

GROUP COMPLETIONS VIA HILBERT SCHEMES

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ABSTRACT. Let X be a projective variety, homogeneous under a linear algebraic group. We show that the diagonal of X belongs to a unique irreducible component \mathcal{H}_X of the Hilbert scheme of $X \times X$. Moreover, \mathcal{H}_X is isomorphic to the “wonderful completion” of the connected automorphism group of X ; in particular, \mathcal{H}_X is non-singular. We describe explicitly the degenerations of the diagonal in $X \times X$, that is, the points of \mathcal{H}_X ; these subschemes of $X \times X$ are reduced and Cohen-Macaulay.

INTRODUCTION

Let G be an adjoint semisimple algebraic group over an algebraically closed field. Then G admits a canonical smooth completion \bar{G} , such that: (a) the action of $G \times G$ by left and right multiplication on G extends to \bar{G} , (b) the boundary $\bar{G} - G$ is a union of smooth irreducible divisors intersecting transversally along the unique closed $G \times G$ -orbit, and (c) the partial intersections of these boundary divisors are exactly the orbit closures. This “wonderful” completion was constructed by De Concini and Procesi [7] in characteristic zero, and then by Strickland [18] in arbitrary characteristics, via representation theory.

In this article, we obtain algebro-geometric realizations of \bar{G} , as follows. Choose a parabolic subgroup P of G and consider the projective variety $G/P = X$. Regard the diagonal in $X \times X$ as a point of the Hilbert scheme $\text{Hilb}(X \times X)$. The group $G \times G$ acts on $\text{Hilb}(X \times X)$ via its natural action on $X \times X$; if G acts faithfully on X , then the $G \times G$ -orbit of the diagonal is isomorphic to G . We show that the closure of this orbit is isomorphic to \bar{G} (Theorem 3). If moreover G is the full connected automorphism group of X , this realizes \bar{G} as the irreducible component \mathcal{H}_X of $\text{Hilb}(X \times X)$ through the diagonal (Lemma 2).

In the case where G is the projective linear group $\text{PGL}(n + 1)$, the wonderful completion \bar{G} is the classical “space of complete collineations”, and the most natural choice for X is the projective space \mathbb{P}^n . In fact, the realization of \bar{G} as the orbit closure of the diagonal in the Hilbert scheme of $\mathbb{P}^n \times \mathbb{P}^n$ is due to M. Thaddeus [19], in the more

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general setting of “complete collineations of a given rank”. The approach of Thaddeus is inspired by Kapranov’s work on the moduli space of stable punctured curves of genus 0, see [10], [11]. It proceeds through several other interesting constructions of \bar{G} , either as an iterated blow-up or as a quotient, that do not seem to extend to other groups.

For this reason, we follow an alternative approach based on results of [4] and [5]. A flat family over \bar{G} was constructed there, whose general fibers are the $G \times G$ -translates of the diagonal in $X \times X$. The resulting morphism $\bar{G} \rightarrow \text{Hilb}(X \times X)$ turns out to be an isomorphism over its image (Theorem 3). As a consequence, we describe the fibers over \bar{G} of the universal family of $\text{Hilb}(X \times X)$, that is, the scheme-theoretic degenerations of the diagonal. These subschemes turn out to be reduced and Cohen-Macaulay. They are obtained by moving a union of Schubert varieties in $X \times X$ by the diagonal action of a Levi subgroup, see Proposition 10. The case where X is the full flag variety of G turns out to be nicer, e.g., all degenerations are Gorenstein. (This does not even extend to $X = \mathbb{P}^2$, see the remark at the end of Section 3.)

This approach also yields algebro-geometric realizations of the “wonderful symmetric varieties” of [7], [8]; moreover, the Hilbert scheme may be replaced everywhere with the Chow variety, in characteristic zero (Theorem 12 and Corollary 15). This generalizes results of Thaddeus [19] concerning complete quadrics and complete skew forms.

As another generalization, one may consider a compact Kähler manifold X and its connected automorphism group G ; then the $G \times G$ -orbit closure \mathcal{H}_X of the diagonal in the Douady space of $X \times X$ is again an equivariant compactification of G , which may be of interest (this construction is used in [17] to compactify the G -action on X). The description of \mathcal{H}_X is easy if X is homogeneous: for, by a theorem of Borel and Remmert (see [1] 3.9), X is then the product of its Albanese variety $A(X)$, a complex torus, with a flag variety Y as above. And one checks that $\mathcal{H}_{Y \times A(X)} \cong \mathcal{H}_Y \times \mathcal{H}_{A(X)} \cong \mathcal{H}_Y \times A(X)$.

More generally, for any integer $m \geq 2$, the irreducible component of the “small” diagonal in the Hilbert or Douady space of X^m is an equivariant compactification of the homogeneous space G^m/G , where $G = \text{Aut}^0(X)$ is embedded diagonally in G^m . Are some of these completions smooth? In the case where $G = \text{PGL}(n+1)$, could they be related to those constructed by Lafforgue [14]?

This work is organized as follows. Section 1 contains preliminary constructions and remarks concerning connected automorphism groups of arbitrary projective varieties. Our main result, Theorem 3, is stated and proved in Section 2, and then extended to wonderful symmetric varieties. As an application, all degenerations of the diagonal in $G/P \times G/P$ are described in Section 3. In Section 4, we adapt our constructions and results to Chow varieties, assuming that the characteristic of k is zero. We obtain a

slightly stronger version of Theorem 3 under this assumption, with a much simpler proof.

We use [16] as a general reference for linear algebraic groups, and [13] for Hilbert schemes.

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1. COMPLETIONS OF CONNECTED AUTOMORPHISM GROUPS

In this section, we consider an arbitrary projective algebraic variety X over an algebraically closed field k . Recall that the automorphism group $\text{Aut}(X)$ is an algebraic group scheme over k with Lie algebra $\text{Hom}(\Omega_X^1, \mathcal{O}_X)$, where Ω_X^1 denotes the sheaf of Kähler differentials of X (see [13] I.2.16.4). Let G be a closed connected subgroup scheme of $\text{Aut}(X)$, with Lie algebra \mathfrak{g} . We shall assume that G is smooth; this assumption is always fulfilled in characteristic zero.

The group $G \times G$ acts on $X \times X$, and thus on its Hilbert scheme $\text{Hilb}(X \times X)$. Let $\mathcal{H}_{X,G}$ be the $G \times G$ -orbit closure in $\text{Hilb}(X \times X)$ of the diagonal, $\text{diag}(X)$; we endow $\mathcal{H}_{X,G}$ with its reduced subscheme structure.

Lemma 1. *With preceding notation, $\mathcal{H}_{X,G}$ is a projective $G \times G$ -equivariant completion of the homogeneous space $(G \times G)/\text{diag}(G)$.*

Proof. Clearly, the isotropy group of $\text{diag}(X)$ in $G \times G$ equals $\text{diag}(G)$, as a set. Thus, it suffices to check that the isotropy Lie algebra of $\text{diag}(X)$ is $\text{diag}(\mathfrak{g})$. To see this, recall that the Zariski tangent space to $\text{Hilb}(X \times X)$ at $\text{diag}(X)$ is

$$T_{\text{diag}(X)}\text{Hilb}(X \times X) = \text{Hom}(\mathcal{I}_{\text{diag}(X)}/\mathcal{I}_{\text{diag}(X)}^2, \mathcal{O}_{\text{diag}(X)}) = \text{Hom}(\Omega_X^1, \mathcal{O}_X).$$

Moreover, the differential of the morphism

$$G \times G \rightarrow \text{Hilb}(X \times X), (g, h) \mapsto (g, h) \cdot \text{diag}(X)$$

at the identity element of $G \times G$, is the map

$$\mathfrak{g} \times \mathfrak{g} \rightarrow \text{Hom}(\Omega_X^1, \mathcal{O}_X), (x, y) \mapsto x - y.$$

□

Note that $(G \times G)/\text{diag}(G)$ is isomorphic to G via $(g, h) \mapsto gh^{-1}$; moreover, the subscheme $(g, h) \cdot \text{diag}(X)$ is the graph of $hg^{-1} = (gh^{-1})^{-1}$. Thus, the closed points of $\mathcal{H}_{X,G}$ are the graphs of elements of G , together with their limits as closed subschemes.

If $\text{Aut}(X)$ is smooth, then we can take $G = \text{Aut}^0(X)$; we then set $\mathcal{H}_{X,G} = \mathcal{H}_X$. By the preceding argument, the Zariski tangent spaces at $\text{diag}(X)$ of \mathcal{H}_X and of $\text{Hilb}(X \times X)$ coincide; we thus obtain

Lemma 2. *With preceding notation, $\text{diag}(X)$ is a non-singular point of $\text{Hilb}(X \times X)$. Moreover, \mathcal{H}_X is the irreducible component of $\text{Hilb}(X \times X)$ through $\text{diag}(X)$.*

The $G \times G$ -action on $\mathcal{H}_{X,G}$ can be interpreted in terms of two well-known operations on $\text{Hilb}(X \times X)$. The first one is the ‘‘convolution’’ product $*$, defined by

$$Z_1 * Z_2 = p_{13*}(p_{12}^* Z_1 \cap p_{23}^* Z_2)$$

where $p_{ij} : X \times X \times X \rightarrow X \times X$ denotes the projection to the (i, j) -factor, and \cap denotes scheme-theoretic intersection. The second one is the ‘‘transposition’’, the involution of $\text{Hilb}(X \times X)$ induced by the involution $(x, y) \mapsto (y, x)$ of $X \times X$; this involution will be denoted $Z \mapsto Z^{-1}$.

For every automorphism g of X , with graph Γ_g , we have $\Gamma_g^{-1} = \Gamma_{g^{-1}}$. Moreover, left (resp. right) convolution with Γ_g is the action of (g^{-1}, id) (resp. (id, g)) on $\text{Hilb}(X \times X)$. Thus, the transposition extends to $\mathcal{H}_{X,G}$ the inverse map on G , and the $G \times G$ -action on $\mathcal{H}_{X,G}$ can be viewed as left and right convolution with graphs of elements of G .

However, $\mathcal{H}_{X,G}$ is generally not invariant under convolution (thus, convolution is not a morphism.) For example, if $X = \mathbb{P}^1$ and $G = \text{PGL}(2)$, then $\mathcal{H}_{X,G} = \mathcal{H}_X$ contains the reduced subscheme $Z_x = \mathbb{P}^1 \times \{x\} \cup \{x\} \times \mathbb{P}^1$ for every $x \in \mathbb{P}^1$, and $Z_x * Z_x$ equals $\mathbb{P}^1 \times \mathbb{P}^1$.

Finally, we shall need a ‘‘twisted’’ version of $\mathcal{H}_{X,G}$. Let σ be an automorphism of the algebraic group G , such that the fixed point subgroup scheme

$$G^\sigma = \{g \in G \mid \sigma(g) = g\}$$

is smooth (e.g., G is linear and σ is semisimple). Consider another variety Y endowed with an action of G and with an isomorphism

$$f : X \rightarrow Y,$$

such that $f(g \cdot x) = \sigma(g) \cdot f(x)$ for all $(g, x) \in G \times X$. Let

$$\Gamma_f \subseteq X \times Y$$

be the graph of f . The stabilizer of Γ_f under the diagonal action of G on $X \times Y$ is G^σ . Let $\mathcal{H}_{f,G}$ be the G -orbit closure of Γ_f in the Hilbert scheme of $X \times Y$, then $\mathcal{H}_{f,G}$ is a projective G -equivariant completion of the homogeneous space G/G^σ ; its points are graphs of G -conjugates of f , together with their limits as subschemes.

Note that the isomorphism

$$id \times f^{-1} : X \times Y \rightarrow X \times X$$

maps Γ_f to $\text{diag}(X)$. Thus, $\mathcal{H}_{f,G}$ embeds into $\mathcal{H}_{X,G}$ as a closed subvariety, invariant under the ‘‘twisted’’ action of G defined by

$$g * Z = (g, \sigma^{-1}(g)) \cdot Z.$$

The restriction to the open orbit $G \cdot \Gamma_f \cong G/G^\sigma$ of both embeddings identifies to the map

$$G/G^\sigma \rightarrow G, \quad gG^\sigma \mapsto g\sigma^{-1}(g^{-1}),$$

equivariant for the G -action on G/G^σ by left multiplication, and for the twisted G -action on G .

Example 1. Consider a finite dimensional k -vector space V endowed with a nondegenerate quadratic form q ; we assume that the characteristic of k is not 2. This defines a symmetric isomorphism $V \rightarrow V^*$ and hence an isomorphism $f : \mathbb{P}(V) \rightarrow \mathbb{P}(V^*)$, together with an involutive automorphism σ of $\mathrm{PGL}(V)$ mapping every element to the inverse of its adjoint with respect to q . Clearly, we have $f(g \cdot x) = \sigma(g) \cdot f(x)$ for all $(g, x) \in \mathrm{PGL}(V) \times \mathbb{P}(V)$, and σ is semisimple with fixed point subgroup the projective orthogonal group, $\mathrm{PO}(V, q)$. Thus, we obtain an equivariant completion \mathcal{H} of the homogeneous space $\mathrm{PGL}(V)/\mathrm{PO}(V, q)$. The latter is the space of non-singular quadrics in $\mathbb{P}(V)$, whereas \mathcal{H} is the closure of the locus of graphs of symmetric isomorphisms $\mathbb{P}(V) \rightarrow \mathbb{P}(V^*)$, in the Hilbert scheme of $\mathbb{P}(V) \times \mathbb{P}(V^*)$.

It was shown by Thaddeus that \mathcal{H} is isomorphic to the “space of complete quadrics” in $\mathbb{P}(V)$, see [19] §10. Moreover, replacing the quadratic form q by a non-degenerate skew form yields the “space of complete skew forms”, see [19] §11. Both spaces are classical examples of wonderful symmetric varieties, see [7]; in fact, Thaddeus’ result will be generalized to all wonderful symmetric varieties in the next section.

2. WONDERFUL COMPLETIONS VIA HILBERT SCHEMES OF FLAG VARIETIES

In this section, we consider a connected linear algebraic group G and a parabolic subgroup P such that G acts faithfully on $G/P = X$. Then G is semisimple adjoint. Moreover, the group scheme $\mathrm{Aut}^0(X)$ is smooth and is semisimple adjoint as well; this group equals G as a rule, and all exceptions are known (see [6]). Thus, we may consider the equivariant completion $\mathcal{H}_{X,G}$ (resp. \mathcal{H}_X); we aim at proving that it is the wonderful completion of G (resp. of $\mathrm{Aut}^0(X)$). For this, we review results of [4] and [5].

Let \bar{P} be the closure of P in the wonderful completion \bar{G} . Since \bar{P} is invariant under the action of $P \times P$ on \bar{G} , we can form the associated fiber bundle

$$p : G \times G \times^{P \times P} \bar{P} \rightarrow (G \times G)/(P \times P) = X \times X,$$

a locally trivial $G \times G$ -equivariant fibration with fiber \bar{P} . Moreover, the inclusion of \bar{P} into \bar{G} extends uniquely to a $G \times G$ -equivariant map

$$\pi : G \times G \times^{P \times P} \bar{P} \rightarrow \bar{G}.$$

The product map

$$p \times \pi : G \times G \times^{P \times P} \bar{P} \rightarrow X \times X \times \bar{G}$$

is a closed immersion; its image is the “incidence variety”

$$\{(gP, hP, x) \mid (g, h) \in G \times G, x \in (g, h)\bar{P}\}.$$

Thus, the fibers of π identify to closed subschemes of $X \times X$ via p_* ; this identifies the fiber at the identity element of G , to $\text{diag}(X)$. By [4] 1.6, π is equidimensional; moreover, by [5] §7, \bar{P} is Cohen-Macaulay, and the fibers of π are reduced. It follows that π is flat, with Cohen-Macaulay fibers.

By the universal property of the Hilbert scheme and the definition of $\mathcal{H}_{X,G}$, we thus obtain a morphism

$$\varphi : \bar{G} \rightarrow \mathcal{H}_{X,G}$$

that maps every $\gamma \in \bar{G}$ to the subscheme $p_*(\pi^*\gamma)$ of $X \times X$.

Theorem 3. *With preceding notation, φ is an isomorphism. As a consequence, the restriction to $\mathcal{H}_{X,G}$ of the universal family over $\text{Hilb}(X \times X)$ identifies to π , so that its fibers are reduced Cohen-Macaulay subschemes of $X \times X$.*

Proof. Choose opposite Borel subgroups B, B^- of G , with common torus T and unipotent radicals U, U^- . By the Bruhat decomposition, the product map

$$U \times T \times U^- \rightarrow G$$

is an open immersion. By [7] and [18], this map extends to an open immersion

$$U \times \bar{T}_0 \times U^- \rightarrow \bar{G}_0,$$

where \bar{T}_0 is an affine open subset of the closure of T in \bar{G} . Moreover, \bar{T}_0 is isomorphic to an affine space, and it meets every $G \times G$ -orbit in \bar{G} along a unique $T \times T$ -orbit. As a consequence, \bar{G}_0 is an $B \times B^-$ -invariant open affine subset of \bar{G} , and \bar{T}_0 meets the closed $G \times G$ -orbit (isomorphic to $G/B^- \times G/B$) at the unique point $z_0 = B^-/B^- \times B/B$.

It suffices to show that $\varphi : \bar{G} \rightarrow \mathcal{H}_{X,G}$ is bijective, and restricts to an isomorphism over \bar{G}_0 . For the latter assertion (the main point of the proof), we shall show that every subscheme $Z \in \varphi(\bar{G}_0)$ intersects transversally certain Schubert varieties in $X \times X$. Mapping Z to its common points with these Schubert varieties, and to the Zariski tangent spaces of Z at these points, will yield appropriate morphisms from $\varphi(\bar{G}_0)$ to U, U^- and \bar{T}_0 , and hence an inverse to $\varphi|_{\bar{G}_0}$.

We need more notation, and results from [5]. Choose for B a Borel subgroup of P ; let L be the Levi subgroup of P containing the maximal torus T . Let W be the Weyl group of (G, T) and let Φ be the root system of (G, T) , with the subset Φ^+ of positive roots defined by B and the corresponding basis Δ . Every $\alpha \in \Delta$ defines a simple reflection $s_\alpha \in W$. The s_α ($\alpha \in \Delta$) generate W ; this defines the length function ℓ on that group. Let w_Δ be the unique element of maximal length; then $B^- = w_\Delta B w_\Delta$. Let I be the subset of Δ consisting of all simple roots of (L, T) , and let Φ_I be the

corresponding root subsystem of Φ . The Weyl group of (L, T) is the Weyl group W_I of the root system Φ_I ; it is generated by the s_α , $\alpha \in I$. Let

$$W^I = \{w \in W \mid w(I) \subseteq \Phi^+\},$$

then W^I consists of all elements of W of minimal length in their right W_I -coset; it is a system of representatives of the quotient W/W_I . The latter identifies to the double coset space $B \backslash G/P$, via $w \mapsto BwP/P$.

Recall that the T -fixed points in G/P are the $e_w = wP/P$ for $w \in W^I$, and the B -orbits (the Bruhat cells) are the

$$C_w = B \cdot e_w \cong U/U \cap {}^wP$$

(where ${}^wP = wPw^{-1}$); the Schubert varieties are their closures

$$X_w = \overline{B \cdot e_w}.$$

We shall also need the opposite Bruhat cells

$$C_w^- = B^- \cdot e_w,$$

and the opposite Schubert varieties

$$X_w^- = \overline{B^- \cdot e_w} = w_\Delta X_{w_\Delta w w_I}$$

(note that W^I is invariant under $w \mapsto w_\Delta w w_I$).

By [4] 1.6, we have

$$p(\pi^{-1}(z_0)) = \bigcup_{w \in W^I} X_w^- \times X_w$$

as sets. Thus, the same holds for the scheme $\varphi(z_0) = Z_0$, since it is reduced.

Lemma 4. *The isotropy group of Z_0 in $G \times G$ equals $B^- \times B$ (as sets). Moreover, φ is bijective.*

Proof. The isotropy group $\text{Stab}_{G \times G}(Z_0)$ contains $B^- \times B$; thus, it can be written as $Q^- \times Q$ for parabolic subgroups Q^- containing B^- , and Q containing B . Since Q is connected, it stabilizes all Schubert varieties X_w ($w \in W^I$). As a consequence, Q stabilizes all Bruhat cells.

If $Q \neq B$, then we can choose $\alpha \in \Delta$ such that s_α has a representative in Q . Let $w \in W^I$, then $Bs_\alpha BwP = BwP$, so that $s_\alpha w \in wW_I$ by the Bruhat decomposition. Thus, $w^{-1}(\alpha) \in \Phi_I$. Since this holds for all w in a system of representatives of W/W_I , it follows that the orbit $W\alpha$ is contained in Φ_I . Therefore, Φ_I contains the intersection of Φ with the linear span of $W\alpha$. As a consequence, P contains a non-trivial closed normal subgroup of G . But this contradicts the assumption that G acts faithfully on G/P .

Thus, $Q = B$; likewise, $Q^- = B^-$. In other words, the restriction of φ to the closed $G \times G$ -orbit is bijective. Using [12] §7, it follows that φ is the normalization of its image; in particular, φ is finite. Moreover, by [7] and [8], the $G \times G$ -isotropy group

of every point of \bar{G} is connected, and the conjugacy class of that group determines the orbit uniquely (an explicit description of these isotropy groups will be recalled in Section 3). Therefore, φ is bijective. \square

As a consequence, $\varphi(\bar{G}_0) = (U \times U^-) \cdot \varphi(\bar{T}_0)$ is an open subset of $\varphi(\bar{G}) = \mathcal{H}_{X,G}$. We now investigate the structure of this subset, in a succession of lemmas.

Lemma 5. *For all $Z \in \varphi(\bar{T}_0)$ and $w \in W^I$, the scheme-theoretic intersection of Z with $X_w \times X_w^-$ is supported at the unique point (e_w, e_w) , and this intersection is transversal.*

Proof. Note that $\varphi(1) = \text{diag}(X)$ contains the $T \times T$ -fixed point (e_w, e_w) ; therefore, this point belongs to every $Z \in \varphi(\bar{T})$.

We first determine the intersection $Z_0 \cap (X_w \times X_w^-)$. It is invariant under $T \times T$, and contains (e_w, e_w) . Let (e_u, e_v) be another fixed point. Then, by the description of Z_0 , there exists $x \in W^I$ such that: $e_u \in X_x^- \cap X_w$ and $e_v \in X_x \cap X_w^-$. Thus, $u \leq ww_I$ for the Bruhat ordering on W , so that $u \leq w$. Moreover, $e_u \in w_\Delta X_{w_\Delta x w_I}$, so that $x \leq u$. Likewise, we have $w \leq v \leq x$. Therefore, we must have $x = w = u = v$, and (e_w, e_w) is the unique $T \times T$ -fixed point of the set $Z_0 \cap (X_w \times X_w^-)$. It follows that (e_w, e_w) is the unique point of that set. Since X_w and X_w^- intersect transversally at e_w , we see that Z_0 and $X_w \times X_w^-$ intersect transversally at (e_w, e_w) .

To extend this to all $Z \in \varphi(\bar{T}_0)$, consider the universal family \mathcal{U} of $\text{Hilb}(X \times X)$, and its pullback $\mathcal{U}_{\bar{T}_0}$ to \bar{T}_0 . This is a closed subscheme of $X \times X \times \bar{T}_0$; the scheme-theoretic intersection

$$\mathcal{U}_{\bar{T}_0} \cap (X_w \times X_w^- \times \bar{T}_0)$$

is invariant under the natural action of $T \times T$, and the second projection

$$q : \mathcal{U}_{\bar{T}_0} \cap (X_w \times X_w^- \times \bar{T}_0) \rightarrow \bar{T}_0$$

is equivariant and proper. By the preceding discussion, the scheme-theoretic fiber of q at is the closed point (e_w, e_w, z_0) . Since each $T \times T$ -orbit closure in \bar{T}_0 contains z_0 , it follows that all fibers of q are finite. Thus, q is finite, and $\mathcal{U}_{\bar{T}_0} \cap (X_w \times X_w^- \times \bar{T}_0)$ is affine. Now a version of Nakayama's lemma implies that q is an isomorphism. \square

By Lemma 5 and the structure of \bar{G}_0 , every $Z \in \varphi(\bar{G}_0)$ intersects $X_w \times X_w^-$ transversally, at a unique point of $C_w \times C_w^-$; let $p_w(Z)$ be that point.

Lemma 6. *Every $p_w : \varphi(\bar{G}_0) \rightarrow C_w \times C_w^-$ is a $B \times B^-$ -equivariant morphism, and the image of the product map*

$$\prod_{w \in W^I} p_w : \varphi(\bar{G}_0) \rightarrow \prod_{w \in W^I} C_w \times C_w^-$$

is closed and isomorphic to $U \times U^-$.

Proof. Consider again the universal family of $\text{Hilb}(X \times X)$, and its restriction $\mathcal{U}_{\varphi(\bar{G}_0)}$ to $\varphi(\bar{G}_0)$. Then the scheme-theoretic intersection

$$\mathcal{U}_{\varphi(\bar{G}_0)} \cap (X_w \times X_w^- \times \varphi(\bar{G}_0))$$

projects isomorphically to $\varphi(\bar{G}_0)$, for every fiber is a point. It follows that p_w is a morphism; it is clearly $B \times B^-$ -equivariant.

Thus, the product morphism $\prod_{w \in W^I} p_w$ is equivariant as well. To show that its image is isomorphic to $U \times U^-$, it suffices to check that the map

$$\iota : U \rightarrow \prod_{w \in W^I} C_w, \quad u \mapsto (u \cdot e_w)_{w \in W^I}$$

is a closed immersion. For this, identify each Bruhat cell C_w with the homogeneous space $U/U \cap {}^w P$. Arguing as in the proof of Lemma 4, we see that the intersection of the subgroups $U \cap {}^w P$ ($w \in W^I$) is trivial, and that the same holds for the intersection of their Lie algebras. Thus, the orbit map ι is an immersion. Moreover, its image is closed, as an orbit of a unipotent group acting on an affine variety. \square

By Lemma 5, every $Z \in \varphi(\bar{T}_0)$ contains (e_w, e_w) as a non-singular point, and the Zariski tangent space

$$T_{(e_w, e_w)} Z = t_w(Z)$$

is transversal to

$$T_{(e_w, e_w)}(X_w \times X_w^-) = T_{(e_w, e_w)}(C_w \times C_w^-).$$

Let Grass_w be the Grassmanian of linear subspaces of $T_{(e_w, e_w)}(X \times X)$, and let $\text{Grass}_{w,0}$ be the open affine subset consisting of those subspaces that are transversal to $T_{(e_w, e_w)}(C_w \times C_w^-)$.

Lemma 7. *Every $t_w : \varphi(\bar{T}_0) \rightarrow \text{Grass}_{w,0}$ is a morphism, and the image of the product map*

$$\prod_{w \in W^I} t_w : \varphi(\bar{T}_0) \rightarrow \prod_{w \in W^I} \text{Grass}_{w,0}$$

is isomorphic to \bar{T}_0 .

Proof. Consider once more the universal family of $\text{Hilb}(X \times X)$, and its restriction $\mathcal{U}_{\varphi(\bar{T}_0)}$ to $\varphi(\bar{T}_0)$. Let \mathfrak{m} be the ideal sheaf of the closed point (e_w, e_w) of $X \times X$ and consider the scheme-theoretic intersection

$$\mathcal{U}_{\varphi(\bar{T}_0)} \cap (\text{Spec}(\mathcal{O}_{X \times X}/\mathfrak{m}^2) \times \varphi(\bar{T}_0)),$$

with projection map q to $\varphi(\bar{T}_0)$. Then the fiber of q at Z equals $\text{Spec}(\mathcal{O}_Z/\mathfrak{m}^2\mathcal{O}_Z)$; its length is finite and independent of Z , since (e_w, e_w) is a non-singular point of that scheme. Thus, q is finite and flat; it follows that the assignment

$$Z \mapsto (\mathfrak{m}\mathcal{O}_Z/\mathfrak{m}^2\mathcal{O}_Z)^* = t_w(Z)$$

is a morphism. Note that t_w maps $\text{diag}(X)$ to the diagonal of $T_{e_w}X$.

For the second assertion, fix $w \in W^I$ and set

$$V = T_{e_w}X, \quad V_+ = T_{e_w}C_w \text{ and } V_- = T_{e_w}C_w^-.$$

Then V is the direct sum of its subspaces V_+ and V_- . Thus, every subspace of $V \times V = T_{(e_w, e_w)}X \times X$, transversal to the subspace $V_+ \times V_- = T_{(e_w, e_w)}C_w \times C_w^-$, can be written as a graph

$$\{(v_- + f_+(v_+ + v_-), v_+ + f_-(v_+ + v_-)) \mid v_+ \in V_+, v_- \in V_-\},$$

with uniquely defined linear maps $f_{\pm} : V \rightarrow V_{\pm}$. This defines isomorphisms

$$\text{Grass}_{w,0} \cong \text{Hom}(V, V_+) \times \text{Hom}(V, V_-) \cong \text{End}(V),$$

mapping $\text{diag}(V)$ to the identity. Moreover, for every $t \in T$, the point

$$\begin{aligned} (t, 1) \cdot \text{diag}(V) &= \{(t \cdot (v_- + v_+), v_+ + v_-) \mid v_+ \in V_+, v_- \in V_-\} \\ &= \{(v_- + t \cdot v_+, v_+ + t^{-1} \cdot v_-) \mid v_+ \in V_+, v_- \in V_-\} \end{aligned}$$

is mapped to the endomorphism

$$\text{diag}(t, t^{-1}) : V \rightarrow V, \quad v_+ + v_- \mapsto t \cdot v_+ + t^{-1} \cdot v_-.$$

Since each weight space of V is one-dimensional, it follows that the eigenvalues of $\text{diag}(t, t^{-1})$ are non-zero regular functions on $\varphi(\bar{T}_0)$, eigenvectors of T (acting on the left) with weights: the T -weights of V_- and the opposites of the T -weights of V_+ . Let Φ_w be the set of weights of these functions; for $\chi \in \Phi_w$, let f_{χ} be the corresponding function. Then f_{χ} is an eigenvector of T (acting on the right) of weight $-\chi$.

The set of weights of $V = T_{e_w}G/P$ (resp. V_+ , V_-) equals $w(\Phi^- - \Phi_I)$ (resp. $\Phi^+ \cap w(\Phi^- - \Phi_I)$; $\Phi^- \cap w(\Phi^- - \Phi_I)$). Thus, we have

$$\Phi_w = \Phi^- \cap w(\Phi^- - \Phi_I) = \Phi^- - w(\Phi_I).$$

Moreover, for every simple root α , there exists $w \in W^I$ such that $-\alpha$ belongs to $\Phi^- - w(\Phi_I)$, by the proof of Lemma 4; and the functions f_{α} ($\alpha \in \Delta$) generate the coordinate ring of \bar{T}_0 , by [8] §3. Thus, the composition of φ with $\prod_{w \in W^I} t_w$ is a closed immersion. \square

Lemmas 6 and 7 imply that the restriction

$$\varphi : \bar{G}_0 \cong U \times U^- \times \bar{T}_0 \rightarrow \varphi(\bar{G}_0)$$

is an isomorphism. Since φ is bijective and \bar{G}_0 meets all $G \times G$ -orbits in \bar{G} , it follows that φ is an isomorphism. \square

Next we extend Theorem 3 to symmetric spaces; for this, we assume that the characteristic of k is not 2. Let σ be an automorphism of order 2 of G , with fixed point subgroup G^{σ} . By [7] and [8], the adjoint symmetric space G/G^{σ} admits a canonical G -equivariant completion $\overline{G/G^{\sigma}}$, a “wonderful symmetric variety”. We shall

need the following realization of $\overline{G/G^\sigma}$ in \bar{G} , obtained by Littelmann and Procesi in characteristic zero (see [15] 3.2).

Lemma 8. *The map $G/G^\sigma \rightarrow G$, $gG^\sigma \mapsto \sigma(g)g^{-1}$ extends to a closed embedding of $\overline{G/G^\sigma}$ into \bar{G} .*

Proof. Since the approach of [15] 3.2 does not extend to arbitrary characteristics in a straightforward way, we provide an alternative argument. First we review some results from [8].

Let P be a parabolic subgroup of G such that ${}^\sigma P$ and P are opposite, and that P is minimal for this property. Choose a maximal σ -split subtorus S of P , a maximal torus T of P containing S , and a Borel subgroup B of P containing T . Then T is σ -stable, so that σ acts on the character group of T , and on the root system Φ ; the opposite Borel subgroup B^- is contained in ${}^\sigma P$, and contains $R_u({}^\sigma P)$. Moreover, the natural map $R_u(P) \times S/S^\sigma \rightarrow G/G^\sigma$ extends to an open immersion

$$R_u(P) \times (\overline{S/S^\sigma})_0 \rightarrow \overline{G/G^\sigma},$$

where $(\overline{S/S^\sigma})_0$ is isomorphic to affine space where S acts linearly with weights $\alpha - \sigma(\alpha)$, $\alpha \in \Delta$.

The isomorphism $U \times U^- \times \bar{T}_0 \rightarrow \bar{G}_0$ restricts to a closed immersion

$$\iota : R_u(P) \times \bar{S}_0 \rightarrow \bar{G}_0, (g, \gamma) \mapsto (g, \sigma(g)) \cdot \gamma,$$

where $\bar{S}_0 = \bar{S} \cap \bar{T}_0$. Note that \bar{S}_0 is isomorphic to $(\overline{S/S^\sigma})_0$, equivariantly for the action of S on \bar{S}_0 by $(g, \gamma) \mapsto g^2 \cdot \gamma$, and for the natural action of S on $(\overline{S/S^\sigma})_0$. Moreover, $\iota(R_u(P) \times S)$ is contained in the image of G/G^σ in G (for every $g \in S$ can be written as $\gamma^2 = \gamma\sigma(\gamma)^{-1}$ for some $\gamma \in S$).

Let \bar{G}_σ be the closure in \bar{G} of the image of G/G^σ , and let $\bar{G}_{\sigma,0} = \bar{G}_\sigma \cap \bar{G}_0$. Then ι induces an isomorphism

$$R_u(P) \times \bar{S}_0 \rightarrow \bar{G}_{\sigma,0}.$$

It follows that the rational map $\overline{G/G^\sigma} \rightarrow \bar{G}_\sigma$ is defined on $(\overline{G/G^\sigma})_0$ and maps it isomorphically to $\bar{G}_{\sigma,0}$. Since $(\overline{G/G^\sigma})_0$ meets all G -orbits in $\overline{G/G^\sigma}$, this rational map is an isomorphism. \square

We return to the situation where P is a parabolic subgroup of G such that G acts faithfully on G/P . Then $\sigma(P)$ is another parabolic subgroup of G , and σ induces an isomorphism

$$f : G/P \rightarrow G/\sigma(P),$$

such that $f(g \cdot x) = \sigma(g) \cdot f(x)$ for all $(g, x) \in G \times G/P$. Thus, we can consider the equivariant completion $\mathcal{H}_{f,G}$ of the symmetric space G/G^σ , constructed at the end of Section 1. The discussion in that section, together with Theorem 3, yields immediately the following result.

Corollary 9. *With preceding notation, $\mathcal{H}_{f,G}$ is isomorphic to the wonderful completion of G/G^σ .*

3. THE DEGENERATIONS OF THE DIAGONAL OF A FLAG VARIETY

We still consider an adjoint semisimple group G and a parabolic subgroup P , such that G acts faithfully on $G/P = X$. In this section, we shall describe the points of $\mathcal{H}_{X,G}$ viewed as closed subschemes of $X \times X$, that is, the partial degenerations of the diagonal (the total degenerations being the points of the closed orbit). By Theorem 3, this amounts to describing the set-theoretical fibers of the map

$$\pi : G \times G \times^{P \times P} \bar{P} \rightarrow \bar{G},$$

embedded into $G/P \times G/P$ via the projection

$$p : G \times G \times^{P \times P} \bar{P} \rightarrow G/P \times G/P.$$

Moreover, by equivariance, it suffices to determine these fibers $p(\pi^{-1}(x))$ at representatives x of the (finitely many) orbits of $G \times G$ in \bar{G} . This will be achieved in Proposition 10 below, in terms of combinatorics of Weyl groups; geometric applications will be given after the proof of that Proposition.

We use the notation $B, B^-, T, \Phi, \Phi^+, \Delta, W, \ell, w_\Delta, W_I, W^I, X_w, X_w^-$ introduced in the proof of Theorem 3. For every (possibly empty) subset J of Δ , let λ_J be the unique one-parameter subgroup of T such that

$$\langle \lambda_J, \alpha \rangle = \begin{cases} 0 & \text{if } \alpha \in J; \\ 1 & \text{if } \alpha \in \Delta \setminus J. \end{cases}$$

Let x_J be the limit in \bar{G} of $\lambda_J(t)$ as $t \rightarrow 0$. By [8] §3 (see also the Appendix in [4]), the x_J ($J \subseteq \Delta$) are a system of representatives of the $G \times G$ -orbits in \bar{G} ; note that $\lambda_\Delta = 0$, so that x_Δ is the identity element of G . This sets up an order-preserving bijection between subsets J of Δ and $G \times G$ -orbits \mathcal{O}_J (ordered by inclusion of their closures). In particular, the closed orbit is \mathcal{O}_\emptyset .

Every λ_J determines two opposite parabolic subgroups P_J, P_J^- of G , where P_J (resp. P_J^-) consists of those $g \in G$ such that $\lambda_J(t)g\lambda_J(t)^{-1}$ has a limit in G as $t \rightarrow 0$ (resp. $t \rightarrow \infty$). Note that P_J contains $B = P_\emptyset$, whereas P_J^- contains $B^- = Q_\emptyset$. The common Levi subgroup $L_J = P_J \cap P_J^-$ is the centralizer of the image of λ_J ; its root system is Φ_J . Moreover, $B_J = B \cap L_J$ is a Borel subgroup of L_J .

Now the isotropy group scheme $\text{Stab}_{G \times G}(x_J)$ is smooth; it is the semi-direct product of the unipotent radical of $P_J^- \times P_J$ with $\text{diag}(L_J)(C_J \times C_J)$, where C_J is the center of L_J . In particular, the isotropy group of x_\emptyset is $B^- \times B$.

Since $C_J = \{t \in T \mid \alpha(t) = 1 \forall \alpha \in J\}$ and J is part of the basis Δ of the character group of T , the group C_J is connected; as a consequence, $\text{Stab}_{G \times G}(x_J)$ is connected as well.

Let I be the subset of Δ such that $P = P_I$. Let

$${}^J W^I = \{w \in W \mid w(I) \subseteq \Phi^+ \text{ and } w^{-1}(J) \subset \Phi^+\}.$$

By [3] IV.1, Exercice 3, this subset of W consists of all elements w of minimal length in their double coset $W_J w W_I$; moreover, it is a system of representatives of the double coset space $W_J \backslash W / W_I$, or, equivalently, of $P_J \backslash G / P = P_J \backslash X$ by [3] IV.2.6, Proposition 2.

For every w in ${}^J W^I$, we have $L_J \cap {}^w B = B_J$; thus, $L_J \cap {}^w P$ is a parabolic subgroup of L_J .

Proposition 10. *With preceding notation, every irreducible component of the fiber $p(\pi^{-1}(x_J))$ can be written as*

$$Z_w = \text{diag}(L_J) \cdot (w_J X_{w_J w}^- \times X_w)$$

for a unique w in ${}^J W^I$. Moreover, $w_J X_{w_J w}^- \times X_w$ is invariant under the diagonal action of $L_J \cap {}^w P$, and the variety

$$\tilde{Z}_w = L_J \times {}^{L_J \cap {}^w P} (w_J X_{w_J w}^- \times X_w)$$

maps birationally to Z_w under the natural morphism $\rho_w : \tilde{Z}_w \rightarrow Z_w$.

Proof. As a first step, we show the equality in \bar{G} :

$$\bar{P} \cap \overline{\mathcal{O}_J} = \bigcup_{w \in {}^I W^J} \overline{(P \times P)(w, w) \cdot x_J}$$

(decomposition into irreducible components). For this, note that we have

$$\bar{P} = \overline{B w_I B} = \overline{B w_I w_\Delta B^- w_\Delta} = (1, w_\Delta) \cdot \overline{B w_I w_\Delta B^-}.$$

Moreover, by [4] Theorem 2.1, the irreducible components of $\overline{B w_I w_\Delta B^-} \cap \overline{\mathcal{O}_J}$ are the closures

$$\overline{(B \times B^-)(w_I w_\Delta v, v) \cdot x_J},$$

where $v \in W$ satisfies

$$v \in W^J \text{ and } \ell(w_I w_\Delta) = \ell(w_I w_\Delta v) + \ell(v).$$

Thus, we obtain the following decomposition into irreducible components:

$$\bar{P} \cap \overline{\mathcal{O}_J} = \bigcup (B \times B)(w_I w_\Delta v, w_\Delta v) \cdot x_J,$$

the union over all v as above. Since every irreducible component of $\bar{P} \cap \overline{\mathcal{O}_J}$ is invariant under $P \times P$, we can rewrite this as

$$\bar{P} \cap \overline{\mathcal{O}_J} = \bigcup \overline{(P \times P)(w_\Delta v, w_\Delta v) \cdot x_J}.$$

Set $w = w_\Delta v$. Then, since W^J is invariant under the map $v \mapsto w_\Delta v w_J$, we have

$$v \in W^J \Leftrightarrow w w_J \in W^J.$$

This in turn amounts to: w is the unique element of maximal length in its right W_J -coset. On the other hand, we have

$$\ell(w_I w_\Delta) = \ell(w_I w_\Delta v) + \ell(v) \Leftrightarrow \ell(w_I w) = \ell(w) - \ell(w_I) \Leftrightarrow w_I w \in {}^I W,$$

which is equivalent to: w is the element of maximal length in its left W_I -coset. So we have proved that

$$\bar{P} \cap \overline{\mathcal{O}_J} = \bigcup \overline{(P \times P)(w, w) \cdot x_J},$$

the union over all $w \in W$ of maximal length in their right W_J -coset and in their left W_I -coset. But every irreducible component $\overline{(P \times P)(w, w) \cdot x_J}$ depends only on the double coset $W_I w W_J$ (for the isotropy group of x_J contains $\text{diag}(L_J)$). Moreover, this double coset contains a unique element of maximal length (for $W_I w W_J$ is the set of $B \times B$ -orbits in $P w P_J$, and the latter contains a unique open $B \times B$ -orbit). Thus, the preceding union is over the set $W_I \backslash W / W_J$, or, equivalently, over ${}^I W^J$. This completes the first step of the proof.

As a second step, we show that

$$p(\pi^{-1}(x_J)) = \bigcup_{w \in {}^J W^I} \overline{\text{Stab}_{G \times G}(x_J) \cdot (e_w, e_w)}$$

(decomposition into irreducible components). For this, given $(g, h) \in G$, note that

$$(gP, hP) \in p(\pi^{-1}(x_J)) \Leftrightarrow (g^{-1}, h^{-1}) \cdot x_J \in \bar{P} \cap (G \times G) \cdot x_J.$$

Note also that ${}^I W^J$ and ${}^J W^I$ are exchanged by $w \mapsto w^{-1}$. The assertion follows from these remarks, together with the first step.

Now recall that

$$\text{Stab}_{G \times G}(x_J) = \text{diag}(L_J)(R_u(P_J^-)C_J \times R_u(P_J)C_J).$$

Moreover, for every w in ${}^J W^I$, we have

$$R_u(P_J)C_J w P = R_u(P_J)w P = B w P,$$

for $B = R_u(P_J)B_J$ and $B_J \subseteq {}^w P$. As a consequence, $B w P$ is invariant under left multiplication by the group $L_J \cap {}^w P$. Likewise, since $w_J B^- w_J = R_u(P_J^-)B_J$, we have

$$R_u(P_J^-)C_J w P = R_u(P_J^-)w P = w_J B^- w_J w P,$$

and this subset is $L_J \cap {}^w P$ -invariant. Thus,

$$\text{Stab}_{G \times G}(x_J) \cdot (e_w, e_w) = \text{diag}(L_J) \cdot (w_J C_{w_J w}^- \times C_w).$$

Together with the second step, this proves all assertions of the Proposition, except for birationality of ρ_w . But \tilde{Z}_w contains

$$\begin{aligned} \tilde{Z}_w^0 &= L_J \times {}^{L_J \cap {}^w P} (R_u(P_J^-) \cdot e_w \times R_u(P_J) \cdot e_w) \\ &= L_J \times {}^{L_J \cap {}^w P} ((R_u(P_J^-)/R_u(P_J^-) \cap {}^w P) \times (R_u(P_J)/R_u(P_J) \cap {}^w P)) \end{aligned}$$

as an open subset, mapped under ρ_w onto

$$Z_w^0 = \text{Stab}_{G \times G}(x_J) \cdot (e_w, e_w).$$

We shall show that the restriction

$$\rho_w : \tilde{Z}_w^0 \rightarrow Z_w^0$$

is an isomorphism.

For this, we describe the (set-theoretical) isotropy group of (e_w, e_w) in $\text{Stab}_{G \times G}(x_J)$, that is, $\text{Stab}_{wP \times wP}(x_J)$. Since

$$\text{Stab}_{G \times G}(x_J) = (R_u(P_J^-) \times R_u(P_J)) \cdot (C_J \times C_J) \cdot \text{diag}(L_J),$$

this isotropy group is contained in

$$({}^wP \cap P_J^-) \times ({}^wP \cap P_J) = ({}^wP \cap R_u(P_J^-))({}^wP \cap L_J) \times ({}^wP \cap R_u(P_J))({}^wP \cap L_J),$$

and contains $({}^wP \cap R_u(P_J^-)) \times ({}^wP \cap R_u(P_J))$. It follows that

$$\text{Stab}_{wP \times wP}(x_J) = (R_u(P_J^-) \cap {}^wP) \times (R_u(P_J) \cap {}^wP) \cdot (C_J \times C_J) \cdot \text{diag}(L_J \cap {}^wP).$$

The isotropy Lie algebra of (e_w, e_w) in $\text{Stab}_{\mathfrak{g} \times \mathfrak{g}}(x_J)$ is described similarly; it follows that

$$Z_w^0 \cong \text{Stab}_{G \times G}(x_J) / \text{Stab}_{wP \times wP}(x_J)$$

and that $\rho_w : \tilde{Z}_w^0 \rightarrow Z_w^0$ is bijective and separable, hence an isomorphism. \square

Corollary 11. *With preceding notation, the set of irreducible components of $\pi^{-1}(x_J)$ is in bijection with the set of P_J -orbits in X . As a consequence, the fibers of π at all boundary points of \tilde{G} are reducible.*

Proof. The first assertion is a direct consequence of Proposition 11. If $\pi^{-1}(x_J)$ is irreducible, then P_J acts transitively on X , and hence its unipotent radical acts trivially. Since G acts faithfully on X , it follows that $P_J = G$, that is, $J = \emptyset$. \square

Example 2. Let X be the full flag variety of G , that is, $P = B$, or, equivalently, I is empty. Then ${}^JW^I$ equals JW , and $L_J \cap {}^wP = B_J$, so that the statement of Proposition 10 simplifies slightly as follows:

$$p(\pi^{-1}(x_J)) = \bigcup_{w \in {}^JW} \text{diag}(L_J) \cdot (w_J X_{w_J w}^- \times X_w)$$

(decomposition into irreducible components), and the map

$$\rho : L_J \times {}^{B_J} \bigcup_{w \in {}^JW} w_J X_{w_J w}^- \times X_w \rightarrow p(\pi^{-1}(x_J))$$

is birational.

Example 3. Let $X = \mathbb{P}^n = \mathbb{P}(k^{n+1})$, then $G = \text{PGL}(n+1)$. Let (v_1, \dots, v_{n+1}) be the standard basis of k^{n+1} ; let T (resp. B, B^-) be the image in G of the group of

diagonal (resp. upper triangular, lower triangular) matrices in $\mathrm{GL}(n+1)$. The Weyl group W is the permutation group of $1, \dots, n+1$, and the simple reflections are the transpositions $(i, i+1)$ where $1 \leq i \leq n$. Identifying the simple roots with the corresponding reflections, we have

$$I = \{(2, 3), \dots, (n, n+1)\} \text{ and } W^I = \{w \in W \mid w(2) < w(3) < \dots < w(n+1)\}.$$

Thus, every $w \in W^I$ is uniquely determined by $w(1)$; for $1 \leq i \leq n+1$, we denote w_i the element of W^I such that $w_i(1) = i$. Then e_{w_i} is the line kv_i , and the Schubert variety X_{w_i} is simply the subspace $\mathbb{P}(kv_1 + \dots + kv_i)$, whereas the opposite Schubert variety $X_{w_i}^-$ equals $\mathbb{P}(kv_i + \dots + kv_{n+1})$.

Let

$$J = \{(j_1, j_1 + 1), \dots, (j_r, j_r + 1)\}$$

be an arbitrary set of simple roots, where $1 \leq j_1 < \dots < j_r \leq n$. For $0 \leq i \leq r$, let V_i be the subspace of k^{n+1} spanned by $v_{j_{i+1}}, \dots, v_{j_{i+1}}$ and let $V_{\leq i} = V_0 \oplus \dots \oplus V_i$, $V_{\geq i} = V_i \oplus \dots \oplus V_r$. Then

$$k^{n+1} = V = V_0 \oplus \dots \oplus V_r,$$

and P_J (resp. P_J^- ; L_J) is the stabilizer in $\mathrm{PGL}(V)$ of all subspaces $\mathbb{P}(V_{\leq i})$ (resp. $\mathbb{P}(V_{\geq i})$; $\mathbb{P}(V_i)$) for $0 \leq i \leq r$. One checks that

$${}^J W^I = \{w_1, w_{j_1+1}, \dots, w_{j_r+1}\}$$

and that $L_J \cap P^{w_i}$ is the stabilizer in L_J of the line $kv_{j_{i+1}}$, for every w_i in ${}^J W^I$.

Now the irreducible components of $p(\pi^{-1}(x_J))$ are the

$$Z_i = \{(x, y) \in \mathbb{P}(V) \times \mathbb{P}(V) \mid x \in \mathbb{P}(V_{< i} + \ell), y \in \mathbb{P}(V_{> i} + \ell) \text{ for some line } \ell \text{ in } V_i\},$$

for $0 \leq i \leq r$, where $Z_i = Z(w_{j_{i+1}})$. Note that Z_i meets Z_{i+1} along their divisor $\mathbb{P}(V_{\leq i}) \times \mathbb{P}(V_{> i})$. We have, with obvious notation,

$$\tilde{Z}_i = \{(x, y, \ell) \in \mathbb{P}(V) \times \mathbb{P}(V) \times \mathbb{P}(V_i) \mid x \in \mathbb{P}(V_{< i} + \ell), y \in \mathbb{P}(V_{> i} + \ell)\},$$

and $\rho_i : \tilde{Z}_i \rightarrow Z_i$ is the first projection; note that \tilde{Z}_i is non-singular.

If $\dim(V_i) \geq 2$ and $0 < i < r$, then the exceptional locus of ρ_i is

$$\mathbb{P}(V_{< i}) \times \mathbb{P}(V_{> i}) \times \mathbb{P}(V_i).$$

This subset has codimension 2 in \tilde{Z}_i , and is mapped by ρ_i to $\mathbb{P}(V_{< i}) \times \mathbb{P}(V_{> i})$. It follows that Z_i is singular along $\mathbb{P}(V_{< i}) \times \mathbb{P}(V_{> i})$. On the other hand, if $\dim(V_i) = 1$, then $\tilde{Z}_i \cong Z_i = \mathbb{P}(V_{\leq i}) \times \mathbb{P}(V_{\geq i})$ is non-singular. Finally, Z_0 (resp. Z_r) is isomorphic to the blow-up of $\mathbb{P}(V_{> 0})$ (resp. of $\mathbb{P}(V_{< r})$) in $\mathbb{P}(V)$.

In particular, the ‘‘minimal’’ degenerations of the diagonal in $\mathbb{P}(V) \times \mathbb{P}(V)$ correspond to the decompositions $V = V_0 \oplus V_1$. Each such degeneration has 2 irreducible components: the blow-up Z_0 of $\mathbb{P}(V_1)$ in $\mathbb{P}(V)$, viewed as a subvariety of $\mathbb{P}(V_0) \times \mathbb{P}(V)$, and the blow-up Z_1 of $\mathbb{P}(V_0)$ in $\mathbb{P}(V)$, viewed in $\mathbb{P}(V) \times \mathbb{P}(V_1)$. These components are glued along their common exceptional divisor $\mathbb{P}(V_0) \times \mathbb{P}(V_1)$.

Remark. If X is the full flag variety, then every fiber of π is Gorenstein (for \bar{B} is Gorenstein, by [5] §5). But this fails e.g. for $X = \mathbb{P}^n$ where n is even: let indeed Z be the total degeneration of the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$. One checks easily that the restriction $\text{Pic}(\mathbb{P}^n \times \mathbb{P}^n) \rightarrow \text{Pic}(Z)$ is an isomorphism. Assuming that Z is Gorenstein, its canonical sheaf is thus isomorphic to $\mathcal{O}_Z(p, q)$ for unique integers p and q . By symmetry, we have $p = q$; by duality, we obtain

$$\chi(Z, \mathcal{O}_Z(-m, -m)) = \chi(Z, \mathcal{O}_Z(m + p, m + p))$$

for all integers m . Moreover, the Hilbert polynomials of Z and of \mathbb{P}^n are equal, so that

$$\chi(Z, \mathcal{O}_Z(m, m)) = \chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2m)) = \frac{(2m + 1) \cdots (2m + n)}{n!}.$$

Since n is even, this yields a contradiction with the previous equality.

4. WONDERFUL COMPLETIONS VIA CHOW VARIETIES

In this section, we assume that the characteristic of the ground field k is zero. As in Section 1, we consider a projective algebraic variety X together with a closed connected subgroup G of $\text{Aut}(X)$. We shall modify the constructions of Section 1 by replacing Hilbert schemes with Chow varieties; the latter can be defined as follows.

Choose a closed embedding of X into some projective space \mathbb{P} . Let $\text{Chow}_{n,d}(X, \mathbb{P})$ be the set of Chow forms of effective cycles of X with dimension n and degree d . Then $\text{Chow}_{n,d}(X, \mathbb{P})$ is a projective algebraic set; moreover, the disjoint union of the $\text{Chow}_{n,d}(X, \mathbb{P})$ over all (n, d) is independent on the projective embedding of X (see [2], Corollaire, p. 115; this may fail in positive characteristics, see [13] I.4). We call this union the Chow variety of X and denote it $\text{Chow}(X)$; this definition differs from that in [13] I.3, where the seminormalization of $\text{Chow}(X)$ is considered.

Taking the fundamental class of a subscheme defines the Hilbert-Chow morphism, from the Hilbert scheme of X (endowed with its reduced scheme structure) onto the Chow variety; see [13] I.3.15, I.3.23.3.

Let $\mathcal{C}_{X,G}$ be the $G \times G$ -orbit closure of the diagonal in $\text{Chow}(X \times X)$. This is a projective equivariant completion of G ; its points are graphs of elements of G , together with their limits as cycles. The Hilbert-Chow morphism from $\text{Hilb}(X \times X)$ to $\text{Chow}(X \times X)$ restricts to a $G \times G$ -equivariant birational morphism from $\mathcal{H}_{X,G}$ to $\mathcal{C}_{X,G}$. Moreover, the action of $G \times G$ on $\mathcal{C}_{X,G}$ can be interpreted in terms of the (partially defined) composition of correspondences; for the latter, see [9] 16.1.

Consider now a semisimple adjoint group G and a parabolic subgroup P such that G acts faithfully on $G/P = X$. Then we saw that all closed points of $\mathcal{H}_{X,G}$ are reduced; thus, the Hilbert-Chow morphism restricts to a bijective morphism

$$HC : \mathcal{H}_{X,G} \rightarrow \mathcal{C}_{X,G}.$$

Composing with $\varphi : \bar{G} \rightarrow \mathcal{H}_{X,G}$, we obtain a morphism

$$\psi : \bar{G} \rightarrow \mathcal{C}_{X,G},$$

mapping every $x \in \bar{G}$ to the cycle $p_*(\pi^*[x])$, where all multiplicities equal 1. We shall show that ψ is an isomorphism; by Theorem 3, this amounts to showing that HC is an isomorphism. Actually, we shall obtain a direct proof of the following equivalent statement.

Theorem 12. *Let X be a projective variety, homogeneous under a connected group G of automorphisms. Then $\mathcal{C}_{X,G}$ is equivariantly isomorphic to \bar{G} .*

Proof. We begin with the following observation.

Lemma 13. *With the preceding notation and assumptions, every connected component of $\text{Chow}(X)$ contains a unique closed G -orbit, and admits a G -equivariant embedding into the projectivization of a G -module.*

Proof. Let \mathcal{C} be a connected component of $\text{Chow}(X)$. Then all points of \mathcal{C} , viewed as cycles in X , are algebraically equivalent. But algebraic equivalence in X coincides with rational equivalence, and the Chow group of X is freely generated by the classes of the Schubert varieties (see [9] Example 19.1.11). Therefore, every cycle in X is algebraically equivalent to a unique cycle with B -stable support. Thus, \mathcal{C} contains a unique fixed point of B , and hence a unique closed G -orbit.

For the second assertion, choose an equivariant embedding of X into the projectivization \mathbb{P} of a G -module. Then \mathcal{C} is contained in some $\text{Chow}_{n,d}(X, \mathbb{P})$. The latter is contained in the projectivization of a G -module, by the construction of the Chow variety. \square

Applying Lemma 13 to $X \times X$, we see that $\mathcal{C}_{X,G}$ contains a unique closed $G \times G$ -orbit. The latter is isomorphic to $G/B \times G/B$, by Lemma 4. Thus, Theorem 12 is a consequence of

Lemma 14. *Let V be a $G \times G$ -module. Let $x \in \mathbb{P}(V)$ satisfy the following conditions:*
(i) The isotropy group $\text{Stab}_{G \times G}(x)$ equals $\text{diag}(G)$.
(ii) The orbit closure $\overline{(G \times G) \cdot x}$ contains a unique closed $G \times G$ -orbit, and the latter is isomorphic to $G/B \times G/B$.

Then the map

$$G \cong (G \times G)/\text{diag}(G) \rightarrow \overline{(G \times G) \cdot x}, \quad (g, h) \mapsto (g, h) \cdot x$$

extends to an isomorphism $\bar{G} \rightarrow \overline{(G \times G) \cdot x}$.

Proof. By [12] §7, the map $G \rightarrow \overline{(G \times G) \cdot x}$ extends to an equivariant birational morphism

$$\bar{G} \rightarrow \overline{(G \times G) \cdot x}.$$

We shall construct an inverse to that morphism.

By (ii), there exists a unique line ℓ in V such that the corresponding point of $\mathbb{P}(V)$ belongs to $\overline{(G \times G) \cdot x}$ and has isotropy group $B \times B$. Thus, ℓ consists of eigenvectors of $B \times B$ of weight (λ, μ) , where λ and μ are regular dominant weights of B . Let $V_{\lambda, \mu}$ be the $G \times G$ -submodule of V generated by ℓ . By complete reductibility, we may choose a $G \times G$ -equivariant projection $V \rightarrow V_{\lambda, \mu}$. Then the corresponding rational map

$$f : \mathbb{P}(V) \dashrightarrow \mathbb{P}(V_{\lambda, \mu})$$

is $G \times G$ -equivariant and defined at ℓ , and hence defined everywhere on $\overline{(G \times G) \cdot x}$. The image of x under f is a fixed point of $\text{diag}(G)$ in $\mathbb{P}(V_{\lambda, \mu})$. In particular, the $G \times G$ -module $V_{\lambda, \mu}$ contains an eigenvector of $\text{diag}(G)$. Thus, this module is the space of endomorphisms of the simple G -module V_μ with highest weight μ ; moreover, the image of x in $\mathbb{P}(V_{\lambda, \mu}) = \mathbb{P}\text{End}(V_\mu)$ is the line spanned by the identity map. By [7] 3.4, the $G \times G$ -orbit closure of that line is isomorphic to \bar{G} . Thus, f restricts to an equivariant birational morphism from $\overline{(G \times G) \cdot x}$ onto \bar{G} . \square

\square

Together with Zariski's main theorem, Theorem 12 implies that the equivariant birational morphisms $HC : \mathcal{H}_{X,G} \rightarrow \mathcal{C}_{X,G}$ and $\varphi : \bar{G} \rightarrow \mathcal{H}_{X,G}$ are isomorphisms as well. This yields an alternative proof of Theorem 3 in characteristic zero; its only overlap with the proof of Section 2 is the simple Lemma 4.

Finally, we extend Theorem 12 to symmetric spaces. As at the end of Section 2, consider an automorphism σ of order 2 of G , the corresponding isomorphism $f : G/P \rightarrow G/\sigma(P)$, and the closure $\mathcal{C}_{f,G}$ of the G -orbit of the graph Γ_f in the Chow variety of $G/P \times G/\sigma(P)$. Then Lemma 8 and Theorem 12 imply readily

Corollary 15. *With preceding notation, $\mathcal{C}_{f,G}$ is isomorphic to the wonderful completion of G/G^σ .*

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