STOCHASTIC INTEGRAL REPRESENTATION OF UNBOUNDED OPERATORS IN FOCK SPACES

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Abstract. — The problem of quantum stochastic integral representation of operators in Fock spaces has been studied mainly by Parthasarathy and Sinha [P-S] and Attal [A1]. They obtained results concerning processes of bounded operators. In this article we extend their results to processes of unbounded operators on the Fock space. We apply these conditions to the characterization of quantum noises and to the characterization of contractive cocycles.

0. Introduction and notations

For any complex separable Hilbert space h, we denote by $\Gamma(h)$ the boson Fock space over h. We write $\Phi = \Gamma(L^2(\mathbf{R}_+))$. We denote for f in $L^2(\mathbf{R}_+)$ by $e(f)$ the associated coherent or exponential vector in Φ; the exponential domain is denoted $\mathcal{E}$ (see [M3] for more details). Recall that $e(0)$ is the vacuum vector in Φ. We denote $\Phi_{t]} = \Gamma\big(L^2([0,t])\big)$, $\Phi_{[s,t]} = \Gamma\big(L^2([s,t])\big)$ and $\Phi_{[t} = \Gamma\big(L^2([t,+\infty[)\big)$. Let h_0 be a separable Hilbert space and

$$\mathcal{H} = h_0 \otimes \Phi, \ \mathcal{H}_{t]} = h_0 \otimes \Phi_{t]} .$$

We have the well known "continuous tensor product" structure

$$\mathcal{H} \simeq \mathcal{H}_{t]} \otimes \Phi_{[t} .$$

The annihilation, creation and conservation operators are defined on the domain $\mathcal{E}$ by the relations

$$\begin{aligned} a^-(f)e(g) &= \langle f, g\rangle e(g) \\ a^+(f)e(g) &= \frac{d}{d\lambda} e(g+\lambda f)\big|_{\lambda=0} \\ \lambda(T)e(g) &= \frac{d}{d\lambda} e(e^{\lambda T} g)\big|_{\lambda=0} \qquad , \end{aligned}$$

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the derivations being understood in the strong sense, for f, g in $L^2(\mathbf{R}_+)$ and $T \in \mathcal{B}(L^2(\mathbf{R}_+))$, the algebra of all bounded operators on $L^2(\mathbf{R}_+)$. The operators $a^-(f)$ and $a^+(f)$ are adjoint to each others on $\mathcal{E}$. If T^* is the adjoint of T then $\lambda(T^*)$ and $\lambda(T)$ are adjoint to each other on $\mathcal{E}$. If $f = 1_{[0,t]}$ and if T is the operator of multiplication by f, then $a^-(f)$, $a^+(f)$ and $\lambda(T)$ are respectively denoted by a_t^-, a_t^- and a_t^0. We put $a_t^\times = tI$.

Let $D_0 \subset h_0$ a dense linear manifold. We put $\widetilde{\mathcal{E}} = D_0 \underline{\otimes} \mathcal{E}$, the algebraic tensor product between D_0 and $\mathcal{E}$. A family of operators $(X_t)_{t\geq 0}$ defined on $D_0\underline{\otimes}\mathcal{E}$ is called an *adapted process with respect to* D_0 if the following conditions are fulfilled:

i) for all $t > 0$, $X_t\big(u \otimes e(f1_{[0,t]})\big) \in \mathcal{H}_{t]}$ and
$X_t(u \otimes e(f)) = X_t\big(u \otimes e(f1_{[0,t]})\big) \otimes e(f1_{[t,+\infty[})$ in $\mathcal{H}_{t]} \otimes \Phi_{[t}$;

ii) for all $u \in D_0$, $f \in L^2(\mathbf{R}_+)$, the map $\mathbf{R}_+ \to \mathcal{H} : t \mapsto X_t(u \otimes e(f))$ is strongly measurable.

We denote again a_t^-, a_t^+, a_t^0 and $a_t^\times$ the ampliation of the annihilation, creation, number and time process to $\mathcal{H}$. Let us now recall some elements of the Hudson-Parthasarathy's quantum stochastic calculus ([H-P1]). Let $(H_t^\varepsilon)_{t\geq 0}$, $\varepsilon \in \{-,+,\times,0\}$ be adapted processes with respect to D_0 such that for all $u \in D_0$, $f \in L^2(\mathbf{R}_+)$ and for all $t > 0$

$$\int_0^t \Big\{|f(s)|^2\|H_s^0(u\otimes e(f))\|^2 + \|H_s^+(u\otimes e(f))\|^2 + \|H_s^\times(u\otimes e(f))\| + |f(s)|\,\|H_s^-(u\otimes e(f))\|\Big\}\, ds < +\infty\,. \tag{0.1}$$

Then the stochastic integral $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon$ is defined as the unique adapted process with respect to D_0 satisfying the relation:

$$\langle u\otimes e(f), T_t(v\otimes e(g))\rangle = \int_0^t \langle u\otimes e(f), \big\{H_s^\times(v\otimes e(g)) + g(s)H_s^-(v\otimes e(g)) + \overline{f(s)}H_s^+(v\otimes e(g)) + \overline{f(s)}g(s)H_s^0(u\otimes e(g))\big\}\rangle ds\,. \tag{0.2}$$

Let Π_t be, for $t \geq 0$, the orthogonal projection onto $\mathcal{H}_{t]}$. If $H^\times = 0$, then we have for $s < t$, $\Pi_s T_t \Pi_s = T_s \Pi_s$ and $(T_t)_{t\geq 0}$ is a martingale. Otherwise we say that $(T_t)_{t\geq 0}$ is a semi-martingale. Now recall the extension of Hudson-Parthasarathy's quantum stochastic calculus due to Attal-Meyer ([A-M]).

One defines $(D_t)_{t\geq 0}$ on $\mathcal{H}$ in the following way: if $u \in h_0$ and $f \in L^2(\mathbf{R}_+)$, $D_t(u \otimes e(f)) = f(t)u \otimes e(f1_{[0,t]})$. Thus we have that for $(u_i)_{1\leq i\leq n}$ in h_0, $(f_i)_{1\leq i\leq n}$ in $L^2(\mathbf{R}_+)$,

$$\Big\|\sum_{i=1}^n u_i \otimes e(f_i)\Big\|^2 = \Big\|\sum_{i=1}^n u_i\Big\|^2 + \int_0^{+\infty} \Big\|D_t\Big(\sum_{i=1}^n u_i\otimes e(f_i)\Big)\Big\|^2 dt$$

so that $(D_t)_{t\geq 0}$ defines a bounded process from $\mathcal{H}$ to $L^2(\mathbf{R}_+, \mathcal{H})$. In fact D_t is the ampliation to $h_0 \otimes \Phi$ of the D_t defined on Φ by Attal in [A2].

Let $(X_t)_{t\geq 0}$ the curve in Φ defined by the following relation: for all f in $L^2(\mathbf{R}_+)$, $\langle e(f), X_t\rangle = \int_0^t \bar{f}(s)\, ds$. If $(y_t)_{t\geq 0}$ is a curve in $\mathcal{H}$ such that $y_t \in \mathcal{H}_{t]}$ for all $t \geq 0$ and $\int_0^{+\infty} \|y_s\|^2\, ds < +\infty$, we can define $\int_0^{+\infty} y_s\, d\chi_s$ by the following relation: for all $u \in h_0$, for all $f \in L^2(\mathbf{R}_+)$

$$\langle u \otimes e(f), \int_0^{+\infty} y_s\, d\chi_s\rangle = \int_0^{+\infty} \overline{f(s)} \langle u \otimes e(f), y_s\rangle\, ds\,.$$

By the same way, we define $\int_0^t y_s\, d\chi_s$ and we have $\Pi_t\big(\int_0^{+\infty} y_s\, d\chi_s\big) = \int_0^t y_s\, d\chi_s$ and

$$u \otimes e(f) = u \otimes e(0) + \int_0^{+\infty} f(s) u \otimes e(f 1_{[0,s]})\, d\chi_s\,. \tag{0.3}$$

So for all F in $\mathcal{H}$, we have $F = \Pi_0(F) + \int_0^{+\infty} D_t F\, d\chi_t$. With these definitions and properties, Attal and Meyer in [A-M] prove that if $T_t = \sum\limits_{\varepsilon} \int_0^t H_s^\varepsilon\, da_s^\varepsilon$ with $(H_t^\varepsilon)_{t\geq 0}$, $\varepsilon \in \{-,+,\times,0\}$ satisfying (0.1), then for all $F \in D_0 \underline{\otimes}\, \mathcal{E}$, we have

$$\begin{aligned} T_t \Pi_t F = \int_0^t T_s D_s F\, d\chi_s + \int_0^t H_s^0 D_s F\, d\chi_s + \int_0^t H_s^+ \Pi_s F\, d\chi_s \\ + \int_0^t H_s^- D_s F\, ds + \int_0^t H_s^\times \Pi_s F\, ds\,. \end{aligned} \tag{0.4}$$

Conversly if $(H_t^\varepsilon)_{t\geq 0}$, $\varepsilon \in \{-,+,0,\times\}$ are adapted operators such that for all $F \in D_0 \underline{\otimes}\, \mathcal{E}$, for all $t \geq 0$,

$$\int_0^t \|H_s^0 D_s F\|^2\, ds < +\infty, \quad \int_0^t \|H_s^- D_s F\|\, ds < +\infty\,,$$
$$\int_0^t \|H_s^+ \Pi_s F\|^2\, ds < +\infty, \quad \int_0^t \|H_s^\times \Pi_s F\|\, ds < +\infty\,,$$

and if $(T_t)_{t\geq 0}$ is a process satisfying (0.4) and such that $\int_0^t \|T_s D_s F\|^2\, ds < +\infty$, then $T_t = \sum\limits_{\varepsilon} \int_0^t H_s^\varepsilon\, da_s^\varepsilon$. In fact, Attal and Meyer used equation (0.4) for processes defined on domains which can be different of $\widetilde{\mathcal{E}}$. A good space of operator processes can be defined by looking at processes of bounded operators satisfying for all $t > 0$

$$\int_0^t \|H_s^\times\|\, ds < +\infty, \quad \int_0^t \|H_s^\varepsilon\|^2\, ds < +\infty \text{ for } \varepsilon = +,- \text{ and } \sup_{0\leq s\leq t} \|H_s^0\| < +\infty\,. \tag{0.6}$$

Clearly these operators satisfy (0.1) and thus $T_t = \sum\limits_{\varepsilon} \int_0^t H_s^\varepsilon\, da_s^\varepsilon$ is well defined on $\mathcal{H}$. S. Attal studies this space of operators in [A1] and denotes it $\mathcal{S}'$. If we add the condition that for all $t \geq 0$, T_t is bounded, then we obtain the space $\mathcal{S}$ of processes of [A1]. In fact, we see for example that a_t^ε, for $\varepsilon = +,-,0$, belongs to $\mathcal{S}'$ but not to $\mathcal{S}$. It is easy to see that if $T_t = \sum\limits_{\varepsilon} \int_0^t H_s^\varepsilon\, da_s^\varepsilon$ with $(H_t^\varepsilon)_{t\geq 0}$, $\varepsilon \in \{-,0,+,\times\}$ satisfying (0.6) then we have for all $t > 0$, for all F in $\mathcal{H}$, for all $0 < r < s < t$,

$$\begin{cases} \|T_t\Pi_rF - T_s\Pi_rF\| \leq \|\Pi_rF\|\Big\{\big(\int_s^t \|H_\tau^+\|^2 d\tau\big)^{1/2} + \int_s^t \|H_\tau^\times\| d\tau\Big\} \\ \|T_t^*\Pi_rF - T_s^*\Pi_rF\| \leq \|\Pi_rF\|\Big\{\big(\int_s^t \|H_\tau^-\|^2 d\tau\big)^{1/2} + \int_s^t \|H_\tau^\times\| d\tau\Big\} \\ \|\Pi_sT_t\Pi_rF - T_s\Pi_rF\| \leq \|\Pi_rF\| \int_s^t \|H_\tau^\times\|\, d\tau\,. \end{cases}$$

So there exists φ in $L^1_{\text{loc}}(\mathbf{R}_+)$ such that, for all $r < s < t$, for all F in $\mathcal{H}$

$$(0.7) \qquad \begin{cases} (i) & \|T_t\Pi_rF - T_s\Pi_rF\|^2 \leq \|\Pi_rF\|^2 \int_s^t \varphi(\tau)\, d\tau \\ (ii) & \|T_t^*\Pi_rF - T_s^*\Pi_rF\|^2 \leq \|\Pi_rF\|^2 \int_s^t \varphi(\tau)\, d\tau \\ (iii) & \|\Pi_sT_t\Pi_rF - T_s\Pi_rF\| \leq \|\Pi_rF\| \int_s^t \varphi(\tau)\, d\tau\,. \end{cases}$$

Parthasarathy and Sinha in [P-S] prove that a bounded martingale satisfying (0.7) *(i)* and *(ii)* (they called it a *regular martingale*) belongs to $\mathcal{S}$. Attal in [A1] proved that adapted processes of bounded operators satisfying (0.7), *(i), (ii)* and *(iii)* belong to $\mathcal{S}$ and even more: *(i), (ii)* and *(iii)* characterize $\mathcal{S}$. Elements of $\mathcal{S}$ are called *regular semi-martingales.*

In this paper, we extend these results about stochastic integral representation of process of operators to the case of unbounded operators.

In the first part we are interested to an another domain $\widetilde{\mathcal{J}}$ in $\mathcal{H}$ larger than $\widetilde{\mathcal{E}}$ and we will show that a large class of stochastic integral are defined on this domain.

In the second part, we prove a result about stochastic integral representation of quasimartingales satisfying some regularity conditions and defined on $\widetilde{\mathcal{J}}$.

In the third part, we will show some consequences of this theorem.

In the fourth part, we apply the theorem to two situations. First we prove that a closable "noise" (see [C]) defined on $\mathcal{J}$ is equal to the sum of creation, annihilation and number processes. Secondly we prove that under a condition of weak differentiability ([A-J-L]), an adapted contractive cocycle $(V_t)_{t\geq 0}$ is the solution of a quantum stochastic differential equation of the form, $V_t = I + \sum\limits_{\varepsilon} \int_0^t V_sL_\varepsilon\, da_s^\varepsilon$ where $(L_\varepsilon)_{\varepsilon\in\{-,+,0,\times\}}$ are operators on h_0.

1. A new domain for quantum stochastic integral

Let $\mathcal{J}$ be the linear manifold generated by $e(0)$ and the vectors of the form $\int_0^{+\infty} g(s)e(f\,1_{[0,s]})\, d\chi_s$ where g, f belongs to $L^2(\mathbf{R}_+)$.

By (0.3), we have $\mathcal{E} \subset \mathcal{J}$. We denote by $j(g,f) = \int_0^{+\infty} g(s)e(f\,1_{[0,s]})\, d\chi_s$. We have $D_t j(g,f) = g(t)e(f\,1_{[0,t]})$ for almost all t in $\mathbf{R}_+$.

LEMMA (1.1). — *Let $f \in L^2(\mathbf{R}_+)$, $T \in \mathcal{B}(L^2(\mathbf{R}_+))$. Then $\mathcal{J}$ is included in the domain of $a^-(f)$, $a^+(f)$ and $\lambda(T)$.*

Proof. — The symmetric Fock space Φ is the direct sum of all the symmetric chaoses $L^2_{\text{sym}}((\mathbf{R}_+)^n)$ with the following representation $F = \sum_n \frac{f_n}{n!}$ with $f_n \in L^2_{\text{sym}}((\mathbf{R}_+)^n)$ and such that $\|F\|^2 = \sum_n \frac{\|f_n\|^2}{n!} < +\infty$. We have that $e(f) = \sum_n \frac{f^{\otimes n}}{n!}$ if $f \in L^2(\mathbf{R}_+)$.

The domain of $a^{\pm}$ consists of those elements $\sum_n \frac{f_n}{n!}$ of Φ such that $\sum_n \frac{\|a^{\pm} f_n\|^2}{n!} < +\infty$.

If $g, f \in L^2(\mathbf{R}_+)$, then $j(g,f) = \sum_{n=1}^{+\infty} \frac{\varphi_n}{n!}$ with

$$\varphi_n(t_1, \dots, t_n) = f(t_1) \cdots f(t_{n-1})\, g(t_n) \quad \text{for} \quad t_1 < t_2 < \cdots < t_n\,. \tag{1.2}$$

It is thus easy to see that $\mathcal{J}$ is included in the domain of $a^{\pm}$. We can write with (1.2) that for $t_1 < t_2 < \cdots < t_n$ and $n \geq 1$

$$\begin{aligned} \big(\lambda(T)\varphi_n\big)(t_1, \dots, t_n) = \sum_{k=1}^{n-1} \Big\{ & T(1_{[0,t_n]} f)(t_k) f(t_1) \cdots f(t_{k-1}) f(t_{k+1}) \cdots f(t_{n-1}) g(t_n) \\ & + T(1_{[t_n, +\infty[}\, g)(t_k) f(t_1) \cdots f(t_{k-1}) f(t_{k+1}) \cdots f(t_n) \Big\} \\ & + T(1_{[0,t_{n-1}]} f)(t_n) g(t_{n-1}) f(t_{n-2}) \cdots f(t_1) \\ & + T(1_{[t_{n-1}, +\infty[}\, g)(t_n) f(t_1) \cdots f(t_{n-1})\,. \end{aligned} \tag{1.3}$$

So as T is a bounded operator on $L^2(\mathbf{R}_+)$, we see easily that $\sum_{n \geq 1} \frac{\|\lambda(T)\varphi_n\|^2}{n!} < +\infty$ and $j(g,f)$ belongs to the domain of $\lambda(T)$. ∎

DEFINITION (1.4). — *A quadruplet $\big((H_t^{\varepsilon})_{t \geq 0}\big)_{\varepsilon \in \{-,0,+,\times\}}$ of adapted processes of operators defined on $D_0 \underline{\otimes} \mathcal{E}$ is called regular if the following conditions are satisfied:*

i) *for all $t > 0$, for all f in $L^2(\mathbf{R}_+)$, for all $u \in D_0$*

$$\sup_{s \in [0,t]} \|H_s^0 \Pi_s(u \otimes e(f))\| < +\infty, \qquad \int_0^t \|H_s^- \Pi_s(u \otimes e(f))\|^2\, ds < +\infty$$

ii) *the processes $(H_t^{\times})_{t \geq 0}$ and $(H_t^+)_{t \geq 0}$ are defined on $D_0 \underline{\otimes} \mathcal{J}$ and for all $u \in D_0$, $f \in L^2(\mathbf{R}_+)$, there exists $\varphi(\cdot, f, u)$ in $L^1_{\text{loc}}(\mathbf{R}_+)$ such that for all $g \in L^2(\mathbf{R}_+)$, for almost all t in $\mathbf{R}_+$*

$$\begin{aligned} &\|H_t^+ \Pi_t(u \otimes j(g,f))\|^2 \leq \|g 1_{[0,t]}\|^2 \varphi(t,f,u) \\ &\|H_t^{\times} \Pi_t(u \otimes j(g,f))\| \leq \|g 1_{[0,t]}\| \varphi(t,f,u) \\ &\int_0^t \|H_s^+ (u \otimes e(0)\|^2\, ds < +\infty \\ &\int_0^t \|H_s^{\times} (u \otimes e(0)\|\, ds < +\infty\,. \end{aligned}$$

A regular quadruplet $((H_t^\varepsilon)_{t\geq 0},\ \varepsilon \in \{-,0,+,\times\})$ satisfies (0.1) and by consequence the operator $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon\, da_s^\varepsilon$ is an adapted process with respect to D_0.

LEMMA (1.5). — *Let $((H_t^\varepsilon)_{t\geq 0}, \varepsilon \in \{-,0,+,\times\})$ be a regular quadruplet and $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon\, da_s^\varepsilon$. For all $t \geq 0$, T_t can be extended to $D_0 \underline{\otimes} \mathcal{J}$ in the sense that for all f, g in $L^2(\mathbf{R}_+)$, for all u in D_0*

$$\begin{aligned} T_t \Pi_t(u \otimes j(g,f)) = &\int_0^t g(s) T_s \Pi_s(u \otimes e(f)\, d\chi_s + \int_0^t g(s) H_s^0 \Pi_s(u \otimes e(f))\, d\chi_s \\ &+ \int_0^t H_s^+ \Pi_s(u \otimes j(g,f))\, d\chi_s + \int_0^t g(s) H_s^- \Pi_s(u \otimes e(f))\, ds \\ &+ \int_0^t H_s^\times \Pi_s(u \otimes j(g,f))\, ds\,. \end{aligned} \tag{1.6}$$

Moreover for all u in D_0, f in $L^2(\mathbf{R}_+)$, there exists an increasing function $\alpha(\cdot, u, f) : \mathbf{R}_+ \to \mathbf{R}_+$ increasing such that for all g in $L^2(\mathbf{R}_+)$,

$$\|T_t \Pi_t(u \otimes j(g,f))\| \leq \|g 1_{[0,t]}\| \alpha(t,u,f)\,. \tag{1.7}$$

Proof. — By (0.4), we have that for all u in D_0, f in $L^2(\mathbf{R}_+)$;

$$\begin{aligned} T_t \Pi_t(u \otimes e(f)) = &\int_0^t f(s) T_s \Pi_s(u \otimes e(f))\, d\chi_s + \int_0^t f(s) H_s^0 \Pi_s(u \otimes e(f))\, d\chi_s \\ &+ \int_0^t H_s^+ \Pi_s(u \otimes e(f))\, d\chi_s + \int_0^t f(s) H_s^- \Pi_s(u \otimes e(f)\, ds \\ &+ \int_0^t H_s^\times \Pi_s(u \otimes e(f))\, ds\,, \end{aligned}$$

and thus by using standard estimates (such as for example: (9.7) of [M3], p. 138), we have that for all $t > 0$, $\sup_{0\leq s\leq t} \|T_s \Pi_s(u \otimes e(f))\| < +\infty$ and thus for all u in D_0, g, f in $L^2(\mathbf{R}_+)$, $\int_0^t T_s D_s(u \otimes j(g,f))\, d\chi_s$ is well defined. By using Attal-Meyer's result, we can define $(T_t)_{t\geq 0}$ on $D_0 \underline{\otimes} \mathcal{J}$ by (1.6). This implies that

$$\begin{aligned} \|T_t \Pi_t(u \otimes j(g,f))\| \leq &\|g 1_{[0,t]}\| \Big\{ \sup_{0\leq s\leq t} \|T_s \Pi_s(u \otimes e(f))\| + \sup_{0\leq s\leq t} \|H_s^0 \Pi_s(u \otimes e(f))\| \\ &+ \Big(\int_0^t \|H_s^- \Pi_s(u \otimes e(f))\|^2\, ds \Big)^{1/2} + \int_0^t \varphi(s,u,f)\, ds \Big\}\,, \end{aligned}$$

where $\varphi(s,u,f)$ is given by the hypothesis on the quadruplet $((H_t^\varepsilon)_{t\geq 0}, \varepsilon \in \{-,0,+,\times\})$. ■

COROLLARY.

(1) All the processes of $\mathcal{S}'$ are defined on $D_0 \underline{\otimes} \mathcal{J}$.

(2) If $T_t = \sum_{\varepsilon} \int_0^t H_s^\varepsilon \, da_s^\varepsilon$ *with* $(H_s^\varepsilon)_{s\geq 0}$ *regular then by (1.7) we can define* $\int_0^t T_s \, da_s^\varepsilon$ *for all* $\varepsilon > 0$. *For all* $n \in \mathbf{N}$, $T_t = \int_{t_1<\cdots<t_n\leq t} da_{t_1}^{\varepsilon_1} \cdots da_{t_n}^{\varepsilon_n}$ *where* $\varepsilon_i \in \{-,0,+,\times\}$ *for* $i = 1,\ldots,n$ *is well defined on* $h\underline{\otimes}\mathcal{J}$.

We see by formula (0.2), (0.4) and (1.5) that it is not necessary to have quadruplet of processes $(H^\varepsilon)_{\varepsilon\in\{-,0,+,\times\}}$ defined for all $t > 0$.

Definition (1.8). — *A quadruplet* $(H^\varepsilon)_{\varepsilon\in\{-,0,+,\times\}}$ *is called regular on* D_0 *if the following conditions are satisfied:*

$$(i) \qquad \begin{aligned} H^\times &: D_0 \underline{\otimes} \mathcal{J} \longrightarrow L^1_{\text{loc}}(\mathbf{R}_+, \mathcal{H}) \\ H^+ &: D_0 \underline{\otimes} \mathcal{J} \longrightarrow L^2_{\text{loc}}(\mathbf{R}_+, \mathcal{H}) \\ H^0 &: D_0 \underline{\otimes} \mathcal{E} \longrightarrow L^\infty_{\text{loc}}(\mathbf{R}_+, \mathcal{H}) \\ H^- &: D_0 \underline{\otimes} \mathcal{E} \longrightarrow L^2_{\text{loc}}(\mathbf{R}_+, \mathcal{H}) \end{aligned}$$

are linear operators.

(ii) For all u *in* D_0, f *in* $L^2(\mathbf{R}_+)$, *there exist* $\varphi(\cdot, u, f) \in L^1_{\text{loc}}(\mathbf{R}_+)$ *such that for almost all* τ *in* $\mathbb{R}_+$, *for all* g *in* $L^2(\mathbf{R}_+)$

$$\begin{aligned} \|H_\tau^+ \Pi_\tau(u \otimes j(g,f))\|^2 &\leq \|g 1_{[0,\tau]}\|^2 \, \varphi(\tau, u, f) \\ \|H_\tau^\times \Pi_\tau(u \otimes j(g,f))\| &\leq \|g 1_{[0,\tau]}\| \, \varphi(\tau, u, f) \, . \end{aligned}$$

(iii) Adaptation property: for all u *in* D_0, f *in* $L^2(\mathbf{R}_+)$, g *in* $L^2(\mathbf{R}_+)$ *for almost all* t *in* $\mathbf{R}_+$, *for all* ε,

$$H_t^\varepsilon \Pi_t(u \otimes e(f)) \in \mathcal{H}_{t]}, \;\; H_t^+ \Pi_t(u \otimes j(g,f)), \;\; H_t^\times \Pi_t(u \otimes j(g,f)) \in \mathcal{H}_{t]},$$

$$H_t^\varepsilon(u \otimes e(f)) = H_t^\varepsilon \Pi_t(u \otimes e(f)) \otimes e(f 1_{[t,+\infty[}) \text{ in } \mathcal{H}_{t]} \otimes \Phi_{[t} \text{ if } \varepsilon = +, \times,$$

$$H_t^\varepsilon(u \otimes j(g,f)) = H_t^\varepsilon \Pi_t(u \otimes j(g,f)) + H_t^\varepsilon \Pi_t(u \otimes e(f 1_{[0,t]})) \otimes j(g 1_{[t,+\infty[}, f 1_{[t,+\infty[}) \, .$$

If (H^ε) is a regular quadruplet, we can define $T_t = \sum_{\varepsilon} \int_0^t H_s^\varepsilon \, da_s^\varepsilon$ on $D_0\underline{\otimes}\mathcal{J}$ by (1.6). Furthermore, we have for all g, f in $L^2(\mathbf{R}_+)$, for all $s \leq t$,

$$\|\Pi_s T_t \Pi_t(u \otimes j(g,f)) - T_s \Pi_s(u \otimes j(g,f))\| \leq \int_s^t \gamma(\tau) \, d\tau \tag{1.9}$$

where γ belongs to $L^1_{\text{loc}}(\mathbf{R}_+)$ and depends of u, g, f.

Definition (1.10). — *A curve* $(x_t)_{t\geq 0}$ *in* $\mathcal{H}$ *is a quasimartingale if for all* $t \geq 0$, $x_t \in \mathcal{H}_{t]}$ *and for all* $s < t$, *it exists* $c(s,t)$ *such that for all subdivisions* $t_0 = s < t_1 < \cdots < t_n = t$ *of* $[s,t]$, *we have* $\sum_{i=0}^{n-1} \|\Pi_{t_i}(x_{t_{i+1}} - x_{t_i})\| \leq c(s,t)$.

Enchev in [E] (see also [M1]) proved that a quasimartingale $(x_t)_{t\geq 0}$ can always be written $x_t = M_t + A_t$ where $(M_t)_{t\geq 0}$ is a martingale, *i.e.* for all $s < t$, $\Pi_s M_t = M_s$ and $(A_t)_{t\geq 0}$ is an adapted process with finite variation in the norm sens and such that $\|A_t - A_s\| \leq c(s,t)$.

Definition (1.11). — *A curve $(x_t)_{t\geq 0}$ in $\mathcal{H}$ is an absolutely continuous quasimartingale if $(x_t)_{t\geq 0}$ is a quasimartingale and if the quantity $c(s,t)$ given by definition (1.10) is of the form $c(s,t) = \int_s^t \psi(\tau)\, d\tau$ for some ψ in $L^1_{\text{loc}}(\mathbf{R}_+)$.*

In this case, we have that $A_t = \int_0^t a(\tau)\, d\tau$ with for almost all τ, $a(\tau)$ in $\mathcal{H}_{\tau]}$ and $\|a(\tau)\| \leq \psi(\tau)$. So if $(T_t)_{t\geq 0}$ is representable as stochastic integral on $D_0 \underline{\otimes} \mathcal{J}$, then by (1.9), for all u in D_0, for all f, g in $L^2(\mathbf{R}_+)$, $(T_t \Pi_t (u \otimes j(g,f)))_{t\geq 0}$ is an absolutely continuous quasimartingale.

Definition (1.12). — *Let $(T_t)_{t\geq 0}$ be an adapted process of operators defined on $D_0 \underline{\otimes} \mathcal{J}$. $(T_t)_{t\geq 0}$ is a quasimartingale (resp. an absolutely continuous quasimartingale) if for all F in $D_0 \underline{\otimes} \mathcal{J}$, $(T_t \Pi_t F)_{t\geq 0}$ is a quasimartingale (resp. an absolutely continuous quasimartingale).*

We have seen in lemma (1.1) that if K is a bounded operator of $L^2(\mathbf{R}_+)$ then $\lambda(K)$ is an operator whose domain contains $\mathcal{J}$. What about the representability of the associated martingale $(\lambda_t(K))_{t\geq 0}$?

Proposition (1.13). — *$(\lambda_t(K))_{t\geq 0}$ belongs to $\mathcal{S}'$ if and only if K is an operator of multiplication by a bounded function.*

Proof. — For f in $L^2(\mathbf{R}_+)$, we have $\lambda_t(K) e(f) = a^+_{1_{[0,t]} K(f 1_{[0,t]})}(e(f))$ thus

$$\lambda_t(K)\Pi_r e(f) - \lambda_s(K)\Pi_r e(f) = a^+_{1_{[s,t]} K(f 1_{[0,r]})}(e(f 1_{[0,r]}))$$

and therefore

$$\|\lambda_t(K)\Pi_r e(f) - \lambda_s(K)\Pi_r e(f)\|^2 = \int_s^t |K(f 1_{[0,r]})(\tau)|^2\, d\tau\, \|e(f 1_{[0,r]})\|^2 .$$

By using (0.7), $(\lambda_t(K))_{t\geq 0} \in \mathcal{S}'$ implies that there exists φ in $L^1_{\text{loc}}(\mathbf{R}_+)$ such that for all $r < s < t$, for all f in $L^2(\mathbf{R}_+)$,

$$\int_s^t |K(f 1_{[0,r]})(\tau)|^2\, d\tau \leq \int_s^t \varphi(\tau)\, d\tau .$$

So $K(f 1_{[0,r]}) 1_{]r,+\infty[} = 0$ and by looking at the adjoint, we must have that for all $r > 0$, for all f in $L^2(\mathbf{R}_+)$, $K(f 1_{]r,+\infty[}) 1_{[0,r]} = 0$. It is easy to see that these conditions imply that for all $a < b$, for all f in $L^2(\mathbf{R}_+)$, $K(f 1_{[a,b]}) = 1_{[a,b]} K f$ and so K commutes with the operator of multiplication by indicators and by consequence it is an operator of multiplication by a bounded function k, and $\lambda(K) = a^0_k = \int_0^{+\infty} k(s)\, da^0_s$.

Lemma (1.14). — *Let K be an Hilbert-Schmidt operator on $L^2(\mathbf{R}_+)$ given by a kernel φ. Then*

$$\lambda_t(K) = \int_0^t H^+_\tau\, da^+_\tau + \int_0^t H^-_\tau\, da^-_\tau$$

where $H^+_\tau = a^-_{\overline{\varphi(\tau,\cdot)}\,1_{[0,\tau[}}$ and $H^-_\tau = a^+_{\varphi(\tau,\cdot)1_{[0,\tau[}}$.

Proof. — Let H^+ and H^- defined as above. Then they give rise to a regular quadruplet (definition (1.8)) and the operator $T_t = \int_0^t H^+_\tau\, da^+_\tau + \int_0^t H^-_\tau\, da^-_\tau$ is defined on $\mathcal{J}$. By (0.2),

$$\begin{aligned}
\langle e(f), T_t e(g)\rangle &= \int_0^t \left\{\bar f(s)\langle e(f), H^+_s e(g)\rangle + g(s)\langle e(f), H^-_s e(g)\rangle\right\} ds \\
&= \langle e(f), e(g)\rangle \int_0^t \left\{\bar f(s)\int_0^s g(\tau)\varphi(s,\tau)\, d\tau + g(s)\int_0^s \overline{f(\tau)}\,\varphi(\tau,s)\, d\tau\right\} ds \\
&= \langle e(f), e(g)\rangle \int_0^t \bar f(s) K(g1_{[0,t]})(s)\, ds \\
&= \langle e(f), a^+_{1_{[0,t]}K(g1_{[0,t]})} e(g)\rangle = \langle e(f), \lambda_t(K) e(g)\rangle\ . \qquad \blacksquare
\end{aligned}$$

This lemma justifies the fact that we have to consider H^+ and H^- as operators from $\mathcal{E}$ (or $\mathcal{J}$) to $L^2(\mathbf{R}_+, \Phi)$ for $\varphi(\tau,\cdot)$ is defined only for almost all τ.

An interesting question is the following: what are the bounded operator K on $L^2(\mathbf{R}_+)$ such that $\lambda_t(K)$ is representable as a stochastic integral on $\mathcal{E}$ or $\mathcal{J}$?

The above results prove that Hilbert-Schmidt operator or multiplication by a bounded function give representable operators, but we can see with the example of Journe in [J-M] that if K is the Hilbert transform, then $(\lambda_t(K))_{t\geq 0}$ is not representable, for it is not a quasimartingale.

We will say that an operator T is representable on $\mathcal{J}$ if the associated martingale $(T_t)_{t\geq 0}$ is representable on $\mathcal{J}$, *i.e.* if there exist a regular quadruplet $(H^\varepsilon)_{\varepsilon\in\{-,0,+,\times\}}$, a constant λ in $\mathbf{R}$ such that for all $t > 0$, $T_t = \lambda \mathrm{Id} + \sum_{\varepsilon\in\{-,0,+,\times\}} \int_0^t H^\varepsilon_s\, da^\varepsilon_s$.

Proposition (1.15). — *Let K a bounded operator in $L^2(\mathbf{R}_+)$. $\lambda(K)$ and $\lambda(K^*)$ are representable on $\mathcal{J}$ if and only if there exist φ in $L^2_{\mathrm{loc}}(\mathbf{R}_+\times\mathbf{R}_+)$, k in $L^\infty_{\mathrm{loc}}(\mathbf{R}_+)$ such that*

$$Kf = \int_0^{+\infty} f(s)\varphi(s,\cdot)\, ds + kf\ .$$

Proof. — We suppose that $\lambda(K)$ and $\lambda(K^*)$ are representable on $\mathcal{J}$, the converse result being already proved.

Let g, f be in $L^2(\mathbf{R}_+)$. We easily see by (1.3) that if $s < t$,

$$\Pi_s\lambda_t(K)\Pi_t j(g,f) - \lambda_s(K)\Pi_s j(g,f) = \Pi_s a^+_{K(g1_{[s,t]})} e(f)\ .$$

By hypothesis,

$$\Pi_s\lambda_t(K)\Pi_t j(g,f) - \lambda_s(K)\Pi_s j(g,f) = \int_s^t g(\tau)\Pi_s H^-_\tau \Pi_\tau e(f)\ d\tau\ .$$

This implies

$$\Pi_s a^+_{K(g1_{[s,t]})} e(f) = \int_s^t g(\tau)\Pi_s H^-_\tau \Pi_\tau e(f)\ d\tau\ . \tag{1.16}$$

Let f be 0, we have

$$\int_0^s K(g1_{[s,t]})(\tau)\ d\chi_\tau = \int_s^t g(\tau)\Pi_s H^-_\tau e(0)\ d\tau\ . \tag{1.17}$$

So for all $s > 0$, for almost all $\tau > s$, $\Pi_s H^-_\tau e(0)$ belongs to the first chaos and so for almost all τ, $H^-_\tau e(0)$ belongs to the first chaos. Let $\varphi(\tau,\cdot)$ be the associated function in $L^2([0,\tau])$. So by (1.17), for all $s < t$, for all g in $L^2(\mathbf{R}_+)$,

$$1_{[0,s]}K(g1_{[s,t]}) = \int_s^t g(\tau)\varphi(\tau,\cdot)1_{[0,s]}(\cdot)\ d\tau \tag{1.18}$$

and by (1.16) for all f in $L^2(\mathbf{R}_+)$, for all g in $L^2(\mathbf{R}_+)$, for all $s < t$,

$$\int_s^t g(\tau)a^+_{\varphi(\tau,\cdot)} e(f)\ d\chi_\tau = \int_s^t g(\tau)\Pi_s H^-_\tau \Pi_\tau e(f)\ d\tau$$

and so $H^-_\tau = a^+_{\varphi(\tau,\cdot)}$.

Furthermore, for all $T > 0$, $\int_0^T \|H^-_\tau e(0)\|^2\ d\tau < +\infty$ and thus

$$\int_0^T \Big(\int_0^\tau |\varphi(\tau,s)|^2\ ds\Big)\ d\tau < +\infty\ .$$

By looking at $\lambda(K)^* = \lambda(K^*)$, we have for almost all $\tau > 0$, the existence of $\varphi^*(\tau,\cdot)$ in $L^2([0,\tau])$ such that for all $T > 0$, $\int_0^T \Big(\int_0^\tau |\varphi(\tau,s)|^2\ ds\Big)\ d\tau < +\infty$ and satisfying for all g in $L^2(\mathbf{R}_+)$, for all $s < t$,

$$1_{[0,s]}K^*(g1_{[s,t]}) = \int_s^t g(\tau)1_{[0,s]}\varphi^*(\tau,\cdot)\ d\tau\ . \tag{1.19}$$

We extend φ to $\mathbf{R}^2_+$ by defining $\varphi(s,t) = \overline{\varphi^*(t,s)}$ for $s < t$. So for all $T > 0$,

$$\int_0^T\int_0^T |\varphi(s,t)|^2\ ds\,dt < +\infty$$

and for all g in $L^2(\mathbf{R}_+)$, for all $s < t$, (1.18) and (1.19) imply

$$(1.20) \qquad \begin{cases} 1_{[0,s]}K(g1_{[s,t]}) = 1_{[0,s]}\int_s^t g(\tau)\varphi(\tau,\cdot)\, d\tau \\ 1_{[s,t]}K(g1_{[0,s]}) = 1_{[s,t]}\int_0^s g(\tau)\varphi(\tau,\cdot)\, d\tau \,. \end{cases}$$

We can conclude like in the proof of (1.13), that (1.20) implies the existence of k in $L^\infty_{\mathrm{loc}}(\mathbf{R}_+)$ such that for all f, for all $T > 0$,

$$K(g1_{[0,T]}) = \int_0^T g(\tau)\varphi(\tau,\cdot)\, d\tau + 1_{[0,T]}kg \,.$$

Remark. — All the same, we have a kind of representability property for $\lambda(K)$ by Maassen's kernel. If K is a bounded operator in $L^2(\mathbf{R}_+)$ with a kernel φ not necessary in $L^2_{\mathrm{loc}}(\mathbf{R}_+ \times \mathbf{R}_+)$ (for example the case of the Hilbert operator), we can write $\lambda(K) = \iint_{\mathbf{R}_+\times\mathbf{R}_+} \varphi(s,t)\, da_s^+\, da_t^-$.

2. Integral representation

DEFINITION (2.1). — *Let $T = (T_t)_{t\geq 0}$ be an absolutely continuous quasimartingale on $D_0 \otimes \mathcal{J}$. We say that T is a regular quasimartingale if for all f in $L^2(\mathbf{R}_+)$, for all u in D_0, there exists $\psi(\cdot,u,f)$ in $L^1_{\mathrm{loc}}(\mathbf{R}_+)$ such that for all g in $L^2(\mathbf{R}_+)$, for all $r < s < t$,*

(2.2) $\|\Pi_s T_t \Pi_r u \otimes j(g,f) - T_s\Pi_r u \otimes j(g,f)\| \leq \|g1_{[0,r]}\| \int_s^t \psi(\tau,u,f)\, d\tau$

(2.3) $\|T_t\Pi_r u \otimes j(g,f) - T_s\Pi_r u \otimes j(g,f)\|^2 \leq \|g1_{[0,r]}\|^2 \int_s^t \psi(\tau,u,f)\, d\tau$

(2.4) The mapping $L^2(\mathbf{R}_+) \to \mathcal{H}$, $g \mapsto T_t\Pi_t(u \otimes j(g,f))$ is closable.

Remark (2.5). — If for all $t > 0$, T_t is a closable operator then (2.4) is satisfied. The hypothesis (2.4) implies by using the closed graph theorem that for all f in $L^2(\mathbf{R}_+)$, for all u in D_0, the mapping $L^2(\mathbf{R}_+) \to \mathcal{H}$, $g \mapsto T_t\Pi_t(u \otimes j(g,f))$ is bounded.

THEOREM (2.6). — *A process $T = (T_t)_{t\geq 0}$ of operators is a regular quasimartingale if and only if there exists a unique regular quadruplet $(H^\varepsilon)_{\varepsilon\in\{-,0,+,\times\}}$ such that for all $t > 0$, $T_t = T_0 + \sum_\varepsilon \int_0^t H_s^\varepsilon\, da_s^\varepsilon$ on $\mathcal{D}_0\underline{\otimes}\mathcal{J}$.*

Proof. — Let $T_t = T_0 + \sum_\varepsilon \int_0^t H_s^\varepsilon\, da_s^\varepsilon$ on $\mathcal{D}_0 \otimes \mathcal{J}$ with $(H^\varepsilon)_{\varepsilon\in\{-,0,+,\times\}}$ being a regular quadruplet. Equation(1.9) implies that $T = (T_t)$ is an absolutely continuous quasimartingale on $\mathcal{D}_0 \otimes \mathcal{J}$. By using (1.6) and (1.7), we prove easily that (2.2), (2.3) and (2.4) are satisfied and by consequence T is a regular quasimartingale.

We now prove the converse result. Let $T = (T_t)_{t\geq 0}$ be a regular quasimartingale. We can change T_t in $T_t - T_0$ without modifying the hypothesis, so we can suppose $T_0 = 0$. Let u in D_0 and f in $L^2(\mathbf{R}_+)$ be fixed. By assumption, for all g in $L^2(\mathbf{R}_+)$, $(T_t\Pi_t(u\otimes e(0)))_{t\geq 0}$ and $(T_t\Pi_t(u\otimes j(g,f)))_{t\geq 0}$ are absolutely continuous quasimartingale and by (1.11) there exists adapted curves from $\mathbf{R}_+$ to $\mathcal{H}$, $\tau\mapsto m_\tau(g,u,f)$, $\tau\mapsto a_\tau(g,u,f)$, $\tau\mapsto m_\tau(u)$ and $\tau\mapsto a_\tau(u)$. More over for all $t>0$, $\int_0^t \|m_\tau(g,u,f)\|^2\,d\tau < +\infty$, $\int_0^t \|a_\tau(g,u,f)\|\,d\tau < +\infty$, $\int_0^t \|m_\tau(u)\|^2\,d\tau < +\infty$ and $\int_0^t \|a_\tau(u)\|\,d\tau < +\infty$ and

$$(2.7)\qquad \begin{cases} T_t\Pi_t(u\otimes e(0)) = \int_0^t m_\tau(u)\,d\chi_\tau + \int_0^t a_\tau(u)\,d\tau \\ T_t\Pi_t(u\otimes j(g,f)) = \int_0^t m_\tau(g,u,f)\,d\chi_\tau + \int_0^t a_\tau(g,u,f)\,d\tau\,. \end{cases}$$

Let $r>0$ be fixed. By (2.2), we have that $\|a_\tau(g1_{[0,r]},u,f)\| \leq \|g1_{[0,r]}\|\psi(\tau,u,f)$ for a.a. $\tau > r$. We denote by γ the lifting from $\mathcal{L}^\infty(\mathbf{R}_+,dt;\mathcal{H})$ to $\mathcal{B}^\infty(\mathbf{R}_+;\mathcal{H})$ the Banach space of everywhere bounded Borel functions on $\mathbf{R}_+$ with the uniform norm (see [M3], p. 293). The function $\tau\mapsto \frac{a_\tau(g1_{[0,r]},u,f)}{\psi(\tau,u,f)}$ (which we assign the value 0 if $\psi(\tau,u,f)=0$) belongs to $\mathcal{L}^\infty(]r,+\infty[,dt;\mathcal{H})$, thus we can define a map $]r,+\infty[\to\mathcal{H}$, $\tau\mapsto A_{r,\tau}(g,u,f)$ with $A_{r,\tau}(g,u,f) = \psi(\tau,u,f)\gamma\big(\frac{a(g1_{[0,r]},u,f)}{\psi(\cdot,u,f)}\big)(\tau)$ for $\tau>r$, and we have that for all $\tau>r$, for all g in $L^2(\mathbf{R}_+)$,

$$\|A_{r,\tau}(g,u,f)\| \leq \|g1_{[0,r]}\|\,\psi(\tau,u,f)\,.$$

Consequently, the mapping $L^2([0,r])\to\mathcal{H}$, $g\mapsto A_{r,\tau}(g,u,f)$ defines a bounded linear operator. As $\Pi_r\Pi_{r'} = \Pi_{r'}$ if $r'<r$, and by using the lifting γ, we have that for all $\tau>r>r'$, for all g in $L^2([0,r'])$,

$$A_{r',\tau}(g,u,f) = A_{r,\tau}(g,u,f)\,.$$

This allows to define $A_\tau(g,u,f)$ for all g in $\bigcup_{r<\tau} L^2([0,r])$, and we have that $\|A_\tau(g,u,f)\| \leq \|g\|\psi(\tau,u,f)$. This defines $A_\tau(g,u,f)$ for g in $L^2([0,\tau])$. One easily checks that $A_\tau(g,u,f)$ belongs to $\mathcal{H}_{\tau]}$. We put $H^\times_\tau\Pi_\tau(u\otimes j(g,f)) = A_\tau(g1_{[0,\tau]},u,f)$ and $H^\times_\tau\Pi_\tau(u\otimes e(0)) = a_\tau(u)$. We have constructed an operator $H^\times : D_0\otimes\mathcal{J}\to L^1_{\mathrm{loc}}(\mathbf{R}_+,\mathcal{H})$ which is adapted and such that for all u in D_0, for all f in $L^2(\mathbf{R}_+)$, for a.a. τ in $\mathbf{R}_+$, for all g in $L^2(\mathbf{R}_+)$,

$$(2.8)\qquad \|H^\times_\tau\Pi_\tau(u\otimes j(g,f))\| \leq \|g1_{[0,\tau]}\|\psi(\tau,u,f)\,.$$

Now define $K_\tau(u\otimes j(g,f)) = a_\tau(g,u,f) - H^\times_\tau\Pi_\tau(u\otimes j(g,f))$ and $K_\tau(u\otimes e(0)) = 0$. The operator $K : D_0\underline{\otimes}\mathcal{J}\to L^1_{\mathrm{loc}}(\mathbf{R}_+,\mathcal{H})$ satisfies for all r in $\mathbf{R}^*_+$, for all u in D_0, for all f in $L^2(\mathbf{R}_+)$ for a.a. $\tau>r$, for all g in $L^2(\mathbf{R}^*_+)$,

$$(2.9)\qquad K_\tau\Pi_r(u\otimes j(g,f)) = 0 \text{ and } K_\tau\Pi_r(u\otimes e(0)) = 0\,.$$

By (2.7), we have moreover that for all $r<s<t$,

$$T_t\Pi_r(u\otimes j(g,f)) - T_s\Pi_r(u\otimes j(g,f)) = \int_s^t m_\tau(g1_{[0,r]},u,f)\,d\chi_\tau + \int_s^t a_\tau(g1_{[0,r]},u,f)\,d\tau\,.$$

So by (2.3) we have, for a.a. $\tau > r$, $\|m_\tau(g1_{[0,r]}, u, f)\|^2 \leq \|g1_{[0,r]}\|^2 \tilde{\psi}(\tau, u, f)$ where $\tilde{\psi}(\cdot, u, f) \in L^1_{\text{loc}}(\mathbf{R}_+)$. Using the same method as above, we construct $H^+ : \mathcal{D}_0 \underline{\otimes} \mathcal{J} \to L^2_{\text{loc}}(\mathbf{R}_+, \mathcal{H})$ adapted such that for all u in D_0, f in $L^2(\mathbf{R}_+)$, a.a. τ in $\mathbf{R}_+$ for all g in $L^2(\mathbf{R}_+)$,

$$(2.10) \qquad \|H^+_\tau \Pi_\tau(u \otimes j(g, f))\| \leq \|g1_{[0,\tau]}\| \tilde{\psi}(\tau, u, f) .$$

Now define $R_\tau(u \otimes j(g, f)) = m_\tau(g, u, f) - H^\times_\tau \Pi_\tau(u \otimes j(g, f))$ and $R_\tau(u \otimes e(0)) = 0$. The operator $R : \mathcal{D}_0 \underline{\otimes} \mathcal{J} \to L^2_{\text{loc}}(\mathbf{R}_+, \mathcal{H})$ satisfies for all r in $\mathbf{R}^*_+$, u in D_0, f in $L^2(\mathbf{R}^*_+)$, for a.a. $\tau > r$, for all g in $L^2(\mathbf{R}_+)$,

$$(2.11) \qquad K_\tau(\Pi_r(u \otimes j(g, f))) = 0 \text{ and } K_\tau \Pi_r(u \otimes e(0)) = 0 .$$

We have thus

$$(2.12) \qquad T_t \Pi_t(u \otimes j(g, f)) = \int_0^t R_\tau(u \otimes j(g, f))\, d\chi_\tau + \int_0^t H^+_\tau \Pi_\tau(u \otimes j(g, f))\, d\chi_\tau + \int_0^t K_\tau(u \otimes j(g, f))\, d\tau + \int_0^t H^\times_\tau \Pi_\tau(u \otimes j(g, f))\, d\tau .$$

Let the linear operator $L_{u,f}$, $L^2(\mathbf{R}_+) \mapsto L^1_{\text{loc}}(\mathbf{R}_+, \mathcal{H}) \times L^1_{\text{loc}}(\mathbf{R}_+, \mathcal{H}) \times L^2_{\text{loc}}(\mathbf{R}_+, \mathcal{H}) \times L^2_{\text{loc}}(\mathbf{R}_+, \mathcal{H})$, $g \mapsto \big(H^+_. \Pi_.(u \otimes j(g, f)), K_.(u \otimes j(g, f)), H^\times_. \Pi_.(u \otimes j(g, f)), R_.(u \otimes j(g, f))\big)$.

By using (2.4), (2.8), (2.10) and (2.12), we see that $L_{u,f}$ is closable and so by the closed graph theorem, it is a bounded operator and there exists for all $t > 0$, u in D_0, f in $L^2(\mathbf{R}_+)$, $c_t(u, f)$ in $\mathbf{R}$ such that for all g in $L^2(\mathbf{R}_+)$,

$$(2.13) \qquad \begin{cases} \int_0^t \|K_\tau(u \otimes j(g, f))\|\, d\tau \leq \|g\| c_t(u, f) \\ \int_0^t \|R_\tau(u \otimes j(g, f))\|^2\, d\tau \leq \|g\|^2 c_t(u, f) \end{cases}$$

Because of (2.9) and (2.11), we can define processes $H^- : D_0 \underline{\otimes} \mathcal{E} \to L^1_{\text{loc}}(\mathbf{R}_+, \mathcal{H})$, $L : D_0 \underline{\otimes} \mathcal{E} \to L^2_{\text{loc}}(\mathbf{R}_+, \mathcal{H})$ by: for u in D_0 and f in $L^2(\mathbf{R}_+)$,

$$H^-_\tau \Pi_\tau(u \otimes e(f)) = K_\tau(u \otimes \int_0^T e(f1_{[0,s]})\, d\chi_s)$$

$$L_\tau \Pi_\tau(u \otimes e(f)) = R_\tau(u \otimes \int_0^T e(f1_{[0,s]})\, d\chi_s)$$

where T is any real $> \tau$.

By definition of K and R, H^- and L are adapted processes. Let g in $L^2(\mathbf{R}_+)$ be a step function given by $g = \sum_i \lambda_i 1_{[t_i, t_{i+1}]}$ for $0 \leq t_0 < t_1 < \cdots < t_n$.

We have

$$\begin{aligned} R_\tau(u \otimes j(g, f)) &= \sum_i \lambda_i \Big\{ R_\tau\Big(u \otimes \int_0^{t_{i+1}} e(f1_{[0,s]})\, d\chi_s\Big) - R_\tau\Big(u \otimes \int_0^{t_i} e(f1_{[0,s]})\, d\chi_s\Big) \Big\} \\ &= \sum_{i / t_i < \tau < t_{i+1}} \lambda_i L_\tau \Pi_\tau(u \otimes e(f)) \\ &= g(\tau) L_\tau \Pi_\tau(u \otimes e(f)) . \end{aligned}$$

By the same computation, we have $K_\tau(u\otimes j(g,f)) = g(\tau)H^-_\tau\Pi_\tau(u\otimes e(f))$. Moreover (2.13) implies that for all step function g in $L^2(\mathbf{R}_+)$, for all $T>0$,

$$\int_0^T |g(\tau)|^2 \|L_\tau\Pi_\tau(u\otimes e(f))\|^2 \, d\tau \le \|g\|^2 c_T(u,f)$$

and

$$\int_0^T |g(\tau)| \|H^-_\tau\Pi_\tau(u\otimes e(f))\| \, d\tau \le \|g\| c_T(u,f)\,.$$

Consequently, for all $T>0$, we have

$$\sup_{0<\tau<T} \|L_\tau\Pi_\tau(u\otimes e(f))\|^2 \le c_T(u,f)$$

and

$$\int_0^T \|H^-_\tau\Pi_\tau(u\otimes e(f))\|^2 \, d\tau \le c_T(u,f)^2\,,$$

and for all g in $L^2(\mathbf{R}_+)$,

$$R_\tau(u\otimes j(g,f)) = g(\tau)L_\tau\Pi_\tau(u\otimes e(f))$$

and

$$K_\tau(u\otimes j(g,f)) = g(\tau)H^-_\tau\Pi_\tau(u\otimes e(f))\,.$$

So by (2.12),

$$\begin{aligned} T_t\Pi_t(u\otimes j(g,f)) = &\int_0^t L_\tau D_\tau\big(u\otimes j(g,f)\big)\, d\chi_\tau + \int_0^t H^+_\tau\Pi_\tau\big(u\otimes j(g,f)\big)\, d\chi_\tau \\ &+ \int_0^t H^-_\tau D_\tau\big(u\otimes j(g,f)\big)\, d\tau + \int_0^t H^\times_\tau\Pi_\tau\big(u\otimes j(g,f)\big)\, d\tau \end{aligned}$$

and

$$\sup_{0<t<T} \|T_t\Pi_t(u\otimes j(g,f))\| \le \|g\|\tilde{c}_T(u,f)\,.$$

We define $H^0 : D_0\underline{\otimes}\mathcal{E} \to L^\infty_{\text{loc}}(\mathbf{R}_+,\mathcal{H})$ adapted by

$$H^0_\tau\Pi_\tau(u\otimes e(g)) = -T_\tau\Pi_\tau(u\otimes e(f)) + L_\tau\Pi_\tau(u\otimes e(f))\,.$$

We finally conclude by the result of S. Attal and P.A. Meyer in [A-M] that $T_t = \sum\limits_{\varepsilon} \int_0^t H^\varepsilon_s \, da^\varepsilon_s$ on $D_0\underline{\otimes}\mathcal{J}$.

We now have to prove the uniqueness. Let $(H^\varepsilon)_{\varepsilon\in\{-,0,+,\times\}}$ a regular quadruplet such that for all u in D_0, g, f in $L^2(\mathbf{R}_+)$, $\Big(\sum\limits_{\varepsilon}\int_0^t H^\varepsilon_s\, da^\varepsilon_s\Big)(u\otimes j(g,f)) = 0$. Let $T_t = \sum\limits_{\varepsilon}\int_0^t H^\varepsilon_s\, da^\varepsilon_s$. For all u in D_0, g, f in $L^2(\mathbf{R}_+)$, r, $r' < s < t$,

$$\Pi_{r'}(T_t\Pi_t - T_s\Pi_s)\Pi_r(u\otimes j(g,f)) = 0$$

and so $\int_s^t \Pi_{r'}H^\times_\tau\Pi_r(u\otimes j(g,f))\, d\tau = 0$.

So for all $r > 0$, $r' > 0$, for a.a. $\tau > r, r'$, $\Pi_{r'} H^\times_\tau(u \otimes j(g,f)) = 0$ thus for all $r > 0$, for a.a. $\tau > r$, $H^\times_\tau \Pi_r(u \otimes j(g,f)) = 0$. But by assumption, for all $r > 0$, for a.a. $\tau > r$

$$\left\| H^\times_\tau \Pi_\tau\big(u \otimes j(g,f) - \Pi_r(u \otimes j(g,f))\big)\right\| \leq \Big(\int_r^\tau |g(s)|^2\, ds\Big)^{1/2} \varphi(\tau, u, f)$$

and thus for a.a. τ, $H^\times_\tau \Pi_\tau(u \otimes j(g,f)) = 0$.

We also have that $H^\times_\tau \Pi_\tau(u \otimes e(0)) = 0$ and so $H^\times = 0$. If we look at $(T_t\Pi_t - T_s\Pi_s)\Pi_r$, we prove by the same method that $H^+ = 0$. So for all $t > 0$, u in D_0, f, g in $L^2(\mathbf{R}_+)$,

$$\int_0^t g(\tau) H^0_\tau \Pi_\tau(u \otimes e(f))\, d\chi_\tau + \int_0^t g(\tau) H^-_\tau \Pi_\tau(u \otimes e(f))\, d\tau = 0$$

and thus for a.a. τ

$$\begin{cases} g(\tau) H^0_\tau \Pi_\tau(u \otimes e(f)) = 0 \\ g(\tau) H^-_\tau \Pi_\tau(u \otimes e(f)) = 0 \end{cases}$$

and by consequence $H^0 = 0$ and $H^- = 0$. ∎

3. Some consequences

a) Operators commuting with projections on $\mathcal{H}_{t]}$.

Proposition 3.1. — *Let T be a closable operator defined on $D_0 \underline{\otimes} \mathcal{J}$ such that for all $t > 0$, $T\Pi_t = \Pi_t T$: then there exist an unique $H : D_0 \underline{\otimes} \mathcal{E} \to L^\infty(\mathbf{R}_+, \mathcal{H})$ and $Z : D_0 \to h_0$ such that $T = \int_0^{+\infty} H_s\, da^0_s + Z \otimes I$.*

Moreover, if T is bounded then H_t is bounded for all t and $t \mapsto \|H^0_t\|$ belongs to $L^\infty(\mathbf{R}_+)$.

Proof. — We define $(T_t)_{t \geq 0}$ as usual by $T_t\Pi_t F = T\Pi_t F = \Pi_t TF$. So if $s < t$, we have $\Pi_s T_t \Pi_t = \Pi_s T \Pi_t = T\Pi_s = T_s$ and $T_0 = Z \otimes \mathrm{Id}$. The hypotheses of theorem (2.6) are satisfied with $\psi = 0$ and so there exists $H : D_0 \underline{\otimes} \mathcal{E} \to L^\infty_{\mathrm{loc}}(\mathbf{R}_+, \mathcal{H})$ such that for all $t > 0$, $T_t = T_0 + \int_0^t H_s\, da^\varepsilon_s$ on $D_0 \underline{\otimes} \mathcal{J}$. If $u \in D_0$, f, g in $L^2(\mathbf{R}_+)$, we have:

$$\Pi_t T_t(u \otimes j(g,f)) = T_0(u \otimes j(g,f)) + \int_0^t g(\tau)\big(T\Pi_\tau(u \otimes e(f)) + H_\tau \Pi_\tau(u \otimes e(f))\big)\, d\chi_\tau$$

and so for all g in $L^2(\mathbf{R}_+)$, for all $t > 0$, for all $F \in D_0 \underline{\otimes} \mathcal{E}$

$$(3.2) \qquad \int_0^t |g(\tau)|^2 \|T\Pi_\tau F + H_\tau \Pi_\tau F\|^2\, d\tau \leq \|(T - T_0)\Big(\int_0^{+\infty} g(\tau)\Pi_\tau F\, d\chi_\tau\Big)\|^2 .$$

By consequence, $\int_0^{+\infty} |g(\tau)|^2 \|T\Pi_\tau F + H_\tau \Pi_\tau F\|^2 \, d\tau < +\infty$ and $\sup_{\tau \in \mathbf{R}^+} \|\Pi_\tau TF + H_\tau \Pi_\tau F\| < +\infty$ and $\sup_{\tau \in \mathbf{R}^+} \|H_\tau \Pi_\tau F\| < +\infty$. This implies that $H : D_0 \underline{\otimes} \mathcal{E} \to L^\infty(\mathbf{R}_+, \mathcal{H})$ and $T = Z \otimes \mathrm{Id} + \int_0^{+\infty} H_s \, da_s^0$. Furthermore, if T is bounded, (3.2) implies that for all g in $L^2(\mathbf{R}_+)$, for all F in $D_0 \underline{\otimes} \mathcal{E}$,

$$\int_0^{+\infty} |g(\tau)|^2 \|\Pi_\tau TF + H_\tau \Pi_\tau\|^2 \, d\tau \leq \|g\|^2 \, \|F\|^2 \, \|T - T_0\|^2$$

and so for all F in $D_0 \underline{\otimes} \mathcal{E}$, for a.a. τ in $\mathbf{R}_+$,

$$\|\Pi_\tau TF + H_\tau \Pi_\tau F\| \leq \|F\| \, \|T - T_0\|$$

and so

$$\|H_\tau \Pi_\tau F\| \leq 3\|T\| \, \|F\| \, . \qquad \blacksquare$$

Remark 3.3. — In fact, we have the more precise result: T is an operator defined on $D_0 \underline{\otimes} \mathcal{J}$ satisfying (2.4) and for all $t > 0$ $T\Pi_t = \Pi_t T$ if and only if there exist $H : D_0 \underline{\otimes} \mathcal{E} \to L^\infty(\mathbf{R}_+, \mathcal{H})$ and $Z : D_0 \to h_0$ such that $T = \int_0^{+\infty} H_s \, da_s^0 + Z \otimes I$.

b) About uniqueness.

Let $H : D_0 \underline{\otimes} \mathcal{J} \to L^p_{\mathrm{loc}}(\mathbf{R}_+, \mathcal{H})$, $(p = 1 \text{ or } 2)$ such that for all $u \in D_0$, for all f in $L^2(\mathbf{R}_+)$, there exists $\varphi(\cdot, u, f)$ in $L^p_{\mathrm{loc}}(\mathbf{R}_+)$ such that for a.a. τ, for all g in $L^2(\mathbf{R}_+)$,

$$\|H_\tau \Pi_\tau (u \otimes j(g, f))\| \leq \|g 1_{[0,\tau]}\| \varphi(\tau, u, f) \, .$$

This condition on H implies a kind of right continuity for H. That is: for a.a. τ in $\mathbf{R}_+$, for all $(\tau_n)_{n \geq 0}$ increasing to (τ)

$$H_\tau \Pi_{\tau_n} (u \otimes j(g, f)) \underset{n \to +\infty}{\longrightarrow} H_\tau \Pi_\tau (u \otimes j(g, f)) \, . \tag{3.4}$$

For example, if we look at the Malliavin's gradient, $\nabla : \mathcal{J} \to L^2(\mathbf{R}_+, \Phi)$ defined by $\nabla_\tau j(g, f) = g(\tau) \Pi_\tau e(f) + f(\tau) j(g 1_{]\tau, +\infty}, f)$, we have that $\nabla_\tau \Pi_{\tau_n} j(g, f) = 0$ if $\tau_n < \tau$ and (3.4) is not fulfilled.

But we have that $\int_0^t \nabla_u da_u^+ = a_t^0$.

c) About the results of Parthasarathy and Sinha, Attal and Meyer.

We have recalled their results in (0.6) and (0.7).

Let $(T_t)_{t \geq 0}$ be a process of bounded adapted operators which satisfy (0.7). Then for all F, G in $\mathcal{H}$, for all $r < s < t$, we have

$$\begin{aligned} \langle T_t^* \Pi_r F - T_s^* \Pi_r F, G \rangle &= \langle T_t^* \Pi_r F - T_s^* \Pi_r F, \Pi_t G - \Pi_s G \rangle + \langle T_t^* \Pi_r F - T_s^* \Pi_r F, \Pi_s G \rangle \\ &= \langle \Pi_r F, \Pi_r (T_t \Pi_s - T_s \Pi_s) G \rangle \, . \end{aligned}$$

So (1.7) implies

$$\left|\langle \Pi_r F, \Pi_r(T_t\Pi_s - T_s\Pi_s)G\rangle\right| \le \|\Pi_r F\|\left\{\sqrt{\int_s^t \varphi(\tau)\, d\tau}\|\Pi_t G - \pi_s G\| + \int_s^t \varphi(\tau)\, d\tau \|\Pi_r G\|\right\}$$

and thus,

$$\|\Pi_r(T_t\Pi_s - T_s\Pi_s)G\| \le \sqrt{\int_s^t \varphi(\tau)\, d\tau}\|\Pi_t G - \pi_s G\| + \int_s^t \varphi(\tau)\, d\tau \|\Pi_r G\| \,. \tag{3.5}$$

(0.7) and (3.5) implies that the hypothesis of theorem (2.1) are satisfied and so we have the existence of $(H_t^\varepsilon)_{t\ge 0}$.

By using the inequalities (0.7) and (3.5), we have that (H_τ^ε), $\varepsilon \in \{+,-,\times\}$ are bounded operators which satisfy (0.6). By using proposition 3.1, we see that H_τ^0 is bounded and that $\tau \to \|H_\tau^0\|$ belongs to $L^\infty_{\mathrm{loc}}(\mathbf{R}_+)$.

4. "Noises" defined on $\mathcal{J}$

Definition 4.1. — *A process of operators $(T_t)_{t\ge 0}$ defined on $\mathcal{J}$ is a noise if for all $s < t$, $T_t - T_s = \mathrm{Id} \otimes K_{s,t} \otimes \mathrm{Id}$ on $\Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}$ with $K_{s,t}$ being an operator on $\Phi_{[s,t]}$.*

Theorem 4.2. — *Let $(T_t)_{t\ge 0}$ be a noise on $\mathcal{J}$ such that each T_t is closable, then there exist $A : \mathbf{R}_+ \to \mathbf{C}$, $f \in L^2_{\mathrm{loc}}(\mathbf{R}_+)$, $g \in L^2_{\mathrm{loc}}(\mathbf{R}_+)$ and $k \in L^\infty_{\mathrm{loc}}(\mathbf{R}_+)$, such that*

$$T_t = A(t)\mathrm{Id} + a^+_{f1_{[0,t]}} + a^-_{g1_{[0,t]}} + a^0_{k1_{[0,t]}}\,.$$

Proof.

$\boxed{1}$ We define $x_t = T_t e(0)$. We can see that for all $s < t$, $x_t - x_s$ belongs to $\Phi_{[s,t]}$ and in the same way as in [C], $x_t = A(t)e(0) + \int_0^t f(s)\, d\chi_s$ with f in $L^2_{\mathrm{loc}}(\mathbf{R}_+)$.

We define $S_t = T_t - A(t)\mathrm{Id} - a^+_{f1_{[0,t]}}$, so we have that $(S_t)_{t\ge 0}$ is a noise on $\mathcal{J}$ and $S_t e(0) = 0$.

$\boxed{2}$ T_t is closable on $\mathcal{J}$ so for all f in $L^2(\mathbf{R}_+)$, the mapping $L^2(\mathbf{R}_+) \to \Phi$, $g \mapsto T_t j(g,f)$ is bounded because it is linear and closable on all $L^2(\mathbf{R}_+)$. So there exists $h_{t,f}$ in $L^2([0,t])$ such that for all g in $L^2(\mathbf{R}_+)$, $\Pi_0 S_t j(g,f) = \int_0^t g(\tau)h_{t,f}(\tau)\, d\tau$. But if $a < b < t$,

$$\begin{aligned}\Pi_0 S_t j(g1_{[a,b]}, f) &= \Pi_0(S_b - S_a)j(g1_{[a,b]}, f) + \Pi_0 S_a j(g1_{[a,b]}, f)\\ &= \Pi_0 S_b j(g,f) - \Pi_0 S_a j(g,f)\,.\end{aligned}$$

This implies that $1_{[0,a]}h_{a,f} = 1_{[0,a]}h_{b,f}$ and we define h_f by $h_f(s) = h_{t,f}(s)$ if $s < t$, so $h_f \in L^2_{\mathrm{loc}}(\mathbf{R}_+)$ and $\Pi_0 S_t j(g,f) = \int_0^t g(\tau)h_f(\tau)\, d\tau$.

We have to prove that $(S_t)_{t\geq 0}$ is an absolutely continuous quasimartingale

$$(\Pi_s S_t \Pi_t - S_s \Pi_s) j(g,f) = (\Pi_0(S_t - S_s)\Pi_t j(g,f))\Pi_s e(f)$$

and so $\|(\Pi_s S_t \Pi_t - S_s\Pi_s) j(g,f)\| \leq \|e(f)\| \int_0^t |g(\tau)|\,|h_f(\tau)|\, d\tau$. Moreover if $r < t$, $S_t\Pi_r - S_s\Pi_r = 0$ on $\mathcal{J}$, so $(T_t)_{t\geq 0}$ is a regular quasimartingale.

So by theorem (2.1), there exist H^- and H^0 such that $S_t = \int_0^t H_s^- \, da_s^- + \int_0^t H_s^0 \, da_s^0$. We define $\ell(\tau) = \Pi_0 H_\tau^- e(0)$ and $k(\tau) = \Pi_0 H_\tau^0 e(0)$. Let f, g in $L^2(\mathbf{R}_+)$ and $r < s < t$,

$$\begin{aligned}
&S_t\Pi_t j(g, f1_{[0,r]}) - S_s\Pi_s j(g, f1_{[0,r]}) \\
&\quad = \int_s^t g(\tau) S_\tau \Pi_r e(f)\, d\chi_\tau + \int_s^t g(\tau) H_\tau^0 \Pi_r e(f)\, d\chi_\tau \\
&\qquad\qquad + \int_s^t g(\tau) H_\tau^- \Pi_r e(f)\, d\tau \\
&\quad = (S_t - S_s)\Pi_t j(g, f1_{[0,r]}) + S_s(\Pi_t - \Pi_s) j(g, f1_{[0,r]}) \\
&\quad = S_s \Pi_r e(f) \otimes \int_s^t g(\tau)\, d\chi_\tau + \Pi_r e(f) \otimes (S_t - S_s) \int_s^t g(\tau)\, d\chi_\tau\, .
\end{aligned}$$

So

$$\begin{aligned}
&\int_s^t g(\tau) H_\tau^0 \Pi_r e(f)\, d\chi_\tau + \int_s^t g(\tau) H_\tau^- \Pi_r e(f)\, d\tau \\
&\qquad = \Pi_r e(f) \otimes (S_t - S_s) \int_s^t g(\tau)\, d\chi_\tau \\
&\qquad = \Pi_r e(f) \otimes \Big(\int_s^t g(\tau) H_\tau^0 e(0)\, d\chi_\tau + \int_s^t g(\tau) H_\tau^- e(0)\, d\tau \Big)
\end{aligned}$$

so for a.a., τ $\begin{cases} H_\tau^0 e(0) = k(\tau) e(0) \\ H_\tau^- e(0) = \ell(\tau) e(0) \end{cases}$ and for all $r > 0$,

for a.a. $\tau > 0$, $\begin{cases} H_\tau^0 \Pi_r e(f) = k(\tau) \Pi_r e(f) \\ H_\tau^- \Pi_r e(f) = \ell(\tau) \Pi_r e(f) \end{cases}$.

We define $\widetilde{S}_t = a^-_{1_{[0,t]}\ell} + a^0_{1_{[0,t]}k}$ and $R_t = S_t - \widetilde{S}_t$. We fix $t = 1$. Let $a < b$. We have for f, g in $L^2(\mathbf{R}_+)$,

$$R_1 j(g1_{[a,b]}, f1_{[0,a]}) = \int_a^b g(\tau) R_\tau \Pi_a e(f)\, d\chi_\tau\, . \tag{4.3}$$

Let $j_n = \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} g(\tau) \Pi_{\frac{i}{n}} e(f)\, d\chi_\tau$. Then $j_n \underset{n\to+\infty}{\longrightarrow} \Pi_1 j(g,f)$ and we can prove by (4.3) that $R_1 j_n \underset{n\to+\infty}{\longrightarrow} \int_0^1 g(\tau) R_\tau \Pi_\tau e(f)\, d\chi_\tau$.

As $\widetilde{S}_1 j_n \underset{n\to+\infty}{\longrightarrow} \widetilde{S}_1 \Pi_1 j(g,f)$ and

$$(A(1)\mathrm{Id} + a^+_{f1_{[0,1]}}) j_n \underset{n\to+\infty}{\longrightarrow} (A(1)\mathrm{Id} + a^+_{f1_{[0,t]}}) \Pi_1 j(g,f)\, ,$$

the closability of T_1 implies that $R_1 j_n \to R_1\Pi_1 j(g,f)$ and so

$$R_1\Pi_1 j(g,f) = \int_0^1 g(\tau) R_\tau \Pi_t e(f)\, d\chi_\tau$$

and so for all $t \geq 0$, $R_t = 0$. ∎

Remark. — This is not a new result. We prove in [C] that if $(T_t)_{t\geq 0}$ is a noise on $\mathcal{E}$ such that all T_t are closable and for all f in $L^2(\mathbf{R}_+)$, the mapping $\mathbf{R}_+ \to \mathbb{C}$, $t \mapsto \Pi_0 T_t\big(e(f) - e(0)\big)$ has finite variations on compact sets, then the conclusion of theorem 4.2 is valid.

The condition "T_t closable on $\mathcal{J}$" implies that for all f in $L^2(\mathbf{R}_+)$, there exists h_f in $L^2_{[loc}(\mathbf{R}_+)$ such that for all g in $L^2(\mathbf{R}_+)$, $\Pi_0 T_t j(g,f) = \int_0^t g(\tau) h_f(\tau)\, d\tau$ and so as $\Pi_0 T_t\big(e(f) - e(0)\big) = \Pi_0 T_t j(f,f)$, $t \mapsto \Pi_0 T_t\big(e(f) - e(0)\big)$ has finite variations on compact sets. So we can apply the above result of [C].

5. Application to contractive cocycles in Fock space

By solving quantum stochastic differential equations of the form

$$dV_t = \sum_\varepsilon V_t L_\varepsilon\, da_t^\varepsilon,\ \ V_0 = \mathrm{Id} \tag{5.1}$$

in which $L_+ = L$, $L_0 = W - I$, $L_- = -L^*W$, $L_\times = iK - \frac{1}{2}L^*L$ and W, K, L are respectively fixed unitary, bounded selfadjoint and bounded operators, one obtains unitary valued Markovian cocycles (with respect to time shift on Fock space), or covariant adapted evolutions, whose reduced semigroup is continuous in norm [HP2]. Conversely, such Markovian cocycles are all solutions of quantum stochastic differential equations of the form (5.1). These results are shown in [H-L] by using the stochastic integral representation theorem for regular Fock martingales [P-S] and basic techniques of the Hudson-Parthasarathy calculus [HP1].

Clearly norm continuity is not satisfied by the most important semigroups. Thus it is interesting to weaken this assumption in order to establish a quantum stochastic analogue of Stone's theorem on one-parameter group of unitary operators. Journe in [J] investigated the strongly continuous case and showed, under a regularity condition, that a cocycle can be reconstructed from the infinitesimal generator by a recursive procedure on the finite particle subspaces. In general the generator will fail to have a common dense domain and the cocycle will not satisfy any q.s.d.e. Accardi, Journe and Lindsay in [A-J-L] prove that this cannot happen when the cocycle V satisfies a weak differentiability condition.

In the present part we show, by using the stochastic integral representation theorem (2.1), that under a condition on the cocycle (which is necessary and sufficient and weaker that the weak differentiability condition of [A-J-L]) that the unitary cocycle V satisfies a q.s.d.e. of the form (5.1).

Let S_t denote the right shift on $L^2(\mathbf{R}_+)$, so that for $t \geq 0$

$$(S_t f)(x) = \begin{cases} f(x-t) & \text{if } x \geq t \\ 0 & \text{otherwise .} \end{cases}$$

Let $\Gamma(S_t)$ be the second quantizations of S_t. $\Gamma(S_t)$ is isometric. For all $s \geq 0$ and all bounded operator X in $\mathcal{B}(\mathcal{H})$, the operator $\Gamma(S_s)X\Gamma(S_s)^*$ maps $h_0 \otimes \mathcal{H}_{[t}$ into itself. The canonical extension to $\mathcal{H}$ will be denoted by $\overline{\Gamma(S_s)X\Gamma(S_s)^*}$.

Definition 5.2. — *A family $(V_t)_{t\geq 0}$ of contraction is a cocycle if it satisfies $V_0 = I$ and for all $s, t \geq 0$, $V_{s+t} = V_s\overline{\Gamma(S_s)V_t\Gamma(S_s)^*}$.*

Let $(P_t)_{t\geq 0}$ defined on h_0 by $P_t u = \Pi_0 V_t u$. The next lemma is proved in [H-L].

Lemma 5.3. — *$(P_t)_{t\geq 0}$ is a semigroup of contractions on h_0.*

Definition 5.4. — *Let $(V_t)_{t\geq 0}$ be an adapted cocycle and $(T_t)_{t\geq 0}$ be defined by: for $u \in h_0$, $T_t u = \Pi_0 V_t(u \otimes \chi_t)$. We will say that $(V_t)_{t\geq 0}$ satisfy assumption (H) if*

(1) $(P_t)_{t\geq 0}$ is strongly continuous with a generator denoted Z on a domain denoted $\mathcal{D}(Z)$.

(2) there exists a dense domain $D \subset \mathcal{D}(Z)$ such that for all u in D, $\left\{\frac{T_t u}{t}\right\}_{t\geq 0}$ is bounded.

Definition 5.5. — *An adapted cocycle $(V_t)_{t\geq 0}$ is said to be weakly differentiable if there exists a dense domain D^V of h_0 such that for all u in D^V, for all v in h_0, for all $f, g \in L^2(\mathbf{R}_+) \cap \mathcal{C}_0(\mathbf{R}_+)$, the mapping $t \mapsto \langle v \otimes e(f), V_t(u \otimes e(g))\rangle$ is $\mathcal{C}^1$ on $\mathbf{R}_+$.*

Lemma 5.6. — *An adapted weakly differentiable cocycle $(V_t)_{t\geq 0}$ satisfies assumption (H).*

Proof. — If we take $f = g = 0$, we see that for all u in D^V, $P_t u - u \underset{t\to 0}{\longrightarrow} 0$ weakly in h_0 and as $(P_t)_{t\geq 0}$ is a contraction, $(P_t)_{t\geq 0}$ is strongly continuous so we have (1) of (H) with $D^V \subset \mathcal{D}(Z)$.

By using Banach Steinhaus theorem, we see easily that weak differentiability implies that for all u in D^v, for all $f \in L^2(\mathbf{R}_+) \cap \mathcal{C}^0$, $\Pi_0 \frac{V_t}{t}\big(u \otimes e(f) - u\big)$ is bounded, and so (2) of (H) is satisfied.

THEOREM 5.7. — *Let $(V_t)_{t\geq 0}$ be an adapted contractive cocycle satisfying hypothesis (H). Then there exist $(L_\varepsilon)_{\varepsilon\in\{-,+,\times,0\}}$ on h_0 with domain D such that $V_t = I + \int_0^t V_s L_\varepsilon \, da_s^\varepsilon$.*

Proof.

$\boxed{1}$ In order to apply theorem (2.1), we have to prove that for all F in $\mathcal{J}$, for all u in D, $(V_t\Pi_t(u\otimes F))_{t\geq 0}$ is an absolutely continuous quasimartingale.

First case: $F = e(0)$, $x_t = V_t(u\otimes e(0))$. Let $s<t$,

$$\begin{aligned}
\|\Pi_s x_t - x_s\| &= \|\Pi_s V_t(u\otimes e(0)) - V_s(u\otimes e(0))\|\\
&= \|\Pi_s V_s\overline{\Gamma(S_s)V_{t-s}\Gamma(S_s)^*}(u\otimes e(0)) - V_s(u\otimes e(0))\|\\
&\leq \|\Pi_s\overline{\Gamma(S_s)V_{t-s}\Gamma(S_s)^*}(u\otimes e(0)) - u\otimes e(0)\|\\
&= \|\Pi_0\Gamma(S_s)V_{t-s}\Gamma(S_s)^*(u\otimes e(0)) - u\otimes e(0)\|\\
&= \|P_{t-s}u - u\| \leq \|Zu\|(t-s)\,.
\end{aligned}$$

Second case: $x_t = V_t(u\otimes\chi_t)$

$$\begin{aligned}
\|\Pi_s x_t - x_s\| &= \|\Pi_s V_s\overline{\Gamma(S_s)V_{t-s}\Gamma(S_s)^*}(u\otimes\chi_t) - V_s(u\otimes\chi_s)\|\\
&\leq \|\Pi_s\overline{\Gamma(S_s)V_{t-s}\Gamma(S_s)^*}(u\otimes\chi_s) - u\otimes\chi_s\|\\
&\qquad\qquad + \|\Pi_s\overline{\Gamma(S_s)V_{t-s}\Gamma(S_s)^*}(u\otimes\chi_t-\chi_s\|\\
&\leq \|(P_{t-s}u-u)\otimes\chi_s\| + \|\pi_0\Gamma(S_s)V_{t-s}\Gamma(S_s)^*(u\otimes\chi_t-\chi_s)\|\\
&\leq \sqrt{s}(t-s)\|Zu\| + \|T_{t-s}u\|\,.
\end{aligned}$$

So there exists $C(u)\in\mathbf{R}$ such that $\|\Pi_s\chi_t - \chi_s\| \leq (t-s)C(u)$.

Third case: $F = \int_0^{+\infty} g(s)\,d\chi_s$ with $g\in L^2(\mathbf{R}_+)$ being a step function. Let $y_t(g) = \Pi_0 V_t(u\otimes\int_0^t g(\tau)\,d\chi_\tau)$ and $0\leq t_0<t_1<\cdots<t_{n-1}\leq t_n = t$ such that $1_{[0,t]}g = \sum_{i=0}^{n-1}\lambda_i 1_{[t_i,t_{i+1}]}$. So

$$\begin{aligned}
y_{t_k}(g) &= \Pi_0 V_{t_k}\Big(u\otimes\int_0^{t_{k-1}} g(\tau)\,d\chi_\tau\Big) + \lambda_{k-1}\Pi_0 V_{t_k}\big(u\otimes\chi_{t_k} - \chi_{t_{k-1}}\big)\\
&= \Pi_0\overline{\Gamma(S_{t_{k-1}})V_{t_k-t_{k-1}}\Gamma(S_{t_{k-1}})^*}\Big(u\otimes\int_0^{t_{k-1}} g(\tau)\,d\chi_\tau\Big)\\
&\qquad + \lambda_{k-1}\Pi_0\overline{\Gamma(S_{k-1})V_{t_k-t_{k-1}}\Gamma(S_{t_{k-1}})^*}(u\otimes\chi_{t_k} - \chi_{t_{k-1}})\\
&= y_{t_{k-1}}(g) + \Pi_0 V_{t_{k-1}}\Big(\big(P_{t_k-t_{k-1}}u - u\big)\otimes\int_0^{t_{k-1}} g(\tau)\,d\chi_\tau\Big)\\
&\qquad + \lambda_{k-1}P_{t_{k-1}}T_{t_k-t_{k-1}}u
\end{aligned}$$

and thus there exists $C(u)$ independent of g such that for all $k = 0, \dots, n-1$,

$$\|y_{t_k}(g) - y_{t_{k-1}}(g)\| \leq (t_k - t_{k-1})C(u) + |\lambda_{k-1}|(t_k - t_{k-1})C(u) .$$

This implies that $\|y_t(g)\| \leq \big(t + \int_0^t |g(\tau)|\, d\tau\big)\, C(u)$. And so for all u in D, there exist $C(u)$ such that for all g in $L^2(\mathbf{R}_+)$, for all t,

$$\Big\|\Pi_0 V_t(u \otimes \int_0^t g(\tau)\, d\chi_\tau)\Big\| \leq \Big(t + \int_0^t |g(\tau)|\, d\tau\Big) C(u) .$$

If $x_t = V_t\big(u \otimes \int_0^t g(\tau)\, d\chi_\tau\big)$,

$$\begin{aligned}\|\Pi_s x_t - x_s\| &\leq \|(P_{t-s}u - u) \otimes \int_0^s g(\tau)\, d\chi_\tau\| + \Big\|\Pi_0 V_{t-s}\Big(u \otimes \int_0^{t-s} g(s+\tau)\, d\chi_\tau\Big)\Big\| \\ &\leq \|g\|(t-s)\|Zu\| + \|y_{t-s}(g(s+\cdot))\| .\end{aligned}$$

Fourth case: $F = \int_0^{+\infty} g(s)\Pi_s e(f)\, d\chi_s$ with f, g in $L^2(\mathbf{R}_+)$.

$$x_t = V_t\Pi_t(u \otimes F) = V_t\Big(u \otimes \int_0^t g(s)\, d\chi_s\Big) + V_t\Big(u \otimes \int_0^t g(s)\big(\Pi_s e(f) - e(0)\big)\, d\chi_s\Big) .$$

So we only have to study

$$\begin{aligned}&\Big\|\Pi_s V_t\Big(u \otimes \int_0^t g(\tau)\big(\Pi_\tau e(f) - e(0)\big)\, d\tau\Big) - V_s\Big(u \otimes \int_0^s g(\tau)\big(\Pi_\tau e(f) - e(0)\big)\, d\tau\Big)\Big\| \\ &\qquad \leq \Big\|(P_{t-s}u - u) \otimes \int_0^s g(\tau)(\Pi_r e(f) - e(0))\, d\chi_\tau\Big\| \\ &\qquad\qquad + \Big\|\Pi_s \overline{\Gamma(S_s) V_{t-s} \Gamma(S_s)^*}\Big(u \otimes \int_s^t g(\tau)\big(\Pi_\tau e(f) - e(0)\big)\, d\chi_\tau\Big)\Big\| .\end{aligned}$$

But

$$\begin{aligned}\int_s^t g(\tau)\big(\Pi_\tau e(f) - e(0)\big)\, d\chi_\tau =& \big(\Pi_s e(f) - e(0)\big) \otimes \int_s^t g(\tau)\, d\chi_\tau \\ &+ \Pi_s e(f) \otimes \int_s^t g(\tau)\big(\Pi_\tau e(f 1_{[s,+\infty[}) - e(0)\big)\, d\chi_\tau\end{aligned}$$

and so

$$\begin{aligned}&\Big\|\Pi_s \overline{\Gamma(S_s) V_{t-s} \Gamma(S_s)^*}\big(u \otimes \int_s^t g(\tau)(\Pi_\tau e(f) - e(0))\, d\chi_\tau\big)\Big\| \\ &\qquad\qquad \leq \|\Pi_s e(f) - e(0)\|\, \Big\|\Pi_0 V_{t-s}\big(u \otimes \int_s^t g(\tau + s)\, d\chi_\tau\big)\Big\| \\ &\qquad\qquad\quad + \|\Pi_s e(f)\|\, \Big\|\int_s^t g(\tau)\big(\Pi_\tau e(f 1_{[s,+\infty[}) - e(0)\big)\, d\chi_\tau\Big\|\end{aligned}$$

and we can conclude by the third case because

$$\Big\|\int_s^t g(\tau)\big(\Pi_\tau e(f 1_{[s,+\infty[}) - e(0)\big)\, d\chi_\tau\Big\|^2 \leq \int_s^t |g(\tau)|^2\, d\tau\Big(e^{\int_s^t |f(\tau)|^2\, d\tau} - 1\Big) .$$

$\boxed{2}$ (2.2) and (2.3) are always satisfied by $(V_t)_{t\geq 0}$ because for all F in Φ and u in $\mathcal{D}(Z)$, for all $r < s < t$,

$$\|\Pi_s V_t(u\otimes \Pi_r F) - V_s(u\otimes \Pi_r F)\| \leq \|P_{t-s}u - u\|\,\|\Pi_r F\| \leq (t-s)\|Zu\|\,\|\Pi_r F\|$$

and

$$\begin{aligned}\|V_t(u\otimes \Pi_r F) - V_s(u\otimes \Pi_r F)\|^2 &\leq \|\Pi_r F\|^2 \times (-2)\Re(\langle u - P_{t-s}u, u\rangle)\\ &\leq 2\|\Pi_r F\|^2 (t-s)\|Zu\|\,\|u\|\,.\end{aligned}$$

$\boxed{3}$ So by theorem 2.1 we have the representation, $V_t = I + \sum_{\varepsilon} \int_0^t H_s^\varepsilon\, da_s^\varepsilon$. By using cocycle property, we can prove as in the proof of [H-L] that $H_s^\varepsilon = V_s L_\varepsilon$ with $(L_\varepsilon)_{\varepsilon\in\{\times,-,+,0\}}$ defined on D and $L_X = Z$. ∎

Remark. — One can prove that if $(P_t)_{t\geq 0}$ is continuous in norm then (H) is satisfied.

One can also prove that if $Z = iH + B$ with H selfadjoint and B bounded then (2) of (H) is satisfied too.

The hypothesis (2) of (H) is equivalent to:

$$\text{for all } u \text{ in } h_0, \text{for all } v \text{ in } D,\ \ t \longmapsto \langle u, V_t(v\otimes \chi_t)\rangle \text{ is } \mathcal{C}^1.$$

Fagnola in [F] gives a characterization theorem for weakly differentiable (contractive, isometric and unitary) Makovian cocycles in the Boson Fock space. He studies the converse result: given $(L_\varepsilon)_{\varepsilon\in\{\times,+,-,0\}}$ satisfying some condition, does there exist an unique solution of (5.1) which is a weakly differentiable cocycle?

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