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TOPOLOGICAL GEODESICS AND VIRTUAL RIGIDITY

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Abstract. We show that closed 3-manifolds whose fundamental group is word-hyperbolic, infinite and residually finite, are virtually rigid. This result is based on the notion, which we introduce, of topological geodesics in such manifolds. We also prove basic existence and uniqueness results for topological geodesics.

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A basic question in topology is to what extent the homotopy type of a manifold determines the manifold. For aspherical manifolds, in particular irreducible 3-manifolds with infinite fundamental group, conjecturally pairs of homotopy equivalent manifolds are always homeomorphic.

One of the fundamental theorems in 3-manifold topology, due to Waldhausen, asserts that this is so for so-called Haken 3-manifolds. These include irreducible manifolds with non-trivial boundary. Following Gabai [5], we prove rigidity results by deleting solid tori and reducing to the case of manifolds with boundary. Partial results along these lines have been obtained also by Dubois ([4]) which simplified the previous proof by Gabai.

We show here that a large class of 3-manifolds are *virtually rigid*, i.e., pairs of homotopy equivalent manifolds have finite covers which are homeomorphic.

Theorem 0.1. *Suppose M is a closed, irreducible 3-manifold with $\pi_1(M)$ infinite, residually finite and word-hyperbolic. Then if $f\colon M \to N$ is a homotopy equivalence, there exist finite covers M' and N' of M and N and a lift $f'\colon M' \to N'$ of f which is homotopic to a homeomorphism.*

Remark. We only use word-hyperbolicity to show the tameness of certain covers of M, which also follows under some weaker hypotheses (see section 6).

Our proof is a generalisation of that of Gabai [5] asserting the same when N is hyperbolic. While this has subsequently been strengthened to showing rigidity for such manifolds (see [6] and [7]), the methods used are rather special to hyperbolic manifolds and are unlikely to generalise.

Our main tool is the notion, which we introduce, of a *topological geodesic* in a 3-manifold. We prove basic existence and uniqueness results for these, which are perhaps of independent interest. Further applications of these will be studied elsewhere.

Suppose henceforth that M is a closed 3-manifold with $\pi_1(M)$ word-hyperbolic and infinite. In particular, the universal cover $\tilde{M}$ of M is homeomorphic to $\mathbb{R}^3$ (see [2]).

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Definition 0.1. An embedded curve γ in M is a topological geodesic if a component $\tilde{\gamma}$ of its inverse image in $\tilde{M}$ is unknotted.

Remark. This is equivalent to saying that every component of its inverse image is unknotted.

Under the hypothesis that M is word-hyperbolic, we have the following existence theorem.

Theorem (see theorem 1.2). *Let M be an irreducible 3-manifold with $\pi_1(M)$ word-hyperbolic and infinite. Then every conjugacy class in $\pi_1(M)$ is represented by a topological geodesic.*

If we further assume that $\pi_1(M)$ is residually finite, we have a uniqueness result.

Theorem (see theorem 4.3). *Let M be an irreducible 3-manifold with $\pi_1(M)$ word-hyperbolic, residually finite and infinite. Suppose c and c' are homotopic topological geodesics in M representing a primitive class in $\pi_1(M)$ (i.e., not a multiple of any other class), then there exists a finite cover M' of M such that c and c' lift to isotopic curves in M'.*

Remark. Topological geodesics arise naturally while studying elliptic 3-manifolds. For instance, whether homotopy lens spaces are lens spaces is equivalent to the existence of the analogue of a topological geodesic. Further, lens spaces can be distinguished by considering the homotopy classes of topological geodesics in a given manifold.

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1. Definition and Existence

We assume henceforth that M is a closed, irreducible 3-manifold with $\pi_1(M)$ word-hyperbolic, residually finite and infinite. We make (as in the introduction) the following definition.

Definition 1.1. An embedded curve γ in M is a topological geodesic if a component $\tilde{\gamma}$ of its inverse image in $\tilde{M}$ is unknotted.

The existence of geodesics is based on the following lemma.

Lemma 1.1. *For $\gamma \in \pi_1(M)$, let $M_\gamma = \tilde{M}/\langle g \rangle$ be the quotient of $\tilde{M}$ by the the group of deck transformations generated by γ. Then $\tilde{M}/\langle g \rangle = S^1 \times \mathbb{R}^2$*

Proof. As $\pi_1(M)$ is word-hyperbolic, the universal cover $\tilde{M}$ has a compactification to B^3, and the action by deck transformations extends to B^3. The action of γ has two fixed points p and q, and γ acts properly discontinuously on $B^3 - \{p, q\}$ with quotient $D^2 \times S^1$. The result follows as M_γ is the interior of this manifold. □

Theorem 1.2. *Let M be an irreducible 3-manifold with $\pi_1(M)$ word-hyperbolic and infinite. Then given $\gamma \in \pi_1(M)$, there is a topological geodesic c that represents γ.*

Proof. We simply take the image in M of a curve c in M_γ that is a core of the solid torus constructed above. It is easy to ensure that the image is embedded. □

Remark. The above theorem shows that there exists a topological geodesic representing every element in the fundamental group, rather than each conjugacy class of elements, which is the case for geodesics in a Riemannian manifold with negative sectional curvature. But given a geodesic representing an element, there is an obvious construction of a geodesic representing any conjugate element. So the real issue is that the topological geodesic representing a conjugacy class should be *unique*.

2. Thick tubes

In this section, we show that a geodesic c corresponding to a primitive element of $\pi_1(M)$ has a *thick tube* around it in a finite cover, i.e., a solid torus Q that contains c and so that $d(\partial Q, c)$ is large and $T - int(N(c)) = T^2 \times [0,1]$. The precise statement is below. Here, and henceforth, we fix a metric on M and use the pullback metric on all its covers

The construction is based on the fact due to Darren Long (see [8]) that in word-hyperbolic, or more generally atoroidal, groups, maximal cyclic groups are separable.

Theorem (D. Long)**.** *Let G be a residually finite group, $\gamma \in G$ an element that generates a maximal abelian subgroup, and S a finite set disjoint from $\langle\gamma\rangle$. Then there exists a subgroup $H \subset G$ with finite index such that $\gamma \in H$ but $S \cap H = \emptyset$*

We construct a thick tube in the cover $\tilde{M}/\langle\gamma\rangle = S^1 \times \mathbb{R}^2$. This embeds in a cover where all other small elements are separated from such a group. In this cover we have a short curve (geodesic) in one homotopy class, but all closed curves that are not homotopic to a power of this are long. The precise statement and proof are below.

Theorem 2.1. *Let $\gamma \in \pi_1(M)$ be a primitive element and let $K \in \mathbb{R}$. Then there is a geodesic $c \subset M$ and a finite cover M' of M such that c lifts to M' and there is an embedded solid torus Q that contains c and so that $d(\partial Q, c)$ is large and $Q - int(N(c)) = T^2 \times [0,1]$*

Proof. Consider the cover $M_\gamma = \tilde{M}/\langle g\rangle$ of M and pick base-points p of M and p' of M_γ. As M_γ is homeomorphic to $S^1 \times R^2$, there is a curve c and a solid torus Q embedded in M_γ, with c based at p', so that c and Q are as required. We shall find a finite cover M' of M which is covered by M_γ so that the image of Q embeds in M'.

As Q is compact, there is a finite collection X of inverse images of p in M_γ such that if each of these map to distinct points in an intermediate cover M' between M_γ and M, then Q embeds in this cover. Choose paths joining p' to each point in X and let S' be the set of elements in $\pi_1(M)$ that are represented by the images of these paths. Let $S = S' \cup \{xy^{-1} : x, y \in S\}$.

By the above theorem, there exists a subgroup H separating γ from S. Let M' be the corresponding cover. It is easy to see that the elements of X map to different points in M', and hence Q embeds in M'. □

Definition 2.1. A thick tube around a geodesic c is an embedded solid torus Q that contains c such that $Q - int(N(c)) = T^2 \times [0,1]$.

Observe that the components of the inverse image of a thick tube around a geodesic in $\tilde{M}$ are unknotted.

3. Virtual rigidity

Gabai's proof of virtual rigidity (see [5]) now generalises. Briefly, if there are knots K and K' in M and N whose complements are irreducible and so that, possibly after changing f by a homotopy, $f(N(K)) \subset N(K')$ and $f(M - int(N(K))) \subset N - int(N(K'))$, then we can apply Waldhausen's result to the knot exteriors. Gabai's proof proceeds by showing that given a geodesic γ in M (which plays the role of K) with a thick tube around it, the image of this contains a solid torus W (which plays the role of $N(K')$), so that f can be deformed to a map that takes $M - int(N(\gamma))$ to $N - int(W)$. Waldhausen's theorem now shows rigidity.

As f is a homotopy equivalence between compact manifolds, there exists C such that $d(x,y) \geq C \implies f(x) \neq f(y)$. By the methods of the previous section, given $C \in \mathbb{R}$ there is a finite cover M' of M, a geodesic $\gamma \subset M'$ and solid tori V_i,$1 \leq i \leq 4$ such that $d(\gamma, \partial V_1) > C$ and $d(\partial V_i, \partial V_{i+1}) > C$, $1 \leq i \leq 3$. Let $S_i = \partial V_i$. Let $V_0 = \gamma$. Replace M by M' and N by its cover with the same fundamental group as M' (under the identification $\pi_1(M) \overset{f_*}{=} \pi_1(N)$.

The rest of the proof is exactly the same as that of Gabai (hyperbolicity is not used beyond this stage - only that the tori are far apart and topologically of a standard form). We outline below the main steps.

Let $g\colon N \to M$ be the homotopy inverse of f. Let $K = f(S_2)$ and $J = N(K) \cup$ (components of $N - K$ disjoint from $f(S_1) \cup f(S_3)$). Then ∂J has two components, one of which bounds a region disjoint from J containing $f(S_1)$ and the other bounds a region disjoint from J containing $f(S_3)$. Further, J is irreducible and $[K]$ generates $H_2(J) = \mathbb{Z}$. All this follows from the fact that $d(x,y) \geq C \implies f(x) \neq f(y)$.

Next, J contains a homologically non-trivial torus which bounds in N a solid torus W containing $f(\gamma)$. This is constructed using the fact that the Thurston norm equals the singular norm, and that we have a singular torus T. Further, $g\colon T \to N - int(W)$ and the inclusion $T \to N - int(W)$ induce injections between the fundamental groups.

Now we can deform f and g so that they restrict to give a homotopy equivalence between $M - V_2$ and $N - W$. Waldhausen's theorem now gives the result.

4. Uniqueness

Lemma 4.1 (Engulfing lemma). *Any curve d in M homotopic to a geodesic c is contained in a tube around the geodesic c in some finite cover.*

Proof. As c and d are homotopic, there exists an annulus A with boundary components c and d. On passing to a cover with a sufficiently thick tube Q around c, the annulus A is contained in the tube Q. Hence d has a lift that is contained in Q. □

Lemma 4.2 (Core lemma). *If c and c' are homotopic geodesics and c' is contained in a thick tube Q around c, then c' is isotopic to c.*

Proof. Consider $\pi_1(Q - c')$. The inverse image of the solid torus Q in $\tilde{M}$ contains an unknotted cylinder D that contains a unique component C' of the inverse image of c'. The group $\pi_1(D - C') = \pi_1(\tilde{M} - C')$ is the kernel of the map $\pi_1(Q - c') \to \mathbb{Z}$ that maps the longitude of the torus ∂Q (hence those of c and c'), to 1 and the meridian to 0. As C' is an unknot (because c' is a geodesic) this kernel is $\mathbb{Z}$, hence $\pi_1(Q - c') = \mathbb{Z}^2$. This implies that c' is isotopic to c. □

Theorem 4.3. *Let M be an irreducible 3-manifold with $\pi_1(M)$ word-hyperbolic, residually finite and infinite. Suppose c and c' are homotopic topological geodesics in M representing a primitive class in $\pi_1(M)$, then there exists a finite cover M' of M such that c and c' lift to isotopic curves in M'.*

Proof. As c and c' are homotopic, there is an annulus A bounding c and c'. By passing to a cover M' with a sufficiently thick tube Q around c, we may ensure that A is contained in Q. The core lemma now implies that c and c' are isotopic. □

5. Homotopy versus isotopy

It is known that for an irreducible manifold homotopic self-homeomorphisms are isotopic provided that the manifold is hyperbolic (a consequence of Gabai's rigidity [6]) or Seifert fibered (by the Scott theorem) or lens spaces.

Theorem 5.1. *Let M be irreducible and with word hyperbolic fundamental group. If $f : M \to M$ is a homeomorphism homotopic to the identity then there is a finite cover M' of M and a lift $f' : M' \to M'$ of f such that f' is isotopic to the identity.*

Proof. Let γ be a topological geodesic correspoding to a primitive class in $\pi_1(M)$. Then the geodesics $f(\gamma)$ and γ are homotopic hence there exists a finite cover M' such that $f(\gamma)$ and γ lift to isotopic curves. Since f is homotopic to identity there is no obstruction in lifting it to a homotopy equivalence f' of the finite covering M'. One can further assume (by means of some isotopy on M') that $f(\gamma') = \gamma'$ (pointwise).

The proof of the core lemma shows that $M' - \gamma'$ is atoroidal hence hyperbolic (since Haken). Furthermore the restriction $f'|_{M'-\gamma'}$ is a homeomorphism. Therefore $f'|_{M'-\gamma'}$ is homotopic to an isometry of $M'-\gamma'$ (by Mostow rigidity) and hence isotopic to an isometry (by Waldhausen's theorem).

Let j be this isometry. Since the isometry group of a hyperbolic manifold of finite volume is finite it follows that j is of finite order. Further j has an extension g (by identity) to all of M', by asking g to keep pointwise γ'. In particular g is a periodic homeomorphism of M'.

Let consider the lift h of g to the universal covering $\widetilde{M} = \mathbf{R}^3$ (which is also the universal covering of M'). The action of h extends continuously to the compactification (over the boundary sphere) to a homeomorphism of the ball B^3. The action by deck transformations extends to one by homeomorphisms of the compactification obtained by adding the boundary of the group $\pi_1(M)$, because this is word hyperbolic. In our case the boundary is the sphere at infinity.

But f is homotopic to identity, hence the action induced on the boundary is trivial. This shows that h is the identity on the boundary sphere.

Lemma 5.2. *A periodic homeomorphism of B^3 which restricts to identity on the boundary is identity.*

Proof. We will use a theorem of Newman ([10]) improved by Smith (see [12]) in its variant stated in ([3],Thm.9.5, p.157). It states that a compact Lie group acting effectively on a connected topological manifold has a nowhere dense fixed point set.

One considers the finite cyclic group action induced by our periodic homeomorphism on the ball B^3. This action extends to the sphere S^3 by the identity outside the upper hemisphere. Then the fixed point set contains a 3-ball and the previous result shows that the action cannot be effective, and hence h must be the identity map. □

In particular f' is isotopic to the identity. □

6. About the hypothesis

While we have assumed word-hyperbolicity for the sake of definiteness, we actually need much less. Assuming $\pi_1(M)$ is atoroidal, we also need everywhere that $\tilde{M} = \mathbb{R}^3$, or equivalently that $\tilde{M}^3$ is *tame* (i.e, is the interior of a compact manifold). Existence and uniqueness use the tameness of M_γ. Finally, virtual rigidity needs tameness for *some* M_γ.

There are contractible 3-manifolds, namely Whitehead manifolds, different from $\mathbb{R}^3$. There are also 3-manifolds different from $S^1 \times R^2$ that have fundamental group $\mathbb{Z}$ and universal cover $\mathbb{R}^3$ (see, for example, [11]). Conjecturally these manifolds do not admit free co-compact actions, but so far this is known only under additional assumptions on the group (for example word-hyperbolicity).

Tameness is implied by hypothesis weaker than word-hyperbolicity. Specifically one could state Lemma 1.1 under the following hypothesis:

Lemma 6.1. *Assume γ is an element of infinite order and $\pi_1(M)$ is a CAT(0) group (i.e. it acts properly discontinuously and co-compactly by isometries on a geodesic CAT(0) metric space), or more generally a semi-hyperbolic group. Then the conclusion of lemma 1.1 holds true.*

Proof. Any infinite cyclic subgroup $\mathbb{Z} \subset \pi_1(M)$ yields a flat in X by the Flat Torus Theorem (see [1, 13]) and so it is a quasi-convex subgroup (see [1], Thm. 9.11 for details). The main theorem of [9] implies that $\tilde{M}/\langle\gamma\rangle$ is a missing boundary manifold, hence a solid torus. In the semi-hyperbolic case an infinite cyclic group lies in the center of its centralizer (which is a finite extension of the infinite cyclic group) and the latter is both semi-hyperbolic and a quasi-convex subgroup of $\pi_1(M)$. □

Corollary 6.2. *In the hypotheses of theorems 0.1, 1.2 and 4.3, we may replace word-hyperbolic by semi-hyperbolic.*

Results of M.Bridson extend the result to solv-manifolds and thus to all fundamental groups of geometric 3-manifolds. Notice that M.Kapovich has announced that atoroidal CAT(0) 3-manifold groups are word hyperbolic, improving a previous theorem of L.Mosher.

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