The $\overline{\partial}$-equation with vanishing and growth conditions at the boundary on some pseudoconvex domains

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1 Introduction

Let (X, g) be a Kähler manifold with strictly positive holomorphic bisectional curvature. For example, the complex projective space $\mathbb{P}^n$ satisfies these requirements. Let $\Omega \subset\subset X$ be a $\mathcal{C}^\infty$-smooth pseudoconvex domain.

We will show some vanishing and separation theorems for the $\overline{\partial}$-cohomology groups with exact support in $\overline{\Omega}$. This will be done by means of basic L^2-estimates on Ω with powers of the inverse of the boundary distance as weight functions. Sobolev-estimates for elliptic operators whose symbol can be controlled by some power of the boundary distance will be deduced in order to prove regularity results for the minimal L^2-solutions of the $\overline{\partial}$-operator.

Similar results under more restrictive assumptions on the ambient Kähler manifold X (namely compacity and the vanishing of certain cohomology groups), but under weaker assumptions on the regularity of Ω, have been obtained by Henkin and Iordan [He-Io] using completely different methods. As a corollary, one can show that some smooth forms, which satisfy the tangential Cauchy-Riemann equations on the boundary of Ω, extend as $\overline{\partial}$-closed forms to $\overline{\Omega}$. One should also consult the related work [Oh], which initiated this paper.

By duality, we can solve the $\overline{\partial}$-equation for extensible currents on Ω. These currents were at first considered by Martineau [Ma]. The analogous

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result in the strongly pseudoconvex case can be found in [Sa] with a very different proof. As an application, we solve the $\overline{\partial}$-equation in bidegree $(0,1)$ for smooth forms admitting a distribution boundary value.

The last part is dedicated to the solvability of the $\overline{\partial}$-equation for smooth forms with polynomial growth, i.e. for forms whose norm can be controlled by some power of the inverse of the boundary distance. This is done again by L^2-methods and a priori estimates for the minimal solution. We apply this result to show the vanishing of the Čech-cohomology groups of the sheaf of germs of holomorphic functions admitting a distribution boundary value. All these results seem to be new.

2 Elliptic operators of polynomial growth at the boundary

In this section, we will study the regularity of the equation $Lu = f$, where L is an elliptic operator on a $\mathcal{C}^\infty$-smooth bounded domain in $\mathbb{R}^n$ whose principal symbol can be controlled by some power of the boundary distance.

More precisely, let $U \subset\subset \mathbb{R}^n$ be a $\mathcal{C}^\infty$-smooth domain, and let $L = \sum_{|\alpha|=m} a_\alpha(x) D^\alpha + \sum_{|\beta|<m} b_\beta(x) D^\beta$ be a differential operator of order m with smooth coefficients $a_\alpha,\ b_\beta \in \mathcal{C}^\infty(U)$ on U.

We say that L is an *elliptic operator of polynomial growth on U* if there exist $k, l \in \mathbb{N}$ such that

$$|\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha| \gtrsim d^k(x)\ |\xi|^m \text{ for every } \xi \in \mathbb{R}^n \tag{1}$$

and

$$|D^\gamma a_\alpha(x)| \lesssim d^{-l-|\gamma|}(x),\ |D^\gamma b_\beta(x)| \lesssim d^{-l-|\gamma|}(x) \tag{2}$$

for all multiindices $\alpha,\ \beta,\ \gamma$, where d is the boundary distance function of U.

Here we write $a \lesssim b$ (resp. $b \gtrsim a$), if there exists an *absolute* constant $C > 0$ such that $a \leq C \cdot b$ (resp. $b \geq C \cdot a$).

Before we proceed to the precise statement of the main theorem of this section, we must recall the basic definitions of the Sobolev norms $\| \ \|_s$ of order s on $\mathbb{R}^n$, $s \in \mathbb{R}$,

$$\|u\|_s^2 \ = \ \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi$$

where $u \in \mathcal{D}(\mathbb{R}^n)$ is a smooth function with compact support in $\mathbb{R}^n$ and $\hat{u}$ is the Fourier transform of u.

It is clear that for any $s \in \mathbb{N}$ and $u \in \mathcal{D}(\mathbb{R}^n)$, $\|u\|_s^2 \ \sim \ \sum_{|\alpha|\leq s} \|D^\alpha u\|_0^2$. For $u \in \mathcal{C}^\infty(U)$ and $s \in \mathbb{N}$, $\|u\|_s^2$ is then defined by $\|u\|_s^2 \ = \ \sum_{|\alpha|\leq s} \|D^\alpha u\|_0^2$, $\|u\|_0^2 = \int_U |u|^2 d\lambda$.

We define $\mathcal{C}^r(\mathbb{R}^n, \overline{U}) := \{f \in \mathcal{C}^r(\mathbb{R}^n) \mid \text{supp} f \subset \overline{U}\}$.

Theorem 2.1
Let L be a differential operator of polynomial growth of order m with smooth coefficients on a $\mathcal{C}^\infty$-smooth domain $U \subset\subset \mathbb{R}^n$.
Then we have the following a priori estimate

$$\|u\|_s^2 \lesssim \|d^{-ts} L u\|_{s-m}^2 + \|d^{-Ts^2} u\|_0^2$$

for some $t, T \in \mathbb{N}$ and $s \gg 1$, $u \in \mathcal{C}^\infty(U)$.
In particular, if $\int_U |u(x)|^2 d^{-N}(x) d\lambda(x) < +\infty$ and $Lu \in \mathcal{C}^N(\mathbb{R}^n, \overline{U})$, then $u \in \mathcal{C}^{s(N)}(\mathbb{R}^n, \overline{U})$ where $s(N) \sim \sqrt{N}$ for all $N \gg 1$.

Proof: It clearly suffices to show the a priori estimate. The last assertion then follows from the Sobolev lemma (see [Fo]).

We prove the estimate by simply expliciting the dependence on d of all the constants involved in the classical proof of the hypoellipticity of uniformely elliptic operators (see [Fo]).

Let us fix $x_0 \in U$ and let $B_\delta(x_0)$ be the ball of radius $\delta \ll 1$ centered at x_0. Let u be a smooth function with support in $B_\delta(x_0)$.

First, we assume that $b_\beta = 0$ for every multiindex β. Then we have

$$(\widehat{L_{x_0} u})(\xi) = (2\pi i)^m \sum_{|\alpha|=m} a_\alpha(x_0) \ \xi^\alpha \ \hat{u}(\xi)$$

where $L_{x_0} = L(x_0)$ is the differential operator with frozen coefficients at x_0.

This implies

$$\begin{aligned}(1+|\xi|^2)^s|\hat{u}(\xi)|^2 &\leq 2^m(1+|\xi|^2)^{s-m}\ (1+|\xi|^{2m})\ |\hat{u}(\xi)|^2\\ &\lesssim (1+|\xi|^2)^{s-m}\ |\hat{u}(\xi)|^2+d^{-2k}(x_0)\ (1+|\xi|^2)^{s-m}\ |(\widehat{L_{x_0}u})(\xi)|^2\end{aligned}$$

by (1). Integrating both sides and using the inequality $\|u\|_{s-m}\leq\|u\|_{s-1}$, one obtains

$$\|u\|_s^2\lesssim d^{-2k}(x_0)\ \|L_{x_0}u\|_{s-m}^2+\|u\|_{s-1}^2.$$

Hence there exists $C_0>0$ such that

$$\|u\|_s^2\leq C_0\ d^{-2k}(x_0)\ (\|L_{x_0}u\|_{s-m}^2+\|u\|_{s-1}^2). \tag{3}$$

We now wish to estimate

$$\|L_xu-L_{x_0}u\|_{s-m}^2=\|\sum_\alpha(a_\alpha(x)-a_\alpha(x_0))\ D^\alpha u\|_{s-m}^2.$$

The estimates (2) yield

$$|a_\alpha(x)-a_\alpha(x_0)|\leq C_1\ d^{-l-1}(x_0)\ |x_0-x|$$

for some $C_1>0$ and all α,x,x_0.

Set $\delta=(8C_0C_1^2n^md^{-2k-2l-2}(x_0))^{-\frac{1}{2}}$ and fix $\phi\in\mathcal{D}(B_{2\delta}(0))$ with $0\leq\phi\leq 1$ and $\phi\equiv 1$ on $B_\delta(0)$. Suppose u is a smooth function supported in $B_\delta(x_0)$. Then

$$(a_\alpha(x)-a_\alpha(x_0))\ D^\alpha u(x)=\phi(x_0-x)\ (a_\alpha(x)-a_\alpha(x_0))\ D^\alpha u(x)$$

and

$$\sup_x|\phi(x_0-x)(a_\alpha(x)-a_\alpha(x_0))|^2\leq 4C_1^2d^{-2l-2}(x_0)\delta^2=(2n^mC_0d^{-2k}(x_0))^{-1}.$$

Hence by (2)

$$\begin{aligned}\|(a_\alpha(x)-a_\alpha(x_0))\ D^\alpha u\|_{s-m}^2 \quad\leq\quad & (2n^mC_0d^{-2k}(x_0))^{-1}\ \|u\|_s^2\\ &+C_2d^{-s_1s-s_0}(x_0)\ \|u\|_{s-1}^2\end{aligned}$$

for some $C_2>0,\ s_0,s_1\in\mathbb{N}$.

Thus, since there are at most n^m multiindices α with $|\alpha|=m$, we have

$$\|L_xu-L_{x_0}u\|_{s-m}^2\leq(2C_0d^{-2k}(x_0))^{-1}\ \|u\|_s^2+n^mC_2d^{-s_1s-s_0}(x_0)\ \|u\|_{s-1}^2$$

Combining this with (3), we then obtain

$$\|u\|_s^2 \leq d^{-2k}(x_0)\ \|Lu\|_{s-m}^2 + d^{-m_0 s-k_0}(x_0)\|u\|_{s-1}^2$$

for some $m_0, k_0 \in \mathbb{N}$.

Next, we consider the case $b_\beta \not\equiv 0$. Replacing m_0, k_0 by larger integers if necessary, we can absorb the additional terms of Lu in the term $d^{-m_0 s-k_0}(x_0)\|u\|_{s-1}^2$ and still have the estimate

$$\|u\|_s^2 \leq d^{-2k}(x_0)\ \|Lu\|_{s-m}^2 + d^{-m_0 s-k_0}(x_0)\|u\|_{s-1}^2$$

We emphasize that all the constants involved are independent of $x_0 \in U$.

Next, we cover U by balls $B_{\delta_i}(x_i)$ of the above type, $i \in \mathbb{N}$. It is easy to construct a partition of the unity $(\theta_i)_{i\in\mathbb{N}}$ with respect to this covering satisfying $\sum_{|\alpha|\leq s}|D^\alpha\theta_i|^2 \leq \theta_i|P_s(\delta_i^{-1})|$ where P_s is a polynomial of degree s in one variable. One has

$$\|\theta_i u\|_s^2 \leq d^{-2k}(x_i)\ \|L\theta_i u\|_{s-m}^2 + d^{-m_0 s-k_0}(x_i)\|\theta_i u\|_{s-1}^2$$

for every smooth function u on U.

Replacing m_0, k_0 by larger integers if necessary, we get

$$\begin{aligned}
\|\theta_i u\|_s^2 &\leq C\ \{d^{-k_0}(x_i)\|\theta_i Lu\|_{s-m}^2 + d^{-m_0 s-k_0}(x_i)\|\theta_i u\|_{s-1}^2 \\
&\quad + d^{-m_0 s-k_0}(x_i)\int_U \theta_i|u|^2 d\lambda\} \\
&\leq Md^{-m_0 s-k_0}(x_i)\ \{\sum_{|\alpha|\leq s-m}\int_U \theta_i|D^\alpha(Lu)|^2 d\lambda \\
&\quad + \|\theta_i u\|_{s-1}^2 + \int_U \theta_i|u|^2 d\lambda\}
\end{aligned} \tag{4}$$

for some $C, M > 0$.

Moreover,

$$\begin{aligned}
&Md^{-m_0 s-k_0}(x_i)\ \|\theta_i u\|_{s-1}^2 \\
&\quad = Md^{-m_0 s-k_0}(x_i)\int_{\mathbb{R}^n}(1+|\xi|^2)^{s-1}|\widehat{\theta_i u}(\xi)|^2 d\lambda \\
&\quad = Md^{-m_0 s-k_0}(x_i)\int_{\{1+|\xi|^2\geq 2Md^{-m_0 s-k_0}(x_i)\}}(1+|\xi|^2)^{s-1}|\widehat{\theta_i u}(\xi)|^2 d\lambda
\end{aligned}$$

$$+Md^{-m_0 s-k_0}(x_i)\int_{\{1+|\xi|^2\leq 2Md^{-m_0 s-k_0}(x_i)\}}(1+|\xi|^2)^{s-1}|\widehat{\theta_i u}(\xi)|^2 d\lambda$$

$$\leq \frac{1}{2}\|\theta_i u\|_s^2 + C' d^{-m_0 s^2+m_0 s-k_0 s+k_0}(x_i)\ \|\theta_i u\|_0^2$$

for some $C' > 0$. Thus, by (4),

$$\|\theta_i u\|_s^2 \lesssim \sum_{|\alpha|\leq s-m}\int_U \theta_i d^{-2ts}|D^\alpha(Lu)|^2 d\lambda + \int_U \theta_i d^{-2Ts^2}|u|^2 d\lambda$$

for some $t, T \in \mathbb{N}$ and $s \gg 1$. So

$$\|u\|_s^2 = \|\sum_i \theta_i u\|_s^2 \leq \sum_i \|\theta_i u\|_s^2 \lesssim \|d^{-ts}Lu\|_{s-m}^2 + \|d^{-Ts^2}u\|_0^2$$

which completes the proof. □

Let us denote by $B_N(U)$, $N \in \mathbb{N}$, the space of $\mathcal{C}^\infty$-functions u on U such that $\sup_{x\in U}|u(x)d^N(x)| < +\infty$. Then we have the following theorem:

Theorem 2.2
Let L be a differential operator of polynomial growth of order m with smooth coefficients on a $\mathcal{C}^\infty$-smooth domain $U \subset\subset \mathbb{R}^n$.
Then we have the following a priori estimate

$$\sup_{x\in U}|d^{N+t}u|^2 \lesssim \sum_{|\alpha|\leq[\frac{n}{2}]+1-m}\|d^N D^\alpha(Lu)\|_0^2 + \|d^N u\|_0^2$$

for some $t \in \mathbb{R}$.
In particular, there exists $t_0 \in \mathbb{N}$ such that if $\int_U |u(x)|^2 d^{2N}(x)d\lambda < +\infty$ and $Lu \in B_N(U)$, then $u \in B_{N+t_0}(U)$.

Proof: By the Sobolev lemma (see [Fo]), it suffices to prove the a priori estimate

$$\|d^{N+t}u\|_{[\frac{n}{2}]+1}^2 \lesssim \sum_{|\alpha|\leq[\frac{n}{2}]+1-m}\int_U |d^N D^\alpha(Lu)|^2 d\lambda + \|d^N u\|_0^2$$

for some $t \in \mathbb{N}$. Let us prove this estimate.

Replacing u by $d^N u$ and s by $[\frac{n}{2}]+1$ in (4), we have

$$\begin{aligned}
\|\theta_i d^N u\|^2_{[\frac{n}{2}]+1} &\leq Md^{-c_0}(x_i)\ \{ \sum_{|\alpha|\leq[\frac{n}{2}]+1-m} \int_U \theta_i |D^\alpha(Ld^N u)|^2 d\lambda \\
&\quad + \|d^N u\|^2_{[\frac{n}{2}]} + \int_U \theta_i |d^N u|^2 d\lambda\} \\
&\leq M' d^{-c_0}(x_i)\ \{ \sum_{|\alpha|\leq[\frac{n}{2}]+1-m} \int_U \theta_i |d^N D^\alpha(Lu)|^2 d\lambda \\
&\quad + d^{-c_1}(x_i)\|d^N u\|^2_{[\frac{n}{2}]} + \int_U \theta_i |d^N u|^2 d\lambda\}
\end{aligned}$$

for some $M' > 0,\ c_0, c_1 \in \mathbb{N}$.

Following the proof of theorem 2.1, there exists $M'' > 0$ such that

$$M' d^{-c_0-c_1}(x_i)\|d^N u\|^2_{[\frac{n}{2}]} \leq \frac{1}{2}\|d^N u\|^2_{[\frac{n}{2}]+1} + M'' d^{-([\frac{n}{2}]+1)(c_0+c_1)}(x_i)\|d^N u\|^2_0.$$

Thus

$$\begin{aligned}
\|\theta_i d^N u\|^2_{[\frac{n}{2}]+1} &\lesssim \sum_{|\alpha|\leq[\frac{n}{2}]+1-m} \int_U \theta_i |d^{N-t} D^\alpha(Lu)|^2 d\lambda \\
&\quad + \int_U \theta_i |d^{N-t} u|^2 d\lambda
\end{aligned}$$

for some $t \in \mathbb{N}$ and $N \geq 1$.

So

$$\|d^{N+t} u\|^2_{[\frac{n}{2}]+1} \lesssim \sum_{|\alpha|\leq[\frac{n}{2}]+1-m} \int_U |d^N D^\alpha(Lu)|^2 d\lambda + \|d^N u\|^2_0$$

for $N \geq 1$. □

3 Some L^2-cohomology groups of the $\overline{\partial}$-operator on pseudoconvex domains

For the rest of this paper, we will denote by (X, g) an n-dimensional Kähler manifold with strictly positive holomorphic bisectional curvature and by $\Omega \subset\subset X$ a $\mathcal{C}^\infty$-smooth pseudoconvex domain.

Let $\delta(z)$ be the distance from $z \in \Omega$ to the boundary of Ω with respect to the metric g. δ is then of class $\mathcal{C}^\infty$ near $\partial\Omega$ and smooth up to the boundary. We denote by ω_g the Kähler form of the metric g on X.

According to a theorem of Elencwajg (see [El], [Su]; [Ta] for the case $X = \mathbb{P}^n$), the function $-\log\delta$ is plurisubharmonic in Ω, and moreover $i\partial\overline{\partial}(-\log\delta) \geq c\ \omega_g$ for some $c > 0$. This implies that Ω admits a strongly plurisubharmonic exhaustion function, therefore Ω is a Stein manifold.

Moreover, Ohsawa and Sibony ([Oh-Si]) have shown that Ω even admits a bounded strongly plurisubharmonic exhaustion function. More precisely, they have shown that for some $\epsilon > 0$, $r := -\delta^\epsilon$ is strongly plurisubharmonic in Ω and $i\partial\overline{\partial}r \geq \mathrm{cte}\ |r|\ \omega_g$.

As $-i\partial\overline{\partial}\log(-r) = \frac{1}{-r}i\partial\overline{\partial}r + \frac{1}{r^2}i\partial r \wedge \overline{\partial}r$, $-\log(-r)$ is strongly plurisubharmonic near $\partial\Omega$. Using the smoothing theorem for strongly plurisubharmonic functions, we can thus find a smooth strongly plurisubharmonic function φ on Ω which coincides with $-\log(-r)$ near $\partial\Omega$.

We define $\omega = i\partial\overline{\partial}\varphi$. Near $\partial\Omega$, we have $\omega \geq \frac{\mathrm{cte}}{-r}\ |r|\ \omega_g \geq \mathrm{cte}\ \omega_g$. ω is then a complete Kähler metric on Ω. From now on, Ω will be equipped with this metric. Moreover, we define $\tilde{r} = -\exp(-\varphi)$. Then $\tilde{r}$ coincides with r near $\partial\Omega$.

Let (E, h) be a hermitian vector bundle on X, and let $N \in \mathbb{Z}$. We denote by $L^2_{p,q}(\Omega, E, N)$ the Hilbert space of (p,q)-forms u with values in E which satisfy

$$\|u\|^2_N := \int_\Omega |u|^2_{\omega,h}(-\tilde{r})^N \mathrm{dV}_\omega < +\infty.$$

Here dV_ω is the canonical volume element associated to the metric ω, and $|\ |_{\omega,h}$ is the norm of (p,q)-forms induced by ω and h.

Let $\overline{\partial}^*_N$ (resp. ∂^*_N) be the Hilbert adjoint of $\overline{\partial}$ (resp. ∂) with respect to the canonical scalar product $\ll\ ,\ \gg_N$ of (p,q)-forms with values in E induced by $\|\ \|_N$.

We recall the Bochner-Kodaira-Nakano identity (see [De2]):

$$\Box''_N = \Box'_N + [i\Theta(E_N), \Lambda]$$

where $\Box''_N = \overline{\partial}\overline{\partial}^*_N + \overline{\partial}^*_N\overline{\partial}$, $\Box'_N = \partial\partial^*_N + \partial^*_N\partial$, $\Theta(E_N)$ is the curvature of the bundle $E_N = (E, (-r)^N h)$ and Λ is the adjoint of multiplication by ω.

$$i\Theta(E_N) = iN\partial\overline{\partial}\varphi \otimes \mathrm{Id}_E + i\Theta(E) = N\omega \otimes \mathrm{Id}_E + i\Theta(E),$$

thus this term can be made $\geq \omega \otimes \mathrm{Id}_E$ by taking $N \gg 1$. Therefore if u is an (n,q)-form with values in E, we obtain

$$< [i\Theta(E_N), \Lambda]u, u >_N \geq \quad |u|_N^2$$

if $N \gg 1$ (see [De2] for the exact calculation of this curvature term).

Then the Bochner-Kodaira-Nakano identity yields

$$\|u\|_N^2 \ \leq \text{cte } (\|\overline{\partial}u\|_N^2 + \|\overline{\partial}_N^* u\|_N^2)$$

for all (n,q)-forms u with compact support in Ω, $q \geq 1$. By definition, this means that E_N is $W^{n,q}$ -elliptic for $q \geq 1$, $N \gg 1$ (see [An-Ve]).

The $W^{n,q}$ -ellipticity immediately implies a vanishing theorem whose proof is standard (see [An-Ve], [De2]).

Proposition 3.1

Let $\Omega \subset\subset X$ be a $\mathcal{C}^\infty$-smooth pseudoconvex domain in a Kähler manifold with strictly positive holomorphic bisectional curvature. Let (E,h) be a hermitian vector bundle on X. For $1 \leq q \leq n$ and $N \gg 1$ we have

$$L^2_{n,q}(\Omega, E, N) \cap \mathrm{Ker}\overline{\partial} = \overline{\partial} L^2_{n,q-1}(\Omega, E, N) \cap L^2_{n,q}(\Omega, E, N).$$

By duality (see also lemma 2 in [An-Ve]), E_N^* ist $W^{0,q}$ -elliptic for $q \leq n-1$. We thus obtain

Proposition 3.2

Let $\Omega \subset\subset X$ be a $\mathcal{C}^\infty$-smooth pseudoconvex domain in a Kähler manifold with strictly positive holomorphic bisectional curvature. Let (E,h) be a hermitian vector bundle on X. For $1 \leq q \leq n-1$ and $N \ll -1$ we have

$$L^2_{0,q}(\Omega, E, N) \cap \mathrm{Ker}\overline{\partial} = \overline{\partial} L^2_{0,q-1}(\Omega, E, N) \cap L^2_{0,q}(\Omega, E, N).$$

4 The $\overline{\partial}$-equation with exact support on some pseudoconvex domains

In this section, we will show some vanishing and separation theorems for the $\overline{\partial}$-cohomology groups with values in a vector bundle E supported in $\overline{\Omega}$:

$$\begin{aligned} H^{p,q}(X,\overline{\Omega},E) &= \{f \in \mathcal{C}^{\infty}_{p,q}(X,E) \mid \mathrm{supp} f \subset \overline{\Omega}\} \cap \mathrm{Ker}\overline{\partial}/ \\ & \quad \overline{\partial}\{f \in \mathcal{C}^{\infty}_{p,q-1}(X,E) \mid \mathrm{supp} f \subset \overline{\Omega}\}. \end{aligned}$$

This is done by solving the $\overline{\partial}$-equation in the L^2-sense as in section 3 and then applying the results of section 2 to the operator $\Box_N = \overline{\partial}\overline{\partial}^*_N + \overline{\partial}^*_N\overline{\partial}$ for $N < 0$.

For later use, let us first compare the spaces $L^2_{p,q}(\Omega, E, N)$ with the Hilbert spaces of square-integrable E-valued forms with respect to the pointwise norm $| \ |_{\omega_g}$ and the volume element dV_{ω_g}.

Lemma 4.1
Let u be a (p,q)-form on Ω with values in E. We have

$$\delta^{2(p+q)} \ |u|^2_{\omega_g} \ dV_{\omega_g} \lesssim |u|^2_{\omega} \ dV_{\omega} \lesssim \delta^{-2n} \ |u|^2_{\omega_g} \ dV_{\omega_g}.$$

Proof: We have $\omega = \frac{1}{-r} i\partial\overline{\partial}r + \frac{1}{r^2} i\partial r \wedge \overline{\partial}r$ with $r = -\delta^{\epsilon}$ for some $\epsilon > 0$. Since $\partial\Omega$ is $\mathcal{C}^{\infty}$-smooth, all the derivatives of δ are bounded. Therefore we have $|\partial r|_{\omega_g} \lesssim \delta^{\epsilon-1}$ and $|\partial\overline{\partial}r|_{\omega_g} \lesssim \delta^{\epsilon-2}$. Thus $\omega \lesssim \delta^{-2}\omega_g$.

In [Oh-Si] it was shown that $i\partial\overline{\partial}r \gtrsim |r| \ \omega_g$. Thus $\omega_g \lesssim \omega \lesssim \delta^{-2}\omega_g$. The lemma is then proved in the same way as lemma 3.3 in [De1]. $\Box$

From the above lemma, it follows in particular that

$$\mathcal{C}^s_{p,q}(X,\overline{\Omega},E) \subset L^2_{p,q}(\Omega, E, -s_0 s + s_1)$$

for some $s_0, s_1 \in \mathbb{R}$.

Theorem 4.2
Let $u \in L^2_{p,q}(\Omega, E, -N)$ be such that $\Box_{-N} u \in \mathcal{C}^N_{p,q}(X,\overline{\Omega})$.
Then $u \in \mathcal{C}^{s(N)}_{p,q}(X,\overline{\Omega})$ where $s(N)$ is a function proportional to $\sqrt{N}$, $N \gg 1$.

Proof: The above theorem will be a consequence of the results of section 2. Indeed, we will show that $\Box_{-N}$ is an elliptic operator of polynomial growth on Ω.

In order to avoid too many sums over too many indices, we will assume that E is the trivial bundle and restrict our attention to $(0,1)$-forms. The general case is handled analogously.

Let $z_0 \in \partial\Omega$ and let $(z_1, \ldots, z_n)$ be local holomorphic coordinates of X in a neighborhood U of z_0.

We have $\omega = i\sum_{j,k=1}^n \omega_{jk} dz_j \wedge d\overline{z}_k$ with $\omega_{jk} = \frac{1}{-r}\frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} + \frac{1}{r^2}\frac{\partial r}{\partial z_j}\frac{\partial r}{\partial \overline{z}_k}$.

Let $L_1, \ldots, L_n$ be an orthonormal basis of $T^{1,0}X_{|U\cap\Omega}$ with respect to ω, i.e. $L_k = \sum_{j\leq k} l_{jk}\frac{\partial}{\partial z_j}$ where the l_{jk} have to be determined by the condition $\sum_{l\leq k}\sum_{i\leq j} l_{lk}\overline{l}_{ij}\omega_{li} = \delta_{jk}$. It is therefore clear that the l_{jk} are functions involving only powers of r and the derivatives of order at most 2 of r.

Let $\epsilon_i, \ldots, \epsilon_j \in (T^{1,0}X)^*_{|U\cap\Omega}$ be the dual basis of $L_1, \ldots, L_n$.

For $u = \sum_{j=1}^n u_j \overline{\epsilon}_j$ we then have

$$\overline{\partial}u = \sum_{j,k} \overline{L}_k(u_j)\overline{\epsilon}_k \wedge \overline{\epsilon}_j - \sum_{j,k,l} c^l_{kj} u_l \overline{\epsilon}_k \wedge \overline{\epsilon}_j$$

where c^l_{jk} can be determined by the condition $[\overline{L}_j, \overline{L}_k] = \sum_l c^l_{jk}\overline{L}_l$, because we have $\overline{\partial}\epsilon_l(\overline{L}_k, \overline{L}_j) = -\overline{\epsilon}_l([\overline{L}_k, \overline{L}_j])$ by the Cartan formula for $\overline{\partial}$. Therefore also the c^l_{jk} are functions involving only powers of r and the derivatives of order at most 3 of r.

Now let $v = \sum_{j,k} v_{kj}\overline{\epsilon}_k \wedge \overline{\epsilon}_j$ be a smooth $(0,2)$-form with compact support in $U\cap\Omega$. Then we have

$$\begin{aligned}
\ll \overline{\partial}u, v \gg_{-N} &= 2^n \int_\Omega (\sum_{k,j} \overline{L}_k(u_j)\overline{v}_{kj} - \sum_{k,j,l} c^l_{kj} u_l \overline{v}_{kj})(-r)^{-N}\det(\omega_{\alpha\beta})d\lambda \\
&= 2^n \int_\Omega \sum_{k,j,l} (\overline{l}_{lk}\frac{\partial u_j}{\partial \overline{z}_l}\overline{v}_{kj} - c^l_{kj} u_l \overline{v}_{kj})(-r)^{-N}\det(\omega_{\alpha\beta})d\lambda \\
&= -2^n \sum_{k,j,l}\int_\Omega \{u_j \frac{\partial}{\partial \overline{z}_l}(\overline{l}_{lk}\overline{v}_{kj}(-r)^{-N}\det(\omega_{\alpha\beta})) \\
&\quad -c^l_{kj} u_l \overline{v}_{kj}(-r)^{-N}\det(\omega_{\alpha\beta})\} d\lambda
\end{aligned}$$

Thus

$$\overline{\partial}^*_{-N} v \;=\; -\sum_{k,j,l} (L_k(v_{kj}) + v_{kj}\frac{\partial l_{lk}}{\partial z_l} + N v_{kj}(-r)^{-1}\frac{\partial r}{\partial z_l}$$
$$+ v_{kj} l_{lk} \frac{\partial}{\partial z_l}(\det(\omega_{\alpha\beta}))\det(\omega_{\alpha\beta})^{-1} + v_{kj}\overline{c}^l_{kj})\overline{\epsilon}_j$$

Hence

$$\begin{aligned}\Box_{-N} u &= \sum_{j,k} L_k \overline{L}_k(u_j)\overline{\epsilon}_j + \text{lower order terms}\\ &= \sum_{i,j,k,l} l_{lk}\overline{l}_{ik}\frac{\partial^2 u_j}{\partial z_l \partial \overline{z}_j}\overline{\epsilon}_j + \text{lower order terms}\end{aligned}$$

where the lower order terms involve only derivatives of order ≤ 1 of u and multiplication by functions involving only powers of r and derivatives of some finite order of r.

One can now define Sobolev norms for forms on Ω with values in E using the metric g. By lemma 4.1, $u \in L^2_{p,q}(\Omega, E, -N)$ satisfies $\int_\Omega |u|^2_{\omega_g} \delta^{-N+2n} dV_{\omega_g} < +\infty$. The theorem then follows from the estimates in section 2 by taking into account that the local estimates can be patched together by choosing a suitable partition of the unity as in the proof of theorem 2.1. $\Box$

We are now ready to prove the main theorem of this section.

Theorem 4.3
Let $\Omega \subset\subset X$ be a $\mathcal{C}^\infty$-smooth pseudoconvex domain in a Kähler manifold with strictly positive holomorphic bisectional curvature. Let E be a holomorphic vector bundle on X. Then we have

$$H^{p,q}(X, \overline{\Omega}, E) = 0 \qquad \textit{for } 1 \leq q \leq n-1$$

and

$$H^{p,n}(X, \overline{\Omega}, E) \textit{ is separated for the usual } \mathcal{C}^\infty - \textit{topology}.$$

Moreover,

$$\overline{\partial}(\mathcal{C}^\infty_{p,n-1}(X, \overline{\Omega}, E)) =$$
$$\bigcap_{N\in\mathbb{N}} \{f \in \mathcal{C}^\infty_{p,n}(X, \overline{\Omega}, E) \mid \int_\Omega f \wedge h = 0 \;\forall h \in L^2_{n-p,0}(\Omega, E^*, N) \cap \mathrm{Ker}\overline{\partial}\}.$$

Proof: Replacing the vector bundle E by $\Lambda^p(T^{1,0}X)^* \otimes E$, it is no loss of generality to assume $p = 0$.

We will begin by proving the following claim:
Let $f \in \mathcal{C}^k_{0,q}(X,\overline{\Omega},E) \cap \mathrm{Ker}\overline{\partial}$, $1 \leq q \leq n-1$, $k \gg 1$. Then there exists $u \in \mathcal{C}^{j(k)}_{0,q-1}(X,\overline{\Omega},E)$ such that $\overline{\partial} u = f$ with $j(k) \sim \sqrt{k}$.

Proof of the claim: Let $f \in \mathcal{C}^k_{0,q}(X,\overline{\Omega},E) \cap \mathrm{Ker}\overline{\partial}$, $1 \leq q \leq n-1$, $k \gg 1$. Proposition 2.2 and the remark after lemma 4.1 imply that there exists $u \in L^2_{0,q-1}(\Omega, E, -s_0 k + s_1)$ such that $\overline{\partial} u = f$ in Ω for some $s_0, s_1 \in \mathbb{R}$. Moreover, choosing the minimal solution, we may assume $\overline{\partial}^*_{-s_0k+s_1} u = 0$, i.e. $\Box_{-s_0k+s_1} u = \overline{\partial}^*_{-s_0k+s_1} f$. By looking at the computation of $\overline{\partial}^*_{-s_0k+s_1}$ as in the proof of theorem 4.2, ist is clear that for some $m_0, m_1 \in \mathbb{N}$, $\overline{\partial}^*_{-s_0k+s_1} f \in \mathcal{C}^{m_0 k + -m_1}_{0,q}(X,\overline{\Omega},E)$.

Applying theorem 4.2, we then have $u \in \mathcal{C}^{j(k)}_{0,q-1}(X,\overline{\Omega},E)$ with $j(k) \sim \sqrt{k}$.

Let us now prove the theorem.

$H^{0,1}(X,\overline{\Omega},E) = 0$ follows immediately from the above claim and the hypoellipticity of $\overline{\partial}$ in bidegree $(0,1)$.

Now assume $1 < q \leq n-1$ and let $f \in \mathcal{C}^\infty_{0,q}(X,\overline{\Omega},E) \cap \mathrm{Ker}\overline{\partial}$. By induction, we will construkt $u_k \in \mathcal{C}^k_{0,q-1}(X,\overline{\Omega},E)$ such that $\overline{\partial} u_k = f$ and $|u_{k+1} - u_k|_{j(k)-1} < 2^{-k}$. It is then clear that $(u_k)_{k\in\mathbb{N}}$ converges to $u \in \mathcal{C}^\infty_{0,q-1}(X,\overline{\Omega},E)$ such that $\overline{\partial} u = f$.

Suppose that we have constructed $u_1, \dots, u_k$. By the above claim, there exists $\alpha_{k+1} \in \mathcal{C}^{k+1}_{0,q-1}(X,\overline{\Omega},E)$ such that $f = \overline{\partial}\alpha_{k+1}$. We have $\alpha_{k+1} - u_k \in \mathcal{C}^k_{0,q-1}(X,\overline{\Omega},E) \cap \mathrm{Ker}\overline{\partial}$. Once again by the above claim, there exists $g \in \mathcal{C}^{j(k)}_{0,q-2}(X,\overline{\Omega},E)$ satisfying $\alpha_{k+1} - u_k = \overline{\partial} g$.

Since $\mathcal{C}^\infty_{0,q-2}(X,\overline{\Omega},E)$ is dense in $\mathcal{C}^{j(k)}_{0,q-2}(X,\overline{\Omega},E)$, there exists $g_{k+1} \in \mathcal{C}^\infty_{0,q-2}(X,\overline{\Omega},E)$ such that $|g - g_{k+1}|_{j(k)} < 2^{-k}$.

Define $u_{k+1} = \alpha_{k+1} - \overline{\partial} g_{k+1} \in \mathcal{C}^{k+1}_{0,q-1}(X,\overline{\Omega},E)$. Then $\overline{\partial} u_{k+1} = f$ and $|u_{k+1} - u_k|_{j(k)-1} = |\overline{\partial} g - \overline{\partial} g_{k+1}|_{j(k)-1} \leq |g - g_{k+1}|_{j(k)} < 2^{-k}$.
Thus u_{k+1} has the desired properties.

It remains to show that

$$\overline{\partial}(\mathcal{C}^\infty_{0,n-1}(X,\overline{\Omega},E)) =$$

$$\bigcap_{N\in\mathbb{N}} \{f \in \mathcal{C}^\infty_{0,n}(X,\overline{\Omega},E) \mid \int_\Omega f \wedge h = 0 \ \forall h \in L^2_{n,0}(\Omega, E^*, N) \cap \mathrm{Ker}\overline{\partial}\}.$$

This clearly implies that $H^{0,n}(X,\overline{\Omega},E)$ is separated.

It follows from Cauchy's formula that a holomorphic section $u \in L^2_{n,0}(\Omega,E^*,N)$ is of polynomial growth (see lemma 4.4 below). Stoke's formula then implies that the left side of the above equation is contained in the right side.

Now, let us take $f \in \bigcap_{N\in\mathbb{N}}\{f \in \mathcal{C}^\infty_{0,n}(X,\overline{\Omega},E) \mid \int_\Omega f\wedge h = 0\ \forall h \in L^2_{n,0}(\Omega,E^*,N)\cap \mathrm{Ker}\overline{\partial}\}$.

We first show that for each $N\in\mathbb{N},\ N\gg 1$, there exists $\beta_N \in L^2_{0,n-1}(\Omega,E,-N)$ satisfying $\overline{\partial}\beta_N = f$.

To see this, we define the linear operator

$$L_f : \quad \overline{\partial}L^2_{n,0}(\Omega,E^*,N) \longrightarrow \mathbb{C}$$
$$\overline{\partial}\varphi \longmapsto \int_\Omega f\wedge\varphi.$$

First of all, notice that L_f is well-defined because of the moment conditions imposed on f.

By proposition 2.1, $\overline{\partial}(L^2_{n,0}(\Omega,E^*,N))\cap L^2_{n,1}(\Omega,E^*,N)$ is a closed subspace of $L^2_{n,1}(\Omega,E^*,N)$. Applying Banach's open mapping theorem, we know that L_f is a continuous linear operator and therefore extends to a continuous linear operator on the Hilbert space $L^2_{n,1}(\Omega,E^*,N)$ by the Hahn-Banach theorem. By the theorem of Riesz, there exists $\beta_N \in L^2_{0,n-1}(\Omega,E,-N)$ such that for every $\varphi\in L^2_{n,0}(\Omega,E^*,N)$ we have

$$\int_\Omega \beta_N\wedge\overline{\partial}\varphi = L_f(\varphi) = \int_\Omega f\wedge\varphi,$$

i.e. $\overline{\partial}\beta_N = f$.

Now the proof follows the same lines as above and we construct $(u_k)_{k\in\mathbb{N}} \in \mathcal{C}^k_{0,n-1}(X,\overline{\Omega},E)$ converging to $u\in\mathcal{C}^\infty_{0,n-1}(X,\overline{\Omega},E)$ such that $\overline{\partial}u = f$, which concludes the proof. □

Lemma 4.4
Let $u\in L^2_{p,0}(\Omega,E,N)\cap\mathrm{Ker}\overline{\partial}$. Then $|u|_{\omega_g} \lesssim \delta^{-(\epsilon N+2p+2n)}$ for some $\epsilon>0$.

Proof: Let $\zeta\in\partial\Omega$ and $(z_1,\dots,z_n)$ be holomorphic coordinates on a neighborhood U of ζ and let $(e_\lambda)_{1\le\lambda\le r}$ $(r = \mathrm{rank}E)$ be a local orthonormal frame of (E,h). Then we have $u = \sum_{1\leq\lambda\leq r}\sum_{|I|=p} h_I dz_I \otimes e_\lambda$ on $U\cap\Omega$ with $h_I \in \mathcal{O}(U\cap\Omega)$ and $\int_{U\cap\Omega} |h_I|^2 \delta^{\epsilon N+2p} dV_{\omega_g} < +\infty$ by lemma 4.1 and the assumption $u \in L^2_{p,0}(\Omega, E, N)$, where ϵ is such that $-r = \delta^\epsilon$.

Let $x \in U\cap\Omega$ and $B_\alpha \subset\subset U\cap\Omega$ be the ball of radius $\alpha = \frac{1}{2}\delta(x)$ centered at x. From Cauchy's formula, we get

$$\begin{aligned} |h_I(x)|^2 &\leq \frac{1}{\pi^n}\frac{1}{\alpha^{2n}}\int_{B_\alpha} |h_I|^2 d\lambda \\ &\lesssim \delta^{-2n}(x) \sup_{x\in B_\alpha(x)} \delta^{-(\epsilon N+2p)}(x) \int_{U\cap\Omega} |h_I|^2 \delta^{\epsilon N+2p} d\lambda \end{aligned}$$

Therefore $|h_I(x)|^2 \leq$ constant $\delta^{-(\epsilon N+2p+2n)}(x)$ with a constant not depending on x, which proves the lemma. □

Corollary 4.5
Let $\Omega \subset\subset X$ be a $\mathcal{C}^\infty$-smooth pseudoconvex domain in a Kähler manifold with strictly positive holomorphic bisectional curvature. Let E be a holomorphic vector bundle on X. Let $f \in \mathcal{C}^\infty_{p,q}(\partial\Omega, E) \cap \mathrm{Ker}\overline{\partial}_b$ satisfy the tangential Cauchy-Riemann equations on $\partial\Omega$, $q \leq n-2$.
Then there exists $F \in \mathcal{C}^\infty_{p,q}(\overline{\Omega}, E)$ such that $F_{|\partial\Omega} = f$ and $\overline{\partial}F = 0$.
For $q = n-1$, the same holds true if there exists $\tilde{f} \in \mathcal{C}^\infty_{p,n-1}(\overline{\Omega}, E)$ such that $\tilde{f}_{|\partial\Omega} = f$, such that $\overline{\partial}\tilde{f}$ vanishes to infinite order on $\partial\Omega$ and such that $\int_\Omega \overline{\partial}\tilde{f} \wedge h = 0$ for all $h \in L^2_{n-p,0}(\Omega, E^*, N) \cap \mathrm{Ker}\overline{\partial}$, for all $N \in \mathbb{N}$.

Proof: There exists $\tilde{f} \in \mathcal{C}^\infty_{p,n-1}(\overline{\Omega}, E)$ such that $\tilde{f}_{|\partial\Omega} = f$ and such that $\overline{\partial}\tilde{f}$ vanishes to infinite order on $\partial\Omega$. Applying theorem 4.3, one can find a solution u to the equation $\overline{\partial}u = \overline{\partial}\tilde{f}$ in such a way that u is of class $\mathcal{C}^\infty$ on $\overline{\Omega}$ and vanishes on $\partial\Omega$. $F = \tilde{f} - u$ is then the desired extension of f to $\overline{\Omega}$. □

Corollary 4.6 (see [He-Io])
Let $\Omega \subsetneq X$ be a $\mathcal{C}^\infty$-smooth pseudoconvex domain in a compact Kähler manifold with strictly positive holomorphic bisectional curvature. Let E be a holomorphic vector bundle on X. Assume that $H^{p,q}(X,E) = 0$ and put $D = X \setminus \overline{\Omega}$.
Then for every $\overline{\partial}$-closed form $f \in \mathcal{C}^\infty_{p,q}(\overline{D}, E)$, which is smooth up to the boundary, there exists $u \in \mathcal{C}^\infty_{p,q-1}(\overline{D}, E)$ such that $\overline{\partial}u = f$, $1 \leq q \leq n-2$.
For $q = n-1$, the same holds true if there exists $\tilde{f} \in \mathcal{C}^\infty_{p,n-1}(X, E)$ such that $\tilde{f}_{|\overline{D}} = f$, such that $\overline{\partial}\tilde{f}$ vanishes to infinite order on $\partial\Omega$ and such that $\int_\Omega \overline{\partial}\tilde{f} \wedge h = 0$ for all $h \in L^2_{n-p,0}(\Omega, E^*, N) \cap \mathrm{Ker}\overline{\partial}$, for all $N \in \mathbb{N}$.

Proof: By corollary 4.5, there exists $\tilde{f} \in \mathcal{C}^\infty_{p,n-1}(X,E)$ such that $\tilde{f}_{|\overline{D}} = f$ and $\overline{\partial}\tilde{f} = 0$. As $H^{p,q}(X,E) = 0$, we have $\tilde{f} = \overline{\partial}u$ for some $u \in \mathcal{C}^\infty_{p,q-1}(X,E)$. Then $u_{|\overline{D}}$ has the desired properties. □

5 The $\overline{\partial}$-equation for extensible currents on some pseudoconvex domains

The results of the preceding section will allow us to solve the $\overline{\partial}$-equation for extensible currents by duality.

A current T defined on Ω is said to be *extensible*, if T is the restriction to Ω of a current defined on X.

It was shown in [Ma] that if Ω satisfies $\overset{\circ}{\overline{\Omega}}$ (which is always satisfied in our case), the vector space $\check{\mathcal{D}}'_{p,q}(\Omega)$ of extensible currents on Ω of bidimension (p,q) is the topological dual of $\mathcal{C}^\infty_{p,q}(X,\overline{\Omega})$.

Theorem 5.1
Let $\Omega \subset\subset X$ be a $\mathcal{C}^\infty$-smooth pseudoconvex domain in a Kähler manifold with strictly positive holomorphic bisectional curvature.
Let $T \in \check{\mathcal{D}}'_{p,q}(\Omega)$ be an extensible current on Ω of bidimension (p,q), $q \leq n-1$ such that $\overline{\partial}T = 0$. Then there exists $S \in \check{\mathcal{D}}'_{p,q+1}(\Omega)$ satisfying $\overline{\partial}S = T$.

Proof: Let $T \in \check{\mathcal{D}}'_{p,q}(\Omega)$ be an extensible current on Ω of bidimension (p,q), $q \leq n-1$ such that $\overline{\partial}T = 0$.

Consider the operator

$$L_T : \quad \begin{array}{ccc} \overline{\partial}\mathcal{C}^\infty_{p,q}(X,\overline{\Omega}) & \longrightarrow & \mathbb{C} \\ \overline{\partial}\varphi & \longmapsto & < T,\varphi > \end{array}$$

We first notice that L_T is well-defined. Indeed, let $\varphi \in \mathcal{C}^\infty_{p,q}(X,\overline{\Omega})$ be such that $\overline{\partial}\varphi = 0$.
If $q = 0$, the analytic continuation principle for holomorphic functions yields $\varphi = 0$, so $< T,\varphi >= 0$.
If $1 \leq q \leq n-1$, one has $\varphi = \overline{\partial}\alpha$ with $\alpha \in \mathcal{C}^\infty_{p,q-1}(X,\overline{\Omega})$ by theorem 4.3. As $\mathcal{D}^{p,q-1}(\Omega)$ is dense in $\mathcal{C}^\infty_{p,q-1}(X,\overline{\Omega})$, there exists $(\alpha_j)_{j\in\mathbb{N}} \in \mathcal{D}^{p,q-1}(\Omega)$ such that $\overline{\partial}\alpha_j \underset{j\to+\infty}{\longrightarrow} \overline{\partial}\alpha$ in $\mathcal{C}^\infty_{p,q-1}(X,\overline{\Omega})$.

Hence $< T, \varphi > = < T, \overline{\partial}\alpha > = \lim_{j \to +\infty} < T, \overline{\partial}\alpha_j > = 0$, because $\overline{\partial} T = 0$.

By theorem 4.3, $\overline{\partial}\mathcal{C}^\infty_{p,q}(X, \overline{\Omega})$ is a closed subspace of $\mathcal{C}^\infty_{p,q+1}(X, \overline{\Omega})$, thus a Fréchet space. Using Banach's open mapping theorem, L_T is in fact continuous, so by the Hahn-Banach theorem, we can extend L_T to a continuous linear operator $\tilde{L}_T : \mathcal{C}^\infty_{p,q+1}(X, \overline{\Omega}) \longrightarrow \mathbb{C}$, i.e. $\tilde{L}_T$ is an extensible current on Ω satisfying

$$< \overline{\partial}\tilde{L}_T, \varphi > = (-1)^{p+q} < \tilde{L}_T, \overline{\partial}\varphi > = (-1)^{p+q} < T, \varphi >$$

for every $\varphi \in \mathcal{C}^\infty_{p,q}(X, \overline{\Omega})$. Therefore $T = (-1)^{p+q}\overline{\partial}\tilde{L}_T$. □

For the notion of differential forms admitting distribution boundary values, which is used in the following corollary, we refer the reader to [Lo-To].

Corollary 5.2
Let $\Omega \subset\subset X$ be a $\mathcal{C}^\infty$-smooth pseudoconvex domain in a Kähler manifold with strictly positive holomorphic bisectional curvature. Let f be a smooth $\overline{\partial}$-closed $(0,1)$-form on Ω admitting a distribution boundary value on $\partial\Omega$. Then there exists a smooth function g on Ω admitting a distribution boundary value on $\partial\Omega$ such that $\overline{\partial} g = f$ on Ω.

Proof: As f admits a distribution boundary value, we may view f as an extensible $\overline{\partial}$-closed current on Ω (see [Lo-To]). Applying theorem 5.1, there exists an extensible current S of bidegree $(0,0)$ on Ω such that $\overline{\partial} S = T$.

The hypoellipticity of $\overline{\partial}$ in bidegree $(0,1)$ yields that S is in fact a $\mathcal{C}^\infty$-smooth function on Ω. But a $\mathcal{C}^\infty$-smooth function S, extensible as a current, such $\overline{\partial} S$ admits a distribution boundary value, admits itself a distribution boundary value (see lemma 4.3 in [Sa]). □

6 The $\overline{\partial}$-equation with polynomial growth on some pseudoconvex domains

For $N \in \mathbb{N}$, we define

$$B^{p,q}_N(\Omega, g) = \{u \in \mathcal{C}^\infty_{p,q}(\Omega) \mid \sup_{x \in \Omega} |u|_g(x)\delta^N(x) < +\infty\}.$$

We say that $u \in \mathcal{C}^\infty_{p,q}(\Omega)$ is *of polynomial growth* if $u \in B^{p,q}_N(\Omega, g)$ for some $N \in \mathbb{N}$ and we denote $B^{p,q}_{slow}(\Omega, g) = \bigcup_{N \in \mathbb{N}} B^{p,q}_N(\Omega, g)$.

Theorem 6.1
Let $\Omega \subset\subset X$ be a $\mathcal{C}^\infty$-smooth pseudoconvex domain in a Kähler manifold with strictly positive holomorphic bisectional curvature. Let $f \in \mathcal{C}^\infty_{p,q}(\Omega)$ be of polynomial growth with $\overline{\partial} f = 0$ on Ω, $0 \le p \le n$, $1 \le q \le n$. Then there exists $u \in \mathcal{C}^\infty_{p,q-1}(\Omega)$ of polynomial growth such that $\overline{\partial} u = f$ on Ω.

Proof: By lemma 4.1, there exists $N \gg 1$ such that $f \in L^2_{p,q}(\Omega, \mathbb{C}, N)$. Applying proposition 3.1, there exists $u \in L^2_{p,q-1}(\Omega, \mathbb{C}, N)$ satisfying $\overline{\partial} u = f$. Choosing the minimal L^2-solution, it is no loss of generality to assume $\overline{\partial}^*_N u = 0$, i.e. $\Box_N u = \overline{\partial}^*_N f$.

Taking into account the computation of $\overline{\partial}^*_N$ as in the proof of theorem 4.2 (which works for positive N as well), there exists $L \gg 1$ such that $\overline{\partial}^*_N f \in B^{p,q}_L(\Omega, g)$. By the same arguments as in the proof of theorem 4.2, we can apply theorem 2.2 to the operator $\Box_N$ to deduce that u is of polynomial growth. $\Box$

Corollary 6.2
Let $\Omega \subset\subset X$ be a $\mathcal{C}^\infty$-smooth pseudoconvex domain in a Kähler manifold with strictly positive holomorphic bisectional curvature. Then we have

$$H^q(\Omega, \check{\mathcal{O}}_\Omega) = 0$$

for every $q \ge 1$, where $\check{\mathcal{O}}_\Omega$ is the sheaf of germs on $\overline{\Omega}$ of holomorphic functions admitting a distribution boundary value.

Proof: We will show that

$$0 \longrightarrow \check{\mathcal{O}}_\Omega \longrightarrow B^{0,0}_{slow}(\Omega, g) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} B^{0,n}_{slow}(\Omega, g) \longrightarrow 0$$

is an exact sequence of sheaves on $\overline{\Omega}$.Then the de Rham-Weil theorem yields

$$H^q(\Omega, \check{\mathcal{O}}_\Omega) \cong \frac{\mathrm{Ker}(B^{0,q}_{slow}(\Omega, g) \xrightarrow{\overline{\partial}} B^{0,q+1}_{slow}(\Omega, g))}{\mathrm{Im}(B^{0,q-1}_{slow}(\Omega, g) \xrightarrow{\overline{\partial}} B^{0,q}_{slow}(\Omega, g))}$$

and by theorem 6.1, the right side of the above isomorphism is 0 for $q \ge 1$.

First of all, $\mathrm{Ker}(B^{0,0}_{slow}(\Omega,g) \xrightarrow{\overline{\partial}} B^{0,1}_{slow}(\Omega,g)) = \check{\mathcal{O}}_\Omega$ was proved in [Lo-To].

To prove the exactness of the rest of the above sequence, we need a version of Dolbeault's lemma. Indeed, we have to prove that for every $z_0 \in \overline{\Omega}$ and every open set $V \ni z_0$ of X and $f \in B^{p,q}_{slow}(V \cap \Omega, g)$ such that $\overline{\partial} f = 0$ on $V \cap \Omega$, there exists an open set $U \ni z_0$ contained in V such that $f_{|U\cap\Omega} = \overline{\partial} g$, where g is of polynomial growth on $U \cap \Omega$.

If $z_0 \notin \partial\Omega$, this follows from the usual Dolbeault lemma by taking U to be a small ball contained in V.

Let us now consider the case $z_0 \in \partial\Omega$. We may (see [Oh-Si]) assume that $\partial\Omega = \{r = 0\}$ where r is a smooth function, strictly plurisubharmonic in Ω.

Let $\varphi := \|z - z_0\|^2 - \alpha$ for $\alpha \ll 1$ such that $\{\varphi < 0\} \subset V$. For $\beta \ll 1$, we put $\phi := \max_\beta(r, \varphi)$, which is a smooth function. Here $\max_\beta(t,s) := \frac{t+s}{2} + \chi_\beta(\frac{t-s}{2})$ for $t, s \in \mathbb{R}$, and χ_β is a non-negative real $\mathcal{C}^\infty$-function on $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, $\chi_\beta(x) = \chi_\beta(-x)$, $|x| \leq \chi_\beta(x) \leq |x| + \beta$, $|\chi'_\beta| \leq 1$, $\chi''_\beta \geq 0$ and $\chi_\beta(x) = |x|$ if $|x| \geq \frac{\beta}{2}$.

We claim that ϕ is strictly plurisubharmonic in Ω. Indeed, as $\chi''_\beta \geq 0$, we have

$$i\partial\overline{\partial}\phi \geq \frac{i}{2}(1 + \chi'_\beta(\frac{r-\varphi}{2}))\partial\overline{\partial} r + \frac{i}{2}(1 - \chi'_\beta(\frac{r-\varphi}{2}))\partial\overline{\partial}\varphi.$$

As $|\chi'_\beta| \leq 1$, and $i\partial\overline{\partial} r > 0$, $i\partial\overline{\partial}\varphi > 0$ in Ω, we clearly have $i\partial\overline{\partial}\phi > 0$ in Ω.

Now define $U = \{\phi < 0\}$. Then U is a $\mathcal{C}^\infty$-smooth pseudoconvex domain $\subset\subset X$ such that a small part of ∂U is contained in a small part of $\partial\Omega$ around z_0. Thus U satisfies the assumptions of theorem 6.1, therefore the desired Dolbeault lemma follows from theorem 6.1, which concludes the proof of the corollary. □

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