DOLBEAULT ISOMORPHISM FOR CR MANIFOLDS

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Let X be a complex manifold of complex dimension n. For $p \in \mathbb{N}$ such that $0 \leqslant p \leqslant n$, we denote by Ω_X^p the sheaf of germs of holomorphic p-forms on X. If the cohomology groups of the sheaf Ω_X^p are denoted by $H^r(X, \Omega_X^p)$ and the Dolbeault cohomology groups for $\mathscr{C}^\infty$-smooth differential forms and for currents are respectively denoted by $H_\infty^{p,r}(X)$ and $H_{\mathrm{cur}}^{p,r}(X)$, it follows from the Dolbeault lemma for $\overline{\partial}$ and the de Rham-Weil isomorphism that for $0 \leqslant r \leqslant n$

$$H^r(X, \Omega_X^p) \simeq H_\infty^{p,r}(X) \simeq H_{\mathrm{cur}}^{p,r}(X) .$$

The natural map $H_\infty^{p,r}(X) \to H_{\mathrm{cur}}^{p,r}(X)$ which is actually an isomorphism is called *the Dolbeault isomorphism.*

If M is an oriented $\mathscr{C}^\infty$-smooth CR manifold, it is then natural to ask which relations may exist between the cohomology groups of the sheaf of germs of CR $\mathscr{C}^\infty$-smooth p-forms on M and the cohomology groups of the tangential Cauchy-Riemann complex for $\mathscr{C}^\infty$-smooth differential forms and for currents. Following the method used in the complex case we may hope to get some answer when there exists a Poincaré lemma for the tangential Cauchy-Riemann operator.

Let M be an oriented locally embeddable $\mathscr{C}^\infty$-smooth CR manifold of real dimension $2n-k$ and CR-dimension $n-k$, and assume moreover M is q-concave, $0 \leqslant q \leqslant n-k$, *i.e.* the Levi form of M has at least q positive eigenvalues in all directions. On M we may consider for all p, $0 \leqslant p \leqslant n$, the tangential Cauchy-Riemann complexes

$$[\mathscr{E}^{p,*}](M) : 0 \longrightarrow [\mathscr{E}^{p,0}](M) \xrightarrow{\overline{\partial}_b} [\mathscr{E}^{p,1}](M) \longrightarrow \cdots \longrightarrow [\mathscr{E}^{p,n-k}](M) \longrightarrow 0$$

$$[\mathscr{D}'^{p,*}](M) : 0 \longrightarrow [\mathscr{D}'^{p,0}](M) \xrightarrow{\overline{\partial}_b} [\mathscr{D}'^{p,1}](M) \longrightarrow \cdots \longrightarrow [\mathscr{D}'^{p,n-k}](M) \longrightarrow 0$$

of $\mathscr{C}^\infty$-smooth differential forms and currents, whose cohomology groups are respectively denoted by $H_\infty^{p,r}(M)$ and $H_{\mathrm{cur}}^{p,r}(M)$. Moreover we denote by $H^r(M, \Omega_M^p)$ the Čech cohomology groups with coefficients in the sheaf Ω_M^p of germs of CR $\mathscr{C}^\infty$-smooth p-forms on M.

Let us recall that by results of Henkin [5], Nacinovich [13] and Nacinovich & Valli [14], the Poincaré lemma holds for $\overline{\partial}_b$ in bidegree (p, r) if $1 \leqslant r \leqslant q-1$ and $n-k-q+1 \leqslant r \leqslant n-k$,

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for $q \geqslant 2$, and that each CR distribution extends locally to a holomorphic function as soon as $q \geqslant 1$.

The main result of this paper is the following theorem.

THEOREM 0.1. — *Let M be an orientable locally embeddable $\mathscr{C}^\infty$-smooth q-concave CR manifold of real dimension $2n-k$ and CR dimension $n-k$, $1 \leqslant q \leqslant n$ and p an integer such that $0 \leqslant p \leqslant n$. Then the natural map*

$$H_\infty^{p,r}(M) \longrightarrow H_{\mathrm{cur}}^{p,r}(M)$$

is an isomorphism if $0 \leqslant r \leqslant q-1$ and $n-k-q+2 \leqslant r \leqslant n-k$, is only injective if $r = q$ and only surjective if $r = n-k-q+1$.

For small degrees, *i.e.* $0 \leqslant r \leqslant q$, the theorem is a direct consequence of the Poincaré lemma for the tangential Cauchy-Riemann operator and of the de Rham-Weil isomorphism as in the complex case.

Note that the surjectivity is proved by Hill and Nacinovich in [7] for high degrees, *i.e.* $r \geqslant n-k-q+1$, when M is of hypersurface type, *i.e.* $k=1$. The proof of the result in high degrees is the object of the present paper. Since M is q-concave, $1 \leqslant q \leqslant n-k$, by a result from Hill and Nacinovich [9], without loss of generality we may assume for the proof that M is globally and generically embedded in an n-dimensional complex manifold.

In [8], Hill and Nacinovich have considered the natural map $H^r(M,\Omega_M^p) \longrightarrow H_\infty^r(M)$ and given for it analogous surjectivity and injectivity properties.

As a consequence of Theorem 0.1 and Theorem 0.1 in [11] we get a vanishing theorem of Malgrange's type for the cohomology of the tangential Cauchy-Riemann complex of currents.

COROLLARY 0.2. — *Let M be an orientable locally embeddable $\mathscr{C}^\infty$-smooth, non compact, connected, 1-concave CR manifold of real dimension $2n-k$ and CR dimension $n-k$. Then for $0 \leqslant p \leqslant n$*

$$H_{\mathrm{cur}}^{p,n-k}(M) = 0\,.$$

1. Cohomological preliminaries

Let A be a ring and $X = (X^{p,q})_{p,q\in\mathbb{Z}}$ a double complex of A-modules, which is defined by the morphisms

$$d' = d'_{p,q} : X^{p,q} \to X^{p+1,q} \text{ and } d'' = d''_{p,q} : X^{p,q} \to X^{p,q+1}$$

satisfying the relations

$$d'^2 = 0, d''^2 = 0 \text{ and } d' \circ d'' = d'' \circ d'.$$

For a given $p \in \mathbb{Z}$, we may consider the simple complex $X_I^p = \{X^{p,q}, d''\}$ and in the same way for a given $q \in \mathbb{Z}$, the simple complex $X_{II}^q = \{X^{p,q}, d'\}$, the associated cohomology groups are denoted respectively by $H_{II}^q(X_I^p)$ and $H_I^p(X_{II}^q)$. Moreover the morphism d' (resp. d'') induces a morphism $d'_* : H_{II}^q(X_I^p) \to H_{II}^q(X_I^{p+1})$ (resp. $d''_* : H_I^p(X_{II}^q) \to H_I^p(X_{II}^{q+1})$), with $d'^2_* = 0$

(resp. $d''^2_* = 0$), and the cohomology groups of this complex are denoted by $H_I^p H_{II}^q(X)$ (resp. $H_{II}^q H_I^p(X)$).

Assume X satisfies the following finiteness property:

$$\text{for any } \ell \in \mathbb{Z} \text{ the set } \{(p,q) \in \mathbb{Z} \times \mathbb{Z} \mid p+q=\ell, X^{p,q} \neq 0\} \text{ is finite.} \tag{1}$$

Then it is possible to associate to X a simple complex $s(X)$ by setting $s(X)^\ell = \bigoplus_{p+q=\ell} X^{p,q}$ and for

$$x_\ell = (x_{p,q})_{p+q=\ell} \in s(X)^\ell, \;\; dx_\ell = (d' x_{p-1,q+1} + (-1)^p d'' x_{p,q})_{p+q=\ell}$$

whose cohomology groups are $H^\ell(s(X))$.

Note that the condition (1) is fulfilled in particular if

$$X^{p,q} = 0 \text{ for } p \notin \{0,\dots,k\}, \text{where } k \in \mathbb{N} \text{ is fixed.} \tag{2}$$

Theorem 1.1. — *Let $f : X \to Y$ be a morphism of double complexes, where X and Y both satisfy* (1). *Assume f induces an isomorphism*

$$H_I^p H_{II}^q(X) \longrightarrow H_I^p H_{II}^q(Y), \quad for\ any\ pair\ (p,q). \tag{3}$$

or

$$H_{II}^q H_I^p(X) \longrightarrow H_{II}^q H_I^p(Y), \quad for\ any\ pair\ (p,q). \tag{4}$$

Then the map $s(f) : s(X) \to s(Y)$ induced by f between the associated simple complexes induces for any $r \in \mathbb{Z}$ an isomorphism

$$H^r(s(X)) \longrightarrow H^r(s(Y)).$$

Remark. — Under the hypothesis (3), the theorem is proved in ([10], chap. I). To get the result under the hypothesis (4) we have to apply the previous case to the new double complex $\hat{X}$ defined by $\hat{X}^{p,q} = X^{q,p}$, $\hat{d}'_{p,q} = (-1)^q d''_{q,p}$ and $\hat{d}''_{p,q} = (-1)^p d'_{q,p}$.

For section 3 we need a different version of Theorem 1.1.

Theorem 1.2. — *Let $f : X \to Y$ be a morphism of double complexes, where X and Y both satisfy* (2), *and $q \in \mathbb{Z}$ is a fixed integer. Assume f induces an isomorphism*

$$f_{II}^{p,r} : H_{II}^r(X_I^p) \longrightarrow H_{II}^r(Y_I^p) \ \ for\ 0 \leqslant p \leqslant k\ \ and\ \ r \geqslant q+1$$

and a surjective map

$$f_{II}^{p,q} : H_{II}^q(X_I^p) \longrightarrow H_{II}^q(Y_I^p) \ \ for\ 0 \leqslant p \leqslant k.$$

Then the map $s(f) : s(X) \to s(Y)$ induced by f between the associated simple complexes induces an isomorphism

$$H^r(s(X)) \longrightarrow H^r(s(Y)) \ \ for\ r \geqslant q+k+1$$

and a surjective map

$$H^{q+k}(s(X)) \longrightarrow H^{q+k}(s(Y)).$$

Proof. — The first assertion follows from Theorem 1.1 applied to truncated complexes associated to X and Y. Now let us consider the second assertion. Let $b \in s(Y)^{q+k}$ such that $db = 0$, *i.e.* $b = (b_{0,q+k}, \dots, b_{k,q})$ with $d'' b_{0,q+k} = 0$ and $d' b_{j-1,q+k-j+1} + (-1)^j d'' b_{j,q+k-j} = 0$ for $j = 1, \dots, k$, we have to find $a \in s(X)^{q+k}$ and $c \in s(Y)^{q+k-1}$ such that $da = 0$ and $s(f)(a) = b + dc$. We shall construct $a = (a_{0,q+k}, \dots, a_{k,q})$ and $c = (c_{0,q+k-1}, \dots, c_{k,q-1})$ by induction on the first index.

As $d'' b_{0,q+k} = 0$, by surjectivity of the map $f_{II}^{0,q+k}$, we get $a_{0,q+k} \in X^{0,q+k}$ and $c_{0,q+k-1} \in Y^{0,q+k-1}$ with $d'' a_{0,q+k} = 0$ and $s(f)(a_{0,q+k}) = b_{0,q+k} + d'' c_{0,q+k-1}$.

Now assume that, for some j, $0 \leqslant j \leqslant k-1$, we already got $a_{0,q+k}, \dots, a_{j,q+k-j}$ and $c_{0,q+k-1}, \dots, c_{j,q+k-j-1}$ with $d'' a_{0,q+k} = 0$, $d' a_{p-1,q+k-p+1} + (-1)^p d'' a_{p,q+k-p} = 0$ and

$$(*_p) \quad s(f)(a_{p,q+k-p}) = b_{p,q+k-p} + d' c_{p-1,q+k-p} + (-1)^p d'' c_{p,q+k-p-1}$$

for $p = 0, \dots, j$ (here $c_{-1,q+k} = 0$). Let us construct $a_{j+1,q+k-j-1}$ and $c_{j+1,q+k-j-2}$. As $db = 0$, we have

$$d'' b_{j+1,q+k-j-1} = (-1)^j d' b_{j,q+k-j}.$$

Using $(*_j)$ we get

$$d''(b_{j+1,q+k-j-1} + d' c_{j,q+k-j-1}) = (-1)^j d' s(f)(a_{j,q+k-j}).$$

Hence the injectivity of $f_{II}^{j+1,q+k-j}$ gives the existence of $\tilde{a}_{j+1,q+k-j-1}$ such that

$$d'' \tilde{a}_{j+1,q+k-j-1} = (-1)^j d' a_{j,q+k-j}$$

and

$$d''[s(f)(\tilde{a}_{j+1,q+k-j-1}) - b_{j+1,q+k-j-1} - d' c_{j,q+k-j-1}] = 0.$$

Using the surjectivity of $f_{II}^{j+1,q+k-j-1}$ we can find $\hat{a}_{j+1,q+k-j-1}$ and $c_{j+1,q+k-j-2}$ such that $d'' \hat{a}_{j+1,q+k-j-1} = 0$ and

$$s(f)(a_{j+1,q+k-j-1}) - b_{j+1,q+k-j-1} - d' c_{j,q+k-j-1} = s(f)(\hat{a}_{j+1,q+k-j-1}) + (-1)^{j+1} d'' c_{j+1,q+k-j-2}.$$

It remains to set $a_{j+1,q+k-j-1} = \tilde{a}_{j+1,q+k-j-1} - \hat{a}_{j+1,q+k-j-1}$.

Note that to construct $a_{k,q}$ we used only the injectivity of $f_{II}^{k,q+1}$ and the surjectivity of $f_{II}^{k,q}$. □

We consider now two special cases of Theorem 1.1.

Proposition 1.3. — *Let $A = (A^{p,q})_{p,q\in\mathbb{Z}}$ be a double complex such that $A^{p,q} = 0$ if $p \notin \{0, \dots, k\}$ and $\alpha = (\alpha^{p,q})_{p,q\in\mathbb{Z}}$ a double complex such that $\alpha^{p,q} = 0$ if $p \neq k$ for a given $k \in \mathbb{N}^*$. We assume that for each $q \in \mathbb{Z}$,*

$$H_I^p(A_{II}^q) = 0 \text{ if } p \leqslant k-1 \text{ and } H_I^k(A_{II}^q) = \alpha^{k,q},$$

then the simple associated complexes $s(A)$ and $s(\alpha)$ are quasi-isomorphic.

Moreover $s(\alpha)$ is quasi-isomorphic to the translated complex $\alpha^{k,\cdot}[-k]$ and the inverse of the natural isomorphism between $H^{n+k}(s(A))$ and $H^n(\alpha^{k,\cdot})$ is given by $\overline{x}^n \mapsto \overline{a}^{n+k}$, where $\overline{a}^{n+k}$ is the cohomology class of an $a^{n+k} = (a_{0,n+k}, \dots, a_{k,n})$ such that

$$d' a_{p-1,n+k-p+1} + (-1)^p d'' a_{p,n+k-p} = 0 \text{ for } p = 1, \dots, k$$

and the image of $a_{k,n}$ by the natural projection from $A^{k,n}$ on to $\alpha^{k,n}$ is x^n.

Proposition 1.4. — *Let* $B = (B^{p,q})_{p,q\in\mathbb{Z}}$ *be a double complex such that* $B^{p,q} = 0$ *if* $p \notin \{0,\dots,k\}$ *for a given* $k \in \mathbb{N}^*$ *and* $\beta = (\beta^{p,q})_{p,q\in\mathbb{Z}}$ *a double complex such that* $\beta^{p,q} = 0$ *if* $p \neq 0$. *We assume that for each* $q \in \mathbb{Z}$,

$$H_I^p(B_{II}^q) = 0 \ \text{ if } \ p \geqslant 1 \ \text{ and } \ H_I^0(B_{II}^q) = \beta^{0,q},$$

then the simple associated complexes $s(B)$ *and* $s(\beta)$ *are quasi-isomorphic.*

Moreover $s(\beta)$ *is quasi-isomorphic to* $\beta^{0,\cdot}$ *and the inverse of the natural isomorphism between* $H^n(\beta^{0,\cdot})$ *and* $H^n(s(B))$ *is given by* $\overline{b}^n \to \overline{y}^n$ *where* $\overline{b}^n$ *is the cohomology class of a* $b^n = (b_{0,n},\dots,b_{k,n-k})$ *such that* $(b_{0,n},\dots,b_{k,n-k})$ *and* $(y,0,\dots,0)$ *are in the same cohomology class of* $H^n(s(B))$.

2. Geometrical preliminaries

CR manifolds.

Let X be a complex manifold of complex dimension n. If M is a $\mathscr{C}^\infty$-smooth real submanifold of real codimension k in X, we denote by $T_\tau^{\mathbb{C}}(M)$ the complex tangent space to M at $\tau \in M$. Such a manifold M can be represented locally in the form

$$M = \{z \in \Omega \mid \rho_1(z) = \cdots = \rho_k(z) = 0\} \tag{5}$$

where the ρ_ν's, $1 \leqslant \nu \leqslant k$, are real $\mathscr{C}^\infty$ functions in an open subset Ω of X. In this representation we have

$$T_\tau^{\mathbb{C}} M = \Big\{\zeta \in \mathbb{C}^n \mid \sum_{j=1}^n \frac{\partial \rho_\nu}{\partial z_j}(\tau)\zeta_j = 0, \quad \nu = 1,\dots,k\Big\} \tag{6}$$

and $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M) \geqslant n-k$ for $\tau \in M\cap\Omega$, where $(z_1,\dots,z_n)$ are local holomorphic coordinates in a neighborhood of τ.

The submanifold M is called *CR* if the number $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M)$ is independent of the point $\tau \in M$, and *CR generic* if $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M) = n-k$ for every $\tau \in M$. In the local representation (5), M is generic if and only if $\overline{\partial}\rho_1 \wedge \ldots \wedge \overline{\partial}\rho_k \neq 0$ on M.

Definition 2.1. — *Let* M *be a* $\mathscr{C}^\infty$*-smooth CR generic submanifold of* X. *We say that* M *is* q-concave, $0 \leqslant q \leqslant n-k$, *if for each* $\tau \in M$, *each local representation of* M *of type* (5) *in a neighborhood of* τ *in* X *and each* $x \in \mathbb{R}^k\setminus\{0\}$, *the quadratic form on* $T_\tau^{\mathbb{C}} M$ *defined by* $\sum_{\alpha,\beta} \frac{\partial^2 \rho_x}{\partial z_\alpha \partial \overline{z}_\beta}(\tau)\zeta_\alpha\overline{\zeta}_\beta$, *where* $\rho_x = x_1\rho_1 + \cdots + x_k\rho_k$ *and* $\zeta \in T_\tau^{\mathbb{C}} M$, *has at least* q *negative eigenvalues.*

Tangential Cauchy-Riemann complexes.

Let M be an oriented $\mathscr{C}^\infty$-smooth *CR* generic submanifold of real codimension k in an n-dimensional complex manifold X. We denote by $\mathscr{I}_M$ the ideal sheaf in the Grassmann algebra $\mathscr{E}$ of germs of complex valued $\mathscr{C}^\infty$-forms on X, that is locally generated by functions which vanish on M and by their anti-holomorphic differentials.

On X we have the Dolbeault complexes for the sheaves of germs of smooth forms:

$$\mathscr{E}^{p,*} : 0 \longrightarrow \mathscr{E}^{p,0} \xrightarrow{\overline{\partial}} \mathscr{E}^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{E}^{p,n} \longrightarrow 0,$$

where $\mathscr{E}^{p,j}$ is the sheaf of germs of complex valued $\mathscr{C}^\infty$-forms of bidegree (p, j) on X, $0 \leqslant p, j \leqslant n$. We set $\mathscr{I}_M^{p,j} = \mathscr{I}_M \cap \mathscr{E}^{p,j}$. Since $\overline{\partial}\mathscr{I}_M^{p,j} \subset \mathscr{I}_M^{p,j+1}$, for each $0 \leqslant p \leqslant n$, we have subcomplexes

$$\mathscr{I}_M^{p,*} : 0 \longrightarrow \mathscr{I}_M^{p,0} \xrightarrow{\overline{\partial}} \mathscr{I}_M^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{I}_M^{p,n} \longrightarrow 0$$

of the complex $\mathscr{E}^{p,*}$ and hence quotient complexes $[\mathscr{E}^{p,*}]$, defined by the exact sequences of fine sheaves complexes

$$0 \longrightarrow \mathscr{I}_M^{p,*} \longrightarrow \mathscr{E}^{p,*} \longrightarrow [\mathscr{E}^{p,*}] \longrightarrow 0.$$

The induced differentials are denoted by $\overline{\partial}_M$. We write the quotient complex as

$$[\mathscr{E}^{p,*}] : 0 \longrightarrow [\mathscr{E}^{p,0}] \xrightarrow{\overline{\partial}_M} [\mathscr{E}^{p,1}] \xrightarrow{\overline{\partial}_M} \cdots \xrightarrow{\overline{\partial}_M} [\mathscr{E}^{p,n}] \longrightarrow 0,$$

it is called the *tangential Cauchy-Riemann complex of $\mathscr{C}^\infty$-smooth forms.* If U is an open subset of X, the cohomology groups of $[\mathscr{E}^{p,*}]$ on $M \cap U$ are denoted by $H_\infty^{p,j}(M \cap U)$.

Let $\mathscr{F}_M$ denote the ideal sheaf of germs of smooth complex valued differential forms on X that are flat on M, and set $\mathscr{F}_M^{p,j} = \mathscr{F}_M \cap \mathscr{E}^{p,j}$. Note that $\overline{\partial}\mathscr{F}_M^{p,j} \subset \mathscr{F}_M^{p,j+1}$, therefore $\mathscr{F}_M^{p,*}$ is a subcomplex of $\mathscr{E}^{p,*}$ and the short exact sequence of fine sheaves complex

$$0 \longrightarrow \mathscr{F}_M^{p,*} \longrightarrow \mathscr{E}^{p,*} \longrightarrow W_M^{p,*} \longrightarrow 0$$

defines the complex

$$W_M^{p,*} : 0 \longrightarrow W_M^{p,0} \xrightarrow{\overline{\partial}} W_M^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} W_M^{p,n} \longrightarrow 0$$

of Whitney germs of forms on M.

It follows from the formal Cauchy-Kowalewsky Theorem for CR generic submanifolds of a complex manifold (cf. [1]) that the complexes $[\mathscr{E}^{p,*}]$ and $W_M^{p,*}$ are quasi-isomorphic (see e.g. [13]), more precisely

THEOREM 2.2. — *If M is a CR generic submanifold of real codimension k in X, then for all $p, j \geqslant 0$ and every open subset U of X with $M \cap U \neq \emptyset$, the maps*

$$H^j(M \cap U, W_M^{p,*}) \longrightarrow H_\infty^{p,j}(M \cap U),$$

induced by the natural map $W_M^{p,*} \to [\mathscr{E}^{p,*}]$, are isomorphisms.

In order to define the current $\overline{\partial}_M$-cohomology groups on $M \cap U$, we first consider the spaces $[\mathscr{D}^{p,j}](M \cap U)$ of sections of $[\mathscr{E}^{p,j}]$ having compact support in $M \cap U$ with their usual inductive limit topology.

We define $[\mathscr{D}'^{p,j}](M \cap U)$ as the topological dual of $[\mathscr{D}^{n-p,n-k-j}](M \cap U)$. In this way we obtain, for each $0 \leqslant p \leqslant n$, a complex of sheaves

$$[\mathscr{D}'^{p,*}] : 0 \longrightarrow [\mathscr{D}'^{p,0}] \xrightarrow{\overline{\partial}_M} [\mathscr{D}'^{p,1}] \xrightarrow{\overline{\partial}_M} \cdots \xrightarrow{\overline{\partial}_M} [\mathscr{D}'^{p,n}] \longrightarrow 0$$

whose cohomology on $M \cap U$ we denote by $H^{p,j}_{\mathrm{cur}}(M \cap U)$.

Let $\mathscr{D}'$ be the sheaf of currents on X, we denote by $\mathscr{D}'_M$ the subsheaf of $\mathscr{D}'$ of currents with support contained in M. Dualizing the formal Cauchy-Kowalewsky Theorem, (cf. [7]), it follows that the complexes $[\mathscr{D}'^{\,p,*}]$ and the translated complexes $\mathscr{D}'^{\,p,*}_M[k]$ are quasi-isomorphic, more precisely

THEOREM 2.3. — *If M is a CR generic submanifold of real codimension k in X, then for all $p,\ j \geqslant 0$ and every open subset U of X with $M \cap U \neq \emptyset$, there are natural isomorphisms*

$$H^{p,j}_{\mathrm{cur}}(M \cap U) \longrightarrow H^{p,j+k}(\mathscr{D}'_M(U)).$$

Simplicial complexes associated to a CR manifold.

Here we denote by M a $\mathscr{C}^\infty$-smooth CR generic submanifold of real codimension k in an n-dimensional complex manifold X. Following the reductions in sections 2 and 3 of [8], without loss of generality we may assume that M is globally defined by

$$M = \{z \in X \mid \rho_1(z) = \cdots = \rho_k(z) = 0\} \tag{7}$$

where the ρ_ν's, $1 \leqslant \nu \leqslant k$, are real $\mathscr{C}^\infty$ functions in X satisfying

$$\partial\rho_1 \wedge \ldots \wedge \partial\rho_k \neq 0 \text{ in } X. \tag{8}$$

For each $\nu = 1,\ldots,k$, we set $\varphi_\nu = \rho_\nu + \psi \sum_{j=1}^k \rho_j^2$ and $\varphi_0 = -\sum_{j=1}^k \rho_j + \psi \sum_{j=1}^k \rho_j^2$, where ψ is a positive function of class $\mathscr{C}^\infty$ in X. Next let

$$\sigma = \left\{\lambda = (\lambda_0,\ldots,\lambda_k) \in \mathbb{R}^{k+1} \mid \lambda_i \geqslant 0, \sum_{i=0}^k \lambda_i = 1\right\}$$

be the standard k-simplex, with boundary $\partial\sigma$. For $\lambda \in \partial\sigma$, we set $\varphi_\lambda = \sum_{\nu=0}^k \lambda_\nu \varphi_\nu$. By choosing ψ sufficiently large on each compact subset of M and possibly shrinking X, if M is q-concave, we can arrange that

(9) for every $\lambda \in \partial\sigma$, the Levi form $\mathscr{L}_{\varphi_\lambda}$ of φ_λ has at least $q + k$ positive eigenvalues,

(10) for every ordered collection of k integers $0 \leqslant i_1 < \ldots < i_k \leqslant k$,

$$\partial\varphi_{i_1} \wedge \ldots \wedge \partial\varphi_{i_k} \neq 0 \ \text{ on } \ X,$$

(11) if we set $\Omega_\nu = \{z \in X \mid \varphi_\nu(z) < 0\}$, $\nu = 0,\ldots,k$, then

$$M = \bigcap_{\nu=0}^k \overline{\Omega}_\nu,\ X \backslash M = \bigcup_{\nu=0}^k \Omega_\nu \ \text{ and } \ X = \bigcup_{\nu=0}^k \overline{\Omega}_\nu.$$

Let E be a $\mathscr{C}^\infty$-differentiable vector bundle over X. Given any open subset U in X, we denote by $\Gamma(U,E)$ the space of smooth sections of E over U. If A is a closed subset of U, we denote by $\mathscr{F}_A(U,E)$ the space of sections $f \in \Gamma(U,E)$ that are flat on A. The space $W(A,E)$ of Whitney sections of E over A is defined by the exact sequence

$$0 \longrightarrow \mathscr{F}_A(U,E) \longrightarrow \Gamma(U,E) \longrightarrow W(A,E) \longrightarrow 0.$$

We shall say that two closed subsets A and B of U are *regularly situated* if and only if the sequence

$$0 \longrightarrow W(A\cup B,E) \longrightarrow W(A,E) \oplus W(B,E) \longrightarrow W(A\cap B,E) \longrightarrow 0$$

is exact for any vector bundle E over Ω.

If we consider the system of closed sets $\overline{\mathscr{U}} = \{\overline{\Omega}_0,\dots,\overline{\Omega}_k\}$ defined by (11) then for any choice of $0 \leqslant i_0,\dots,i_r \leqslant k$ and $0 \leqslant j_0,\dots,j_s \leqslant k$, $0 \leqslant r,s \leqslant k$, the closed sets $\overline{\Omega}_{i_0} \cap \dots \cap \overline{\Omega}_{i_r}$ and $\overline{\Omega}_{j_0} \cap \dots \cap \overline{\Omega}_{j_s}$ are regularly situated because of the transversality condition (10) (cf. [15]). Then we can define the space $\mathscr{C}^s_\infty(\overline{\mathscr{U}},E)$, $o \leqslant s \leqslant k$, of alternating cochains of the form

$$f^s = (f_{j_0\cdots j_s}) \quad \text{with} \quad f_{j_0\cdots j_s} \in W(\overline{\Omega}_{j_0} \cap \dots \cap \overline{\Omega}_{j_s}, E)$$

with the coboundary operator

$$(\delta f^s)_{j_0\cdots j_{s+1}} = \sum_{h=0}^{s+1} (-1)^h f_{j_0\cdots \hat{j}_h \cdots j_{s+1}} |_{\Omega_{j_0}\cap\dots\cap\Omega_{j_{s+1}}} .$$

It is easy to verify that

$$\mathscr{C}^0_\infty(\overline{\mathscr{U}},E) \xrightarrow{\delta} \mathscr{C}^1_\infty(\overline{\mathscr{U}},E) \xrightarrow{\delta} \mathscr{C}^2_\infty(\overline{\mathscr{U}},E) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathscr{C}^k_\infty(\overline{\mathscr{U}},E) \longrightarrow 0$$

is a complex. Moreover it follows from Proposition 1 in [13] that the sequence

$$0 \longrightarrow \Gamma(X,E) \xrightarrow{\delta^0} \mathscr{C}^0_\infty(\overline{\mathscr{U}},E) \xrightarrow{\delta} \mathscr{C}^1_\infty(\overline{\mathscr{U}},E) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathscr{C}^k_\infty(\overline{\mathscr{U}},E) \longrightarrow 0, \tag{12}$$

where δ^0 is defined by $\delta^0 f = (f|_{\overline{\Omega}_s})$, is exact.

Let E be a $\mathscr{C}^\infty$-differentiable vector bundle over X. Given any open subset U in X we denote by $\mathscr{D}'(U,E)$ the space of distribution sections of E over U. For a closed subset A of X we denote by $\mathscr{D}'_A(U,E)$ the subspace of distribution sections in $\mathscr{D}'(U,E)$ having support contained in $A \cap U$.

The maps $U \mapsto \mathscr{D}'(U,E)$ and $U \mapsto \mathscr{D}'_A(U,E)$ from the open sets of X to the category of vector spaces define soft sheaves over X, that we will denote by $\mathscr{D}'(E)$ and $\mathscr{D}'_A(E)$ respectively. Let us fix an open subset $\Omega \subset X$, then the sheaf $\check{\mathscr{D}}'_\Omega(E)$ of extensible distributions in Ω with values in E is defined by the exact sequence

$$0 \longrightarrow \mathscr{D}'_{X\setminus\Omega}(E) \longrightarrow \mathscr{D}'(E) \longrightarrow \check{\mathscr{D}}'_\Omega(E) \longrightarrow 0. \tag{13}$$

Then also $\check{\mathscr{D}}'_\Omega(E)$ is a soft sheaf and for any open set U in X the space $\check{\mathscr{D}}'_\Omega(U,E)$ of sections of $\check{\mathscr{D}}'_\Omega(E)$ over U can be identified to the space of distribution sections of E over $\Omega \cap U$ that are restrictions of distribution sections of E over U. In [12], Martineau proves that if $\mathring{\overline{\Omega}} = \Omega$, then for every open set U in X, the space $\check{\mathscr{D}}'_\Omega(U,E)$ is the dual of the subspace of $\Gamma(U,E^*)$ of smooth sections of E^* over U having compact support contained in $\overline{\Omega}$.

Consider the system of open sets $\mathscr{U} = \{\Omega_0,\dots,\Omega_k\}$ defined by (11), the transversality condition (10) implies that it is co-regular, *i.e.* for every open set U in X the system of closed sets $\{U\setminus\Omega_0, \ldots, U\setminus\Omega_k\}$ is regularly situated. Now we can define the space $\check{\mathscr{D}}'^s(\mathscr{U}, E)$, $0 \leqslant s \leqslant k$, of alternating cochains of the form

$$T^s = (T_{j_0\cdots j_s}) \quad \text{with} \quad T_{j_0\cdots j_s} \in \check{\mathscr{D}}'_{\Omega_{j_0}\cap\ldots\cap\Omega_{j_s}}(X, E)$$

with the coboundary map

$$(\delta T^s)_{j_0\cdots j_{s+1}} = \sum_{h=0}^{s+1}(-1)^h T_{j_0\cdots \hat{j}_h\cdots j_{s+1}} |_{\Omega_{j_0}\cap\ldots\cap\Omega_{j_{s+1}}}$$

and we get a complex

$$\check{\mathscr{C}}^0(\mathscr{U}, E) \xrightarrow{\delta} \check{\mathscr{C}}^1(\mathscr{U}, E) \xrightarrow{\delta} \check{\mathscr{C}}^2(\mathscr{U}, E) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \check{\mathscr{C}}^{k-1}(\mathscr{U}, E) \longrightarrow 0.$$

Moreover it follows from Proposition 2 in [14] that the sequence

$$(14) \qquad 0 \longrightarrow \check{\mathscr{D}}'_{X\setminus M}(X, E) \xrightarrow{\delta^0} \check{\mathscr{C}}^0(\mathscr{U}, E) \xrightarrow{\delta} \check{\mathscr{C}}^1(\mathscr{U}, E) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \check{\mathscr{C}}^{k-1}(\mathscr{U}, E) \longrightarrow 0$$

where δ^0 is defined by $\delta^0 T = (T\, |_{\Omega_j})$, is exact.

Combining the exact sequences (13) and (14) we get the exact sequence

$$(15) \qquad 0 \longrightarrow \mathscr{D}'_M(X, E) \longrightarrow \mathscr{D}'(X, E) \xrightarrow{\delta^0} \check{\mathscr{C}}^0(\mathscr{U}, E) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \check{\mathscr{C}}^{k-1}(\mathscr{U}, E) \longrightarrow 0.$$

Let $E^{p,q}$, $0 \leqslant p, q \leqslant n$, be the vector bundle of (p, q)-forms on X. For each $0 \leqslant p \leqslant n$ and $0 \leqslant s \leqslant k-1$, using the $\overline{\partial}$-operator we associate to the systems $\overline{\mathscr{U}}$ and $\mathscr{U}$ the complexes $(\mathscr{C}^s_\infty(\overline{\mathscr{U}}, E^{p,*}), \overline{\partial})$ and $(\check{\mathscr{C}}^s(\mathscr{U}, E^{p,*}), \overline{\partial})$. It is easy to verify that then the coboundary maps δ are morphisms of complexes. We define the complexes $(\mathscr{C}^{-1}_\infty(\overline{\mathscr{U}}, E^{p,*}), \overline{\partial})$ and $(\check{\mathscr{C}}^{-1}(\mathscr{U}, E^{p,*}), \overline{\partial})$ by setting

$$\mathscr{C}^{-1}_\infty(\overline{\mathscr{U}}, E^{p,*}) := \Gamma(X, E^{p,*}) \quad \text{and} \quad \check{\mathscr{C}}^{-1}(\mathscr{U}, E^{p,*}) := \mathscr{D}'(X, E^{p,*}).$$

With the notations of section 1, we can summarize the previous results in the following way, since $\mathscr{C}^k_\infty(\overline{\mathscr{U}}, E^{p,*}) = W(M, E^{p,*})$.

To simplify the notations we set $W^{p,*}_M(X) = W(M, E^{p,*})$ and $\mathscr{D}'^{p,q}_M(X) = \mathscr{D}'(X, E^{p,*})$.

Proposition 2.4. — *For each* $0 \leqslant p \leqslant n$,

1. *The double complex* $C_\infty := (\mathscr{C}^s_\infty(\overline{\mathscr{U}}, E^{p,q}))_{\substack{-1\leqslant s\leqslant k-1\\ 0\leqslant q\leqslant n}}$ *satisfies for all* q

$$H^s_I(C^q_{\infty,II}) = 0 \quad \textit{if} \quad s \leqslant k-2 \quad \textit{and} \quad H^{k-1}_I(C^q_{\infty,II}) = W^{p,q}_M(X).$$

2. *The double complex* $\check{C} := (\check{\mathscr{C}}^s(\mathscr{U}, E^{p,q}))_{\substack{-1\leqslant s\leqslant k-1\\ 0\leqslant q\leqslant n}}$ *satisfies for all* q

$$H^s_I(\check{C}^q_{II}) = 0 \quad \textit{if} \quad s \geqslant 0 \quad \textit{and} \quad H^{-1}_I(\check{C}^q_{II}) = \mathscr{D}'^{p,q}_M(X).$$

As a $\mathscr{C}^\infty$-smooth form on the closure of an open subset of X defines an extensible current, there is a natural map between the double complexes C_∞ and $\check{C}$. Considering Propositions 1.3 and 1.4, it follows from Proposition 2.4 that this natural map induces a map between $H^r(M, W^{p,*}_M)$ and $H^{r+k}(M, \mathscr{D}'^{p,*}_M)$.

PROPOSITION 2.5. — *The map between $H^r(M, W_M^{p,*})$ and $H^{r+k}(M, \mathscr{D}'^{p,*}_M)$, $0 \leqslant r \leqslant n-k$, induces by the natural map between C_∞ and $\check{C}$ is given by $[f] \mapsto (-1)^{\frac{k(k-1)}{2}}[T]$, where T is the current defined by $\langle T, \varphi\rangle = \int_M f \wedge \varphi$ for $\varphi \in \mathscr{D}^{n-p,n-k-r}(X)$ and $[\cdot]$ denotes the cohomology class of the element.*

Proof. — To prove this result we have first to introduce some new sets. Let Γ_i be the subset of Ω_i defined for $i = 0, \ldots, k$ by

$$\Gamma_i = \Omega_i \cap \{z \in X \mid \varphi_i(z) < \varphi_j(z), \quad \forall\, j \neq i\}.$$

For $r \leqslant s \leqslant k$, we set

$$\Gamma_{i_1 \ldots i_s} = \Omega_{i_1} \cap \ldots \cap \Omega_{i_s} \cap \{z \in X \mid \varphi_{i_1}(z) = \cdots = \varphi_{i_s}(z) < \varphi_j(z), \quad \forall\, j \neq i_1, \ldots, i_s\}$$

and $\Gamma_{01\ldots k} = M$. We define an orientation on the $\Gamma_{i_1\cdots i_s}$'s by taking on Γ_i, $0 \leqslant i \leqslant k$, the orientation induced by X and on $\Gamma_{i_1\cdots i_s}$ the orientation of the boundary of $\Gamma_{i_1\cdots \hat{i}_\nu \cdots i_s}$ multiplied by $(-1)^{\nu+1}$. These sets have the following properties : let $I = \{i_1 \cdots i_s\}$ be a multi-index of length $|I| = s$, $1 \leqslant s \leqslant k+1$, then

$$\dim_{\mathbb{R}} \Gamma_I = 2n - |I| + 1 \quad \text{and} \quad \sum_{\nu \notin I} \Gamma_{\nu I} = \partial \Gamma_I. \tag{16}$$

Let $f \in W_M^{p,r}(X)$ be a $\overline{\partial}$-closed Withney section of $E^{p,r}$ over M. Going back to Proposition 1.3, it follows from 1 in Proposition 2.4 that there exists a family of cochains $\gamma^s \in \mathscr{C}^s_\infty(\overline{\mathscr{U}}, E^{p,r+k-s-1})$, $-1 \leqslant s \leqslant k-1$, such that

$$\delta\gamma^s + (-1)^s \overline{\partial}\gamma^{s+1} = 0 \quad \text{for} \quad -1 \leqslant s \leqslant k-2 \quad \text{and} \quad \delta\gamma^{k-1} = f. \tag{17}$$

Recall that $\gamma^s = \{\gamma_I, |I| = s+1\}$, where $\gamma_I \in W(\overline{\Omega}_{i_0} \cap \ldots \cap \overline{\Omega}_{i_s}, E^{p,r+k-s-1})$ if $I = i_0 \cdots i_s$, $0 \leqslant s \leqslant k-1$ and $\gamma^{-1} \in \Gamma(X, E^{p,r+k})$.

Using the natural map between Withney sections and extensible currents the cochain γ^s defines an element, still denoted γ^s, in $\check{\mathscr{C}}^s(\mathscr{U}, E^{p,r+k-s-1})$. Following Proposition 1.4 associated to Proposition 2.4 we have to prove that $((-1)^{\frac{k(k-1)}{2}} T, 0, \ldots, 0)$ and $(\gamma^{-1}, \ldots, \gamma^{k-1})$ are in the same cohomology class of the simple complex associated to the double complex $\check{C}$. Then the proposition will be an immediate consequence of the next lemma. □

In the lemma and in its proof all summations are taken over increasing indices.

LEMMA 2.6. — *The families of the cochains $S^{\ell-1} \in \check{\mathscr{C}}^{\ell-1}(\mathscr{U}, E^{p,r+k-\ell-1})$, $0 \leqslant \ell \leqslant k$ defined for $1 \leqslant \ell \leqslant k-1$ by $S^{\ell-1} = \{S_I, |I| = \ell\}$, where*

$$S_I = (-1)^\ell \sum_{\nu=1}^{k-\ell} (-1)^{1+\sum\limits_{\lambda=1}^{\nu} \lambda} \sum_{|J|=\nu} \int_{\Gamma_J} \gamma_{IJ} \wedge$$

and by

$$S^{-1} = (-1)^\ell \sum_{\nu=1}^{k} (-1)^{1+\sum\limits_{\lambda=1}^{\nu} \lambda} \sum_{|J|=\nu} \int_{\Gamma_J} \gamma_J \wedge$$

and $S^{k-1} = 0$ satisfy

(i) $\delta S^{\ell-2} + (-1)^{\ell-2} \overline{\partial} S^{\ell-1} = \gamma^{l-1}$ *for* $1 \leqslant \ell \leqslant k$

(ii) $\overline{\partial} S^{-1} = \gamma^{-1} - (-1)^{\frac{k(k-1)}{2}} \int_M f \wedge$.

Proof. — We shall use the following notation : let K and $\widetilde{K}$ be two multi-indexes such that $\widetilde{K} \subset K, |\widetilde{K}| = |K| - 1$ we set $\sigma(K, \widetilde{K}) = \mu + 1$ if the missing index in $\widetilde{K}$ is the index that was at rank μ in K. Consider the assertion *(i)*. Let $\varphi \in \mathscr{D}^{n-p,n-k-r+\ell+1}(\Omega_{i_1} \cap \ldots \cap \Omega_{i_\ell})$ and $I = (i_1 \cdots i_\ell)$, $1 \leqslant \ell \leqslant k$, then

$$\langle (\delta S^{\ell-2})_I, \varphi \rangle = \sum_{\substack{|\widetilde{I}|=|I|-1 \\ \widetilde{I}\subset I}} (-1)^{\sigma(I,\widetilde{I})} \langle S_{\widetilde{I}}, \varphi \rangle$$

and

$$\langle \overline{\partial} S_I, \varphi \rangle = (-1)^{p+r+k-\ell} \langle S_I, \overline{\partial}\varphi \rangle.$$

Using the definition of the $S_I's$ and the Stokes formula associated to (16) we get

$$\langle (\delta S^{\ell-2})_I + (-1)^{\ell-2} \overline{\partial} S_I, \varphi \rangle = A_I + B_I + C_I$$

where we have

$$\begin{aligned}
A_I &= (-1)^{\ell-1} \sum_{|J|=1} \int_{\Gamma_J} \sum_{\substack{|\widetilde{I}|=\ell-1 \\ \widetilde{I}\subset I}} (-1)^{\sigma(I,\widetilde{I})} \gamma_{\widetilde{I}J} \wedge \varphi + \sum_{|J|=1} \int_{\Gamma_J} \overline{\partial}\gamma_{IJ} \wedge \varphi \\
B_I &= (-1)^{\ell-1}(-1)^{1+\sum_{\lambda=1}^{k-\ell+1}\lambda} \sum_{|J|=k-\ell+1} \int_{\Gamma_J} \sum_{\substack{|\widetilde{I}|=\ell-1 \\ \widetilde{I}\subset I}} (-1)^{\sigma(I,\widetilde{I})} \gamma_{\widetilde{I}J} \wedge \varphi \\
&\quad + (-1)^{k-\ell}(-1)^{1+\sum_{\lambda=1}^{k-\ell}\lambda} \sum_{|J|=k-\ell+1} \int_{\Gamma_J} \sum_{\substack{|\widetilde{J}|=k-\ell \\ \widetilde{J}\subset J}} (-1)^{\sigma(J,\widetilde{J})} \gamma_{I\widetilde{J}} \wedge \varphi \\
C_I &= \sum_{\nu=2}^{k-\ell} (-1)^{\ell-1}(-1)^{1+\sum_{\lambda=1}^{\nu}\lambda} \sum_{|J|=\nu} \int_{\Gamma_J} \sum_{\substack{|\widetilde{I}|=\ell-1 \\ \widetilde{I}\subset I}} (-1)^{\sigma(I,\widetilde{I})} \gamma_{\widetilde{I}J} \wedge \varphi \\
&\quad + \sum_{\nu=2}^{k-\ell} (-1)^{\nu-1}(-1)^{1+\sum_{\lambda=1}^{\nu-1}\lambda} \sum_{|J|=\nu} \int_{\Gamma_J} \sum_{\substack{|\widetilde{J}|=\nu-1 \\ \widetilde{J}\subset J}} (-1)^{\sigma(J,\widetilde{J})} \gamma_{I\widetilde{J}} \wedge \varphi \\
&\quad - \sum_{\nu=2}^{k-\ell} (-1)^{\nu}(-1)^{1+\sum_{\lambda=1}^{\nu}\lambda} \sum_{|J|=\nu} \int_{\Gamma_J} \overline{\partial}\gamma_{IJ} \wedge \varphi.
\end{aligned}$$

By definition of the operator δ we have

$$(\delta\gamma)_{IJ} = \sum_{\substack{|\widetilde{I}|=|I|-1 \\ \widetilde{I}\subset I}} (-1)^{\sigma(I,\widetilde{I})} \gamma_{\widetilde{I}J} + (-1)^{|I|} \sum_{\substack{|\widetilde{J}|=|J|-1 \\ \widetilde{J}\subset J}} (-1)^{\sigma(J,\widetilde{J})} \gamma_{I\widetilde{J}}.$$

So the relations (17) and the fact that $\overline{\partial} f = 0$ imply

$$\begin{aligned}
C_I &= (-1)^{\ell}\sum_{\nu=2}^{k-\ell}(-1)^{\frac{\nu(\nu+1)}{2}}\sum_{|J|=\nu}\int_{\Gamma_J}((\delta\gamma^{\nu+\ell-2})_{IJ}+(-1)^{\nu+\ell-2}\overline{\partial}\gamma_{IJ})\wedge\varphi=0\\
B_I &= (-1)^{\ell}(-1)^{\frac{(k-\ell+2)(k-\ell+1)}{2}}\sum_{|J|=k-\ell+1}\int_{\Gamma_J}(\delta\gamma^{k-1})_{IJ}\wedge\varphi=0\\
A_I &= \int_{\Omega}\gamma_I\wedge\varphi-\sum_{j=0}^{k}\int_{\Gamma_j}((\delta\gamma^{\ell})_{jI}+(-1)^{\ell}\overline{\partial}\gamma_{jI})\wedge\varphi=\int_{\Omega}\gamma_I\wedge\varphi.
\end{aligned}$$

Let us prove now the assertion *(ii)*. Let $\varphi\in\mathscr{D}^{n-p,n-k-r}(X)$, using again the Stokes formula associated to (16), by definition of S^{-1} we get

$$\begin{aligned}
\langle\overline{\partial}S^{-1},\varphi\rangle &= (-1)^{p+r+k}\langle S^{-1},\overline{\partial}\varphi\rangle\\
&= \sum_{\nu=1}^{k}(-1)^{\sum_{\lambda=1}^{\nu}\lambda}\sum_{|I|=\nu+1}\int_{\Gamma_I}(\delta\gamma^{\nu-1})_I\wedge\varphi-\sum_{\nu=1}^{k}(-1)^{\sum_{\lambda=1}^{\nu+1}\lambda}\sum_{|I|=\nu}\int_{\Gamma_I}\overline{\partial}\gamma_I\wedge\varphi\\
&= (-1)^{\frac{(k+1)(k+2)}{2}}\int_M\delta\gamma^{k-1}\wedge\varphi+\int_{\Omega}\gamma^{-1}\wedge\varphi\\
&\qquad+\sum_{\nu=2}^{k}(-1)^{1+\frac{\nu(\nu+1)}{2}}\sum_{|I|=\nu}\int_{\Gamma_I}\Big((\delta\gamma^{\nu-2})_I+(-1)^{\nu}\overline{\partial}\gamma_I\Big)\wedge\varphi
\end{aligned}$$

Then it follows from (17) that

$$\int_{\Omega}\gamma^{-1}\wedge\varphi-\langle\overline{\partial}S^{-1},\varphi\rangle=(-1)^{\frac{k(k-1)}{2}}\int_M f\wedge\varphi.$$

□

Remark. — We may notice that Proposition 2.4 and 2.5 are still true without q-concavity hypothesis on M, the only thing we have used is the genericity of M.

3. The isomorphism theorem for a single wedge

We denote by X a complex manifold of complex dimension n and by E a holomorphic vector bundle over X. We are going to generalize the notion of q-convex extension (cp.,e.g., [6]) and prove that some results on q-convex extensions are still valid in our case.

DEFINITION 3.1. — *Let X be an n-dimensional complex manifold, q an integer such that $1\leqslant q\leqslant n$ and D a domain in X. We say that X is a* wedge q-convex extension *of D if the following conditions hold:*

(i) D meets all the connected components of X.

(ii) There exist some applications $\rho_j:\mathbb{R}\times U\to\mathbb{R}$ for $j=1,\dots,k$, where U is a neighborhood of $X\setminus D$ such that

a) The maps $t\mapsto\rho_j(t,z)$, $j=1,\dots,k$, are continuous from $\mathbb{R}$ into $\mathscr{C}^{\infty}(U,\mathbb{R})$.

b) For all $t \in \mathbb{R}$, $d\rho_1(t,\cdot) \wedge \cdots \wedge d\rho_k(t,\cdot) \neq 0$ *on* U *and, for each* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *with* $\lambda_j \geqslant 0$ *and* $\sum_{j=1}^{k} \lambda_j = 1$, *the complex Hessian of* $\rho_\lambda(t,\cdot) = \sum_{j=1}^{k} \lambda_j \rho_j(t,\cdot)$ *has at least* $(n-q+1)$ *positive eigenvalues throughout* U.

c) For all $z \in U$ *and all* $j = 1, \ldots, k$, $\rho_j(\cdot, z)$ *is a decreasing function.*

d) $D \cap U = \{z \in U \mid \rho_j(0,z) < 0, \quad j = 1, \ldots, k\}$ *and for each* $t > 0$, *the set* $\{z \in U \mid \rho_j(t,z) < 0\} \cap \complement\overline{D}$ *is relatively compact in* X.

Remark. — If D is a domain in X such that X is an $n-q$-convex extension in the sense of [6][1] then it is clear that X is a wedge q-convex extension of D, it is sufficient to consider one function $\rho(z) - t$, where ρ is the function from Definition 12.1 in [6] (provided ρ is $\mathscr{C}^\infty$) .

Definition 3.2. — *A domain* $D \subset\subset \mathbb{C}^n$ *is called a* local q-convex wedge, $1 \leqslant q \leqslant n$, *if there exists a finite number of* $\mathscr{C}^\infty$ *functions* $\rho_1, \ldots, \rho_k$ *in a neighborhood* $U_{\overline{D}}$ *of* $\overline{D}$ *such that*

$$D = \{z \in U_{\overline{D}} \mid \rho_j(z) < 0, \quad j = 1, \ldots, \ell\}$$

and the following condition holds : if $z \in \partial D$ *and* $1 \leqslant k_1 < \cdots < k_p \leqslant \ell$ *with* $\rho_{k_1}(z) = \cdots = \rho_{k_p}(z) = 0$ *then* $d\rho_{k_1}(z) \wedge \cdots \wedge d\rho_{k_p}(z) \neq 0$ *and for all* $\lambda_1, \ldots, \lambda_p \geqslant 0$ *with* $\lambda_1 + \cdots + \lambda_p = 1$, *the Levi form at* z *of the function* $\lambda_1 \rho_{k_1} + \cdots \lambda_p \rho_{k_p}$ *has at least* $n-q+1$ *positive eigenvalues.*

Definition 3.3. — *Let* X *be an* n*-dimensional complex manifold and* q *an integer such that* $1 \leqslant q \leqslant n$. *If* A_1, A_2, V *are domains in* X, *then we say* $[A_1, A_2, V]$ *is a* wedge q-convex extension element *in* X *if the following conditions are fulfilled*

(i) $A_1 \subset A_2$ *and* $A_2 \backslash A_1 \subset\subset V \subset\subset X$;

(ii) V *is contained in some local coordinate patch and there exists a domain* $D \subset\subset V$ *such that* $D_1 = A_1 \cap D$ *and* $D_2 = A_2 \cap D$ *are biholomorphic to local* q*-convex wedges and* $\overline{D_2 \backslash A_1} \cap \overline{A_1 \backslash D_2} = \varnothing$.

Lemma 3.4. — *Let* $\rho_j : \mathbb{R} \times X \to \mathbb{R}$, $j = 1, \ldots, \ell$, *be some applications with the properties a), b) and c) from Definition 3.1. We set for all* $\alpha \in \mathbb{R}$

$$D_\alpha = \{z \in X \mid \rho_j(\alpha, z) < 0, \quad j = 1, \ldots, \ell\}$$

and we assume there exists $\alpha_0 > 0$ *such that if* $|\alpha| \leqslant \alpha_0$, $|\alpha'| < \alpha_0$ *and* $\alpha' < \alpha$, *then* $D_\alpha \backslash \overline{D}_{\alpha'}$ *is relatively compact in* X.

Then we can find a real number $\varepsilon > 0$ *such that for all* α, β *with* $-\varepsilon \leqslant \alpha \leqslant 0 \leqslant \beta \leqslant \varepsilon$, *there exists a finite number of domains* $(A_i)_{0 \leqslant i \leqslant N}$ *such that*

$$D_\alpha = A_0 \subset A_1 \subset \cdots \subset A_N = D_\beta$$

and for each $j, 1 \leqslant j \leqslant N$, A_j *can be obtained from* A_{j-1} *by a wedge* q*-convex extension element.*

[1] Note that *q-convexity in the sense of Andreotti-Grauert* in [6] is called $(n-q)$*-convexity.*

Proof. — This lemma is a generalization of Lemma 12.3 in [6]. The proof is the same: we define A_p by

$$A_p = \Big\{z \in X \mid \rho_j(\alpha, z) - (\rho_j(\beta, z) - \rho_j(\alpha, z))(\sum_{\nu=1}^{p} \chi_\nu) < 0, \quad j = 1, \dots, \ell\Big\}$$

where $(\chi_\nu)_{1 \leqslant \nu \leqslant N}$ is the partition of unity from the proof of Lemma 12.3 in [6]. □

Replacing in the proof of Lemma 12.4 of [6], the Henkin-Lieb integral operators by the integral operators defined by Barkatou [2] to solve the $\overline{\partial}$-equation with $\mathscr{C}^\infty$ estimates in local q-convex wedges we get

LEMMA 3.5. — *Let $[A_1, A_2, V]$ be a wedge q-convex extension element in X, $1 \leqslant q \leqslant n$.*

i) If $q \leqslant r \leqslant n$ and if U is a neighborhood of $\overline{A_2 \backslash A_1}$, then for any $f_1 \in Z^\infty_{0,r}(\overline{A}_1, E)$, there exist $f_2 \in Z^\infty_{0,r}(\overline{A}_2, E)$ and $u \in \mathscr{C}^\infty_{0,r-1}(\overline{A}_1, E)$, such that $f_1 - f_2 = \overline{\partial} u$ on A_1 and $f_1 = f_2$ on $\overline{A}_1 \backslash U$.

ii) If $q + 1 \leqslant r \leqslant n$, and if U is a neighborhood of $\overline{A_2 \backslash A_1}$, then for any $f \in Z^\infty_{0,r}(\overline{A}_2, E)$ such that $f = \overline{\partial} u_1$ on A_1 with $u_1 \in \mathscr{C}^\infty_{0,r-1}(\overline{A}_1, E)$, there exists $u_2 \in \mathscr{C}^\infty_{0,r-1}(\overline{A}_2, E)$ such that $f = \overline{\partial} u_2$ on A_2 and $u_1 = u_2$ on $\overline{A}_1 \backslash U$.

Following the methods used in paragraph 12 of [6], we deduce from Lemmas 3.4 and 3.5 that Theorem 12.14 in [6] is still partially true for wedge q-convex extensions.

THEOREM 3.6. — *Let D be a domain in X such that X is a wedge q-convex extension of D, $1 \leqslant q \leqslant n$. Then the restriction map $H^{0,r}(X, E) \to H^{0,r}(D, E)$ is an isomorphism for $q + 1 \leqslant r \leqslant n$, and is surjective if $r = q$.*

DEFINITION 3.7. — *A domain D in X with piecewise $\mathscr{C}^\infty$-smooth boundary is called a* strictly q-convex wedge in X, *$1 \leqslant q \leqslant n$, if there exists a finite number of functions ρ_j, $j = 1, \dots, \ell$, of class $\mathscr{C}^\infty$ in a neighborhood $U_{\partial D}$ of ∂D such that*

$$D \cap U_{\partial D} = \{z \in U_{\partial D} \mid \rho_j(z) < 0, \quad j = 1, \dots, \ell\}$$

and the following condition is fulfilled: for all $z \in U_{\partial D}$, $d\rho_1(z) \wedge \cdots \wedge d\rho_\ell(z) \neq 0$ and for each $\lambda = (\lambda_1, \dots, \lambda_\ell)$ with $\lambda_j \geqslant 0$ and $\sum_{j=1}^{\ell} \lambda_j = 1$, the complex Hessian of $\rho_\lambda = \sum_{j=1}^{\ell} \lambda_j \rho_j$ has at least $(n - q + 1)$-positive eigenvalues on $U_{\partial D}$

PROPOSITION 3.8. — *Let D be a strictly q-convex wedge in X, $0 \leqslant q \leqslant n - 1$, and $\rho_1, \dots, \rho_\ell$ some defining functions for D in a neighborhood $U_{\partial D}$ of ∂D. For $\varepsilon > 0$, set $D_\varepsilon = D \cup \{z \in U_{\partial D} \mid \rho_j(z) < \varepsilon, \ j = 1, \dots, \ell\}$, then for sufficiently small ε, D_ε is a wedge q-convex extension of D.*

Proof. — Let $(K_t)_{t \geqslant 0}$ be an increasing family of compact subsets of $U_{\partial D}$ such that $\partial D \subset \cup_{t \geqslant 0} K_t$, $\partial D \cap \overset{\circ}{K}_0 \neq \varnothing$, and if $t < t'$ then $K_t \subset \overset{\circ}{K}_{t'}$. Define by $(\chi_t)_{t \geqslant 0}$ an increasing family of $\mathscr{C}^\infty$-functions with compact support in $\overset{\circ}{K}_t$ such that $0 \leqslant \chi_t \leqslant 1$ on K_t and $\chi_t \equiv 1$ on $K_{t-1/2}$. One can choose the family $(K_t)_{t \geqslant 0}$ and the functions $(\chi_t)_{t \geqslant 0}$ such that the map $t \mapsto \chi_t$ is continuous from $[1, +\infty]$ into $\mathscr{C}^\infty(U_{\partial D}, \mathbb{R})$ and the Levi forms of the χ_t's are uniformly bounded. For $j = 1, \dots, \ell$, set $\rho_j(t, z) = \rho_j(z) - \varepsilon \frac{t}{|t|} \chi_{|t|}(z)$ for $|t| \geqslant 1$, $\rho_j(0, z) = \rho_j(z)$ and

$\rho_j(t,z) = \rho_j(z) - \varepsilon\frac{t^2}{|t|}\chi_1(z)$ for $|t| \leqslant 1$, then the functions $\rho_j(t,z)$, $j = 1,\dots,\ell$, satisfy all the conditions of Definition 2.1 as soon as ε is sufficiently small, moreover $D_\varepsilon = D \cup \cup_{t\geqslant 0}\{z \in U_{\partial D} \mid \rho_j(t,z) < 0,\ j = 1,\dots,\ell\}$ and consequently D_ε is a wedge q-convex extension of D. □

Using Lemmas 3.4 and 3.5, in the same way as in the proof of Theorem 12.15 in [6] we can prove easily

THEOREM 3.9. — *Let D be a strictly q-convex wedge with piecewise $\mathscr{C}^\infty$-smooth boundary in X, $1 \leqslant q \leqslant n$. Then the restriction map $H^{0,r}_\infty(\overline{D},E) \to H^{0,r}_\infty(D,E)$ is an isomorphism if $q+1 \leqslant r \leqslant n$ and is surjective if $r = q$.*

Moreover if $\widetilde{D}$ is a neighborhood of $\overline{D}$ such that $\widetilde{D}$ is a wedge q-convex extension of D, then the restriction map $H^{0,r}_\infty(\widetilde{D},E) \to H^{0,r}_\infty(\overline{D},E)$ is an isomorphism if $q+1 \leqslant r \leqslant n$ and is surjective if $r = q$.

Note that the second assertion of Theorem 3.9 is proved in [8].

In order to extend Theorem 3.9 to current cohomology we need a local result on the Cauchy-Riemann equation for extensible currents.

PROPOSITION 3.10. — *Let $D \subset\subset \mathbb{C}^n$ be a local q-convex wedge, $1 \leqslant q \leqslant n$. For r with $q \leqslant r \leqslant n$, let $T \in \check{\mathscr{D}}'^{0,r}_D(\mathbb{C}^n,E)$ be an extensible current on D such that $\overline{\partial}T = 0$ on D, then there exists $S \in \check{\mathscr{D}}'^{0,r-1}_D(\mathbb{C}^n,E)$ such that $\overline{\partial}S = T$ on D.*

Recall that by the Hahn-Banach theorem (cp. [12]) $\check{\mathscr{D}}'^{0,r}_D(\mathbb{C}^n,E)$ is the dual of the space $\mathscr{D}^{n,n-r}_{\overline{D}}(\mathbb{C}^n,E^*)$ of E^*-valued $\mathscr{C}^\infty$-smooth $(n,n-r)$-forms with support contained in $\overline{D}$. In the proof of Proposition 3.10, we shall proceed by duality and we need the following lemma.

LEMMA 3.11. — *Let $D \subset\subset \mathbb{C}^n$ be a local q-convex wedge, $1 \leqslant q \leqslant n$. Then for $0 \leqslant r \leqslant n-q-1$*

$$\overline{\partial}\mathscr{D}^{0,r}_{\overline{D}}(\mathbb{C}^n,E) = \{f \in \mathscr{D}^{0,r+1}_{\overline{D}}(\mathbb{C}^n,E) \mid \overline{\partial}f = 0\}$$

and hence is a Fréchet space and moreover $\overline{\partial}\mathscr{D}^{0,n-q}_{\overline{D}}(\mathbb{C}^n,E)$ is also a Fréchet space.

Proof. — If D is defined by $D = \{z \in U_{\overline{D}} \mid \rho_j(z) < 0,\ \ j = 1,\dots,\ell\}$ then for sufficiently small $\varepsilon > 0$, $D_\varepsilon = \{z \in U_{\overline{D}} \mid \rho_j(z) < \varepsilon,\ j = 1,\dots,\ell\}$ is also a local q-convex wedge and by smoothing the boundary of D_ε we get a q-convex domain with $\mathscr{C}^\infty$-smooth boundary $\widetilde{D}_\varepsilon$ such that $D \subset \widetilde{D}_\varepsilon \subset D_\varepsilon$. Moreover as $\widetilde{D}_\varepsilon$ is q-complete in the sense of Andreotti-Grauert, we have

$$H^{0,r+1}_c(\widetilde{D}_\varepsilon) = 0 \ \text{ for } \ 0 \leqslant r \leqslant n-q-1$$

and

$$H^{0,n-q+1}_c(\widetilde{D}_\varepsilon) \ \text{ is Hausdorff}$$

which implies that, if $f \in \mathscr{D}^{0,r+1}_{\overline{D}}(\mathbb{C}^n,E)$ verifies $\overline{\partial}f = 0$, if $0 \leqslant r \leqslant n-q-1$, and $\int_{\mathbb{C}^n} f \wedge g = 0$ for all $\overline{\partial}$-closed form $g \in \mathscr{C}^\infty_{n,n-r-1}(\widetilde{D}_\varepsilon)$, if $r = n-q$, then there exists a form $h \in \mathscr{D}^{0,r}(\widetilde{D}_\varepsilon)$ such that $\overline{\partial}h = f$ in $\mathbb{C}^n$.

If $r = 0$, h is holomorphic in $\mathbb{C}^n \setminus \overline{D}$ and vanishes in $\mathbb{C}^n \setminus \widetilde{D}_\varepsilon$, hence $h \equiv 0$ on $\mathbb{C}^n \setminus \overline{D}$ by analytic continuation and therefore $\operatorname{supp} h \subset \overline{D}$ which prove the lemma in this case.

Assume now $1 \leqslant r \leqslant n-q$, then h is $\overline{\partial}$-closed on $\mathbb{C}^n \setminus \overline{D}$ and vanishes in a neighborhood of $\mathbb{C}^n \setminus \widetilde{D}_\varepsilon$. It follows from [3] that there exists a $\mathscr{C}^\infty$-smooth $(0, r-1)$-form u in $\mathbb{C}^n \setminus D$, which vanishes in a neighborhood of $\mathbb{C}^n \setminus \widetilde{D}_\varepsilon$ and satisfies $\overline{\partial} u = h$ in $\mathbb{C}^n \setminus \overline{D}$. Let $\tilde{u}$ be $\mathscr{C}^\infty$-smooth extension of u to $\mathbb{C}^n$ then $\operatorname{supp}(h - \overline{\partial}\tilde{u}) \subset \overline{D}$ and $\overline{\partial}(h - \overline{\partial}\tilde{u}) = f$. So we proved that

$$\overline{\partial}\mathscr{D}^{0,r}_{\overline{D}}(\mathbb{C}^n, E) = \{ f \in \mathscr{D}^{0,r+1}_{\overline{D}}(\mathbb{C}^n, E) \mid \overline{\partial} f = 0\}, \text{ if } 0 \leqslant r \leqslant n-q-1$$

and

$$\overline{\partial}\mathscr{D}^{0,n-q}_{\overline{D}}(\mathbb{C}^n, E) = \{ f \in \mathscr{D}^{0,n-q+1}_{\overline{D}}(\mathbb{C}^n, E) \mid \int_{\mathbb{C}^n} f \wedge g = 0, \ \forall g \in \mathscr{C}^\infty_{n,q-1}(\widetilde{D}_\varepsilon), \overline{\partial} g = 0\}$$

and consequently $\overline{\partial}\mathscr{D}^{0,r}_{\overline{D}}(\mathbb{C}^n, E)$ is a Fréchet space for $0 \leqslant r \leqslant n-q$. □

Proof of Proposition 3.10. — Let $T \in \check{\mathscr{D}}'^{0,r}_{D}(\mathbb{C}^n, E)$ be an extensible current on D such that $\overline{\partial} T = 0$ on D. We define a linear form L_T on $\overline{\partial}\mathscr{D}^{n,n-r}_{\overline{D}}(\mathbb{C}^n, E^*)$ by setting $L_T(\overline{\partial}\varphi) = \langle T, \varphi \rangle$. Note that if $\overline{\partial}\varphi = \overline{\partial}\varphi'$, then $\varphi - \varphi'$ is an $(n, n-r)$-form of class $\mathscr{C}^\infty$, $\overline{\partial}$-closed with support contained in $\overline{D}$. By Lemma 3.11, if $1 \leqslant n-r \leqslant n-q$, there exists $\theta \in \mathscr{D}^{n,n-r-1}_{\overline{D}}(\mathbb{C}^n, E^*)$ such that $\varphi - \varphi' = \overline{\partial}\theta$. Since $\mathscr{D}^{n,n-r-1}(D, E^*)$ is dense in $\mathscr{D}^{n,n-r-1}_{\overline{D}}(\mathbb{C}^n, E^*)$, there exists a sequence $(\theta_j)_{j\in\mathbb{N}} \subset \mathscr{D}^{n,n-r-1}(D, E^*)$ which converges to θ in $\mathscr{D}^{n,n-r-1}_{\overline{D}}(\mathbb{C}^n, E^*)$. Then, as $\overline{\partial} T = 0$ on D we get

$$\langle T, \varphi - \varphi' \rangle = \langle T, \overline{\partial}\theta \rangle = \lim_{j\to\infty} \langle T, \overline{\partial}\theta_j \rangle = 0$$

and therefore L_T is well defined. Now if $n - r = 0$, $\varphi - \varphi'$ is an holomorphic function with compact support and by analytic continuation $\varphi = \varphi'$ and L_T is also well defined.

It follows from the open mapping theorem that L_T is continuous, since by Lemma 3.11, $\overline{\partial}\mathscr{D}^{n,n-r}_{\overline{D}}(\mathbb{C}^n, E^*)$ is a Fréchet space. Now we may apply the Hahn-Banach theorem and extends L_T to a continuous linear form on $\mathscr{D}^{n,n-r+1}_{\overline{D}}(\mathbb{C}^n, E^*)$ which can be identified to an extensible $(0, r-1)$-current $\widetilde{S}$ on D satisfying for all $\varphi \in \mathscr{D}^{n,n-r}_{\overline{D}}(\mathbb{C}^n, E^*)$

$$\langle \widetilde{S}, \overline{\partial}\varphi \rangle = \langle T, \varphi \rangle$$

in particular we get $\overline{\partial}\widetilde{S} = (-1)^r T$ on D and $S = (-1)^r \widetilde{S}$ is a solution of $\overline{\partial} S = T$ on D. □

Remark. — Proposition 3.10 is proved in [16] in the case when D is a completely strictly q-convex domain with $\mathscr{C}^\infty$-smooth boundary of a complex manifold.

As in the proof of Lemma 12.4 of [6], from Proposition 3.10 we obtain

Lemma 3.12. — *Let $[A_1, A_2, V]$ be a wedge q-convex extension element in X, $1 \leqslant q \leqslant n$.*

i) If $q \leqslant r \leqslant n$ and if U is a neighborhood of $\overline{A_2 \setminus A_1}$, then for any $\overline{\partial}$-closed current $T_1 \in \check{\mathscr{D}}^{0,r}_{A_1}(X, E)$ there exists a $\overline{\partial}$-closed current $T_2 \in \check{\mathscr{D}}^{0,r}_{A_2}(X, E)$ and $S \in \check{\mathscr{D}}^{0,r-1}_{A_1}(X, E)$ such that

$$T_1 - T_2 = \overline{\partial} S \text{ on } A_1 \text{ and } T_1 = T_2 \text{ on } A_1 \setminus U.$$

ii) If $q+1 \leqslant r \leqslant n$ and if U is a neighborhood of $\overline{A_2 \setminus A_1}$, then for any $\overline{\partial}$-closed current $T \in \check{\mathscr{D}}^{0,r}_{A_2}(X, E)$ such that $T = \overline{\partial} S_1$ on A_1 with $S_1 \in \check{\mathscr{D}}^{0,r-1}_{A_1}(X, E)$, there exists $S_2 \in \check{\mathscr{D}}^{0,r-1}_{A_2}(X, E)$ such that

$$T = \overline{\partial} S_2 \text{ on } A_2 \text{ and } S_1 = S_2 \text{ on } A_1 \setminus U.$$

If U is an open subset of X, we denote by $\check{H}^{0,r}_{\mathrm{cur}}(U,E)$ the cohomology groups of extensible currents on U, *i.e.* the quotient space

$$\check{\mathscr{D}}'^{0,r}_U(X,E)\cap\ker\overline{\partial}/\overline{\partial}\check{\mathscr{D}}'^{0,r-1}_U(X,E).$$

Using Lemmas 3.4 and 3.12 similar to the proof of Theorem 12.15 in [6], we get easily

THEOREM 3.13. — *Let D be a strictly q-convex wedge with piecewise $\mathscr{C}^\infty$-smooth boundary in X, $1\leqslant q\leqslant n$. Then the natural map $\check{H}^{0,r}_{\mathrm{cur}}(D,E)\to H^{0,r}_{\mathrm{cur}}(D)$, where $\check{H}^{0,r}_{\mathrm{cur}}(D,E)$ denotes the Dolbeault cohomology of the extensible currents on D, is an isomorphism if $q+1\leqslant r\leqslant n$ and is surjective if $r=q$.*

Moreover if $\widetilde{D}$ is a neighborhood of $\overline{D}$ such that $\widetilde{D}$ is a wedge q-convex extension of D then the restriction map $H^{0,r}_{\mathrm{cur}}(\widetilde{D},E)\to\check{H}^{0,r}_{\mathrm{cur}}(D,E)$ is an isomorphism if $q+1\leqslant r\leqslant n$ and is surjective if $r=q$.

From Theorems 3.9 and 3.13, we deduce the following isomorphism corollary

COROLLARY 3.14. — *Let D be a strictly q-convex wedge with piecewise $\mathscr{C}^\infty$-smooth boundary in X, $1\leqslant q\leqslant n$ then the natural map between $H^{0,r}_\infty(\overline{D},E)$ and $\check{H}^{0,r}_{\mathrm{cur}}(D)$ is an isomorphism if $q+1\leqslant r\leqslant n$ and is surjective if $r=q$.*

Proof. — Let $\widetilde{D}$ be a neighborhood of $\overline{D}$ such that $\widetilde{D}$ is a wedge q-convex extension of D then we have the following commutative diagram

$$\begin{array}{ccc} H^{0,r}_\infty(\widetilde{D}) & \longrightarrow & H^{0,r}(\overline{D}) \\ \downarrow & & \downarrow \\ H^{0,r}_{\mathrm{cur}}(\widetilde{D}) & \longrightarrow & \check{H}^{0,r}_{\mathrm{cur}}(D) \end{array}$$

where the horizontal maps are the restriction maps and the vertical ones are the natural maps. By the classical Dolbeault isomorphism for complex manifold the first vertical map is an isomorphism for all $0\leqslant r\leqslant n$, moreover by Theorems 3.9 and 3.13 the horizontal maps are isomorphisms if $q+1\leqslant n$ and surjective if $r=q$, therefore the second vertical map is an isomorphism if $q+1\leqslant r\leqslant n$ and is surjective if $r=q$. □

4. Dolbeault isomorphism in *CR* manifolds

Let M be an oriented $\mathscr{C}^\infty$-smooth CR generic submanifold of real codimension k in an n-dimensional complex manifold X. Following section 2 we can associate to M a family $\{\Omega_0,\Omega_1,\dots,\Omega_k\}$ of open subsets of X. If moreover M is q-concave then by (9) the open subsets $\Omega_{j_0}\cap\dots\cap\Omega_{j_s}$, $0\leqslant j_0,\dots,j_s\leqslant k$, are strictly $(n-q-k+1)$-convex wedges in X. Now let us consider the double complexes C_∞ and $\check{C}$ associated to $\overline{\mathscr{U}}=\{\overline{\Omega}_0,\dots,\overline{\Omega}_k\}$ and $\mathscr{U}=\{\Omega_0,\dots,\Omega_k\}$. It follows from Corollary 3.14 that the double complexes C_∞ and $\check{C}$ satisfy the hypotheses of Theorem 1.2, *i.e.* the natural map between $H^r_{II}(C^s_{\infty,I})$ and $H^r_{II}(\check{C}^s_I)$, $-1\leqslant s\leqslant k-1$, is an isomorphism if $r\geqslant n-k-q+2$ and is surjective if $r=n-k-q+1$. Therefore the natural map between the associated simple complexes induces an isomorphism

$$H^r(s(C_\infty))\longrightarrow H^r(s(\check{C})),\quad r\geqslant n-k-q+2 \tag{18}$$

and a surjective map

$$H^{n-k-q+1}(s(C_\infty)) \longrightarrow H^{n-k-q+1}(s(\check{C}))\,. \tag{19}$$

Moreover from Propositions 1.3, 1.4, 2.4 and 2.5 we get

$$H^r(M, W_M^{p,*}) \longrightarrow H^{r+k}(s(C_\infty)) \longrightarrow H^{r+k}(s(\check{C})) \longrightarrow H^{r+k}(M, \mathscr{D}_M'^{\,p,*}) \tag{20}$$

for $0 \leqslant p \leqslant n$ and $0 \leqslant r \leqslant n-k$, where the first and the last map are isomorphisms and the composed map is given by $[f] \mapsto (-1)^{\frac{k(k-1)}{2}}[T]$, where T is the current defined by $\langle T, \varphi\rangle = \int_M f \wedge \varphi$ for $\varphi \in \mathscr{D}^{n-p,n-k-r}(X)$ and $[\,\cdot\,]$ denotes the cohomology class of the element.

Now associating Theorems 2.2 and 2.3 with (18), (19) and (20) we obtain

Theorem 4.1. — *Let M be an oriented $\mathscr{C}^\infty$-smooth CR generic, q-concave, submanifold, real codimension k in an n-dimensional complex manifold X, $1 \leqslant q \leqslant n-k$. For all integer p such that $0 \leqslant p \leqslant n$, the natural map*

$$H_\infty^{p,r}(M) \longrightarrow H_{\mathrm{cur}}^{p,r}(M)$$

is an isomorphism if $r \geqslant n-k-q+2$ and is surjective if $r = n-k-q+1$.

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