# Polynomial invariants of links satisfying cubic skein relations

Paolo Bellingeri and Louis Funar*
*Institut Fourier, BP 74, Univ.Grenoble I, Mathématiques,*
*38402 Saint-Martin-d'Hères cedex, France,*
*e-mail:* paolo.bellingeri@ujf-grenoble.fr, louis.funar@ujf-grenoble.fr

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**Abstract**

The aim of this paper is to define two link invariants satisfying cubic skein relations. In the hierarchy of polynomial invariants determined by explicit skein relations they are the next level of complexity after Jones, HOMFLY, Kauffman and Kuperberg's $G_2$ quantum invariants. Our method consists in the study of Markov traces on a suitable tower of quotients of cubic Hecke algebras extending Jones approach.

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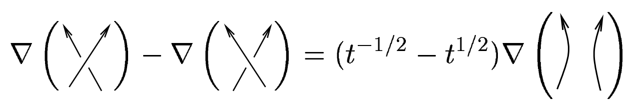
**Keywords:** skein relation, cubic Hecke algebras, Markov trace.

## 1 Introduction

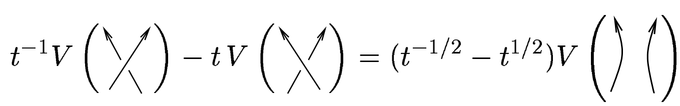
### 1.1 Preliminaries

John Conway showed that the Alexander polynomial of a knot, when suitably normalized, satisfies the following skein relation:



Given a knot diagram one can always change some of the crossings such that the modified diagram represents the unknot. Therefore one can use the skein relation for a recursive computation of $\nabla$, although this algorithm is rather time consuming (exponential).

In the mid eighties Jones discovered another invariant verifying a different but quite similar skein relation, namely:



which was further generalized to a 2-variable invariant by replacing the factor $(t^{1/2} - t^{-1/2})$ with a new variable $x$. The latter one was shown to specialize to both Alexander and Jones polynomials.

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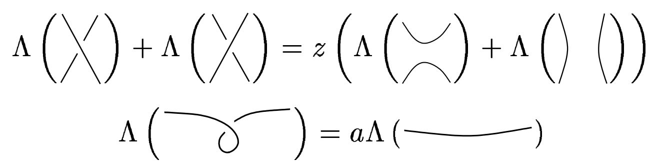
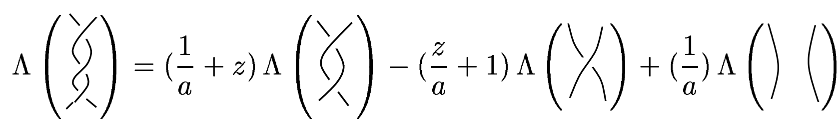
The Kauffman polynomial is another extension of Jones polynomial which satisfies a skein relation in the realm of unoriented diagrams. Specifically the formulas



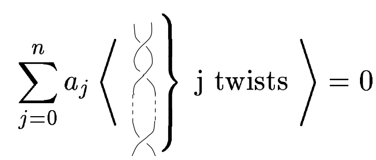
define a regular isotopy invariant of links, which can be renormalized (by using the writhe of the oriented diagram) in order to become a link invariant. Remark that some elementary manipulations show that $\Lambda$ verifies a cubical skein relation:



It is not known whether this relation is sufficient for a recursive computation of $\Lambda$ (we say then that the skein relations are complete). A conjecture of Montesinos and Nakanishi (see problem 1.59 [10]) claims this holds true.

These invariants were generalized to quantum invariants associated to Lie (super Lie, etc) algebras and their representations. Turaev ([17]) identified the HOMFLY and Kauffman polynomials with the invariants obtained from the series $A_n$ and $B_n, C_n, D_n$ respectively. Kuperberg ([11]) defined the $G_2$ quantum invariant of knots by means of skein relations making use of trivalent graphs diagrams and exploited further these ideas for spiders of rank 2 Lie algebras. The skein relations satisfied by the quantum invariants coming from simple Lie algebras were approached also via weight systems and the Kontsevich integral in ([14, 15]) for the classical series and in ([1, 2]) for the case of $g_2$.

Notice that any link invariant coming from some R-matrix $R$ verifies a skein relation of the type



which can be derived from the polynomial equation satisfied by the matrix $R$.

Let us mention that the skein relations are somewhat related to the representation theory of the Hopf algebra associated to $R$. In particular there are no other invariants whose skein relations are completely known and one expects that the invariants obtained from other super Lie algebras or by cabling the previous ones satisfy skein relations of degree at least 4 (as the $G_2$ invariant does).

This makes the search for an explicit set of complete skein relations, in which at least one relation is cubical, particularly difficult and interesting. This problem was first considered in [9] and solved in a particular case. The aim of this paper is to complete the previous results by constructing a deformation of the previously considered quotients (of the cubic Hecke algebras) and of the Markov traces supported by these algebras. In particular the link invariants obtained this way will be recursively computable and different from the HOMFLY, Kauffman polynomials and their cablings. This does not preclude the possibility to be a linear combination of the last ones.

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## 1.2 The main result

The aim of this paper is to define two link invariants by means of (a complete set of) skein relations. More precisely we will prove the following Theorem (see section 5):

**Theorem 1.1 (Main Theorem)** *There exist two link invariants $I_{(\alpha,\beta)}$ and $I^{(z,\delta)}$ which are (uniquely) determined by the two skein relations shown in picture 1 and their value for the unknot (which traditionally it is 1). These invariants take values in*

$$\frac{\mathbb{Z}[\alpha,\, \beta,\, (\alpha^2-2\beta)^{\pm\epsilon/2},\, (\beta^2+2\alpha)^{\pm\epsilon/2}]}{(H_{(\alpha,\beta)})},$$

*and respectively*

$$\frac{\mathbb{Z}[z^{\pm\epsilon/2}, \delta^{\pm\epsilon/2}]}{(P^{(z,\delta)})},$$

*where $\epsilon - 1 \in \{0,1\}$ is the number of components mod 2 and*

$$\begin{aligned} H_{(\alpha,\beta)} \quad := \quad & 8\alpha^6 - 8\alpha^5\beta^2 + 2\alpha^4\beta^4 + 36\alpha^4\beta - 34\alpha^3\beta^3 + 17\alpha^3 + 8\alpha^2\beta^5 + 32\alpha^2\beta^2 - \\ & -36\alpha\beta^4 + 38\alpha\beta + 8\beta^6 - 17\beta^3 + 8, \end{aligned}$$

*and respectively*

$$P^{(z,\delta)} := z^{23} + z^{18}\delta - 2z^{16}\delta^2 - z^{14}\delta^3 - 2z^9\delta^4 + 2z^7\delta^5 + \delta^6 z^5 + \delta^7.$$

*Here $(Q)$ denotes the ideal generated by the element $Q$ in the algebra under consideration.*

The polynomials $A, B, C..., P$ corresponding to $I_{(\alpha,\beta)}$ are given in the table below. In order to obtain those corresponding to $I^{(z,\delta)}$ it suffices to set $w = (-z^4/(\delta z))^{1/2}$ and replace $\alpha = -(z^7 + \delta^2)/(z^4\delta)$ and $\beta = (\delta - z^2)/z^3$ in the other entries of table 1.

| | |
|---|---|
| $w = ((\alpha^2 + 2\beta)/(2\alpha - \beta^2))^{1/2}$ | $A = (\beta^2 - \alpha)$ |
| $B = (\alpha^2 - \alpha\beta^2 - \beta)$ | $C = (\alpha^2 - \alpha\beta^2)$ |
| $D = (1 + 2\alpha\beta + \alpha^2\beta^2 - \alpha^3)$ | $E = (1 + \alpha\beta + \alpha^2\beta^2 - \alpha^3)$ |
| $F = (1 + 2\alpha\beta - \beta^3)$ | $G = (\alpha\beta^3 - 2\alpha - 2\alpha^2\beta)$ |
| $H = (\alpha\beta^3 - 2\alpha - 2\alpha^2\beta + \beta^2)$ | $I = (\alpha^4 - \alpha^3\beta^2 - 2\alpha^2\beta - 3\alpha)$ |
| $L = (2\alpha^3\beta + 3\alpha^2 - \alpha^2\beta^3 - \alpha\beta^2)$ | $M = (\beta^4 - 2\beta - 3\alpha\beta^2 + \alpha^2)$ |
| $N = (1 + 4\alpha\beta + 3\alpha^2\beta^2 - \alpha^3 - \alpha\beta^4 - \beta^3)$ | $O = (1 + 3\alpha\beta + 3\alpha^2\beta^2 - \alpha^3 - \alpha\beta^4)$ |
| $P = (3\beta^2 - \beta^5 - 2\alpha - 3\alpha^2\beta + 4\alpha\beta^3)$ | |

Table 1

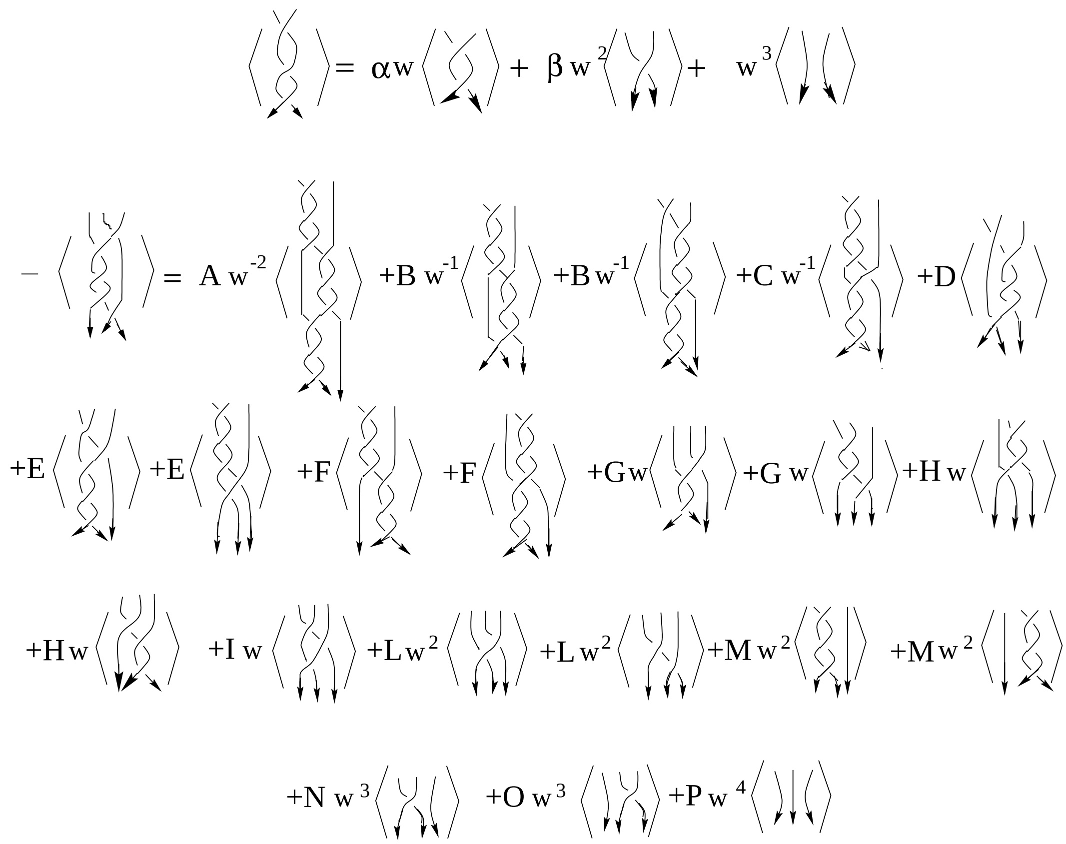


Figure 1: The skein relations

## 1.3 Conjectures and speculations

There are three essentially distinct link invariants which come from Markov traces on the cubic Hecke algebras. For each quadratic factor $P_i$ of the cubic polynomial $Q$ one has a Markov trace which factors through $H(P_i, n)$, yielding a reparameterized HOMFLY invariant. The two others are the Kauffman polynomial and $I_{(\alpha,\beta)}$ (or $I^{(z,\delta)}$). It would be very interesting to find whether there exists some relation among them. First of way one expects there exists a lift of the invariant we described to a genuine two-parameter invariant.

**Conjecture 1.1** *There exists a Markov trace on $H(Q, n)$ taking values in an algebraic extension of $\mathbb{Z}[\alpha, \beta]$ lifting the Markov trace underlying $I_{(\alpha,\beta)}$ . In other words the non-determinacy $H_{(\alpha,\beta)}$ in $I_{(\alpha,\beta)}$ can be removed.*

Notice that the polynomials $H$ and $P$ define irreducible planar algebraic curves which are non-rational. In particular one cannot express explicitly the invariants as one variable polynomial invariants.

How far are these invariants from the usual Kauffman and HOMFLY polynomials is hard to determine in the present state. One might expect they give rise to some nice weight systems for particular values of the parameters, which should be compared with those coming from Lie algebras.

It seems however that:

**Conjecture 1.2** *The cubic invariants $I_{(\alpha,\beta)}$ and $I^{(z,\delta)}$ can be obtained from the colored (i.e. cablings of) HOMFLY and Kauffman invariants.*

## 1.4 Cubic Hecke algebras

The first skein relation from figure 1 suggests (and follows from) considerations on cubic quotients of braid group algebras $\mathbb{C}[B_n]$. Specifically let us define a cubic Hecke algebra by analogy with the usual Hecke algebra ([7]) as

$$H(Q, n) = \mathbb{C}[B_n]/(Q(b_j)\,;\; j = 1, \ldots, n-1)\,,$$

where $Q(b_j) = b_j^3 - \alpha\, b_j^2 - \beta\, b_j - 1,\ \alpha,\, \beta \in \mathbb{C}$ and $B_n$ is standardly presented as

$$B_n = \langle b_1, \ldots, b_{n-1} \,|\, b_i b_j = b_j b_i,\ |i-j| > 1 \ \text{ and } \ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1},\ i < n-1 \,\rangle.$$

Our aim is to construct Markov traces on the tower of cubic Hecke algebras since Markov traces define link invariants. Let us stress that for $Q(0) \neq 0$ one has (see also [8]):

- $\dim_{\mathbb{C}} H(Q, 3) = 24$, and $H(Q, 3)$ is a deformation of the group algebra of the binary tetrahedral group $< 2, 3, 3 >$ of order 24 (isomorphic to $SL(2, \mathbf{Z}_3)$).
- $\dim_{\mathbb{C}} H(Q, 4) = 96$, and $H(Q, 4)$ is a deformation of the group algebra of $< -2, 3|4 >$.
- $\dim_{\mathbb{C}} H(Q, 5) = 600$ and $H(Q, 5)$ is a deformation of the group algebra of $GL(2, \mathbf{Z}_5)$.
- $\dim_{\mathbb{C}} H(Q, n) = \infty$ for $n \geq 6$.

Thus a direct definition of the trace on $H(Q, n)$ for $n = 6$ is highly a nontrivial matter, in particular it would involve the explicit solution of the conjugacy problem in these algebras which seems out of reach for the authors.

In order to deal with finite dimensional algebras one introduces smaller quotients $K_n(\alpha,\, \beta)$ by adding one more relation living in $H(Q,3)$. The exact form of this relation is

$$b_2\, b_1^2\; b_2 + A\; b_1^2\, b_2^2\, b_1^2 + B\; b_1\, b_2^2\, b_1^2 + B\; b_1^2\, b_2^2\, b_1\; + C\; b_1^2\, b_2\, b_1^2 + D\; b_1\, b_2^2\; b_1 + E\; b_1\, b_2\, b_1^2 + E\; b_1^2\, b_2\, b_1 + F\; b_2^2\, b_1^2 + F\; b_1^2\, b_2^2 + G\; b_2\, b_1^2 + G\; b_1^2\, b_2 + H\; b_2^2\, b_1 + H\; b_1\, b_2^2 + I\; b_1\, b_2\, b_1 + L\; b_2\, b_1 + L\; b_1\, b_2 + M\; b_1^2 + M\; b_2^2 + N\; b_1 + O\; b_2 + P = 0$$

where $A, B, \ldots, P$ are the polynomials from table 1.

**Remark 1.1** *The algebras $K_n(\alpha,\, \beta)$ are finite dimensional for any $n$.*

Let us explain the heuristics behind that choice for the additional relation. The algebra $H(Q,3)$ is semisimple (for generic $Q$) and decomposes as $\mathbb{C}^3 \oplus M_2^{\oplus 3} \oplus M_3$, where $M_m$ is the algebra of $m \times m$ matrices. As explained in section 2.2 the usual quadratic Hecke algebra $H_q(3)$ arises when the factor $\mathbb{C} \oplus M_2^{\oplus 2} \oplus M_3$ is killed. It is known that Jones and HOMFLY polynomials can be derived by the unique Markov trace on the tower $H_q(n)$. In a similar way the Birman-Wenzl algebra, which yields the Kauffman polynomial ([9]) is obtained when we quotient by $\mathbb{C} \oplus M_2^2$. In our situation the extra relation kills exactly the factor $\mathbb{C}^3$.

The geometric interpretation of these relations is now obvious: the first skein relation in figure 1 is the cubical relation corresponding to the quotients $H(Q,n)$ and the second skein relation defines the algebras $K_n(\alpha,\, \beta)$.

Our main theorem is a consequence of the more technical result below (see sections 2,3,4).

**Theorem 1.2** *For exactly four values of the $(z,\, \bar{z})$ there exists an unique Markov traces $\mathcal{T}$ on $K_n(\alpha,\, \beta)$ with parameters $(z,\, \bar{z})$ i.e. verifying:*

1. $\mathcal{T}(xy) = \mathcal{T}(yx)$,
2. $\mathcal{T}(xb_{n-1}) = z\mathcal{T}(x)$,
3. $\mathcal{T}(xb_{n-1}^{-1}) = \bar{z}\mathcal{T}(x)$.

*The first couple $(z,\, \bar{z})$ is*

$$z = (2\alpha - \beta^2)/(\alpha\beta + 4),\; \bar{z} = -(\alpha^2 + 2\beta)/(\alpha\beta + 4),$$

*and the corresponding trace is $\mathcal{T}_{\alpha,\beta} : K_n(\alpha,\, \beta) \to \mathbb{Z}[\alpha,\, \beta,\, 1/(\alpha\beta + 4)]/(H_{(\alpha,\beta)})$.*
*The other three solutions are not rational functions on the parameters and we prefer to give $\alpha$, $\beta$ and $\bar{z}$ as functions of $z, \delta$ $(\delta = z^2(\beta z + 1))$. More precisely we set*

$$\mathcal{T}^{(z,\delta)} : K_*(\alpha,\, \beta) \to \mathbb{Z}[z^{\pm 1}, \delta^{\pm 1}]/(P^{(z,\delta)}),$$

*where*

$$\beta = (\delta - z^2)/z^3,\; \alpha = -(z^7 + \delta^2)/(z^4\delta)\; \bar{z} = -z^4/\delta.$$

## 1.5 Outline of the proof

We will prove by recurrence on $n$ that a Markov trace on $K_n(\alpha,\, \beta)$ extends to a Markov trace on $K_{n+1}(\alpha,\, \beta)$. Since there is a nice system of generators for $K_{n+1}(\alpha,\, \beta)$ constructed out of one for $K_n(\alpha,\, \beta)$, such an extension, if ever exists, it must be unique. This is a consequence of the form of the additional relation. However the most difficult step is to prove that the canonical extension is a well-defined linear functional and satisfies the trace commutativity. The method of proof is greatly inspired by [3]. One defines a graph whose vertices are linear combinations on the elements of the Abelian semigroup associated to the free group in $n-1$ letters (in first instance) and whose edges correspond to elements which differ by exactly one relation (from the set of relations defining $K_n(\alpha, \beta)$).

One gives an orientation on part of the edges of this graph and look for the existence of minimal elements in each connected component of the graph. If there is an unique minimal element in each component then one is able to derive a basis for $K_\infty(\alpha, \beta)$. In order to achieve the uniqueness one adds sufficiently many relations, which are formal consequences of the basic ones.

The usual procedure to obtain the existence of minimal elements is to consider the lexicographic order on the free semigroup on $n-1$ letters and to use the relations as replacements of some word by (a linear combination of) smaller ones.

One defines therefore a reduction process for words by introducing the following orientations on some edges. The arrows show the orientation, if exactly one monomial is changed using one of the rules

$$\begin{aligned}
&ab_j^3b \to \alpha\, ab_j^2b + \beta\, ab_jb + ab,\\
&ab_{j+1}b_jb_{j+1}b \to ab_jb_{j+1}b_jb,\\
&ab_{j+1}b_j^2b_{j+1}b \to aS_jb,\\
&ab_{j+1}b_j^2b_{j+1}^2b \to aC_jb,\\
&ab_{j+1}^2b_j^2b_{j+1}b \to aD_jb,
\end{aligned}$$

where $S_j, C_j$ and $D_j$ are of the form $\sum_i P_i\, b_j^{a_i} b_{j+1}^{b_i} b_j^{c_i}$, $P_i$ polynomials in $\alpha$, $\beta$ and $a_i, b_i, c_i = 0, 1, 2$. Several edges remain unoriented. They correspond to a change in a monomial of type

$$ab_ib_jb \to ab_jb_ib \text{ whenever } \mid i-j \mid > 1.$$

Remark that the extra relations (which obviously hold in $H(Q,n)$) make the reduction process ambiguous. The reason for introducing them is to insure the existence of descending paths to some minimal points even if closed oriented loops may be found in the graph. It remains then to check the existence and uniqueness of minimal elements up to unoriented paths in this semi-oriented graph by means of so-called Pentagon Lemma (see section 2). When this approach will be not successful we shall enlarge our graph to a tower of graphs modeling not one algebra $K_n(\alpha, \beta)$ but the functionals on the whole tower $\cup_{n=2}^{\infty} K_n(\alpha, \beta)$ satisfying a recurrence condition which permits to reduce further the minimal elements. Here the Colored Pentagon Lemma (see section 2) can be applied and the problem is reduced to some algebraic computations. We will find that the main obstructions lie in $K_4(\alpha, \beta)$ as it could be expected from the study of quadratic Hecke algebras. When we wish to check the commutativity condition for the functional be actually a Markov trace another obstruction appears in $K_4(\alpha, \beta)$. Then there are only two types of obstructions to the existence of a Markov traces:

- CPC obstructions (Colored Pentagon Condition);
- commutativity obstructions.

These finitely many obstructions have been checked by using the computer and all of them lie in the principal ideal generated by $H_{(\alpha,\beta)}$ (respectively $P^{(z,\delta)}$).

### 1.6 Properties of the invariants

In the next section we will compute these obstructions and derive the existence of the two traces $\mathcal{T}_{(\alpha,\beta)}$ and $\mathcal{T}^{(z,\delta)}$. When we have a Markov trace $\mathcal{T}$, there is a natural way to get a link invariant, by setting:

$$I(x) = \left(\frac{1}{z\bar{z}}\right)^{\frac{n-1}{2}} \left(\frac{\bar{z}}{z}\right)^{\frac{e(x)}{2}} \mathcal{T}(x),$$

where $x \in B_n$ is a braid representative of the link $L$ and $e(x)$ is the exponent sum of $x$. Therefore we find two invariants $I_{(\alpha,\beta)}$ and $I^{(z,\delta)}$. We find that:

- they distinguish all knots with number crossing at most 10 that have the same HOMFLY polynomial (and then they are independent from HOMFLY). However, like HOMFLY and Kauffman polynomials, they seem to not distinguish among mutants knots (in particular they don't separate Kinoshita-Terasaka and Conway knots).
- $I_{(\alpha,\beta)} = I_{(-\beta,-\alpha)}$ for amphicheiral knots, and $I_{(\alpha,\beta)}$ detects the chirality of all the knots with number crossing at most 10, where HOMFLY fails.
- $I_{(\alpha,\beta)}$ and $I^{(z,\delta)}$ have a *cubical* behaviour.

Let us explain briefly what we meant by *cubical behaviour.*

**Definition 1.1** *A Laurent polynomial $\sum_{j\in\mathbb{Z}} c_j a^j$ is a $(n,k)$-polynomial (for $n, k \in \mathbb{N}$) if $c_j = 0$ for $j \neq hn + k$, for all $h \in \mathbb{Z}$.*

**Remark 1.2**
- *The HOMFLY polynomial can be written as $\sum_{k\in z} R_k(l)m^k$ and respectively as $\sum_{k\in z} S_k(m)l^k$, where $R_k(l)$ and $S_k(m)$ are $(2,k)$-Laurent polynomials with $R_{2k+1}(l) = S_{2k+1}(m) = 0$.*
- *The Kauffman polynomial can be written as $\sum_{k\in z} U_k(l)m^k$ (respectively as $\sum_{k\in z} T_k(m)l^k$), where $U_k(l)$ and $T_k(m)$ are $(2,k+1)$-Laurent polynomials.*

In this respect the HOMFLY and Kauffman polynomials have a quadratic behaviour.

**Proposition 1.1** *$I_{(\alpha,\beta)}$ and $I^{(z,\delta)}$ have a cubical behaviour, i.e. for each link $L$ there exists some $l \in \{0,1,2\}$ so that*

$$I_{(\alpha,\beta)}(L) = \frac{\sum_{k\in\mathbb{N}} P_k(\beta)\alpha^k}{\sum_{k\in\mathbb{N}} Q_k(\beta)\alpha^k} = \frac{\sum_{k\in\mathbb{N}} M_k(\alpha)\beta^k}{\sum_{k\in\mathbb{N}} N_k(\alpha)\beta^k},$$

*where $P_k, Q_k, M_k, N_k$ are $(3, k+l)$-polynomials, and*

$$I^{(z,\delta)}(L) = \sum_{k\in\mathbb{Z}} H_k(\delta)z^k = \sum_{k\in\mathbb{Z}} G_k(z)\delta^k,$$

*where $H_k, G_k$ are $(3,k)$-Laurent polynomials.*

# 2 Markov traces on $K_n(\alpha,\,\beta)$

## 2.1 Cubic Hecke algebras

The generalized Hecke algebras were introduced by analogy with the classical case [16] as the quotients

$H(Q,n) = \mathbb{C}\,[B_n]\,/(Q(b_j); j = 1, n-1)$

of the group algebra of the braid group by the ideal generated by $Q(b_j)$ where $Q$ is a polynomial having $Q(0) \neq 0$.

The structure of these algebras is well-known in the quadratic case (see [7]). They are finite dimensional semi-simple modules of dimension $n!$. In the general case we notice that some new features arise. In particular $dim_{\mathbb{C}} H(Q,n) = \infty$ if $deg(Q) > 6$, and $n \geq 3$ (see [4], [8]).

The cubic Hecke algebras are the quotients

$$H(Q,n) = \mathbb{C}[B_n]/(Q(b_j)\,;\; j = 1,\ldots,n-1)$$

of the group algebra of the braid group by the ideal generated by $Q(b_j)$, cubic polynomial with parameters $\alpha$ and $\beta$, i.e. $Q(b_j) = b_j^3 - \alpha\, b_j^2 - \beta\, b_j - \gamma,\ \alpha,\ \beta,\ \gamma \in \mathbb{C}$. From now on, one considers $\gamma = 1$ in the cubic polynomial since $H(Q,\infty)$ and $H(\gamma^{-1}Q,\infty)$ are isomorphic. In [9] it was shown that:

**Proposition 2.1** *For all cubic polynomials $Q$ with $Q(0) \neq 0$ one has $dim_{\mathbb{C}} H(Q,3) = 24$. A convenient base of the vector space $H(Q,3)$ is*

*$e_1 = 1$, $e_2 = b_1$, $e_3 = b_1^2$, $e_4 = b_2$, $e_5 = b_2^2$, $e_6 = b_1b_2$, $e_7 = b_2b_1$, $e_8 = b_1^2b_2$, $e_9 = b_2b_1^2$, $e_{10} = b_1b_2^2$, $e_{11} = b_2^2b_1$, $e_{12} = b_1^2b_2^2$, $e_{13} = b_2^2b_1^2$, $e_{14} = b_1b_2b_1$, $e_{15} = b_1^2b_2b_1$, $e_{16} = b_1b_2b_1^2$, $e_{17} = b_1b_2^2b_1^2$, $e_{18} = b_1^2b_2b_1^2$, $e_{19} = b_1^2b_2^2b_1$, $e_{20} = b_1b_2^2b_1$, $e_{21} = b_1^2b_2^2b_1^2$, $e_{22} = b_2b_1^2b_2$, $e_{23} = b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2$, $e_{24} = b_2b_1^2b_2b_1^2 = b_1b_2b_1^2b_2b_1 = b_1^2b_2b_1^2b_2$.*

We refer also to [9] for the following identities:

$b_{j+1}b_j^2b_{j+1}b_j = b_jb_{j+1}b_j^2b_{j+1}$,

$b_{j+1}^2b_j^2b_{j+1} = b_jb_{j+1}^2b_j^2 + \alpha(b_{j+1}b_j^2b_{j+1} - b_jb_{j+1}^2b_j) + \beta(b_j^2b_{j+1} - b_jb_{j+1}^2)$,

$b_{j+1}b_j^2b_{j+1}^2 = b_j^2b_{j+1}^2b_j + \alpha(b_{j+1}b_j^2b_{j+1} - b_jb_{j+1}^2b_j) + \beta(b_{j+1}b_j^2 - b_{j+1}^2b_j)$.

## 2.2 The homogeneous quotient of rank 3

The quotient $P(\infty)$ of $H(Q,\infty)$ is homogeneous if any identity $F(b_i, b_{i+1}, \ldots, b_j) = 0$, which holds in $P(\infty)$ remains valid under the translation of indices i.e. also $F(b_{i+k}, b_{i+k+1}, \ldots, b_{j+k}) = 0$, for $k \in \mathbb{Z}, k \geq 1-i$. One considers the Markov traces supported by the quotients $K_n(\alpha,\,\beta) = H(Q,n)/I_n$, where $I_n$ is the (two-sided) ideal generated by:

$$b_jb_{j-1}^2b_j + (\beta^2-\alpha)b_{j-1}^2b_j^2b_{j-1}^2 + (\alpha^2-\alpha\beta^2-\beta)b_{j-1}b_j^2b_{j-1}^2 + (\alpha^2-\alpha\beta^2-\beta)b_{j-1}^2b_j^2\, b_{j-1} + (\alpha^2-\alpha\beta^2)b_{j-1}^2b_jb_{j-1}^2 + (1+2\alpha\beta+\alpha^2\beta^2-\alpha^3)b_{j-1}b_j^2b_{j-1} + (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)b_{j-1}b_jb_{j-1}^2 + (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)b_{j-1}^2b_jb_{j-1} + (1+2\alpha\beta-\beta^3)b_j^2b_{j-1}^2 + (1+2\alpha\beta-\beta^3)b_{j-1}^2b_j^2 + (\alpha\beta^3-2\alpha-2\alpha^2\beta)b_jb_{j-1}^2 + (\alpha\beta^3-2\alpha-2\alpha^2\beta)b_{j-1}^2b_j + (\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)b_j^2b_{j-1} + (\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)b_{j-1}b_j^2 + (\alpha^4-\alpha^3\beta^2-2\alpha^2\beta-3\alpha)b_{j-1}b_jb_{j-1} + (2\alpha^3\beta+3\alpha^2-\alpha^2\beta^3-\alpha\beta^2)b_jb_{j-1} + (2\alpha^3\beta+3\alpha^2-\alpha^2\beta^3-\alpha\beta^2)b_{j-1}b_j + (\beta^4-2\beta-3\alpha\beta^2+\alpha^2)b_{j-1}^2 + (\beta^4-2\beta-3\alpha\beta^2+\alpha^2)b_j^2 + (1+4\alpha\beta+3\alpha^2\beta^2-\alpha^3-\alpha\beta^4-\beta^3)b_{j-1} + (1+3\alpha\beta+3\alpha^2\beta^2-\alpha^3-\alpha\beta^4)b_j + 3\beta^2-\beta^5-2\alpha-3\alpha^2\beta+4\alpha\beta^3,$$

where $j = 1,\ldots,n-1$. Then $K_\infty(\alpha,\,\beta)$ is a homogeneous quotient of $H(Q,\infty)$.

**Remark 2.1** *$H(Q,3)$ is a semisimple algebra which decomposes generically as $\mathbb{C}^3 \oplus M_2^{\oplus 3} \oplus M_3$, where $M_n$ is the algebra of $n \times n$ matrices. The morphism into $\mathbb{C}^3$ is obtained via the abelianization map and that into $M_2$ is part of the projection onto the quadratic Hecke algebra defined by a divisor of $Q$ (which is $\mathbb{C}^2 \oplus M_2$). One identifies then $K_3(\alpha,\, \beta) \cong M_2^{\oplus 3} \oplus M_3$.*

In fact it suffices to show that the ideal $I_3$ is a vector space of dimension 3. Let $R$ be the span of $R_0, R_1, R_2$, where

$R_0 := b_2b_1^2b_2 + (\beta^2-\alpha)\, b_1^2b_2^2b_1^2 + (\alpha^2-\alpha\beta^2-\beta)b_1b_2^2b_1^2 + (\alpha^2-\alpha\beta^2-\beta)b_1^2b_2^2b_1^2 + (\alpha^2-\alpha\beta^2)b_1^2b_2b_1^2 + (1+2\alpha\beta+\alpha^2\beta^2-\alpha^3)b_1b_2^2b_1 + (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)b_1b_2b_1^2 + (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)b_1^2b_2b_1 + (1+2\alpha\beta-\beta^3)b_2^2b_1^2 + (1+2\alpha\beta-\beta^3)b_1^2b_2^2 + (\alpha\beta^3-2\alpha-2\alpha^2\beta)b_2b_1^2 + (\alpha\beta^3-2\alpha-2\alpha^2\beta)b_1^2b_2 + (\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)b_2^2b_1 + (\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)b_1b_2^2 + (\alpha^4-\alpha^3\beta^2-2\alpha^2\beta-3\alpha)b_1b_2b_1 + (2\alpha^3\beta+3\alpha^2-\alpha^2\beta^3-\alpha\beta^2)b_2b_1 + (2\alpha^3\beta+3\alpha^2-\alpha^2\beta^3-\alpha\beta^2)b_1b_2 + (\beta^4-2\beta-3\alpha\beta^2+\alpha^2)b_1^2 + (\beta^4-2\beta-3\alpha\beta^2+\alpha^2)b_2^2 + (1+4\alpha\beta+3\alpha^2\beta^2-\alpha^3-\alpha\beta^4-\beta^3)b_1 + (1+3\alpha\beta+3\alpha^2\beta^2-\alpha^3-\alpha\beta^4)b_2 + 3\beta^2-\beta^5-2\alpha-3\alpha^2\beta+4\alpha\beta^3$,

$R_1 := b_1R_0 = b_1b_2b_1^2b_2 - \beta b_1^2b_2^2b_1^2 + (1+\alpha\beta)b_1b_2^2b_1^2 + (1+\alpha\beta)b_1^2b_2^2b_1^2 + (1+\alpha\beta)b_1^2b_2b_1^2\ (-\alpha^2\beta - 2\alpha)b_1b_2^2b_1 + (-\alpha^2\beta-2\alpha)b_1b_2b_1^2 + (-\alpha^2\beta-2\alpha)b_1^2b_2b_1 + (\beta^2-\alpha)b_2^2b_1^2 + (\beta^2-\alpha)b_1^2b_2^2 + (\alpha^2-\alpha\beta^2)b_2b_1^2 + (\alpha^2-\alpha\beta^2)b_1^2b_2 + (\alpha^2-\alpha\beta^2-\beta)b_2^2b_1 + (\alpha^2-\alpha\beta^2-\beta)b_1b_2^2 + (\alpha^3\beta+\beta+3\alpha^2)b_1b_2b_1 + (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)b_2b_1 + (1+\alpha\beta+\alpha^2\beta^2-\alpha^3)b_1b_2 + (1+2\alpha\beta-\beta^3)b_1^2 + (1+2\alpha\beta-\beta^3)b_2^2 + (\alpha\beta^3-2\alpha-2\alpha^2\beta+\beta^2)b_1 + (\alpha\beta^3-2\alpha-2\alpha^2\beta)b_2 + \beta^4-2\beta-3\alpha\beta^2+\alpha^2$,

$R_2 := b_1R_1 = b_1^2b_2b_1^2b_2 + b_1^2b_2^2b_1^2 - \alpha b_1b_2^2b_1^2 - \alpha b_1^2b_2^2b_1^2 - \alpha b_1^2b_2b_1^2 + \alpha^2b_1b_2^2b_1 + (\alpha^2+\beta)b_1b_2b_1^2 + (\alpha^2+\beta)b_1^2b_2b_1 + (-\beta)b_2^2b_1^2 + (-\beta)b_1^2b_2^2 + (1+\alpha\beta)b_2b_1^2 + (1+\alpha\beta)b_1^2b_2 + (1+\alpha\beta)b_2^2b_1 + (1+\alpha\beta)b_1b_2^2 + (-\alpha^3\beta-\alpha\beta+1)b_1b_2b_1 + (-\alpha^2\beta-2\alpha)b_2b_1 + (-\alpha^2\beta-2\alpha)b_1b_2 + (\beta^2-\alpha)b_1^2 + (\beta^2-\alpha)b_2^2 + (-\alpha\beta^2+\alpha^2-\beta)b_1 + (-\alpha\beta^2+\alpha^2)b_2 + 1 + 2\alpha\beta - \beta^3$.

**Lemma 2.1** *As vector spaces $R \cong I_3$ in $H(Q,3)$.*

*Proof:* Remark that

$$b_1R_0 = R_0b_1 = R_1,\ b_1R_1 = R_1b_1 = R_2,\ b_1R_2 = R_2b_1 = R_0 + \beta R_1 + \alpha R_2.$$

and after some messy computations (computer aided) we obtain that

$$b_2R_0 = R_0b_2 = R_1,\ b_2R_1 = R_1b_2 = R_2,\ b_2R_2 = R_2b_2 = R_0 + \beta R_1 + \alpha R_2.$$

From these relations we find that $xR_0y \in R$ for all $x, y \in H(Q,3)$, hence $I_3 \subset R$. The other inclusion is trivial. □

## 2.3 Uniqueness of Markov trace on $K_n(\alpha,\, \beta)$

From now on we will work with the group ring $\mathbb{Z}[\alpha,\beta][B_\infty]$ instead of $\mathbb{C}[B_\infty]$.

**Definition 2.1** *Let $z, \bar{z} \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$, $R$ be a $\mathbb{Z}[\alpha,\beta,z,\bar{z}]-$module and $H$ an ideal of $R$.*
*i) $\mathcal{T}$ is an admissible functional on $K_\infty(\alpha,\, \beta)$ (taking values in $R/H$) if the following conditions are fulfilled:*

$$\mathcal{T}(xb_ny) = z\mathcal{T}(xy) \quad \textit{for any} \ \ x, y \in K_n(\alpha,\, \beta),$$

$$\mathcal{T}(xb_n^{-1}y) = \bar{z}\mathcal{T}(xy) \quad for\ any\ \ x, y \in K_n(\alpha,\, \beta).$$

*ii) An admissible functional* $\mathcal{T}$ *is a Markov trace if*

$$\mathcal{T}(ab) = \mathcal{T}(ba) \ \ for\ any\ \ a, b \in K_n(\alpha,\, \beta).$$

**Remark 2.2** *Markov traces on the quadratic Hecke algebras (see [16]) have the following multiplicative property:* $\mathcal{T}(xb_n) = \mathcal{T}(x)\mathcal{T}(b_n)$*, for* $x \in H(Q, n)$*, which implies that:* $\mathcal{T}(xy) = \mathcal{T}(x)\mathcal{T}(y)$*, for* $x \in H(Q, n), y \in < 1, b_n, b_{n+1}, ..., b_{n+k} >$*.*

However we cannot expect that this property will extend to higher level algebras and Markov traces on them.

**Definition 2.2** *The Markov trace* $\mathcal{T}$ *is multiplicative if* $\mathcal{T}(xb_n^k) = \mathcal{T}(x)\mathcal{T}(b_n^k)$ *holds when* $x \in H(Q, n), k \in Z$*.*

**Remark 2.3** *In the case of cubic Hecke algebras the Markov traces are multiplicative. In fact using the identity* $b_n^2 = \alpha b_n + \beta + b_n^{-1}$ *we derive then the multiplicativity for* $k = 2$*, and by recurrence for all* $k$*. In particular if* $\mathcal{T}$ *is a Markov trace it follows that* $\mathcal{T}(ab_n^2 b) = t\mathcal{T}(ab) \ \ a, b \in B_n$ *, where* $t = \alpha z + \beta + \bar{z}$*.*

One can state now the unique extension property of Markov traces.

**Proposition 2.2** *For fixed* $(z, t) \in (\mathbb{C}^*)^2$ *there exists at most one Markov trace on* $K_n(\alpha,\, \beta)$ *with parameters* $(z, t)$*.*

*Proof:* Define recursively the modules $L_n$ by
$L_2 = H(Q, 2)$,
$L_3 = \mathbb{C} < b_1^i b_2^j b_1^k;\ i, j, k \in \{0, 1, 2\} >$,
$L_{n+1} = \mathbb{C} < ab_n^{\varepsilon} b \,|\, a, b \in$ basis of $L_n,\ \varepsilon \in \{1, 2\} > \oplus L_n$.
We need the following result

**Lemma 2.2** *Under the natural projection* $\pi$ *on* $K_n(\alpha,\, \beta)$*,* $L_n$ *surjects onto* $K_n(\alpha,\, \beta)$*.*

*Proof:* For $n = 2$ it is clear. For $n = 3$ we know that $b_2 b_1^2 b_2,\ b_1 b_2 b_1^2 b_2,\ b_1^2 b_2 b_1^2 b_2 \in \pi(L_3)$.

Consider now $w \in K_{n+1}(\alpha,\, \beta)$ represented by a word in the $b_i$'s having only positive exponents. We assume that the degree of the word in the variable $b_n$ is minimal among all linear combinations of words (with positive exponents) representing $w$.

If the degree is less or equal to 1 there is nothing to prove.

If the degree is 2 then either $w = ub_n^2 v$, $u, v \in K_n(\alpha,\, \beta)$ so using the induction hypothesis we are done, or else $w = ub_n z b_n v$, where $u, z, v \in K_n(\alpha,\, \beta)$. Therefore $z = xb_{n-1}^{\varepsilon} y$ where $x, y \in K_{n-1}(\alpha,\, \beta)$ by the induction and $\varepsilon \in \{0, 1, 2\}$. If $\varepsilon = 0$ then $w$ can be reduced to $uzb_n^2 v$. If $\varepsilon = 1$ then $w = ub_n x b_{n-1} y b_n v = uxb_{n-1} b_n b_{n-1} yv$ hence the degree of $w$ can be lowered by 1, which contradicts our assumption. If $\varepsilon = 2$ then $w = uxb_n b_{n-1}^2 b_n yv$. One derives
$b_n b_{n-1}^2 b_n \in \mathbb{C} < b_{n-1}^i b_n^j b_{n-1}^k,\ i, j, k \in \{0, 1, 2\} >$,
hence we reduced the problem to the case when $w$ is a word of type $u' b_n^2 v'$.

If the degree of $w$ is at least 3 we will contradict the minimality. In fact $w$ contains either a subword $w' = b_n^a u b_n^b$, $u \in K_n(\alpha,\, \beta)$ and $a + b \geq 3$, or else a subword $w'' = b_n u b_n v b_n$, $u, v \in K_n(\alpha,\, \beta)$.

In the first case using the induction we can write $u = xb_{n-1}^{\varepsilon}y$, $x, y \in K_{n-2}(\alpha,\ \beta)$.

If $\varepsilon = 0$ then $w' = b_n^{a+b}xy = \alpha b_n^{a+b-1}xy + \beta b_n^{a+b-2}xy + b_n^{a+b-3}xy$, hence the degree of $w$ can be lowered by 1.

If $\varepsilon = 1$ then $w' = b_n^{a-1}xb_nb_{n-1}b_nyb_n^{b-1} = b_n^{a-1}xb_{n-1}b_nb_{n-1}yb_n^{b-1}$, and again its degree can be reduced by one unit.

If $\varepsilon = 2$ then $a$ or $b$ equals 2. Set $a = 2$. We can write

$w' = xb_n^2b_{n-1}^2b_nyb_n^{b-1} = xb_{n-1}b_n^2b_{n-1}^2yb_n^{b-1} + \alpha(b_nb_{n-1}^2b_n - b_{n-1}b_n^2b_{n-1})yb_n^{b-1}+$
$+\beta(b_{n-1}^2b_n - b_{n-1}b_n^2)yb_n^{b-1}$.

still contradicting the minimality of the degree of $w$.

In the second case we can write also $u = xb_{n-1}^{\varepsilon}y$, $v = rb_{n-1}^{\delta}s$ with $x, y, r, s \in K_{n-1}(\alpha,\ \beta)$.

If $\varepsilon$ or $\delta$ equals 1 then, after some obvious commutation the word $w''$ contains the subword $b_nb_{n-1}b_n$ which can be replaced by $b_{n-1}b_nb_{n-1}$ hence lowering its degree.

If $\varepsilon = \delta = 2$ then $w'' = xb_nb_{n-1}^2b_nyrb_{n-1}^2b_ns$. We use the homogeneity to replace $b_nb_{n-1}^2b_n$ by a sum of elements of type $b_{n-1}^ib_n^jb_{n-1}^k$. Each term of the expression of $w''$ which comes from a factor having $j < 2$ has the degree less than it had before. The remaining terms are $xb_{n-1}^ib_n^2b_{n-1}^kyrb_{n-1}^2b_ns$, so they contains a subword $b_n^2ub_n$ whose degree we already know that it can be reduced as above. This proves our claim. □

Now the Markov traces $\mathcal{T}$ on $H(Q, \infty)$ are multiplicative hence $\mathcal{T}(xb_n^{\varepsilon}y) = \mathcal{T}(b_n^{\varepsilon})\mathcal{T}(yx)$ holds, and $K_n(\alpha,\ \beta)$ it is an algebra hence $yx \in K_n(\alpha,\ \beta)$ . Therefore the extension of $\mathcal{T}$, by recursion, from $K_n(\alpha,\ \beta)$ to $K_{n+1}(\alpha,\ \beta)$ if ever exists it is unique. This ends the proof of our proposition. □

# 3 CPC Obstructions

## 3.1 The pentagonal condition

The following Lemma is also a consequence of the previous one:

**Lemma 3.1** *There is a surjection of $(K_n(\alpha,\ \beta), K_n(\alpha,\ \beta))$-bimodules*

$K_n(\alpha,\ \beta) \oplus K_n(\alpha,\ \beta) \otimes_{K_{n-1}(\alpha,\beta)} K_n(\alpha,\ \beta) \oplus K_n(\alpha,\ \beta) \otimes_{K_{n-1}(\alpha,\beta)} K_n(\alpha,\ \beta) \longrightarrow K_{n+1}(\alpha,\ \beta)$

*given by*

$x \oplus y \otimes z \oplus u \otimes v \to x + yb_nz + ub_n^2v.$

In particular admissible functionals are unique up to the choice of $\mathcal{T}(1) \in R$. Look now at the algebra $K_*(\alpha,\ \beta)$. We wish to use the following transformations on the words (one way):

(C0)(j+1) $ab_{j+1}^3b \to aE_{j+1}b$,

(C1)(j) $ab_{j+1}b_jb_{j+1}b \to ab_jb_{j+1}b_jb$,

(C2)(j) $ab_{j+1}b_j^2b_{j+1}b \to aS_jb$,

(C12)(j) $ab_{j+1}b_j^2b_{j+1}^2b \to aC_jb$,

(C21)(j) $ab_{j+1}^2b_j^2b_{j+1}b \to aD_jb$,

where $E_{j+1} = \alpha b_{j+1}^2 + \beta b_{j+1} + 1$, $S_j = b_{j+1}b_j^2b_{j+1} - R_{(0,j)}$ , $C_j = b_j^2b_{j+1}^2b_j + \alpha(b_{j+1}b_j^2b_{j+1} - b_jb_{j+1}^2b_j) + \beta(b_{j+1}b_j^2 - b_{j+1}^2b_j)$ and $D_j = b_jb_{j+1}^2b_j^2 + \alpha(b_{j+1}^2b_j^2b_{j+1}^2 - b_jb_{j+1}^2b_j) + \beta(b_j^2b_{j+1} - b_jb_{j+1}^2)$, $j = 0, \ldots, n-2$. Our aim is to reduce the degree of $b_{n-1}$ as much as possible in $K_n(\alpha,\ \beta)$. According to the previous Lemma every word in $K_n(\alpha,\ \beta)$ is equivalent to a sum of words of type $\sum_i x_ib_{n-1}^{\varepsilon_i}y_i$. Unfortunately we are forced to use also the transformations

$b_ib_j \leftrightarrow b_jb_i$ for $\mid i - j \mid > 1$,

which have to be used in both directions.

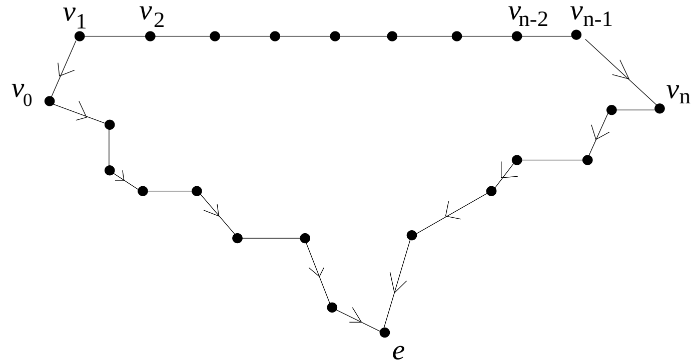


Figure 2: The pentagon condition

Assume this is the reduction process we want to carry out. We eventually obtain a sum $\sum_i x_i b_{n-1}^{\varepsilon_i} y_i$ with $x_i, y_i \in K_{n-1}(\alpha, \beta)$. Of course this *normal form* for the word we started with is not unique since we may perform again permutations of its letters in each term. But if any two such normal forms are equivalent under permutations of its letters (i.e. of the letters $b_i b_j$ with $\mid i-j \mid > 1$) then we will get an almost canonical description of the basis of $K_n(\alpha, \beta)$. Indeed the last assumption is equivalent to say that the surjection of the previous Lemma is an isomorphism. Unfortunately this is not the case. One can however obtain the obstructions to the uniqueness of this almost canonical form as follows.

We return now to the module of admissible functionals. The last group of relations enables us to make a further reduction, namely

$ab_{n-1}b \to z\, ab,$

$ab_{n-1}b \to t\, ab.$

This way we can reduce a word to a linear combination of words lying in $K_{n-1}(\alpha, \beta)$. Assume that we are using a recurrence on $n$. This means that each element of $K_{n-1}(\alpha, \beta)$ can be uniquely reduced to an element of $R$ (the value of the functional on the element). Thus it suffices to check the obstructions directly on the values in order to obtain that the functional is well-defined. One formalizes this at follows. Let $\Gamma$ be a semi-oriented graph. This means that some of its edges are oriented and the remaining ones are unoriented. A path $v_1 v_2 ... v_n$ is a semi-oriented path if either $v_j \to v_{j+1}$ or else $v_j v_{j+1}$ is unoriented, for all $j$. If all edges of the chain are unoriented we say that its endpoints are unoriented equivalents.

**Definition 3.1** *$v_1 v_2 ... v_n$ is an open pentagon configuration in $\Gamma$ (o.p.c.) if $v_2 \to v_1$, $v_2 v_3 ... v_{n-1}$ is an unoriented path and $v_{n-1} \to v_n$.*

We state first the *pentagon condition* for semi-oriented graphs:

**Definition 3.2** *$\Gamma$ verifies the pentagon condition (PC) if for any open pentagon configuration $v_1 v_2 ... v_n$ there exist semi-oriented paths $v_1 x_1 x_2 ... x_m e$ and $v_n y_1 y_2 ... y_p e$ having the same endpoint (see figure 2).*

Set now $x \leq y$ if there exists an semi-oriented path from $y$ to $x$ in $\Gamma$. Of course $\leq$ is not always a partial order relation. A necessary and sufficient condition is that no closed semi-oriented loops exist in $\Gamma$. One says that $x$ is minimal if $y \leq x$ implies that $y$ is unoriented equivalent to $x$.

**Lemma 3.2** *Suppose that the (PC) holds. If a connected component $C$ of the graph $\Gamma$ has a minimal element $m_C$ then it is unique up to unoriented equivalence.*

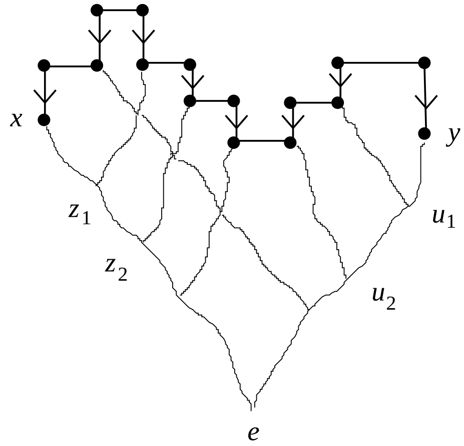


Figure 3: Proof of Pentagon Lemma

*Proof:* Consider two minimal elements $x$ and $y$ which lie in $C$. Then there exists some path $xx_0x_1...x_ny$ joining them. Since $x$ is minimal the closest oriented edge (if ever exists ) is ingoing, and the same is true for $y$. If this path is not unoriented again from minimality there are at least two oriented edges. Therefore open pentagon configurations (i.e. those configurations where (PC) applies) exist. We apply then (PC) iteratively whenever such configurations exist or has appeared. When this process stops we find two semi-oriented $xz_1z_2...z_pe$ and $yu_1u_2...u_se$ having the same endpoint $e$. So $e \leq x$ and $e \leq y$. Again from minimality these paths must be unoriented so $x$ and $y$ are unoriented equivalent (see figure 3). □

**Remark 3.1** *A priori one cannot say too much about the existence of such minimal elements. If $\leq$ had been a partial order with descending chain condition then the existence of minimal elements would be standard. However in the present case even if $\leq$ is not a partial order the existence of minimal elements can be established.*

### 3.2 The colored pentagon condition: the definition of $\Gamma_n$

Suppose now we have a sequence of disjoint graphs $\Gamma_n$. In every $\Gamma_n$ there exists a distinguished subset of vertices $V_n^0$ which are minimal elements in their connected components. Suppose that each connected component admits at least one minimal element. Each such vertex from $V_n^0$ has exactly one outgoing edge going to a vertex of $\Gamma_{n-1}$. We color these new edges in red. Set $\Gamma_n^*$ for the union of all $\Gamma_j$, $j \leq n$ and with the red edges added in each rank $j$.

**Definition 3.3** $\Gamma_n^*$ *is* coherent *if any connected component of $\Gamma_n$ has an unique minimal element (with respect to $\Gamma_n^*$) in $\Gamma_0$ up to unoriented equivalence.*

We state now the *colored* version of the Pentagon Lemma for this type of graphs.

**Definition 3.4** *We say that $\Gamma_n$ verifies the colored pentagon condition (CPC) if for any open pentagon configuration $v_1v_2...v_n$ in $\Gamma_n$ then there exist bicoloured semi-oriented paths (in $\Gamma_n^*$) from $v_2$ and $v_{n-1}$ having the same endpoint. In addition if $xy$ is an unoriented edge in $\Gamma_n$ with $x, y \in V_n^0$ then there exist semi-oriented paths in $\Gamma_n^*$ starting with red edges and having the same endpoint (see the figure 4).*

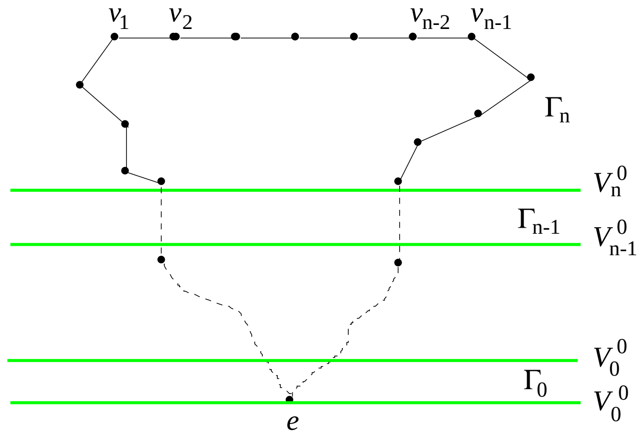


Figure 4: The colored Pentagon Condition

**Lemma 3.3** *Suppose that $\Gamma_{n-1}^*$ is coherent and the (CPC) condition is fulfilled. Then $\Gamma_n^*$ is coherent.*

*Proof:* The proof is similar to that of Pentagon Lemma. □

Now we are ready define our graph $\Gamma_n$. Its vertices are the elements of the ring algebra $\mathbb{Z}[\alpha, \beta, z, \bar{z}]F_n$, where $F_n$ is the free monoid $F_n$ in the $n$ letters $\{b_1, b_2, ..., b_n\}$. The vertices of $\Gamma_0$ will be the elements of $\mathbb{Z}[\alpha, \beta, z, \bar{z}]$. Two vertices $v = \sum_i \alpha_i x_i$ and $w = \sum_i \beta_i y_i$, $\alpha_i, \beta_i \in \mathbb{Z}[\alpha, \beta, z, \bar{z}]$ and $x_i$, $y_i \in F_n$, are related by an oriented edge if exactly one monomial of $v$ is changed following one of the rules

(C0)(j) $ab_j^3 b \to aE_j b$,
(C1)(j) $ab_{j+1}b_j b_{j+1} b \to ab_j b_{j+1} b_j b$,
(C2)(j) $ab_{j+1}b_j^2 b_{j+1} b \to aS_j b$,
(C12)(j) $ab_{j+1}b_j^2 b_{j+1}^2 b \to aC_j b$,
(C21)(j) $ab_{j+1}^2 b_j^2 b_{j+1} b \to aD_j b$,

where $E_j, S_j, C_j, D_j$ as above. An unoriented edge between $v$ and $w$ corresponds to a change in a monomial of $v$ of type

($P_{ij}$) $ab_i b_j b \to ab_j b_i b$ whenever $\mid i - j \mid > 1$.

Remark that the use of (C12) and (C21) is somewhat ambiguous since we may always use (C2) for a subword. Their role is to break in some sense the closed oriented loops in $\Gamma_n$. Consider now the following sets of words in the $b_i$'s:

$W_0 = \{1\}$,
$W_{n+1} = W_n \cup W_n b_{n+1} Z_n \cup W_n b_{n+1}^2 Z_n$.
$Z_n = \{b_n^{i_0} b_{n-1}^{i_1} ... b_{n-p}^{i_p}; i_1, i_2, ..., i_p \in \{1, 2\}, p = 0, n-1\}$.

Let $V_n^0$ be the set of vertices corresponding to elements of the $\mathbb{Z}[\alpha, \beta, z, \bar{z}]$-module generated by $W_n$.

**Lemma 3.4** *Each connected component of $\Gamma_n$ has a minimal element in $V_n^0$, not necessarily unique.*

*Proof:* We prove our claim by induction on $n$. For $n = 0$ it is obvious. Let now $w$ be a word in the $b_i$'s having only positive exponents. If its degree in $b_n$ is zero or one we apply the induction hypothesis and we are done. If the degree is 2 and it contains the subword $b_n^2$ we are able to apply the induction hypothesis. One can also suppose that no exponents greater than 2 occur by using (C0) several times. If the degree is 2 then $w = xb_n y b_n z$ with $x, y, z \in F_{n-1}$. The induction applied to $y$ implies that $w \geq xb_n a b_{n-1}^{\varepsilon} bz$ with $a, b \in F_{n-1}$. Then several transforms of type ($P_{nj}$) and (C$\varepsilon$)

will do the job. Consider now that the degree is strictly greater than 2. So we have a subword of type

$b_n^\alpha x b_n^\beta$ with $3 \leq \alpha + \beta \leq 4$

or else one of the type

$b_n x b_n y b_n$.

The second case reduce to the first one as above. Next say that $x \geq a b_{n-1}^\varepsilon b$, $a, b \in F_{n-2}$. Several applications of $(P_{nj})$ leads us to consider the word $b_n^\alpha b_{n-1}^\varepsilon b_n^\beta$. If $\varepsilon = 1$ we apply two times (C1) and we are done. Otherwise we shall apply (C$\alpha\beta$) and then (C1) if $\alpha \neq \beta$ or both (C12) and (C21) and then (C1) if $\alpha = \beta = 2$. This proves that every vertex descends to $V_n^0$. But these vertices have not outgoing edges as can be easily seen. When we use the unoriented edges some new vertices have to be added. But it is easy to see that these also does not have outgoing edges. Since any vertex has a semi-oriented path ending in $V_n^0$ we are done. □

**Remark 3.2** *The moves (C12) and (C21) are really necessary for the conclusion of the previous Lemma to remain valid. For instance look at the case $\alpha = \beta = 0$. From $b_{j+1} b_j^2 b_{j+1}^2$ only (C2) can be applied and in the linear combination we obtain the factor $b_{j+1}^2 b_j^2 b_{j+1}$. If we continue, then at each stage we shall find one of these two monomials. When all possible reductions are performed at the second stage we recover the word $b_{j+1} b_j^2 b_{j+1}^2$ so we have a closed oriented loop in the graph. This connected component has no minimal element unless we enlarge the graph by using the extra transformations (C12) and (C21). For general $\alpha, \beta$ a similar argument holds, and it can be checked by a computer program. If one does not use (C12) or (C21) then the reduction process for $b_{j+1} b_j^2 b_{j+1}^2$ yields at the sixth stage a sum of words generating an oriented loop.*

### 3.3 The bicoloured graph $\Gamma_n^*(H)$: the sub-module H

We are able now to define the bicoloured graph $\Gamma_n^*(H)$. The red edges are defined as follows. Each minimal vertex $v = \sum_{i,k} \alpha_{(i,k)} x_{(i,k)} b_n^k y_{(i,k)}$, where $k = 0, 1, 2$, is joined by a red edge to $w = \sum_{i,k} \alpha_{(i,k)} u_k x_{(i,k)} y_{(i,k)}$, which is a vertex of $\Gamma_{n-1}$, where we set $u_0 = 1, u_1 = z, u_2 = t$. Finally $\Gamma_0(H)$ is the graph having the vertices corresponding to the module $R$ and two vertices are connected by an unoriented edge iff the corresponding elements lie in the same coset of $R/H$, $H$ being a certain submodule of $R$. The submodule $H$ is necessary because going on different descending paths we might obtain different elements. Then, we have to find whether there exists $H$ so that $\Gamma_n^*(H)$ is coherent.

We will test the conditions of coherence of each $\Gamma_n^*(H)$ by recurrence on $n$. Notice that for $n = 1, 2$ there are no conditions on $H$. Our strategy is to make use of the Colored Pentagon Lemma in the following way. For those configurations that we cannot prove the (PC) holds directly we shall check that the (CPC) (which is weaker since it concerns all the tower $\Gamma_n^*(H)$) is verified.

Consider an open pentagon configuration (abbreviated o.p.c. ) $[w_0, w_1, ..., w_n]$. This means that $w_1 \to w_0$, $w_1, ..., w_{n-1}$ are unoriented equivalent and $w_{n-1} \to w_n$. We say that this o.p.c. is irreducible if none of the vertices $w_1, w_2, ..., w_{n-1}$ has an outgoing edge.

**Lemma 3.5** *i) In order to verify (PC) it suffices to restrict to irreducible configurations.*

*ii) It suffices to verify (PC) only for monomials from $F_n$.*

*iii) Suppose $w_j' = A w_j B$, for $j = 0, n$ (so $A, B$ are not touched by any transform) in the o.p.c.. If (PC) holds for $[w_0, w_1, ..., w_n]$ it also holds for $[w_0', w_1', ..., w_n']$.*

*iv) Suppose that (PC) holds for $[w_0, w_1, ..., w_n]$ and for $[y_0, y_1, ..., y_m]$. Then for all $A, B, C$ the (PC) is valid also for*

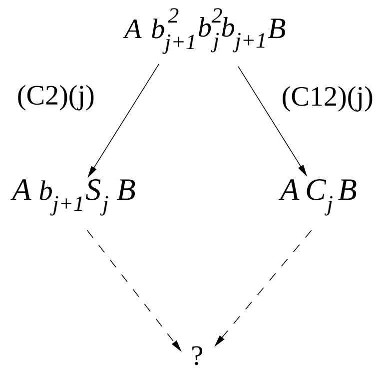


Figure 5: The o.p.c. for $Ab_{j+1}^2 b_j^2 b_{j+1} B$

$[Aw_0By_1C, Aw_1By_1C, ..., Aw_{n-1}By_1C, Aw_{n-1}By_2C, Aw_{n-1}By_3C, ...$
$.., Aw_{n-1}By_{m-1}C, Aw_{n-1}By_mC]$.

*In fact when we fix the endpoints of the o.p.c. we can mix the unoriented edges of each subjacent o.p.c. in any order we want. Let* $(i_k, j_k) \in \{0,1,...,n\} \times \{0,1,...,m\}, k = 1,p$ *such that* $i_0 = 0 < i_1 \le i_2 \le ... \le i_p, j_p = m > j_{p-1} \ge ... \ge 0$, *and* $i_{k+1} - i_k + j_{k+1} - j_k = 1$ *for all* $k$. *Then the o.p.c.* $\left[Aw_{i_0}By_{j_0}C, Aw_{i_1}By_{j_1}C, ..., Aw_{i_p}By_{j_p}C\right]$ *fulfills the (PC).*

*Proof:* i) We may always decompose a configuration into irreducible ones and iterate the construction.

ii) The reduction transforms on different monomials commute with each other so we are done.

iii) Obvious.

iv) The reductions of $x_{n-1}$ and $y_1$ commute again with each other. □

Thus the top line of a o.p.c. corresponds to a word $w_1$ and a sequence of permutations of its letters giving in order $w_2, w_3, ..., w_{n-1}$. We may suppose that $w = w_1$ has no proper subwords $w_1'$ which fulfill the following two conditions:

i) Set $w = Aw'B$. Then each of the considered permutations acts only on the letters of $A$, of $B$ or $w'$. Thus the transform $w"$ of $w'$ is equivalent to $w'$.

ii) The reduction transforms performed at $w_1$ and $w_2$ acts actually on $w'$ and $w"$.

Now we can study the (PC) for irreducible configurations as in Lemma 3.5.
The first step is to check if the (PC) condition holds when the top line is trivial ($n = 2$) and there are two or more outgoing edges. For instance, see figure 5. $A$ and $B$ are subwords not touched by reductions and on the subword $b_{j+1}^2 b_j^2 b_{j+1}$ one can apply (Cij) or (C2). $C_j$ and $S_j$ are as in section 3.1 .

**Lemma 3.6** *If the top line is trivial then the (PC) holds.*

*Proof:* By Lemma 3.5 we have a finite number of cases to test. These are the words of the form $abc$, where $ab$ and $bc$ are subwords belong to the set $\{b_{j+1}^3,\ b_{j+1}b_jb_{j+1},\ b_{j+1}b_j^2b_{j+1},\ b_{j+1}^2b_j^2b_{j+1},\ b_{j+1}b_j^2b_{j+1}^2\}$, $j = 1, \ldots, n-2$. The number of cases to study can be easily reduced, since

- If $b$ is the identity, the (PC) trivially holds.
- By homogeneity of the reductions (C $\varepsilon$)(j) it suffices to consider $j = 1$.

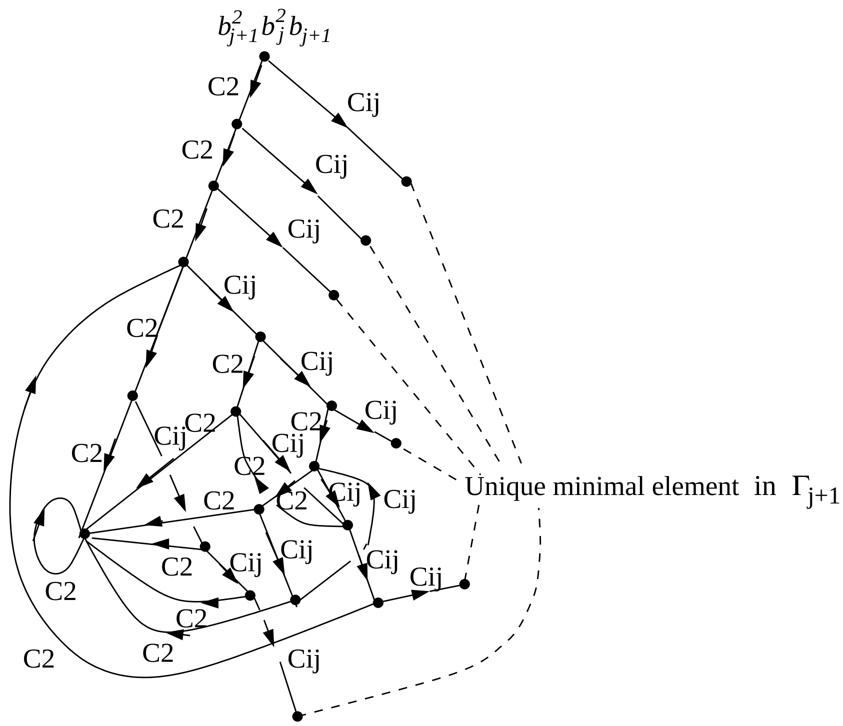


Figure 6: The graph underlying to $b_{j+1}^2 b_j^2 b_{j+1}$

- For a word $w = w_1, \ldots, w_l$ its symmetric is the word $w^* = w_l, \ldots, w_1$. If the (PC) holds for $abc$, (PC) holds also for the symmetric word $(abc)^*$ (this result follows from the form of reductions).
- Several cases, as $b_{j+1}^3 b_j b_{j+1}$, can be easily tested at hand.

The non trivial cases appear when a (Cij)-move (and then a (C2)-move) can be applied. Actually, we have to check only $b_{j+1}^2 b_j^2 b_{j+1}$, since $b_{j+1} b_j^2 b_{j+1}^2$ is its symmetric and the cases $b_{j+1}^{\varepsilon_1} b_j^2 b_{j+1}^{\varepsilon_2}$ ($\varepsilon_i = 2, 3$) are consequences of these ones. Then, we start from the situation depicted in figure 5 ($A, B$ are empty words). If we apply (Cij) whether is possible on $b_{j+1} S_j$, after a long and messy computation we find the same minimal element associated to $C_j$. □

**Remark 3.3** *Using a computer program one can get the oriented graph associated to* $b_{j+1}^2 b_j^2 b_{j+1}$ *(figure 6). The vertices are of the type* $\sum c_j w_j$, $c_j$ *polynomials in* $\alpha$, $\beta$ *and* $w_j$ *words in* $b_j$, $b_{j+1}$. *An oriented edge between an outgoing vertex a and an ingoing vertex b indicates that the reduction procedure applied to a yields b. When there are no subwords* $b_{j+1}^2 b_j^2 b_{j+1}$ *or* $b_{j+1} b_j^2 b_{j+1}^2$ *the edges are spotted. As we already noticed in Remark 3.2, if we apply six times the procedure without (Cij) we find a loop.*

Let us study the case when the top line is non trivial. By Lemma 3.6 we can suppose that $w_1$ and $w_{n-1}$ have each one exact one outgoing edge. In particular, when a (Cij)-move can be applied, we choose always the edge (Cij) in the reduction process.

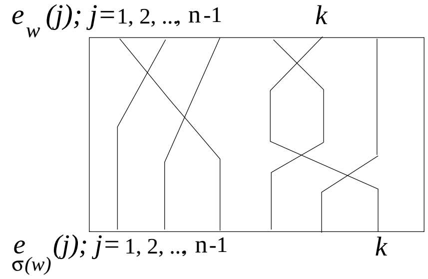


Figure 7: The complete diagram associated to an o.p.c.

Now the top line is determined by the sequence of transpositions of the letters of $w$. Let $l$ be the length of $w$. Otherwise this is the same to giving a permutation $\sigma \in S_l$ with a prescribed decomposition into transpositions. Set $T_j$ for the transposition which interchanges the letters on the positions $j$ and $j+1$. Notice that for a fixed $w$ not all $\sigma$ are suitable. In fact only a subset of the group of permutations, which we call permitted, may work. Say $P(w)$ is the set of permitted permutations. If $e_w : \{1, 2, ..., l\} \longrightarrow \{1, 2, ..., n-1\}$ is the evaluation map

$e_w(j) =$index of the letter lying in position $j$ on $w$

then $T_j\sigma$ is permitted (where $\sigma \in P(w)$) iff

$\mid e_{\sigma(w)}(j) - e_{\sigma(w)}(j+1) \mid > 1$.

Say that two permitted permutations $\sigma$ and $\sigma'$ are equivalent if for the o.p.c. corresponding to $\sigma$ and $\sigma'$ the (PC) is valid or not for both in same time.

**Lemma 3.7** *i) Suppose that $\sigma_1 T_j T_i \sigma_2 \in P(w)$, $\mid i - j \mid > 1$. Then $\sigma_1 T_i T_j \sigma_2 \in P(w)$ and these two permutations are equivalent.*

*ii) Suppose that $\sigma_1 T_{i+1} T_i T_{i+1} \sigma_2 \in P(w)$. Then $\sigma_1 T_i T_{i+1} T_i \sigma_2 \in P(w)$ and these two permutations are equivalent. The converse is still true.*

*iii) If $\sigma_1 T_i^2 \sigma_2 \in P(w)$ then $\sigma_1\sigma_2$ is permitted and equivalent to the previous one.*

*Proof:* The existence in the first case is equivalent to $\mid e_{\sigma_2(w)}(j) - e_{\sigma_2(w)}(j+1) \mid > 1$ and $\mid e_{\sigma(w)}(i) - e_{\sigma(w)}(i+1) \mid > 1$, so it is symmetric. In the second case also it is equivalent to $\mid e_{\sigma_2(w)}(j+\varepsilon_1) - e_{\sigma_2(w)}(j+\varepsilon_2) \mid > 1$ for all $\varepsilon_j \in \{0, 1, 2\}$, so it is again symmetric. The equivalence is trivial. □

One uses a graphical representation for the decomposition of $\sigma$ into transpositions similar to the braid pictures (see picture 7), where we specify on the top and bottom lines of the diagram the values of the evaluation maps.

This picture encodes all information about the o.p.c. because the two words $w$ and $\sigma(w)$ have unique reduction. For the moment one draws only those trajectories of the six (to ten) elements which enter in the two blocks which reduces. Suppose for instance that the two reduction moves are two (C0). So $w = xiiiy$ and $\sigma(w) = x'jjjy'$. Say that $i = j$. The trajectories of the $i'$s may be disjointed since the transposition acting on the couple $ii$ is trivial in fact. So the possible trajectories fit into 4 cases which may be seen in picture 8.a,b,c,d.

Suppose now we have two trajectories of $i$ and $j \neq i$ which intersects. First of way we derive that $\mid i - j \mid > 1$. Orient all the arcs from the top to the bottom.

**Lemma 3.8** *i) Suppose that the arcs labeled $i$ and $j$ have algebraic intersection number 0. Then we can replace the diagram by an equivalent one where the arcs are disjoint.*

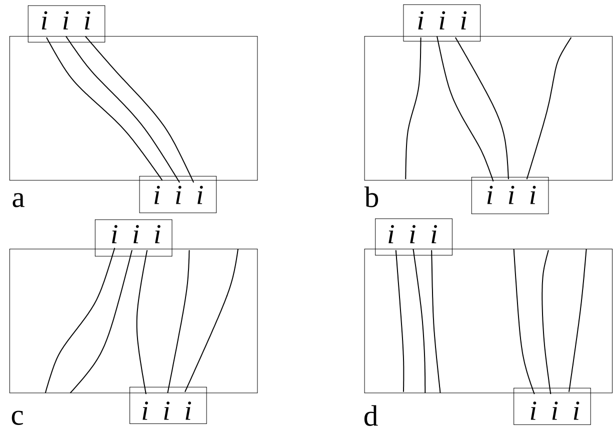


Figure 8: The essential trajectories for (C0)(i)-(C0)(i)

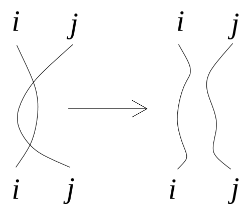


Figure 9: Disjointing trajectories

*ii) Suppose that the arcs labeled $i$ and $j$ have algebraic intersection number 1. Then we can replace the diagram by an equivalent one where the arcs have exactly one intersection point.*

*Proof:* We consider the diagram is that from figure 9.

We can assume that the biangle in the middle is minimal, hence it does not contain any other biangle. In fact we can apply repeatedly the disjointedness procedure only for minimal biangles. Such biangle have two walls: one coming from $i$ and the other from $j$. From minimality no other arc cross twice the same wall (see picture 10).

Let consider the region $L$ and $R$ such that: the set of arcs labeled by something not commuting with $j$ is contained in $L$, and those labeled by some $k$ not commuting with $i$ are contained in $R$. Then the situation is that from picture 11.

Thus all arcs which cross the biangle are labeled by some $k$ which commutes with both $i$ and $j$. The same commutation transforms may be performed whenever we make the arcs $i$ and $j$ disjoint. □

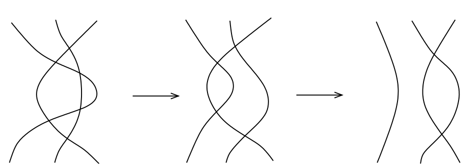

Figure 10: Non minimal biangle procedure

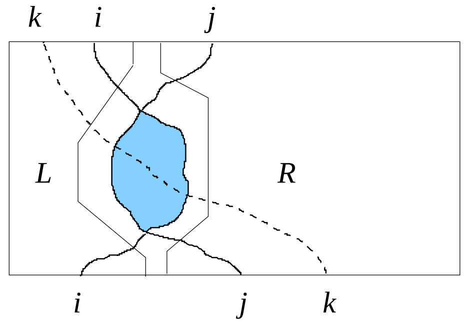


Figure 11: The regions $R$ and $L$

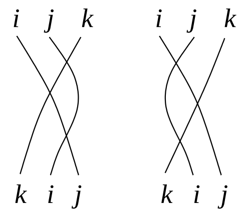


Figure 12: Equivalent diagrams

A similar reasoning permits to say that the diagrams from picture 12 are equivalent.
When the triangle in the middle is not touched by any arc then it is a simple consequence of lemma 3.8 ii). If it is minimal, any arc which cross it is labeled by something which commutes with $j$.
Remark now the similitude of pictures 9 and 12 with the Reidemester's moves on link diagrams. So we can actually isotopy our arcs leaving the endpoints fixed and keeping the tangent (in a $C^1$-approximation of arcs) away from the horizontal.
Now we can continue our discussion on the trajectories of $i'$s and $j'$s. If $\mid i-j \mid= 1$ the trajectories are disjoint so there are as in picture 13.

If $i$ and $j$ commutes there are essentially sixteen diagrams (up to isotopy) which can be seen in picture 14.

In order to represent graphically the possible diagrams for the (C1), (C2), (C12), (C21) moves we shall picture the trajectories of a couple of neighbor points having the same label as a single thicker trajectory. This may be done since every arc crossing the dashed region (see figure 15) between the trajectories of the the two $i'$s has a label commuting with $i$. In addition the trajectories of $i$ and $i+1$ are disjoint.

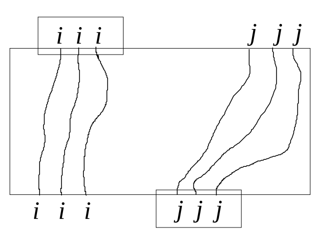


Figure 13: The diagram for (C0)(i)-(C0)(j) when $\mid i-j \mid> 1$

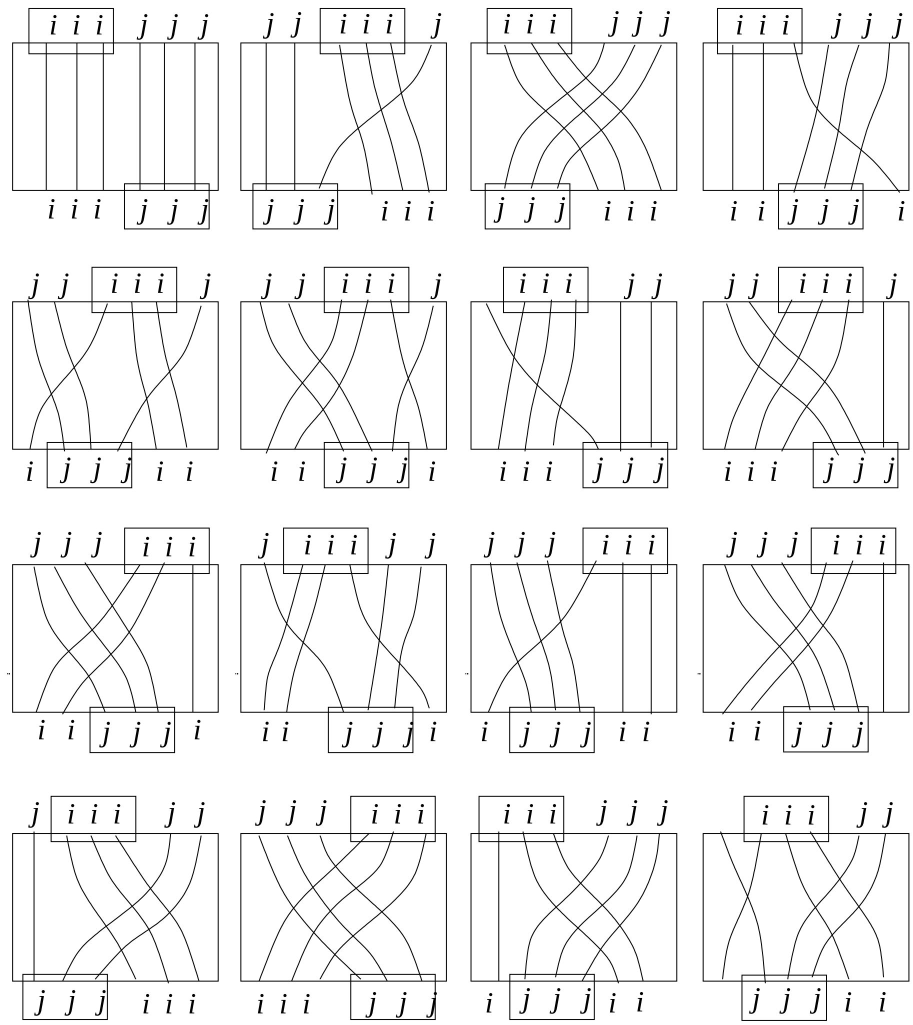


Figure 14: The 16 diagrams for (C0)(i) -(C0)(j) in the commuting case

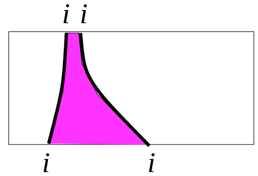


Figure 15: The graphical representation of the dashed region

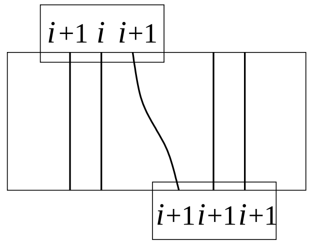


Figure 16: The new diagram for (C1)(i)-(C0)(i+1)

Suppose we are in the case (C1)(i)-(C0)(j). For $j \neq i-1, i, i+1, i+2$ the twelve diagrams from above appear appropriately labeled. For $j = i-1, i, i+2$ some diagrams are not realized because the arcs labeled by $i-1$ and $i$ does not intersect, so several cases have to be left. For $j = i+1$ another diagram have to be considered, that from figure 16.

The same situation we encounter when we describe the possible trajectories for the couple of reduction transforms (C2)-(C0), (C12)-(C0), (C21)-(C0). A simple analysis shows that in the remaining cases the only new diagrams are those from figure 17.

The other ones are obtained from the previous twelve using the suitable labeling, and taking into account the constraints of disjointedness imposed by the labels. We say now that a diagram is *interactive* if there is some marked arc relating the top and bottom blocks where the reduction transforms act. Our task will be to eliminate the non-interactive diagrams where the (PC) trivially holds.

**Lemma 3.9** *The usual (PC) is valid in* $\Gamma_n$ *for non-interactive diagrams.*

*Proof:* We consider first the case where no crossings of the essential arcs exist. The typical case is that from picture 13. We draw now all trajectories as in figure 18. We have the dashed regions $U$ and $V$ which are bounded by the $i'$s arcs and respectively $j'$s arcs.

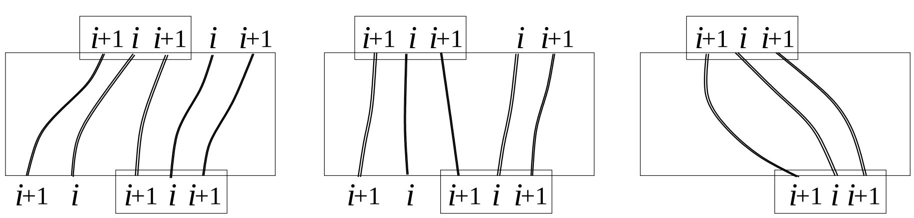


Figure 17: The new diagrams for (Cx)(i)-(Cy)(i) $x, y \neq 0$

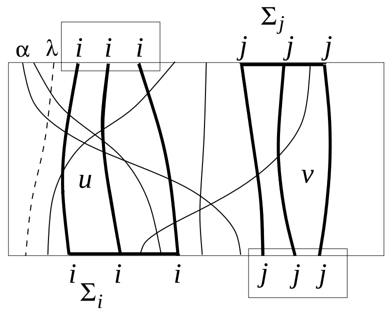


Figure 18: The whole picture of a non-interactive diagram without crossings

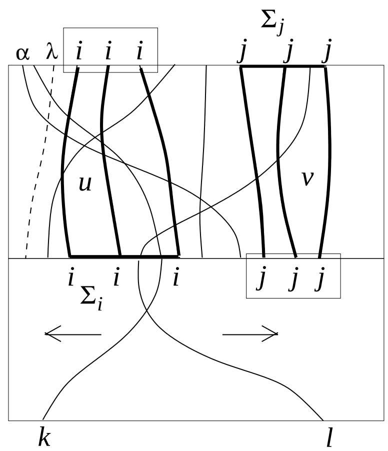


Figure 19: The simplification of a non-interactive diagram without crossings of essential arcs

Everything crossing the regions $U$ and $V$ commutes with $i$ and $j$ respectively. We claim first that $U$ and $V$ are tangent to the end lines from left and right respectively. If not there exists some arc labeled $\lambda$ lying to the left of $U$. Assume that this arc is the first from the left having this property. In particular $\lambda$ commutes with every label $\alpha$ which stands to the left of $\lambda$. Thus we may perform these commutation transforms at any moment, to get $\lambda$ on the first position. Since $\lambda$ does not cross $U$ we may leave it on the the first position replacing the o.p.c. by an equivalent one. Thus the new configuration corresponds to a word which is not minimal with respect to the reduction procedure (see Lemma 3.5 and the subsequent comments).

Let now $\Sigma_i$ be the convex hull of the three points labeled $i$ coming from essential arcs and lying on the bottom line. Similarly set $\Sigma_j$ for the convex hull of the $j'$s on the top line. Every arc which arrive on $\Sigma_i$ must cross $U$ hence is labeled by some $k$ commuting with $i$. We can move these endpoints using the commutation rules from the left or the right according to the following principle: if the start point of the arc labeled $k$ is in the left of the block of $i'$'s on the top line, then we move to the left. Otherwise we move to the right. The only problem which we can have is in the following case: the start point of some $k$ is in the left of the arc labeled $l$, both arrive on $\Sigma_i$, but this time the endpoint of $l$ is in the left of $k$. A topological argument shows that these two arcs cross each other. Therefore $k$ and $l$ are commuting and we can perform our transforms as it was said (see figure 19).

Finally we recover a diagram which this time has crossings but is equivalent to the standard one

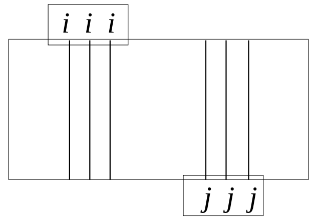


Figure 20: The standard non-interactive diagram

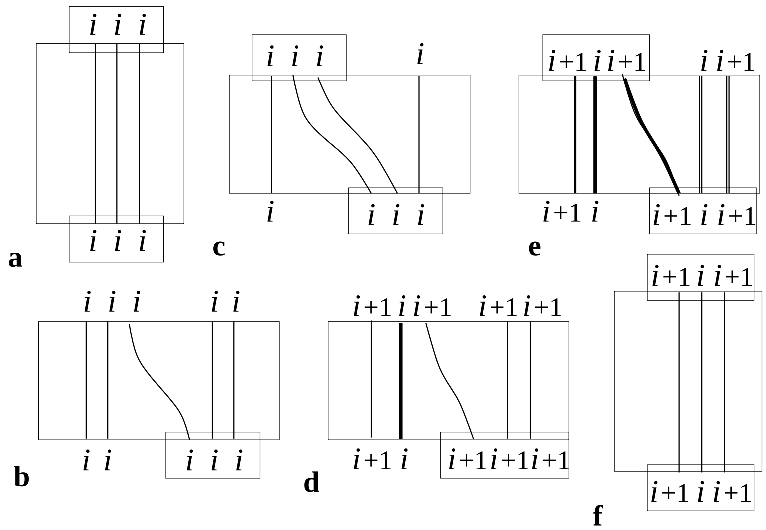


Figure 21: The normal forms of interactive configurations

of picture 20.

Without loss of generality we can set $\alpha = \beta = 0$ in the reduction transforms in order to simplify the notation. Suppose now that the reduction transforms $AiiiB \to AB$ and $CjjjD \to CD$ are also performed. We may use the simplification transforms (commutations which are still valid even if the $i$ or the $j$ are collapsed) for above for each word: to $AB$ in the part of $j'$s and to $CD$ in the part of $i'$s. Due to the particular form of the standard diagram we shall get (see the picture 18) the words $UjjjV$ and $U'iiiV'$ respectively, with $UV = U'V'$. So again the use of a reduction transform will get the same word. Thus the (PC) is satisfied for these configurations. It is almost the same reasoning for the other non-interactive diagrams without crossings.
It remains the case when crossings of essential arcs appear. But the commutation transforms may be also be performed in such way that the starting points of $j'$s on the top line will be all on the same part with respect to the $iii$ block. In other words we make $\Sigma_j$ and the block $iii$ disjoint. The same is true for the bottom line. The worst case is again when $iii$ is in the left of $\Sigma_j$ on the top line and down the situation is reversed. But again $i$ and $j$ commutes with everything starts or arrives on the convex hulls of $iii \cup \Sigma_j$ and $jjj \cup \Sigma_i$. So we can rearrange them to obtain the same order in the top and bottom lines. This ends the proof of the Lemma. □

So it remains to look at the interactive configurations. The same reasoning permits us to restrict to the normal forms drawn in figure 21.a-f.

Some of the trajectories may be thick trajectories.

The cases **a**,**b**,**c**,**d**,**f** are trivially verified because only the consistency of relations defining $K_3(\alpha,\beta)$ is involved.

Let us check a subcase of **d**, corresponding to (C$\varepsilon$)-C(0). The monomial has the form $w = b_{i+1}b_i^{\varepsilon}b_{i+1}xb_{i+1}^2$ which is unoriented equivalent to $w' = b_{i+1}b_i^{\varepsilon}xb_{i+1}^3$. Here $x$ commutes with $b_{i+1}$ and so we may suppose it lies in $F_{i-1}$. Therefore $x \to x_0 b_{i-1}^{j_1} b_{i-2}^{j_2} ... b_{i-p}^{j_p}$, with $x_0 \in F_{i-2}$. So again we can restrict to the case $x_0 = 1$. Consider the case $\varepsilon = 2$ (the others are trivial). Set $q = b_{i-2}^{j_2}...b_{i-p}^{j_p}$. We have the following situation

$$\begin{array}{ccc} & w \;\text{——}\; w' & \\ \swarrow & & \searrow \\ S_j b_{i-1}^{j_1} b_{i+1}^2 q & & b_{i+1} b_i^2 E_j b_{i-1}^{j_1} q \end{array}$$

where $S_j, E_j$ as above. From Lemmas 3.5 and 3.6 it follows that (PC) holds for

$$\begin{array}{ccc} & b_{i+1} b_i^2 b_{i+1}^3 b_{i-1}^{j_1} q & \\ \swarrow & & \searrow \\ S_j b_{i+1}^2 b_{i-1}^{j_1} q & & b_{i+1} b_i^2 E_j b_{i-1}^{j_1} q \end{array}$$

Since $S_j b_{i-1}^{j_1} b_{i+1}^2 q$ is unoriented equivalent to $S_j b_{i+1}^2 b_{i-1}^{j_1} q$ we have done. All other cases but **e** are similar.

In the case **e** the situation is different. Using the commutation rules, as above we must preserve the term $b_{i-1}^{j_1}$. So we must check the configurations

$w = x b_{i+1}^{\alpha} b_i^{\varepsilon} b_{i+1}^{\beta} b_{i-1}^{\mu} b_i^{\delta} b_{i+1}^{\gamma} b_{i-2}^{j_2} ... b_{i-p}^{j_p}$,

where $x \in F_{i-1}$. At this point one cannot prove that the (PC) holds. In fact it does not hold since the surjection of Lemma 3.1 has a nontrivial kernel in rank $n = 3$. Fortunately we proved that the configurations that don't verify (CPC) come from a finite number of obstructions. Therefore one can define $H$ as the ideal containing all these obstructions, and see whether it is nontrivial.

**Lemma 3.10** *It suffices to consider $x = 1, p = 1$.*

*Proof:* One observes that any admissible functional $\mathcal{T}$ on $K_\infty(\alpha,\beta)$ satisfies:

$\mathcal{T}(xuv) = \mathcal{T}(u)\mathcal{T}(xv)$ for $x, v \in H(Q,m)$ and $u \in < 1, b_m, b_{m+1}, ..., b_{m+k} >$.

In fact for $k = 0$ this is the multiplicativity of $\mathcal{T}$. For $k > 0$ then in the reduction process one replaces $u$ by $\alpha b_m^{\varepsilon}$ where $\mathcal{T}(u) = \alpha\mathcal{T}(b_m^{\varepsilon})$. When reducing again one derives $\mathcal{T}(xuv) = \alpha\mathcal{T}(b_m^{\varepsilon})\mathcal{T}(xv)$.

Further the (CPC) is equivalent to the existence of an admissible functional. □

We have therefore to check the o.p.c. corresponding to following couples

$$b_3^{\xi}\, b_2^{\epsilon}\, b_1^{\nu}\, b_3^{\mu}\, b_2^{\delta}\, b_3^{\gamma} \text{ and } b_3^{\xi}\, b_2^{\epsilon}\, b_3^{\mu}\, b_1^{\nu}\, b_2^{\delta}\, b_3^{\gamma} \quad \xi, \epsilon, \mu, \nu, \delta, \gamma = 1 \text{ or } 2$$

Then the only possible obstructions to the existence of Markov trace come out from these couples. In section 5 we study these obstructions and we find the ideal $H$ in $R$ containing them.

# 4 The computation of obstructions

## 4.1 Commutativity obstructions

We are now concerned with the commutativity constraints:

$$\mathcal{T}(ab) = \mathcal{T}(ba) \text{ for all } a,\, b.$$

At the first stage (i.e. $K_3(\alpha,\,\beta)$) we obtain the identities

$$\mathcal{T}(b_2\, b_1^2\, b_2) = \mathcal{T}(b_1^2\, b_2^2),\ \mathcal{T}(b_1\, b_2\, b_1^2\, b_2) = \mathcal{T}(b_2\, b_1\, b_2\, b_1^2).$$

Thus the following equations should be satisfied:

$$\mathcal{T}(R_0) = \mathcal{T}(R_1) = 0.$$

i.e.
$(-\beta^3+3\alpha\beta+4)t^2+(3\alpha^2-7\alpha\beta^2-6\beta+2\beta^4)t+(3\beta^2-\beta^5-2\alpha-3\alpha^2\beta+4\alpha\beta^3)+(2\alpha\beta^3+\beta^2-6\alpha^2\beta-10\alpha)zt+(-3\alpha^3+7\alpha^2\beta^2+9\alpha\beta+4-\beta^3-2\alpha\beta^4)z+(3\alpha^3\beta+7\alpha^2-\alpha^2\beta^3-\alpha\beta^2+2\beta)z^2 = (\beta^2-2\alpha)t^2+(4+5\alpha\beta-2\beta^3)t+(\beta^4-2\beta-3\alpha\beta^2+\alpha^2)+(2\beta+5\alpha^2-2\alpha\beta^2)zt+(\beta^2+2\alpha\beta^3-5\alpha^2\beta-6\alpha)z+(4+\alpha^2\beta^2+\alpha\beta-2\alpha^3)z^2 = 0$

These yield the following values for the parameters:

- either

$$z = \frac{-\beta^2+2\alpha}{\alpha\beta+4}, \qquad t = \frac{\alpha^2+2\beta}{\alpha\beta+4},$$

- or else

$$t = \frac{2\alpha z-2z^2+\beta}{2+\beta z}, \quad \text{where } z \text{ verifies } (\alpha\beta+1)z^3+(\alpha+\beta^2)z^2+2\beta z+1=0.$$

One checks then the commutativity constraints by induction on $n$. It suffices to consider $b \in \{b_1,...,b_n\}$ and $a$ lying in a system of generators of $K_{n+1}(\alpha,\beta)$, let us say $W_n$ (section 3.2). For $b=b_i$, $i<n$ it is obvious. It remains to check whenever $\mathcal{T}(ab_n)=\mathcal{T}(b_na)$. We have three cases
i) $a \in K_n(\alpha,\beta)$.
ii) $a = xb_ny$, $x,y \in K_n(\alpha,\beta)$.
iii) $a = xb_n^2y$, $x,y \in K_n(\alpha,\beta)$.

which will be discussed in combination with the six subcases
1) $x \in K_{n-1}(\alpha,\beta)$, and $y \in K_{n-1}(\alpha,\beta)$.
2) $x \in K_{n-1}(\alpha,\beta)$, and $y = ub_{n-1}v$, $u,v \in K_{n-1}(\alpha,\beta)$.
3) $x \in K_{n-1}(\alpha,\beta)$, and $y = ub_{n-1}^2v$, $u,v \in K_{n-1}(\alpha,\beta)$.
4) $x = rb_{n-1}s$, $r,s \in K_{n-1}(\alpha,\beta)$, $y = ub_{n-1}v$, $u,v \in K_{n-1}(\alpha,\beta)$.
5) $x = rb_{n-1}s$, $r,s \in K_{n-1}(\alpha,\beta)$, $y = ub_{n-1}^2v$, $u,v \in K_{n-1}(\alpha,\beta)$.
6) $x = rb_{n-1}^2s$, $r,s \in K_{n-1}(\alpha,\beta)$, $y = ub_{n-1}^2v$, $u,v \in K_{n-1}(\alpha,\beta)$.

Now (*,i), (1,ii) and (1,iii) are trivial.

(2,ii) $\mathcal{T}(b_nxb_nub_{n-1}v) = tz\mathcal{T}(xuv) = \mathcal{T}(xb_nub_{n-1}vb_n)$.

(2,iii) $\mathcal{T}(b_n x b_n^2 u b_{n-1} v) = (\alpha t + \beta z + 1)\mathcal{T}(xub_{n-1}v) = (\alpha t + \beta z + 1) z \mathcal{T}(xuv)$
$= \mathcal{T}(xub_{n-1}b_n b_{n-1}^2 v) = \mathcal{T}(xb_n^2 u b_{n-1} v b_n)$.
(3,ii) $\mathcal{T}(b_n x b_n u b_{n-1}^2 v) = t^2 \mathcal{T}(xuv) = \mathcal{T}(b_n^2 b_{n-1}^2)\mathcal{T}(xuv) = \mathcal{T}(b_n b_{n-1}^2 b_n)\mathcal{T}(xuv) =$
$= \mathcal{T}(xub_n b_{n-1}^2 b_n v) = \mathcal{T}(xb_n u b_{n-1}^2 v b_n)$.
(3,iii) $\mathcal{T}(b_n x b_n^2 u b_{n-1}^2 v) = (\alpha t + \beta z + 1)\mathcal{T}(xub_{n-1}^2 v) = (\alpha t + \beta z + 1) t \mathcal{T}(xuv)$
$= \mathcal{T}(xuv)\mathcal{T}(b_n^2 b_{n-1}^2 b_n) = \mathcal{T}(xb_n^2 u b_{n-1}^2 v b_n)$.

For the other cases, we need also to know the form of $su$. Set $su = pb_{n-2}^{\varepsilon} q$ with $p, q \in K_{n-2}(\alpha, \beta)$ where $\varepsilon = 0, 1$ or $2$. We can show by a direct computation that the equalities hold also for $(4, ii), (4, iii), (6, ii)$, and $(6, iii)$. Using Maple we have found that in the cases $(5, ii)$ and $(5, iii)$ for $su = pb_{n-2}^2 q$ there are only two new equations, which are not consequences of the identities $\mathcal{T}(R_0) = \mathcal{T}(R_1) = 0$). Specifically we have three obstructions in each case, namely the polynomial coefficients of $\mathcal{T}(rpb_{n-2}^2 qv)$, $\mathcal{T}(rpb_{n-2} qv)$ and $\mathcal{T}(rpqv)$.

- from $(5, ii)$ we have
  - the coefficient of $\mathcal{T}(rpb_{n-2}^2 qv)$ yields the equation $L := 3\alpha\beta^4 + 5\alpha^2\beta^5 - 2\alpha\beta + 2\alpha^4\beta - 7\alpha^3\beta^3 - 7\alpha^2\beta^2 - \alpha\beta^7 + \alpha^3 + (13\alpha^3\beta^2 - 10\alpha^2\beta^4 + 13\alpha^2\beta - 6\alpha\beta^3 - 2\alpha^4 + 3\alpha + 2\alpha\beta^6)t + (-6\alpha^3\beta - \alpha\beta^5 - 6\alpha^2 + 3\alpha\beta^2 + 5\alpha^2\beta^3)t^2 + (-16\alpha^4\beta^2 - 5\alpha\beta^2 - 2\alpha^2 + 3\alpha^5 + 2\alpha\beta^5 - 13\alpha^3\beta + 11\alpha^3\beta^4 - 2\alpha^2\beta^6)z + (-2\alpha\beta^4 + 15\alpha^4\beta + 2\alpha^2\beta^5 - 11\alpha^3\beta^3 + 15\alpha^3 + 6\alpha\beta)zt + (-3\alpha - \alpha^3\beta^5 + 6\alpha^4\beta^3 - 3\alpha^3\beta^2 + 2\alpha^2\beta^4 - 9\alpha^5\beta - 9\alpha^2\beta - 10\alpha^4)z^2 = 0$,
  - the coefficient of $\mathcal{T}(rpb_{n-2} qv)$ vanishes is equivalent to $M := \alpha - \alpha^4 + 6\alpha^2\beta - 2\alpha^5\beta - 2\alpha\beta^3 + 7\alpha^4\beta^3 + 11\alpha^3\beta^2 + \alpha\beta^6 - 7\alpha^2\beta^4 - 5\alpha^3\beta^5 + \alpha^2\beta^7 + (-21\alpha^3\beta - 2\alpha^2\beta^6 + 2\alpha\beta^2 + 14\alpha^2\beta^3 - 13\alpha^4\beta^2 - 7\alpha^2 + 10\alpha^3\beta^4 - 2\alpha\beta^5 + 2\alpha^5)t + (-7\alpha^2\beta^2 + 6\alpha^4\beta + 10\alpha^3 + \alpha\beta^4 + \alpha^2\beta^5 - 5\alpha^3\beta^3)t^2 + (-3\alpha^6 + 2\alpha^3\beta^6 + 5\alpha\beta + 11\alpha^2\beta^2 + 16\alpha^5\beta^2 + 8\alpha^3 + 25\alpha^4\beta - 11\alpha^4\beta^4 - 4\alpha\beta^4 - 10\alpha^3\beta^3)z + (11\alpha^4\beta^3 - 14\alpha^2\beta + 10\alpha^3\beta^2 - \alpha + 4\alpha\beta^3 - 15\alpha^5\beta - 27\alpha^4 - 2\alpha^3\beta^5)zt + (4\alpha\beta^2 - 4\alpha^2\beta^3 + \alpha^4\beta^5 + 19\alpha^5 - \alpha^3\beta^4 + 4\alpha^2 - 3\alpha^4\beta^2 + 21\alpha^3\beta - 6\alpha^5\beta^3 + 9\alpha^6\beta)z^2 = 0$,
  - the coefficient of $\mathcal{T}(rpqv)$ from which one derives $N := 12\alpha^2\beta^3 + \alpha\beta^8 - 6\alpha^2\beta^6 - 2\alpha^2 + 3\alpha\beta^2 + 11\alpha^3\beta^4 - 4\beta^5\alpha - 6\alpha^4\beta^2 - 7\alpha^3\beta + (-21\alpha^3\beta^3 + 7\alpha\beta^4 + 5\alpha^3 + 10\alpha^4\beta - 2\alpha\beta^7 - 2\alpha\beta - 17\alpha^2\beta^2 + 12\alpha^2\beta^5)t + (-4\alpha^4 + 10\alpha^3\beta^2 - 3\alpha + \alpha\beta^6 + 5\alpha^2\beta - 6\alpha^2\beta^4 - 3\alpha\beta^3)t^2 + (3\alpha + 3\alpha\beta^3 + 2\alpha^2\beta^7 + 16\alpha^3\beta^2 - 2\alpha\beta^6 - 7\alpha^4 - 13\alpha^5\beta + 5\alpha^2\beta - 13\alpha^3\beta^5 + 25\alpha^4\beta^3)z + (\alpha^2 - 12\alpha^3\beta + 10\alpha^5 + 13\alpha^3\beta^4 - \alpha^2\beta^3 - 2\alpha^2\beta^6 + 2\alpha\beta^5 - 24\alpha^4\beta^2 - 5\alpha\beta^2)zt + (5\alpha^3 + 4\alpha^3\beta^3 + 14\alpha^5\beta^2 + 8\alpha^4\beta + 7\alpha^2\beta^2 + \alpha^3\beta^6 + 5\alpha\beta - 2\alpha^2\beta^5 - 6\alpha^6 - 7\alpha^4\beta^4)z^2 = 0$.

- from $(5, iii)$ one obtains the obstructions:
  - the coefficient of $\mathcal{T}(rpb_{n-2}^2 qv)$ yields $-\alpha L = 0$,
  - the coefficient of $\mathcal{T}(rpb_{n-2} qv)$ yields $-\alpha M = 0$,
  - the coefficient of $\mathcal{T}(rpqv)$ yields $-\alpha N = 0$.

## 4.2 The CPC obstructions for n=4

As pointed out in section 3 the coherence of $\Gamma_n^*(H)$ depends on the following couples:

$$b_3^{\xi}\, b_2^{\epsilon}\, b_1^{\nu}\, b_3^{\mu}\, b_2^{\delta}\, b_3^{\gamma} \text{ et } b_3^{\xi}\, b_2^{\epsilon}\, b_3^{\mu}\, b_1^{\nu}\, b_2^{\delta}\, b_3^{\gamma} \quad \xi, \epsilon, \mu, \nu, \delta, \gamma = 1 \text{ or } 2$$

Recall that for a word $w = w_1, \ldots, w_l$ its symmetric is the word $w^* = w_l, \ldots, w_1$. Since $\mathcal{T}(w) = \mathcal{T}(w^*)$ holds one can reduce ourselves to the study of 24 couples. The couples that we must check are the following:

- $(1.i)$ : $b_3\, b_2\, P_i\, b_2^2\, b_3$ and $b_3\, b_2\, P_i'\, b_2^2\, b_3$,
- $(2.i)$ : $b_3\, b_2\, P_i\, b_2\, b_3^2$ and $b_3\, b_2\, P_i'\, b_2\, b_3^2$,
- $(3.i)$ : $b_3\, b_2^2\, P_i\, b_2\, b_3^2$ and $b_3\, b_2^2\, P_i'\, b_2\, b_3^2$,
- $(4.i)$ : $b_3^2\, b_2^2\, P_i\, b_2^2\, b_3$ and $b_3^2\, b_2^2\, P_i'\, b_2^2\, b_3$,
- $(5.i)$ : $b_3^2\, b_2\, P_i\, b_2^2\, b_3^2$ and $b_3^2\, b_2\, P_i'\, b_2^2\, b_3^2$,
- $(6.i)$ : $b_3^2\, b_2^2\, P_i\, b_2\, b_3$ and $b_3^2\, b_2^2\, P_i'\, b_2\, b_3$,

where $P_1 = b_1\, b_3$, $P_2 = b_1^2\, b_3$, $P_3 = b_1\, b_3^2$, $P_4 = b_1^2\, b_3^2, P_1' = b_3\, b_1$, $P_2' = b_3\, b_1^2$, $P_3' = b_3^2\, b_1$, $P_4' = b_3^2\, b_1^2$. From now on we denote the corresponding couples by the respective label $(i, j)$. For general $\alpha$, $\beta$ the computation is very long and and we needed a computer program. For more information about the code, see Remark 6.2.

One finds 15 different obstructions from these CPC obstructions, and the following identities among the obstructions: $(5.2) = -\alpha(3.2)$, $(6.2) = \alpha(1.2)$, $(1.4) = -\alpha(1.2)$. Thus, we must consider the couples $(1, 2)$, $(2, 4)$, $(3, 2)$, $(3, 3)$, $(3, 4)$, $(4, 1)$, $(4, 2)$, $(4, 3)$, $(4, 4)$, $(5, 3)$, $(5, 4)$, $(6, 4)$.

The exact form of the obstructions will be given in the next section.

# 5 The existence of Markov traces

## 5.1 Statements

**Theorem 5.1** *There exists an unique Markov trace*

$$\mathcal{T}_{(\alpha,\, \beta)} : K_*(\alpha,\, \beta) \to \frac{\mathbb{Z}[\alpha,\, \beta,\, (4+\alpha\beta)^{-1}]}{(H_{(\alpha,\, \beta)})}$$

*with parameters* $z = (2\alpha - \beta^2)/(\alpha\beta + 4)$ *and* $\bar{z} = -(\alpha^2 + 2\beta)/(\alpha\beta + 4)$*, where*

$H_{(\alpha\,,\beta)} := 8\alpha^6 - 8\alpha^5\beta^2 + 2\alpha^4\beta^4 + 36\alpha^4\beta - 34\alpha^3\beta^3 + 17\alpha^3 + 8\alpha^2\beta^5 + 32\alpha^2\beta^2 - 36\alpha\beta^4 + 38\alpha\beta + 8\beta^6 - 17\beta^3 + 8.$

It is convenient now to put $\delta = z^2(\beta z + 1)$, so that the obstructions below in the second case become Laurent polynomials in $z$ and $\delta$.

**Theorem 5.2** *Set* $\alpha = -(z^7 + \delta^2)/(z^4\delta)$, $\beta = (\delta - z^2)/z^3$ *and* $\bar{z} = -z^2/(\beta z + 1) = -z^4/\delta$. *There exists an unique Markov trace with parameters* $(z, \bar{z})$

$$\mathcal{T}^{(z,\, \delta)} : K_{(\alpha,\, \beta)} \to \frac{\mathbb{Z}[z^{\pm 1},\, \delta^{\pm 1}]}{(P^{(z,\, \delta)})}$$

*where* $P^{(z,\, \delta)} = z^{23} + z^{18}\delta - 2z^{16}\delta^2 - z^{14}\delta^3 - 2z^9\delta^4 + 2z^7\delta^5 + \delta^6 z^5 + \delta^7$.

## 5.2 Proof of Theorem 5.1

The parameters $z, t$ have to satisfy the condition

$$\mathcal{T}(R_0) = \mathcal{T}(R_1) = 0$$

Consider first the simple solutions $z = (2\alpha - \beta^2)/(\alpha\beta + 4)$ and $t = (\alpha^2 + 2\beta)/(\alpha\beta + 4)$. We set $\mathcal{T}_{(\alpha,\, \beta)}$ for the admissible functional associated to these values of the parameters. Notice that in this case $\bar{z} = -t$. Set $u := 1/(\alpha\beta + 4)$, $z_0 := 2\alpha - \beta^2$ and $t_0 := \alpha^2 + 2\beta =: -\bar{z}_0$.

#### 5.2.1 The commutativity obstructions

The equations encountered above for $(5, ii)$ amount to

- $u^2\beta\mathrm{H}_{(\alpha,\beta)} = 0$,
- $-u^2(\alpha\beta + 2)\mathrm{H}_{(\alpha,\beta)} = 0$,
- $u^2(\alpha - \beta^2)\mathrm{H}_{(\alpha,\beta)} = 0$.

#### 5.2.2 CPC obstructions

- (1.2): $-u^3\alpha(\alpha - \beta^2)\mathrm{H}_{(\alpha,\beta)}W$,
- (2.4): $u^3(\alpha - \beta^2)(\alpha^2 + \beta)\mathrm{H}_{(\alpha,\beta)}W$,
- (3.2): $u^3(-\alpha^2\beta^2 + 2 + \alpha\beta + \alpha^3)\mathrm{H}_{(\alpha,\beta)}W$,
- (3.3): $u^3(\alpha\beta + 2)\mathrm{H}_{(\alpha,\beta)}W$,
- (3.4): $u^3\alpha\beta(\alpha - \beta^2)\mathrm{H}_{(\alpha,\beta)}W$,
- (4.1): $-u^3(\alpha - \beta^2)(\alpha^2 + \beta)\mathrm{H}_{(\alpha,\beta)}W$,
- (4.2): $u^3\alpha(\alpha^3 + 2 + 2\alpha\beta - \alpha^2\beta^2 - \beta^3)\mathrm{H}_{(\alpha,\beta)}W$,
- (4.3): $u^3\alpha(\alpha^3 - \alpha^2\beta^2 - 2 - \beta^3)\mathrm{H}_{(\alpha,\beta)}W$,
- (4.4): trivial,
- (5.3): $-u^3(\beta^2 + 2\alpha + 2\alpha^2\beta)\mathrm{H}_{(\alpha,\beta)}W$,
- (5.4): $u^3\alpha(-\alpha^3\beta^2 - \beta^2 - \alpha^2\beta + \alpha^4)\mathrm{H}_{(\alpha,\beta)}W$,
- (6.4): $-u^3\alpha(\beta + 2\alpha^2)(\alpha - \beta^2)\mathrm{H}_{(\alpha,\beta)}W$,

where $W = (\alpha + 2 - \beta)(\alpha^2 - 2\alpha + 4 + \alpha\beta + 2\beta + \beta^2) = \alpha^3 + 8 - \beta^3 + 6\alpha\beta$.

### 5.3 Proof of Theorem 5.2

There are three more solutions of $\mathcal{T}(R_0) = \mathcal{T}(R_1) = 0$, given by $t = \frac{2\alpha z - 2z^2 + \beta}{2 + \beta z}$, where $z$ verifies $(\alpha\beta + 1)z^3 + (\alpha + \beta^2)z^2 + 2\beta z + 1 = 0$. In this case the obstructions are better expressed as rational functions on $z$ and $\beta$.

#### 5.3.1 The commutativity obstructions

- $-ZB_1/(z^7(z\beta + 1)^4) = 0$,
- $-ZB_2/(z^9(z\beta + 1)^5) = 0$,
- $ZB_3/(z^7(z\beta + 1)^5) = 0$.

### 5.3.2 The CPC obstructions

- (1.2): $-ZB_4B_5B_6/(z^{13}(z\beta+1)^8)$,
- (2.4): $-ZB_4B_6B_7/(z^{15}(z\beta+1)^9)$,
- (3.2): $ZB_4B_8/(z^{15}(z\beta+1)^9)$,
- (3.3): $-ZB_4B_9/(z^{11}(z\beta+1)^7)$,
- (3.4): $ZB_4B_5B_6\beta/(z^{13}(z\beta+1)^8)$,
- (4.1): $ZB_4B_6B_7/(z^{15}(z\beta+1)^9)$,
- (4.2): $ZB_4B_5B_{10}/(z^{17}(z\beta+1)^{10})$,
- (4.3): $ZB_4B_5B_{11}/(z^{17}(z\beta+1)^{10})$,
- (4.4): trivial,
- (5.3): $-ZB_4B_{12}/(z^{13}(z\beta+1)^8)$,
- (5.4): $-ZB_4B_5B_{13}/(z^{19}(z\beta+1)^{11})$,
- (6.4): $-ZB_4B_5B_6B_{14}/(z^{17}(z\beta+1)^{10})$,

where $Z, B_1, \ldots, B_{14}$ are the following polynomials in $z, \beta$:

- $Z = 1 + 7z\beta + 21z^2\beta^2 + z^3 + 35z^3\beta^3 + 35z^4\beta^4 + 21z^5\beta^5 + 7z^6\beta^6 + z^7\beta^7 + z^9\beta^6 + 8z^8\beta^5 + 23z^7\beta^4 + 32z^6\beta^3 + 23z^5\beta^2 + 8z^4\beta - 2z^6 + z^9 - z^9\beta^3 - 5z^8\beta^2 - 6z^7\beta$,
- $B_1 = 3z^3 + z^4\beta + 1 + z\beta$,
- $B_2 = 5z^3 + 10z^4\beta + 6z^5\beta^2 + z^6\beta^3 + 4z^6 + 2z^7\beta + 1 + 3z\beta + 3z^2\beta^2 + z^3\beta^3$,
- $B_3 = \beta + 2z\beta^2 + 4z^3\beta + 5z^4\beta^2 + z^5\beta^3 + z^2\beta^3 - 2z^5$,
- $B_4 = (z\beta + z^2\beta + 1 + z - z^2)(z\beta + 1 + 2z^3)(z^4\beta^2 - z^3\beta^2 + z^2\beta^2 + 1 + 2z\beta - z - 2z^2\beta + 2z^2 + 3z^3\beta + z^3 + z^4\beta + z^4)$,
- $B_5 = 1 + z^3 + z^2\beta^2 + 2z\beta$,
- $B_6 = z^3\beta^3 + 1 + 2z\beta + 2z^2\beta^2 + z^3$,
- $B_7 = 1 + 4z\beta + 6z^2\beta^2 + 2z^3 + 4z^3\beta^3 + z^4\beta^4 + z^6\beta^3 + 4z^5\beta^2 + 5z^4\beta + z^6)$,
- $B_8 = z^2\beta^3 + \beta + 2z\beta^2 - 2z^2 - z^3\beta$,
- $B_9 = 1 + 6z\beta + 16z^2\beta^2 + 3z^3 + 25z^3\beta^3 + 25z^4\beta^4 + 16z^5\beta^5 + 6z^6\beta^6 + z^7\beta^7 + 3z^8\beta^5 + 13z^7\beta^4 + 24z^6\beta^3 + 24z^5\beta^2 + 13z^4\beta + z^7\beta + z^6 + z^9$,
- $B_{10} = 1 + 6z\beta + 16z^2\beta^2 + 3z^3 + 25z^3\beta^3 + 25z^4\beta^4 + 16z^5\beta^5 + 6z^6\beta^6 + z^7\beta^7 + z^9\beta^6 + 7z^8\beta^5 + 20z^7\beta^4 + 31z^6\beta^3 + 28z^5\beta^2 + 14z^4\beta + z^6 + z^9 + z^9\beta^3 + 2z^8\beta^2 + 2z^7\beta$,
- $B_{11} = 6z\beta + 16z^2\beta^2 + 3z^3 + 10z^8\beta^2 + 5z^8\beta^5 + z^7\beta^7 + z^9\beta^6 + 12z^7\beta + 12z^7\beta^4 + 19z^6\beta^3 + 20z^5\beta^2 + 12z^4\beta + 6z^6\beta^6 + 3z^9\beta^3 + 5z^6 + z^9 + 1 + 25z^3\beta^3 + 25z^4\beta^4 + 16z^5\beta^5$,

- $B_{12} = 2\beta + 4z^5\beta^3 - 2z^5 + 2z^4\beta^5 + 8z\beta^2 + 12z^2\beta^3 - 2z^2 + 8z^3\beta^4 + 3z^4\beta^2 - 2z^3\beta + z^6\beta^4$,
- $B_{13} = 1 + 8z\beta + 29z^2\beta^2 + 63z^3\beta^3 + 80z^6\beta^3 + 29z^7\beta^7 + 13z^9\beta^6 + 17z^9\beta^3 + 91z^4\beta^4 + 57z^5\beta^2 + 23z^4\beta + 4z^3 + 6z^6 + 4z^9 + 91z^5\beta^5 + 63z^6\beta^6 + 39z^8\beta^5 + 70z^7\beta^4 + 30z^8\beta^2 + 22z^7\beta + z^{12} + z^9\beta^9 - z^{12}\beta^6 + z^{10}\beta^4 + 2z^{10}\beta^7 + 8z^8\beta^8 - 3z^{11}\beta^5 + 3z^{11}\beta^2 + 7z^{10}\beta$,
- $B_{14} = 2 + 8z\beta + 12z^2\beta^2 + 4z^3 + 8z^3\beta^3 + 2z^4\beta^4 + z^6\beta^3 + 6z^5\beta^2 + 9z^4\beta + 2z^6$.

Notice that $Z(z,\beta) = P^{(z,\delta)}(z,\delta)$.

### 5.3.3 Corollaries

**Corollary 5.1** • *There exists an unique Markov trace*

$$\mathcal{T} : K_*(0,\, 2\lambda) \to \frac{\mathbb{Z}[\lambda]}{(8\lambda^6 - 17\lambda^3 + 1)},$$

*with parameters* $z = -\lambda^2$, $t = \lambda$ *and* $\bar{z} = -\lambda$,

- *respectively*

$$\mathcal{T} : K_*(-2\lambda,\, 0) \to \frac{\mathbb{Z}[\lambda]}{(8\lambda^6 - 17\lambda^3 + 1)},$$

*with parameters* $z = -\lambda$, $t = \lambda^2$ *and* $\bar{z} = -\lambda^2$.

We have a similar situation for the other three solutions. In fact for $\alpha = 0$, we derive $z = -(t-\beta)^2$, where $t$ satisfies $(t^3 - 4\beta t^2 + 5\beta^2 t + 1 - 2\beta^3) = 0$. In particular $\bar{z}^3 - \beta\bar{z}^2 + 1 = 0$ because $\bar{z} = t - \beta$.

**Corollary 5.2** • *There exists an unique Markov trace*

$$\mathcal{T} : K_*(0,\, \frac{\lambda^3+1}{\lambda^2}) \to \frac{\mathbb{Z}[\lambda^{\pm 1}]}{(\lambda^9 - 2\lambda^6 + \lambda^3 + 1)},$$

*with parameters* $z = -\lambda^2$, $\bar{z} = \lambda$ *and* $t = \dfrac{2\lambda^3+1}{\lambda^2}$,

- *and respectively*

$$\mathcal{T} : K_*(-\frac{\lambda^3+1}{\lambda^2},\, 0) \to \frac{\mathbb{Z}[\lambda^{\pm 1}]}{(\lambda^9 - 2\lambda^6 + \lambda^3 + 1)},$$

*with parameters* $z = \lambda$, $\bar{z} = -\lambda^2$ *and* $t = -\dfrac{2\lambda^3+1}{\lambda^2}$.

# 6 The invariants

## 6.1 The definition of $I_{(\alpha,\beta)}$

As in section 5.2 we set $z = (2\alpha - \beta^2)/(\alpha\beta + 4)$, $t = (\alpha^2 + 2\beta)/(\alpha\beta + 4)$, $u := 1/(\alpha\beta + 4)$, $z_0 := 2\alpha - \beta^2$ and $t_0 := \alpha^2 + 2\beta =: -\bar{z}_0$ (notice that in this case $\bar{z} = -t$).

**Definition 6.1** *Let us set for an oriented link L*

$$I_{(\alpha,\beta)}(L) = \left(\frac{1}{z\bar{z}}\right)^{\frac{n-1}{2}} \left(\frac{\bar{z}}{z}\right)^{\frac{e(x)}{2}} \mathcal{T}_{(\alpha,\beta)}(x) \in \frac{\mathbb{Z}[\alpha,\, \beta,\, z_0^{\pm\epsilon/2}, \bar{z}_0^{\pm\epsilon/2}]}{(H_{(\alpha,\beta)})},$$

*where $x \in B_n$ is any braid whose closure is isotopic to L. Here $\epsilon - 1$ is the number of components mod 2.*

**Lemma 6.1** *$I_{(\alpha,\beta)}$ is well-defined.*

*Proof:* Since $b_j^{-1} = b_j^2 - \alpha b_j - \beta$, we can suppose that $x$ is a positive braid. All the elements in $\Gamma_0(H)$ associated to $x$ are polynomials in the variables $z$, $t$ of degree at most $n-1$. The substitutions $z = uz_0$ and $t = ut_0$ imply that, if $\mathcal{T}_{(\alpha,\beta)}(x)$ and $\mathcal{T}_{(\alpha,\beta)}(x)'$ are representatives of the trace of $x$, then $\mathcal{T}_{(\alpha,\beta)}(x)' - \mathcal{T}_{(\alpha,\beta)}(x) = u^{n-1}G(\alpha,\,\beta)\mathrm{H}_{(\alpha,\beta)}$, where $G(\alpha,\,\beta)$ is a polynomial in $\alpha$, $\beta$. It follows

$$I_{(\alpha,\beta)}(L) = \left(\frac{1}{z_0\bar{z}_0}\right)^{\frac{n-1}{2}} \left(\frac{\bar{z}_0}{z_0}\right)^{\frac{e(x)}{2}} \widetilde{\mathcal{T}}_{(\alpha,\beta)}(x),$$

where

$$\widetilde{\mathcal{T}}_{(\alpha,\beta)}(x) := u^{-n+1}\mathcal{T}_{(\alpha,\beta)}(x) \in \frac{\mathbb{Z}[\alpha,\, \beta]}{(\mathrm{H}_{(\alpha,\beta)})}.$$

□

## 6.2 The cubical behaviour

**Proposition 6.1** *For any link K there exists some $l \in \{0,\, 1,\, 2\}$ such that*

$$I_{(\alpha,\beta)}(K) = \frac{\sum_{k\in\mathbb{N}} P_k(\beta)\alpha^k}{\sum_{k\in\mathbb{N}} Q_k(\beta)\alpha^k} = \frac{\sum_{k\in\mathbb{N}} M_k(\alpha)\beta^k}{\sum_{k\in\mathbb{N}} N_k(\alpha)\beta^k}$$

*where $P_k, Q_k, M_k, N_k$ are $(3, k+l)$-polynomials.*

*Proof:* We will show that $M_k, N_k$ are $(3, k+l)$-polynomials. The other case is analogous. Suppose first that $x \in B_n^+$, where $B_n^+$ is the set of positive braids and $n$ is such that $x \notin B_{n-1}^+$. Then $e(x) = |x|$ where $|x|$ means the length of $x$. In the process computing the value of the trace on the word $x$ we make two types of reductions: either one uses the relations in some $K_n(\alpha,\, \beta)$, or else one replaces $ab_lb$ by $zab$ (respectively $ab_l^2b$ by $tab$), where $a, b$ are subwords, and this way one lowers the rank $n$. Using the relations the word $y$ is replaced by $\sum_s(\sum_{k\in\mathbb{N}} D_{k,s}(\alpha)\beta^k)y_s$ where the $y_s$ are a finite number of words in $B_n$ and the coefficients $D_{k,s}(\alpha)$ are $(3, k+e(x)-l_s)$-polynomials where $l_s = |y_s|$. In the second case the word $w$ is replaced by $zw' + tw''$ where $|w'| = |w| - 1$ and $|w''| = |w| - 2$. When we introduce the $z$ and $t$ as functions on $\alpha$ and $\beta$ one finds that

$$\mathcal{T}_{(\alpha,\beta)}(x) = \sum_{k\in\mathbb{N}} u^{s_k} V_k(\alpha)\beta^k,$$

where $0 \le s_k \le n-1$ and $V_k(\alpha)$ are $(3, k+e(x))$-polynomials. In particular

$$\widetilde{\mathcal{T}}_{(\alpha,\beta)}(x) = \sum_{k\in\mathbb{N}} u^{s_k-n+1} V_k(\alpha)\beta^k.$$

Now $u^{s_k-n+1} = \sum_{k\in\mathbb{N}} Y_k(\alpha)\beta^k$ where $Y_k(\alpha)$ are $(3,k)$-polynomials. Thus it follows

$$\widetilde{\mathcal{T}}_{(\alpha,\beta)}(x) = \sum_{k\in\mathbb{N}} L_k(\alpha)\beta^k,$$

where $L_k(\alpha)$ are $(3, k+e(x))$-polynomials.

The same is true for non necessarily positive $x \in B_n$, by getting rid of the negative exponents (using the cubic relation).

Taking into account the normalization factor in front of the trace we obtain the claim. □

**Corollary 6.1** *$I_{(\alpha,0)}(K) = \sum_{i\in\mathbb{N}} a_{3i}\alpha^{3i}$ and, respectively, $I_{(0,\beta)}(K) = \sum_{i\in\mathbb{Z}} b_{3i}\beta^{3i}$, where $a_{3i}$, $b_{3i} \in \mathbb{Z}[\frac{1}{2}]$.*

### 6.3 Chirality and other properties of $I_{(\alpha,\beta)}$

**Lemma 6.2** *Set $x^* \in B_n$ for the word one obtains from $x$ when each $b_j^\epsilon$ is replaced by $b_j^{-\epsilon}$. Then $\mathcal{T}_{(\alpha,\beta)}(x) = \mathcal{T}_{(-\beta,-\alpha)}(x^*)$ holds true. Consequently for amphicheiral $K$, $I_{(\alpha,\beta)}(K) = I_{(-\beta,-\alpha)}(K)$ is fulfilled.*

*Proof:* Let $Q(b_j)^*$ (respectively $R_0^*$) denotes the image of $Q(b_j)$ (respectively $R_0$) after the substitutions $\alpha \to -\beta$, $\beta \to -\alpha$ and $b_l \to b_l^{-1}$ for $l = 1,\ldots,n-1$. It is easy to check that $Q(b_j)^* = b_j^{-3}Q(b_j) = 0$. Using a computer we verified that $R_0^* = R_1 = 0$. Since $\mathrm{H}_{(\alpha,\beta)} = \mathrm{H}_{(-\beta,-\alpha)}$ we are done. □

The following properties have been checked with a computer program:

- $I_{(\alpha,\beta)}$ is independent from HOMFLY and in particular it distinguishes knots that have the same HOMFLY polynomial. The knots 5.1 and 10.132 have the same HOMFLY polynomial but different $I_{(\alpha,0)}$ and $I_{(0,\beta)}$ invariants. This holds true for the other three couples of prime knots with number crossing $\leq 10$ that HOMFLY fails to distinguish, i.e. (8.8, 10.129), (8.16, 10.156), (10.25, 10.56).
- $I_{(\alpha,\beta)}$ detects the chirality of those knots with crossing number at most 10, where HOMFLY fails i.e. the knots 9.42, 10.48, 10.71, 10.91, 10.104 and 10.125).
- $I_{(\alpha,\beta)}$ does not distinguish a well-known pair of mutant knots, the Conway knot ($C$) and the Kinoshita-Terasaka knot ($KT$).

### 6.4 The definition of $I^{(z,\delta)}$

**Definition 6.2** *Let us set for an oriented link L*

$$I^{(z,\delta)}(L) = \left(\frac{1}{z\bar{z}}\right)^{\frac{n-1}{2}} \left(\frac{\bar{z}}{z}\right)^{\frac{e(x)}{2}} \mathcal{T}^{(z,\delta)}(x) \in \frac{\mathbb{Z}[z^{\pm\epsilon/2}, \delta^{\pm\epsilon/2}]}{(P^{(z,\delta)})},$$

*where $x \in B_n$ is any braid whose closure is isotopic to $L$ and $\alpha$, $\beta$, $t$, $\bar{z}$ as in Theorem 5.2. Here $\epsilon - 1$ is the number of components mod 2, $\epsilon \in \{1,2\}$.*

**Remark 6.1** *This invariant doesn't detect the amphicheirality of knots. Also $I^{(z,\delta)}$ does not distinguish the Conway knot and the Kinoshita-Terasaka knot.*

**Proposition 6.2**

$$I^{(z,\delta)}(K) = \sum_{k\in\mathbb{Z}} H_k(\delta) z^k = \sum_{k\in\mathbb{Z}} G_k(z)\delta^k,$$

*where $H_k, G_k$ are $(3,k)$-Laurent polynomials.*

*Proof:* The proof is analogous to the proof of Proposition 6.1. □

**Remark 6.2** *For evaluating obstructions and traces of braids we used a Delphi code. The input is a word (or a linear combinations) and we restricted to words representing 5-braids. One transforms first the word to a sum of positive words, by using the cubic relations. Furthermore the transformations $C_i$ and $Cij$ are used in order to reduce the shape of the word as much as possible. When it gets stalked one allows permutations of the letters. The final result is the value of the trace on the braid element. The program is available on* www-fourier.ujf-grenoble.fr/~bellinge.html.

# 7 Appendix

The values of the polynomials for $I_{(\alpha,0)}(K)$ and $I_{(0,\beta)}(K)$ are displayed below for all knots with no more than 8 crossings. The second column is a braid representative for the knot. A bold entry in the table is the coefficient of $\alpha^0$ (respectively $\beta^0$). The other entries are the non zero coefficients of $\alpha^{3k}$ and $\beta^{3k}$ respectively, for $k \in \mathbb{Z}$. For example,

$$I_\alpha(6.2) = [-5 - \frac{19}{4}\alpha^3 - \frac{1}{2}\alpha^6]; \;\; I_\beta(6.2) = [-16\beta^{-3} + 19 - 2\beta^3].$$

The entry "A" in the last column means that the knot is amphicheiral.

| | | | | |
|---|---|---|---|---|
| 3.1 | $b_1^3$ | **-1** $-1/4$ | $-8$ **2** | |
| 4.1 | $b_1b_2^{-1}b_1b_2^{-1}$ | $8$ **10** $1$ | $-8$ **10** $-1$ | $A$ |
| 5.1 | $b_1^5$ | **0** $7/8$ $1/8$ | $-24$ **4** | |
| 5.2 | $b_1^2b_2^2b_1^{-1}b_2$ | **2** $17/8$ $1/4$ | $-8$ **2** | |
| 6.1 | $b_1^{-1}b_2b_1^{-1}b_3b_2^{-1}b_3b_2$ | $-8$ **-16** $-10$ $-1$ | **1** | |
| 6.2 | $b_1^{-1}b_2b_1^{-1}b_2^3$ | **-5** $-19/4$ $-1/2$ | $-16$ **19** $-2$ | |
| 6.3 | $b_1^{-1}b_2^2b_1^{-2}b_2$ | **-3** $-1/2$ | **-3** $1/2$ | $A$ |
| 7.1 | $b_1^7$ | **0** $-5/8$ $-9/16$ $-1/16$ | $-56$ **8** | |
| 7.2 | $b_1^{-1}b_3^3b_2b_1^2b_3^{-1}b_2$ | **-3** $-11/2$ $-21/8$ $-1/4$ | $-64$ $-64$ **-6** | |
| 7.3 | $b_1^2b_2b_1^{-1}b_2^4$ | **-1** $-7/4$ $-19/16$ $-1/8$ | $-64$ $48$ **-4** | |
| 7.4 | $b_1^2b_2b_3^2b_1^{-1}b_2b_3^{-1}b_2$ | **0** $-17/8$ $-9/4$ $-1/4$ | $-64$ $+128$ **-78** $8$ | |
| 7.5 | $b_1^4b_2b_1^{-1}b_2^2$ | **0** $-9/8$ $-9/8$ $-1/8$ | $-24$ **4** | |
| 7.6 | $b_1b_2^{-1}b_1^{-2}b_3b_2^3b_3$ | **-4** $-37/8$ $-1/2$ | $-24$ **20** $-2$ | |
| 7.7 | $b_1b_3^{-1}b_2b_3^{-1}b_2b_1^{-1}b_2b_3^{-1}b_2$ | $-8$ **-20** $-21/2$ $-1$ | **-19** $37/2$ $-2$ | |
| 8.1 | $b_1^{-1}b_2b_3b_2^{-1}b_1^{-1}b_4^2b_3b_2b_4^{-1}$ | $16$ **43** $37$ $12$ $1$ | $-64$ $144$ **-88** $9$ | |
| 8.2 | $b_1^{-1}b_2^5b_1^{-1}b_2$ | **4** $59/8$ $23/8$ $1/4$ | $-24$ **36** $-4$ | |
| 8.3 | $b_1^{-2}b_2^{-1}b_1b_4^2b_3b_4^{-1}b_2^{-1}b_3$ | $-8$ **-8** $-1$ | $8$ **-8** $1$ | $A$ |
| 8.4 | $b_1^3b_3b_2^{-1}b_3^{-2}b_1b_2^{-1}$ | $8$ **8** $3/4$ | $8$ **-24** $19$ $-2$ | |
| 8.5 | $b_1^3b_2^{-1}b_1^3b_2^{-1}$ | **1** $3$ $19/8$ $1/4$ | $-24$ **36** $-4$ | |
| 8.6 | $b_1^{-1}b_2b_1^{-1}b_3^{-1}b_2^3b_3^2$ | **5** $21/2$ $21/4$ $1/2$ | **1** | |
| 8.7 | $b_1^4b_2^{-2}b_1b_2^{-1}$ | **3** $9/4$ $1/4$ | $16$ **-25** $3$ | |
| 8.8 | $b_1^{-1}b_2b_1^2b_3^{-1}b_2^2b_3^{-2}$ | **3** $17/4$ $1/2$ | $16$ **-21** $5/2$ | |
| 8.9 | $b_1^{-1}b_2b_1^{-3}b_2^3$ | **-7** $-9$ $-1$ | **-7** $9$ $-1$ | $A$ |
| 8.10 | $b_1^{-1}b_2^2b_1^{-2}b_2^3$ | **1** $2$ $1/4$ | $8$ **-8** $1$ | |
| 8.11 | $b_1^{-1}b_2^2b_3^{-1}b_2b_3^2b_1^{-1}b_2$ | $8$ **21** $147/8$ $6$ $1/2$ | $-64$ $136$ **-79** $8$ | |
| 8.12 | $b_1b_2^{-1}b_3b_4^{-1}b_3b_4^{-1}b_2b_1b_3^{-1}b_2^{-1}$ | $24$ **44** $21$ $2$ | $-24$ **44** $-21$ $2$ | $A$ |
| 8.13 | $b_1^2b_2b_3^{-1}b_2b_1^{-1}b_3^{-2}b_2$ | $8$ **12** $21/4$ $-1/2$ | $8$ **-28** $39/2$ $-2$ | |
| 8.14 | $b_1^2b_2^2b_1^{-1}b_3^{-1}b_2b_3^{-1}b_2$ | **6** $85/8$ $21/4$ $1/2$ | $-8$ **18** $-2$ | |
| 8.15 | $b_1^2b_2^{-1}b_1b_3^2b_2^2b_3$ | **0** $-17/8$ $-9/4$ $-1/4$ | $64$ $-32$ **4** | |
| 8.16 | $b_1^2b_2^{-1}b_1^2b_2^{-1}b_1b_2^{-1}$ | **-3** $3/2$ $1/4$ | **-7** $1$ | |
| 8.17 | $b_1^{-1}b_2b_1^{-1}b_2^2b_1^{-2}b_2$ | **-11** $-19/2$ $-1$ | **-11** $19/2$ $-1$ | $A$ |
| 8.18 | $b_1b_2^{-1}b_1b_2^{-1}b_1b_2^{-1}b_1b_2^{-1}$ | $-8$ **-16** $-10$ $-1$ | $8$ **-16** $10$ $-1$ | $A$ |
| 8.19 | $b_1b_2b_1b_2b_1b_2^2b_1$ | **0** $3/8$ $1/16$ | $64$ $-64$ **1** | |
| 8.20 | $b_1^3b_2b_1^{-3}b_2$ | **5** $9/2$ $1/2$ | $-8$ **0** | |
| 8.21 | $b_1b_2^{-2}b_1^2b_2^3$ | **1** $-1$ $-1/8$ | $8$ **0** | |

Table 2

## References


[1] A.-B. Berger and I.Stassen: Skein relations for the link invariants coming from exceptional Lie algebras, *J. Knot Theory Ramifications*, 8(1999), 835–853.

[2] A.-B. Berger and I.Stassen: The skein relation for the $(g_2, V)$-link invariant, *Commentarii Math. Helvetici*, 75(2000), 134-155.

[3] G.M. Bergmann: The diamond lemma for ring theory, *Advances in Math.*, 29:178–218, 1978.

[4] D. Bessis: Groupes de tresses et elements reguliers, *J.Reine Angew. Math.*, 518(2000), 1-40.

[5] J. Birman and H. Wenzl: Braids, link polynomials and a new algebra, Trans.A.M.S. 313(1989), 249-273.

[6] M. Broué, G. Malle and J.Michel: Towards spetses I, *Transformation Groups*, 4(1999), 157-218.

[7] N. Bourbaki: *Groupes et algebres de Lie IV, V, VI*, Masson, Paris, 1982.

[8] H.S.M. Coxeter: Factors of braid groups, *Proc. 4-th Canadian Math. Congress, Banff*, 1957, Univ. Toronto Prss 1959, 95-122.

[9] L. Funar: On cubic Hecke algebras. *Commun. Math. Phys.*, 173(1995), 513-558.

[10] R. Kirby: *Problems in low dimensional topology* in "Geometric Topology", Georgia International Topology Conference (W.H.Kazez, Editor) AMS-IP Studies in Advanced Mathematics,**2**, part 2, 35-473, 1995.

[11] G. Kuperberg: The quantum $G_2$ link invariant, *International Journal of Math.*, 5(1994), 61-85.

[12] G. Kuperberg: Spiders for rank 2 Lie algebras, *Commun. Math. Phys.*, 180(1996), 109-151.

[13] S. Lambroupoulou: Knot theory related to generalized and cyclotomic Hecke algebras of type $\mathcal{B}$, *J. Knot Theory Ramifications* 8(1999), 621-658.

[14] T.T.Q. Le and J. Murakami: Kontsevich's integral for the Kauffman polynomial, *Nagoya Math. J.* 142 (1996), 39–65.

[15] T.T.Q. Le and J. Murakami: Kontsevich's integral for the Homfly polynomial and relations between values of multiple zeta functions, *Topology Appl.* 62 (1995), 193-206.

[16] V. Jones: Hecke algebras and a polynomial invariant for knots, *Ann.of Math.*, 126:335–388, 1987.

[17] V. Turaev: The Yang-Baxter equation and invariants of links, *Inventiones Math.* 92(1988), 527-553.