The graded cobordism group of codimension-one immersions

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Abstract

The cobordism group $N(M^n)$ of codimension-one immersions in the n-manifold M^n has a natural filtration induced from any cellular decomposition. The problem addressed in this paper is the explicit computation of the graded group $gr^*N(M^n)$. Previous results were obtained by Benedetti and Silhol ([1]) which proved that for an orientable 3-manifold M^3 the cobordism group $N(M^3)$ is isomorphic to $H_1(M^3, \mathbf{Z}/2\mathbf{Z}) \oplus H_2(M^3, \mathbf{Z}/2\mathbf{Z}) \oplus \mathbf{Z}/8\mathbf{Z}$, endowed with a twisted product. One develops new techniques to approach the higher dimensional case and gets the answer for dimension at most 7. As a corollary we find the group of cubulations modulo cubical moves (see [3]) under the same dimensional restrictions.

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1 Introduction

The classification of manifold immersions in codimension at least one up to cobordism was reduced to a homotopy problem by the results of [17, 18]. These techniques are however awkward to apply if one wants to get effective results. The classification up to regular homotopy is also a homotopy problem which is closely related to the previous one. For instance two immersed surfaces in $\mathbf{R}^3$ of the same topological type are regularly homotopic if and only if they are cobordant (see [15]). This subject received recently more attention (see e.g. [13]).

The group P_n of codimension-one immersions in the n-sphere up to cobordism is the the n-th stable homotopy group of $\mathbf{RP}^\infty$ and they were computed by Liulevicius ([11]) for $n \leq 9$. Explicit classifications for regular homotopy equivalence of immersed surfaces in 3-manifolds were first obtained by Hass and Hughes in [8] and Pinkall (see [15]).

The cobordism group $N(M^3)$ of immersed surfaces in the 3-manifold M^3 was computed geometrically by Benedetti and Silhol ([1]). Let M^3 be a compact oriented 3-manifold and f a smooth codimension-one immersion of a (compact) surface F^2 in M^3. Fixing a Spin structure on M^3 one has a Pin structure induced on F^2 which defines a $\mathbf{Z}/4\mathbf{Z}$-valued quadratic form on $H_1(F^2, \mathbf{Z}/2\mathbf{Z})$, by counting how the immersion f twists the regular neighborhoods of 1-cycles in F^2. There is then an isomorphism between $N(M^3)$ and $H_1(M^3, \mathbf{Z}/2\mathbf{Z}) \oplus H_2(M^3, \mathbf{Z}/2\mathbf{Z}) \oplus \mathbf{Z}/8\mathbf{Z}$, the last being endowed with a twisted product. The isomorphism sends an immersion f into the triple consisting of the homology class of the double points locus, the homology class of the image of f, and the Arf invariant of the quadratic form from above. A similar result holds for nonorientable 3-manifolds M^3, but the factor $\mathbf{Z}/8\mathbf{Z}$ is replaced now by $\mathbf{Z}/2\mathbf{Z}$ (see [5]). Notice that the factor $\mathbf{Z}/8\mathbf{Z}$ is nothing but $N(S^3)$ so the two results above can be stated in a unitary way by considering $H^3(M^3, \mathbf{Z}/8\mathbf{Z})$.

One would like to have a similar description for the group $N(M^n)$ in all dimensions n. More motivation for that is the result of [3] which relates the cobordism group $N(M^n)$ to the set $CB(M)$ of cubulations of the manifold M^n modulo a set of combinatorial moves analogous to Pachner's move on simplicial complexes. This is improved in the present paper where it is shown that the two groups are actually isomorphic. We refer to [3] for an extensive discussion of this problem, due to Habegger (see problem 5.13 from [9]).

One remarks first that there is a natural grading gr^* on $N(M^n)$ induced by a cellular decomposition. The Atiyah-Hirzebruch spectral sequence (see [6]) has its second term $E_2^{p,q} = H^q(M^n, P_p)$ and converges to the graded $N(M^n)$. However one has only very few informations about the differentials in this sequence, hence the direct use of this approach fails. One develops then a combinatorial way to settle this question.

One obtains thus the extension of the computations of the graded group $gr^*N(M^n)$ up to dimension 4, and up to dimension 7 under a mild homological condition. However the techniques one uses are different from those of Benedetti and Silhol though as they are still geometric in nature.

The main theorem is the following:

Theorem 1.1. *Let M be a closed n-manifold, $n \leq 7$. Then*

$$gr^*(N(M)) = H^1(M, P_1) \times EH^2(M) \times H^3(M, P_3) \times \cdots \times H^n(M, P_n).$$

The subgroup $EH^2(M) \subseteq H^2(M, P_2)$ is defined in section 6.1 and is computed for any $n \leq 4$, and in some cases for $n = \{5, 6, 7\}$, as follows:

$$EH^2(M) = \begin{cases} H^2(M, P_2) & \text{if } n \leq 3 \text{ or } n = 4 \text{ and } M \text{ non-orientable} \\ \{x \in H^2(M, P_2);\ x \cup x = 0\} & \text{if } n = 4 \text{ and } M \text{ orientable or } n > 4 \text{ and condition } (*) \text{ holds} \end{cases}$$

where the condition $(*)$ for n-manifold for $n = 5, 6, 7$ is that M is orientable and $Ext(H_3(M), \mathbf{Z}/8\mathbf{Z}) = 0$.

The whole theory is explicitly developed for closed manifold. However the present methods can be applied to non-compact manifolds simply by substituting the ordinary cohomology with the cohomology with compact support.

A sketch of proof. We briefly summarize the guiding line of the paper. We introduce in section 3 the natural filtration of $N(M)$ that gives rise to the graded group, and prove that F^k can be interpreted geometrically as the subgroup of $N(M)$ of those immersions avoiding the k-skeleton up to cobordism. This property is independent of the cellular decomposition. In section 4 we define in a geometric way an injective homomorphism

$$\widetilde{\chi}^k : F^k/F^{k+1} \longrightarrow H^k(M, P_k)/NEH^k(M)$$

where $NEH^k(M) \subseteq H^k(M, P_k)$ is a subgroup of cocycles related to some particular null-cobordant immersions. We then introduce an obstruction theory that permits to study the inverse map to $\widetilde{\chi}^k$, in particular to determine its image. In section 6.1 the theory is applied to explicit computations that provide the image of $\widetilde{\chi}^k$ up to $n = 7$, and in section 6.2 it is proven that $NEH^k(M) = 0$ for all k in all n-manifold M up to $n = 7$. These computations prove the main theorem.

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2 The groups P_n and Q_n in low dimension

In this paper we mainly deal with the cobordism group of immersions in manifolds of dimension less or equal than 7. This is due to the fact that the groups $P_n := N(S^n) = N(\mathbf{R}^n)$ are particularly simple for $n < 7$, as is shown in the following table (see [11]):

n	1	2	3	4	5	6	7	8	9
P_n	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/8\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/16\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 3}$	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 4}$

The simplest case is $n = 5$, but the cases $n = 1, 2, 4, 6$ are also very easy to handle. Indeed consider the classical invariant

$$\theta_n : P_n \longrightarrow \mathbf{Z}/2\mathbf{Z}$$

that associates to an immersion the number of n-tuple points modulo 2, or equivalently, the homology class modulo 2 of the set of n-tuple points (as an element of $H_0(S^n, \mathbf{Z}/2\mathbf{Z})$). It is well-known that θ_n is an isomorphism for $n = 1, 2, 4, 6$ (see for example [2]). The group P_2 is generated by the immersion 8 which looks like the figure eight in the plane, while the group P_4 is generated by an immersion of S^3 with a single quadruple point.

Proposition 2.1. *Let $f : F \longrightarrow S^n$ be a a codimension-one immersion, and let n be such that either θ_n is an isomorphism or P_n is trivial. Then f bounds an immersion in a manifold whose boundary is S^n if and only if it represents the trivial element of P_n.*

Proof. If f is trivial in P_n it bounds in $S^n \times I$ which is a collar of the boundary of D^n. Vice versa suppose that there exists a $(n+1)$-manifold N with boundary S^1 and a generic codimension-one immersion g in N transverse to the boundary and such that $g \cap \partial N = f$. If $P_n = 0$ then f bounds in D^n. Assume then that θ_n is an isomorphism. This means that f represents the trivial element of P_n if and only if it has an even number of n-tuple points. But this follows from the fact that the curve of n-tuple points of g bounds the set of n-tuple points of f. □

This proposition does not apply, for example, for $n = 3$. The group $P_3 = \mathbf{Z}/8\mathbf{Z}$ is generated by the left Boy immersion, which has a single triple point. The even elements have no triple points. Canonical representatives for these classes are the immersions in $\mathbf{R}^3$ obtained by rotating an 8 on the xz plane with the double point in $(1, 0, 0)$ around the z axis, while rotating it in its own plane of half a twist, a whole twist and three halves of twists (see [15]). These immersions, whose invariants are 2,4 and 6 respectively, have a circle of double points. The immersion similarly constructed that makes no rotations is null-cobordant. Define the group Q_n to be the subgroup of P_n of those immersions in S^n which bound in a $(n+1)$-manifold with boundary S^n. From proposition 2.1 it immediately follows:

Corollary 2.2. *For any n the group Q_n is contained in* $\ker \theta_n$.

In computing Q_3 we will make use first of a natural way of producing codimension-one immersions.

Definition 2.3. Let N be a n-manifold (possibly with boundary) and S an embedded codimension-k submanifold. Assume that the structure group normal bundle ν to S in M can be reduced to the group of symmetries of an element $f \in P_k$, and such a reduction is chosen. There is then a canonical embedding of f in each fiber $\mathbf{R}^k$ of ν giving rise to a sub-fibration of ν with fiber f. The total space of the last fibration is an immersion in the tubular neighborhood of S which has fiber f and will be called the *immersion obtained by decorating S with f.*

For example in an orientable n-manifold any simple closed curve has trivial normal bundle, hence it can be decorated by any element of P_{n-1}.

Proposition 2.4. $Q_3 = 2P_3$.

Proof. Let M be the non-orientable bundle in S^3 on S^1, let e^4 be a 4-ball in M intersecting S^1 and $N = M \setminus int(e^4)$. Let γ be $S^1 \setminus (int(e^4) \cap S^1)$. Its normal bundle is trivial hence one can decorate it with left Boy immersions. Then the resulting immersion g is such that its intersection f with $\partial e \cong S^3$ is not trivial. When seen from e^4 (with any of its possible orientations) the two connected components of f have the same orientation. Hence f determines a non-trivial element of P_3 that has invariant 2 or 6, according to the orientation one have chose on ∂e. But it is clear from its construction that f bounds in $N = M \setminus int(e^4)$. □

The proof of this proposition easily extends to the following statement:

Proposition 2.5. *For any n one has $2P_n \subseteq Q_n$. Moreover any $f \in 2P_n$ bounds in any non-orientable $(n+1)$-manifold with boundary S^n.* $\square$

Set then $\mathcal{P}_n = P_n/Q_n$ for the group of immersions in S^n up to cobordism in manifolds bounding two spheres. We proved the following:

n	1	2	3	4	5	6
$\mathcal{P}_n$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}/2\mathbf{Z}$

Remark 2.6. The immersion with invariant 4 bounds in an orientable 4-manifold. Let L be a sphere corresponding to a complex line in $\mathbf{CP}^2$. There exists a normal vector field on L with a single zero. Let e^4 be a 4-ball containing this zero. The normal field then trivializes the normal bundle to $L \setminus (L \cap e^4)$ in $N = \mathbf{CP}^2 \setminus int(e^4)$, so one can define the immersion g obtained by decorating $L \setminus (L \cap e^4)$ with an 8, by means of this trivialization. A straightforward computation shows that $f = g \cap \partial e^4$ restricted to the boundary of e^4 is an immersion with Arf invariant 4, and clearly f bounding in N.

3 A natural filtration of N(M)

The point of view from which we are able to tackle the computation of $N(M)$ is that of splitting it in pieces. At first sight the splitting depends on the cellular decomposition of M.

Definition 3.1. Let M be a n-manifold and let

$$M_0 \subseteq \cdots \subseteq M_k \subseteq \cdots \subseteq M_n$$

be a skeleton decomposition. Let $F^k \subset N(M)$, for $k \geq 1$, be the set of immersions that up to cobordism do not intersect M_{k-1}. An immersion f whose class belongs to F^k will be said *k-admissible* if $f \cap M_{k-1} = \emptyset$.

Remark that F^k is a subgroup of $N(M)$ hence one has a filtration of $N(M)$. One has

$$N(M) = F^0 \supseteq \cdots \supseteq F^k \supseteq \cdots \supseteq F^n$$

where F^0 was added for convenience of notation. This filtration comes in fact in a natural way from the algebraic-topological definition of $N(M)$.

Proposition 3.2. *Under the Pontryagin-Thom construction we have*

$$F^k = \{\varphi \in [M, Q\mathbf{RP}^\infty] \text{ such that } \varphi_{|M_{k-1}} \sim *\} = \ker([M, Q\mathbf{RP}^\infty] \to [M_{k-1}, Q\mathbf{RP}^\infty])$$

where $\sim$ means homotopy and $*$ is the trivial based loop.

Proof. If an immersion f does not intersect M_{k-1} then the map φ_f associated by the Pontryagin-Thom construction is constant on M_{k-1}. On the other side given a map φ that is null-homotopic when restricted to M_{k-1} consider the homotopy

$$H : M_{k-1} \times I \longrightarrow Q\mathbf{RP}^\infty$$

such that $H(-,0) = \varphi_{|M_{k-1}}$ and $H(-,1)$ is the constant map on M_{k-1}. The inclusion of the $(k-1)$-skeleton being a cofibration implies (see [14]) that H extends to $M \times I$. Thus it gives a homotopy between φ and a map φ' defined on all of M, whose restriction to M_{k-1} is constant. Consider a closed regular neighborhood N of M_{k-1} in M. There exists then a global retraction $r : M \to M$ so that $r(N) = M_{k-1}$. The map $\varphi'' = \varphi' \circ r$ is thus constant on N (in particular on ∂N) and is homotopic to φ. Therefore the Thom-Pontryagin construction for the manifold with boundary $M - int(N)$ associates to the map $\varphi''_{|M\setminus int(N)}$ an immersion in $M \setminus int(N)$. When looking at that immersion as contained in M it results as a k-admissible representative of the class associated to φ, hence the claim is proved.

We give a second proof introducing a recursive technique that will be often exploited in the sequel. We assume that the cellular decomposition is in particular a cubulation, see section 8 for a precise definition.

Consider a class with a representative immersion f for which the map φ_f defined by the Pontryagin-Thom construction belongs to $\ker(N(M^n) \to [M_{k-1}, Q\mathbf{RP}^\infty])$. We want to deform f up to cobordism in such a way that the new representative does not intersect M_{k-1}. There exists a homotopy H of M so that the restriction of $\varphi_f \circ H|_{M_{k-1}}$ is null-homotopic. One uses now a recurrence on the degree k. If $k = 1$ then it is obvious since the immersion can miss the 0-skeleton by general position. Assume the claim is true for degree at most $k-1$. Then there exists a representative immersion so that $f \cap M^n_{k-2} = \emptyset$. This means that the intersection $f \cap e^{k-1}$ with any $(k-1)$-cell is a closed immersed submanifold lying in the interior of the cell. By hypothesis we can assume that $\varphi_f|_{M_{k-1}} = 1$, where 1 denotes the constant (trivial) map. This means that $\varphi_{f\cap e^{k-1}} = \varphi_f|_{e^{k-1}} = 1$. The Thom-Pontryagin theory implies that the immersion $f \cap e^{k-1}$ is null cobordant. Consider a small regular neighborhood V of e^k in M, which is a product $e^k \times B^{n-k}$. Since $f \cap e^{k-1}$ is null cobordant there exists an immersion g in $e^{k-1} \times I$ providing a null-cobordism for $f \cap e^{k-1}$. One uses $g \times \partial B^{n-k}$ to change the immersion in V so that the new immersion misses e^{k-1}. □

Now it is a classical result that such a filtration is independent on the cellular decomposition. Indeed $[\cdot, Q\mathbf{RP}^\infty]$ is the 0-th degree of the generalized cohomology theory associated to the suspension spectrum of $Q\mathbf{RP}^\infty$

$$h^q(X) = \lim_{n\to\infty} [\Sigma^n X, \Sigma^{n+q}\mathbf{RP}^\infty] \qquad , q \in \mathbf{Z},$$

and so the Atiyah-Hirzebruch spectral sequence converges to the graded group associated to the filtration of proposition 3.2. The filtration being independent on the cellular decomposition for realizable generalized cohomology theories is then illustrated in the first and third chapter of the book of Hilton [6].
From now on we will often choose to use cubical decompositions of the manifold M. This will permit to perform recursive constructions in an easier way because a cubulation of M induces in an obvious way one for $M \times I$.

4 Cohomological invariants

Recall that a k-admissible immersion f is such that $f \cap M_{k-1} = \emptyset$. In particular for any k-cell e^k the intersection $f \cap e^k$ is contained in $int(e^k)$. If e^k is oriented then $f \cap e^k$ detects an element of P_k. One introduces then the following geometric definition. To any k-admissible immersion f there is associated a cochain $\chi^k_f \in C^k(M, P_k)$ the following way:

$$\chi^k_f(e^k) := f \cap e^k \in P_k.$$

Proposition 4.1. *For k-admissible f the cochain χ^k_f is a cocycle.*

Proof. Given a $(k+1)$-cell e^{k+1} recall that $\delta\chi^k_f(e^{k+1}) = \chi^k_f(\partial e^{k+1})$ holds. Now denote $\partial e^{k+1} = \sum_{i\in\mathcal{I}(e^{k+1})} \varepsilon_i e^k_i$ where $\mathcal{I}(e^{k+1})$ is a finite set, e^k_i are oriented k-cells (not necessarily different from each other) and $\varepsilon_i = \pm 1$.

Consider the cell as a closed $(k+1)$-disk attached to M_k by means of an attaching map, that results to be a homeomorphism when restricted to any connected component of the preimage of any e_i^k. Since $f \cap M_{k-1}$ is empty one can then pull back $f \cap e^{k+1}$ in the disk. The restriction of the resulting immersion to the boundary of the disk is then

$$\sum_{i \in \mathcal{I}(e^{k+1})} \varepsilon_i (f \cap e_i^k) = \sum_{i \in \mathcal{I}(e^{k+1})} \varepsilon_i \chi_f^k(e_i^k) \in N(S^k)$$

and is trivial since it bounds in the disk. But this is $\chi_f^k(\partial e^{k+1})$, hence χ_f^k is a cocycle. □

The argument of the previous proof will be repeatedly used. It is easy to visualize it when the cellular decomposition is a cubulation. For an immersion f and a $(k+1)$-cube e^{k+1} the intersection $f \cap e^{k+1}$ is a cobordism to the empty set of $f \cap \partial e^{k+1}$. Thus the last one is trivial as an element of P_k. When f is k-admissible $f \cap \partial e^{k+1}$ splits as the sum (with signs) of χ_f^k(faces).

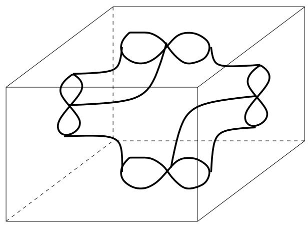

Figure 1: A k-admissible immersion restricted to a $(k+1)$-cube

Assume from now on that the cellular decomposition is a cubulation. It is easy to see that if f and f' are k-admissible immersions admitting a cobordism g, which does not intersect $M_{k-1} \times I$, then $\chi_f^k = \chi_{f'}^k$. In fact for any k-cube e^k the intersection $g \cap (e^k \times I)$ is a cobordism between $f \cap e^k$ and $f' \cap e^k$. The most natural question is whether f and f' admit a k-admissible cobordism, that is a cobordism not intersecting $M_{k-2} \times I$. This leads to the following result:

Proposition 4.2. *Let f and f' be k-admissible cobordant immersions admitting a cobordism that is k-admissible in $M \times I$. Then χ_f^k and $\chi_{f'}^k$ are cohomologous.*

Proof. Assume f and f' transverse to the cubulation, and take a cobordism g between them that is transverse to the standard cubulation of $M \times I$ associated with the cubulation of M. Our aim is to define a coboundary between χ_f^k and $\chi_{f'}^k$ by means of the same definition we used for χ^k, but now applied to g. Remark that for any k-cube e^k the intersection $g \cap (e^k \times I)$ is a cobordism to the empty set of $g \cap \partial(e^k \times I)$, thus the last one is then a trivial element in P_k. Since g is k-admissible this element splits as the sum of $f \cap e^k - f' \cap e^k = \chi_f^k - \chi_{f'}^k$ and of $g \cap (\partial e \times I)$, see figure 1. We claim that the last summand is the coboundary of a $(k-1)$-cochain of M.

Define a cochain ψ in $C^{k-1}(M, P_k)$ this way. For any oriented $(k-1)$-cell e^{k-1} consider the class of the immersion $g \cap (e^{k-1} \times I)$ where the orientation of $e^{k-1} \times I$ is such that it induces on e^{k-1} the opposite of its orientation. With this convention $g \cap (e^{k-1} \times I)$ is a well-defined element of P_k.

It is then easy to see that $\delta\psi(e^k) = g \cap (\partial e^k \times I)$, hence $\delta\psi = \chi_{f'}^k - \chi_f^k$. □

It is immediate that two cobordant immersions which are 1-admissible have a 1-admissible cobordism between them. For general k we are not able to prove the analogous statement. We will face this problem gradually. If g is a generic cobordism between k-admissible immersions, and e^s is an s-cube of M we denote by $g(e^s)$ the immersion $g \cap (e^s \times I)$, where the orientation of $e^s \times I$ is such that it induces on e^s the opposite of its orientation. Now if g is s-admissible then $g(e^s)$ is actually an immersion in the open $(s+1)$-disk $int(e^s) \times (0,1)$, hence $g(e^s)$ represents an element of P_{s+1}. Further if $g(e^{k-2})$ is empty for any $(k-2)$-cube then g is k-admissible.

Proposition 4.3. *Let f and f' be 2-admissible cobordant immersions. Then there exists a 2-admissible cobordism between them.*

Proof. Let g be a generic cobordism between f and f', transverse to the cubulation of $M \times I$ associated to a chosen cubulation of M. We want to prove that we can deform g until $g(e^0) = \emptyset$ for any e^0.

Remark that $g(e^0)$ represents an element of P_1. We claim that, up to modify g if M is compact, this element is trivial for any e^0. First remark that if e^0 and f^0 are two vertices of the cubulations that are connected by a edge e^1, then $g(e^0)$ and $g(f^0)$ are the same element of P_1. This is because $g(e^1)$ provides a cobordism between them and this proves in fact, M being connected, that there is a well-defined $g^0 \in P_1$ such that $g(e^0) = g^0$ for any vertex e^0 of M. Now suppose M is not compact. Since the domain of g is compact its image cannot intersect all edges of the type $e^0 \times I$, hence g^0 is the trivial element of P_1, and the claim is proved in this case. If M is compact, then g^0 might be non-trivial. But then consider a new cobordism obtained by adding a $M \times \{t\}$ to g. Call g the new cobordism and now g^0 is trivial as required.

So we are ready to get rid of intersections of type $g(e^0)$. For any vertex e^0 there is a diffeomorphism of a neighborhood $U(e^0) \times I$ with $B^n \times I$ such that $g \cap D^n \times I$ is the inclusion of an even number of disks at levels $p_1, \dots p_{2n}$, since g can be assumed to be transverse to $e^0 \times I$. In this model cut the corresponding disks of radius 1/2, and connect the holes in pairs by means of cylinders $1/2 S^{n-1} \times [p_{2i-1}, p_{2i}]$. The immersion obtained by repeating this construction in any vertex and then smoothing, that we still call g, satisfies $g(e^0) = \emptyset$ for any vertex e^0, hence g is 2-admissible. □

In general, given a s-admissible cobordism g the elements $g(e^{s-1})$ for $s > 1$ might be nontrivial. If they are trivial however it is possible to deform g to a $(s+1)$-admissible cobordism. Consider the standard cubulation of $M \times I$ associated to a cubulation of M. We will say that cubes of the form $e \times I$ are *vertical*, cubes of the form $e \times \{0\}$ are *at the bottom* and cubes of the form $e \times \{1\}$ are *at the top*. In general cubes of the form $e \times \{t\}$ are *horizontal*.

Lemma 4.4. *Let f and and f' be $(s+1)$-admissible immersions, let g be a cobordism between them that is s-admissible and such that for any $(s-1)$-cell e^{s-1} of M the immersion $g(e^{s-1})$ represents a trivial element of P_s. Then g can be deformed to a $(s+1)$-admissible cobordism g'.*

If g is a s-admissible cobordism between f and f', but f alone is $(s+1)$-admissible, then g can be modified to a $(s+1)$-admissible cobordism g' between f and an immersion coinciding with f' outside a neighborhood of M_s.

Proof. By a construction analogous to that of proposition 4.3 it is possible to get rid of intersections. We put again ourselves in a model, as follows: the normal bundle to e^{s-1} in M is trivial, take a trivialized neighborhood $U(e^{s-1}) = e^{s-1} \times B^{n-s+1}$ and take its product with the interval I. From transversality one can suppose that $g \cap U(e^{s-1}) \times I$ has the structure of a product $g(e^{s-1}) \times B^{n-s+1}$. Consider a cobordism to the empty set of $g(e^{s-1})$, let h be the embedding of this cobordism in $e^{s-1} \times 1/4B^1 \times I \subset U(e^{s-1}) \times I$ and take the product $h \times 1/2S^{n-s-1}$. Remark that the image of this product does not intersect $e^{s-1} \times I$, and that it intersects g in $g(e^{s-1}) \times 1/2S^{n-s}$. Thus one can excise $g(e^{s-1}) \times 1/2B^{n-s+1}$ and glue back $h \times 1/2S^{n-s}$. After repeating this construction for any $(s-1)$-cube and smoothing one gets a cobordism g' between f and f' satisfying $g'(e^{s-1}) = \emptyset$ for any e^s.

If f and f' are both $(s+1)$-admissible then g is clearly $(s+1)$-admissible, since the s-skeleton of $M \times I$ are the vertical $e^{s-1} \times I$ plus the two horizontal copies of M_s.

If f is $(s+1)$-admissible, but f' intersects the s-skeleton, then we deform g' further. Remark that for any s-cube e^s of M the cobordism $g' \cap (e^s \times I)$ is a cobordism to the empty set for $f' \cap e^s$. Hence this last immersion represents the trivial element of P_s. Then a surgery similar to the one of the first part of the proof leads to the excision of all of the intersections $g' \cap (e^s \times \{1\})$, and this proves the claim. □

Since we cannot claim than any null-cobordant k-admissible immersion admits a k-admissible cobordism to the empty set one introduces the following definition.

Definition 4.5. Set $NEH^k(M)$ for the subset of $H^k(M, P_k)$ of those cohomology classes represented by some k-admissible null-cobordant immersions.

Lemma 4.6. *$NEH^k(M)$ is a subgroup of $H^k(M, P_k)$.*

Proof. Given α and β cohomology classes represented by k-admissible null-cobordant immersions it is obvious that $\alpha + \beta$ is represented the same way.

As for $-\alpha$, let f be a k-admissible immersion such that $\chi_f^k = \alpha$, and let g be a cobordism to the empty set of f. Let g' be the cobordism between f in $M \times \{1\}$ and the empty set in $M \times \{0\}$ obtained by composing g with the reflection of I given by $t \mapsto 1 - t$. In a single $e^k \times I$ consider the following construction. Put a representative immersion of $-\alpha(e^k)$ in $e^k \times \{0\}$, put in $e^k \times \{1/3\}$ the same representative plus two copies of $f \cap e^k$ slightly isotoped, and in $e^k \times \{2/3\}$ a single copy of $f \cap e^k$. Then fill $e^k \times [1, 1/3]$ with $-\alpha(e^k) \times [1, 1/3]$ plus two copies of $g' \cap (e^k \times I)$ rescaled of $1/3$, fill $e^k \times [1/3, 2/3]$ with a cobordism between $-\alpha(e^k)$ plus a copy of $f \cap e^k$ and the empty set and with $f \cap e^k \times [1/3, 2/3]$, and finally fill $e^k \times [2/3, 1]$ with $g' \cap (e^k \times I)$ (rescaled of $1/3$). Remark that this immersion is not a cobordism between $-\alpha(e^k)$ and $\alpha(e^k)$, since g and g' possibly intersect $(\partial e^k) \times I$. Now remark that the collection of immersions so defined glue together to a cobordism in $M_k \times I$, that restricted to $M_k \times \{0\}$ represents $-\alpha$. Consider on $M \times \{1\}$ the empty immersion and in $M_k \times I$ the collection of immersions defined before. This cobordism can be completed by lemma 4.7 to a cobordism $\tilde{g}$ between an immersion $\tilde{f} = \tilde{g} \cap (M \times \{0\})$ representing $-\alpha$ and the empty set, hence $-\alpha \in NEH^k(M)$. □

The following lemma, that provides the technical step of the previous proof, will be repeatedly used in this section. It is an easy algebraic-topological argument, that has however an important geometric interpretation.

Lemma 4.7. *An immersion g traced in $M \times \{0\} \cup M_s \times I$ extends to an immersion traced in the whole of $M \times I$. Moreover if g is l-admissible, for $l \leq s$, the extension is still l-admissible.*

Proof. If we see an immersion traced in $M \times \{0\} \cup M_s \times I$ as a continuous map from $M \times \{0\} \cup M_s \times I$ to $Q\mathbf{RP}^\infty$ this immediately follows from the fact that (M, M_s) is a cofibration. This construction has however an easy geometrical interpretation, see figure 2. Consider a $(s+1)$-cube e^{s+1}. We want to extend the immersion traced on $g \cap (e^{s+1} \times \{0\} \cup \partial e^{s+1} \times I)$ to $e^{s+1} \times I$. First remark that the resulting immersion on the boundary of the cube at the top is null-cobordant, since it bounds in the disk $e^{s+1} \times \{0\} \cup \partial e^{s+1} \times I$, hence a null-cobordism can be traced on the cube at the top. The resulting immersion in $\partial(e^{s+1} \times I)$ represents an element of P_{s+1}. Up to adding (in the interior of the cube at the top) another immersion we can assume this element is trivial, hence g can be extended to the interior of $e^{s+1} \times I$. Recursively g is extended to $M \times I$.

The second statement follows obviously from the construction. □

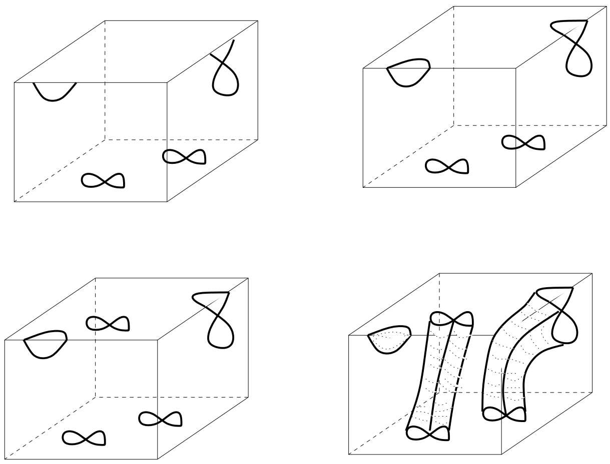

Figure 2: Extending an immersion to the interior of a cube with a free face

We prove that χ^k is a well-defined invariant in the group $H^k(M, P_k)/NEH^k(M)$.

Proposition 4.8. *Let f and f' be k-admissible cobordant immersions. Then $\chi^k_f - \chi^k_{f'}$ represents an element in $NEH^k(M)$, hence χ^k is a well-defined cobordism invariant in the group $H^k(M, P_k)/NEH^k(M)$.*

Proof. The technical step is to modify g, far from $M \times \{0\}$, in such a way that it doesn't intersect $M_{k-2} \times I$. This is only possible, in general, up to modifying $g \cap M \times \{1\}$. By proposition 4.3 we might assume that g is 2-admissible. Assume by a recurrence hypothesis that g is a s-admissible cobordism between f and $f' + f_0$, where f_0 is a (null-cobordant) s-admissible immersion.

We want to deform g in such a way that $g(e^{s-1})$ becomes trivial for any e^s, in order to apply lemma 4.4. We build up an auxiliary immersion $\tilde{g}_s$ in $M \times I$. On the vertical cells $e^{s-1} \times I$ put a copy $g(e^{s-1})$, and on the bottom cells $e^j \times \{0\}$ fix the empty immersion for any j. Consider then the resulting collection of immersions as an immersion in $M \times \{0\} \cup M_{s-1} \times I$ and extend it to a cobordism $\tilde{g}_s$ by means of lemma 4.7. Then $\tilde{g}_s$ satisfies the following properties:

$$\begin{aligned} &\tilde{g}_s(e^{s-1}) = g(e^{s-1}) \\ &\tilde{g}_s \cap M \times \{0\} = \emptyset \\ &\tilde{g}_s \cap M_{s-2} \times I = \emptyset. \end{aligned}$$

Remark now that there exists an integer a (the order of P_s minus 1) such that $g + a\tilde{g}_s$ has the property that $(g + a\tilde{g}_s)(e^{s-1})$ is trivial for any e^{s-1}. Call again g this cobordism and apply lemma 4.4. The resulting cobordism g' between f and $g' \cap M \times \{1\}$ is then $(s+1)$-admissible, and $g' \cap M \times \{1\}$ is of the form $f' + f'_0$ where f'_0 is (null-cobordant and) $(s+1)$-admissible.

Repeat this construction until $s = k - 1$ and the resulting k-admissible cobordism, that we again call g, is such that

$$g \cap M \times \{1\} = f' + f_0$$

where f_0 is (null-cobordant and) k-admissible. Then by proposition 4.2 $\chi^k_{f'+f_0}$ differs from χ^k_f by a coboundary. But since

$$\chi^k_{f'+f_0} = \chi^k_{f'} + \chi^k_{f_0}$$

one gets the claim. □

The invariant

$$\chi^k : F^k \longrightarrow \frac{H^k(M, P_k)}{NEH^k(M)}$$

defined by these propositions will be called the *k-th cohomological invariant*.
Remark that if f is actually in F^{k+1} then χ^k_f is trivial, hence we are left with a well-defined homomorphism from F^k/F^{k+1} to $H^k(M, P_k)/NEH^k(M)$. This homomorphism is in fact injective.

Proposition 4.9. *Let M be an n-manifold. For any $k = 1, \dots, n$ the kernel of the k-th cohomological invariant is F^{k+1}.*

Proof. Fix a cubulation in M. Let f be a k-admissible immersion such that $\chi^k_f = 0 \in H^k(M, P_k)$. This means there is an element $\gamma \in C^{k-1}(M, P_k)$ such that $\delta\gamma = \chi^k_f$. One builds up a cobordism g between f and a $(k+1)$-admissible immersion.

Consider the standard cubulation of $M \times I$ associated to the given cubulation of M. Put f in the bottom $M \times \{0\}$. For any $(k-1)$-cell of M, say e^{k-1}, put in the vertical cell $e^{k-1} \times I$ (with the orientation that induces on e^{k-1} the opposite of its orientation) the element $\gamma(e^{k-1}) \in P_k$. One fixes also the cobordism on the top $M_k \times \{1\}$. Consider a k-cell e^k of M. One defines the cobordism on $e^k \times I$ by remarking that (from the definition of γ) the union of all immersions already defined in $\partial(e^k \times I)$ is null-cobordant. One can choose therefore a cobordism to the empty set. This can be done recursively on the whole of $M_k \times I$. Remark that the resulting immersion g does not intersect $M_k \times \{1\}$.

Now g is defined on $M \times \{0\} \cup M_k \times I$, and applying lemma 4.7 gives rise to a cobordism g in $M \times I$, that provides a cobordism between f and $g \cap (M \times \{1\})$; and this last does not intersect the k-skeleton, by construction.

A similar construction can be performed if $\chi^k_f \in NEH^k(M)$. Let f_0 be a null-cobordant map such that $\chi^k_{f_0} \sim \chi^k_f$. Put f in the whole bottom $M \times \{0\}$, $f_0 \cap M_k$ in the intermediate $M_k \times \{1/2\}$ and trace on the vertical $M_k \times [1/2, 1]$ the intersection of a cobordism to the empty set of f_0. The $(k-1)$-cochain that cobounds $\chi^k_{f_0}$ and χ^k_f provides as before a cobordism between $f \cap M_k$ and $f_0 \cap M_k$, which we put in the vertical $M_k \times [0, 1/2]$. Over all this is a cobordism between $f \cap M_k$ and the empty set and so it can be extended by lemma 4.7 to a cobordism g between f and a map $g \cap (M \times \{1\})$. By construction the last one does not intersect the k-skeleton. □

Corollary 4.10. *Let M be an n-manifold. For any $k = 1, \dots, n$ the induced homomorphism*

$$\tilde{\chi}^k : \frac{F^k}{F^{k+1}} \longrightarrow \frac{H^k(M, P^k)}{NEH^k(M)}$$

is injective.

This corollary shows that the power of these new invariants is considerable. Indeed they describe the *graded group of $N(M)$ associated to the filtration*

$$gr(N(M)) = F^1/F^2 \times \cdots \times F^{n-1}/F^n \times F^n$$

as a subgroup of $H^1(M, P_1)/NEH^1(M) \times \cdots \times H^n(M, P_n)/NEH^n(M)$, that is:

Theorem 4.11. *The cohomological invariants induce an injective homomorphism*

$$\tilde{\chi} : gr(N(M)) \longrightarrow H^1(M, P_1)/NEH^1(M) \times \cdots \times H^n(M, P_n)/NEH^n(M). \ \square$$

We end this section with an important remark. The cohomological invariants reduce, under suitable hypothesis, to the restriction of James-Hopf invariants. These are classical cobordism invariants, see [7] and [10].

Definition 4.12. Let M be a n-manifold and (F, f) a generic codimension-one immersion. For $i = 1, \ldots, n$ consider the locus of $(n-i)$-tuple points of f, that is, the points of M that have a number of preimages equal or bigger than $n - i$. This set is in fact a i-cycle modulo 2, whose homology class is invariant up to cobordism. We denote $JH_i(f) \in H_i(M, \mathbf{Z}/2\mathbf{Z})$ this class and call it *i-th James-Hopf invariant.*

These invariants are particularly meaningful for those k such that P_k is non-trivial. For example given a codimension-k embedded submanifold S of an n-manifold M such that the normal bundle is reducible to the symmetry group of an element $f \in P_k$ with a single k-tuple point, the immersion obtained by decorating S with f has the homology class modulo 2 of S as k-th James-Hopf invariant. It is known ([18]) that the groups P_k are 2-torsion.

Proposition 4.13. *Assume that k is such that $\theta_k : P_k \longrightarrow \mathbf{Z}/2\mathbf{Z}$ is the reduction modulo 2 of P_k. Then for any n-manifold the $(n-k)$-th James-Hopf invariant restricted to F^k is the Poincaré dual to the reduction modulo 2 of χ^k.*

Proof. Let f be a k-admissible immersion generic and transverse to the decomposition of M. Then for any k-cell e^k of M the number $PDJH_{n-k}(e^k)$ is the number of k-tuple points of $f \cap e^k$, modulo 2, hence, by the hypothesis on k, is the reduction modulo 2 of $f \cap e^k$ as an element of P_k, that is $\theta_k(\chi^k_f(e^k))$. The following diagram then commutes

$$\begin{array}{ccc} F^k & \xrightarrow{\chi_k} & H^k(M, P_k) \\ \Big\downarrow{\scriptstyle JH_{n-k}|_{F^k}} & & \swarrow{\scriptstyle \theta_k^*} \\ H_{n-k}(M, \mathbf{Z}/2\mathbf{Z}) & & \\ \Big\downarrow{\scriptstyle PD} & & \\ H^k(M, \mathbf{Z}/2\mathbf{Z}) & & \end{array}$$

and since θ_k^* is reduction modulo 2 in cohomology the claim follows. □

This proves at once the following:

Proposition 4.14. *If k is such that θ_k is an isomorphism then $NEH^k(M) = 0$.* □

5 Obstruction theory

A more detailed study of the groups $NEH^k(M)$ is in order. From proposition 4.3 and lemma 4.4 one might guess that the vanishing of $NEH^k(M)$ is correlated with constructions that make the intersections $g(e)$ of a cobordism g with the vertical walls of $M \times I$ null-cobordant. This was made possible for example in the proof of proposition 4.8 by means of the construction of an auxiliary immersion $\tilde{g}$ with prescribed image on $M \times \{0\} \cup M_{s-1} \times I$. As we saw the possibility of obtaining such an auxiliary immersion amounts, at an accurate analysis, to the fact that (M, M_{s-1}) is a cofibration. However in order to leave the image of g fixed also in $M \times \{1\}$ we need a more delicate construction, that will be developed in section 5 and applied in section 6.2.

A second obvious motivation for developing this theory is the computation of the image of χ^k, that is, the subgroup of cohomology classes that are represented by an immersion.

In this section we describe the general obstructions for a cochain in $C^k(M, P_k)$ to be realizable as an immersion. The basic idea is a recursive construction. Given $\xi \in C^k(M, P_k)$ we first put in the interior of every k-cube an immersion representing $\xi(e^k) \in P_k$, then try to extend this codimension-one immersion in M_k to a codimension-one immersion in M_{k+1}. This will be called the *first extendibility*. If this construction can be repeated until the n-th skeleton i.e. the immersion can be further extended to a *second extension*, and so on then the original cochain is said to be *realizable* or *extendible*. If one can reach the s-th stage one says that the cochain is *s-extendible*.

We adapt to this context Eilenberg's obstruction theory, see [19], §V.5. A cochain in $C^k(M, P_k)$ can be thought of as a map φ_k defined from M_k to $Q\mathbf{RP}^\infty$, that restricted to any k-cell is geometrically represented by an element of P_k. The problem to which we apply Eilenberg's theory is that of extending this map over the next skeleton.

5.1 A review of obstruction theory

Given a s-simple space Y, a CW-complex X and a map $\varphi : X_s \longrightarrow Y$ the *obstruction to extending f to the $(s+1)$-skeleton* is a cochain $c^{s+1}(\varphi) \in C^{s+1}(X, \pi_s(Y))$, assigning to each $(s+1)$-cell e^{s+1} the map $\varphi|_{\partial e^{s+1}}$. Its fundamental properties are stated in the following theorem (see [19], §V.5):

Theorem 5.1. *1. φ is extendible to X_{s+1} if and only if $c^{s+1}(\varphi)$ is the trivial cochain.*

2. $c^{s+1}(\varphi)$ is a cocycle, hence represents an element of $H^{s+1}(X, \pi_s(Y))$.

3. $\varphi|_{X_{s-1}}$ is extendible to X_{s+1} if and only if $c^{s+1}(\varphi)$ is trivial in $H^{s+1}(X, \pi_s(Y))$.

The problem of further extending φ is codified in a sequence of obstruction maps. However for any extension there exists an obstruction cocycle, hence the obstruction to further extend $\varphi|_{X_{s-1}}$ becomes a set of cohomology classes. Assume that it is extendible to the $(s+l-1)$-th skeleton, and let $\mathcal{O}^{s+l}(\varphi)$ be the set of cohomology classes given by

$$\mathcal{O}^{s+l}(\varphi) := \{c^{s+l}(\varphi_{s+l-1}) | \varphi_{s+l-1} \text{ is an extension of } \varphi|_{X_{s-1}} \text{ to the } (s+l-1)\text{-}thskeleton\} \subset H^{s+l}(X, \pi_{s+l-1}(Y))$$

Theorem 5.2. *Assume the map $\varphi|_{X_{s-1}}$ is extendible to the $(s+l-1)$-skeleton. Then it is extendible to the $(s+l)$-skeleton if and only if the 0 class belongs to $\mathcal{O}^{s+l}(\varphi)$.*

5.2 Geometrical interpretation

We first recall that

$$[S^s, Q\mathbf{RP}^\infty] = P_s$$

hence the theory applies directly with coefficients in the groups P_s.

Fixed a k-cochain $\xi \in C^k(M, P_k)$ we consider it as a map defined on M_k taking values on $Q\mathbf{RP}^\infty$. Consider then the obstruction

$$\begin{array}{rccc} c^{k+1}(\xi): & C_{k+1}(M) & \longrightarrow & \pi_k(Q\mathbf{RP}^\infty) \\ & e^{k+1} & \mapsto & \xi(\partial e^{k+1}) \end{array}$$

and remark that since $\pi_k(Q\mathbf{RP}^\infty) = P_k$ the obstruction c^{k+1} is nothing but the ordinary coboundary of cochains with coefficients in P_k. Hence by property 1 of theorem 5.1 it follows that:

Proposition 5.3. *ξ is 1-extendible if and only if it is a cocycle.*

Remark 5.4. This condition has the geometrical interpretation that was already illustrated in figure 1.

Define now $\mathcal{O}_k^{k+l}$, for $l \geq 2$, to be the map that associates to $\xi \in H^k(M, P_k)$ the set $\mathcal{O}^{k+l}(\varphi_\xi) \subset H^{k+l}(X, P_{k+l-1})$, φ_ξ being any extension of ξ to the $(k+1)$-skeleton, and define ker $\mathcal{O}_k^s$ to be the subset of $H^k(M, P_k)$ of cocycles ξ such that $\mathcal{O}^s(\varphi_\xi)$ contains the trivial element of $H^s(X, P_{s-1})$. Remark that ker $\mathcal{O}_k^{s-1} \subseteq$ ker $\mathcal{O}_k^s$, and that $\mathcal{O}_k^s$ is in fact only defined on ker $\mathcal{O}_k^{s-1}$. From theorem 5.2 one obtains the following proposition:

Proposition 5.5. *$\xi \in H^k(M, P_k)$ is l-extendible if and only if it belongs to $\ker \mathcal{O}_k^{k+l}$. In particular it is realizable if and only if it belongs to $\ker \mathcal{O}_k^n$.* □

Denote by $EH^k(M)$ the subgroup of $H^k(M, P_k)$ of extendible cocycles. Proposition 5.5 translates into:

$$EH^k(M) = ker \mathcal{O}_k^n.$$

Proposition 5.6. *Let $\xi \in EH^k(M)$; then the immersion f_ξ realizing ξ is a well-defined element in F^k/F^{k+1}.*

Proof. This follows from proposition 4.9. If f' and f both realize ξ in particular they have the same k-th cohomological invariant $[\xi] \in EH^k(M)/NEH^k(M)$, so they differ by an element of F^{k+1}. □

6 Explicit computations

We apply the obstruction theory to the computation of both $EH^k(M)$ and $NEH^k(M)$ (the subgroup of $EH^k(M)$ of *null-extendible cocycles* cocycles realizable as a null-cobordant k-admissible immersion). The computations prove theorem 1.1.

6.1 Extendible cocycles

The results are summarized in the following table, where $N^k := H^k(M, P_k)/EH^k(M)$ denotes the *non-extendible cocycles.*

dim M	orientability	N^1	N^2	N^3	N^4	N^5	N^6	N^7
2	both	0	0	·	·	·	·	·
3	both	0	0	0	·	·	·	·
4	orientable	0	$H^2(M, \mathbf{Z}/2\mathbf{Z})/Q^2(M, \mathbf{Z}/2\mathbf{Z})$	0	0	·	·	·
4	non-orientable	0	0	0	0	·	·	·
5	both	0	$H^2(M, \mathbf{Z}/2\mathbf{Z})/Q^2(M, \mathbf{Z}/2\mathbf{Z})^{(*)}$	0	0	0	·	·
6	both	0	$H^2(M, \mathbf{Z}/2\mathbf{Z})/Q^2(M, \mathbf{Z}/2\mathbf{Z})^{(*)}$	0	0	0	0	·
7	both	0	$H^2(M, \mathbf{Z}/2\mathbf{Z})/Q^2(M, \mathbf{Z}/2\mathbf{Z})^{(*)}$	0	0	0	0	0

(*) Under the condition M orientable and $Ext(H_3(M, \mathbf{Z}), \mathbf{Z}/8\mathbf{Z}) = 0$.

In this table $Q^2(M, \mathbf{Z}/2\mathbf{Z})$ denotes the subgroup of $H^2(M, \mathbf{Z}/2\mathbf{Z})$ defined by

$$Q^2(M, \mathbf{Z}/2\mathbf{Z}) = \{x \in H^2(M, \mathbf{Z}/2\mathbf{Z});\ x \cup x = 0\}.$$

Remark that if M is a manifold this is the quadric of $H^2(M, \mathbf{Z}/2\mathbf{Z})$ associated with the intersection form. Moreover for any dimension n one has:

dim M	orientability	N^1	N^s	N^{n-1}	N^n
n	both	0	?	0	0

We first prove the results from the last table.

Proposition 6.1. *For any dimension n and for any n-manifold M*

$$EH^1(M) = H^1(M, P_1) = H^1(M, \mathbf{Z}/2\mathbf{Z}),$$

$$EH^{n-1}(M) = H^{n-1}(M, P_{n-1}),$$

$$EH^n(M) = H^n(M, P_n) = \begin{cases} P_n & \text{if } M \text{ is orientable} \\ P_n/2P_n & \text{if } M \text{ is non-orientable.} \end{cases}$$

Proof. Since $P_1 = \mathbf{Z}/2\mathbf{Z}$ one can use Poincaré duality with coefficients P_1 in both orientable and non-orientable context. Now represent the Poincaré dual to ξ by an embedding and remark that this embedding realizes ξ. That n and $(n-1)$-cohomology classes extend follows immediately from obstruction theory. □

Remark 6.2. This result can be interpreted geometrically. That every n-class is extendible follows easily from the fact that taking a representative cocycle ξ a putting in any n-cube e^n an immersion representing $\xi(e^n)$ already realizes the cocycle.

Let now M be an orientable n-manifold, let ξ be a cohomology class in $H^{n-1}(M, P_{n-1})$, and consider its Poincaré dual $PD(\xi)$. By the universal coefficient theorem $PD(\xi)$ can be thought of as an element of $H_1(M, \mathbf{Z}) \otimes P_{n-1}$, hence as a combination of the type $\sum \gamma_i \otimes f_i$ with $\gamma_i \in H_1(M, \mathbf{Z})$ and $f_i \in P_{n-1}$. To each $\gamma_i \otimes f_i$ we associate the following immersion. Take a simple closed loop representing γ_i. By decorating γ_i with f_i, which is always possible since the normal bundle to γ_i is trivial, one obtains an immersion realizing the Poincaré dual of $\gamma_i \otimes f_i$. Obviously the sum of such immersions realizes ξ.

Let n be such that P_{n-1} is either trivial or $\mathbf{Z}/2\mathbf{Z}$. Then if M is a non-orientable n-manifold the same construction applies. Indeed, these conditions on P_{n-1} both mean that any immersion in $\mathbf{R}^{n-1}$ (up to a cobordism) admits a reflection in its symmetry group, since $f = -f$. Hence this construction applies, since also curves with non-orientable normal bundle can be always decorated with immersions in P_{n-1}.

We are ready now to prove the results from the main table. Let us first concentrate on codimension-two cocycles. If $k = n-2$ there is only one obstruction, namely $\mathcal{O}^n_{n-2}$. This lives in the cyclic group $H^n(M, P_n)$. One describes now $\mathcal{O}^n_{n-2}(\xi)$ explicitly as an element of P_n or of $P_n/2P_n$, depending on M being orientable or not.

Lemma 6.3. *Let M be any n-manifold. Every $\xi \in H^{n-2}(M, P_{n-2})$ is extendible to an immersion defined in $M \setminus int(e^n)$, where e^n is an n-ball.*

Proof. Perform a first extension f of ξ to M_{n-1}. Remark that $M \setminus int(e^n)$ collapses simplicially on a subset S_{n-1} of M_{n-1}. Fix a way of building up $M \setminus int(e^n)$ from this subset, that is, order the set of n-cells in such a way that $e^n_{(1)}$ is attached to S_{n-1} and $e^n_{(i)}$ is attached to $S_{n-1} \cup \bigcup_{j=1}^{i-1} e^n_{(j)}$. For any i call *free face* of $e^n_{(i)}$ the one to which first a cell of M will be attached. Remark that every n-cell but e^n has a free face. When

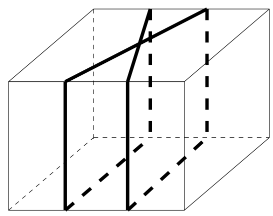

Figure 3: An immersion not extendible over the ball

attaching the first n-cell $e^n_{(1)}$ extend the immersion $f \cap \partial e^n_{(1)}$ this way. If $f \cap \partial e^n_{(1)}$ is trivial in P_{n-1} then the extension is the cobordism to the empty set, if it is non-trivial, see figure 6.1, then add a representative of $-f \cap \partial e^n_{(1)}$ on the free face of $e^n_{(1)}$. Call again f the new extension of ξ, and perform recursively the same construction. At the end one is left with f defined on $M \setminus int(e^n)$. □

Lemma 6.4. *Let M be an n-manifold, $\xi \in H^{n-2}(M, P_{n-2})$ and f be any extension of ξ to $M \setminus int(e^n)$. If M is orientable then $f \cap \partial e^n \in P_{n-1}$ depends only on ξ. If M is non-orientable then $f \cap \partial e^n \in P_{n-1}/2P_{n-1}$ depends only on ξ.*

Proof. Given any extension f of ξ to M_{n-1} the set of extension modulo cobordism relative to M_{n-2} is acted on by $C_{n-1}(M, P_{n-1})$, by the action $(\alpha * f) \cap e^n = \alpha(e^n) \cup (f \cap e^n)$. This action is transitive. Remark that if two extensions f and f' both extends to $M \setminus int(e^n)$ then their difference α is must be such that for any n-cell $e' \neq e$ it holds $\alpha(\partial e') = 0$.

Assume now that M is orientable. Then at the cochain level $\partial e = \sum_{e' \neq e} \partial e'$, then $\alpha(\partial e) = 0$, hence $f \cap \partial e = f' \cap \partial e$.

If M is non-orientable the equation $\partial e = \sum_{e' \neq e} \partial e'$ only holds modulo 2, hence one can say that $\alpha(\partial e) \in 2P_{n-1}$.

Assume now that ξ is represented by a different cocycle, hence by a different immersion in M_{n-2}. Since the two cocycles are cohomologous the two immersions are cobordant, hence there exists a cobordism in $M_{n-2} \times I$ between the two representatives. This cobordism can be extended to a cobordism between extensions to M_{n-1}, since any cube $e^{n-1} \times I$ has a free face, say, $e^{n-1} \times \{1\}$, and in an analogous way to a cobordism between two extensions f and f' to $M \setminus int(e^n)$. This proves that $f \cap \partial e^n$ and $f' \cap \partial e^n$ are cobordant.

That $f \cap \partial e$ does not depend neither on e nor on the process of collapsing is then straightforward, hence the claim. □

Proposition 6.5. *Let M be an n-manifold and $\xi \in H^{n-2}(M, P_{n-2})$. Then $\xi \in EH^{n-2}(M)$ if and only if, given any extension f of ξ to $M \setminus int(e^n)$,*

$$f \cap \partial e = 0 \in \begin{cases} P_{n-1} & \textit{if } M \textit{ is orientable} \\ P_{n-1}/2P_{n-1} & \textit{if } M \textit{ is non-orientable.} \end{cases}$$

Proof. The case M orientable follows immediately from the preceding lemmas. As for the non-orientable case recall the proof of theorem 2.4 and theorem 2.5 and remark that if $f \cap \partial e \in 2P_{n-1}$ then its opposite g bounds in $M \setminus int(e)$ a cobordism not intersecting M_{n-2}. □

Corollary 6.6. *Let n be such that $Q_{n-1} = 0$. Then for any n-manifold*

$$EH^{n-2}(M) = H^{n-2}(M, P_{n-2}).$$

More generally, if n is such that $Q_{n-1} \subseteq 2P_{n-1}$, then the same holds true for non-orientable n-manifold.

Proof. This immediately follows from proposition 6.5 and the fact that for any extension f of ξ to $M \setminus int(e^n)$ the immersion $f \cap \partial e$ belongs to Q_{n-1}. □

This yields the claimed values for N^{n-2} in all cases but for orientable 4-manifolds. In this case a geometric construction is in order.

Proposition 6.7. *Let M be an orientable 4-manifold, let $\xi \in H^2(M, \mathbf{Z}/2\mathbf{Z})$. Then $\xi \in EH^2(M)$ if and only if $x \cup x = 0$.*

Proof. Assume first that $\xi \cup \xi = 0$, that is, $\xi \in Q^2(M, \mathbf{Z}/2\mathbf{Z})$. Take a smoothly embedded representative F of $PD(\xi)$, and take a generic normal field ν to F in M. The hypothesis on ξ implies that ν has an even number of isolated, simple zeroes $z_1, \ldots, z_{2s}$. Around each zero z_i take a small disk D_i in F such that $\nu|_{\partial D_i}$ has degree 1 or -1. Then cut off all of the disks and connect the remaining holes in pairs with tubes all contained in a tubular neighborhood of F. The resulting surface F' still represents $PD(\xi)$ and admits a nowhere vanishing normal field of directions. Since the group of symmetries of the 8 in $\mathbf{R}^2$ is equal to the group of symmetries of a line, the existence of the field of directions means that it is possible to decorate F' with 8's. The resulting codimension-one immersion is in F^2 and has second cohomological invariant equal to ξ.

On the opposite direction, assume by absurd that there exists $\xi \in EH^2(M)$ with $\xi \cup \xi \neq 0$. Consider an embedded surface representing $PD(\xi)$, then choose a normal field to F. Up to changing F in its homology class we can assume as in the previous step that F admits a normal field of directions with a single isolated degenerate point z. Let e^4 be a 4-disk around z in M. One can extend ξ to an immersion defined in $M \setminus int(e^4)$ by decorating F with 8's following the normal field, call f this immersion. Now $f \cap \partial e^4$ is a non-trivial element of $Q_3 = 2P_3$. If it was trivial, the normal field of directions could be extended to the whole of F, which is not possible. By a local analysis, one can reduce to the situation of remark 2.6, hence one obtains that $f \cap \partial e^4$ is in fact the element $4 \in P_3$. But from proposition 6.5, since M is orientable, $\xi \notin EH^2(M)$. □

Remark 6.8. We showed in theorem 2.4 that $Q_3 = 2P_3$ and in remark 2.6 that $4P_3$ contains immersions that bound in an orientable manifold. The theory of this section shows that $4P_3$ is the subgroup of immersions bounding in an orientable manifold, that is, immersions with invariant 2 and 6 do *not* bound in any orientable manifold.

This settles the table for $n \leq 4$. One prove now that all obstructions involved in the table are trivial except for $\mathcal{O}_2^4$. This fact is due to the particular properties of P_s and Q_s for $s \leq 6$ and $s \neq 4$. In general, the more the groups P_s are simple the more the extensions $\mathcal{O}_k^{s+1}$ are easy to compute. The easiest case is of course $s = 5$.

Proposition 6.9. *For $k \leq 4$ the extension $\mathcal{O}_k^6$ is trivial.* □

The easiest next step is a property of the first obstruction $\mathcal{O}_k^{k+2}$ for some values of k.

Proposition 6.10. *Let k be such that θ_{k+1} is an isomorphism. Then the first obstruction $\mathcal{O}_k^{k+2}$ is trivial.*

Proof. For $\xi \in H^k(M, P_k)$ consider any first extension f to M_{k+1}. Remark that $f \cap e^{k+1}$ cannot be considered as an element of P_{k+1}, since $f \cap \partial e^{k+1}$ is not trivial. However the number of $(k+1)$-tuple points modulo 2 of $f \cap e^{k+1}$ is well-defined. Then let $\alpha \in C^{k+1}(M, P_{k+1})$ be the cochain that associates to e^{k+1} this number. Remark that the composition with θ_{k+1} induces a natural isomorphism between $C^{k+1}(M, P_{k+1})$ and $C^{k+1}(M, \mathbf{Z}/2\mathbf{Z})$. So we can consider $\alpha * f$, with the action defined in the proof of lemma 6.3. This immersion extends ξ and is extendible to M_{k+2}, since for any $(k+2)$-cell e^{k+2}

$$\theta_{k+1}((\alpha * f) \cap \partial e^{k+2}) = \theta_{k+1}(f \cap \partial e^{k+1}) + \sum_{e \in \partial e^{k+1}} \alpha(e) = 0$$

that is, $(\alpha * f) \cap \partial e^{k+2} = 0 \in P_{k+1}$. □

The triviality of almost all of the obstructions involved in the table follow then from generalizing the previous results.

The proof of proposition 6.10 actually extends to the following result, that is in fact the strongest triviality result in this section. Given an immersion however traced on a s-skeleton, if θ_s is an isomorphism then in each s-cube one can force the parity of s-tuple points to be even, and since any $(s+1)$-cube has an even number of faces, this permits extension to the $(s+1)$-skeleton.

Theorem 6.11. *Let s be such that θ_s is an isomorphism. Then the obstruction $\mathcal{O}_k^{s+1}$ is trivial, for any $k \leq s-1$.*

Proof. The action defined in the proof of proposition 6.10 can be defined on the set of extensions from any skeleton to the following one, hence the proof applies. □

On the other side, the construction of the unique obstruction for codimension-two cocycles can be performed in a more general context. Specifically, by an easy adaptation of the arguments of lemmas 6.3 and 6.4 one proves the following proposition:

Proposition 6.12. *Let n be such that $Q_{n-1} = 0$. Then for any n-manifold and any $k \leq n-2$ the last obstruction $\mathcal{O}_k^n$ is trivial. If n is such that $Q_{n-1} \subseteq 2P_{n-1}$ then the same holds for any non-orientable n-manifold.* □

We are then left to study N^2 for n-manifolds with $n > 4$, since $\mathcal{O}_2^4$ is the only nontrivial obstruction involved in the table.

Theorem 6.13. *Let M be a closed orientable n-manifold such that*

$$Ext(H_3(M, \mathbf{Z}), \mathbf{Z}/8\mathbf{Z}) = 0.$$

Then

$$EH^2(M) \subseteq Q^2(M, \mathbf{Z}/2\mathbf{Z}) \subseteq ker\mathcal{O}_2^4$$

Proof. One shows first that $Ext(H_3(M, \mathbf{Z}), \mathbf{Z}/2\mathbf{Z}) = 0$ implies $EH^2(M) \subseteq Q^2(M, \mathbf{Z}/2\mathbf{Z})$. This condition on Ext is in fact a consequence of $Ext(H_3(M, \mathbf{Z}), \mathbf{Z}/8\mathbf{Z}) = 0$. Let $\xi \in EH^2(M)$. The hypothesis on M implies by the universal coefficient theorem that $H^4(M, \mathbf{Z}/2\mathbf{Z}) = Hom(H_4(M, \mathbf{Z}), \mathbf{Z}/2\mathbf{Z})$. We want to prove that $\xi \cup \xi = 0$ by showing that for all $c \in H_4(M, \mathbf{Z})$ one obtains $(\xi \cup \xi)(c) = 0$. Fix a class $c \in H_4(M, \mathbf{Z})$, and represent it by an embedded orientable 4-submanifold F (see [16], theorem II.27). One obtains easily $i^*(\xi) \cup i^*(\xi) = 0$, where i is inclusion of F in M. In fact $i^*(\xi)$ is an extendible element of $H^2(F, \mathbf{Z}/2\mathbf{Z})$, and the claim follows from the characterization of theorem 6.7. Hence $0 = i^*(\xi) \cup i^*(\xi) = i^*(\xi \cup \xi) = PD(\xi \cup \xi)(c)$.

Now we show that $Ext(H_3(M,\mathbf{Z}),\mathbf{Z}/8\mathbf{Z})=0$ implies that $Q^2(M,\mathbf{Z}/2\mathbf{Z})\subseteq ker\mathcal{O}_2^4$. Let $\xi\in Q^2(M,\mathbf{Z}/2\mathbf{Z})$, then $\mathcal{O}_2^4(\xi)\in H^4(M,\mathbf{Z}/8\mathbf{Z})=Hom(H_4(M,\mathbf{Z}),\mathbf{Z}/8\mathbf{Z})$. For any $c\in H_4(M,\mathbf{Z})$ take as before a representative F orientable and embedded in M. Remark that $i^*(\xi)\cup i^*(\xi)=0$, hence $\mathcal{O}_2^4(i^*(\xi))=0\subset H^4(F,\mathbf{Z}/8\mathbf{Z})$. By functorial properties of obstruction cocycles (see [19] page 230) $\mathcal{O}_2^4(i^*(\xi))=i^*(\mathcal{O}_2^4(\xi))$, and this last is the set $\mathcal{O}_2^4(\xi)\subset H^4(M,\mathbf{Z}/8\mathbf{Z})=Hom(H_4(M,\mathbf{Z}),\mathbf{Z}/8\mathbf{Z})$ evaluated on $c\in H_4(M,\mathbf{Z})$ and composed with Poincaré duality. We proved therefore that this set contains 0 for any $c\in H_4(M,\mathbf{Z})$, hence $\mathcal{O}_2^4(\xi)$ contains the trivial cocycle and so ξ extends to the 4-skeleton. □

Corollary 6.14. *If M is a closed orientable n-manifold, $n\leq 7$, and $Ext(H_3(M,\mathbf{Z}),\mathbf{Z}/8\mathbf{Z})=0$ then*

$$EH^2(M)=Q^2(M,\mathbf{Z}/2\mathbf{Z}).$$

Proof. This is because for $n\leq 7$ the obstruction $\mathcal{O}_2^4$ is the only one that has not been proven to be trivial yet, hence $EH^2(M)=\ker\,\mathcal{O}_2^4$. By theorem 6.13 this is however trivial under these hypotheses, hence the claim. □

The results summarized in the table are then proved.
Finally remark that the crucial property of θ_s that makes proposition 6.10 and theorem 6.11 work can be abstracted to the following definition:

Definition 6.15. For any $f\in P_{s-1}$ denote by $N(B^s,f)$ the group of immersions g in B^s such that $g\cap\partial B^s$ are cobordant to f. We say s is *simple* if for any $f\in P_{s-1}$ there exists an isomorphism $i_f:N(B^s,f)\to P_s$ such that

$$\begin{array}{ccc} N(B^s,f)\times N(B^s,-f) & \longrightarrow & N(S^s) \\ \Big\downarrow{\scriptstyle i_f\times i_{-f}} & & \Big\downarrow{\scriptstyle\sim} \\ P_s\times P_s & \xrightarrow[\cdot]{} & P_s. \end{array}$$

Hence the following theorem holds:

Theorem 6.16. *For simple s the obstruction $\mathcal{O}_k^{s+1}$ is trivial for any $k\leq s-1$.* □

6.2 Null-extendible cocycles

We apply obstruction theory to computations concerning $NEH^k(M)\subseteq EH^k(M)$, the subgroup of cocycles that are extendible to null-cobordant immersions. We actually obtained the following satisfactory statement:

Theorem 6.17. *Let M be an n-manifold, with $n\leq 7$. Then for any $k=1,\dots,n$*

$$NEH^k(M)=0.$$

Proof. We apply proposition 4.2. We have to prove that for any $k\leq n\leq 7$ given a cobordism g between k-admissible immersions one can obtain form g a k-admissible cobordism. By lemma 4.4 it is enough to show recursively that if g is s-admissible for $s\leq k-1$ then one can obtain from it a s-admissible cobordism g' such that for any $(s-1)$-cube e^{s-1} of M the intersection $g'(e^{s-1})$ is a trivial element of P_s. Recall that $g'(e^{s-1})$ actually represents an element of P_s since g' is s-admissible. We already know by proposition 4.3 that any cobordism g between 2-admissible immersions can be considered to be 2-admissible.

Assume then that f and f' are 3-admissible immersions, and let g be a 2-admissible cobordism between them. For any 1-edge e^1 of M the immersion $g(e^1)$ represents a well-defined element of $P_2 = \mathbf{Z}/2\mathbf{Z}$. We define a 2-cocycle in $M \times I$ this way

$$\Lambda(e) = g \cap e.$$

This means that if e is horizontal $\Lambda(e) = 0$ and if e is vertical of the form $e^1 \times I$ then $\Lambda(e) = g(e^1)$. Remark that Λ is closed as a cocycle in $M \times (0,1)$, since f and f' being 3-admissible implies that $g \cap (M_2 \times \{0,1\})$ is empty.

We claim that Λ is an extendible cocycle (with compact support) in $M \times (0,1)$. If it is so, then any associated immersion h in $M \times (0,1)$ is such that $g' = g \cup h$ is a 2-admissible cobordism such that $g'(e^1) = 0$ for all e^1, hence can be deformed to a 3-admissible cobordism.

We first build a particular extension of Λ to the vertical 3-skeleton of $M \times I$ this way. Given any vertical 3-cube remark that Λ evaluates non-trivially on an even number of its 4 vertical 2-faces (possibly none). Trace a vertical 8 on any vertical face with non-trivial Λ, and connect the 8's in pairs by means of tubes whose section is a vertical 8. If all of the 4 faces are traced, the pairs must be of adjacent faces. Now remark that this first extension is further extendible to the vertical 4-skeleton. Indeed consider any vertical 4-cube. The collection of its 6 vertical 3-faces (that can be visualized as an $S^2 \times (0,1)$) contains a disjoint union of immersions each representing an element of $2P_3$. But each of these is easily seen to be the trivial element, since the top of the 8 describes a curve which bounds a disk, hence has trivial linking number with the curve of double points. Hence Λ is extendible to the vertical 4-skeleton. The following obstructions are all trivial by theorem 6.11, so Λ is extendible to $M \times (0,1)$ as was claimed.

Now given a s-admissible cobordism g between $(s+1)$-admissible immersions, $3 < s \leq 7$, define the same way the s-cochain Λ closed in $M \times (0,1)$ and by directly applying theorem 6.11 extend it to an immersion h in $M \times (0,1)$. Up to adding h an appropriate number of times one obtains a cobordism g' that is still s-admissible but such that $g'(e^{s-1}) = 0$ for all $(s-1)$-cube e^{s-1} of M, hence that can be deformed to a $(s+1)$-admissible cobordism. □

7 The group structure on orientable 4-manifold

The graded group $gr^*(N(M))$ is isomorphic to $N(M)$ as a set, but looses its group structure. We give a result concerning the group structure when M is an orientable 4-manifold.

We first remark that for any n-manifold the *total James-Hopf invariant* JH, that is, the product of the James-Hopf invariants composed with Poincaré duality, becomes a homomorphism of groups with $\oplus_{j=1}^{n} H^j(M, P_j)$ endowed with the group structure coming from that of algebra

$$(\alpha * \beta)_j = \alpha_j + \beta_j + \sum_{s+t=j} \alpha_s \beta_t.$$

In dimension 3 the invariant JH provides completely the group structure, up to immersions that are contained in a ball. These last form a subgroup that can be detected by a form of the Arf invariant of P_3 (though Benedetti and Silhol provided a deeper invariant). Up to immersions in a ball, any class can be realized as the decoration of an embedded representative and any immersion can be split in a unique way as the sum of immersions obtained by decorating a submanifold, that is, JH is surjective and injective. Neither of those properties hold for orientable 4-manifolds. Indeed from the main theorem self-intersection of 2-classes is the (only) obstruction for decorating an embedded representative. Moreover decorating a simple curve with an element of $2P_3$ provides an immersion in F^3 with trivial JH but non-trivial χ^3.

We define the map k from $H_1(M, \mathbf{Z}/4\mathbf{Z}) = H_1(M, \mathbf{Z}) \otimes \mathbf{Z}/4\mathbf{Z}$ in $N(M)$ that associates to $\gamma \otimes m$ an embedded curve decorated by an immersion with invariant $2m$. This map results to be well-defined. Moreover the image of JH is the subgroup of $\oplus_{j=1}^4 H^j(M, P_j)$ whose support is $H^1(M, \mathbf{Z}/2\mathbf{Z}) \oplus Q^2(M, \mathbf{Z}/2\mathbf{Z}) \oplus H^3(M, \mathbf{Z}/2\mathbf{Z}) \oplus H^4(M, \mathbf{Z}/2\mathbf{Z})$ (see [4] for more details). The following holds.

Proposition 7.1. *There is a short exact sequence of groups*

$$0 \to H_1(M, \mathbf{Z}/4\mathbf{Z}) \stackrel{k}{\to} N(M) \stackrel{JH}{\to} H^1(M, \mathbf{Z}/2\mathbf{Z}) * Q^2(M, \mathbf{Z}/2\mathbf{Z}) * H^3(M, \mathbf{Z}/2\mathbf{Z}) * H^4(M, \mathbf{Z}/2\mathbf{Z}) \to 0.$$

Proof. That k is injective follows from the fact that images of different cycles have different χ^3 and this last is injective. We show exactness in the middle term. That $JH(k(\alpha \otimes m))$ is trivial for any $\alpha \otimes n \in H_1(M, \mathbf{Z}) \otimes \mathbf{Z}/4\mathbf{Z}$ since any representative f of $k(\alpha \otimes m)$ with $2m$ canonical has no triple points nor quadruple points, and the locus of its double points is a surface representing the trivial element of $H_2(M, \mathbf{Z}/2\mathbf{Z})$. The whole of f retracts on the decorated curve in fact, hence also $JH_3(f)$ is trivial. Now suppose that $JH(f) = (0, 0, 0, 0)$. We must prove that f is in the image of the map k. Since $\chi^1(f) = JH_3(f) = 0$, from lemma 4.9 f belongs to F_2. Assume then that f is 2-admissible. Now f has trivial second cohomological invariant, since $\chi^2(f) = PD(JH_2(f))$, hence in particular f belongs to F^3, see proposition 4.9. Assume that f is 3-admissible and consider $\chi^3(f) \in H^3(M, P_3)$. This class has trivial reduction modulo 2, since

$$\chi^3(f)(mod\ 2) = PD(JH_1(f)) = 0,$$

so there is an element $\kappa_f \in H_1(M, \mathbf{Z}/4\mathbf{Z})$ such that $\chi^3(f) = 2PD(\kappa_f)$. It is easy to see that $f = k(\kappa_f)$. □

Remark 7.2. The image of $PD(JH_2)$ being $Q^2(M, \mathbf{Z}/2\mathbf{Z})$ implies in particular that for any pair of 1-cocycles modulo 2 α and β the relation $\alpha^2\beta^2 = 0$ holds. This fact has an elementary geometric proof. Represent the dual of α and β by means of embedded hypersurfaces A and B. Remark that if A is orientable then α^2 is 0, and if A is non-orientable its self-intersection is the orientation cycle of A, hence can be represented by an orientable surface F in A. Call C the curve intersection between F and B. Then a representative of the dual of $\alpha^2\beta^2$ is the intersection between C and B. But the normal bundle to B restricted to C is orientable since it is the normal bundle to C in F, that is travial since F is orientable, hence $C \cdot B = 0$.

8 The group of cubulations

A *cubical complex* is a complex K consisting of Euclidean cubes, such that the intersection of two cubes is a finite union of cubes from K, once a cube is in K all its faces are still in K, and no identifications of faces of the same cube are allowed. A *cubulation* of a manifold is specified by a cubical complex PL homeomorphic to the manifold. In order to apply the state-sum machinery to these decompositions we need an analogue of Pachner's theorem. Specifically, N.Habegger asked (see problem 5.13 from R.Kirby's list ([9])) the following: Suppose M and N are PL-homeomorphic cubulated n-manifolds. Are they related by the following set of moves: excise B and replace it by B', where B and B' are complementary balls (union of n-cubes) in the boundary of the standard $(n+1)$-cube? These moves will be called *bubble* moves in the sequel. Obstructions for two cubulations be equivalent have been defined in [3].

A *marked* cubulation is a cubulation C of the manifold M, endowed with a PL-homeomorphism $|C| \longrightarrow M$ of its subjacent space $|C|$, considered up to isotopy. If a bubble move is performed on C, then there is a natural marking induced for the bubbled cubulation. Thus it makes sense to consider the set $\widetilde{CB}(M)$

of marked cubulations mod bubble moves. We associate to each marked cubulation C a codimension-one generic immersion $\varphi_C : N_C \longrightarrow M$ (the cubical complex N_C is also called the derivative complex). Each cube is divided into 2^n equal cubes by n hyperplanes which we call sections. When gluing together cubes in a cubical complex the sections are glued accordingly. The union of the hyperplane sections form the image of a codimension-one generic immersion. In the differentiable case one uses a suitable smoothing when gluing the faces.

It was proved in [3] that the map $C \to \varphi_C$ induces a surjection $I : \widetilde{CB}(M) \longrightarrow N(M)$ and it was conjectured that I is injective, for which we give a simple proof below.

Theorem 8.1. *The map $I : \widetilde{CB}(M) \longrightarrow N(M)$ is a bijection.*

Proof. Consider two immersions which are cobordant in $M \times [0,1]$ and are associated to cubulations of M. One can put the cobordism g in general position with respect to the slices $M \times \{t\}$. Then for all but finitely many t the intersection $g \cap M \times \{t\}$ is a normal crossings immersion in M, whose homeomorphism type is locally constant. Thus a small perturbation near t does not affect the combinatorics of the cubulation. The other points are either the critical points of the height function or the n-multiple points of g (which one can suppose they lay at different heights). As g is normal crossings the local structure of g around the n-multiple point is that of n-hyperplanes passing through the origin in $\mathbf{R}^{n+1}$. The dual picture (at the cubulation level) is the standard cube. Up to a small isotopy the topological change which occurs at such a point is the dual of a bubble move (see also [3] p.689) at the cubulation level. On the other hand the critical points for the height function can be assumed to be isolated. Then the topological change corresponds to adding or deleting a small embedded sphere, which also can be realized on the cubulation level. □

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