SPIN^c-QUANTIZATION AND THE K-MULTIPLICITIES OF THE DISCRETE SERIES

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ABSTRACT. We express the K-multiplicities of a representation of the discrete series associated to a coadjoint orbit $\mathcal{O}$ in terms of Spin^c-index on symplectic reductions of $\mathcal{O}$.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to show that the '*quantization commutes with reduction*' principle of Guillemin-Sternberg [16] holds for the coadjoint orbits that parametrize the discrete series of a real connected semi-simple Lie group.

Consider a real connected semi-simple Lie group G with finite center, which admits a discrete series. Let K be a maximal compact subgroup, and let T be a maximal torus in K: then T is also a Cartan subgroup of G. Following Harish-Chandra, the discrete series of G are parametrized by a subset $\widehat{G}_d$ in the dual $\mathfrak{t}^*$ of the Lie algebra of T. For each $\lambda \in \widehat{G}_d$, let us denote Θ_λ the trace of the representation of G associated to the coadjoint orbit $G\cdot\lambda$: it is a generalized function on G, invariant by conjugation, and which admits a restriction to K denoted $\Theta_\lambda|_K$.

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The following definition was first introduced by Cannas da Sylva, Karshon and Tolman [11].

Definition. *Let (M, ω, Φ) be a Hamiltonian manifold for the action of a Lie group H. The manifold M is oriented by its symplectic form ω. We say that (M, ω, Φ) is* Spin^c *prequantized if M carries an H-equivariant* Spin^c*-structure with determinant line bundle $L_{2\omega}$ which is a prequantum line bundle over $(M, 2\omega)$.*

We fix now $\lambda \in \widehat{G}_d$, and we consider the regular coadjoint orbit $G \cdot \lambda$ equipped with its Kirillov-Kostant-Souriau symplectic form ω. The action of G on $G \cdot \lambda$ is Hamiltonian with moment map $\Phi_G : G \cdot \lambda \hookrightarrow \mathfrak{g}^*$ equal to the inclusion. Here the orbit $G \cdot \lambda \simeq G/T$ is Spin^c prequantized : the corresponding determinant line bundle is $L_{2\omega} = G \times_T \mathbb{C}_{2\lambda}$. The action of K on $G \cdot \lambda$ is Hamiltonian, and the induced moment map $\Phi : G \cdot \lambda \to \mathfrak{k}^*$ is proper. Hence the reduced spaces $(G \cdot \lambda)_\xi = \Phi^{-1}(K \cdot \xi)/K$ are compact for all $\xi \in \mathfrak{k}^*$. The Convexity Theorem tells us also that $\Delta = \Phi(G \cdot \lambda) \cap \mathfrak{t}^*_+$ is a convex polyhedral subset of the Weyl chamber $\mathfrak{t}^*_+$ containing λ [2, 15, 23, 26].

Let ρ_c be half the sum of the positive compact roots, and let $\mu \in \Lambda^*_+$ be an integral dominant weight, such that $\tilde{\mu} := \mu + \rho_c$ is a regular value of Φ.We show then that the symplectic quotient $((G\cdot\lambda)_{\tilde{\mu}}, \omega_{\tilde{\mu}})$ is Spin^c prequantized, and we define an integer $\mathcal{Q}((G \cdot \lambda)_{\mu+\rho_c})$ as the index of the corresponding Spin^c Dirac operator on $(G \cdot \lambda)_{\mu+\rho_c}$. Then we are able to define the number $\mathcal{Q}((G \cdot \lambda)_{\mu+\rho_c})$ for arbitrary μ by shift desingularization. In particular $\mathcal{Q}((G \cdot \lambda)_{\mu+\rho_c}) = 0$ if $\mu + \rho_c$ does not belong to the relative interior Δ^o of Δ.

The central result of this paper is

Theorem. *We have the decomposition*

$$\Theta_\lambda|_K = \sum_{\mu \in \Lambda^*_+ \cap (\Delta^o - \rho_c)} \mathcal{Q}\left((G \cdot \lambda)_{\mu+\rho_c}\right) \chi^K_\mu \ ,$$

where χ^K_μ is the character of the irreducible K-representation with highest weight μ.

We give now an outline of the proof. For the precise definitions, see section 2.

Let (M, ω, Φ) be a *compact* Hamiltonian K-manifold. Let J be an almost complex structure on M, and let κ be the canonical line bundle associated to J. In section 2, we extend the '*quantization commutes with reduction*' principle when we tensor a prequantum line bundle L by the bundle of half forms $\kappa^{1/2}$. The product $\tilde{L} = L \otimes \kappa^{1/2}$ may exist even if neither L nor $\kappa^{1/2}$ exists . A complex line bundle $\tilde{L}$ over M is referred as 'κ-prequantum' if $\tilde{L}^2 \otimes \kappa^{-1}$ is a prequantum line bundle over $(M, 2\omega)$. If M carries a κ-prequantum line bundle $\tilde{L}$, (M, ω, Φ) is Spin^c prequantized by the Spin^c structure defined by J and twisted by $\tilde{L}$. Let $\mu \in \Lambda^*_+$ be a dominant weight, such that $\mu + \rho_c$ is a regular value of Φ. We show then that the symplectic quotient $(M_{\mu+\rho_c}, \omega_{\mu+\rho_c})$ is Spin^c prequantized, and we define $\mathcal{Q}(M_{\mu+\rho_c}) \in \mathbb{Z}$ as the index of the corresponding Spin^c Dirac operator. The number $\mathcal{Q}(M_{\mu+\rho_c})$ is defined in the general case by shift desingularization. In particular $\mathcal{Q}(M_{\mu+\rho_c}) = 0$ if $\mu + \rho_c$ does not belong to the relative interior Δ^o of the moment polytope $\Delta = \Phi(M) \cap \mathfrak{t}^*_+$.

Let $\varepsilon = \pm 1$ be the 'quotient' of the orientation $o(J)$ defined by the almost complex structure and the orientation $o(\omega)$ defined by the symplectic form. If the infinitesimal stabilizers for the K-action on M are *abelian*, we show that

$$(1) \qquad RR^{K}(M, \tilde{L}) = \varepsilon \sum_{\mu \in \Lambda^*_+ \cap (\Delta^o - \rho_c)} \mathcal{Q}(M_{\mu+\rho_c})\, \chi^{K}_{\mu} \ .$$

The computation of the multiplicity of the trivial representation in $RR^{K}(M, L \otimes \kappa^{1/2})$ was already known when K is a torus [18, 38].

In section 3, we extend (1) to a non-compact setting. First we define a generalized Riemann-Roch character $RR^{K}_{\Phi}(M, -)$ when (M, ω, Φ) is an Hamiltonian manifold where the function $\| \Phi \|^2 \colon M \to \mathbb{R}$ has a *compact* set of critical points (which is denoted $\mathrm{Cr}(\| \Phi \|^2)$). If moreover the moment map is assumed proper, we prove in section 4 that Equality (1) extends to

$$(2) \qquad RR^{K}_{\Phi}(M, \tilde{L}) = \varepsilon \sum_{\mu \in \Lambda^*_+ \cap (\Delta^o - \rho_c)} \mathcal{Q}(M_{\mu+\rho_c})\, \chi^{K}_{\mu} \ ,$$

for every κ-prequantum line bundle, when the infinitesimal stabilizers are abelian. Here some additional assumptions are needed to control the data on the non-compact manifold M (see Assumption 3.4).

In the last section we consider, for $\lambda \in \widehat{G}_d$, the case of the coadjoint orbit $G \cdot \lambda \simeq G/T$ with the canonical K-action. The moment map Φ is proper and the critical set $\mathrm{Cr}(\| \Phi \|^2)$ coincides with $K \cdot \lambda$, hence is compact. Thus the generalized Riemann-Roch character $RR^{K}_{\Phi}(G \cdot \lambda, -)$ is well defined. On $G \cdot \lambda$, there is a (integrable) complex structure for which the line bundle $\tilde{L} := G \times_T \mathbb{C}_{\lambda - \rho} \to G/T$ is κ-prequantum[1]. With the help of the Blattner formula we show that

$$(3) \qquad RR^{K}_{\Phi}(M, \tilde{L}) = (-1)^{\frac{\dim(G/K)}{2}}\, \Theta_\lambda|_K \ .$$

Since $\varepsilon = (-1)^{\frac{\dim(G/K)}{2}}$ in this context, the Theorem follows from (2) and (3), if one verifies that Assumption 3.4 holds on $G \cdot \lambda$: this is done in the last subsection of this paper.

Notation

Throughout the paper K will denote a compact, connected Lie group, and $\mathfrak{k}$ its Lie algebra. We let T be a maximal torus in K, and $\mathfrak{t}$ be its Lie algebra. The integral lattice $\Lambda \subset \mathfrak{t}$ is defined as the kernel of $\exp : \mathfrak{t} \to T$, and the real weight lattice $\Lambda^* \subset \mathfrak{t}^*$ is defined by : $\Lambda^* := \hom(\Lambda, 2\pi\mathbb{Z})$. Every $\mu \in \Lambda^*$ defines a 1-dimensionnal T-representation, denoted $\mathbb{C}_\mu$, where $t = \exp X$ acts by $t^\mu := e^{\imath \langle \mu, X \rangle}$. We let W be the Weyl group of (K, T), and we fix the positive Weyl chambers $\mathfrak{t}_+ \subset \mathfrak{t}$ and $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$. For any dominant weight $\mu \in \Lambda^*_+ := \Lambda^* \cap \mathfrak{t}^*_+$, we denote V_μ the K-irreducible representation with highest weight μ, and χ^{K}_{μ} its character. We denote by $R(K)$ (resp. $R(T)$) the ring of characters of finite-dimensional K-representations (resp. T-representations). We denote by $R^{-\infty}(K)$ (resp. $R^{-\infty}(T)$) the set of generalized characters of K (resp. T). An element $\chi \in R^{-\infty}(K)$ is of the form

[1]Here ρ is half the sum of the positive roots for the T-action on the Lie algebra $\mathfrak{g}$ of G. When $\lambda \in \widehat{G}_d$, $\lambda - \rho$ is integrable.

$\chi = \sum_{\mu\in\Lambda^*_+} \mathrm{m}_\mu\, \chi^K_\mu$, where $\mu \mapsto \mathrm{m}_\mu, \Lambda^*_+ \to \mathbb{Z}$ has at most polynomial growth. In the same way, an element $\chi \in R^{-\infty}(T)$ is of the form $\chi = \sum_{\mu\in\Lambda^*} \mathrm{m}_\mu\, t^\mu$, where $\mu \mapsto \mathrm{m}_\mu, \Lambda^* \to \mathbb{Z}$ has at most polynomial growth. We denote by $w\circ\mu = w(\mu+\rho_c)-\rho_c$ the affine action of the Weyl group on the set of weights. The holomorphic induction map $\mathrm{Hol}^K_T : R^{-\infty}(T) \longrightarrow R^{-\infty}(K)$ is characterized by the following properties by the following properties: i) $\mathrm{Hol}^K_T(t^\mu) = \chi^K_\mu$ for every dominant weight $\mu \in \Lambda^*_+$; ii) $\mathrm{Hol}^K_T(t^{w\circ\mu}) = (-1)^w \mathrm{Hol}^K_T(t^\mu)$ for every $w \in W$ and $\mu \in \Lambda^*$; iii) $\mathrm{Hol}^K_T(t^\mu) = 0$ if $W \circ \mu \cap \Lambda^*_+ = \emptyset$. Some additional notation will be introduced later :

- G : connected real semi-simple Lie group with finite center
- $\mathbb{T}_\beta$: torus generated by $\beta \in \mathfrak{k}$
- M^γ : submanifold of points fixed by $\gamma \in \mathfrak{k}$
- $\mathbf{T}M$: tangent bundle of M
- $\mathbf{T}_K M$: set of tangent vectors orthogonal to the K-orbits in M
- Φ : moment map
- $\Delta = \Phi(M) \cap \mathfrak{t}^*_+$: moment polytope
- $\mathcal{H}$: vector field generated by Φ
- $\mathrm{Char}(\sigma)$: characteristic set of the symbol σ
- $\mathrm{Thom}^\Phi_K(M, J)$: Thom symbol pushed by Φ
- $RR^K_\Phi(M, -)$: generalized Riemann-Roch character
- $\mathrm{m}_\mu(E)$: multiplicity of $RR^K_\Phi(M, E)$ relatively to $\mu \in \Lambda^*_+$.

2. Spin^c-quantization of compact Hamiltonian K-manifolds

Let M be a compact Hamiltonian K-manifold with symplectic form ω and moment map $\Phi : M \to \mathfrak{k}^*$ characterize by the relation : $d\langle\Phi, X\rangle = -\omega(X_M, -)$ for all $X \in \mathfrak{k}$. In the process of quantization one tries to associate a unitary representation of K to these data. Here we associate to the data (M, ω, Φ) a virtual character of K, defined as the equivariant index of a Spin^c Dirac operator.

In this section, we recall first the known facts about Spin^c-quantization, in particular the *quantization commutes with reduction* principle, and we give a modified version when M satisfies the *corrected quantification condition* (see [41], page 202). In the next section we extend these procedures to the setting where M is non-compact and the moment map is proper.

In the Kostant-Souriau framework, M is prequantized if there is a K-equivariant Hermitian line bundle L with a K-invariant Hermitian connection ∇^L of curvature $-\imath\omega$. The line bundle L is called a prequantum line bundle for the Hamiltonian K-manifold (M, ω, Φ). Recall that the data (∇, Φ) are related by the Kostant formula

$$\mathcal{L}^L(X) - \nabla^L_{X_M} = \imath\langle f_G, X\rangle,\ X \in \mathfrak{k}\ . \tag{2.1}$$

Here $\mathcal{L}^L(X)$ is the infinitesimal action of X on the section of $L \to M$ and X_M is the vector field on M generated by $X \in \mathfrak{k}$: $X_M(m) := \frac{d}{dt}\exp(-tX).m|_{t=0}$, for $m \in M$.

When M is compact and prequantized by L, we can associate a virtual representation $RR^K(M, L)$ of K in the following manner. Choose an invariant almost complex structure J on M that is compatible with ω, in the sense that $\omega(-, J-)$

defines a Riemannian structure. The almost complex structure J defines a K-equivariant Spin^c-structure, which we twist by the line bundle L. Any choice of Hermitian connection on $\mathbf{T}M$ defines a Dirac operator $\mathcal{D}_L^+$ for the twisted Spin^c-structure and we define $RR^K(M,L)$ as its equivariant index

$$RR^K(M,L) := \mathrm{Index}_M^G(\mathcal{D}_L^+) \quad \text{in} \quad R(K).$$

Following [29], we call a point $\mu \in \mathfrak{k}^*$ a *quasi-regular* value of μ if all the K_μ-orbits in $\Phi^{-1}(\mu)$ have the same dimension. For any quasi-regular value $\mu \in \mathfrak{k}^*$ the reduced space $M_\mu := \Phi^{-1}(\mu)/K_\mu$ is a symplectic orbifold. For any dominant weight $\mu \in \Lambda_+^*$ which is a quasi-regular value of Φ,

$$L_\mu := (L|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/K_\mu$$

is a prequantum orbifold-line bundle over M_μ. The definition of Spin^c-index carries over to the orbifold case, hence $RR(M_\mu, L_\mu) \in \mathbb{Z}$ is defined. In [29], this is extended further to the case of singular symplectic quotients, using partial (or shift) desingularization. The following Theorem was conjectured by Guillemin-Sternberg [16] and is known as "quantization commutes with reduction" [28, 29].

Theorem 2.1. *(Meinrenken, Meinrenken-Sjamaar). Let (M,ω,Φ) be a compact Hamiltonian K-manifold prequantized by L. Let $RR^K(M,-)$ be the quantization defined by means of a compatible almost complex structure on M. We have the following equality in $R(K)$*

$$RR^K(M,L) = \sum_{\mu\in\Lambda_+^*} RR(M_\mu, L_\mu)\, \chi_\mu^K \ , \tag{2.2}$$

where χ_μ^K is the trace of the K-irreducible representation with highest weight μ.

Other proofs can be found in [33, 36]. For an introduction and further references see [34, 39].

Let us consider the basic example of coadjoint orbits for K. They are parametrized by the Weyl chamber $\mathfrak{t}_+^*$. For $a \in \mathfrak{t}_+^*$, the coadjoint orbit $\mathcal{O}^a := K \cdot a$ carries the Kirillov-Kostant-Souriau symplectic form and the canonical K-action is Hamiltonian with moment map equal to the inclusion map $\mathcal{O}^a \hookrightarrow \mathfrak{k}^*$. One can show that $\mathcal{O}^a$ can be prequantized if and only if a is a real infinitesimal weight for the maximal torus, i.e. a belongs to the set of dominant weights Λ_+^*. Then the line bundle $\mathbb{C}_{[a]} := K \times_{K_a} \mathbb{C}_a$ is the unique prequantum line bundle over $\mathcal{O}^a$. Since a compatible almost complex structure on $\mathcal{O}^a$ is integrable, the term $RR^K(\mathcal{O}^a, \mathbb{C}_{[a]})$ is computed by the Borel-Weil-Bott theorem :

$$RR^K(\mathcal{O}^a, \mathbb{C}_{[a]}) = \chi_a^K \tag{2.3}$$

for any $a \in \Lambda_+^*$. We can also use (2.2) to compute $RR^K(\mathcal{O}^a, \mathbb{C}_{[a]})$ since for $M = \mathcal{O}^a$, the reduced space M_μ is empty for $\mu \neq a$, and equal to $\{pt\}$ for $\mu = a$.

For our purpose we need another version of the 'quantization commutes with reduction' principle when M satisfies the *corrected quantization condition* (see Definition 2.2). Now we do not assume that the almost complex structure is compatible with the symplectic form.

The tangent bundle $\mathbf{T}M$ endowed with J is a complex vector bundle over M, and we consider its complex dual $\mathbf{T}_\mathbb{C}^*M := \hom_\mathbb{C}(\mathbf{T}M, \mathbb{C})$. We suppose first that the canonical line bundle $\kappa := \det \mathbf{T}^*_{\mathbb{C}} M$ admits a K-equivariant square root $\kappa^{1/2}$. If M is prequantized by L, a standard procedure in the geometric quantization literature is to tensor L by the bundle of half-forms $\kappa^{1/2}$ [41]. We consider the index $RR^{^K}(M, L \otimes \kappa^{1/2})$ instead of $RR^{^K}(M, L)$. In many contexts, the tensor product $\tilde{L} = L \otimes \kappa^{1/2}$ has a meaning even if L nor $\kappa^{1/2}$ exist.

Definition 2.2. *An Hamiltonian K-manifold (M, ω, Φ), equipped with an almost complex structure, is κ-prequantized by an equivariant line bundle $\tilde{L}$ if $L_{2\omega} := \tilde{L}^2 \otimes \kappa^{-1}$ is a prequantum line bundle for $(M, 2\omega, 2\Phi)$.*

The basic examples are the regular coadjoint orbits of K. Let ρ_c be half the sum of the positive roots relative to the choice of the Weyl chamber $\mathfrak{t}^*_+$. For any $\mu \in \Lambda^*_+$, consider the regular coadjoint orbit $\mathcal{O}^{\mu+\rho_c} := K \cdot (\mu + \rho_c)$ with the compatible complex structure. The line bundle[2] $\mathbb{C}_{[\mu]} = K \times_T \mathbb{C}_\mu$ is the unique κ-prequantum line bundle over $\mathcal{O}^{\mu+\rho_c}$: here the hypothetic prequantum line bundle and bundle of half-forms $\kappa^{1/2}$ should be respectively $K \times_T \mathbb{C}_{\mu+\rho_c}$ and $K \times_T \mathbb{C}_{-\rho_c}$; both of them exist if and only if μ and ρ_c are integral. And we have

$$RR^{^K}(\mathcal{O}^{\mu+\rho_c}, \mathbb{C}_{[\mu]}) = \chi^{_K}_{\mu} \tag{2.4}$$

for any $\mu \in \Lambda^*_+$.

Definition 2.2 can be rewritten in the Spin^c setting. The almost complex structure induces a Spin^c structure P with determinant line bundle $\det_{\mathbb{C}} \mathbf{T}M$. If (M, ω, J) is κ-prequantized by $\tilde{L}$ one can twist P by $\tilde{L}$, and then define a new Spin^c structure with determinant line bundle $\det_{\mathbb{C}} \mathbf{T}M \otimes \tilde{L}^2 = L_{2\omega}$.

Definition 2.3. *A symplectic manifold (M, ω) is* Spin^c*-prequantized if there exists a* Spin^c *structure with determinant line bundle $L_{2\omega}$ which is a prequantum line bundle on $(M, 2\omega)$. If a compact Lie group acts on M, the* Spin^c*-structure is required to be equivariant. Here we take the symplectic orientation on M.*

In this paper, we work under the assumption that the infinitesimal stabilizers for the K-action on M are *abelian*. Then the principal face of (M, Φ) is the interior of the Weyl chamber and it contains the relative interior Δ^o of the moment polytope $\Delta := \Phi(M) \cap \mathfrak{t}^*_+$ (see subsection 4.2). A quasi-regular value $\xi \in \Delta^o$ of Φ is called *generic* if $\Phi^{-1}(\xi)$ is of maximal dimension. The following Proposition is the central point for computing the K-multiplicities of $RR^{^K}(M, \tilde{L})$ in terms of the reduced spaces $M_{\mu+\rho_c} := \Phi^{-1}(\mu + \rho_c)/T$, $\mu \in \Lambda^*_+$.

Proposition 2.4. *Let (M, ω, Φ, J) be κ-prequantized by $\tilde{L}$, and let $\mu \in \Lambda^*_+$.*

- *If $\mu + \rho_c \notin \Delta^o$, we set $\mathcal{Q}(M_{\mu+\rho_c}) = 0$.*
- *If $\mu + \rho_c$ is a generic quasi-regular value of Φ, the reduced space $M_{\mu+\rho_c}$ inherits a canonical symplectic structure which is* Spin^c*-prequantized. We denote $\mathcal{Q}(M_{\mu+\rho_c}) \in \mathbb{Z}$ the index of the corresponding* Spin^c *Dirac operator.*
- *If $\mu + \rho_c \in \Delta^o$, we take ξ generic and quasi-regular sufficiently closed to $\mu + \rho_c$. The reduced space $M_\xi := \Phi^{-1}(\xi)/T$ inherits a* Spin^c*-structure with determinant line bundle $(L_{2\omega}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-2(\mu+\rho_c)})/T$. The index $\mathcal{Q}(M_\xi)$ of the corresponding* Spin^c *Dirac operator on M_ξ does not depend of ξ, when ξ is sufficiently closed to $\mu + \rho_c$: it is denoted $\mathcal{Q}(M_{\mu+\rho_c})$.*

[2]Here we shall not confuse $\mathbb{C}_{[\mu]} \to \mathcal{O}^{\mu+\rho_c}$ which is isomorphic to the line bundle $K \times_T \mathbb{C}_\mu \to K/T$, with $\mathbb{C}_{[\mu]} \to \mathcal{O}^\mu$ which is isomorphic to the line bundle $K \times_{K_\mu} \mathbb{C}_\mu \to K/K_\mu$.

The following '*quantization commutes with reduction*' Theorem holds for the κ-prequantum line bundles.

Theorem 2.5. *Let (M,ω,Φ) be a compact Hamiltonian K-manifold equipped with an almost complex structure J. Let $\tilde{L}$ be a κ-prequantum line bundle over M, and let $RR^{K}(M,-)$ be the quantization map defined by J. If the infinitesimal stabilizers for the action of K on M are* abelian, *we have the following equality in $R(K)$*

$$RR^{K}(M,\tilde{L}) = \varepsilon \sum_{\mu\in\Lambda^*_+} Q(M_{\mu+\rho_c})\,\chi^K_\mu \ , \tag{2.5}$$

where $\varepsilon=\pm1$ is the 'quotient' of the orientation $o(J)$ defined by the almost complex structure and the orientation $o(\omega)$ defined by the symplectic form.

Theorem 2.5 will be proved in a stronger form in the next section.

Let us now give an example where the stabilizers for the action of K on M are *not abelian*, and where (2.5) does not hold. Suppose that the group K is not abelian, so we can consider a face $\sigma\neq\{0\}$ of the Weyl chamber. Let $\rho_{c,\sigma}$ be half the sum of the positive roots which vanish on σ, and consider the coadjoint orbit $M:=K.(\rho_c-\rho_{c,\sigma})$ equipped with its compatible complex structure. Since $\rho_c-\rho_{c,\sigma}$ belongs to σ, the trivial line bundle $M\times\mathbb{C}\to M$ is κ-prequantum[3], and the image of the moment map $\Phi: M\to\mathfrak{k}^*$ does not intersect the interior of the Weyl chamber. So $M_{\mu+\rho_c}=\emptyset$ for every μ, thus the RHS of (2.5) is equal to zero. But the LHS of (2.5) is $RR^{K}(M,\mathbb{C})$ which is equal to 1, the character of the trivial representation. □

Theorem 2.5 can be extended in two directions. First one can bypass the condition on the stabilizers by the following trick. Starting from a κ-prequantum line bundle $\tilde{L}\to M$, one can form the product $M\times(K\cdot\rho_c)$ with the coadjoint orbit through ρ_c. The Kunneth formula gives

$$RR^{K}(M\times(K\cdot\rho_c),\tilde{L}\boxtimes\mathbb{C}) = RR^{K}(M,\tilde{L})\otimes RR^{K}(K\cdot\rho_c,\mathbb{C}) = RR^{K}(M,\tilde{L})$$

since $RR^{K}(K\cdot\rho_c,\mathbb{C})=1$. Now we can apply Theorem 2.5 to compute the multiplicities of $RR^{K}(M\times(K\cdot\rho_c),\tilde{L}\boxtimes\mathbb{C})$ since $\tilde{L}\boxtimes\mathbb{C}$ is a κ-prequantum line bundle over $M\times(K\cdot\rho_c)$, and the stabilizers for the K-action on $M\times(K\cdot\rho_c)$ are abelian. Finally we see that the multiplicity of the irreducible representation with highest weight μ in $RR^{K}(M,\tilde{L})$ is equal to $\varepsilon\mathcal{Q}((M\times(K\cdot\rho_c))_{\mu+\rho_c})$.

On the other hand, we can extend Theorem 2.5 to the Spin^c setting. It will be treated in a forthcoming paper.

3. Quantization of non-compact Hamiltonian K-manifolds

In this section (M,ω,Φ) denotes a Hamiltonian K-manifold, not necessarilly compact, but with *proper* moment map Φ. Then the reduced spaces $M_{\mu+\rho_c}$ are *compact* symplectic orbifolds (after desingularization if it's necessary).

[3]Here the hypothetic prequantum line bundle and bundle of half-forms $\kappa^{1/2}$ should be respectively $K\times_{K_\sigma}\mathbb{C}_{(\rho_c-\rho_{c,\sigma})}$ and $K\times_{K_\sigma}\mathbb{C}_{-(\rho_c-\rho_{c,\sigma})}$

Let J be an almost complex structure over M, and let $\tilde{L}$ be a κ-prequantum line bundle (see Def. 2.2). We prove in subsection 4.3 that Proposition 2.4 still holds in this context, so the infinite sum

$$\sum_{\mu\in\Lambda^*_+} \mathcal{Q}(M_{\mu+\rho_c})\,\chi^K_\mu \tag{3.6}$$

is a well defined element of $\widehat{R}(K) := \hom_{\mathbb{Z}}(R(K),\mathbb{Z})$.

The aim of this section is to realize this sum as the index of a *transversally elliptic* symbol naturally associated to the data $(M,\Phi,J,\tilde{L})$.

3.1. **Transversally elliptic symbols.** Here we give the basic definitions of the theory of transversally elliptic symbols (or operators) defined by Atiyah in [1]. For an axiomatic treatment of the index morphism see Berline-Vergne [9, 10] and for a short introduction see [33].

Let M be a *compact* manifold provided with an action of a compact connected Lie group K, with Lie algebra $\mathfrak{k}$. For any $X\in\mathfrak{k}$, we denote X_M the following vector field : for $m\in M$, $X_M(m) := \frac{d}{dt}\exp(-tX).m|_{t=0}$. Let $p:\mathbf{T}M\to M$ be the projection, and let $(-,-)_M$ be a K-invariant Riemannian metric.

If E^0,E^1 are K-equivariant vector bundles over M, a K-equivariant morphism $\sigma\in\Gamma(\mathbf{T}M,\hom(p^*E^0,p^*E^1))$ is called a *symbol.* The subset of all $(m,v)\in\mathbf{T}M$ where $\sigma(m,v):E^0_m\to E^1_m$ is not invertible will be called the *characteristic set* of σ, and denoted $\mathrm{Char}(\sigma)$.

Let $\mathbf{T}_K M$ be the following subset of $\mathbf{T}M$:

$$\mathbf{T}_K M \ = \{(m,v)\in\mathbf{T}M,\ (v,X_M(m))_{_M}=0\quad\text{for all } X\in\mathfrak{k}\}\,.$$

A symbol σ is *elliptic* if σ is invertible outside a compact subset of $\mathbf{T}M$ ($\mathrm{Char}(\sigma)$ is compact), and is *transversally elliptic* if the restriction of σ to $\mathbf{T}_K M$ is invertible outside a compact subset of $\mathbf{T}_K M$ ($\mathrm{Char}(\sigma)\cap\mathbf{T}_K M$ is compact). An elliptic symbol σ defines an element in the equivariant K-theory with compact support of $\mathbf{T}M$, which is denoted $\mathbf{K}_K(\mathbf{T}M)$, and the index of σ is a virtual finite dimensional representation of K [4, 5, 6, 7].

A *transversally elliptic* symbol σ defines an element of $\mathbf{K}_K(\mathbf{T}_K M)$, and the index of σ is defined (see [1] for the analytic index and [9, 10] for the cohomological one) and is a trace class virtual representation of K. Remark that any elliptic symbol of $\mathbf{T}M$ is transversally elliptic, hence we have a restriction map $\mathbf{K}_K(\mathbf{T}M)\to\mathbf{K}_K(\mathbf{T}_K M)$. We have the following commutative diagram

$$\begin{array}{ccc} \mathbf{K}_K(\mathbf{T}M) & \longrightarrow & \mathbf{K}_K(\mathbf{T}_K M) \\ \mathrm{Index}^K_M\downarrow & & \downarrow\mathrm{Index}^K_M \\ R(K) & \longrightarrow & R^{-\infty}(K)\ . \end{array} \tag{3.7}$$

Using the *excision property*, one can easily show that the index map $\mathrm{Index}^K_{\mathcal{U}}$: $\mathbf{K}_K(\mathbf{T}_K\mathcal{U})\to R^{-\infty}(K)$ is still defined when $\mathcal{U}$ is a K-invariant relatively compact open subset of a K-manifold (see [33][section 3.1]).

3.2. **Thom symbol pushed by the moment map.** To a K-invariant almost complex structure J is associated the Thom symbol $\mathrm{Thom}_{_K}(M,J)$, and the corresponding Riemann-Roch character $RR^{^K}$ [33]. Let us recall the definitions.

Consider a K-invariant Riemannian structure q on M such that J is orthogonal relatively to q, and let h be the Hermitian structure on $\mathbf{T}M$ defined by : $h(v,w) = q(v,w) - \imath q(Jv,w)$ for $v, w \in \mathbf{T}M$. The symbol

$$\mathrm{Thom}_K(M,J) \in \Gamma\left(M, \hom(p^*(\wedge_{\mathbb{C}}^{even}\mathbf{T}M),\ p^*(\wedge_{\mathbb{C}}^{odd}\mathbf{T}M))\right)$$

is equal, at $(m,v) \in \mathbf{T}M$, to the Clifford map[4]

$$\mathrm{Cl}_m(v) \ :\ \wedge_{\mathbb{C}}^{even}\mathbf{T}_m M \longrightarrow \wedge_{\mathbb{C}}^{odd}\mathbf{T}_m M, \tag{3.8}$$

where $\mathrm{Cl}_m(v).w = v \wedge w - c_h(v).w$ for $w \in \wedge_{\mathbb{C}}^{\bullet}\mathbf{T}_x M$. Here $c_h(v) : \wedge_{\mathbb{C}}^{\bullet}\mathbf{T}_m M \to \wedge^{\bullet-1}\mathbf{T}_m M$ denotes the contraction map relatively to h. Since the map $\mathrm{Cl}_m(v)$ is invertible for all $v \neq 0$, the symbol $\mathrm{Thom}_K(M,J)$ is elliptic when M is *compact*.

The important point is that $\mathrm{Thom}_K(M,J) \otimes E$ corresponds, for any K-vector bundle E, to the *principal symbol* of the twisted Spin^c Dirac operator $\mathcal{D}_E^+$ [14]. So, when M is a *compact* manifold, the Riemann-Roch character $RR^K : \mathbf{K}_K(\mathbf{T}M) \to R(K)$ is defined by the following relation

$$RR^K(M,E) = \mathrm{Index}_M^K\left(\mathrm{Thom}_K(M,J) \otimes E\right)\ . \tag{3.9}$$

Consider now the case of a *non-compact* Hamiltonian K-manifold (M,ω,Φ). We choose a K-invariant scalar product on $\mathfrak{k}^*$, and we consider the function $\| \Phi \|^2$: $M \to \mathbb{R}$. Let $\mathcal{H}$ be the Hamiltonian vector field of $\frac{-1}{2} \| \Phi \|^2$: the contraction of the symplectic form by $\mathcal{H}$ is equal to the 1-form $\frac{-1}{2} d \| \Phi \|^2$. The vector field $\mathcal{H}$ is in fact defined only by Φ. The scalar product on $\mathfrak{k}^*$ gives an identification $\mathfrak{k}^* \simeq \mathfrak{k}$, hence Φ can be consider as a map from M to $\mathfrak{k}$. We have then

$$\mathcal{H}_m = (\Phi(m))_M|_m, \quad m \in M\ , \tag{3.10}$$

where $(\Phi(m))_M$ is the vector field on M generated by $\Phi(m) \in \mathfrak{k}$.

Definition 3.1. *The Thom symbol pushed by the moment map, which is denoted* $\mathrm{Thom}_K^\Phi(M,J)$*, is defined by the relation*

$$\mathrm{Thom}_K^\Phi(M,J)(m,v) := \mathrm{Thom}_K(M,J)(m, v - \mathcal{H}_m)$$

for any $(x,v) \in \mathbf{T}M$. *Likewise, any equivariant map* $S : M \to \mathfrak{k}$ *defines a Thom symbol* $\mathrm{Thom}_K^S(M,J)$ *pushed by the vector field* $S_M : m \to S(m)_M|_m$ *:* $\mathrm{Thom}_K^S(M,J)(m,v) := \mathrm{Thom}_K(M,J)(m, v - S_M(m))$.

Atiyah first proposed to 'push' the symbol of an elliptic operator by the vector field induced by an S^1-action to localize its index on the fixed point submanifold, giving then another proof of the Lefschetz fixed-point theorem [1][Lecture 6]. The idea was exploited afterwards by Vergne to give a proof of the 'quantization commutes with reduction' theorem in the case of an S^1-action [38]. In [33], we extended this procedure for an action of a compact Lie group. Here, we use this idea to produce a transversally elliptic symbol on a non-compact manifold.

The characteristic set of $\mathrm{Thom}_K^\Phi(M,J)$ corresponds to $\{(m,v) \in \mathbf{T}M,\ v = \mathcal{H}_m\}$, the graph of the vector field $\mathcal{H}$. Since $\mathcal{H}$ belongs to the set of tangent vectors to the K-orbits, we have

$$\begin{aligned} \mathrm{Char}\left(\mathrm{Thom}_K^\Phi(M,J)\right) \cap \mathbf{T}_K M &= \{(m,0) \in \mathbf{T}M,\ \mathcal{H}_m = 0\} \\ &\simeq \{m \in M,\ d \| \Phi \|_m^2 = 0\}\ . \end{aligned}$$

[4]Here $(\mathbf{T}M, J)$ is considered as a complex vector bundle over M.

Hence the symbol $\mathrm{Thom}_K^{\Phi}(M,J)$ is transversally elliptic if and only if the set $\mathrm{Cr}(\|\Phi\|^2)$ of critical points of the function $\|\Phi\|^2$ is *compact.*

Definition 3.2. *Let (M,ω,Φ) be a Hamiltonian K-manifold such that $\mathrm{Cr}(\|\Phi\|^2)$ is compact. For any almost complex structure J, the symbol* $\mathrm{Thom}_K^{\Phi}(M,J)$ *is* transversally elliptic. *For any K-vector bundle $E \to M$, the tensor product* $\mathrm{Thom}_K^{\Phi}(M,J)\otimes E$ *is transversally elliptic and we denote*

$$RR_{\Phi}^{K}(M,E) \in R^{-\infty}(K)$$

its index[5]. *In the same way, an equivariant map $S : M \to \mathfrak{k}$ defines a transversally elliptic symbol* $\mathrm{Thom}_K^{S}(M,J)$ *if and only if $\{m \in M,\ S_M(m) = 0\}$ is compact. If this holds one defines the localized Riemann-Roch character $RR_S^K(M,E) := \mathrm{Index}_M^K(\mathrm{Thom}_K^S(M)\otimes E)$.*

Remark 3.3. *If M is compact the symbols* $\mathrm{Thom}_K(M,J)$ *and* $\mathrm{Thom}_K^{\Phi}(M,J)$ *are homotopic as elliptic symbols, hence the maps $RR^K(M,-)$ and $RR_{\Phi}^K(M,-)$ coincide (see section 4 of* [33]*).*

The sum (3.6) is defined under the assumption that Φ *is proper.* On the other hand the generalized index $RR_{\Phi}^K(M,\tilde{L})$ is defined when $\mathrm{Cr}(\|\Phi\|^2)$ *is compact.* We will prove in section 4 that these quantities coincide under another assumption including the two previous ones. In subsections 3.3 and 3.4, we set up the technical preliminaries that are needed to compute the K-multiplicity of $RR_{\Phi}^K(M,\tilde{L})$.

In section 4, we show that the multiplicity of $RR_{\Phi}^K(M,\tilde{L})$ relatively to the highest weight μ can be computed in a neighborhood of $\Phi^{-1}(\mu+\rho_c)$.

3.3. **Counting the K-multiplicities.** Let E be a K-vector bundle over the Hamiltonian manifold (M,ω,Φ) and suppose that $\mathrm{Cr}(\|\Phi\|^2)$ is compact. One wants to compute the K-multiplicities of $RR_{\Phi}^K(M,E) \in R^{-\infty}(K)$, which are the integers $\mathrm{m}_\mu(E) \in \mathbb{Z}$, $\mu \in \Lambda_+^*$ such that

$$RR_{\Phi}^K(M,E) = \sum_{\mu\in\Lambda_+^*} \mathrm{m}_\mu(E)\,\chi_\mu^K\ . \tag{3.11}$$

For this purpose one can use the classical 'shifting trick'. By definition, one has

$$\mathrm{m}_\mu(E) = \left[RR_{\Phi}^K(M,E)\otimes V_\mu^*\right]^K\ ,$$

where V_μ is the irreducible K-representation with highest weight μ, and V_μ^* is its dual. We know from (2.4) that the K-trace of V_μ is $\chi_\mu^K = RR^K(\mathcal{O}^{\tilde{\mu}},\mathbb{C}_{[\mu]})$, where

$$\tilde{\mu} = \mu + \rho_c\ . \tag{3.12}$$

Hence the K-trace of the dual V_μ^* is equal to $RR^K(\overline{\mathcal{O}^{\tilde{\mu}}},\mathbb{C}_{[-\mu]})$, where $\overline{\mathcal{O}^{\tilde{\mu}}}$ is the manifold $\mathcal{O}^{\tilde{\mu}}$ with opposite symplectic structure and opposite complex structure.

[5] Here we take a K-invariant relatively compact open subset $\mathcal{U}$ of M such that $\mathrm{Cr}(\|\Phi\|^2)\subset\mathcal{U}$. Then the restriction of $\mathrm{Thom}_K^{\Phi}(M)$ to $\mathcal{U}$ defines a class $\mathrm{Thom}_K^{\Phi}(M,J)|_{\mathcal{U}} \in \mathbf{K}_K(\mathbf{T}_K\mathcal{U})$. Since the index map is well defined on $\mathcal{U}$, one take $RR_{\Phi}^K(M,E) := \mathrm{Index}_{\mathcal{U}}^K(\mathrm{Thom}_K^{\Phi}(M,J)|_{\mathcal{U}}\otimes E|_{\mathcal{U}})$. A simple application of the excision property shows us that the definition does not depend on the choice of $\mathcal{U}$. In order to simplify our notation (when the almost complex structure is understood), we will write $RR_{\Phi}^K(M,E) := \mathrm{Index}_M^K(\mathrm{Thom}_K^{\Phi}(M)\otimes E)$.

Let $\mathrm{Thom}_K(\overline{\mathcal{O}^{\tilde\mu}})$ be the equivariant Thom symbol on $\overline{\mathcal{O}^{\tilde\mu}}$. Then the trace of V_μ^* is equal to $\mathrm{Index}^K_{\mathcal{O}^{\tilde\mu}}(\mathrm{Thom}_K(\overline{\mathcal{O}^{\tilde\mu}})\otimes\mathbb{C}_{[-\mu]})$, and finally the Kunneth formula gives

$$\mathrm{m}_\mu(E)=\Big[\mathrm{Index}^K_{M\times\mathcal{O}^{\tilde\mu}}\Big((\mathrm{Thom}^\Phi_K(M)\otimes E)\odot(\mathrm{Thom}_K(\overline{\mathcal{O}^{\tilde\mu}})\otimes\mathbb{C}_{[-\mu]})\Big)\Big]^K .$$

See [1, 33], for the definition of the exterior product $\odot:\mathbf{K}_K(\mathbf{T}_K M)\times\mathbf{K}_K(\mathbf{T}\mathcal{O}^{\tilde\mu})\to\mathbf{K}_K(\mathbf{T}_K(M\times\mathcal{O}^{\tilde\mu}))$.

The moment map relative to the Hamiltonian K-action on $M\times\overline{\mathcal{O}^{\tilde\mu}}$ is

$$\begin{aligned}\Phi_{\tilde\mu}:M\times\overline{\mathcal{O}^{\tilde\mu}}&\longrightarrow\mathfrak{k}^*\\ (m,\xi)&\longmapsto\Phi(m)-\xi\end{aligned}\tag{3.13}$$

For any $t\in\mathbb{R}$, we consider the map $\Phi_{t\tilde\mu}:M\times\overline{\mathcal{O}^{\tilde\mu}}\to\mathfrak{k}^*$, $\Phi_{t\tilde\mu}(m,\xi):=\Phi(m)-t\,\xi$.

Assumption 3.4. *There exists a* compact *subset $\mathcal{K}\subset M$, such that, for every $t\in[0,1]$, the critical set of $\|\Phi_{t\tilde\mu}\|^2$ is contained in $\mathcal{K}\times\mathcal{O}^{\tilde\mu}$.*

If M satisfies Assumption 3.4 at $\tilde\mu$, one has a generalized quantization map $RR^K_{\Phi_{\tilde\mu}}(M\times\overline{\mathcal{O}^{\tilde\mu}},-)$ since $\mathrm{Cr}(\|\Phi_{\tilde\mu}\|^2)$ is compact.

Proposition 3.5. *If M satisfies Assumption 3.4 at $\tilde\mu$, then*

$$\mathrm{m}_\mu(E)=\Big[RR^K_{\Phi_{\tilde\mu}}(M\times\overline{\mathcal{O}^{\tilde\mu}},E\boxtimes\mathbb{C}_{[-\mu]})\Big]^K .$$

Proof : One has to show that the transversally elliptic symbols $\mathrm{Thom}^\Phi_K(M)\odot\mathrm{Thom}_K(\overline{\mathcal{O}^{\tilde\mu}})$ and $\mathrm{Thom}^{\Phi_{\tilde\mu}}_K(M\times\overline{\mathcal{O}^{\tilde\mu}})$ define the same class in $\mathbf{K}_K(\mathbf{T}_K(M\times\mathcal{O}^{\tilde\mu}))$ when M satisfies Assumption 3.4 at $\tilde\mu$.

Let σ_1,σ_2 be respectively the Thom symbols $\mathrm{Thom}_K(M)$ and $\mathrm{Thom}_K(\overline{\mathcal{O}^{\tilde\mu}})$. The symbol $\sigma_I=\mathrm{Thom}^\Phi_K(M)\odot\mathrm{Thom}_K(\overline{\mathcal{O}^{\tilde\mu}})$ is defined by

$$\sigma_I(m,\xi,v,w)=\sigma_1(m,v-\mathcal{H}_m)\odot\sigma_2(\xi,w)\ ,$$

where $(m,v)\in\mathbf{T}M$, $(\xi,w)\in\mathbf{T}\mathcal{O}^{\tilde\mu}$, and $\mathcal{H}$ is defined in (3.10). Let $\mathcal{H}^t$ be the vector field on $M\times\mathcal{O}^{\tilde\mu}$ generated by the map $\Phi_{t\tilde\mu}:M\times\mathcal{O}^{\tilde\mu}\to\mathfrak{k}$. For $(m,\xi)\in M\times\mathcal{O}^{\tilde\mu}$, we have $\mathcal{H}^t_{(m,\xi)}=(\mathcal{H}^{a,t}_{(m,\xi)},\mathcal{H}^{b,t}_{(m,\xi)})$ where $\mathcal{H}^{a,t}_{(m,\xi)}\in\mathbf{T}_mM$ and $\mathcal{H}^{b,t}_{(m,\xi)}\in\mathbf{T}_\xi\mathcal{O}^{\tilde\mu}$. The symbol $\sigma_{II}=\mathrm{Thom}^{\Phi_{\tilde\mu}}_K(M\times\overline{\mathcal{O}^{\tilde\mu}})$ is defined by

$$\sigma_{II}(m,\xi,v,w)=\sigma_1(m,v-\mathcal{H}^{a,1}_{m,\xi})\odot\sigma_2(\xi,w-\mathcal{H}^{b,1}_{(m,\xi)})\ .$$

We connect σ_I and σ_{II} through two homotopies. First consider the symbol A on $[0,1]\times\mathbf{T}(M\times\mathcal{O}^{\tilde\mu})$ defined by

$$A(t;m,\xi,v,w)=\sigma_1(m,v-\mathcal{H}^{a,t}_{m,\xi})\odot\sigma_2(\xi,w-\mathcal{H}^{b,t}_{(m,\xi)})\ ,$$

for $t\in[0,1]$, and $(m,\xi,v,w)\in\mathbf{T}(M\times\mathcal{O}^{\tilde\mu})$. We have $\mathrm{Char}(A)=\{(t;m,\xi,v,w)\mid v=\mathcal{H}^{a,t}_{m,\xi},\ \text{and}\ w=\mathcal{H}^{b,t}_{(m,\xi)}\}$ and

$$\begin{aligned}\mathrm{Char}(A)\bigcap[0,1]\times\mathbf{T}_K(M\times\mathcal{O}^{\tilde\mu})&=\{(t;m,\xi,0,0)\mid(m,\xi)\in\mathrm{Cr}(\|\Phi_{t\tilde\mu}\|^2)\ \}\\ &\subset[0,1]\times\mathcal{K}\times\mathcal{O}^{\tilde\mu}\ ,\end{aligned}$$

where $\mathcal{K}\subset M$ is the compact subset of Assumption 3.4. Hence A defines an homotopy of transversally elliptic symbols. The restriction of A to $t=1$ is equal to σ_{II}. The restriction of A to $t=0$ defines the following transversally elliptic symbol

$$\sigma_{III}(m,\xi,v,w)=\sigma_1(m,v-\mathcal{H}_m)\odot\sigma_2(\xi,w-\mathcal{H}^{b,0}_{(m,\xi)})$$

since $\mathcal{H}^{a,0}_{m,\xi} = \mathcal{H}_m$ for every $(m,\xi) \in M \times \mathcal{O}^{\tilde{\mu}}$. Now consider the symbol B on $[0,1] \times \mathbf{T}(M \times \mathcal{O}^{\tilde{\mu}})$ defined by

$$B(t; m, \xi, v, w) = \sigma_1(m, v - \mathcal{H}_m) \odot \sigma_2(\xi, w - t\,\mathcal{H}^{b,0}_{(m,\xi)}) \ .$$

We have $\mathrm{Char}(B) = \{(t; m, \xi, v, w) \mid v = \mathcal{H}_m, \text{ and } w = t\,\mathcal{H}^{b,0}_{(m,\xi)}\}$ and

$$\mathrm{Char}(B) \bigcap [0,1] \times \mathbf{T}_K(M \times \mathcal{O}^{\tilde{\mu}}) \subset \Big\{ (t; m, \xi, v = \mathcal{H}_m, w = t\,\mathcal{H}^{b,0}_{(m,\xi)}) \quad , \quad \| \mathcal{H}_m \|^2 + t \, \| \mathcal{H}^{b,0}_{(m,\xi)} \|^2 = 0 \Big\} \ .$$

In particular $\mathrm{Char}(B) \cap [0,1] \times \mathbf{T}_K(M \times \mathcal{O}^{\tilde{\mu}})$ is contained in $\{(t; m, \xi, 0, w = t\,\mathcal{H}^{b,0}_{(m,\xi)}) \,,\, m \in \mathrm{Cr}(\| \Phi \|^2)\}$ which is compact since $\mathrm{Cr}(\| \Phi \|^2)$ is compact. So, B defines an homotopy of transversally elliptic symbols between $\sigma_I = B|_{t=0}$ and $\sigma_{III} = B|_{t=1}$. We have finally proved that $\sigma_I, \sigma_{II}, \sigma_{III}$ define the same class in $\mathbf{K}_K(\mathbf{T}_K(M \times \mathcal{O}^{\tilde{\mu}}))$. □

In our shifting trick procedure, when $E = \tilde{L}$ is a κ-prequantum line bundle over M, the line bundle $L \boxtimes \mathbb{C}_{[-\mu]}$ is a κ-prequantum line bundle over $M \times \overline{\mathcal{O}^{\mu+\rho_c}}$. Then Proposition 3.5 tells us that under Assumption 3.4 the K-multiplicities of $RR^K_\Phi(M, \tilde{L})$ have the form

$$\Big[RR^K_\Phi(\mathcal{X}, \tilde{L}_{\mathcal{X}}) \Big]^K \ , \tag{3.14}$$

where $(\mathcal{X}, \omega_{\mathcal{X}}, \Phi)$ is a Hamiltonian K-manifold with $\mathrm{Cr}(\| \Phi \|^2)$ compact, and $\tilde{L}_{\mathcal{X}}$ is a κ-prequantum line bundle over $\mathcal{X}$ relative to a K-invariant almost complex structure.

To compute the quantity (3.14), we exploit the technique of localization developed in [33].

3.4. **Localization of the map RR^K_Φ.** For a detailed account on the procedure of localization that we use here, see sections 4 and 6 of [33]. We first state general facts about the symbol $\mathrm{Thom}^S_K(M,J)$ associated to an equivariant map $S : M \to \mathfrak{k}$. Let $\mathcal{U}$ be a K-invariant open subspace of M. The restriction $\mathrm{Thom}^S_K(M,J)|_{\mathcal{U}} = \mathrm{Thom}^S_K(\mathcal{U},J)$ is transversally elliptic[6] if and only if $\{m \in M,\, S_M(m) = 0\} \cap \mathcal{U}$ is compact. Let $j^{\mathcal{U},\mathcal{V}} : \mathcal{U} \hookrightarrow \mathcal{V}$ be two K-invariant open subspaces of M, where $j^{\mathcal{U},\mathcal{V}}$ denotes the inclusion. If $\{m \in M,\, S_M(m) = 0\} \cap \mathcal{U} = \{m \in M,\, S_M(m) = 0\} \cap \mathcal{V}$ is compact, the excision porperty tells us that

$$j^{\mathcal{U},\mathcal{V}}_* \Big(\mathrm{Thom}^S_K(\mathcal{U},J) \Big) = \mathrm{Thom}^S_K(\mathcal{V},J) \ ,$$

where $j^{\mathcal{U},\mathcal{V}}_* : \mathbf{K}_K(\mathbf{T}_K\mathcal{U}) \to \mathbf{K}_K(\mathbf{T}_K\mathcal{V})$ is the pushforward map (see [33][Section 3]).

Lemma 3.6. (1) *Let $S^0, S^1 : M \to \mathfrak{k}$ be two equivariant maps. Suppose there exist an open subset $\mathcal{U} \subset M$, and a vector field θ on $\mathcal{U}$ such that $(S^0_M, \theta)_M$ and $(S^1_M, \theta)_M$ are > 0 outside a compact subset $\mathcal{K}$ of $\mathcal{U}$. Then, the equivariant symbols $\mathrm{Thom}^{S^1}_K(\mathcal{U},J)$ and $\mathrm{Thom}^{S^0}_K(\mathcal{U},J)$ are transversally elliptic and define the same class in $\mathbf{K}_K(\mathbf{T}_K\mathcal{U})$.*

[6]Then, $Thom^S_K(M,J)|_{\mathcal{U}}$ defines a class in $\mathbf{K}_K(\mathbf{T}_K\mathcal{U})$.

(2) *Let J^0, J^1 be two almost complex structures on $\mathcal{U}$, and suppose that $\{m \in M,\ S_M(m) = 0\} \cap \mathcal{U}$ is compact. The transversally elliptic symbols* $\mathrm{Thom}^S_K(\mathcal{U}, J^0)$ *and* $\mathrm{Thom}^S_K(\mathcal{U}, J^1)$ *define the same class if there exists an homotopy J^t, $t \in [0,1]$ of K-equivariant almost complex structures between J^0 and J^1.*

Proof : The proof of (1) is similar to our deformation process in [31]. Here we consider the maps $S^t := tS^1 + (1-t)S^0$, $t \in [0,1]$, and the corresponding symbols $\mathrm{Thom}^{S^t}_K(\mathcal{U}, J)$. The vector field θ, insures that $\mathrm{Char}(\mathrm{Thom}^{S^t}_K(\mathcal{U}, J)) \cap \mathbf{T}_K\mathcal{U} \subset \mathcal{K}$ is compact. Hence $t \to \mathrm{Thom}^{S^t}_K(\mathcal{U}, J)$ defines an homotopy of transversally elliptic symbols. The proof of (2) is identical to the proof of Lemma 2.2 in [33]. □

In the remaining of this section $(\mathcal{X}, \omega_{\mathcal{X}}, \Phi)$ is a Hamiltonian K-manifold with $\mathrm{Cr}(\| \Phi \|^2)$ compact, and which is equipped with a κ-prequantum line bundle $\tilde{L}_{\mathcal{X}}$ associated to a K-invariant almost complex structure. Our aim here is to give a condition under which $[RR^K_\Phi(\mathcal{X}, \tilde{L})]^K$ only depends of the data in the neighborhood of $\Phi^{-1}(0)$.

For any $\beta \in \mathfrak{k}$, let M^β be the symplectic submanifold of points of M fixed by $\mathbb{T}_\beta$. Following Kirwan [22], the critical set $\mathrm{Cr}(\| \Phi \|^2)$ admits the decomposition

$$\mathrm{Cr}(\| \Phi \|^2) = \bigcup_{\beta \in \mathcal{B}} C^K_\beta, \quad \text{with} \quad C^K_\beta = K.(\mathcal{X}^\beta \cap \Phi^{-1}(\beta)), \tag{3.15}$$

where $\mathcal{B}$ is the subset of $\mathfrak{t}^*_+$ defined by $\mathcal{B} := \{\beta \in \mathfrak{t}^*_+,\ \mathcal{X}^\beta \cap \Phi^{-1}(\beta) \neq \emptyset\}$. Since $\mathrm{Cr}(\| \Phi \|^2)$ is supposed to be compact, $\mathcal{B}$ is finite.

For each $\beta \in \mathcal{B}$, let $\mathcal{U}^\beta \hookrightarrow \mathcal{X}$ be a K-invariant *relatively compact* open neighbourhood of C^K_β such that $\overline{\mathcal{U}^\beta} \cap \mathrm{Cr}(\| \Phi \|^2) = C^K_\beta$. The restriction of the transversally elliptic symbol $\mathrm{Thom}^\Phi_K(\mathcal{X})$ to the subset $\mathcal{U}^\beta$ defines $\mathrm{Thom}^\Phi_K(\mathcal{U}^\beta) \in \mathbf{K}_K(\mathbf{T}_K\mathcal{U}^\beta)$.

Definition 3.7. *For every $\beta \in \mathcal{B}$, we denote $RR^K_\beta(\mathcal{X}, -)$ the Riemann-Roch character localized near C^K_β, which is defined by*

$$RR^K_\beta(\mathcal{X}, E) = \mathrm{Index}^K_{\mathcal{U}^\beta}\left(\mathrm{Thom}^\Phi_K(\mathcal{U}^\beta) \otimes E|_{\mathcal{U}^\beta}\right),$$

for every K-vector bundle $E \to \mathcal{X}$.

By Section 4 of [33], we have

$$RR^K_\Phi(\mathcal{X}, E) = \sum_{\beta \in \mathcal{B}} RR^K_\beta(\mathcal{X}, E) \tag{3.16}$$

for every K-vector bundle $E \to \mathcal{X}$. Fix now a non-zero element $\beta \in \mathcal{B}$. For every connected component $\mathcal{Z}$ of $\mathcal{X}^\beta$, let $\mathcal{N}_{\mathcal{Z}}$ be the normal bundle of $\mathcal{Z}$ in $\mathcal{X}$. Let $\alpha^{\mathcal{Z}}_1, \cdots, \alpha^{\mathcal{Z}}_l$ be the real infinitesimal weights for the action of $\mathbb{T}_\beta$ on the fibers of $\mathcal{N}_{\mathcal{Z}} \otimes \mathbb{C}$. The infinitesimal action of β on $\mathcal{N}_{\mathcal{Z}} \otimes \mathbb{C}$ is a linear map with trace equal to $\sqrt{-1}\sum_i \langle \alpha^{\mathcal{Z}}_i, \beta\rangle$.

Definition 3.8. *Let us denote $\mathbf{Tr}_\beta|\mathcal{N}_{\mathcal{Z}}|$ the following positive number*

$$\mathbf{Tr}_\beta|\mathcal{N}_{\mathcal{Z}}| := \sum_{i=1}^{l} |\langle \alpha^{\mathcal{Z}}_i, \beta\rangle|$$

where $\alpha_1^{\mathcal{Z}}, \cdots, \alpha_l^{\mathcal{Z}}$ are the the real infinitesimal weights for the action of $\mathbb{T}_\beta$ on the fibers of $\mathcal{N}\otimes\mathbb{C}$. For any $\mathbb{T}_\beta$-equivariant real vector bundle $\mathcal{V}\to\mathcal{Z}$ (resp. real $\mathbb{T}_\beta$-equivariant real vector space E), we define in the same way $\mathbf{Tr}_\beta|\mathcal{V}|\geq 0$ (resp. $\mathbf{Tr}_\beta|E|\geq 0$).

Remark 3.9. *If $\mathcal{V}=\mathcal{V}^1+\mathcal{V}^2$, we have $\mathbf{Tr}_\beta|\mathcal{V}|=\mathbf{Tr}_\beta|\mathcal{V}^1|+\mathbf{Tr}_\beta|\mathcal{V}^2|$, and if $\mathcal{V}'$ is an equivariant real subbundle of $\mathcal{V}$, we get $\mathbf{Tr}_\beta|\mathcal{V}|\geq\mathbf{Tr}_\beta|\mathcal{V}'|$. In particular one see that $\mathbf{Tr}_\beta|\mathcal{N}_{\mathcal{Z}}|=\mathbf{Tr}_\beta|\mathbf{T}M|_{\mathcal{Z}}|$, and then, if $E_m\subset\mathbf{T}_mM$ is a $\mathbb{T}_\beta$-invariant real vector subspace for some $m\in\mathcal{Z}$, we have $\mathbf{Tr}_\beta|\mathcal{N}_{\mathcal{Z}}|\geq\mathbf{Tr}_\beta|E_m|$.*

The following Proposition and Corollary give us an essential condition under which the number $[RR^{^K}_\Phi(\mathcal{X},\tilde{L}_{\mathcal{X}})]^K$ only depends on data localized in a neighborhhood of $\Phi^{-1}(0)$.

Proposition 3.10. *Let $\tilde{L}_{\mathcal{X}}$ be a κ-prequantum line bundle over $\mathcal{X}$. Let ρ_c be half the sum of the positive roots. The multiplicity of the trivial representation in $RR^{^K}_\beta(\mathcal{X},\tilde{L}_{\mathcal{X}})$ is equal to zero if*

$$\| \beta \|^2+\frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}_{\mathcal{Z}}|-2(\rho_c,\beta)>0 \tag{3.17}$$

for every connected component $\mathcal{Z}$ of $\mathcal{X}^\beta$ which intersects $\Phi^{-1}(\beta)$. Condition (3.17) always holds if $\beta\in\mathfrak{k}$ is K-invariant or if $\|\beta\|>\|\rho_c\|$.

Since every $\beta\in\mathcal{B}$ belongs to the Weyl chamber, we have $2(\rho_c,\beta)=\mathbf{Tr}_\beta|\mathfrak{k}/\mathfrak{t}|$, and then (3.17) can be rewritten as $\|\beta\|^2+\frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}_{\mathcal{Z}}|-\mathbf{Tr}_\beta|\mathfrak{k}/\mathfrak{t}|>0$. From eq. (3.16), we get

Corollary 3.11. *If condition (3.17) holds for all non-zero $\beta\in\mathcal{B}$, we have*

$$\Big[RR^{^K}_\Phi(\mathcal{X},\tilde{L}_{\mathcal{X}})\Big]^K=\Big[RR^{^K}_0(\mathcal{X},\tilde{L}_{\mathcal{X}})\Big]^K$$

where $RR^{^K}_0(\mathcal{X},-)$ is the Riemann-Roch character localized near $\Phi^{-1}(0)$ (see Definition 3.7). In particular, $\Big[RR^{^K}_\Phi(\mathcal{X},\tilde{L}_{\mathcal{X}})\Big]^K=0$ if condition (3.17) holds for all non-zero $\beta\in\mathcal{B}$, and $0\notin\mathrm{Image}(\Phi)$.

3.5. **Proof of Proposition 3.10.** First when $\beta\in\mathfrak{k}$ is K-invariant, the scalar product (ρ_c,β) vanishes and then (3.17) trivially holds. Let us show that (3.17) holds when $\|\beta\|>\|\rho_c\|$. Let $\mathcal{Z}$ a connected component of $\mathcal{X}^\beta$ which intersects $\Phi^{-1}(\beta)$. Let $m\in\Phi^{-1}(\beta)\cap\mathcal{Z}$, and let $E_m\subset\mathbf{T}_mM$ be the subspace spanned by $X_M(m)$, $X\in\mathfrak{k}$. We have $E_m\simeq\mathfrak{k}/\mathfrak{k}_m$, where $\mathfrak{k}_m:=\{X\in\mathfrak{k},\,X_M(m)=0\}$. Since $\Phi(m)=\beta$, and Φ is equivariant $\mathfrak{k}_m\subset\mathfrak{k}_\beta:=\{X\in\mathfrak{k},\,[X,\beta]=0\}$, so $\mathbf{T}_mM$ contains a $\mathbb{T}_\beta$-equivariant subspace isomorphic to $\mathfrak{k}/\mathfrak{k}_\beta$. So we have $\mathbf{Tr}_\beta|\mathcal{N}_{\mathcal{Z}}|\geq\mathbf{Tr}_\beta|\mathfrak{k}/\mathfrak{k}_\beta|=2(\rho_c,\beta)$, and then

$$\begin{aligned}\|\beta\|^2+\frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}_{\mathcal{Z}}|-2(\rho_c,\beta) &\geq \|\beta\|^2-(\rho_c,\beta)\\ &> 0\end{aligned}$$

since $\|\beta\|>\|\rho_c\|$. □

We prove now that condition (3.17) forces[7] $[RR^{^K}_\beta(\mathcal{X},\tilde{L})]^K$ to be equal to 0. Let $\mathrm{m}_{\beta,\mu}(E)\in\mathbb{Z}$ be the K-multiplicities of the localized Riemann-Roch character

[7]In this section we denotes simply by $\tilde{L}$ the κ-prequantum line bundle over $\mathcal{X}$.

$RR^K_\beta(\mathcal{X},E)$ introduced in Definition 3.7 : $RR^K_\beta(\mathcal{X},E)=\sum_{\mu\in\Lambda^*_+}\mathrm{m}_{\beta,\mu}(E)\,\chi^K_\mu$. We show now that $\mathrm{m}_{\beta,0}(\tilde{L})=0$, by using the formulaes of localization that we proved in [33] for the maps $RR^K_\beta(\mathcal{X},-)$.

First case : $\beta\in\mathcal{B}$ is a non-zero K-invariant element of $\mathfrak{k}^*$.

We show here the following relation on the multiplicities $\mathrm{m}_{\beta,\mu}(\tilde{L})$:

$$\mathrm{m}_{\beta,\mu}(\tilde{L})\neq 0 \implies (\mu,\beta)\geq \|\beta\|^2+\frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}_\mathcal{Z}| \quad \text{for some } \mathcal{Z}\subset\mathcal{X}^\beta, \tag{3.18}$$

in particular $\mathrm{m}_{\beta,0}(\tilde{L})=0$.

Since $\mathbb{T}_\beta$ belongs to the center of K, $\mathcal{X}^\beta$ is a symplectic K-invariant submanifold of $\mathcal{X}$. Let $\mathcal{N}$ be the normal bundle of $\mathcal{X}^\beta$ in $\mathcal{X}$. The K-invariant almost complex structure of $\mathcal{X}$ induces a K-invariant almost complex structure on $\mathcal{X}^\beta$, and a complex structure on the fibers of $\mathcal{N}\to\mathcal{X}^\beta$. We have then a Riemann-Roch character $RR^K_\beta(\mathcal{X}^\beta,-)$ localized on $\mathcal{X}^\beta\cap\Phi^{-1}(\beta)$ with the decomposition $RR^K_\beta(\mathcal{X}^\beta,F)=\sum_\mathcal{Z}RR^K_\beta(\mathcal{Z},F|_\mathcal{Z})$, where the sum is taken over the connected component $\mathcal{Z}\subset\mathcal{X}^\beta$ which intersect $\Phi^{-1}(\beta)$. The torus $\mathbb{T}_\beta$ acts linearly on the fibers of the complex vector bundle $\mathcal{N}$, thus we can associate the polarized complex K-vector bundle $\mathcal{N}^{+,\beta}$ and $(\mathcal{N}\otimes\mathbb{C})^{+,\beta}$ (see Definition 5.5 in [33]): for any real $\mathbb{T}_\beta$-weight α on $\mathcal{N}^{+,\beta}$, or on $(\mathcal{N}\otimes\mathbb{C})^{+,\beta}$, we have

$$(\alpha,\beta)>0\ . \tag{3.19}$$

We proved in section 6.2 of [33], the following localization formula which holds in $\widehat{R}(K)$ for any K-vector bundle E over $\mathcal{X}$:

$$RR^K_\beta(\mathcal{X},E)=(-1)^{r_\mathcal{N}}\sum_{k\in\mathbb{N}}RR^K_\beta\left(\mathcal{X}^\beta,E|_{\mathcal{X}^\beta}\otimes\det\mathcal{N}^{+,\beta}\otimes S^k((\mathcal{N}\otimes\mathbb{C})^{+,\beta})\right)\ . \tag{3.20}$$

Here $r_\mathcal{N}$ is the locally constant function on $\mathcal{X}^\beta$ equal to the complex rank of $\mathcal{N}^{+,\beta}$, and $S^k(-)$ is the k-th symmetric product over $\mathbb{C}$.

Let $i:\mathfrak{t}_\beta\hookrightarrow\mathfrak{t}$ be the inclusion of the Lie algebra of $\mathbb{T}_\beta$, and let $i^*:\mathfrak{t}^*\to\mathfrak{t}^*_\beta$ be the canonical dual map. Let us recall the basic relationship between the $\mathbb{T}_\beta$-weight on the fibers of a K-vector bundle $F\to\mathcal{X}^\beta$ and the K-multiplicities of $RR^K_\beta(\mathcal{X}^\beta,F)\in\widehat{R}(K)$: if the irreducible representation V_μ occurs in $RR^K_\beta(\mathcal{X}^\beta,F)$, then $i^*(\mu)$ is a $\mathbb{T}_\beta$-weight on the fibers of F (see Appendix B in [33]).

If one now uses (3.20), one sees that $\mathrm{m}_{\mu,\beta}(\tilde{L})\neq 0$ only if $i^*(\mu)$ is a $\mathbb{T}_\beta$-weight on the fibers of some $\tilde{L}|_\mathcal{Z}\otimes\det\mathcal{N}^{+,\beta}_\mathcal{Z}\otimes S^k((\mathcal{N}_\mathcal{Z}\otimes\mathbb{C})^{+,\beta})$. Since $(i^*(\mu),\beta)=(\mu,\beta)$, (3.18) will be proved if one shows that each $\mathbb{T}_\beta$-weight γ_z on $\tilde{L}|_\mathcal{Z}\otimes\det\mathcal{N}^{+,\beta}_\mathcal{Z}\otimes S^k((\mathcal{N}_\mathcal{Z}\otimes\mathbb{C})^{+,\beta})$ satisfies

$$(\gamma_z,\beta)\geq\|\beta\|^2+\frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}_\mathcal{Z}|\ . \tag{3.21}$$

Let α_z be the $\mathbb{T}_\beta$-weight on the fiber of the line bundle $\tilde{L}|_\mathcal{Z}\otimes\det\mathcal{N}^{+,\beta}_\mathcal{Z}$. Since any $\mathbb{T}_\beta$-weight on $S^k((\mathcal{N}\otimes\mathbb{C})^{+,\beta})$ satisfies (3.19), (3.21) holds if

$$(\alpha_z,\beta)\geq\|\beta\|^2+\frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}_\mathcal{Z}| \tag{3.22}$$

for every $\mathcal{Z} \subset \mathcal{X}^\beta$ which intersects $\Phi^{-1}(\beta)$. Let $L_{2\omega}$ be the prequantum line bundle on $(M, 2\omega, 2\Phi)$ such that $\tilde{L}^2 = L_{2\omega} \otimes \kappa$ (where κ is by definition equal to $\det(\mathbf{T}^*_{\mathbb{C}} M) \simeq \det(\mathbf{T}M)^{-1}$). We have

$$(\tilde{L}|_{\mathcal{Z}} \otimes \det(\mathcal{N}^{+,\beta}))^2 = L_{2\omega}|_{\mathcal{Z}} \otimes \det(\mathbf{T}M)^{-1}|_{\mathcal{Z}} \otimes \det(\mathcal{N}^{+,\beta})^2 \ .$$

So $2\alpha_{\mathcal{Z}} = \alpha_1 + \alpha_2$ where α_1, α_2 are respectively $\mathbb{T}_\beta$-weights on $L_{2\omega}|_{\mathcal{Z}}$ and $\det(\mathbf{T}M)^{-1}|_{\mathcal{Z}} \otimes \det(\mathcal{N}^{+,\beta})^2$. The Kostant formula (2.1) on $L_{2\omega}|_{\mathcal{Z}}$ gives $(\alpha_1, X) = 2(\beta, X)$ for every $X \in \mathfrak{t}_\beta$, in particular

$$(\alpha_1, \beta) = 2 \parallel \beta \parallel^2 \ . \tag{3.23}$$

On $\mathcal{Z}$, the complex vector bundle $\mathbf{T}M$ has the following decomposition, $\mathbf{T}M|_{\mathcal{Z}} = \mathbf{T}\mathcal{Z} \oplus \mathcal{N}^{-,\beta} \oplus \mathcal{N}^{+,\beta}$, where $\mathcal{N}^{-,\beta}$ is the orthogonal complement of $\mathcal{N}^{+,\beta}$ in $\mathcal{N}$: every $\mathbb{T}_\beta$-weight δ on $\mathcal{N}^{-,\beta}$ verifies $(\delta, \beta) < 0$. So we get the decomposition $\det(\mathbf{T}M)^{-1}|_{\mathcal{Z}} \otimes \det(\mathcal{N}^{+,\beta})^2 = \det(\mathbf{T}\mathcal{Z}) \otimes \det(\mathcal{N}^{-,\beta})^{-1} \otimes \det(\mathcal{N}^{+,\beta})$, which shows

$$(\alpha_2, \beta) = \mathbf{Tr}_\beta |\mathcal{N}_{\mathcal{Z}}| \tag{3.24}$$

since $\mathbb{T}_\beta$ acts trivially on $\mathbf{T}\mathcal{Z}$. Finally (3.22) follows trivially from (3.23) and (3.24).

Second case : $\beta \in \mathcal{B}$ such that $K_\beta \neq K$.

Consider the induced Hamiltonian action of K_β on $\mathcal{X}$, with moment map $\Phi_{K_\beta} : \mathcal{X} \to \mathfrak{k}^*_\beta$. Let $\mathcal{B}'$ be the indexing set for the critical point of $\parallel \Phi_{K_\beta} \parallel^2$ (see (3.15)). Following Definition 3.7, we consider for each $\beta' \in \mathcal{B}'$ the K_β-quantization map $RR^{K_\beta}_{\beta'}(\mathcal{X}, -)$ localised on $C^{K_\beta}_{\beta'} = K_\beta.(\mathcal{X}^{\beta'} \cap \Phi^{-1}_{K_\beta}(\beta'))$. Here β is a K_β-invariant element of $\mathcal{B}'$ with[8] $C^{K_\beta}_\beta = \mathcal{X}^\beta \cap \Phi^{-1}(\beta)$.

Let $\mathrm{Hol}^K_T : R^{-\infty}(T) \to R^{-\infty}(K)$, $\mathrm{Hol}^{K_\beta}_T : R^{-\infty}(T) \to R^{-\infty}(K_\beta)$, and $\mathrm{Hol}^K_{K_\beta} : R^{-\infty}(K_\beta) \to R^{-\infty}(K)$ be the holomorphic induction maps (see Appendix B in [33]). Recall that $\mathrm{Hol}^K_T = \mathrm{Hol}^K_{K_\beta} \circ \mathrm{Hol}^{K_\beta}_T$. The choice of a Weyl chamber determines a complex structure on the real vector space $\mathfrak{k}/\mathfrak{k}_\beta$. We denote $\overline{\mathfrak{k}/\mathfrak{k}_\beta}$ the vector space endowed with the opposite complex structure.

The induction formula that we proved in [33][Section 6] states that, for every equivariant vector bundle E, we have

$$RR^K_\beta(\mathcal{X}, E) = \mathrm{Hol}^K_{K_\beta} \left(RR^{K_\beta}_\beta(\mathcal{X}, E) \wedge^{\bullet}_{\mathbb{C}} \overline{\mathfrak{k}/\mathfrak{k}_\beta} \right) \tag{3.25}$$

Let us first write the decomposition[9] $RR^{K_\beta}_\beta(M, \tilde{L}) = \sum_{\mu \in \Lambda^+_\beta} \mathrm{m}_{\mu,\beta}(\tilde{L}) \chi^{K_\beta}_\mu$, into irreducible characters of K_β. Since β is K_β-invariant we can use the results of the First case. In particular (3.18) tells us that

$$\mathrm{m}_{\beta,\mu}(\tilde{L}) \neq 0 \implies (\mu, \beta) \geq \parallel \beta \parallel^2 + \frac{1}{2} \mathbf{Tr}_\beta |\mathcal{N}_{\mathcal{Z}}| \tag{3.26}$$

for some connected component $\mathcal{Z} \subset \mathcal{X}^\beta$ which intersects $\Phi^{-1}(\beta)$.

Each irreducible character $\chi^{K_\beta}_\mu$ is equal to $\mathrm{Hol}^{K_\beta}_T(t^\mu)$, so from (3.25) we get $RR^K_\beta(M, \tilde{L}) = \mathrm{Hol}^K_T \left((\sum_\mu m_{\mu,\beta}(\tilde{L})\, t^\mu) \Pi_{\alpha \in \mathfrak{R}^+(\mathfrak{k}/\mathfrak{k}_\beta)} (1 - t^{-\alpha}) \right)$ where $\mathfrak{R}^+(\mathfrak{k}/\mathfrak{k}_\beta)$ is the set of positive T-weights on $\mathfrak{k}/\mathfrak{k}_\beta$: so $\langle \alpha, \beta \rangle > 0$ for all $\alpha \in \mathfrak{R}^+(\mathfrak{k}/\mathfrak{k}_\beta)$. Finally ,

[8] On $\mathcal{X}^\beta$, the map Φ_{K_β} and Φ coincide.

[9] We choose a set $\Lambda^*_{+,\beta}$ of dominant weights for K_β that contains the set Λ^*_+ of dominant weight for K.

we see that $RR^K_\beta(M,\tilde{L})$ is a sum of terms of the form $m_{\mu,\beta}(\tilde{L})\,\mathrm{Hol}^K_T(t^{\mu-\alpha_I})$ where $\alpha_I = \sum_{\alpha\in I}\alpha$ and I is a subset of $\mathfrak{R}^+(\mathfrak{k}/\mathfrak{k}_\beta)$. We know that $\mathrm{Hol}^K_T(t^{\mu'})$ is either 0 or the character of an irreducible representation (times ±1) ; in particular $\mathrm{Hol}^K_T(t^{\mu'})$ is equal to ±1 only if $(\mu', X) \leq 0$ for every $X \in \mathfrak{t}_+$ (see Appendix B in [33]). So $[RR^K_\beta(M,\tilde{L})]^K \neq 0$ only if there exists a weight μ such that $m_{\mu,\beta}(\tilde{L}) \neq 0$ and that $\mathrm{Hol}^K_T(t^{\mu-\alpha_I}) = \pm1$. The first condition imposes $(\mu,\beta) \geq \parallel \beta \parallel^2 + \frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}_\mathcal{Z}|$ for some connected component $\mathcal{Z} \subset \mathcal{X}^\beta$, and the second one gives $(\mu,\beta) \leq (\alpha_I,\beta)$. Combining the two we end up with

$$\parallel \beta \parallel^2 + \frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}_\mathcal{Z}| \leq (\alpha_I,\beta) \leq \sum_{\alpha\in\mathfrak{R}^+(\mathfrak{k}/\mathfrak{k}_\beta)} (\alpha,\beta) = 2(\rho_c,\beta) \ ,$$

for some connected component $\mathcal{Z} \subset \mathcal{X}^\beta$ which intersects $\Phi^{-1}(\beta)$. Finally we have proved that $[RR^K_\beta(M,\tilde{L})]^K = 0$ if $\parallel \beta \parallel^2 + \frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}_\mathcal{Z}| > 2(\rho_c,\beta)$ for every component $\mathcal{Z} \subset \mathcal{X}^\beta$ which intersects $\Phi^{-1}(\beta)$. □

4. Quantization commutes with reduction

Let (M,ω,Φ) be a Hamiltonian K-manifold equipped with an almost complex structure J. In this section, we assume that the moment map Φ is *proper* and that the set $\mathrm{Cr}(\parallel \Phi \parallel^2)$ of critical points of $\parallel \Phi \parallel^2$: $M \to \mathbb{R}$ is *compact*. We denote $RR^K_\Phi(M,-)$ the corresponding quantization map (see Definition 3.2). Let Δ be the convex polyhedral set $\Phi(M) \cap \mathfrak{t}^*_+$.

The main result of this section is the following

Theorem 4.1. *Suppose that M satisfies Assumption 3.4 at every $\tilde{\mu} \in \mathfrak{t}^*$, and that the infinitesimal stabilizers for the K-action on M are* abelian. *If $\tilde{L}$ is a κ-prequantum line bundle over (M,ω,Φ,J), we have*

$$RR^K_\Phi(M,\tilde{L}) = \varepsilon \sum_{\mu\in\Lambda^*_+} \mathcal{Q}(M_{\mu+\rho_c})\,\chi^K_\mu \ , \tag{4.27}$$

where $\varepsilon = \pm1$ is the 'quotient' of the orientation $o(J)$ defined by the almost complex structure and the orientation $o(\omega)$ defined by the symplectic form. Here the integer $\mathcal{Q}(M_{\mu+\rho_c})$ are computed by Proposition 2.4. In particular, $\mathcal{Q}(M_{\mu+\rho_c}) = 0$ if $\mu+\rho_c$ does not belong to the relative interior of Δ.

The same result holds in the traditional 'prequantum' case. Suppose that M satisfies Assumption 3.4 at every $\mu \in \mathfrak{t}^*$, and that the almost complex structure J is compatible with ω. If L is prequantum line bundle over (M,ω,Φ), we have $RR^K_\Phi(M,L) = \sum_{\mu\in\Lambda^*_+} RR(M_\mu,L_\mu)\,\chi^K_\mu$.

The remaining part of this section is devoted to the proof of Theorem 4.1: we have to show that the K-multiplicities, $\mathrm{m}_\mu(\tilde{L})$, of $RR^K_\Phi(M,\tilde{L})$ are equal to $\varepsilon\mathcal{Q}(M_{\mu+\rho_c})$, where the quantities $\mathcal{Q}(M_{\mu+\rho_c})$ are defined by Proposition 2.4. Since (M,Φ) satisfies Assumption 3.4 at every $\tilde{\mu}$, we know from Proposition 3.5 that $\mathrm{m}_\mu(\tilde{L}) = [RR^K_{\Phi_{\tilde{\mu}}}(M\times\overline{\mathcal{O}^{\tilde{\mu}}}, \tilde{L}\boxtimes\mathbb{C}_{[-\mu]})]^K$ for every $\mu\in\Lambda^*_+$.

The next Lemma is the first step in computing the multiplicities $\mathrm{m}_\mu(\tilde{L})$.

Lemma 4.2. *Let $RR_0^K(M\times\overline{\mathcal{O}^{\tilde\mu}},-)$ be the Riemann-Roch character localized*[10] *near $\Phi_{\tilde\mu}^{-1}(0)\simeq\Phi^{-1}(\mu+\rho_c)$ (see Def. 3.7). If $\tilde L$ is a κ-prequantum line bundle over M, and if the infinitesimal stabilizers for the K-action are* abelian, *we have*

$$\mathrm{m}_\mu(\tilde L)=\Big[RR_0^K(M\times\overline{\mathcal{O}^{\tilde\mu}},\tilde L\boxtimes\mathbb{C}_{[-\mu]})\Big]^K\ . \tag{4.28}$$

In particular $\mathrm{m}_\mu(\tilde L)=0$ if $\mu+\rho_c$ does not belong to the moment polyhedral subset Δ.

Proof : The lemma will follow from Corollary 3.11, applied to the symplectic manifold $\mathcal{X}:=M\times\overline{\mathcal{O}^{\tilde\mu}}$, with moment map $\Phi_{\tilde\mu}$ and κ-prequantum line bundle $\tilde L\boxtimes\mathbb{C}_{[-\mu]}$. Let $\beta\neq0$ such that $\mathcal{X}^\beta\cap\Phi_{\tilde\mu}^{-1}(\beta)\neq\emptyset$. Let $\mathcal{N}$ be the normal bundle of $\mathcal{X}^\beta$ in $\mathcal{X}$, and let $x\in\mathcal{X}^\beta\cap\Phi_{\tilde\mu}^{-1}(\beta)$. By the criterion of Proposition 3.10, it is sufficient to show that

$$\parallel\beta\parallel^2+\frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}_x|-2(\rho_c,\beta)>0\ . \tag{4.29}$$

Write $x=(m,\xi)$ with $m\in M^\beta$ and $\xi\in(\mathcal{O}^{\tilde\mu})^\beta$. We know that $\mathbf{Tr}_\beta|\mathcal{N}_x|=\mathbf{Tr}_\beta|\mathbf{T}_x\mathcal{X}|=\mathbf{Tr}_\beta|\mathbf{T}_mM|+\mathbf{Tr}_\beta|\mathbf{T}_\xi\mathcal{O}^{\tilde\mu}|$. The tangent space $\mathbf{T}_\xi\mathcal{O}^{\tilde\mu}\simeq\mathfrak{k}/\mathfrak{k}_\xi$ contains[11] a copy of $\mathfrak{k}/\mathfrak{k}_\beta$, so $\mathbf{Tr}_\beta|\mathbf{T}_\xi\mathcal{O}^{\tilde\mu}|\geq\mathbf{Tr}_\beta|\mathfrak{k}/\mathfrak{k}_\beta|=2(\rho_c,\beta)$. On the other hand, $\mathbf{T}_mM$ contains the vector space $E_m\simeq\mathfrak{k}/\mathfrak{k}_m$ spanned by $X_M(m)$, $X\in\mathfrak{k}$. We have assume that the stabilizer subalgebra $\mathfrak{k}_m$ is abelian, and since $\beta\in\mathfrak{k}_m$, we get $\mathfrak{k}_m\subset\mathfrak{k}_\beta$. Thus $\mathfrak{k}/\mathfrak{k}_\beta\subset E_m\subset\mathbf{T}_mM$ and $\mathbf{Tr}_\beta|\mathbf{T}_mM|\geq2(\rho_c,\beta)$. Finally (4.29) is proved since $\frac{1}{2}(\mathbf{Tr}_\beta|\mathbf{T}_mM|+\mathbf{Tr}_\beta|\mathbf{T}_\xi\mathcal{O}^{\tilde\mu}|)\geq2(\rho_c,\beta)$. □

In the next subsection, we recall the basic notions about Spin^c-structures.

4.1. Spin^c structures and symbols.

We refer to Lawson-Michelson [25] for background on Spin^c-structures, and to Duistermaat [14] for a discussion of the symplectic case.

The group Spin_n is the connected double cover of the group SO_n. Let $\eta:\mathrm{Spin}_n\to\mathrm{SO}_n$ be the covering map, and let ε be the element who generates the kernel. The group Spin_n^c is the quotient $\mathrm{Spin}_n\times_{\mathbb{Z}_2}\mathrm{U}_1$, where $\mathbb{Z}_2$ acts by $(\varepsilon,-1)$. There are two canonical group homomorphisms

$$\eta:\mathrm{Spin}_n^c\to\mathrm{SO}_n\quad,\quad\mathrm{Det}:\mathrm{Spin}_n^c\to\mathrm{U}_1\ .$$

Note that $\eta^c=(\eta,\mathrm{Det}):\mathrm{Spin}_n^c\to\mathrm{SO}_n\times\mathrm{U}_1$ is a double covering map.

Let $p:E\to M$ be a oriented Euclidean vector bundle of rank n, and let $\mathrm{P}_{\mathrm{SO}}(E)$ be its bundle of oriented orthonormal frames. A Spin^c-structure on E is a Spin_n^c principal bundle $\mathrm{P}_{\mathrm{Spin}^c}(E)\to M$, together with a Spin^c-equivariant map $\mathrm{P}_{\mathrm{Spin}^c}(E)\to\mathrm{P}_{\mathrm{SO}}(E)$. The line bundle $\mathbb{L}:=\mathrm{P}_{\mathrm{Spin}^c}(E)\times_{\mathrm{Det}}\mathbb{C}$ is called the determinant line bundle associated to $\mathrm{P}_{\mathrm{Spin}^c}(E)$. Whe have then a double covering map[12]

$$\eta_E^c\ :\mathrm{P}_{\mathrm{Spin}^c}(E)\longrightarrow\mathrm{P}_{\mathrm{SO}}(E)\times\mathrm{P}_{\mathrm{U}}(\mathbb{L})\ , \tag{4.30}$$

where $\mathrm{P}_{\mathrm{U}}(\mathbb{L}):=\mathrm{P}_{\mathrm{Spin}^c}(E)\times_{\mathrm{Det}}\mathrm{U}_1$ is the associated U_1-principal bundle over M.

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[10] Note that $RR_0^K(M\times\overline{\mathcal{O}^{\tilde\mu}},-)$ is the *zero map* if $\Phi^{-1}(\mu+\rho_c)=\emptyset$.

[11] Since the stabilizer $\mathfrak{k}_\xi\simeq\mathfrak{t}$ is abelian and $\beta\in\mathfrak{k}_\xi$ we get $\mathfrak{k}_\xi\subset\mathfrak{k}_\beta$.

[12] If P, Q are principal bundle over M respectively for the groups G and H, we denote simply by $P\times Q$ their fibering product over M which is a $G\times H$ principal bundle over M.

A Spin^c-structure on a oriented Riemannian manifold is a Spin^c-structure on its tangent bundle. If a group K acts on the bundle E, preserving the orientation and the Euclidean structure, we define a K-equivariant Spin^c-structure by requiring $P_{\mathrm{Spin}^c}(E)$ to be a K-equivariant principal bundle, and (4.30) to be $(K \times \mathrm{Spin}^c_n)$-equivariant.

Let Δ_{2m} be the complex Spin representation of Spin^c_{2m}. Recall that $\Delta_{2m} = \Delta^+_{2m} \oplus \Delta^-_{2m}$ inherits a canonical Clifford action $\mathbf{c} : \mathbb{R}^{2m} \to \mathrm{End}_{\mathbb{C}}(\Delta_{2m})$ which is Spin^c_{2m}-equivariant, and which interchanges the graduation : $\mathbf{c}(v) : \Delta^{\pm}_{2m} \to \Delta^{\mp}_{2m}$, for evrey $v \in \mathbb{R}^{2m}$. Let

$$\mathcal{S}(E) := P_{\mathrm{Spin}^c}(E) \times_{\mathrm{Spin}^c_{2m}} \Delta_{2m} \tag{4.31}$$

be the spinor bundle over M, with the grading $\mathcal{S}(E) := \mathcal{S}(E)^+ \oplus \mathcal{S}(E)^-$. Since $E = P_{\mathrm{Spin}^c}(E) \times_{\mathrm{Spin}^c_{2m}} \mathbb{R}^{2m}$, the bundle $p^*\mathcal{S}(E)$ is isomorphic to $P_{\mathrm{Spin}^c}(E) \times_{\mathrm{Spin}^c_{2m}} (\mathbb{R}^{2m} \oplus \Delta_{2m})$.

Let $\overline{E}$ be the bundle E with opposite orientation. A Spin^c structure on E induces a Spin^c on $\overline{E}$, with the same determinant line bundle, and such that $\mathcal{S}(\overline{E})^{\pm} = \mathcal{S}(E)^{\mp}$.

Definition 4.3. *Let* S-Thom$(E) : p^*\mathcal{S}(E)^+ \to p^*\mathcal{S}(E)^-$ *be the symbol defined by*

$$\begin{array}{ccc} P_{\mathrm{Spin}^c}(E) \times_{\mathrm{Spin}^c_{2m}} (\mathbb{R}^{2m} \oplus \Delta^+_{2m}) & \longrightarrow & P_{\mathrm{Spin}^c}(E) \times_{\mathrm{Spin}^c_{2m}} (\mathbb{R}^{2m} \oplus \Delta^-_{2m}) \\ {[p; v, w]} & \longmapsto & [p, v, \mathbf{c}(v)w] \ . \end{array}$$

When E is the tangent bundle of a manifold M, the symbol S-Thom(E) *is denoted* S-Thom(M). *If a group K acts equivariantly on the* Spin^c*-stucture, we denote* S-Thom$_K(E)$ *the equivariant symbol.*

The characteristic set of S-Thom(E) is $M \simeq \{\text{zero section of } E\}$, hence it defines a class in $\mathbf{K}(E)$ if M is compact (this class is a free generator of the $\mathbf{K}(M)$-module $\mathbf{K}(E)$ [3]). When $E = \mathbf{T}M$, the symbol S-Thom(M) corresponds to the *principal symbol* of the Spin^c Dirac operator associated to the Spin^c-structure [14]. If moreover M is compact, the number $\mathcal{Q}(M) \in \mathbb{Z}$ is defined as the index of S-Thom(M). If we change the orientation, note that $\mathcal{Q}(\overline{M}) = -\mathcal{Q}(M)$.

Remark 4.4. *It should be noted that the choice of the metric on the fibers of E is not essential in the construction. Let g_0, g_1 be two metric on the fibers of E, and suppose that (E, g_0) admits a* Spin^c*-stucture denoted* $P_{\mathrm{Spin}^c}(E, g_0)$. *The trivial homotopy $g_t = (1-t).g_0 + t.g_1$ between the metrics, induces an homotopy between the principal bundles* $P_{\mathrm{SO}}(E, g_0)$, $P_{\mathrm{SO}}(E, g_1)$ *which can be lifted to an homotopy between* $P_{\mathrm{Spin}^c}(E, g_0)$ *and a* Spin^c*-bundle over* (E, g_1). *When the base M is compact, the corresponding symbols* S-Thom(E, g_0) *and* S-Thom(E, g_1) *define the same class in* $\mathbf{K}(E)$.

These notions extend to the orbifold case. Let M be a manifold with a locally free action of a compact Lie group H. The quotient $\mathcal{X} := M/H$ is an orbifold, a space with finite quotient singularities. A Spin^c structure on $\mathcal{X}$ is by definition a H-equivariant Spin^c structure on the bundle $\mathbf{T}_H M \to M$, where $\mathbf{T}_H M$ is identified with the pullback of $\mathbf{T}\mathcal{X}$ via the quotient map $\pi : M \to \mathcal{X}$. We define in the same way S-Thom$(\mathcal{X}) \in \mathbf{K}_{orb}(\mathbf{T}\mathcal{X})$, such that π^*S-Thom$(\mathcal{X}) =$ S-Thom$_H(\mathbf{T}_H M)$. Here $\mathbf{K}_{orb}$ denotes the K-theory of *proper* vector bundles [21]. The pullback by π induces an isomophism $\pi^* : \mathbf{K}_{orb}(\mathbf{T}\mathcal{X}) \simeq \mathbf{K}_H(\mathbf{T}_H M)$. The number $\mathcal{Q}(\mathcal{X}) \in \mathbb{Z}$ is defined as the index of S-Thom($\mathcal{X}$), or equivalently as the multiplicity of the trivial representation in $\mathrm{Index}^H_M(\text{S-Thom}_H(\mathbf{T}_H M))$.

Consider now the case of a *complex* vector bundle $E \to M$, of complex rank m. The orientation on the fibers of E is given by the complex structure J. Let $\mathrm{P_U}(E)$ be the bundle of unitary frames on E. We have a morphism $\mathrm{j} : \mathrm{U}_m \to \mathrm{Spin}^c_{2m}$ which makes the diagram[13]

(4.32)
$$\begin{array}{ccc} \mathrm{U}_m & \xrightarrow{\ \mathrm{j}\ } & \mathrm{Spin}^c_{2m} \\ & \searrow^{\mathrm{i}\times\det} & \downarrow \eta^c \\ & & \mathrm{SO}_{2m} \times \mathrm{U}_1 \ . \end{array}$$

commutative [25]. Then

$$\mathrm{P}_{\mathrm{Spin}^c}(E) := \mathrm{Spin}^c_{2m} \times_{\mathrm{j}} \mathrm{P_U}(E) \tag{4.33}$$

defines a Spin^c-structure over E, with determinant line bundle equal to $\det_{\mathbb{C}} E$.

Lemma 4.5. *Let M be a manifold equipped with an almost complex structure J. The symbol* S-Thom(M) *defined by the* Spin^c*-structure (4.33), and the Thom symbol* Thom(M, J) *defined in section 3.2 coincide.*

Proof : The Spinor bundle $\mathcal{S}$ is of the form $\mathrm{P}_{\mathrm{Spin}^c}(\mathbf{T}M) \times_{\mathrm{Spin}^c_{2m}} \Delta_{2m} = \mathrm{P_U}(\mathbf{T}M) \times_{\mathrm{U}_m} \Delta_{2m}$. The map $\mathbf{c} : \mathbb{R}^{2m} \to \mathrm{End}_{\mathbb{C}}(\Delta_{2m})$, when restricted to the U_m-equivariant action through j, is equivalent to the Clifford map $\mathrm{Cl} : \mathbb{R}^{2m} \to \mathrm{End}_{\mathbb{C}}(\wedge \mathbb{C}^m)$ (with the canonical action of U_m on $\mathbb{R}^{2m}$ and $\wedge\mathbb{C}^m$). Then $\mathcal{S} = \wedge_{\mathbb{C}} \mathbf{T}M$ endowed with the Clifford action. □

Lemma 4.6. *Let P be a* Spin^c*-structure over M, with bundle of spinors $\mathcal{S}$, and determinant line bundle $\mathbb{L}$. For every Hermitian line bundle $L \to M$, there exists a unique* Spin^c*-structure P_L with bundle of spinors $\mathcal{S} \otimes L$, and determinant line bundle $\mathbb{L} \otimes L^2$ (P_L is called the* Spin^c*-structure P twisted by L).*

Proof : Take $P_L = P \times_{\mathrm{U}_1} \mathrm{P_U}(L)$.

We end up this subsection with the following definitions. Let (M, o) be an oriented manifold. Suppose that

- a connected compact Lie K acts on M
- (M, o, K) carries a K-equivariant Spin^c-structure
- one has an equivariant map $\Psi : M \to \mathfrak{k}$.

Suppose first that M is compact. The symbol $\text{S-Thom}_K(M)$ is then elliptic and defines a quantization map

$$\mathcal{Q}^K(M, -) : \mathbf{K}_K(M) \to R(K)$$

by the relation $\mathcal{Q}^K(M, V) := \mathrm{Index}^K_M(\text{S-Thom}_K(M) \otimes V)$: $\mathcal{Q}^K(M, V)$ is the equivariant index of the Spin^c Dirac operator on M twisted by V.

Let Ψ_M be the equivariant vector field on M defined by $\Psi_M(m) := \Psi(m)_M|_m$.

[13]Here $\mathrm{i} : \mathrm{U}_m \hookrightarrow \mathrm{SO}_{2m}$ is the canonical inclusion map.

Definition 4.7. *The symbol* $\text{S-Thom}_K(M)$ *pushed by the map* Ψ*, which is denoted* $\text{S-Thom}_K^\Psi(M)$*, is defined by the relation*

$$\text{S-Thom}_K^\Psi(M)(m,v) := \text{S-Thom}_K(M)(m, v - \Psi_M(m))$$

for any $(m,v) \in \mathbf{T}M$*. The symbol* $\text{S-Thom}_K^\Psi(M)$ *is transversally elliptic if and only if* $\{m \in M,\ \Psi_M(m) = 0\}$ *is compact. When this holds one defines the localized quantization map* $\mathcal{Q}_\Psi^K(M,V) := \text{Index}_M^K(\text{S-Thom}_K^\Psi(M) \otimes V)$.

We end this section with an adaptation of Lemma 9.4 in Appendix B of [33] to the localized quantization map $\mathcal{Q}_\Psi^K(M,-)$. Let $\beta \in \mathfrak{t}_+$ be a non-zero element in the center of the Lie algebra $\mathfrak{k}$ of K. We suppose here that the subtorus $i : \mathbb{T}_\beta \hookrightarrow K$, which is equal to the closure of $\{\exp(t.\beta),\ t \in \mathbb{R}\}$, acts trivially on M. Let $\mathrm{m}_\mu(V)$, $\mu \in \Lambda_+^*$ be the K-multiplicities[14] of $\mathcal{Q}_\Psi^K(M,V)$.

Lemma 4.8. *If* $\mathrm{m}_\mu(V) \neq 0$, $i^*(\mu)$ *is a weight for the action of* $\mathbb{T}_\beta$ *on* $V \otimes \mathbb{L}^{\frac{1}{2}}$*. If each weight* α *for the action of* $\mathbb{T}_\beta$ *on* $V \otimes \mathbb{L}^{\frac{1}{2}}$ *satisfies* $(\alpha,\beta) > 0$*, then* $[\mathcal{Q}_\Psi^K(M,V)]^K = 0$.

4.2. Spin^c **structures on symplectic reductions.** Let (M,ω,Φ,K) be a Hamiltonian manifold, such that Φ is proper. Let J be a K-invariant almost complex structure on M. On M, we have the orientation $o(J_M)$ defined by the almost complex structure and the orientation $o(\omega_M)$ defined by the symplectic form. We denote $\varepsilon = \pm 1$ their 'quotient'. On the symplectic reductions we will have also two orientations, one induces by ω, and the other induces by J, with the same 'quotient' ε.

There exists a unique relatively open face τ of the Weyl chamber $\mathfrak{t}_+^*$ such that $\Phi(M) \cap \tau$ is dense in $\Phi(M) \cap \mathfrak{t}_+^*$. The face τ is called the principal face of (M,Φ) [26]. All points in the open face τ have the same connected centralizer K_τ. The Principal-cross-section Theorem tells us that $\mathcal{Y}_\tau := \Phi^{-1}(\tau)$ is a symplectic K_τ-manifold, where $[K_\tau, K_\tau]$ acts trivially [26].

Here we work under the assumption that the infinitesimal stabilizers for the K-action on M are *abelian*: for $m \in M$, $\mathfrak{k}_m := \{X \in \mathfrak{k},\ X_M(m) = 0\}$ is an abelian subalgebra of $\mathfrak{k}$. Since $[\mathfrak{k}_\tau, \mathfrak{k}_\tau] \subset \mathfrak{k}_m$ for every $m \in \mathcal{Y}_\tau$, our assumption imposes that $[\mathfrak{k}_\tau, \mathfrak{k}_\tau] = 0$, hence τ is the interior of the Weyl chamber. For the remaining of this section, we denote $\mathcal{Y} = \Phi^{-1}(\text{interior}(\mathfrak{t}_+^*))$ the symplectic slice relative to interior of the Weyl chamber. It is a symplectic manifold with an induced Hamiltonian action of the maximal torus T, with moment map the restriction of Φ to $\mathcal{Y}$: $\Phi : \mathcal{Y} \to \mathfrak{t}^*$. The convexity Theorem tell us that $\Delta := \Phi(M) \cap \mathfrak{t}_+^*$ is a convex polyhedral subset of $\mathfrak{t}_+^*$ (called the moment polytope when M is compact). The relative interior Δ^o of Δ is a dense subset of $\Phi(\mathcal{Y})$.

Proposition 4.9. *The almost complex structure* J *induces :*

i) an orientation $o(\mathcal{Y})$ *on* $\mathcal{Y}$*, and*

ii) a T*-equivariant* Spin^c *structure on* $(\mathcal{Y}, o(\mathcal{Y}))$ *with determinant line bundle* $\det_{\mathbb{C}}(\mathbf{T}M|_{\mathcal{Y}})\,\mathbb{C}_{-2\rho_c}$.

Proof of i) : On $\mathcal{Y}$, we have the decomposition $\mathbf{T}M|_{\mathcal{Y}} = \mathbf{T}\mathcal{Y} \oplus [\mathfrak{k}/\mathfrak{t}]$, where $[\mathfrak{k}/\mathfrak{t}]$ denotes the trivial bundle $\mathcal{Y} \times \mathfrak{k}/\mathfrak{t}$ corresponding of the subspace of $\mathbf{T}M|_{\mathcal{Y}}$ formed by the vector fields generated by the infinitesimal action of $\mathfrak{k}/\mathfrak{t}$. The choice

[14] $\mathcal{Q}_\Psi^K(M,V) = \sum_{\mu \in \Lambda_+^*} \mathrm{m}_\mu(V) \chi_\mu^K$

of the Weyl chamber induces a complex structure on $\mathfrak{k}/\mathfrak{t}$, and hence an orientation $o([\mathfrak{k}/\mathfrak{t}])$: this orientation can be also defined by a symplectic form of the type $\omega_{\mathfrak{k}/\mathfrak{t}}(X,Y) = \langle \xi, [X,Y]\rangle$, where ξ belongs to the interior of the Weyl chamber $\mathfrak{t}^*_+$. Let $o(\mathcal{Y})$ be the orientation on $\mathcal{Y}$ defined by $o(J_M)|_{\mathcal{Y}} = o(\mathcal{Y})o([\mathfrak{k}/\mathfrak{t}])$. On $\mathcal{Y}$, we have also the orientation $o(\omega_{\mathcal{Y}})$ defined by the symplectic form $\omega_{\mathcal{Y}}$. Note that if $o(J_M) = \varepsilon\, o(\omega_M)$, we have also $o(\mathcal{Y}) = \varepsilon\, o(\omega_{\mathcal{Y}})$.

Let $\mathrm{P} := \mathrm{Spin}^c_{2n} \times_{\mathrm{U}_m} \mathrm{P_U}(\mathbf{T}M)$ be the Spin^c structure on M induced by J (see eq. 4.33). When we restrict to $\mathcal{Y}$, $\mathrm{P}|_{\mathcal{Y}}$ is then a Spin^c structure on the bundle $\mathbf{T}\mathcal{Y} \oplus [\mathfrak{k}/\mathfrak{t}]$. Let q be a T-invariant Riemanian structure on $\mathbf{T}\mathcal{Y} \oplus [\mathfrak{k}/\mathfrak{t}]$ such that $\mathbf{T}\mathcal{Y}$ is orthogonal with $[\mathfrak{k}/\mathfrak{t}]$, and q equals the Killing form on $[\mathfrak{k}/\mathfrak{t}]$. Following Remark 4.4, $\mathrm{P}|_{\mathcal{Y}}$ induces a Spin^c structure P' on $(\mathbf{T}\mathcal{Y} \oplus [\mathfrak{k}/\mathfrak{t}], q)$, with the same determinant line bundle $\mathbb{L} = \det_{\mathbb{C}}(\mathbf{T}M|_{\mathcal{Y}})$. Since the $\mathrm{SO}_{2k} \times \mathrm{U}_l$-principal bundle $\mathrm{P_{SO}}(\mathbf{T}\mathcal{Y}) \times \mathrm{U}(\mathfrak{k}/\mathfrak{t})$ is a reduction[15] of the SO_{2n} principal bundle $\mathrm{P_{SO}}(\mathbf{T}\mathcal{Y} \oplus [\mathfrak{k}/\mathfrak{t}])$, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Q} & \longrightarrow & \mathrm{P_{SO}}(\mathbf{T}\mathcal{Y}) \times \mathrm{U}(\mathfrak{k}/\mathfrak{t}) \times \mathrm{P_U}(\mathbb{L}) \\ \downarrow & & \downarrow \\ \mathrm{P}' & \longrightarrow & \mathrm{P_{SO}}(\mathbf{T}\mathcal{Y} \oplus [\mathfrak{k}/\mathfrak{t}]) \times \mathrm{P_U}(\mathbb{L}) \ , \end{array} \tag{4.34}$$

where Q is a $(\eta^c)^{-1}(\mathrm{SO}_{2k} \times \mathrm{U}_l) \simeq \mathrm{Spin}^c_{2k} \times \mathrm{U}_l$-principal bundle. Finally we see that $\mathrm{Q}' = \mathrm{Q}/\mathrm{U}_l$ is a Spin^c structure on $\mathbf{T}\mathcal{Y}$. Since $(\mathrm{U}(\mathfrak{k}/\mathfrak{t}) \times \mathrm{P_U}(\mathbb{L}))/\mathrm{U}_l \simeq \mathrm{P_U}(\mathbb{L} \otimes \mathbb{C}_{-2\rho_c})$, the corresponding determinant line bundle is $\mathbb{L}' = \mathbb{L} \otimes \mathbb{C}_{-2\rho_c}$. □

Let $\mathrm{Aff}(\Delta)$ be the affine subspace generated by Δ, and let $\overrightarrow{\Delta}$ be the subspace of $\mathfrak{t}^*$ generated by $\{m - n \mid m, n \in \Delta\}$. Let T_Δ the subtorus of T with Lie algebra $\mathfrak{t}_\Delta$ equal to the orthogonal (for the duality) of $\overrightarrow{\Delta}$. It is not difficult to see that T_Δ corresponds to the connected component of the principal stabilizer for the T-action on $\mathcal{Y}$.

Here we consider the symplectic reduction $M_\xi := \Phi^{-1}(\xi)/T$, for *generic quasi-regular values* ξ; that is, for $\xi \in \Delta^o$ such that T/T_Δ acts locally freely on $\Phi^{-1}(\xi)$: in this case $\Phi^{-1}(\xi)$ is a smooth submanifold in $\mathcal{Y}$ with a tubular neighborhhood equivariantly diffeomorphic to $\Phi^{-1}(\xi) \times \overrightarrow{\Delta}$.

Proposition 4.10. *Let $\mu \in \Lambda^*_+$ such that $\tilde{\mu} = \mu + \rho_c$ belongs to Δ. Let $\tilde{L}$ be a κ-prequantum line bundle. For every generic quasi-regular value $\xi \in \Delta$, the* Spin^c *structures on $\mathcal{Y}$, when twisted by $\tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu}$, induces a* Spin^c *structure on the reduced space $M_\xi := \Phi^{-1}(\xi)/T$ with determinant line bundle $(L_{2\omega}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-2\tilde{\mu}})/T$. Here we have two choices for the orientations : $o(M_\xi)$ induced by $o(\mathcal{Y})$, and $o(\omega_{M_\xi})$ defined by the induced symplectic form. They are tied by the relation $o(M_\xi) = \varepsilon\, o(\omega_{M_\xi})$.*

Proof : Let $\xi \in \Delta$ be a generic quasi-regular value of Φ, and $\mathcal{Z} := \Phi^{-1}(\xi)$. This is a submanifold of $\mathcal{Y}$ with a trivial action of T_Δ and a locally free action of T/T_Δ. We denote $\pi : \mathcal{Z} \to M_\xi$ the quotient map. We identify $\pi^*(\mathbf{T}M_\xi)$ with the orthogonal complement (relatively to a Riemannian metric) to the trivial bundle[16]

[15] Here $2n = \dim M$, $2k = \dim \mathcal{Y}$ and $2l = \dim(\mathfrak{k}/\mathfrak{t})$, so $n = k + l$.

[16] $[\mathfrak{t}/\mathfrak{t}_\Delta]$ corresponds to the subspace of $\mathbf{T}\mathcal{Z}$ formed by the vector fields generated by the infinitesimal action of $\mathfrak{t}/\mathfrak{t}_\Delta$.

$[\mathfrak{t}/\mathfrak{t}_\Delta]$. On the other hand the tangent bundle $\mathbf{T}\mathcal{Y}$, when restricted to $\mathcal{Z}$, decomposes as $\mathbf{T}\mathcal{Y}|_{\mathcal{Z}} = \mathbf{T}\mathcal{Z} \oplus [\overrightarrow{\Delta}]$, so we have

$$\begin{aligned} \mathbf{T}\mathcal{Y}|_{\mathcal{Z}} &= \pi^*(\mathbf{T}M_\xi) \oplus [\mathfrak{t}/\mathfrak{t}_\Delta] \oplus [\overrightarrow{\Delta}] \\ &= \pi^*(\mathbf{T}M_\xi) \oplus [\mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{C}] \ , \end{aligned} \tag{4.35}$$

with the convention $\mathfrak{t}/\mathfrak{t}_\Delta = \mathfrak{t}/\mathfrak{t}_\Delta \otimes i\mathbb{R}$ and $\overrightarrow{\Delta} = \mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{R}$. Since $\mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{C}$ is canonically oriented by the complex multiplication by i, the orientation $o(\mathcal{Y})$ determines an orientation $o(M_\xi)$ on $\mathbf{T}M_\xi$ through (4.35).

Now we proceed like in Proposition 4.10. Let Q' be the Spin^c structure on $\mathcal{Y}$ introduced in Proposition 4.10, and let Q^μ be Q' twisted by the line bundle $\tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu}$: its determinant line bundle is $\det_{\mathbb{C}}(\mathbf{T}M)|_{\mathcal{Y}} \otimes \mathbb{C}_{-2\rho_c} \otimes (\tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu})^2 = L_{2\omega}|_{\mathcal{Y}} \otimes \mathbb{C}_{-2\tilde{\mu}}$. The $\mathrm{SO}_{2k'} \times \mathrm{U}_{l'}$-principal bundle $\mathrm{P}_{\mathrm{SO}}(\pi^*(\mathbf{T}M_\xi)) \times \mathrm{U}(\mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{C})$ is a reduction[17] of the SO_{2k} principal bundle $\mathrm{P}_{\mathrm{SO}}(\pi^*(\mathbf{T}M_\xi) \oplus [\mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{C}])$; we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Q}'' & \longrightarrow & \mathrm{P}_{\mathrm{SO}}(\pi^*(\mathbf{T}M_\xi)) \times \mathrm{U}(\mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{C}) \times \mathrm{P}_{\mathrm{U}}(\mathbb{L}|_{\mathcal{Z}}) \\ \downarrow & & \downarrow \\ \mathrm{Q}^\mu|_{\mathcal{Z}} & \longrightarrow & \mathrm{P}_{\mathrm{SO}}(\pi^*(\mathbf{T}M_\xi) \oplus [\mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{C}]) \times \mathrm{P}_{\mathrm{U}}(\mathbb{L}|_{\mathcal{Z}}) \ , \end{array} \tag{4.36}$$

where $\mathbb{L} = L_{2\omega}|_{\mathcal{Y}} \otimes \mathbb{C}_{-2\tilde{\mu}}$. Here Q'' is a $(\eta^c)^{-1}(\mathrm{SO}_{2k'} \times \mathrm{U}_{l'}) \simeq \mathrm{Spin}^c_{2k'} \times \mathrm{U}_{l'}$-principal bundle. The Kostant formula (2.1) tells us that the action of T_Δ is trivial on $\mathbb{L}|_{\mathcal{Z}}$, since $\xi - \tilde{\mu} \in \overrightarrow{\Delta}$. Thus the action of T_Δ is trivial on Q''. Finally we see that $\mathrm{Q}_\xi = \mathrm{Q}''/(\mathrm{U}_{l'} \times T)$ is a Spin^c structure on M_ξ with determinant line bundle $\mathbb{L}_\xi = (L_{2\omega}|_{\mathcal{Z}} \otimes \mathbb{C}_{-2\tilde{\mu}})/T$. □

4.3. **Definition of $\mathcal{Q}(M_{\mu+\rho_c})$.** First we give three different ways to define the quantity $\mathcal{Q}(M_{\mu+\rho_c}) \in \mathbb{Z}$ for any $\mu \in \Lambda^*_+$. The compatibility of these different definitions gives a proof of Theorem 4.1. First of all $\mathcal{Q}(M_{\mu+\rho_c}) = 0$ if $\mu + \rho_c \notin \Delta$.

First definition.

If $\mu + \rho_c$ is a *generic quasi regular* value of Φ, $M_{\mu+\rho_c} := \Phi^{-1}(\mu+\rho_c)/T$ is a symplectic orbifold. We know from Proposition 4.10 that $M_{\mu+\rho_c}$ inherits Spin^c-structures, with the same determinant line bundle $(L_{2\omega}|_{\Phi^{-1}(\tilde{\mu})} \otimes \mathbb{C}_{-2\tilde{\mu}})/T$, for the two choices of orientation $o(M_\xi)$ and $o(\omega_{M_\xi})$. We denote $\mathcal{Q}(M_{\mu+\rho_c}) \in \mathbb{Z}$ the index of the Spin^c Dirac operator associated to the Spin^c structure on $(M_\xi, o(\omega_{M_\xi}))$ and $\mathcal{Q}(M_{\mu+\rho_c}, o(M_{\mu+\rho_c}))$ the index of the Spin^c Dirac operator associated to the Spin^c structure on $(M_{\mu+\rho_c}, o(M_{\mu+\rho_c}))$. Since $o(M_\xi) = \varepsilon\, o(\omega_{M_\xi})$, we have $\mathcal{Q}(M_{\mu+\rho_c}, o(M_{\mu+\rho_c})) = \varepsilon\, \mathcal{Q}(M_{\mu+\rho_c})$.

Second definition.

We can also define $Q(M_{\mu+\rho_c})$ by shift 'desingularization' as follows. If $\mu+\rho_c \in \Delta$, one considers generic quasi regular values ξ of Φ, close enough to $\mu+\rho_c$. Following Proposition 4.10, $M_\xi = \Phi_{\mathcal{Y}}^{-1}(\xi)/T$ inherits a Spin^c structure, with determinant line bundle $(L_{2\omega}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-2\tilde{\mu}})/T$. Then we set $\mathcal{Q}(M_{\mu+\rho_c}) := \mathcal{Q}(M_\xi)$, where the RHS is the index of the Spin^c Dirac operator associated to the Spin^c structure on $(M_\xi, o(\omega_{M_\xi}))$. In the same way $\mathcal{Q}(M_{\mu+\rho_c}, o(M_{\mu+\rho_c})) = \mathcal{Q}(M_\xi, o(M_\xi))$. Here one has to show that these quantities are independent of the choice of ξ when ξ

[17]Here $2k = \dim \mathcal{Y}$, $2k' = \dim M_\xi$ and $l' = \dim(\mathfrak{t}/\mathfrak{t}_\Delta)$, so $k = k' + l'$.

is a generic quasi regular value close enough to $\mu + \rho_c$. We will see in fact that $\mathcal{Q}(M_{\mu+\rho_c}) = 0$ when $\mu + \rho_c \in \partial\Delta$ (see 4.38).

Third definition.
Since M is supposed to satisfy Assumption 3.4 at $\tilde{\mu}$, we can use the characterization of the multiplicity $\mathrm{m}_\mu(\tilde{L})$ given in Lemma 4.2. The number $\mathcal{Q}(M_{\mu+\rho_c})$ is the multiplicity of the trivial representation in $\varepsilon\, RR_0^K(M \times \overline{\mathcal{O}^{\tilde{\mu}}}, \tilde{L} \otimes \mathbb{C}_{[-\mu]})$.

We work now with a fixed element $\mu \in \Lambda^*_+$, and during the remaining part of this section, $\mathcal{Y}$ will denotes a small T-invariant open neighborhood of $\Phi^{-1}(\mu + \rho_c)$ in the symplectic slice $\Phi^{-1}(\mathrm{interior}(\mathfrak{t}^*_+))$.

We have to show the compatibility of our definitions, that is

$$\left[RR_0^K(M \times \overline{\mathcal{O}^{\tilde{\mu}}}, \tilde{L} \otimes \mathbb{C}_{[-\mu]})\right]^K = \mathcal{Q}(M_\xi, o(M_\xi)) \tag{4.37}$$

for any generic quasi regular value $\xi \in \mathrm{Aff}(\Delta)$ close enough to $\mu + \rho_c$.

The manifold $(\mathcal{Y}, o(\mathcal{Y}))$ carries a T-invariant Spin^c-structure, so we can consider the localized quantization maps $\mathcal{Q}^T_{\Phi-\tilde{\mu}}(\mathcal{Y}, -)$ and $\mathcal{Q}^T_{\Phi-\xi}(\mathcal{Y}, -)$, since the functions $\| \Phi - \tilde{\mu} \|^2$ and $\| \Phi - \xi \|^2$ have compact critical set on $\mathcal{Y}$ when $\xi \in \mathrm{Aff}(\Delta)$ is close enough to $\tilde{\mu}$. The proof of (4.37) is divided in two steps.

Proposition 4.11. *Let E and F be respectively K-equivariant complex vector bundle over M and $\mathcal{O}^{\tilde{\mu}}$. We have the following equality*

$$RR_0^K(M \times \overline{\mathcal{O}^{\tilde{\mu}}}, E \boxtimes F) = \mathrm{Ind}_T^K\left(\mathcal{Q}^T_{\Phi-\tilde{\mu}}(\mathcal{Y}, E|_{\mathcal{Y}} \otimes F|_{\bar{e}})\right)$$

in $R^{-\infty}(K)$. It gives in particular that

$$\left[RR_0^K(M \times \overline{\mathcal{O}^{\tilde{\mu}}}, E \boxtimes F)\right]^K = \left[\mathcal{Q}^T_{\Phi-\tilde{\mu}}(\mathcal{Y}, E|_{\mathcal{Y}} \otimes F|_{\bar{e}})\right]^T .$$

Proposition 4.12. *If $\xi \in \mathrm{Aff}(\Delta)$ is close enough to $\tilde{\mu}$:*
i) the quantization maps $\mathcal{Q}^T_{\Phi-\tilde{\mu}}(\mathcal{Y}, -)$ and $\mathcal{Q}^T_{\Phi-\xi}(\mathcal{Y}, -)$ are equal,
ii) if ξ is a generic quasi-regular value of Φ we have

$$\left[\mathcal{Q}^T_{\Phi-\tilde{\xi}}(\mathcal{Y}, \tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu})\right]^T = \mathcal{Q}(M_\xi, o(M_\xi)) ,$$

and if $\xi \notin \Delta$, $[\mathcal{Q}^T_{\Phi-\tilde{\xi}}(\mathcal{Y}, \tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu})]^T = 0$.

Propositions 4.11 and 4.12 will be proved in subsections 4.4 and 4.5.

If $\tilde{\mu} \in \partial\Delta$, we can take $\xi \in \mathrm{Aff}(\Delta)$ close enough to $\tilde{\mu}$, with $\xi \notin \Delta$. Propositions 4.11 and 4.11 give

$$\begin{aligned} \left[RR_0^K(M \times \overline{\mathcal{O}^{\tilde{\mu}}}, \tilde{L} \otimes \mathbb{C}_{[-\mu]})\right]^K &= \left[\mathcal{Q}^T_{\Phi-\tilde{\mu}}(\mathcal{Y}, \tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu})\right]^T \\ &= \left[\mathcal{Q}^T_{\Phi-\xi}(\mathcal{Y}, \tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu})\right]^T \\ &= 0 . \end{aligned} \tag{4.38}$$

Hence $\mathcal{Q}(M_{\mu+\rho_c}) = 0$ for every $\mu + \rho_c \in \partial\Delta$.

4.4. **Proof of Proposition 4.11.** The induction formula of Proposition 4.11 is essentially identical to the one we proved in [33]. The main difference is that the almost complex structure is not assumed to be compatible with the symplectic structure.

We identify the coadjoint orbit $\mathcal{O}^{\tilde\mu}$ with $\overline{K/T}$. Let $\mathcal{H}^{\tilde\mu}$ be the Hamiltonian vector field of the function $\frac{1}{2}\parallel \Phi_{\tilde\mu} \parallel^2$: $M \times \overline{K/T} \to \mathbb{R}$. Here $\mathcal{Y}$ denotes a small neighborhood of $\Phi^{-1}(\tilde\mu)$ in the symplectic slice $\Phi^{-1}(\mathrm{interior}(\mathfrak{t}^*_+))$ such that the open subset $\mathcal{U} := (K \times_T \mathcal{Y}) \times \overline{K/T}$ is a neighborhood of $\Phi_{\tilde\mu}^{-1}(0) = K \cdot (\Phi^{-1}(\tilde\mu) \times \{\bar e\})$ which verifies $\overline{\mathcal{U}} \cap \{\mathcal{H}^{\tilde\mu} = 0\} = \Phi_{\tilde\mu}^{-1}(0)$.

By Definition 3.7, the localized Riemann-Roch character $RR_0^{^K}(M \times \overline{K/T}, -)$ is computed by means of the Thom class $\mathrm{Thom}_K^{\Phi_{\tilde\mu}}(\mathcal{U}) \in \mathbf{K}_K(\mathbf{T}_K\mathcal{U})$. On the other hand, the localized quantization map $\mathcal{Q}^{^T}_{\Phi-\tilde\mu}(\mathcal{Y}, -)$ is computed by means of the class $\text{S-Thom}_T^{\Phi-\tilde\mu}(\mathcal{Y}) \in \mathbf{K}_T(\mathbf{T}_T\mathcal{Y})$ (see Definition 4.7). Proposition 4.11 will follow from a simple relation between these two transversally elliptic symbols.

First, one considers the isomorphism

$$\begin{array}{rcl}\phi : \mathcal{U} & \to & \mathcal{U}' \\ ([k;y],[h]) & \to & [k;[k^{-1}h],y]\ ,\end{array} \tag{4.39}$$

with $\mathcal{U}' := K \times_T (\overline{K/T} \times \mathcal{Y})$, and $\phi^* : \mathbf{K}_K(\mathbf{T}_K\mathcal{U}') \to \mathbf{K}_K(\mathbf{T}_K\mathcal{U})$ being the induced isomorphism. Then one consider the inclusion $i : T \hookrightarrow K$ which induces an isomorphism $i_* : \mathbf{K}_T(\mathbf{T}_T(\overline{K/T} \times \mathcal{Y})) \to \mathbf{K}_K(\mathbf{T}_K\mathcal{U}')$ (see [1, 33]). Let $j : \mathcal{Y} \hookrightarrow \overline{K/T} \times \mathcal{Y}$ be the T-invariant inclusion map defined by $j(y) := (\bar e, y)$. We have then a pushforward map $j_! : \mathbf{K}_T(\mathbf{T}_T\mathcal{Y}) \to \mathbf{K}_T(\mathbf{T}_T(\overline{K/T} \times \mathcal{Y}))$. Finally we have produced a map $\Theta := \phi^* \circ i_* \circ j_!$ from $\mathbf{K}_T(\mathbf{T}_T\mathcal{Y})$ to $\mathbf{K}_K(\mathbf{T}_K\mathcal{U})$, such that $\mathrm{Index}^K_{\mathcal{U}}(\Theta(\sigma)) = \mathrm{Ind}_{_T}^{^K}(\mathrm{Index}^T_{\mathcal{Y}}(\sigma))$ for every $\sigma \in \mathbf{K}_T(\mathbf{T}_T\mathcal{Y})$.

Proposition 4.11 is an immediate consequence of the following

Lemma 4.13. *We have the equality*

$$\Theta\left(\text{S-Thom}_T^{\Phi-\tilde\mu}(\mathcal{Y})\right) = \mathrm{Thom}_K^{\Phi_{\tilde\mu}}(\mathcal{U})\ .$$

Proof : Let $\mathcal{S}$ be the bundle of spinors on $K \times_T \mathcal{Y}$: $\mathcal{S} = \mathrm{P} \times_{\mathrm{Spin}^c_{2k}} \Delta_{2k}$, where $\mathrm{P} \to \mathrm{P}_{\mathrm{SO}}(\mathbf{T}(K \times_T \mathcal{Y})) \times \mathrm{P}_{\mathrm{U}}(\mathbb{L})$ is the Spin^c structure induced by the complex structure. By Proposition 4.9, we have the reductions

$$\begin{array}{ccc} \mathrm{Q} & \longrightarrow & \mathrm{P}_{\mathrm{SO}}(\mathbf{T}\mathcal{Y}) \times \mathrm{U}(\mathfrak{k}/\mathfrak{t}) \times \mathrm{P}_{\mathrm{U}}(\mathbb{L}|_{\mathcal{Y}}) \\ \downarrow & & \downarrow \\ \mathrm{P}|_{\mathcal{Y}} & \longrightarrow & \mathrm{P}_{\mathrm{SO}}(\mathbf{T}\mathcal{Y} \oplus [\mathfrak{k}/\mathfrak{t}]) \times \mathrm{P}_{\mathrm{U}}(\mathbb{L}|_{\mathcal{Y}}) \\ \downarrow & & \downarrow \\ \mathrm{P} & \longrightarrow & \mathrm{P}_{\mathrm{SO}}(\mathbf{T}(K \times_T \mathcal{Y})) \times \mathrm{P}_{\mathrm{U}}(\mathbb{L})\ . \end{array} \tag{4.40}$$

Here Q/U_l is the induced Spin^c-structure on $\mathcal{Y}$. Let us denote $p : \mathbf{T}(K \times_T \mathcal{Y}) \to K \times_T \mathcal{Y}$, $p_{\mathcal{Y}} : \mathbf{T}\mathcal{Y} \to \mathcal{Y}$ and $p_{K/T} : \mathbf{T}(K/T) \to K/T$ the canonical projections. Using (4.40), we see that

$$p^*\mathcal{S} = \left(K \times_T p_{\mathcal{Y}}^*\mathcal{S}(\mathcal{Y})\right) \otimes\ p^*_{K/T} \wedge_{\mathbb{C}} \mathbf{T}(K/T)\ ,$$

where $\mathcal{S}(\mathcal{Y})$ is the spinor bundle on $\mathcal{Y}$. Hence we get the decomposition

$$\text{S-Thom}_K(K\times_T\mathcal{Y})=\text{Thom}_K(K/T)\odot K\times_T\text{S-Thom}_T(\mathcal{Y})\ .$$

Let $\Phi'_{\tilde\mu}:=\Phi_{\tilde\mu}\circ\phi^{-1}$ be the moment map on $\mathcal{U}'$, and let $\mathcal{H}'$ be the Hamiltonian vector field of $\|\ \Phi'_{\tilde\mu}\ \|$. The transversally elliptic symbol $\text{Thom}_K^{\Phi_{\tilde\mu}}(\mathcal{U})$ is equal to

$$\Big[\text{Thom}_K(K/T)\odot\text{Thom}_K(\overline{K/T})\odot K\times_T\text{S-Thom}_T(\mathcal{Y})\Big]_{\text{pushed by }\mathcal{H}^{\tilde\mu}}\ ,$$

hence $\sigma_1:=(\phi^{-1})^*\text{Thom}_K^{\Phi_{\tilde\mu}}(\mathcal{U})$ is equal to

$$\Big[\text{Thom}_K(K/T)\odot K\times_T\Big(\text{Thom}_T(\overline{K/T})\odot\text{S-Thom}_T(\mathcal{Y})\Big)\Big]_{\text{pushed by }\mathcal{H}'}\ .$$

Using the decomposition $\mathbf{T}\mathcal{U}'\simeq K\times_T(\mathfrak{k}/\mathfrak{t}\oplus K\times_T(\overline{\mathfrak{k}/\mathfrak{t}})\oplus\mathbf{T}\mathcal{Y})$, we have $\mathcal{H}'(m)=pr_{\mathfrak{k}/\mathfrak{t}}(h\tilde\mu)+R(m)+\mathcal{H}_{\tilde\mu}(y)+S(m)$ for $m=[k;[h],y]\in\mathcal{U}'$, where[18] $R(m)\in\overline{\mathfrak{k}/\mathfrak{t}}$ and $S(m)\in\mathbf{T}_y\mathcal{Y}$ vanishes when $m\in K\times_T(\{\bar e\}\times\mathcal{Y})$, i.e. when $[h]=\bar e$. Here $\mathcal{H}_{\tilde\mu}$ is the Hamiltonian vector field of the function $\frac{1}{2}\parallel\Phi-\tilde\mu\parallel^2\colon\mathcal{Y}\to\mathbb{R}$.

The transversally elliptic symbol σ_1 is equal to the exterior product

$$\sigma_1(m,\xi_1+\xi_2+v)=\mathbf{c}(\xi_1-pr_{\mathfrak{k}/\mathfrak{t}}(h\tilde\mu))\odot\mathbf{c}(\xi_2-R(m))\odot\mathbf{c}(v-\mathcal{H}_{\tilde\mu}-S(m))\ ,$$

with $\xi_1\in\mathfrak{k}/\mathfrak{t}$, $\xi_2\in\overline{\mathfrak{k}/\mathfrak{t}}$, and $v\in\mathbf{T}\mathcal{Y}$.

Now we simplify the symbol σ_1 whithout changing its K-theoretic class. Since $\text{Char}(\sigma_1)\cap\mathbf{T}_K\mathcal{U}'=K\times_T(\{\bar e\}\times\mathcal{Y})$, we transform σ_1 through the K-invariant diffeomorphism $h=e^X$ from a neighborhood of 0 in $\overline{\mathfrak{k}/\mathfrak{t}}$ to a neighborhood of $\bar e$ in $\overline{K/T}$. That gives $\sigma_2\in K_K(\mathbf{T}_K(K\times_T(\mathfrak{k}/\mathfrak{t}\times\mathcal{Y})))$ defined by

$$\begin{aligned}&\sigma_2([k;X,y],\xi_1+\xi_2+v)=\\&\qquad\mathbf{c}(\xi_1-pr_{\mathfrak{k}/\mathfrak{t}}(e^X\tilde\mu))\odot\mathbf{c}(\xi_2-R(m))\odot\mathbf{c}(v-\mathcal{H}_{\tilde\mu}-S(m))\ .\end{aligned}$$

Now trivial homotopies link σ_2 with the symbol σ_3, where we have removed the terms $R(m)$ and $S(m)$, and where we have replaced $pr_{\mathfrak{k}/\mathfrak{t}}(e^X\tilde\mu)=[X,\tilde\mu]+\mathrm{o}([X,\tilde\mu])$ by the term $[X,\tilde\mu]$:

$$\sigma_3([k,X,y],\xi_1+\xi_2+v)=\mathbf{c}(\xi_1-[X,\tilde\mu])\odot\mathbf{c}(\xi_2)\odot\mathbf{c}(v-\mathcal{H}_{\tilde\mu})\ .$$

Now, we get $\sigma_3=i_*(\sigma_4)$ where the symbol $\sigma_4\in K_T(\mathbf{T}_T(\mathfrak{k}/\mathfrak{t}\times\mathcal{Y}))$ is defined by

$$\sigma_4(X,y;\xi_2+v)=\mathbf{c}(-[X,\tilde\mu])\odot\mathbf{c}(\xi_2)\odot\mathbf{c}(v-\mathcal{H}_{\tilde\mu})\ .$$

So σ_4 is equal to the exterior product of $(y,v)\to\mathbf{c}(v-\mathcal{H}_a^\tau)$, which is $\text{S-Thom}_T^{\Phi-\tilde\mu}(\mathcal{Y})$, with the transversally elliptic symbol on $\mathfrak{k}/\mathfrak{t}$: $(X,\xi_2)\to\mathbf{c}(-[X,\tilde\mu])\odot\mathbf{c}(\xi_2)$. But the K-theoretic class of this former symbol is equal to $k_!(\mathbb{C})$, where $k:\{0\}\hookrightarrow\mathfrak{k}/\mathfrak{t}$ (see subsection 5.1 in [33]). This shows that

$$\sigma_4=k_!(\mathbb{C})\odot\text{S-Thom}_T^{\Phi-\tilde\mu}(\mathcal{Y})=j_!(\text{S-Thom}_T^{\Phi-\tilde\mu}(\mathcal{Y}))\ .$$

□

[18]A small computation shows that $R(m)=pr_{\mathfrak{k}/\mathfrak{t}}(h^{-1}(pr_{\mathfrak{t}}(h\tilde\mu)-\Phi(y)))$, and $S(m)=[\tilde\mu-pr_{\mathfrak{t}}(h\tilde\mu)]_{\mathcal{Y}}(y)$.

4.5. **Proof of Proposition 4.12.** In this subsection, $\tilde{\mu} = \mu + \rho_c$ is fixed once for all, $r > 0$ is the smallest non zero critical value of $\| \Phi - \tilde{\mu} \|^2$, and we take $\mathcal{Y} = \Phi^{-1}\{\xi \in \mathrm{Aff}\Delta, \| \xi - \tilde{\mu} \| < \frac{r}{2}\}$.

For $\xi \in \mathrm{Aff}\Delta$ close enough to $\tilde{\mu}$, we consider $\xi_t = t\xi + (1-t)\tilde{\mu}$, $0 \leq t \leq 1$. If one shows that there exists a compact subset $\mathcal{K}$ of $\mathcal{Y}$ such that $\mathrm{Cr}(\| \Phi - \xi_t \|^2) \cap \mathcal{Y} \subset \mathcal{K}$, the familly $\text{S-Thom}_T^{\Phi-\xi_t}(\mathcal{Y})$, $0 \leq t \leq 1$, is an homotopy of transversally elliptic symbols between $\text{S-Thom}_T^{\Phi-\tilde{\mu}}(\mathcal{Y})$ and $\text{S-Thom}_T^{\Phi-\xi}(\mathcal{Y})$. This shows then that $\mathcal{Q}^T_{\Phi-\tilde{\mu}}(\mathcal{Y}, -)$ and $\mathcal{Q}^T_{\Phi-\xi}(\mathcal{Y}, -)$ are equal.

We describe now $\mathrm{Cr}(\| \Phi - \xi_t \|^2) \cap \mathcal{Y}$ using a parametrization introduced in [30][Section 6]. Let $\mathcal{B}_{\tilde{\mu}}$ be the collection of affine subspaces of $\mathfrak{t}^*$ generated by the image under Φ of submanifolds $\mathcal{Z}$ of the following type: $\mathcal{Z}$ is a connected component of $\mathcal{Y}^H$ which intersects $\Phi^{-1}(\tilde{\mu})$, H being a subgroup of T. The set $\mathcal{B}_{\tilde{\mu}}$ is finite since $\Phi^{-1}(\tilde{\mu})$ is compact and thus has a finite number of stabilizers for the T action. Note that $\mathcal{B}_{\tilde{\mu}}$ is reduced to $\mathrm{Aff}\Delta$ if $\tilde{\mu}$ is a generic quasi regular value of Φ. For $A \in \mathcal{B}_{\tilde{\mu}}$, we denote T_A the subtorus of T with Lie algebra equal to the orthogonal (for the duality) of A, and $\beta(-, A)$ denotes the orthogonal projection on A.

Like in [31], we see that

$$\mathrm{Cr}(\| \Phi - \xi \|^2) \cap \mathcal{Y} = \cup_{A \in \mathcal{B}_{\tilde{\mu}}} \left(\mathcal{Y}^{T_A} \cap \Phi^{-1}(\beta(\xi, A))\right) \tag{4.41}$$

if $\| \xi - \tilde{\mu} \| < \frac{r}{2}$. If we take $\mathcal{K} := \{\xi \in \mathrm{Aff}\Delta, \| \xi - \tilde{\mu} \| \leq \frac{r}{3}\}$, we have $\mathrm{Cr}(\| \Phi - \xi \|^2) \cap \mathcal{Y} \subset \mathcal{K}$ for $\| \xi - \tilde{\mu} \| \leq \frac{r}{3}$. Thus point $i)$ is proved.

Now we fix $\xi \in \mathrm{Aff}\Delta$ close enough to $\tilde{\mu}$, and we parametrize $\mathrm{Cr}(\| \Phi - \xi \|^2) \cap \mathcal{Y}$ by $\mathcal{B} = \{\beta(\xi, A) - \xi \mid A \in \mathcal{B}_{\tilde{\mu}}\}$ so that $\mathrm{Cr}(\| \Phi - \xi \|^2) \cap \mathcal{Y} = \cup_{\beta \in \mathcal{B}} \left(\mathcal{Y}^\beta \cap \Phi^{-1}(\beta + \xi)\right)$. If $\mathcal{Q}^T_\beta(\mathcal{Y}, -)$ denotes the quantization map localized near $\mathcal{Y}^\beta \cap \Phi^{-1}(\beta + \xi)$, the excision property tells us, like in 3.16, that

$$\mathcal{Q}^T_{\Phi-\xi}(\mathcal{Y}, -) = \sum_{\beta \in \mathcal{B}} \mathcal{Q}^T_\beta(\mathcal{Y}, -)\ .$$

Note that $0 \in \mathcal{B}$ if and only if $\Phi^{-1}(\xi) \neq \emptyset$. Point ii) of Proposition 4.12 will follow from the following results.

Lemma 4.14. *If ξ is a generic quasi regular value of Φ, we have* $[\mathcal{Q}^T_0(\mathcal{Y}, \tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu})]^T = \mathcal{Q}(M_\xi, o(M_\xi))$.

Lemma 4.15. *Let β be a non-zero element of $\mathcal{B}$. Then* $[\mathcal{Q}^T_\beta(\mathcal{Y}, \tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu})]^T = 0$. *Hence* $[\mathcal{Q}^T_{\Phi-\xi}(\mathcal{Y}, \tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu})]^T = 0$, *if* $\Phi^{-1}(\xi) = \emptyset$.

Other versions of Lemmas 4.14 and 4.15 are already known : in the Spin-case for an S^1-action by Vergne [38], and by the author [33] when the Spin^c-structure comes from an almost complex structure.

We review briefly the arguments, as they work in the same way. We consider the Spin^c structure on $\mathcal{Y}$ defined in Proposition 4.9, that we twist by the line bundle $\tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu}$: it defines a Spin^c structure Q^μ on $\mathcal{Y}$ with determinant line bundle $\mathbb{L}^\mu := L_{2\omega} \otimes \mathbb{C}_{-2\tilde{\mu}}$. We consider then the symbol $\text{S-Thom}_{T,\mu}^{\Phi-\xi}(\mathcal{Y})$ constructed with Q^μ (see Def. 4.7). For $\beta \in \mathcal{B}$, the term $\mathcal{Q}^T_\beta(\mathcal{Y}, \tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu})$ is by definition the T-index of $\text{S-Thom}_{T,\mu}^{\Phi-\xi}(\mathcal{Y})|_{\mathcal{U}^\beta}$, where $\mathcal{U}^\beta$ is a sufficiently small open neighborhood of $\mathcal{Y}^\beta \cap \Phi^{-1}(\beta + \xi)$ in $\mathcal{Y}$.

Proof of Lemma 4.14: A neighborhood $\mathcal{U}^0$ of $\mathcal{Z} := \Phi^{-1}(\xi)$ is diffeomorphic to a neighborhood of $\mathcal{Z}$ in $\mathcal{Z} \times \overrightarrow{\Delta}$, where $\Phi - \xi : \mathcal{Z} \times \overrightarrow{\Delta} \to \overrightarrow{\Delta}$ is the projection to the second factor. We still denote $\mathcal{Q}^\mu$ the Spin^c-structure on $\mathcal{Z} \times \overrightarrow{\Delta}$ equal[19] to $pr^*(\mathcal{Q}^\mu|_{\mathcal{Z}})$. We easily show that $\mathcal{Q}^T_\beta(\mathcal{Y}, \tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu})$ is equal to the T-index of $\sigma_{\mathcal{Z}} = \text{S-Thom}^{\Phi-\xi}_{T,\mu}(\mathcal{Z} \times \overrightarrow{\Delta})$. Let Q'' be the reduction of $\mathcal{Q}^\mu|_{\mathcal{Z}}$ introduced in (4.36). Since $\mathcal{Q}^\mu|_{\mathcal{Z}} = \mathrm{Spin}^c_{2k} \times_{(\mathrm{Spin}^c_{2k'} \times \mathrm{U}_{2l'})} \mathrm{Q}''$, the bundle of spinors $\mathcal{S}$ over $\mathcal{Z} \times \overrightarrow{\Delta}$ decomposes as

$$\mathcal{S} = pr^*\Big(\pi^*\mathcal{S}(M_\xi) \otimes \mathcal{Z} \times \wedge(\mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{C})\Big) .$$

Here $\mathcal{S}(M_\xi)$ is the bundle of spinors on M_ξ induces by the Spin^c-structure $\mathrm{Q}''/\mathrm{U}_{2l'}$, and $\pi : \mathcal{Z} \to M_\xi$ is the quotient map. In the trivial bundle $\mathcal{Z} \times (\mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{C})$, we have identified $\mathcal{Z} \times (\mathfrak{t}/\mathfrak{t}_\Delta \otimes i\mathbb{R})$ with the subspace of $\mathbf{T}\mathcal{Z}$ formed by the vector fields generated by the infinitesimal action of $\mathfrak{t}/\mathfrak{t}_\Delta$, and $\mathcal{Z} \times (\mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{R})$ with $\mathcal{Z} \times \overrightarrow{\Delta} \subset \mathbf{T}(\mathcal{Z} \times \overrightarrow{\Delta})|_{\mathcal{Z}}$. For $(z, f) \in \mathcal{Z} \times \overrightarrow{\Delta}$, let us decompose $v \in \mathbf{T}_{(z,f)}(\mathcal{Z} \times \overrightarrow{\Delta})$ into $v = v_1 + X + \imath Y$, where $v_1 \in \pi^*(\mathbf{T}M_\xi)$, and $X + \imath Y \in \mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{C}$. The map $\sigma_{\mathcal{Z}}(z, f; v_1 + X + \imath Y)$ acts on $\mathcal{S}(M_\xi)_z \otimes \wedge(\mathfrak{t}/\mathfrak{t}_\Delta \otimes \mathbb{C})$ as the product

$$\mathbf{c}_z(v_1) \odot \mathbf{c}(X + i(Y - f)) ,$$

which is homotopic[20] to the transversally elliptic symbol

$$\mathbf{c}_z(v_1) \odot \mathbf{c}(f + iX)$$

So we have proved that $\sigma_{\mathcal{Z}} = j_! \circ \pi^*(\text{S-Thom}(M_\xi))$, where $j_! : K_T(\mathbf{T}_T\mathcal{Z}) \to K_T(\mathbf{T}_T(\mathcal{Z} \times \overrightarrow{\Delta}))$ is induced by the inclusion $j : \mathcal{Z} \hookrightarrow \mathcal{Z} \times \overrightarrow{\Delta}$. The last equality finishes the proof (see [33][Section 6.1]).

Proof of Lemma 4.15 : The equality $[\mathcal{Q}^T_\beta(\mathcal{Y}, \tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu})]^T = 0$ comes from a localization formula on the submanifold $\mathcal{Y}^\beta$ for the map $\mathcal{Q}^T_\beta(\mathcal{Y}, -)$ (see [33, 38]). The normal bundle $\mathcal{N}$ of $\mathcal{Y}^\beta$ in $\mathcal{Y}$ carries a complex structure J on the fibers such that each $\mathbb{T}_\beta$-weight α on $(\mathcal{N}, J)$ satisfies $(\alpha, \beta) > 0$. The principal bundle Q^μ, when restricted to $\mathcal{Y}^\beta$ admits the reduction

$$(4.42) \qquad \begin{array}{ccc} \mathrm{Q}' & \longrightarrow & \mathrm{P_{SO}}(\mathbf{T}\mathcal{Y}^\beta) \times \mathrm{P_U}(\mathcal{N}) \times \mathrm{P_U}(\mathbb{L}|_{\mathcal{Y}^\beta}) \\ \downarrow & & \downarrow \\ \mathrm{Q}^\mu|_{\mathcal{Y}^\beta} & \longrightarrow & \mathrm{P_{SO}}(\mathbf{T}\mathcal{Y}^\beta \oplus \mathcal{N}) \times \mathrm{P_U}(\mathbb{L}^\mu|_{\mathcal{Y}^\beta}) , \end{array}$$

Hence $\mathrm{Q}^\beta := \mathrm{Q}'/\mathrm{U}(l)$ is a Spin^c-structure on $\mathcal{Y}^\beta$ with determinant line bundle equal to $\mathbb{L}^\beta := \mathbb{L}^\mu|_{\mathcal{Y}^\beta} \otimes (\det \mathcal{N})^{-1} = L_{2\omega}|_{\mathcal{Y}^\beta} \otimes \mathbb{C}_{-2\tilde{\mu}} \otimes (\det \mathcal{N})^{-1}$. Let $\mathcal{Q}^T_\beta(\mathcal{Y}^\beta, -)$ be the quantization map defined by Q^β and localized near $\Phi^{-1}(\beta + \xi) \cap \mathcal{Y}^\beta$ by $\Phi - \xi$. Following the argument of [33][Section 6] one obtains

$$\mathcal{Q}^T_\beta(\mathcal{Y}, \tilde{L}|_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu}) = (-1)^l \sum_{k \in \mathbb{N}} \mathcal{Q}^T_\beta(\mathcal{Y}^\beta, \det \mathcal{N} \otimes S^k(\mathcal{N}))$$

where $S^k(\mathcal{N})$ is the k-th symmetric product of $\mathcal{N}$, and $l = \mathrm{rank}_{\mathbb{C}}\mathcal{N}$. Thus, it is sufficient to prove that $[\mathcal{Q}^T_\beta(\mathcal{Y}^\beta, \det \mathcal{N} \otimes S^k(\mathcal{N}))]^T = 0$ for every $k \in \mathbb{N}$. For this

[19]Here $pr : \mathcal{Z} \times \overrightarrow{\Delta} \to \mathcal{Z}$ is the projection to the first factor.

[20]See [33][Section 6.1].

purpose, we use Lemma 4.8. Let α be the $\mathbb{T}_\beta$-weight on $\det\mathcal{N}$. By the Kostant formula, the $\mathbb{T}_\beta$-weight on $L_{2\omega}|_{\mathcal{Y}^\beta}$ is equal to $\beta+\xi$. Hence any $\mathbb{T}_\beta$-weight γ on $\det\mathcal{N}\otimes S^k(\mathcal{N})\otimes(\mathbb{L}^\beta)^{1/2}$ is of the form

$$\gamma=\beta+\xi-\tilde{\mu}+\frac{1}{2}\alpha+\delta$$

where δ is a $\mathbb{T}_\beta$-weight on $S^k(\mathcal{N})$. So $(\gamma,\beta)=(\beta+\xi-\tilde{\mu},\beta)+\frac{1}{2}(\alpha,\beta)+(\delta,\beta)$. But the $\mathbb{T}_\beta$-weights on $\mathcal{N}$ are 'positive' for β, so $(\alpha,\beta)>0$ and $(\delta,\beta)\geq 0$. On the other hand, $\beta+\xi=\beta(\xi,A)$ is the orthogonal projection of ξ on some affine subspace $A\subset\mathfrak{t}^*_+$ which contains $\tilde{\mu}$: hence $(\beta+\xi-\tilde{\mu},\beta)=0$. Finally we have proved that $(\gamma,\beta)>0$. □

5. Quantization and the discrete series

In this section we follow closely the notation of [12]. Let G be a connected, real, semisimple Lie group with finite center. By definition, the *discrete series* of G is the set of isomorphism classes of irreducible, square integrable, unitary representations of G.

Let K be a maximal compact subgroup of G, and T be a maximal torus in K. Harish-Chandra has shown that G has a discrete series if and only if T is a Cartan subgroup of G [19]. For the remainder of this section, we may therefore assume that T is a Cartan subgroup of G. The discrete series are parametrized by a subset $\widehat{G}_d$ in the dual $\mathfrak{t}^*$ of the Lie algebra of T. For any $\lambda\in\widehat{G}_d$, Harish-Chandra has associated an invariant eigendistribution on G, denoted Θ_λ, which is shown to be the global trace of an irreducible, square integrable, unitary representations of G.

On the other hand one can associate to $\lambda\in\widehat{G}_d$, the regular coadjoint orbit $M:=G\cdot\lambda$. It is a symplectic manifold with a Hamiltonian action of K. Since the vectors X_M, $X\in\mathfrak{g}$, span the tangent space at every $\xi\in M$, the symplectic 2-form is determined by

$$\omega(X_M,Y_M)_\xi=\langle\xi,[X,Y]\rangle\ .$$

The corresponding moment map $\Phi:M\to\mathfrak{k}^*$ for the K-action is the composition of the inclusion $i:M\hookrightarrow\mathfrak{g}^*$ with the projection $\mathfrak{g}^*\to\mathfrak{k}^*$. The vector λ determines a choice $\mathfrak{R}^+$ of positive roots for the T-action on $\mathfrak{g}_\mathbb{C}$. We recall now how the choice of $\mathfrak{R}^+$ determines a complex structure on M. The decomposition[21] $\mathfrak{g}_\mathbb{C}=\mathfrak{t}_\mathbb{C}\oplus\sum_{\alpha\in\mathfrak{R}}\mathfrak{g}_{\mathbb{C},\alpha}$ gives the following T-equivariant decomposition of the complexified tangent space of M at λ :

$$\mathbf{T}_\lambda M\otimes\mathbb{C}=\sum_{\alpha\in\mathfrak{R}}\mathfrak{g}_{\mathbb{C},\alpha}=V\oplus\overline{V}\ ,$$

with $V=\sum_{\alpha\in\mathfrak{R}^+}\mathfrak{g}_{\mathbb{C},\alpha}$. We have then a T-equivariant isomorphism $\mathcal{I}:\mathbf{T}_\lambda M\to V$ equal to the composition of the inclusion $\mathbf{T}_\lambda M\hookrightarrow\mathbf{T}_\lambda M\otimes\mathbb{C}$ with the projection $V\oplus\overline{V}\to V$. The T-equivariant complex structure J_λ on $\mathbf{T}_\lambda M$ is determined by the relation $\mathcal{I}(J_\lambda v)=\imath\mathcal{I}(v)$. Hence, the set of real infinitesimal weights for the T-action on $(\mathbf{T}_\lambda M,J_\lambda)$ is $\mathfrak{R}^+$. Since M is an homogeneous space, J_λ integrates into an almost complex structure (which is in fact integrable). J on M. Using the isomorphism $M\simeq G/T$, the canonical line bundle $\kappa=\det_\mathbb{C}(\mathbf{T}M)^{-1}$ is $\kappa=G\times_T\mathbb{C}_{2\rho}$ where ρ is halph the sum of the roots of $\mathfrak{R}_+$.

[21]For $v\in\mathfrak{g}_{\mathbb{C},\alpha}$, $\exp(X).v=e^{\imath\langle\alpha,X\rangle}\,v$ for any $X\in\mathfrak{t}$.

If $\lambda \in \widehat{G}_d$, $\lambda - \rho$ is a weight and

$$\tilde{L} := G \times_T \mathbb{C}_{\lambda-\rho} \to G/T \tag{5.43}$$

is a κ-prequantum line bundle over (M, ω, J). We have shown in [32], that $\mathrm{Cr}(\| \Phi \|^2)$ is compact, equal to the K-orbit $K \cdot \lambda$. The quantization map $RR^K_\Phi(M, -)$ is then well defined. The main result of this section is the following

Theorem 5.1. *We have the following equality of tempered distribution on K*

$$\Theta_\lambda|_K = (-1)^{\frac{\dim(G/K)}{2}} RR^K_\Phi(G \cdot \lambda, \tilde{L}) \ ,$$

where $\Theta_\lambda|_K$ is the restriction of the eigendistribution Θ_λ to the subgroup K.

With Theorem 5.1 in hand we can exploit the result of Theorem 4.1 to compute the K-multiplicities of $\Theta_\lambda|_K$ in term of reduced spaces. Let us fix some notation. For every $\mu \in \Lambda^*_+$, we denote $\mathrm{m}_\mu(\lambda) \in \mathbb{N}$ the multiplicity[22] of $\Theta_\lambda|_K$ relatively to the K-irreducible representation with highest weigt μ :

$$\Theta_\lambda|_K = \sum_{\mu \in \Lambda^*_+} \mathrm{m}_\mu(\lambda)\, \chi^K_\mu \quad \text{in} \quad R^{-\infty}(K) \ . \tag{5.44}$$

The moment map $\Phi : G \cdot \lambda \to \mathfrak{k}^*$ is *proper* since the coadjoint orbit is closed [32]. We show in Lemma 5.5 that the moment polyhedral subset $\Delta = \Phi(G \cdot \lambda) \cap \mathfrak{t}^*_+$ is of dimension $\dim T$. Thus the notions of *generic quasi-regular values* and *regular values* coincide : they concern the elements $\xi \in \Delta$ such that $\Phi^{-1}(\xi)$ is a smooth submanifold with a locally free action of T. We have shown in subsection 4.3 how to define the quantity $Q((G \cdot \lambda)_{\mu+\rho_c}) \in \mathbb{Z}$ as the index of a Spin^c Dirac operator on $\Phi^{-1}(\xi)/T$, where ξ is a regular value of Φ close enough to $\mu + \rho_c$.

Proposition 5.2. *For every $\mu \in \Lambda^*_+$, we have*

$$\mathrm{m}_\mu(\lambda) = \mathcal{Q}\left((G \cdot \lambda)_{\mu+\rho_c}\right) \ ,$$

in particular $\mathrm{m}_\mu(\lambda) = 0$ if $\mu + \rho_c$ does not belong to the relative interior of the moment polyhedral subset Δ.

Proof : A small check of orientations shows that $\varepsilon = (-1)^{\frac{\dim(G/K)}{2}}$. By Theorem 4.1, we have to show that $(G \cdot \lambda, \Phi)$ satisfies Assumption 3.4, and that the infinitesimal K-stabilizers are abelian. The first point will be treated in subsection 5.3. The second point is obvious since $M \simeq G/T$: all the G-stabilizers are conjugate to T, so all the K-stabilizers are abelian. □

Let us fix some notation. Let $\mathfrak{R} \subset \mathfrak{R}_c \subset \Lambda^*$ be respectively the set of (real) roots for the action of T on $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{k}_\mathbb{C}$. We choose a system of positive roots $\mathfrak{R}^+_c$ for $\mathfrak{R}_c$, we denote $\mathfrak{t}^*_+$ the corresponding Weyl chamber, and we let ρ_c be half the sum of the elements of $\mathfrak{R}^+_c$. We denote by B the Killing form on $\mathfrak{g}$. It induces on $\mathfrak{t}$, and then on $\mathfrak{t}^*$, a scalar product (denoted $(-,-)$). An element $\lambda \in \mathfrak{t}^*$ is called *regular* if $(\lambda, \alpha) \neq 0$ for every $\alpha \in \mathfrak{R}$. Let $\mathfrak{R}^+$ be a system of positive roots for $\mathfrak{R}$, and let ρ be half the sum of elements of $\mathfrak{R}^+$. The set $\Lambda^* + \rho$ does not depend of the choice of $\mathfrak{R}^+$: we denote it by Λ^*_ρ.

[22] See subsection 5.1.

The discrete series of G are parametrized by

$$\widehat{G}_d := \{\lambda \in \mathfrak{t}^*, \lambda \text{ regular } \} \cap \Lambda^*_\rho \cap \mathfrak{t}^*_+ \tag{5.45}$$

When G is compact (i.e. $G = K$), the set $\widehat{G}_d$ equals $\Lambda^*_+ + \rho_c$, and it parametrizes the set of irreducible representations of G.

5.1. **K-multiplicities of the discrete series.** In this section, we fix $\lambda \in \widehat{G}_d$. Let $\mathfrak{R}^+$ be the system of positive roots for $\mathfrak{R}$ such that $(\lambda, \alpha) > 0$ for every $\alpha \in \mathfrak{R}^+$. Then $\mathfrak{R}^+_c \subset \mathfrak{R}^+$, and $\rho = \frac{1}{2}\sum_{\alpha\in\mathfrak{R}^+} \alpha$ decomposes in $\rho = \rho_c + \rho_n$ where $\rho_n = \frac{1}{2}\sum_{\alpha\in\mathfrak{R}^+_n} \alpha$ and $\mathfrak{R}^+_n = \mathfrak{R}^+ - \mathfrak{R}^+_c$.

Let π_λ be the (equivalence class of) unitary, square integrable, irreducible G-representation associated to λ, and let Θ_λ be its global character. This G-invariant distribution admits a restriction $\Theta_\lambda|_K$ to K which is equal to the global character of $\pi_\lambda|_K$ (see the Appendix in [13]). For every $\mu \in \Lambda^*_+$, the multiplicity of V_μ in $\pi_\lambda|_K$ is *finite* and denoted $\mathrm{m}_\mu(\lambda)$. We have the following equality of distributions on K : $\Theta_\lambda|_K = \sum_{\mu\in\Lambda^*_+} \mathrm{m}_\mu(\lambda)\, \chi^K_\mu$.

Let $\mathcal{P} : \Lambda^* \mapsto \mathbb{N}$ be the partition function associated to the set $\mathfrak{R}^+_n$: for $\mu \in \Lambda^*$, $\mathcal{P}(\mu)$ is the number of distinct ways we can write $\mu = \sum_{\alpha\in\mathfrak{R}^+_n} n_\alpha \alpha$ with $n_\alpha \in \mathbb{N}$ for all α. The following Theorem is known as the Blattner formula and was first proved by Hecht and Schmid [20].

Theorem 5.3. *For $\mu \in \Lambda^*_+$, we have*

$$\mathrm{m}_\mu(\lambda) = \sum_{w\in W} (-1)^w \mathcal{P}\Big(w(\mu + \rho_c) - (\mu_\lambda + \rho_c)\Big) \ ,$$

where[23] $\mu_\lambda := \lambda - \rho_c + \rho_n$.

With Theorem 5.3 in hand, we can describe $\Theta_\lambda|_K$ through the holomorphic induction map $\mathrm{Hol}^K_T : R^{-\infty}(T) \to R^{-\infty}(K)$. Recall that Hol^K_T is characterized by the following properties: i) $\mathrm{Hol}^K_T(t^\mu) = \chi^K_\mu$ for every dominant weight $\mu \in \Lambda^*_+$; ii) $\mathrm{Hol}^K_T(t^{w\circ\mu}) = (-1)^w \mathrm{Hol}^K_T(t^\mu)$ for every $w \in W$ and $\mu \in \Lambda^*$; iii) $\mathrm{Hol}^K_T(t^\mu) = 0$ if $W \circ \mu \cap \Lambda^*_+ = \emptyset$. Using these properties we have

$$\sum_{\mu\in\Lambda^*} \mathcal{R}(\mu)\, \mathrm{Hol}^K_T(t^\mu) = \sum_{\mu\in\Lambda^*_+} \Big[\sum_{w\in W} (-1)^w \mathcal{R}(w\circ\mu)\Big]\, \chi^K_\mu \ , \tag{5.46}$$

for every map $\mathcal{R} : \Lambda^* \to \mathbb{Z}$.

For a weight $\alpha \in \Lambda^*$, with $(\lambda, \alpha) \neq 0$, let us denote $[1 - t^\alpha]^{-1}_\lambda \in R^{-\infty}(T)$ the oriented inverse of $(1 - t^\alpha)$: $[1 - t^\alpha]^{-1}_\lambda = \sum_{k\in\mathbb{N}} t^{k\alpha}$ if $(\lambda, \alpha) > 0$, and $[1 - t^\alpha]^{-1}_\lambda = -t^{-\alpha}\sum_{k\in\mathbb{N}} t^{-k\alpha}$ if $(\lambda, \alpha) < 0$. Let $A = \{\alpha_1, \cdots, \alpha_l\}$ be a set of weights with $(\lambda, \alpha_i) \neq 0$, $\forall i$. We denote $A^+ = \{\varepsilon_1\alpha_1, \cdots, \varepsilon_l\alpha_l\}$ the corresponding set of polarized weights: $\varepsilon_i = \pm 1$ and $(\lambda, \varepsilon_i\alpha_i) > 0$ for all i. The product $\Pi_{\alpha\in A}[1 - t^\alpha]^{-1}_\lambda$ is well defined (denoted $[\Pi_{\alpha\in A}(1 - t^\alpha)]^{-1}_\lambda$) and a small computation shows that

$$\begin{aligned}
\Big[\Pi_{\alpha\in A}(1 - t^\alpha)\Big]^{-1}_\lambda &= (-1)^r\, t^{-\gamma} \Big[\Pi_{\alpha\in A^+}(1 - t^\alpha)\Big]^{-1}_\lambda \\
&= (-1)^r\, t^{-\gamma} \sum_{\mu\in\Lambda^*} \mathcal{P}_{A^+}(\mu)\, t^\mu \ .
\end{aligned} \tag{5.47}$$

[23] We shall note that $\mu_\lambda \in \Lambda^*_+$ (see [12], section 5).

Here $\mathcal{P}_{A+} : \Lambda^* \mapsto \mathbb{N}$ is the partition function associated to A^+, $\gamma = \sum_{(\lambda,\alpha)<0} \alpha$, and $r = \sharp\{\alpha \in A,\ (\lambda,\alpha) < 0\}$. These notations are compatible with those we used in [33][Section 5]. If V is a complex T-vector space with $V^\lambda := \{v \in V,\ \lambda.v = 0\}$ reduced to 0, the element $\wedge^\bullet_{\mathbb{C}} V \in R(T)$ admits a polarized inverse $[\wedge^\bullet_{\mathbb{C}} V]^{-1}_\lambda = [\Pi_{\alpha\in\mathfrak{R}(V)}(1-t^\alpha)]^{-1}_\lambda$ where $\mathfrak{R}(V)$ is the set of real infinitesimal T-weights on V.

Lemma 5.4. *We have the following equality in* $R^{-\infty}(K)$

$$\Theta_\lambda|_K = \mathrm{Hol}_T^K \left(t^{\mu_\lambda} \left[\Pi_{\alpha\in\mathfrak{R}_n^+}(1-t^\alpha)\right]^{-1}_\lambda \right) .$$

Proof : Let $\Theta \in R^{-\infty}(K)$ be the RHS in the equality of the Lemma. From (5.47), we have $\Theta = \sum_{\mu\in\Lambda^*} \mathcal{P}(\mu)\,\mathrm{Hol}_T^K(t^{\mu+\mu_\lambda}) = \sum_{\mu\in\Lambda^*} \mathcal{P}(\mu-\mu_\lambda)\,\mathrm{Hol}_T^K(t^\mu)$. If we use now (5.46), we see that multiplicity of Θ relative to the highest weight $\mu \in \Lambda^*_+$ is $\sum_{w\in W}(-1)^w \mathcal{P}(w(\mu+\rho_c) - (\mu_\lambda+\rho_c))$. From Theorem 5.3, we conclude that $\Theta_\lambda|_K = \Theta$. □

5.2. **Proof of Theorem 5.1.** In Lemma 5.4 we have used the Blattner formula to write $\Theta_\lambda|_K$ in term of the holomorphic induction map Hol_T^K. Theorem 5.1 is then proved if one shows that $RR_\Phi^K(G\cdot\lambda, \tilde{L}) = (-1)^r \mathrm{Hol}_T^K(t^{\mu_\lambda}[\Pi_{\alpha\in\mathfrak{R}_n^+}(1-t^\alpha)]^{-1}_\lambda)$, with $\mu_\lambda = \lambda - \rho_c + \rho_n$, and $r = \frac{1}{2}\dim(G/K)$. More generally, we show in this section that for any K-equivariant vector bundle $E \to G\cdot\lambda$

$$RR_\Phi^K(G\cdot\lambda, E) = (-1)^r \mathrm{Hol}_T^K\left(E_\lambda . t^{2\rho_n} . \left[\Pi_{\alpha\in\mathfrak{R}_n^+}(1-t^\alpha)\right]^{-1}_\lambda\right) , \tag{5.48}$$

where $E_\lambda \in R(T)$ is the fiber of E at λ.

First we recall why $\mathrm{Cr}(\parallel\Phi\parallel^2) = K\cdot\lambda$ in $M := G\cdot\lambda$ (see [32] for the general case of closed coadjoint orbits). One can work with an adjoint orbit $M := G\cdot\tilde{\lambda}$ through the G-identification $\mathfrak{g}^* \simeq \mathfrak{g}$ given by the Killing form; then $\Phi : M \to \mathfrak{k}$ is just the projection on M to the (orthogonal) projection $\mathfrak{g} \to \mathfrak{k}$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Every $m \in M$ decomposes in $m = x_m + y_m$, with $x_m = \Phi(m)$ and $y_m \in \mathfrak{p}$. The Hamiltonian vector field of $\frac{-1}{2} \parallel\Phi\parallel^2$ is, following (3.10), $\mathcal{H}_m = [x_m, m] = [x_m, y_m]$ then

$$\mathrm{Cr}(\parallel\Phi\parallel^2) = \{\mathcal{H} = 0\} = \{m \in M,\ [x_m, y_m] = 0\} .$$

Now, since $\tilde{\lambda}$ is elliptic, every $m \in M$ is also elliptic. If $m \in \mathrm{Cr}(\parallel\Phi\parallel^2)$, $[m, x_m] = 0$ and m, x_m are elliptic, hence $y_m = m - x_m$ is elliptic and so is equal to 0. Finally $\mathrm{Cr}(\parallel\Phi\parallel^2) = G\cdot\tilde{\lambda}\cap\mathfrak{k} = K\cdot\tilde{\lambda}$. □

After Definition (3.2), the computation of $RR_\Phi^K(G\cdot\lambda, \tilde{L})$ holds on a small K-invariant neighborhood of $K\cdot\lambda$ of $G\cdot\lambda$. Our model for the computation will be

$$\tilde{M} := K \times_T \mathfrak{p}$$

endowed with the canonical K-action. The tangent bundle $\mathbf{T}\tilde{M}$ is isomorphic to $K\times_T(\mathfrak{r}\oplus\mathbf{T}\mathfrak{p})$ where $\mathfrak{r}$ is the T-invariant complement of $\mathfrak{t}$ in $\mathfrak{k}$. One has a symplectic form $\tilde{\Omega}$ on $\tilde{M}$ defined by $\tilde{\Omega}_m(V,V') = \langle\lambda, [X,X'] + [v,v']\rangle$. Here $m = [k,x] \in K\times_T\mathfrak{p}$, and $V = [k,x;X+v]$, $X' = [k,x;X'+v']$ are two tangent vectors[24]. A small

[24] $X, X' \in \mathfrak{r}$ and $v, v' \in \mathfrak{p}$.

computation shows that the K-action on $(K\times_T\mathfrak{p},\tilde{\Omega})$ is Hamiltonian with moment map $\tilde{\Phi}:\tilde{M}\to\mathfrak{k}^*$ defined by

$$\tilde{\Phi}([k,x]) = k\cdot\left(\lambda-\frac{1}{2}pr_{\mathfrak{t}^*}(\lambda\circ \mathrm{ad}(x)\circ\mathrm{ad}(x))\right)\ .$$

Here $\mathrm{ad}(x)$ is the adjoint action of x, and $pr_{\mathfrak{t}^*}:\mathfrak{g}^*\to\mathfrak{t}^*$ is the projection.

Lemma 5.5. *There exists a K-hamiltonian isomorphism $\Upsilon:\mathcal{U}\simeq\tilde{\mathcal{U}}$, where $\mathcal{U}$ is a K-invariant neighborhood of $K\cdot\lambda$ in M, and $\tilde{\mathcal{U}}$ is a K-invariant neighborhood of K/T in $\tilde{M}$. We can impose furthermore that the diffential of Υ at λ is the identity*[25].

Corollary 5.6. *The cone $\lambda+\sum_{\alpha\in\mathfrak{R}_n^+}\mathbb{R}^+\alpha$ coincides with $\Delta=(G\cdot\lambda)\cap\mathfrak{t}^*_+$ in a neighborhood of λ. The polyhedral set Δ is of dimension* $\dim T$.

Proof: The first assertion is an immediate consequence of Lemma 5.5 and of the convexity Theorem [26]. Let $X_o\in\mathfrak{t}$ such that $\xi(X_o)=0$ for all $\overrightarrow{\Delta}$, that is $\alpha(X_o)=0$ for all $\alpha\in\mathfrak{R}_n^+$: X_o commutes with all elements in $\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$, and let Σ be the set of weights for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$: $\mathfrak{g}=\sum_{\alpha\in\Sigma}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha=\{Z\in\mathfrak{g},[X,Z]=\alpha(X)Z$ for all $X\in\mathfrak{a}\}$. Since $[X_o,\mathfrak{a}]=0$, $[X_o,\mathfrak{g}_\alpha]\subset\mathfrak{g}_\alpha$ for all $\alpha\in\Sigma$. But since $[X_o,\mathfrak{p}]=0$, and $\mathfrak{g}_\alpha\cap\mathfrak{k}=0$ for all $\alpha\neq 0$, we see that $[X_o,\mathfrak{g}_\alpha]=0$ for all $\alpha\neq0$. The subalgebra $\mathfrak{g}_0$ is abelian and contains X_o, thus $[X_o,\mathfrak{g}]=0$: X_o belongs to the center $Z(\mathfrak{g})$ of $\mathfrak{g}$. Since G has a *finite center*, $Z(\mathfrak{g})=0$, and then $X_o=0$. Finally we have proved that $\overrightarrow{\Delta}^\perp=0$, or equivalently $\overrightarrow{\Delta}=\mathfrak{t}$. □

Proof of Lemma 5.5 : The *symplectic cross-section Theorem* [17] asserts that the pre-image $\mathcal{Y}:=\Phi^{-1}(\mathrm{interior}(\mathfrak{t}^*_+))$ is a symplectic submanifold provided with an Hamiltonian action of T. The restriction $\Phi|_{\mathcal{Y}}$ is the moment map for the T-action on $\mathcal{Y}$. Moreover, the set $K.\mathcal{Y}$ is a K-invariant neighborhood of $K\cdot\lambda$ in M diffeomorphic to $K\times_T\mathcal{Y}$. Since λ is a fixed T-point of $\mathcal{Y}$, an Hamiltonian model for $(\mathcal{Y},\omega|_{\mathcal{Y}},\Phi|_{\mathcal{Y}})$ in a neighborhood of λ is $(\mathbf{T}_\lambda\mathcal{Y},\omega_\lambda,\Phi_\lambda)$ where ω_λ is the linear symplectic form of the tangent space $\mathbf{T}_\lambda M$ restricted to $\mathbf{T}_\lambda\mathcal{Y}$, and $\Phi_\lambda:\mathbf{T}_\lambda\mathcal{Y}\to\mathfrak{t}^*$ is the unique moment map with $\Phi_\lambda(0)=\lambda$. A small computation shows that $x\to\lambda\circ\mathrm{ad}(x)$ is an isomorphism from $\mathfrak{p}$ to $\mathbf{T}_\lambda\mathcal{Y}$, and $\Phi_\lambda(x)=\lambda-\frac{1}{2}pr_{\mathfrak{t}^*}(\lambda\circ\mathrm{ad}(x)\circ\mathrm{ad}(x))$. □

We still denoted J the almost complex structure transported on $\tilde{\mathcal{U}}\subset K\times_T\mathfrak{p}$ through Υ. Let $\pi:K\times_T\mathfrak{p}\to K/T$, and $\pi_{\tilde{\mathcal{U}}}:\tilde{\mathcal{U}}\to K/T$ be the fibering maps. Remark that for any equivariant vector bundle E over M, the vector bundle $(\Upsilon^{-1})^*(E|_{\mathcal{U}})\to\tilde{\mathcal{U}}$ is isomorphic to $\pi^*_{\tilde{\mathcal{U}}}(K\times_T E_\lambda)$, where $E_\lambda\in R(T)$ is the fiber of E at λ. At this stage, we have after Definition 3.2

$$RR_\Phi^K(G\cdot\lambda,E)=\mathrm{Index}^K_{\tilde{\mathcal{U}}}\left(\mathrm{Thom}_K^{\tilde{\Phi}}(\tilde{\mathcal{U}},J)\otimes\pi^*_{\tilde{\mathcal{U}}}(K\times_T E_\lambda)\right)\ . \tag{5.49}$$

We will now define a simpler representant of the class in $\mathbf{K}_K(\mathbf{T}_K\tilde{\mathcal{U}})$ defined by $\mathrm{Thom}_K^{\tilde{\Phi}}(\tilde{\mathcal{U}},J)$ with the help of Lemma 3.6. Consider the map

$$\begin{aligned}\underline{\lambda}:K\times_T\mathfrak{p}&\longrightarrow\ \mathfrak{k}^*\\ (k,x)&\longmapsto\ k\cdot\lambda\ ,\end{aligned}$$

[25] Note that $\mathbf{T}_\lambda\mathcal{U}$ and $\mathbf{T}_\lambda\tilde{\mathcal{U}}$ are canonically isomorphic to $\mathfrak{r}\oplus\mathfrak{p}$.

and let $\underline{\lambda}_{\tilde{M}}$ be the vector field on $\tilde{M}$ generated $\underline{\lambda}$ (see (3.10)). Note that $\underline{\lambda}_{\tilde{M}}$ never vanishes outside the zero section of $K \times_T \mathfrak{p}$. Let $(-,-)_{\tilde{M}}$ be the riemmanian metric on $\tilde{M}$ defined by $(V, V')_{\tilde{M}} = (X, X') + (v, v')$ for $V = [k, x; X+v]$, $V = [k, x; X'+v']$. A small computation shows that

$$(\tilde{\mathcal{H}}, \underline{\lambda}_{\tilde{M}})_{\tilde{M}} = \parallel \underline{\lambda}_{\tilde{M}} \parallel^2 + o(\parallel \underline{\lambda}_{\tilde{M}} \parallel^2)$$

in the neighborhood of the zero section in $K \times_T \mathfrak{p}$. Hence, if we take $\tilde{\mathcal{U}}$ small enough, $(\tilde{\mathcal{H}}, \underline{\lambda}_{\tilde{M}})_{\tilde{M}} > 0$ on $\tilde{\mathcal{U}} - \{\text{zero section}\}$, hence $\mathrm{Thom}_K^{\tilde{\Phi}}(\tilde{\mathcal{U}}, J) = \mathrm{Thom}_K^{\underline{\lambda}}(\tilde{\mathcal{U}}, J)$ in $\mathbf{K}_K(\mathbf{T}_K \tilde{\mathcal{U}})$ (see Lemma 3.6).

Let us denote $J_\lambda : \mathfrak{r} \oplus \mathfrak{p} \to \mathfrak{r} \oplus \mathfrak{p}$ the complex structure defined by J on the tangent space $\mathbf{T}_\lambda \tilde{M}$. Let $\tilde{J}$ be the K-invariant almost complex structure on $\tilde{M}$, constant on the fibers of $\tilde{M} \to K/T$, and equal to J_λ at λ: if $[k, x] \in K \times_T \mathfrak{p}$, $\tilde{J}_{[k,x]}(V) = [k, x, J_\lambda(X + v)]$ for $V = [k, x, X + v]$.

Since the set $\{\underline{\lambda}_{\tilde{M}} = 0\} = K/T$ is compact, one defines with $\tilde{J}$ and the map $\underline{\lambda}$ the localized Thom symbol

$$\mathrm{Thom}_K^{\underline{\lambda}}(\tilde{M}, \tilde{J}) \in \mathbf{K}_K(\mathbf{T}_K \tilde{M}) \ .$$

Through the canonical identification with the tangent space at $[k, x]$ and $[k, 0]$, on can write $\tilde{J}_{[k,x]} = \tilde{J}_{[k,0]} = J_{[k,0]}$ for any $[k, x] \in \tilde{\mathcal{U}}$. We note that J and $\tilde{J}$ are related on $\tilde{\mathcal{U}}$ by the homotopy J^t of almost complex structures: $J^t_{[k,x]} := J_{[k,tx]}$ for $[k, x] \in \tilde{\mathcal{U}}$. By Lemma 3.6, we conclude that the localized Thom symbols $\mathrm{Thom}_K^{\underline{\lambda}}(\tilde{\mathcal{U}}, J)$ and $\mathrm{Thom}_K^{\underline{\lambda}}(\tilde{M}, \tilde{J})|_{\tilde{\mathcal{U}}}$ define the same class in $\mathbf{K}_K(\mathbf{T}_K \tilde{\mathcal{U}})$, thus (5.49) becomes

$$RR_\Phi^K(G \cdot \lambda, E) = \mathrm{Index}_{\tilde{M}}^K \Big(\mathrm{Thom}_K^{\underline{\lambda}}(\tilde{M}, \tilde{J}) \otimes \pi^*(K \times_T E_\lambda) \Big) \ . \tag{5.50}$$

In order to compute (5.50), we now use the induction morphism

$$i_* : \mathbf{K}_T(\mathbf{T}_T \mathfrak{p}) \longrightarrow \mathbf{K}_K(\mathbf{T}_K(K \times_T \mathfrak{p}))$$

defined by Atiyah in [1] (see [33][Section 3]). The map i_* enjoys two properties: first, i_* is an isomorphism and the K-index of $\sigma \in \mathbf{K}_K(\mathbf{T}_K(K \times_T \mathfrak{p}))$ can be computed with the T-index of $(i_*)^{-1}(\sigma)$.

Let $\sigma : p^*(E^+) \to p^*(E^-)$ be a K-transversally elliptic symbol on $K \times_T \mathfrak{p}$, where $p : \mathbf{T}(K \times_T \mathfrak{p}) \to K \times_T \mathfrak{p}$ is the projection, and E^+, E^- are equivariant vector bundles over $K \times_T \mathfrak{p}$: for any $[k, x] \in K \times_T \mathfrak{p}$, we have a collection of linear maps $\sigma([k, x, X + v]) : E^+_{[k,x]} \to E^-_{[k,x]}$ depending on the tangent vectors $X + v$. The symbol $(i_*)^{-1}(\sigma)$ is defined by

$$(i_*)^{-1}(\sigma)(x, v) = \sigma([1, x, 0 + v]) : E^+_{[1,x]} \longrightarrow E^-_{[1,x]} \ , \tag{5.51}$$

for any $(x, v) \in \mathbf{T}\mathfrak{p}$. For $\sigma = \mathrm{Thom}_K^{\tilde{\Phi}}(\tilde{M}, \tilde{J})$, the vector bundle E^+ (resp. E^-) is $\wedge_{\mathbb{C}}^{odd} \mathbf{T}\tilde{M}$ (resp. $\wedge_{\mathbb{C}}^{even} \mathbf{T}\tilde{M}$). Since the complex structure leaves $\mathfrak{r} \simeq \mathfrak{k}/\mathfrak{p}$ and $\mathfrak{p}$ invariant one gets

$$(i_*)^{-1}(\mathrm{Thom}_K^{\underline{\lambda}}(\tilde{M}, \tilde{J})) = \mathrm{Thom}_T^{\lambda}(\mathfrak{p}, J_\lambda) \ \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{t} \ ,$$

and

$$(i_*)^{-1} \Big(\mathrm{Thom}_K^{\underline{\lambda}}(\tilde{M}, \tilde{J}) \otimes \pi^*(K \times_T E_\lambda) \Big) = \mathrm{Thom}_T^{\lambda}(\mathfrak{p}, J_\lambda) \, E_\lambda \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{t} \ , \tag{5.52}$$

where[26] $\mathrm{Thom}_T^\lambda(\mathfrak{p}, J_\lambda)$ is the T-equivariant Thom symbol on the complex vector space $(\mathfrak{p}, J_\lambda)$ pushed by the constant map $\mathfrak{p} \to \mathfrak{t}$, $x \mapsto \lambda$.

To express the K-index of σ in terms of the T-index of $(i_*)^{-1}(\sigma)$, we need the induction map

$$\mathrm{Ind}_T^K : \mathcal{C}^{-\infty}(T) \longrightarrow \mathcal{C}^{-\infty}(K)^K , \tag{5.53}$$

where $\mathcal{C}^{-\infty}(T)$ is the set of generalized functions on T, and the K invariants are taken with the conjugation action. The map Ind_T^K is defined as follows : for $\phi \in \mathcal{C}^{-\infty}(T)$, we have $\int_K \mathrm{Ind}_T^K(\phi)(k) f(k) dk = \frac{\mathrm{vol}(K,dk)}{\mathrm{vol}(T,dt)} \int_T \phi(t) f|_T(t) dt$, for every $f \in \mathcal{C}^\infty(K)^K$. Theorem 4.1 of Atiyah in [1] tells us that

$$\begin{array}{ccc} K_T(\mathbf{T}_T\mathfrak{p}) & \xrightarrow{\ i_*\ } & K_K(\mathbf{T}_K\tilde{M}) \\ \Big\downarrow {\scriptstyle \mathrm{Index}_{\mathfrak{p}}^T} & & \Big\downarrow {\scriptstyle \mathrm{Index}_{\tilde{M}}^K} \\ \mathcal{C}^{-\infty}(T) & \xrightarrow[\mathrm{Ind}_T^K]{} & \mathcal{C}^{-\infty}(K)^K . \end{array} \tag{5.54}$$

is a commutative diagram (with $\tilde{M} = K \times_T \mathfrak{p}$). In other words, $\mathrm{Index}_{\tilde{M}}^K(\sigma) = \mathrm{Ind}_T^K(\mathrm{Index}_{\mathfrak{p}}^T((i_*)^{-1}(\sigma)))$. With (5.50), (5.52), and (5.54), we find

$$\begin{aligned} RR_\Phi^K(G\cdot\lambda, E) &= \mathrm{Ind}_T^K\left(\mathrm{Index}_{\mathfrak{p}}^T(\mathrm{Thom}_T^\lambda(\mathfrak{p}, J_\lambda))\, E_\lambda \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{t}\right) \\ &= \mathrm{Hol}_T^K\left(\mathrm{Index}_{\mathfrak{p}}^T(\mathrm{Thom}_T^\lambda(\mathfrak{p}, J_\lambda))\, E_\lambda\right) . \end{aligned}$$

(See the Appendix in [33] for the relation $\mathrm{Hol}_T^K(-) = \mathrm{Ind}_T^K(- \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{t})$.) Now we can conclude, since $\mathrm{Index}_{\mathfrak{p}}^T(\mathrm{Thom}_T^\lambda(\mathfrak{p}, J_\lambda))$ is computed in section 5 of [33]:

$$\begin{aligned} \mathrm{Index}_{\mathfrak{p}}^T(\mathrm{Thom}_T^\lambda(\mathfrak{p}, J_\lambda)) &= \left[\Pi_{\alpha\in\mathfrak{R}_n^+}(1-t^{-\alpha})\right]_\lambda^{-1} \\ &= (-1)^r\, t^{2\rho_n} \left[\Pi_{\alpha\in\mathfrak{R}_n^+}(1-t^{\alpha})\right]_\lambda^{-1} , \end{aligned}$$

with $r = \frac{1}{2}\dim(G/K)$. □

5.3. $(G\cdot\lambda, \Phi)$ **satisfies Assumption 3.4.** Let M be a regular elliptic coadjoint orbit for G, and let K be a maximal compact subgroup of G. We denote $\Phi : M \to \mathfrak{k}^*$ the canonical moment map for the K-action. For $\mu \in \mathfrak{k}^*$, let us consider the map $\Phi_\mu : M \times K\cdot\mu \to \mathfrak{k}^*$, $(m,n) \to \Phi(m) - n$. Let $\|\cdot\|$ be the Euclidan norm on $\mathfrak{k}$ defined by the Killing form: it induces an identification $\mathfrak{k} \simeq \mathfrak{k}^*$ and so an Euclidan norm on $\mathfrak{k}^*$.

This section will be devoted to the proof of the following

Proposition 5.7. *The set* $\mathrm{Cr}(\| \Phi_\mu \|^2)$ *of critical point of* $\| \Phi_\mu \|^2$ *is compact in* $M \times K\cdot\mu$. *Precisely, for any* μ, *there exists* $c(\mu) > 0$ *such that*

$$\mathrm{Cr}(\| \Phi_{\mu'} \|^2) \subset \left(M \cap \left\{\xi \in \mathfrak{g}^*, \| \xi \| \leq c(\mu)\right\}\right) \times K\cdot\mu' ,$$

for any μ' *such that* $\| \mu' \| \leq \| \mu \|$.

[26] Our notations use the structure of $R(T)$-module for $\mathbf{K}_T(\mathbf{T}_T\mathfrak{p})$, hence we can multiply $\mathrm{Thom}_T^\lambda(\mathfrak{p}, J_\lambda)$ by $E_\lambda \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{t}$.

Consider the map $M \to \mathfrak{k}^*$, $m \mapsto \Phi(m) - \mu$. One see easily that $\mathrm{Cr}(\| \Phi_\mu \|^2) = K\cdot\Big(\mathrm{Cr}(\| \Phi_\mu \|^2) \cap (M \times \{\mu\})\Big)$, and $\mathrm{Cr}(\| \Phi_\mu \|^2) \cap (M \times \{\mu\}) \subset \mathrm{Cr}(\| \Phi - \mu \|^2) \times \{\mu\}$. Proposition 5.7 will follow from the following

Proposition 5.8. *For any μ, there exists $c(\mu) > 0$ such that*

$$\mathrm{Cr}(\| \Phi - \mu' \|^2) \subset M \cap \Big\{\xi \in \mathfrak{g}^*,\ \| \xi \| \leq c(\mu)\Big\}\ ,$$

for any μ' such that $\| \mu' \| \leq \| \mu \|$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. The Killing form B provides a G-equivariant identification $\mathfrak{g} \simeq \mathfrak{g}^*$ and K-equivariant identifications $\mathfrak{k} \simeq \mathfrak{k}^*$, $\mathfrak{p} \simeq \mathfrak{p}^*$. Then we consider M as a adjoint orbit of G: $M = G \cdot \lambda$ where $\lambda \in \mathfrak{k}$ is a regular element, i.e. $G_\lambda = K_\lambda$ is a maximal torus in K, and the moment map $\Phi : M \to \mathfrak{k}$ is just the restriction to M of the projection $\mathfrak{k} \oplus \mathfrak{p} \to \mathfrak{k}$. The Killing form B defines K-invariant Euclidean structure on $\mathfrak{k}$ and $\mathfrak{p}$ such that

$$B(X,X) = - \| X_1 \|^2 + \| X_2 \|^2\ , \tag{5.55}$$

for $X = X_1 + X_2$, with $X_1 \in \mathfrak{k}$, $X_2 \in \mathfrak{p}$. Using the fact that $m \to B(m,m)$ is constant equal to $- \| \lambda \|^2$, we get $\| \Phi(m) \|^2 = \frac{1}{2} \| m \|^2 + \frac{1}{2} \| \lambda \|^2$ for any $m \in M$. Finally we obtain

$$\| \Phi(m) - \mu \|^2 = \frac{1}{2} \| m \|^2 - 2 < m, \mu > + \mathrm{cst}$$

where $\mathrm{cst} = \frac{1}{2} \| \lambda \|^2 + \| \mu \|^2$. Using the Cartan decomposition $G = K \cdot \exp(\mathfrak{p})$, one can consider [27] $f^\mu : K \times \mathfrak{p} \to \mathbb{R}$, $(k, X) \to \| \Phi(k^{-1} \cdot e^X \cdot \lambda) - \mu \|^2$ which is equal to $\frac{1}{2} \| e^X \cdot \lambda \|^2 - 2 < e^X \cdot \lambda, k \cdot \mu > + \mathrm{cst}$.

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Since $\mathfrak{p} = \cup_{k \in K} k \cdot \mathfrak{a}$, one see that f^μ is related to

$$F^\mu : (K \cdot \lambda) \times (K \cdot \mu) \times \mathfrak{a} \to \mathbb{R}$$

defined by $F^\mu(m, n, X) = \frac{1}{2} \| e^X \cdot m \|^2 - 2 < e^X \cdot m, n >$. In particular, Proposition 5.8 is equivalent to the following

Proposition 5.9. *For any μ, there exists $c(\mu) > 0$ such that*

$$(m, n, X) \in \mathrm{Cr}(F^{\mu'}) \implies \| e^X \cdot m \| \leq c(\mu)\ ,$$

for any μ' such that $\| \mu' \| \leq \| \mu \|$.

Proof : Let $(m, n, X) \in \mathrm{Cr}(F^\mu)$. Then, the identity $\frac{d}{dt} F^\mu(m, n, X + tX)|_{t=0} = 0$ gives

$$< e^X \cdot m, e^X \cdot [X, m] > = 2 < e^X \cdot [X, m], n >\ . \tag{5.56}$$

Lemma 5.10. *For every $(m, n, X) \in K \cdot \lambda \times \mathfrak{k} \times \mathfrak{a}$ satisfying (5.56), we have*

$$\| e^X \cdot [X, m] \| \leq c(n) \| X \|\ ,$$

where $c(n) > c(n')$ if $\| n \| > \| n' \|$.

---

[27] Here $\cdot$ means the adjoint action.

Proof of the Lemma : Let Σ the set of weights for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$: $\mathfrak{g} = \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{Z \in \mathfrak{g},\, [X,Z] = \alpha(X)Z$ for all $X \in \mathfrak{a}\}$. Recall that $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, where θ denotes the Cartan involution.

We decompose $m \in K\cdot\lambda$ into $m = \sum_\alpha m_\alpha$, and $A := < e^X\cdot m, e^X\cdot[X,m] > = \sum_\alpha e^{2\alpha(X)}\alpha(X) \parallel m_\alpha \parallel^2$. Let $\Sigma_m^\pm := \{\alpha\in\Sigma,\, m_\alpha \neq 0 \text{ and } \pm\alpha(X) > 0\}$, then

$$\begin{aligned} A &= \sum_{\alpha\in\Sigma_m^+} e^{2\alpha(X)}\alpha(X) \parallel m_\alpha \parallel^2 + \sum_{\alpha\in\Sigma_m^-} e^{2\alpha(X)}\alpha(X) \parallel m_\alpha \parallel^2 \\ &\geq \sum_{\alpha\in\Sigma_m^+} e^{2\alpha(X)} \frac{\alpha(X)^2}{r \parallel X \parallel} \parallel m_\alpha \parallel^2 - r \parallel X \parallel \sum_{\alpha\in\Sigma_m^-} \parallel m_\alpha \parallel^2 \qquad [1] \end{aligned}$$

with $r := \sup_{\alpha, \|X\|\leq 1} |\alpha(X)|$. But

$$\begin{aligned} \sum_{\alpha\in\Sigma_m^+} e^{2\alpha(X)}\alpha(X)^2 \parallel m_\alpha \parallel^2 &= \parallel e^X\cdot[X,m] \parallel^2 - \sum_{\alpha\in\Sigma_m^-} e^{2\alpha(X)}\alpha(X)^2 \parallel m_\alpha \parallel^2 \\ &\geq \parallel e^X\cdot[X,m] \parallel^2 - r^2 \parallel X \parallel^2 \sum_{\alpha\in\Sigma_m^-} \parallel m_\alpha \parallel^2 \qquad [2] \end{aligned}$$

So, the inequalities [1] and [2] give[28]

$$(5.57) \qquad A \geq \frac{\parallel e^X\cdot[X,m] \parallel^2}{r \parallel X \parallel} - r \parallel X \parallel . \parallel \lambda \parallel^2 \ .$$

Since $2 < e^X\cdot[X,m], n > \leq 2 \parallel e^X\cdot[X,m] \parallel . \parallel n \parallel$, (5.56) and (5.57) yield

$$2 \parallel e^X\cdot[X,m] \parallel . \parallel n \parallel \geq \frac{\parallel e^X\cdot[X,m] \parallel^2}{r \parallel X \parallel} - r \parallel X \parallel . \parallel \lambda \parallel^2 \ .$$

In other words $E := \parallel e^X\cdot[X,m] \parallel$ satisfies the polynomial inequality $E^2 - 2bE - c^2 \leq 0$, with $c = r \parallel X \parallel . \parallel \lambda \parallel$ and $b = r \parallel X \parallel . \parallel n \parallel$. A direct computation gives

$$\parallel e^X\cdot[X,m] \parallel \leq c(n) \parallel X \parallel ,$$

with $c(n) = r(\parallel n \parallel + \sqrt{\parallel n \parallel^2 + \parallel \lambda \parallel^2})$. □

We take now $n \in K\cdot\mu$, hence $c(n) = c(\mu)$. Proposition 5.9 follows from the following claim : for any $c > 0$ there exists $c' > 0$, such that

$$\parallel e^X\cdot[X,m] \parallel \leq c \parallel X \parallel \Longrightarrow \parallel e^X\cdot m \parallel \leq c'$$

holds for every $(m,X) \in K\cdot\lambda\times\mathfrak{a}$.

Suppose that the claim does not hold : there is a sequence $(m_i,X_i)_{i\in\mathbb{N}}$ in $K\cdot\lambda\times\mathfrak{a}$ such that $\parallel e^{X_i}\cdot[X_i,m_i] \parallel \leq c \parallel X_i \parallel$ but $\parallel e^{X_i}\cdot m_i \parallel \to \infty$. We write $X_i = t_i v_i$ with $t_i > 0$ and $\parallel v_i \parallel = 1$. We can assume moreover that $v_i \to v_\infty$ with $\parallel v_\infty \parallel = 1$, and $m_i \to m_\infty \in K\cdot\lambda$ when $i\to\infty$. Let $\Sigma_{m_\infty} \subset \Sigma$ be the subset $\{\alpha\in\Sigma,\, m_{\infty,\alpha} \neq 0\}$. Note that $\Sigma_{m_\infty} \neq \emptyset$ since m_∞ is a regular element in $\mathfrak{k}$ and rank(G)=rank(K). The sequence

$$(I) \qquad e^{t_i v_i}.[v_i,m_i] = \sum_\alpha e^{t_i\alpha(v_i)}\alpha(v_i) m_{i,\alpha}$$

is bounded in $\mathfrak{g}$, and

$$(II) \qquad e^{t_i v_i}.m_i = \sum_\alpha e^{t_i\alpha(v_i)} m_{i,\alpha}$$

[28]Since $\alpha \in \Sigma_m^+ \Leftrightarrow -\alpha \in \Sigma_m^-$, we see that $2\sum_{\alpha\in\Sigma_m^-} \parallel m_\alpha \parallel^2 = \sum_{\alpha\in\Sigma_m} \parallel m_\alpha \parallel^2 = \parallel m \parallel^2 = \parallel \lambda \parallel^2$.

diverges. From the sequence (II), we see that $(t_i)_{i\in\mathbb{N}}$ is not bounded, so can be assumed to be divergent to ∞. Since for every $\alpha \in \Sigma_{m_\infty}$, $\| m_{i,\alpha} \| \geq \varepsilon > 0$ for i large enough, the fact that (I) is bounded implies that $e^{t_i\alpha(v_i)}\alpha(v_i)$ is bounded in $\mathbb{R}$ for all $\alpha \in \Sigma_{m_\infty}$. Since $\theta(m_\infty) = m_\infty$, the subset $\Sigma_{m_\infty} \subset \Sigma$ is stable through the transformation $\alpha \to -\alpha$. So

$$e^{t_i\alpha(v_i)}\alpha(v_i) \quad \text{and} \quad e^{-t_i\alpha(v_i)}\alpha(v_i)$$

are bounded in $\mathbb{R}$, for every $\alpha \in \Sigma_{m_\infty}$. Since $t_i \to \infty$, the later result forces $\alpha(v_i) \to 0$ when $i \to \infty$; hence

$$[v_\infty, m_\infty] = \sum_{\alpha\in\Sigma_{m_\infty}} \alpha(v_\infty) m_{\infty,\alpha} = 0 \ . \tag{5.58}$$

But $m_\infty \in K\cdot\lambda$ is a regular element of G, and rank(G) =rank(K), then $\mathfrak{g}_{m_\infty} = \mathfrak{k}_{m_\infty}$ is a Cartan subalgebra of $\mathfrak{k}$. So (5.58) implies $v_\infty = 0$ which is contradictory. □

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