, $m \in X$ is a (singular) Lagrangian manifold in the symplectic manifold of affine lines in $\mathbb{R}^3$. It can be shown that S is not a complete intersection[2].

1.3 Reduction

If $Z \subset X$ is a co-isotropic manifold of a symplectic manifold X and Z^o the isotropic foliation of Z, $X_R = Z/Z^o$ is the *reduced symplectic manifold.* If L is a Lagrangian submanifold of X, $L_R = L \cap Z/Z^o$ is the reduced Lagrangian manifold in X_R. It is well known that the reduced Lagrangian manifold L_R is a smooth Lagrangian manifold if some *clean intersection property (Morse-Bott)* is satisfied (see [24] p 291-292). Otherwise, L_R can be singular.

In semi-classical analysis, singular reductions occur in the *trace formula;* the classical Lagrangian submanifold of $T^\star\mathbb{R} = \{(t,\tau)\}$ is obtained from the classical flow φ_t of an Hamiltonian H in the following way: $L \subset T^\star(\mathbb{R} \times M \times M) = X$ is defined by

$$L = \{(t,\tau;x,\xi;y,\eta) | \tau = H(x,\xi), (x,\xi) = \varphi_t(y,-\eta)\}$$

and we reduce using the co-normal bundle to the diagonal:

$$Z = N^\star\{(t,x,x) | t \in \mathbb{R},\ x \in M\}\ .$$

We obtain the reduced Lagrangian manifold L_R which is the *microsupport* of the spectral density of the Schrödinger operator $\hat{H}$

$$L_R = \{(T,E) | \exists (x,\xi), H(x,\xi) = E, \varphi_T(x,\xi) = (x,\xi)\}\ .$$

Singularities of L_R corresponds to *bifurcations* of periodic orbits of the Hamiltonian flow.

Reduced Lagrangian manifolds are obtained locally from generating functions (Hörmander's *phase functions*): if $\varphi : X \times \mathbb{R}^N \to \mathbb{R}$ (the phase function), $Z = \{(x,\theta;\xi,0)\} \subset T^\star(X \times \mathbb{R}^N)$, $Z_R = T^\star X$ and L is the graph of $d\varphi$. Semi-classical objects (WKB-Maslov Ansatz) are then given by the following oscillatory integrals:

$$u_h(x) = \int_{\mathbb{R}^N} e^{i\varphi(x,\theta)/h} a(x,\theta) d\theta$$

whose microsupport is the reduced Lagrangian manifold

$$L_\varphi = \{(x, \partial_x\varphi) \mid \partial_\theta\varphi = 0\}\ .$$

[2] I thank very much Marcelo Morales for the computations of these examples

Question 2 Give a characteristic property of singular germs of Lagrangian manifolds which are obtained by reduction of a smooth Lagrangian manifold.

With respect to this question, we propose the 2 following examples:

Example 1.5 *If $L \subset T^\star\mathbb{R}^2$ is the graph of $d\varphi$ with $\varphi(x,y) = y(x^2 - y^2/3)$ and $Z = \{\eta = 0\}$, we get $L_R = \{(x, 2xy) \mid y = \pm x\} = \{\xi^2 - 4x^4 = 0\}$.*

Example 1.6 *We have the following (see also [35]) :*

Proposition 1 *The germ at 0 of the normal bundle of the cusp (example 1.3) cannot be obtained by reduction of a germ of* smooth *Lagrangian manifold.*

Proof.–

The Maslov index of any closed curve inside the smooth part of the germ would be zero. Let us consider the curve

$$\gamma(\theta) = m(\cos\theta, \sin\theta)$$

The Lagrangian vector space tangent to L at the point $\gamma(\theta)$ is generated by the vectors $(0,0;1,-\cos\theta)$, $(\cos^2\theta, \cos\theta; 0, -\sin\theta)$ and by reduction with respect to $\xi = 0$, we get the curve $\theta \to [(\cos\theta; -\sin\theta)]$ inside the Lagrangian Grassmanian of $T^\star\mathbb{R}$ whose Maslov index is ± 2.

□

2 Infinitesimal deformations

We propose below a very naïve approach, restricting ouself to 2D phase spaces: a more precise and algebraic approach in any dimension can be found in [37].

We reduce in what follows to the case $d = 1$.

Definition 2 *We will say that the germs $(< F_0 >, \omega_0)$ and $(< F >, \omega)$ are* equivalent *if there exists a germ of diffeomorphism χ such that $F \circ \chi = EF_0$ ($E(0) \neq 0$) and $\chi^\star(\omega) = \omega_0$. By Darboux theorem, we will often restrict ourself to $\omega = \omega_0$.*

By the codimension of $(< F_0 >, \omega_0)$, we mean the codimension of the set of germs equivalent to $(< F_0 >, \omega_0)$.

More explicitely:

Definition 3 *Given a singular germ of curve $\mathcal{L}$ in $T^\star\mathbb{R}$ given by $F_0 = 0$, the* space of infinitesimal deformations *(as a Lagrangian manifold) of $\mathcal{L} =< F_0 >$ is the space of all germs of functions $\mathcal{E}$.*

A general deformation of (F_0, ω_0) is given by (F_t, ω_t). Using Darboux, we can reduce to deformations $(F_0 + tK + O(t^2), \omega_0)$. K is an arbitrary germ of real valued function.

Definition 4 *A deformation $\mathcal{L}_t =< F_t >$ is* trivial *if there exists a smooth family χ_t of canonical transformations and a smooth family of functions $E_t \in \mathcal{E}$, such that:*

$$F_t \circ \chi_t = E_t F_0 \ .$$

This implies that there exists germs of functions X and Y such that the infinitesimal deformation $K = \frac{dF_t}{dt}_{|t=0}$ satisfies:

$$K = \{X, F_0\} + YF_0 \ .$$

We can now give the definition of a finite codimensional singular germ of curve:

Definition 5 *We will say that* $\mathcal{L} =< F_0 >$ *is* of finite codimension μ *if*

$$\dim\big(\mathcal{E}/\left(\{\mathcal{E}, F_0\} + \mathcal{E}.F_0\right)\big) = \mu < \infty \ , \tag{1}$$

where $\{.|.\}$ *is the Poisson bracket,*

A basis $K_\alpha \in D_{\mathcal{L}},\ \alpha = 1, \cdots, \mu$, *of a supplementary space of* $\{\mathcal{E}, F_0\} + \mathcal{E}.F_0$ *in* $D_{\mathcal{L}}$ *will be called a* (uni)versal deformation of $\mathcal{L}$.

More precisely, we ask that equation (1) is true with $\mathcal{E}(U_j)$ *for a basis* U_j *of neighbourhoods of* O *(with the same functions* K_α*).*

Question 3 *Give a natural extension of the definition 1.1 to the case of systems of operators, i.e. matrix valued germs of functions (see [7]).*

3 Examples

1. **The smooth case:** the differentials dF_j are linearly independent in some neighbourhood of the origine. Then $\mathcal{L}$ is a germ of smooth Lagrangian manifold. This $\mathcal{L}$ is of codimension 0. Moreover Darboux theorem implies that up to canonical transformation $\mathcal{L} =< \xi_1, \cdots, \xi_d >$.

2. **The Morse** ($d = 1$) **case:** let $F_\varepsilon = F_0 + O(\varepsilon)$ where F_0 is a non degenerate quadratic form on $T^\star\mathbb{R}$. By the *lemme de Morse isochore* (see [14]), there exists χ_ε a germ of canonical transformations smoothly depending of ε and a smooth function Φ_ε such that

$$F_\varepsilon \circ \chi_\varepsilon = \Phi_\varepsilon \circ F_0$$

and $\Phi_0'(0) \neq 0$. Hence Φ_ε admits a non degenerate zero $t(\varepsilon)$ and we have

$$F_\varepsilon \circ \chi_\varepsilon(x, \xi) = E_\varepsilon(x, \xi)(F_0(x, \xi) - t(\varepsilon))$$

from which it is clear that $< F_0 - t >$ is a versal deformation of $< F_0 >$.

3. **The Eliasson case** ([21] or the non degenerate case of [33], définition 2.1.). It is an extension of the previous case to several quadratic forms. Let

$$q_1, \cdots q_d$$

be d independent commuting quadratic forms on $T^\star\mathbb{R}^d$ where $(q_1, \cdots, q_d)$ is of type (m_e, m_h, m_f) and $d = m_e + m_h + 2m_f$ where m_e is the number of elliptic forms, m_h the number of hyperbolic one's and m_f the number of focus-focus one's. We have $\mu = d$. This value is minimal for rank 0 singular point of an integrable system.

4. **Cusp** (A_2) : $F_0 = \xi^2 + x^3$ $(d = 1)$ and $\mu = 2$:

$$K_1 = 1,\ K_2 = x \ .$$

We will see that up to canonical transformation any F which admits a non degenerate cusp is equivalent to the standard example $\xi^2 + x^3$.

5. **Quartic oscillator** (A_3^+) $F_0 = \xi^2 + x^4$ $(d = 1)$ and $\mu = 3$: $K_1 = 1,\ K_2 = x,\ K_3 = x^2$.

6. **Quartic anti-oscillator** (A_3^-) $F_0 = \xi^2 - x^4$ or $F_0 = x(x - \xi^2)$ $(d = 1)$ and $\mu = 3$.

7. **Triple crossing** (D_4^-) $F_0 = x\xi(x-\xi)$ $(d=1)$ and $\mu = 4$:

$$K_1 = 1,\ K_2 = x,\ K_3 = \xi,\ K_4 = x\xi\ .$$

8. **Hyperbolic umbilic** (D_4^+) $F_0 = x(x^2+\xi^2)$ $(d=1)$ and $\mu = 4$.

Question 4 *Describe all singular Lagrangian manifolds of small codimension.*

4 Integrable systems

4.1 Singularities of integrable systems

Definition 6 *An* integrable Hamiltonian system *is given by a map (the momentum map):*

$$F = (F_1, \cdots, F_d) : Z^{2d} \to \mathbb{R}^d$$

where the Poisson brackets $\{F_i, F_j\}$ *all vanish identically. We assume that the differentials* $dF_j(z)$ *are linearly independent almost everywhere in* Z.

A singular point z_0 *is a point where the* rank $r(z_0)$ *of the* $dF_j(z_0)$ *is* $< d$.

A singular point is reduced *if his rank vanishes.*

4.2 Singularities of integrable systems and deformations of Lagrangian manifolds

Let $< F(x,\xi;t) = 0 >$, $(t \in \mathbb{R}^N)$ be a deformation of the germ of curve $< F(x,\xi,0) = 0 >$ and assume

$$(\star)\ \frac{\partial F}{\partial t}(0,0;0) \neq 0$$

We can associate to it a germ of completely integrable system in $T^\star\mathbb{R}^N$ where $(t,x) \in \mathbb{R}^N$ in the following way: we choose coordinates $t = (t', t_N)$ such that $\partial_{t_N} F \neq 0$. Then we can rewrite $F(x,\xi,t) = E(x,\xi,t)(H_N(x,\xi;t') - t_N)$. We take the commuting Hamiltonians $t_1, \cdots, t_{N-1}, H_N$ which define an integrable germ.

We can go back to the deformation in the following way: we start with the integrable germ with a singularity of rank $N-1$ and choose $t_1, \cdots, t_{N-1}$ integrals whose differential at the singular point are independent. We reduce the systems and we get for each $a \in \mathbb{R}^{N-1}$ a 2-dimensional curve $< H_N(x,\xi,a) - b >$ which give the previous deformation.

Proposition 2 *The previous correspondence is an isomorphism between germs of integrable systems of rank* $N-1$ *(modulo canonical diffeomorphisms) and* N *parameters deformations of curves (modulo canonical diffeomorphisms) satisfying* $(\star)$.

Moreover, we get that way a correspondence between *universal deformations* (deformations containing the versal deformation) and *stable germs of integrable systems.* A germ of integrable systems will sid to be stable if the singularity is moved into an equivalent one by a small perturbation of the germ of integrable system.

In other words, deformations of codimension $\mu \leq N$ corresponds to generic singularities of integrable systems with N degrees of freedoms and associated separatrix for the momentum map in $\mathbb{R}^N$ are of codimension $N - \mu$.

4.3 Generic singularities of integrable systems with 2 degrees of freedom

From the previous sections, we get the following list of locally stable singularities of integrable systems with 2 degrees of freedom (see [22] for pictures of these separatrices for classical systems).

1. **Rank 1:**

   E (elliptic) $(x_1^2 + \xi_1^2,\ \xi_2)$ $(\mu = 1)$

   H (hyperbolic) $(x_1\xi_1,\ \xi_2)$ $(\mu = 1)$

   C (cusp) $(\xi_1^2 + \xi_2 x_1^3, \xi_2)$ $(\mu = 2)$

2. **Rank 0:**

   EE (elliptic-elliptic), **EH** (elliptic-hyperbolic), **HH** (hyperbolic-hyperbolic), **L** (loxodromic) $(\mu = 2)$.

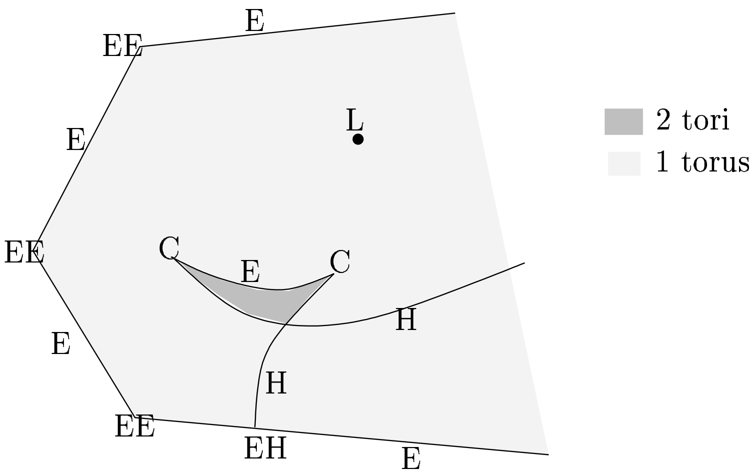

Figure 1: Typical bifurcation diagramm for a 2 degrees of freedom system

Question 5 *Are there other stable singularities? Find the corresponding list for* $d = 3, 4, \cdots$.

5 The symplectic codimension of curves with isolated singularities

For the $d = 1$ case, Bernard Malgrange [3], using the Gauss-Manin connection and results of Sebastiani (see [26]), showed me the following result (see [29] for the hyperelliptic case), also observed in [37]:

Theorem 1 *If* $F_0 : (T^\star\mathbb{R} = \mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *is a germ of analytic function and admits an isolated singularity at* 0 *whose multiplicity is* μ, $< F_0 >$ *is of codimension* μ.

[3] Oral communication

Recall that the multiplicity μ (see [27]) of a germ $F : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ of isolated singularity is the $\mathbb{C}-$dimension of $\mathcal{E}/\mathrm{Jac}(F)$ where $\mathrm{Jac}(F)$ is the ideal generated by the two partial derivatives of F. The germs of the non singular curves $\{F = \varepsilon\}$ ($\varepsilon \neq 0$ and small) have the homotopy type of a bouquet of μ circles (the "vanishing cycles").
Proof.–

Let us denote by Ω^j the germs of differential forms of degree j near 0 in $\mathbb{C}^2$. From the results of Sebastiani (see [26] p.416), we know that

$$\Omega^2/dF_0 \wedge d\Omega^0$$

is a free module of rank μ over $\mathbb{C}\{F_0\}$. We get the consequence that:

$$\Omega^2/\left(dF_0 \wedge d\Omega^0 + F_0\Omega^2\right)$$

is of dimension μ over $\mathbb{C}$. The result follows from the natural identifications of the 2-forms with the functions and of the wedge product $df \wedge dg$ with the Poisson bracket.

□

A simple proof of theorem 1 in the quasi-homogeneous case will be given in section 7.

Theorem 1 admits a very nice geometrical interpretation which we can derive from the paper [29]. If χ is a germ of canonical transformation near the origin, actions integrals over small cycles are preserved. Hence any (uni)versal deformation should be able to reproduce the variations of the action integrals over the vanishing cycles. This is strongly consistent with the fact that μ is also the number of vanishing cycles as shown in [27]. This is exactly the way things work in the quasi-homogeneous case as shown in section 8; we will show there how to get the versal deformation theorem for quasi-homogeneous singularities.

If the singularity is not quasi-homogeneous, $\mathcal{E}F_0 + \{\mathcal{E}, F_0\}$ is no more the Jacobian ideal; indeed Saito proved in [32] that: ($F \in \mathrm{Jac}(F)$) *implies* (F *quasi-homogeneous).* In other words, there are deformations which are trivial as singularities of functions, but not for the symplectic version. There is always a choice of a versal deformation which is valid for both problems: a pair of vector subspaces of the same codimension always admit a commun supplementary subspace.

For example of a non quasi-homogeneous singularity, we can take the singularity called Z_{11} ($\mu = 11$) in [3], which is given by $F_a = x^3\xi + \xi^5 + ax\xi^4$. Different values of a give non-equivalent singularities of functions, but equivalent ideals.

If $F_0 = 0$ is a germ of singular curve, we can associate to it a de Rham complex as in [23]:

$$0 \to \mathcal{E} \to \Omega^1/K \to 0$$

where the non trivial arrow is d and K is the set of 1-form which vanish on the tangent vectors to the smooth stratum of $F_0 = 0$. Then

$$H^1_{de\ Rham}(< F_0 >) = \Omega^1/(K + d\mathcal{E}) \ .$$

There is a subspace of the space of infinitesimal deformations which we can identify with $H^1_{de\ Rham}(< F_0 >)$. If $\alpha \in \Omega^1$ is a germ of 1-form, it gives a deformation of (F_0, ω_0) defined by $(F_0, \omega_0 + \varepsilon d\alpha)$. It is equivalent to fix ω_0 and to deform F_0 by $F_0 + \varepsilon dF_0(X_\alpha)$ where X_α is defined by $\iota(X_\alpha)\omega_0 = \alpha$. It is easy to check that, if $\alpha - dh$ vanishes on C_0, the deformation is trivial and conversely if the deformation is trivial the cohomology class of α vanishes.

We can summarize the situation as follows (see also [37]):

Theorem 2 *We have a decomposition of the infinitesimal versal deformation space into a direct sum of H^1_{dR} and the space of deformations of the ideal $< F_0 >$ whose dimension τ is called the* Tyurina number. *We have:*

$$\mu = \tau + b_1 \ .$$

This decomposition is easily obtained by looking for a deformation $(< F_t >, \omega_t)$ of a pair of a germ of curve and a symplectic form.

6 Versal deformations: the formal case

Theorem 3 *Let $\mathcal{L}_0 =< F_0 >$ $(d = 1)$ be a singular Lagrangian manifold of codimension μ and let us denote by $F_0 + \sum_{\alpha=1}^{\mu} x_\alpha K_\alpha$ a versal deformation of F_0. Let $\mathcal{L}_\varepsilon =< F_\varepsilon >$ with $F_\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k F_k + O(\varepsilon^\infty)$ be a smooth deformation of $\mathcal{L}_0 =< F_0 >$.*

Then there exists a smooth family of canonical transformations χ_ε, a smooth invertible function $E_\varepsilon(x,\xi)$ and smooth functions $a_\alpha(\varepsilon) = O(\varepsilon)$ such that

$$F_\varepsilon \circ \chi_\varepsilon = E_\varepsilon\big(F_0 + \sum_{\alpha=1}^{\mu} a_\alpha(\varepsilon) K_\alpha\big) + O(\varepsilon^\infty)) \ .$$

Question 6 *We may conjecture on the basis of the Morse case and of the proof that the formal series $a_\alpha(\varepsilon)$ are uniquely defined.*

Proof.–

We assume that

$$F_\varepsilon = F_0 + \sum_{\alpha=1}^{\mu} a_\alpha(\varepsilon) K_\alpha + \varepsilon^n R_n + O(\varepsilon^{n+1})$$

We need to find $\chi_\varepsilon = Id + O(\varepsilon^n) = \exp(\varepsilon^n Z) + O(\varepsilon^{n+1})$ where Z is the Hamiltonian vector field of X such that

$$F_\varepsilon \circ \chi_\varepsilon = (1 + \varepsilon^n E)(F_0 + \sum_{\alpha=1}^{\mu} a_\alpha(\varepsilon) K_\alpha + \varepsilon^n (\sum b_\alpha K_\alpha)) + O(\varepsilon^{n+1}) \ .$$

By identification of terms in ε^n, we get the following equation:

$$\{F_0, X\} - EF_0 = -R_n + \sum b_\alpha K_\alpha \ ,$$

which can be solved in a fixed open set by the hypothesis of finite codimension.

□

We assume that $< F_0 >$ is of codimension μ. Let $< F_\varepsilon >$ be a smooth deformation of $< F_0 >$. A basic question is the following one:

do there exists a smooth canonical deformation of the identity χ_ε, a smooth deformation E_ε of the function 1 and smooth functions $a_\alpha(\varepsilon) = O(\varepsilon)$ such that

$$F_\varepsilon \circ \chi_\varepsilon = E_\varepsilon(F_0 + \sum_\alpha a_\alpha(\varepsilon) K_\alpha) \ ? \qquad (2)$$

The transformations χ_ε move then the deformation $< F_\varepsilon >$ of $< F_0 >$ into the universal one $< F_0 + \sum a_\alpha(\varepsilon) K_\alpha >$. The condition of finite codimension allows to solve the linearized problem, so it is natural to ask the following:

Question 7 *Does there exists in this context an implicit function theorem "à la Mather"?*

The answer is yes for quasi-homogeneous singularities in the holomorphic case (see section 8.1).

7 The quasi-homogeneous case

7.1 Definitions

We give the:

Definition 7 *$F = F(x,\xi)$ is $(a,b,n)-$quasi-homogeneous, where a,b and n are integers with a and b coprime, if F is a polynomial satisfying the identity:*

$$F(t^a x, t^b \xi) = t^n F(x,\xi) \ .$$

We denote by $\mathcal{E}^n_{a,b}$ this space of polynomials.

Any monomial $x^p\xi^q$ is in $\mathcal{E}^{pa+qb}_{a,b}$. The algebra $\mathbb{R}[[x,\xi]]$ of formal series with the usual products is gradued by

$$\mathbb{R}[[x,\xi]] = \oplus_{n=0}^{\infty}\mathcal{E}^n_{a,b} \ .$$

Concerning Poisson brackets, we have:

$$\{\mathcal{E}^l_{a,b}, \mathcal{E}^m_{a,b}\} \subset \mathcal{E}^{l+m-(a+b)}_{a,b} \ .$$

If $< F >$ is quasi-homogeneous ($F \in \mathcal{E}^n_{a,b}$) of finite codimension, we can choose quasi-homogeneous K_α and for any k:

$$\mathcal{E}^{n+k}_{a,b} = \{\mathcal{E}^{k+a+b}_{a,b}, F\} + \mathcal{E}^k_{a,b}F + \sum_{K_\alpha \in \mathcal{E}^{n+k}_{a,b}} \mathbb{R}K_\alpha \ .$$

7.2 Normal forms

The main result of this section is the following one (see [23]):

Theorem 4 *Let $< F >$ a singular germ of (smooth or analytic curve) such that there exists a diffeomorphism φ with $< F \circ \varphi > = < F_0 >$, where F_0 is a quasi-homogeneous polynomial with an isolated singularity, then $(< F >, \omega)$ is symplectically equivalent to $(< F_0 >, \pm\omega_0)$.*

Proof.–

We can assume that $F = F_0$ and we have 2 symplectic forms ω and ω_0. Depending on the value at 0 of ω the path $\omega_t = \omega + t(\varepsilon\omega_0 - \omega)$ where $\varepsilon = \pm 1$ is a path of symplectic forms. We need to find a diffeomorphism ψ such that ψ preserves the curves $F_0 = 0$ and $\psi^\star(\omega) = \varepsilon\omega_0$. We can use the homotopy (Moser) trick, following [23]: if $\omega - \varepsilon\omega_0 = d\alpha$, it is enough to find f such that $df - \alpha = 0$ on every vector tangent to the curve $F_0 = 0$. Let us assume that $F_0(t^a x, t^b \xi) = tF_0(x,\xi)$ $(a,b > 0)$ and introduce the vector field $V = ax\partial_x + by\partial_y$ which is tangent to $F_0 = 0$. The flow of V is $\varphi_t(x,y) = (e^{at}x, e^{bt}y)$ and we define f by

$$f(x,y) = \int_{-\infty}^{0} A(e^{at}x, e^{bt}y)dt$$

where $A(x,y) = \alpha(V(x,y))$. Because $A(0,0) = 0$, we see easily that f is analytic (smooth). Moreover $df(V(x,y)) = A(x,y) = \alpha(V(x,y))$.

□

7.3 Using Euler identity

Theorem 5 *If F is a quasi-homogenous isolated singularity of Milnor number μ, i.e.*

$$\dim \left(\mathcal{E}/\mathrm{Jac}(F)\right) = \mu \ ,$$

then $< F >$ is of codimension μ. More precisely

$$\mathrm{Jac}(F) = \mathcal{E}F + \{\mathcal{E}, F\}$$

Proof.–

Let us denote $A = \partial F/\partial x, \ B = \partial F/\partial \xi$, we have by Euler identity:

$$a'xA + b'\xi B = F \ ,$$

with $a' = a/n$ and $b' = b/n$. We want to solve:

$$\{X, F\} + YF = \lambda A + \nu B$$

where λ, ν are given and $X, Y \in \mathcal{E}$ are unknown functions. We get

$$A(\frac{\partial X}{\partial \xi} + a'xY) + B(-\frac{\partial X}{\partial x} + b'\xi Y) = \lambda A + \nu B$$

and it is now enough to solve

$$\frac{\partial X}{\partial x} = -\nu + b'\xi Y, \ \frac{\partial X}{\partial \xi} = \lambda - a'xY \ .$$

The integrability condition is:

$$(a' + b')Y + b'\xi\frac{\partial Y}{\partial \xi} + a'x\frac{\partial Y}{\partial x} = \frac{\partial \lambda}{\partial x} + \frac{\partial \nu}{\partial \xi} \ ,$$

which admits an unique solution Y: we solve first inside formal series, then inside flat functions. We can take for the U_j's a basis of neighbourhoods star-shaped with respect to quasi-homogeneous dilatations.

□

8 Versal deformations for quasi-homogeneous singularities

8.1 Holomorphic case

We will prove the versal deformation theorem for all quasi-homogeneous singularities.

Lemma 1 *Let $F_a(x, \xi)$ $(a \in \mathbb{C}^\mu)$ be a versal deformation of a quasi-homogeneous singularity and γ_j a locally constant basis of the vanishing homology. Then the Jacobian determinant $J(a)$ of $a \to (\int_{\gamma_j(a)} \xi dx)$ which is well defined outsise the discriminant set (the set of a's for which the curve $F_a = 0$ is singular) extends to $\mathbb{C}^\mu$ as a* non vanishing *holomorphic function. If we take the versal deformation generated by monomials, J is constant.*

As a corollary we get that there exists a canonical measure on the versal deformation (because the vanishing homology has a canonical Lebesgue measure). It would be nice to have a geometric definition of that measure.
Proof.–

- We first check that:

$$\frac{\partial}{\partial a_\alpha} \int_{\gamma(a)} \xi dx = \int_{\gamma(a)} K_\alpha dt$$

where dt is the time for the dynamics induced by the Hamiltonian $H_0 + \sum a_\alpha K_\alpha$ on the surface $H_0 + \sum a_\alpha K_\alpha = 0$.

- We then prove using Picard-Lefschetz formula that J is univalent: the Poincaré group of the complement of the discriminant is generated by small loops around the stratum corresponding to 1 vanishing cycle say γ_1. Following such a loop will add to the lines of the Jacobian determinant a linear combination of the first one.

- J is bounded near the codimension 1 stratum of the discriminant. Hence J is holomorphic near the codimension 1 strata and by Hartogs everywhere. J is clearly quasi-homogeneous. Being nonvanishing by [5] p.95, J is quasi-homogeneous of degree 0, hence a non-zero constant.

□

Using the strategy of Pham in [29], we can prove the following:

Theorem 6 *Let $< F_0 >$ a quasi-homogeneous singularity with $F_a = F_0 + \sum a_\alpha K_\alpha$ (K_α monomials) as a versal deformation. Let $< F_t >$ be any analytic deformation of $< F_0 >$. There exists an analytic family of germs of canonical diffeomorphisms χ_t such that*

$$< F_t \circ \chi_t > = < F_{a(t)} >$$

where the functions $a_j(t)$ are analytic.

Proof.–

We will give the proof for A_2 (the cusp), it is then trivial to see how to extend the proof to the general case.

Using Moser's method, the idea is to fit the action integrals. The details run as follows:

- We can assume, using the versal deformation theorem (see [3]), that we start with $F_a = F_0 + a_1 x + a_2$ and $\omega_c = \omega_0 + O(c)$ and think as $t = (a, c)$. We choose λ_c such that $d\lambda_c = \omega_c - \omega_0$ an assume that $\lambda_c = O(|c|)$.

- Let $\delta = \{4a_1^3 + 27a_2^2 = 0\}$ the discriminant set. We want to define a smooth family of holomorphic diffeomorphisms $a \to \varphi_c(a) = a'$ such that $\varphi_0 = Id$ and for all cycles γ_j of $Z_a = \{F_a = 0\}$ we have

$$\int_{\gamma_j(a')} \xi dx = \int_{\gamma_j(a)} \xi dx + \int_{\gamma_j(a)} \lambda_c$$

- This implicit equation can be uniquely solved for c small enough outside δ because the Jacobian determinant of $a \to (\int_{\gamma_j(a)} \xi dx)_{j=1,2}$ is a nonzero constant (see lemma 1).

- Near the stratum of the discriminant where the vanishing cycle is γ_1, the integrals $\int_{\gamma_1}$ and $\int_{\gamma_2} \pm \int_{\gamma_1} \log \int_{\gamma_1}$ are univalent and holomorphic, thanks to the Picard-Lefschetz formula and the Jacobian determinant is the same: so we can also solve.

- Now we have solved the equation outside a set of codimension 2 and we conclude by the fact that holomorphic functions have no singularities of codimension ≥ 2 (Hartog's theorem).
- Performing the reparametrization of the versal deformation we need to show that $(< F_a >, (\varphi_c^{-1})^\star(d\xi \wedge dx))$ and $(< F_a >, d\xi \wedge dx + d\lambda_c)$ are equivalent. The difference of these 2 symplectic forms is $d\beta_c$ where the integral of β_c over all vanishing cycles of all Z_a's vanish.
- It remains now to find $f_{a,c}(x,\xi)$ whose differential on Z_a is β_c. We define $f_{a,c} = g_{a_1,c}$. The restriction of $g_{a_1,c}$ to all $Z_{a_1,b}$ is obtained by integration from a point $m_{a_1,b} \in Z_{a_1,b} \cap \{\|z\| = 1\}$ which can be choosen an analytic function of (a_1, b) of the forms β_c. The smoothness of f outside δ is clear. Moreover f is holomorphic outside δ and bounded hence holomorphic everywhere.
- We can then apply Moser's method.

□

8.2 Smooth case

The same result (versal deformation, theorem 6) is probably true in the smooth case. The strategy could be to use first the formal case, then to use an induction argument on μ.

9 Semi-classics

In this section, we will *quantize* everything in order to get semi-classical objects.

9.1 Semi-classical normal forms

Theorem 7 *Let $< H_0 >$ be of finite codimension μ with a (classical) real versal deformation generated by K_α, $\alpha = 1, \cdots, \mu$. Let $\widehat{H}$ be a pseudo-differential operator on $\mathbb{R}$ whose principal symbol is H_0. There exists then some elliptic pseudo-differential operators $\hat{U}$ and $\hat{V}$ and formal series $a_\alpha(h) = O(h)$ such that we have microlocally near 0*

$$\hat{U}\widehat{H}\hat{V} = \widehat{H_0} + \sum_\alpha a_\alpha(h)\widehat{K_\alpha} + O(h^\infty)$$

where $\widehat{Q}$ is the Weyl quantization of Q. If $\hat{H}$ is self-adjoint, we can choose $\hat{U}$ and $\hat{V}$ so that the a_α's are real valued.

The proof by induction on the powers of h is similar to that of section 6.

9.2 Mixed case

We consider now a smooth family $\widehat{H_\varepsilon}$ of semi-classical Hamiltonians and denote by H_0 the principal symbol of $\widehat{H_0}$. We assume that $< H_0 >$ is of finite codimension μ. The following result is an extension of theorems 3 ($h = 0$) and 7 ($\varepsilon = 0$).

Theorem 8 *There exist elliptic pseudo-differential operators $\widehat{U_\varepsilon}$ and $\widehat{V_\varepsilon}$ and formal series $a_\alpha(\varepsilon, h) = O(|h| + |\varepsilon|)$ such that*

$$\widehat{U_\varepsilon}\widehat{H_\varepsilon}\widehat{V_\varepsilon} = \widehat{H_0} + \sum_\alpha a_\alpha(\varepsilon, h)\widehat{K_\alpha} + O(\varepsilon^\infty + h^\infty) \ .$$

The proof is by induction on the powers of h and for each power of h by induction on the powers of ε.

9.3 The holomorphic quasi-homogeneous case

In the holomorphic quasi-homogeneous case, using the tools of section 8.1, we get a much better result:

Definition 8 *We will say that* $\hat{H}_E = \mathrm{Op}_W\left(\sum h^j H_j(E;x,\xi)\right)$ *is an* analytic family of pseudo-differential operators near 0*, if, for all indices* j*,* $H_j(E,x,\xi)$ *extends to an holomorphic function in some complex neighbourhood* Ω *of* 0 *independent of* j*.*

Theorem 9 *If* $\hat{H}_E$ *is an analytic family of pseudo-differential operators of order* 0 *such that* $H_0(0;x,\xi)$ *is an isolated quasi-homogeneous singularity, there exists, for* E *small enough, an analytic family of unitary Fourier integral operators* U_E*, an analytic family of elliptic pseudo-diffential operators* F_E *and symbols (analytic w.r. to* E*)* $a_\alpha(E,h)$ *such that we have, microlocally near* 0*:*

$$U_E^\star \hat{H}_E U_E = F_E \circ \left(\hat{H}_0 + \sum a_\alpha(E,h)\hat{K}_\alpha) + O(h^\infty)\right) \ .$$

Proof.–

Proceeding by induction on the powers of h, we get the following equation to solve where $X(E;x,\xi)$, $Y(E;x,\xi)$, $c_\alpha(E)$ are the unknown functions:

$$\{H_0+\sum a_\alpha(E)K_\alpha, X\}+Y(H_0+\sum a_\alpha(E)K_\alpha) = R(E;x,\xi)-\sum c_\alpha(E)K_\alpha(x,\xi)$$

This equation express that on the Riemann surface $H_0 + \sum a_\alpha(E)K_\alpha = 0$, $R(E;x,\xi)-\sum c_\alpha(E)K_\alpha(x,\xi)$ is the derivative with respect to the time of the function X. We need first to choose $c_\alpha(E)$ so that the integrals

$$\int_{\gamma_j(E)} (R(E;x,\xi) - \sum c_\alpha(E)K_\alpha(x,\xi))dt$$

all vanish. This is possible outside the discriminant set because of the non vanishing of the determinant $\int_{\gamma_j(E)} K_\alpha dt$ (see lemma 1). The solution is bounded near the discriminant, hence can be extended to an holomorphic function. The proof is then finished using the same arguments as in the proof of theorem 6.

□

10 Singular Bohr-Sommerfeld rules: the general scheme

From the local model and the WKB solutions, we define the scattering matrices and singular holonomies. We show how one can take the principal part of the regular holonomies in order to get the singular holonomies. We can then derive the Bohr-Sommerfeld rules using the same combinatorial recipe as in [13] (maximal trees ...).

10.1 The context

We will assume that $\widehat{H}_E$ is a pseudo-differential operator of order 0 on the real line and denote by H_E his principal symbol. H is supposed to be real valued and we assume that the energy surface $Z = H_0^{-1}(0)$ admits only finite codimension singularities $z_j,\ j = 1, \cdots, N$ with normal forms

$$\widehat{U_j}\widehat{H}\widehat{V_j} = \widehat{H_j} + \sum_{\alpha=1}^{\mu_j} a_{j,\alpha}(E,h)\widehat{K_{j,\alpha}} + O(h^\infty) \ , \tag{3}$$

with $a_{j,\alpha}(E,h)$ symbols in h and $\widehat{K_{j,\alpha}}$ are Weyl quantizations of the real versal deformation.

10.2 Local models and scattering matrices

In this section we want to describe the solutions of the local model which is mapped on our problem near the singular point z_j.

We will omit the index j in this section.

We fix a neighbourhood Ω of 0 in the (y, η) symplectic plane. We denote by $H_a = H_0 + \sum a_\alpha K_\alpha$ the versal deformation of the model and $\widehat{H_a}$ his (Weyl)-quantized version. We will denote $\gamma_l,\ l = 1, \cdots, L = 2L'$ $(L \geq 2)$ the real branches of the germ $Z_a = H_a^{-1}(0)$. We chose to orient the γ_l's according to the dynamics of H_a. There are now L' ingoing and L' outgoing branches. We choose open sets $\Omega_l \subset \Omega$ with empty mutual intersections and such that $\Omega_l \cap \gamma_l$ is a nonempty connected arc. We assume that a is small enough so that $\Omega_l \cap Z_a$ with $Z_a = H_a^{-1}(0)$ is also a nonempty connected arc.

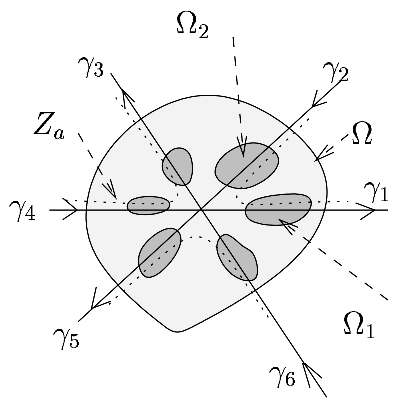

Figure 2: the model problem: $L = 6$

We are looking for the following equation

$$(\widehat{H_0} + \sum a_\alpha \widehat{K_\alpha})u = O(h^\infty) \ , \tag{4}$$

where u is a microfunction in Ω. It is in general not difficult to prove that the space of microfunctions solutions of equation (4) in Ω is a free module of rank $L' = L/2$ over the moderate growth functions of h. We choose microlocal solutions $u_l(a)$ of equation (4) inside Ω_l smoothly dependent of a of the form (in case of no caustics):

$$u_l(a; x) \sim \Big(\sum_{k=0}^{\infty} c_{k,l}(a,x)h^k\Big)e^{iS_l(a;x)/h} \tag{5}$$

with $c_{k,l}$ and S_l smoothly depending on a. Any solution u of equation (4) in Ω restricts to $x_l u_l(a)$ in Ω_l. Given $(x_l) = (x_{in}, x_{out})$ we can express the condition that $x_l u_l(a)$ are the restrictions to Ω_l of some solution u of equation (4) by a matrix

$$x_{out} = \mathcal{S}(a,h)x_{in} \ , \tag{6}$$

where $\mathcal{S}(a,h))$ is called the *scattering matrix*.

10.2.1 Unitarity

We assume that the operator $\hat{H}_a$ is formally self-adjoint. Let us choose Π a pseudo-differential operator of order 0 compactly supported in Ω and equal to Id near the origin. More precisely, we assume that

$$Z_a \cap \{\Pi(Id-\Pi) \neq 0\} \subset \cup_l (\Omega_l \cap Z_a)$$

We define the following inner products on microfunctions in Ω:

$$J_a(u,v) = \frac{i}{h} < [\Pi, \hat{H}_a]u, v >$$

It is clear that

1. If $u, v \in \ker(\hat{H}_a)$, $J(u,v) = O(h^\infty)$
2. If $u_{|\Omega_l} = x_l u_l$ and $v_{|\Omega_l} = y_l u_l$, we have: $J(u,v) = \sum_l x_l \bar{y}_l J(u_l, u_l)$
3. If the principal symbol of u_l is $|dt|^{\frac{1}{2}}$, we have $J(u_l,u_l) = \pm 1 + O(h)$ where we have a $+$ sign if the arc γ_l is ingoing and a $-$ sign if it is outgoing.

From that we deduce that $\mathcal{S}(a,h)$ is unitary (with maybe some domain).

10.3 Singular holonomies

Let γ_0 be a cycle of Z_0, we want to define the singular holonomy (of $\widehat{H_E}$) along γ and compute it. For simplicity we will assume that there exists only one singular point z_1 in γ_0 at which we have a normal form given by equation (3). We can therefore omit the index j. We first cover the cycle γ_0 by open sets $U_1, \cdots U_n$ such that we can find WKB solutions v_j of $\widehat{H_0}v = O(h^\infty)$ inside U_j, points $\zeta_j = (a_j, b_j) \in U_j \cap U_{j+1}$ and such that the Ω_j's covering the singular point z_0 ($j = 1, n$) are the image by the canonical transformation χ of some open sets Ω_l, $l = 1, 2$ introduced in the previous section. We choose $v_1 = \widehat{V_1}u_1$ and $v_n = \widehat{V_1}(u_2)$. We define then then the singular holonomy $\mathrm{HolS}(\widehat{H_0}, \gamma_0)$ by

$$\mathrm{HolS}(\widehat{H_0}, \gamma_0) = \prod_{j=1}^{n-1} \frac{v_j(a_j)}{v_{j+1}(a_j)} \tag{7}$$

It is clear from the theory of WKB-Maslov Ansatz that $\mathrm{HolS}(\widehat{H_0}, \gamma_0) = e^{i(\sum_{k=-1}^{\infty} B_k h^k)}$ so that we go to some *Log* scale and put

$$\mathrm{LHolS} = -i \log \mathrm{HolS} \sim \sum_{k=-1}^{\infty} B_k h^k \ .$$

It is easily checked that singular holonomies are independent of all choices (including χ and the associated FIO's) except for the choosen WKB solutions u_l of the model problem. As we will see singular holonomies and scattering matrices are enough to derive Bohr-Sommerfeld rules.

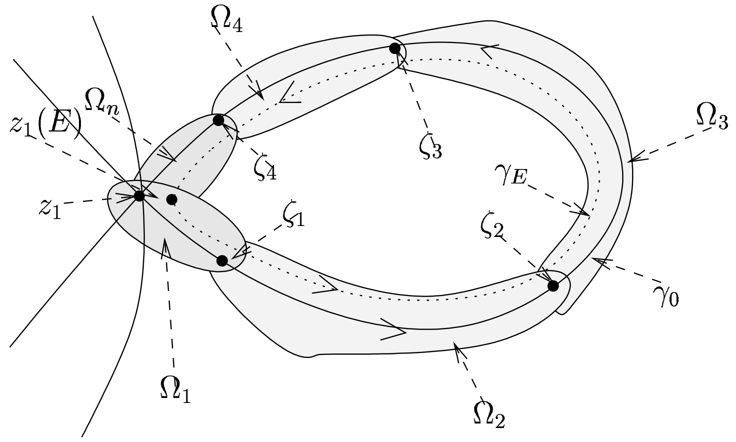

Figure 3: singular holonomy

10.4 Regularisation

We will now choose a deformation $\widehat{H_E}$, $E \geq 0$, of $\widehat{H}$ such that $H_E^{-1}(0) = Z_E$ is smooth and a cycle γ_E of Z_E such that $\gamma_E \to \gamma_0$ as $E \to 0^+$. The goal is to derive $\mathrm{LHolS}(\widehat{H}, \gamma_0)$ as a regularisation of the usual holonomy (the log of) $\mathrm{LHol}(\widehat{H_E}, \gamma_E) \sim \sum_{k=-1}^{\infty} A_k(E) h^k$. In general the A_k's are divergent as $E \to 0^+$ but we can substract the divergent part using the scattering matrix. More precisely, assume $E > 0$, we have then $v_n = s_{1,n}(E,h) v_1$ where $s_{1,n}$ is the corresponding entry of the local scattering matrix. We deduce:

$$\mathrm{LHol}(E,h) = \mathrm{LHolS}(E,h) + \frac{1}{i} \log s_{1,n}(E,h) \ .$$

For fixed $E > 0$, we have then

$$A_k(E) = B_k(E) + \sigma_{1,n}^k(E) \ ,$$

where

$$\frac{1}{i} \log s_{1,n}(E,h) \sim \sum_{k=-1}^{\infty} \sigma_{1,n}(E) h^k \ .$$

We get that way:

$$B_k(0) = \lim_{E \to 0^+} (A_k(E) - \sigma_{1,n}^k(E)) \ .$$

10.5 Singular Bohr-Sommerfeld rules

Once the singular holonomies are defined, the Bohr-Somerfeld rules follow the same combinatorial picture as in [13].

11 The cusp

The saddle-node bifurcation occurs generically for a 1-dimensional system depending on some extra parameter: it is the generic way to change the number of critical points for a Morse function.

Definition 9 *We will say that the planar curve $L =< H >$ admits at z_0* a non degenerate cusp *if z_0 is a degenerate (non Morse) critical point of H such that $H''(z_0)$ is of rank 1 and the polynomial of degree 3 in the Taylor expansion does not vanish on the kernel of $H''(z_0)$.*

11.1 Classics

Theorem 10 *Let H be an Hamiltonian such that $< H >$ admits at z_0 a non degenerate cusp, there exists a canonical transformation χ and a smooth function E non vanishing at z_0 such that $H \circ \chi = EH_0$ with $H_0 = \xi^2 + x^3$:*

$$< H \circ \chi >= H_0 \ .$$

By theorem 4, it is enough to know that H and $\xi^2 + x^3$ are equivalent germs. This result can be proved easily as follows: apply first Morse lemma, we get $\xi^2 + f(x)$ where the third derivative of f does not vanish. See [3] chapter 2.

11.2 Semi-classics

Let $\widehat{H_t} u = 0$ be an analytic family of semi-classical equations such that the principal symbol H_0 of $\widehat{H_0}$ vanishes at z_0 with a non degenerate cusp. Using theorems 6, 8 and 10, we get the following pseudo-differential equation as a microlocal normal form:

$$-h^2 u'' + (x^3 + a(t,h)x + b(t,h))u = O(h^\infty)$$

where $a \sim \sum_{j=0}^{\infty} a_j(t)h^j$ and $b \sim \sum_{j=0}^{\infty} b_j(t)h^j$ are formal series in h.

11.3 Computation of the first coefficients $a_{1,0}$ et $b_{1,0}$

Let us start with F_0 having a cusp at 0. By a rotation, we can assume that

$$F_0 = A\xi^2 + Bx^3 + O(7)$$

where $f = O(N)$ means $F(t^3\xi, t^2 x) = O(t^N)$. By a canonical diagonal linear transformation, we get

$$F_0 = (A^3 B^2)^{1/5}(\xi^2 + x^3 + \alpha x^2 \xi + \beta x \xi^2 + \gamma x^4 + O(9)) \ ,$$

and then, removing the constant prefactor:

$$F_0 = \xi^2 + (x + \frac{\alpha}{3}\xi)^3 + (\beta - \frac{\alpha^2}{3}) x\xi^2 + \gamma x^4 + O(9)) \ ,$$

and putting $\xi_1 = \xi, \ x_1 = x + \frac{\alpha}{3}\xi$:

$$F_0 = \xi_1^2 + x_1^3 + (\beta - \frac{\alpha^2}{3}) x_1 \xi_1^2 + \gamma x_1^4 + O(9)) \ .$$

We want to find χ so that:

$$F_0 \circ \chi = (1 + e x_1)(\xi_1^2 + x_1^3) + O(9)$$

We compute easily

$$x\xi^2 = \frac{3}{7} x(\xi^2 + x^3) + \{S, \xi^2 + x^3\}, x^4 = \frac{4}{7} x(\xi^2 + x^3) + \{S', \xi^2 + x^3\}$$

and we get that way

$$e = \frac{1}{7}(3\beta + 4\gamma - \alpha^2)$$

We have now:

$$(F_0 + tK) \circ \chi = (1 + e x_1) \left(\xi_1^2 + x_1^3 + t \frac{K \circ \chi}{1 + e x_1} \right)) + O(9)$$

and we get by projecting the deformation onto the versal deformation

$$(F_0 + tK) \circ \chi_t = E_t(x_1, \xi_1)\left(\xi_1^2 + x_1^3 + t(a_{1,0}x_1 + b_{1,0})\right) + O(t^2) \ .$$

We put $k_0 = K(0),\ k_1 = \partial_{x_1} K(0)$ and we get

$$a_{1,0} = k_1 - ek_0,\ b_{1,0} = k_0 \ .$$

The same formulae holds for $a_{0,1}$ and $b_{0,1}$ be replacing K by the subprincipal symbol of $\hat{H}_0$.

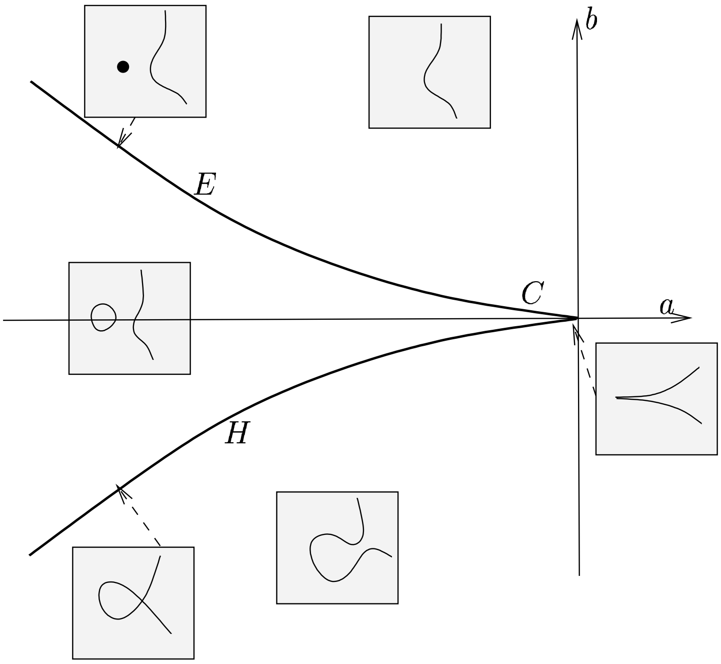

Figure 4: bifurcation diagramm of the cusp

11.4 The model problem

Let $\hat{P}v(y) = -v''(y) + (y^3 + Ay + B)v(y)$ with $A, B \in \mathbb{R}$. We may define the reflexion coefficient $R(A,B)$ in the following way: the equation $\hat{P}v = 0$ admits 2 exact solutions $v_\pm(y)$, smoothly depending on A and B, which admits WKB expansions at infinity ($v_-(y) = \overline{v_+(y)}$) and, as $y \to -\infty$:

$$v_+(y) = |y|^{-\frac{3}{4}} e^{i(\frac{2}{5}|y|^{5/2} + A|y|^{1/2})}\left(1 + \sum_{\alpha=1}^{\infty} a_\alpha(A,B)|y|^{-\alpha/2}\right) + O(\ |y|^{-\infty});$$

existence of solutions with a given asymptotic expansion is a classical fact. They are clearly unique (for a general approach concerning asymptotic solutions, see [6], [31], [36]). There exists an unique function $R(A,B)$ (of modulus 1), called the *reflexion coefficient* or *scattering matrix*, such that $v = v_- + R(A,B)v_+ \in L^2([0,+\infty[, dy)$. This function $R(A,B)$ is the *special function* of our problem. It can be related with Stokes multipliers.

Question 8 *Describe as much as possible the function $R(A,B)$.*

11.5 The semi-classical bifurcation

11.5.1 The scattering matrix

We choose exact solutions $u_{\pm,a,b}$ of equation

$$-h^2u'' + (x^3 + ax + b)u = O \tag{8}$$

(with a and b real valued) which admit the following WKB expansions:

$$u_{\pm,a,b}(x,h) = e^{iS_{a,b}(x)/h}(\sum_{j=0}^{\infty} a_j(x;a,b)h^j) + O(h^\infty)$$

normalized by $S_{a,b}(-1) = 0$ and $\sum_{j=0}^{\infty} a_j(-1;a,b)h^j = 1$. We obtain that way the *semi-classical scattering matrix* $\sigma(a,b;h)$, well defined modulo $O(h^\infty)$ by asking that $u_{+,a,b} + \sigma(a,b;h)u_{-,a,b}$ extends to an admissible function.

11.5.2 Renormalisation

Let us start with the semi-classical model problem given by equation (8) and assume that a and b can be h dependent. We will denote by

$$\|a,b\| = (|a|^3 + b^2)^{5/12}$$

And we will measure the distance to the bifurcation using τ defined by:

$$\|a,b\| = h\tau = \eta$$

We can now use the renormalisation $x = \eta^{2/5}y$ which gives:

$$-\tau^{-2}v''_{y^2} + (y^3 + Ay + B)v = 0$$

with $a = A\eta^{4/5},\ b = B\eta^{6/5}$. Now A and B are of order 1. We have 3 domains:

1. The domain where τ is bounded (w.r. to h) where the bifurcation really takes place and there is no further asymptotics.
2. The Log domain where $1 << \tau = O(|\log h|)$ where we can use the semi-classical asymptotics w.r. to τ including tunneling effect which is not $O(h^\infty)$.
3. The domain where $\tau >> |\log h|$ where we can apply usual formulae without looking at the bifurcation problem: the semi-classical spectrum splits into 2 parts; one associated to the real vanishing circle, the other to the bigg closed cycle.

11.5.3 The bifurcation domain

In this domain ($\|a,b\| = O(h)$), τ is bounded.

If we use $a = Ah^{4/5},\ b = Bh^{6/5}$, the renormalized equation is

$$-v'' + (x^3 + Ax + B)v = O \tag{9}$$

where A and B are bounded.

In this domain, we have the following relationship between R and σ:

$$\sigma(a,b;h) = R(A,B)e^{-\frac{i}{h}(\frac{4}{5}+2Ah^{4/5})}\left(1 + \sum_{\alpha=1}^{\infty}\gamma_\alpha(A,B))h^{\alpha/5}\right) + O(h^\infty)\ , \tag{10}$$

with $A = ah^{-4/5},\ B = bh^{-6/5}$ and the γ_α's can be computed from the a_α's .

11.5.4 The Log domain

In this domain we can compute the τ semi-classical solution using tunneling effect (see [20], [16]).

11.6 Bohr-Sommerfeld rules

From the previous sections, we can compute the singular holonomy using the asymptotic behaviour of $\sigma(a,b;h)$ for (a,b) non zero and $h \to 0$. We can then derive the Bohr-Sommerfeld rules from $R(A,B)$ using equation (10).

References

[1] V. Arnold, Lagrangian manifolds with singularities, asymptotic rays and the open swallowtail. *Functional Analysis and its Applications,* 15(4):235-246, 1981.

[2] V. Arnold, First Steps in Local Symplectic Algebra. *Amer. Math. Soc. Transl.,* (2)194:1-8, 1999.

[3] V. Arnold, A. Varchenko et S. Goussein-Sadé, Singularités des applications différentiables I. *Mir (Moscou), 1986.*

[4] V. Arnold, Mathematical Methods of Classical Mechanics. *Springer, 1989.*

[5] V. Arnold, Dynamical Systems VI. *Springer, 1993.*

[6] W. Balser, Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations. *Springer, 2000.*

[7] P. Braam and H. Duistermaat, Normal forms of real symmetric systems with multiplicity. *Indag. Math., N.S.*, 4(4):407-421, 1993.

[8] E. Caliceti, S. Graffi and M. Maioli, Perturbation of odd anharmonic oscillators. *Commun. Math. Phys.*, 75:51-66, 1980.

[9] E. Caliceti and M. Maioli, Odd anharmonic oscillators and shape resonances. *Annales de l'IHP (Physique théorique)*, 38:175-186, 1983.

[10] Y. Colin de Verdière, M. Lombardi and J. Pollet, The microlocal Landau-Zener formula. *Annales de l'IHP (Physique théorique)*, 71:95-127, 1999.

[11] Y. Colin de Verdière et B. Parisse, Equilibre instable en régime semi-classique : I–Concentration microlocale. *Commun. PDE*, 19:1535-1563, 1994.

[12] Y. Colin de Verdière et B. Parisse, Equilibre instable en régime semi-classique : II–Conditions de Bohr-Sommerfeld. *Annales de l'IHP (Physique théorique)*, 61:347-367, 1994.

[13] Y. Colin de Verdière and B. Parisse, Singular Bohr-Sommerfeld rules. *Commun. Math. Phys.*, 205:459-500, 1999.

[14] Y. Colin de Verdière et J. Vey, Le lemme de Morse isochore. *Topology*, 18:283-293, 1979.

[15] Y. Colin de Verdière and San Vũ Ngọc, Singular Bohr-Sommerfeld Rules for 2d Integrable Systems. *Prépublication no 508 de l'Institut Fourier (mai 2000),* http://www-fourier.ujf-grenoble.fr/~ ycolver.

[16] E. Delabaere, H. Dillinger and F. Pham, Exact semi-classical Expansions for one Dimensional Quantum Oscillators, *Journal Math. Phys.,* 38 (12):6126-6184 (1997).

[17] E. Delabaere, H. Dillinger et F. Pham, Résurgence de Voros et périodes des courbes hyperelliptiques. *Ann. Institut Fourier,* 43(1):163-199, 1993.

[18] E. Delabaere and F. Pham, Unfolding the quartic oscillator. *Ann. of Phys.,* 4261(2):180-218, 1997.

[19] E. Delabaere and D.T. Trinh, Spectral analysis of the complex cubic oscillator. *Preprint Nice* (september, 1999).

[20] R. Dingle, Asymptotic expansions: their Derivation and Interpretation. *Academic Press (1973).*

[21] L. Eliasson, Normal forms for Hamiltonian systems with Poisson commuting integrals. *Comment. Math. Helv.,* 65:4-35, 1990.

[22] A.T. Fomenko, Topological classification of all integrable Hamiltonian differential equations of general type with two degrees of freedom, in "The geometry of Hamiltonian systems, Proc. Workshop", Berkeley/CA (USA) 1989, *Math. Sci. Res. Inst. Publ.,* 22:131-339, 1991.

[23] A.B. Givental', Singular Lagrangian manifolds and their Lagrangian maps. *J. Sov. Math.,* 52(4):3246-3278, 1990.

[24] L. Hörmander, The Analysis of Linear Partial Differential Operators III. *Grundlehren 274, Springer, 1985.*

[25] L. Hörmander, The Analysis of Linear Partial Differential Operators IV. *Grundlehren 275, Springer, 1985.*

[26] B. Malgrange, Intégrales asymptotiques et monodromie. *Ann. Scient. Ec. Norm. Sup,* 7:405-430, 1974.

[27] J. Milnor, Singular Points of Complex Hypersurfaces. *Princeton, 1968.*

[28] Nguyen Hu'u Du'c et F. Pham, Germes de configurations legendriennes stables et fonctions d'Airy-Weber généralisées. *Annales Inst. Fourier,* 41(4):905-936, 1991.

[29] F. Pham, Multiple turning points in exact WKB analysis. *Preprint Nice* (1999).

[30] E. Delabaere and F. Pham, Unfolding the Quartic Oscillator. *Annals of Physics,* 261:180-218, 1997.

[31] J.-P. Ramis, Séries divergentes et théories asymptotiques. *Société Math. de France, Panoramas et Synthèses,* 121:1-74 , 1993.

[32] K. Saito, Quasihomogene isolierte Singularitäten von Hyperflächen. *Invent. Math.,* 14:123-142, 1971.

[33] San Vũ Ngọc, Formes normales semi-classiques de systèmes complètement intégrables. *Asymptotic Analysis,* 24:319-342, 2000.

[34] San Vũ Ngọc, Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type. *Comm. Pure and Applied Math.,* 53:143-217, 2000.

[35] R. Schoen and J. Wolfson, *Minimizing area among Lagrangian surfaces: the mapping problem,* preprint AIM #2000 − 6 (http://www.aimmath.org), 2000.

[36] Y. Sibuya, Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation. *AMS, 1990.*

[37] D. van Straten and C. Sevenheck, Deformation of singular lagrangian subvarieties, *preprint Universität Mainz,* February 2000.

[38] A. Voros, The return of the quartic oscillator (the complex WKB method). *Annales de l'IHP (Physique théorique),* 39:211-338, 1983.

[39] A. Voros, Exact resolution method for general 1D polynomial Schrödinger equation. *J. Phys. A: Math. Gen.,* 33:5993-6007, 1999.

Résumé et mots-clés en français pour l'article "Singular Lagrangian manifolds and semi-classical analysis" ("Variétés lagrangiennes singulières et analyse semi-classique").

Résumé :

Les sous-variétés lagrangiennes des variétés symplectiques sont des objets fondamentaux en mécanique classique et en analyse microlocale. Ces variétés sont souvent singulières (systèmes intégrables, bifurcations, réduction). De nombreux travaux leurs sont consacrés en particulier par Arnold et Givental. Le but de cet article est d'étendre les formes normales classiques et semi-classiques des systèmes complètement intégrables près de singularités non dégénérées (au sens de Morse-Bott) à des systèmes plus singuliers. Il se trouve qu'il y a une façon agréable de faire cela conduisant à des formes normales et à leurs déploiements versels. Nous obtenons ainsi des Ansatz naturels qui généralisent l'Ansatz "BKW-Maslov". Nous donnons plus de détails dans l'exemple le plus simple qui n'est pas de Morse, le cusp, qui correspond à une bifurcation noeud-col.

Mots Clés: variétés lagrangiennes singulières, systèmes hamiltoniens intégrables, bifurcations, règles de Bohr-Sommerfeld, BKW, semi-classique, formes normales, déformations verselles.

Mathematics Subject Classification 2000: 32S05, 32S40, 34E20, 35P20, 53D12, 70H06.