Geometric study of the set $\mathbb{Z}_\beta$ of beta-integers with β a Perron number, a β - number and a Pisot number and mathematical quasicrystals

Jean-Louis VERGER-GAUGRY and Jean-Pierre GAZEAU[†]

Prépublication de l'Institut Fourier n° 513 (2000)

http://www-fourier.ujf-grenoble.fr/prepublications.html

Institut Fourier, UJF Grenoble, UFR de Mathématiques -
CNRS UMR 5582, BP 74 - Domaine Universitaire,
38402 Saint Martin d'Hères, France.
email : jlverger@ujf-grenoble.fr

[†] *LPTMC, Université Paris 7 Denis Diderot -*
mailbox 7020, 2 place Jussieu,
75251 Paris Cedex 05, France.

Abstract

We investigate in a geometrical way the sieving process of $\mathbb{Z}[\beta]$ for obtaining the Delone set $\mathbb{Z}_\beta \subset \mathbb{Z}[\beta]$ of β - integers where β is a Perron number in the context of linear asymptotic invariants associated with a canonical inductive system constructed from β . When β is a Pisot number, we exhibit a canonical cut-and-project scheme, a model set associated with $\mathbb{Z}_\beta$ and so prove that it is a Meyer set. We show how to lift up the elements of $\mathbb{Z}_\beta$ to a subset $\pm\mathscr{L}_{\mathscr{K}}$ of the lattice $\mathbb{Z}^m$ (m =degree β) , lying about the dominant eigenspace of the companion matrix of β . We deduce from this linearized version of $\mathbb{Z}_\beta$ (i) the existence of a finite number of elements $g_1, g_2, \cdots, g_\eta \in \mathbb{Z}_\beta^+$ of small norm such that the semi-group $\mathbb{N}[g_1, g_2, \cdots, g_\eta]$ contains $\mathbb{Z}_\beta^+$ except possibly a finite number of elements close to the origin, (ii) an upper bound for the integer $\mathscr{L}$ taking place in the relation

$$x, y \in \mathbb{Z}_\beta \implies x + y \quad (\text{resp.} x - y) \in \frac{1}{\beta^{\mathscr{L}}}\mathbb{Z}_\beta$$

if $x + y$ and $x - y$ have finite β- expansions.

1. Introduction

Gazeau [43], Burdik et al [2] have shown how to construct a Delone-Meyer set $\mathbb{Z}_\beta$ [37] from the dense finitely generated $\mathbb{Z}$ - module $\mathbb{Z}[\beta] \subset \mathbb{R}$, where $\beta > 1$ is a Pisot-Vijayaraghavan number

---

Math. classification: 11,52.
Keywords: quasicrystal, Perron number, Pisot number, β-numeration.

(called Pisot- or PV-number also) of degree $m > 1$, *i.e.* a real algebraic integer, root of an irreducible polynomial of the form $X^m - a_{m-1}X^{m-1} - a_{m-2}X^{m-2} - \cdots - a_1X - a_0$, $a_i \in \mathbb{Z}$ where all the Galois conjugates $\beta^{(i)}, i = 1, 2, \cdots, m-1$ of $\beta = \beta^{(0)}$ satisfy $|\beta^{(i)}| < 1$ for all $i = 1, 2, \cdots, m-1$. The *sieving* process, obtained algebraically, is based on the β - expansion of real numbers [4] [5], and provides uniformly discrete and relatively dense sets of points in $\mathbb{R}$.

The aims of the present work consist in studying asymptotic linear invariants (cut-and-project schemes, ...) in $\mathbb{R}^m$, canonically associated with Delone-Meyer sets $\mathbb{Z}_\beta$, and the geometrical counterpart of this sieving process. Here β is a Perron number, a β -number or a Pisot number, of degree m . In §3, we redefine $\mathbb{Z}_\beta$ as an inductive limit $\mathscr{K}$. We show in §4 that the asymptotic linear invariants associated with this inductive limit arise from the real Jordan decomposition of $\mathbb{R}^m$ under the action of the companion operator of β , and from the real and complex embeddings of the number field $\mathbb{Q}(\beta)$ when β is an arbitrary Perron number. This allows us to construct explicitely suitable cut-and-project schemes and to define the linearized version of $\mathbb{Z}_\beta$. The latter is called $\pm\mathscr{L}_{\mathscr{K}}$ and is a discrete subset of $\mathbb{Z}^m$. When β is a Pisot number, we investigate in §5 the additive properties of $\pm\mathscr{L}_{\mathscr{K}}$. In particular, we give, thanks to this geometrical approach, new proofs of some recent results using the dynamics of self-similar tilings and the symbolics of β - numeration applied to the aperiodic tilings of the real line [20, 21, 2, 18, 28, 29]. Namely, we provide a geometrical interpretation of the finite sets T and T' in the relations [2]

$$\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+ + T \tag{1}$$

$$\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta + T' \tag{2}$$

and an upper bound of the integer $\mathscr{L}$ in

$$x, y \in \mathbb{Z}_\beta^+ \Longrightarrow x \pm y \in \mathbb{Z}_\beta / \beta^{\mathscr{L}} \tag{3}$$

when $x + y$ and $x - y$ have finite β-expansions. Masakova et al [29], Frougny et al [20, 2, 18] proved that an additive law $\oplus$ can be set on $\mathbb{Z}_\beta$ for which $(\mathbb{Z}_\beta, \oplus)$ is isomorphic (but not isometric) to $(\mathbb{Z}, +)$ for some quadratic Pisot numbers β . Here, we give a geometrical description of a finite subset of $\mathbb{Z}_\beta$ for which any element of $\mathbb{Z}_\beta$, except possibly a finite number of them, can be expressed as an integral nonegative combination of them. In other terms, it generates a semi-group of finite type covering almost entirely $\mathbb{Z}_\beta$. This uses the existence of a covering radius of $\mathscr{L}_{\mathscr{K}}$.

2. Context of Mathematical Quasicrystals and Classification of Delone Sets

Cut-and-project schemes and model sets are nowadays commonly used for modelling quasicrystals with suitable choices of windows and parameters in the internal space [32]. In such a state of matter, there is no average lattice. On the contrary, incommensurate structures do have an average lattice. For both of them, a periodization in a space of higher dimension is performed and crystallographic groups in dimension higher than 3 are necessary to understand the geometry of atomic sites [33]. But the new definition of a *crystal* [34] covers much more than quasicrystals or incommensurate structures. It is basically a Delone set for which the spectral measure is pure point. The definition of a crystal has been extended in 1991 to take into consideration aperiodic crystals in general. There exist various classes of Delone sets which are more general than model sets, so-called mathematical quasicrystals [14] [15] [16], generically named in an attempt to cover the field of aperiodic crystals. Just above the class of model sets is the class of Meyer sets [36] since a Meyer set is always a subset of a certain model set (Theorem 9.1 in [37]). A Meyer set is already not necessarily a *crystal* (see for instance Verger-Gaugry et al [40] [41] with the Thue-Morse quasicrystal). There is a lack of criteria for saying that a Delone set is deprived of diffuse (continuous) measures in its spectrum. Of course, it is a formidable task to find such criteria, if any [38] [39]. This objective seems unreachable at present. Delone sets can be roughly classified by the complexity of local configurations. When G is a finite group acting on $\mathbb{R}^k$, $k \leqslant n$, and X is a G - cluster in $\mathbb{R}^k$ of total length n or $2n$ (total number of points in the finite union of orbits of points which constitute X, viewed as a shellable discrete object), the construction of a cut-and-project scheme is canonical from this G - cluster in $\mathbb{R}^n$ [42]. Here, local order is fairly regular in the sense it is *globally* controlled by a finite symmetry group. This allows to introduce naturally G - clustering in Meyer sets. Such Delone sets inherit local structures in G - clusters by partial repetition and translation. As Meyer sets on the real line, cite the Thue-Morse quasicrystal [40], G-cluster sets [42], $\mathbb{Z}_\beta$ when β is a Pisot number [2], sets of vertices of aperiodic tilings [38] [45] [46] [47] [48] [21], constructions from algebraic numbers [29] [44], etc. A Delone set Λ is said to be a finitely generated Delone set if $\mathbb{Z}[\Lambda - \Lambda]$ is finitely generated [14] [15]. It is said to be a Delone set of finite type if $\Lambda - \Lambda$ is such that its intersection with any closed ball is finite. The class of finitely generated Delone sets is strictly larger than the class of Delone set of finite type. Obviously, the latter one contains the class of Meyer sets.

Hof [30] [31] has developed the mathematics of diffraction for arbitrary Delone sets through the notion of autocorrelation measure. The class of Delone sets which can be called crystals intersects a priori all the classes of Delone sets, i.e. mathematical quasicrystals, which are mentioned above. Model sets are crystals. Indeed, Hof [30] has proved that all model sets, i.e. quasiperiodic point sets obtained with windows of boundaries of Lebesgue measure zero in the internal space, have a pure point spectrum. See also a review article by Moody about diffraction of model sets in [35]. Another direction to study the spectral measure of Delone sets was carried out by Verger-Gaugry et al [41] by decomposing the Fourier transform of the autocorrelation measure of an arbitrary Delone set on lattices which intersect the given Delone set and by characterizing arithmetically the rarefaction laws of points and their critical exponents, associated with their distributions at infinity.

Diffraction laws of the set of β- integers will be reported elsewhere.

3. An inductive system

For all $x \in \mathbb{R}$, denote by $\lfloor x \rfloor$ the usual integer part of x, and by $\{x\} = x - \lfloor x \rfloor$ its usual fractional part. We follow the presentation given in [2] for β - expansions of real numbers. If

$(\alpha_0, \alpha_1, \cdots)$ and $(\gamma_0, \gamma_1, \cdots)$ are finite or infinite sequences of non-negative integers with the same number of terms, we write

$$(\alpha_0, \alpha_1, \cdots) < (\gamma_0, \gamma_1, \cdots)$$

when $\alpha_n < \gamma_n$ for the first n for which $\alpha_n \neq \gamma_n$ (lexicographical order). Let $\beta > 1$ be a real number. We write $(\alpha_0, \alpha_1, \alpha_2, \cdots)(\beta)$ for

$$\alpha_0 + \frac{\alpha_1}{\beta} + \frac{\alpha_2}{\beta^2} + \cdots$$

A representation in base β of a real number $x \geqslant 0$ is an infinite sequence of integers $(x_i)_{k \geqslant i \geqslant -\infty}$, such that

$$x = x_k \beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1 \beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots$$

for a certain integer $k \geqslant 0$, where $x_i \in \{0, 1, \cdots, [\beta]\}$ if $\beta \notin \mathbb{N}$, and $x_i \in \{0, 1, \cdots, \beta - 1\}$ if $\beta \in \mathbb{N}$. Denote $A = \{0, 1, \cdots, [\beta]\}$ in the first case. If β is an integer, A will be $\{0, 1, \cdots, \beta - 1\}$ instead, but we are mostly interested in this contribution by algebraic numbers β of degree greater than or equal to 2 and we will discard this second case. The *integral part* of x is $x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1\beta + x_0$ and the *fractional part* of x is $x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots$. Such a representation of x can be computed by the so-called "greedy algorithm". There exists $k \in \mathbb{Z}$ such that $\beta^k \leqslant x < \beta^{k+1}$. This gives $x_k = [x/\beta^k]$. Let $r_k = \{x/\beta^k\}$. Then, for $k > i \geqslant -\infty$, put $x_i = [\beta r_{i+1}]$ and $r_i = \{\beta r_{i+1}\}$. If $k < 0$ $(x < 1)$, we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$.

On the other hand, Renyi [4] has developed for a non-integer β the notion of "f-expansion" with $f(x) = 1$ for $\beta < x$, $f(x) = x/\beta$ for $0 \leqslant x \leqslant \beta$. This process gives a representation of x in base β for its fractional part: $x = x_0.x_{-1}x_{-2}x_{-3}\ldots$, where $x_0 = \lfloor x \rfloor$, $x_{-1} = \lfloor \beta\{x\} \rfloor$, $x_{-2} = \lfloor \beta\{\beta\{x\}\} \rfloor$, and so on. The sequences $(x_{-1}, x_{-2}, x_{-3}, \ldots)$ (forgetting about the x_0) obtained in this way form a subset in $A^{\mathbb{N}}$ invariant under the shift: $\sigma : (x_{-1}, x_{-2}, x_{-3}, \ldots) \to (x_{-2}, x_{-3}, x_{-4}, \ldots)$, whose closure takes the name of β - shift. $T(x) = \{\beta x\}$ is an ergodic transformation sending $[0, 1)$ onto itself. We will call any closed σ - invariant subset S of $A^{\mathbb{N}}$ a symbolic dynamical system on A. The language $L(S)$ associated with S is the set of words which can be built from S:

$$(y_n)_{n \geqslant 0} \in S \quad \Leftrightarrow \quad y_p \ldots y_q \in L(S) \quad \text{for all } p \leqslant q$$

and the real number whose representation in base β is $y_p \ldots y_q$ is $(y_p, \ldots, y_q)(\beta)$. The Renyi representation of $x = x_0.x_{-1}x_{-2}x_{-3}\ldots$ in base β takes the name of β- expansion of x. From the greedy algorithm, we see that the representation of β in base β is $\beta = 1.\beta$, whereas the β - expansion of β is, say

$$\beta = t_1 + \frac{t_2}{\beta} + \frac{t_3}{\beta^2} + \ldots$$

where $t_1 = \lfloor \beta \rfloor$, $t_i \in A$. Setting $T^0(x) = x$ and inductively $T^n(1) = T^{n-1}(\{\beta\}) = \{\beta T^{n-1}(1)\}$ for $n \geqslant 1$ we have canonically $t_i = \lfloor \beta T^{i-1}(1) \rfloor$ for $i \geqslant 1$. Dividing the above equation by β itself gives the so-called *Rényi* β - expansion of 1, $d_\beta(1) = t_1\beta^{-1} + t_2\beta^{-2} + \cdots = 0.t_1 t_2 \cdots$, while the representation in base β of 1 given by the greedy algorithm is 1. We will say that the β -

expansion of β is finite when $\beta = (t_1, t_2, \cdots, t_m)(\beta)$ with some integer $m \geqslant 1$. In the following, we will denote

$$c_1 c_2 c_3 \cdots \quad = \begin{cases} t_1 t_2 t_3 \cdots & \text{if the } \beta \text{ - development } d_\beta(1) = 0.t_1 t_2 \cdots \text{ is infinite} \\ (t_1 t_2 \cdots t_{m-1}(t_m - 1))^\omega & \text{if } d_\beta(1) \text{ is finite and equal to } 0.t_1 t_2 \cdots t_m \end{cases} \tag{1}$$

where $(\ )^\omega$ means that the word within $(\ )$ is indefinitely repeated. Since $m \geqslant 2$, we always have $c_1 = t_1 = \lfloor \beta \rfloor$.

Theorem 3.1. — *[5] If the β - expansion of β is*

$$\beta = t_1 + \frac{t_2}{\beta} + \frac{t_3}{\beta^2} + \cdots \tag{2}$$

with $t_i \in A$ for all $i \geqslant 1$ and if $(b_0, b_1, \cdots)$ is a sequence of non-negative numbers whose tail, in the case in which the β - expansion of β is finite, does not coincide with $\{c_n\}$, a necessary and sufficient condition for the existence of x with β - expansion

$$x = b_0 + \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \cdots$$

is that

$$(b_n, b_{n+1}, \cdots) < (t_1, t_2, \cdots) \tag{3}$$

for all $n \geqslant 1$. In particular, $(t_n, t_{n+1}, \cdots) < (t_2, t_3, \cdots)$ for all $n \geqslant 2$.

This is a maximality condition for the elements of S. We now extend the β - shift to $A^{\mathbb{Z}}$. This means that we keep this maximality condition to sieve the infinite sequences of integers $(x_i)_{k \geqslant i \geqslant -\infty}$, such that $x = x_k x_{k-1} \cdots x_1 x_0 . x_{-1} x_{-2} \cdots$ is a positive real number. Only certain representations become allowed.

Definition 3.2. — *Denote $\mathbb{Z}_\beta^+ = \{x_k \beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1 \beta + x_0 \mid x_i \in A, k \geqslant 0$, and $(x_j, x_{j-1}, \cdots, x_1, x_0, 0, 0, \cdots) < (c_1, c_2, \cdots)$ for all $j, 0 \leqslant j \leqslant k\}$ the discrete subset of $\mathbb{R}^+$. The set $\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup \left(-\mathbb{Z}_\beta^+\right)$ is called the set of β - integers.*

$\mathbb{Z}_\beta^+$ is the set of integral parts of β - developments. Set $\mathbb{Z}_\beta^- = -\mathbb{Z}_\beta^+$. The Rényi β - expansion of 1 and the maximality condition with $c_1 c_2 \ldots$ are then sufficient to exhaust all the possibilities of enumeration of β - expansions of β - integers. Note that the element 1 belongs to $\mathbb{Z}_\beta^+$, but the Rényi β - expansion of 1 does not; of course, its integral part is 0 and 0 belongs to $\mathbb{Z}_\beta^+$.

For instance, with $\tau = \frac{1+\sqrt{5}}{2}$, $t_1 t_2 \cdots = 110\ldots$ and $c_1 c_2 \cdots = 1010\ldots$. No sequence $x_k x_{k-1} \ldots$ representing an element x of $\mathbb{Z}_\tau$ contains the lexicographically impossible word 11.

Those $w = \sum_{j=-k}^{+\infty} w_{-j} \beta^{-j}$ developed by the greedy algorithm which obey the maximality condition $(w_{-j}, w_{-j-1}, \ldots) < (c_1, c_2, \ldots)$ for all $j \geqslant 1$ and for which $w_{-(p+s)} = w_{-p}$ for all $p \geqslant p_0$, where $p_0 \geqslant 1$, for some positive integer s, are said to have a recurrent tail and are named

β - numbers. Their β - expansion is periodic after a certain rank. Those β - numbers having a tail composed of zeroes after a certain rank are called simple β - numbers [5]. It is known that β - numbers are algebraic integers of degree $p_0 + s$ (with s minimal).

Let us turn now to algebraic numbers. When β is real positive and an algebraic integer, it is the solution of an irreducible (minimal) polynomial of the form,

$$P(X) = X^m - a_{m-1}X^{m-1} - a_{m-2}X^{m-2} - \cdots - a_1 X - a_0 \quad , a_i \in \mathbb{Z} \tag{4}$$

with m = degree $\beta \geqslant 1$. Replacing X by β in the above polynomial and dividing the equality by β^{m-1}, we can see that β is a simple β - number of β - expansion $a_{m-1} + a_{m-2}\beta^{-1} + \cdots + a_1\beta^{-m+2} + a_0\beta^{-m+1}$ when the following condition is fulfilled

$$(a_n, a_{n+1}, \cdots) < (a_{m-1}, a_{m-2}, \cdots, a_0, 0, 0, \cdots) \tag{5}$$

for all $n \leqslant m-2$, with all a_j's positive integers. This condition is restrictive and the coefficients of the above minimal polynomial do not obey necessarily this rule. Note that the results obtained by Parry [5] are valid for polynomials in $\mathbb{Z}[\beta]$ for which β is a root but that are not necessarily irreducible. In other terms, it may exist a polynomial $P_P(X) = X^{m'} - \sum_{n=0}^{m-1} a'_n X^n$ (the letter "P" is for Parry) in the ideal $P(X)\mathbb{Z}[\beta]$ which has a dominant coefficient equal to 1 and the other coefficients a'_j negative such that similar inequalities as (5) are fulfilled with the a'_j's, but such that the inequalities (5) are not fulfilled with the coefficients of $P(X)$. Relations between the minimal polynomial $P(X)$ and $P_P(X)$ were already outlined by Frougny and Solomyak [20].

Links between β - numbers and other algebraic numbers have already been investigated.

THEOREM 3.3. — *(Bertrand [7], Schmidt [8]) Let θ be a Pisot number. We have*

$$x \in \mathbb{Q}(\theta) \iff \text{the } \theta \text{ - expansion of } x \text{ has a recurrent tail} \tag{6}$$

In particular, θ is a θ - number.

Since the set of Pisot numbers is closed (Cassels [9], Chapter VIII, Theorem III) and that the set of simple β - numbers is everywhere dense in $[1; +\infty[$ (Parry [5], Theorem 5), there are many more β - numbers on the positive real line than Pisot numbers. Their behaviour is closely related since the conjugates of a β - number are also bounded : they have an absolute value less than 2 ([5], Theorem 4). Pisot - numbers are interesting since they provide Meyer sets $\mathbb{Z}_\beta$ by the sieving process of $\mathbb{Z}[\beta]$ (Burdik et al [2], Theorem 2.4).

A Perron number β is a real algebraic integer $\beta \geqslant 1$ whose remaining conjugates $\beta^{(i)}$ are of absolute value strictly less than β. A Lind number β has the same definition except we allow at least one conjugate of β to have β as absolute value (Lagarias [14] has named such algebraic integers from Lind's works [17]; similarly Douglas Lind had previously proposed the terminology Perron numbers for some algebraic integers from Perron's works). From [5], all β - numbers $\geqslant 2$ are Perron numbers. In the present contribution, we are dealing with Perron numbers where the conjugates lie within certain discs centred at the origin in the complex plane (for instance, for Pisot numbers, the open unit disc). Perron has proved that, when all the conjugates of β ($\beta > 1, \deg\beta = m$) belong to the open unit disc, then the polynomial $P(X) \in \mathbb{Z}[X]$, satisfying

$P(\beta) = 0, \deg P(X) = m$ and leading coefficient equal to 1, is necessarily irreducible over $\mathbb{Q}$ or over a quadratic imaginary field. Results about irreducibility can be found in Brauer [6]. We will assume throughout this paper that $P(X)$ is irreducible, but it is a weak assumption as soon as $m > 2$ and β a Pisot number.

We now assume that $\beta > 1$ is an algebraic integer, without any restriction on the coefficients of its minimal polynomial $P(X)$. We will construct an inductive system $(\mathscr{A}_n, f_n^p)_{p\geqslant n\geqslant m}$ using the minimal polynomial of β. In the inductive limit $\mathscr{E}$ of this inductive system, we will characterize a discrete subset $\mathscr{K} \subset \mathscr{E}$ associated with the supplementary conditions (3) arising from the existence of β - expansions and represent $\mathbb{Z}_\beta$ from $\mathscr{K}$, that is from some selected points lying in the lattice $\mathbb{Z}^m$ in $\mathbb{R}^m$. The inductive system associated with β is given by a collection of $\mathbb{Z}$ - modules

$$\mathscr{A}_n = \mathbb{Z}^n \text{ for all integer } n \geqslant m$$

and arrows $f_n^p : \mathscr{A}_p \longrightarrow \mathscr{A}_n$, $p \geqslant n$, which are $\mathbb{Z}$ - linear maps, constructed as follows. For any $p = n$, $f_p^p = Id$. For any $n \geqslant m$,

$$f_n^{n+1} : \mathscr{A}_{n+1} \longrightarrow \mathscr{A}_n \ : \ (x_n, x_{n-1}, \cdots, x_1, x_0) \longrightarrow (x'_{n-1}, x'_{n-2}, \cdots, x'_1, x'_0)$$

such that $\sum_{i=0}^{n} x_i\beta^i = \sum_{j=0}^{n-1} x'_j\beta^j$ by replacing β^m by its polynomial expression $a_{m-1}\beta^{m-1} + a_{m-2}\beta^{m-2} + \cdots + a_1\beta + a_0$ in the term of highest degree $x_n\beta^n = (x_n\beta^{n-m})\beta^m$. We obtain $x'_j = x_n a_j + x_j$, for all $j \in \{n-m, n-m+1, \cdots, n-1\}$ and $x'_j = x_j$ for $j \in \{0, 1, \ldots, n-m-1$. We set $f_n^p = f_{p-1}^p \circ f_{p-2}^{p-1} \circ \cdots \circ f_n^{n+1}$ for all $p > n$. These transition mappings are transitive by construction : for all integers i, k, j such that $i \geqslant k \geqslant j \geqslant m$, $f_j^i = f_k^i \circ f_j^k$. There are several ways to reduce the expression $\sum_{i=0}^{n} x_i\beta^i$ instead of only transforming the term of highest degree. However, since $\{1, \beta, \beta^2, \cdots, \beta^{m-1}\}$ is a free system in $\mathbb{Q}(\beta)$ (its discriminant is $(-1)^{\frac{m(m-1)}{2}} N_{\mathbb{Q}(\beta)/\mathbb{Q}}(P'(\beta)) \neq 0$, where $N_{\mathbb{Q}(\beta)/\mathbb{Q}}$ is the usual algebraic norm) and a basis in $\mathbb{Z}[\beta]$, the decomposition $\sum_{i=0}^{n} x_i\beta^i$ into $\sum_{i=0}^{m-1} x''_i\beta^i$ becomes unique at the rank $n = m$.

$$\begin{tikzcd}
 & & \mathscr{E} & & \\
\cdots \arrow[r, "f_{n+1}^{n+2}"] & \mathscr{A}_{n+1} \arrow[ur] \arrow[r, "f_n^{n+1}"] \arrow[d, "g_{n+1}"] & \mathscr{A}_n \arrow[u] \arrow[r, "f_{n-1}^{n}"] \arrow[d, "g_n"] & \cdots \arrow[r, "f_m^{m+1}"] & \mathscr{A}_m \arrow[ull] \arrow[d, "g_m"] \\
\cdots \arrow[r, phantom, "="] & \mathbb{Z}[\beta] \arrow[r, phantom, "="] & \mathbb{Z}[\beta] \arrow[r, phantom, "="] & \cdots \arrow[r, phantom, "="] & \mathbb{Z}[\beta]
\end{tikzcd}$$

Figure 1: The inductive limit $\mathscr{E}$ isomorphic to $\mathbb{Z}[\beta]$.

In the (external) sum $\bigoplus_{n=m}^{+\infty} \mathscr{A}_n$, we have the equivalence relation which identifies $X_i \in \mathscr{A}_i$ and $X_j \in \mathscr{A}_j$:

$$X_i \ \mathscr{R} \ X_j \iff \text{there exists } k, \text{ with } m \leqslant k \leqslant i, j \text{ such that } f_k^i(X_i) = f_k^j(X_j)$$

For all $n \geqslant m$, denote $g_n : \mathscr{A}_n \longrightarrow \mathbb{Z}[\beta] : (x_{n-1}, x_{n-2}, \cdots, x_1, x_0) \longrightarrow \sum_{d=0}^{n-1} x_j\beta^j$. It is a morphism of $\mathbb{Z}$ - modules, and g_m is one-to-one. We have $X_i \ \mathscr{R} \ X_j$ if and only if $f_m^i X_i = f_m^j X_j$, which is equivalent to $g_m(f_m^i X_i) = g_m(f_m^j X_j)$. We deduce the $\mathbb{Z}$ - isomorphism (for $+$)

$$\mathscr{E} \;=\; \lim\ \mathrm{ind}\,\mathscr{A}_n \;\simeq\; \mathbb{Z}[\beta]$$

Since $\mathbb{Z}[\beta]$ and $\mathbb{Z}[X]/\left(P(X)\right)$ are isomorphic as rings, $\mathscr{E}$ is naturally endowed with a ring structure. Consider now the subsets, for all $n \geqslant m$

$$\mathscr{K}_n := \{(x_{n-1}, x_{n-2}, \cdots, x_1, x_0) \in A^n \mid x_{n-1} \neq 0, \text{ and, for all } d \leqslant n-1, \\ (x_d, x_{d-1}, \cdots, x_0, 0, 0, \cdots) < (c_1, c_2, \cdots)\}$$

We have the inclusions $\mathscr{K}_n \subset \mathscr{A}_n$, $n \geqslant m$, and $g_n(\mathscr{K}_n) \subset \mathbb{Z}_\beta^+$ is the set of positive β - expansions with strictly n digits with no fractional part. This set is finite and

$$\mathbb{Z}_\beta \;=\; \bigcup_{n=m}^{+\infty} g_n(\pm\mathscr{K}_n) \tag{7}$$

The diagram in figure 1 is commutative. The subsystem $\left(\mathscr{K}_n, f_n^p\right)$, using the equivalence relation $\mathscr{R}$, gives

$$\mathbb{Z}_\beta \;=\; g_m\left(\bigcup_{n=m}^{+\infty} f_m^n(\pm\mathscr{K}_n)\right)$$

as a Delone subset of $\mathbb{Z}[\beta]$. This provides $\mathscr{K} \;=\; \lim\ \mathrm{ind}\,\mathscr{K}_n \;\subset\; \mathscr{E}$ in the inductive limit. The arrows f_n^p are $\mathbb{Z}$ - linear, but the sets $\mathscr{K}_n, n \geqslant m, \mathscr{K}$ have no algebraic structure a priori. However, Burdik et al [18] have recently shown that $\mathscr{K}$ is endowed with an internal law $\oplus$ when $m = 2$, for some values of the minimal polynomial: when it is $X^2 - aX + 1$, with $a \geqslant 3$ and $X^2 - aX - 1$, with $a \geqslant 1$, and $(\mathscr{K}, \oplus)$ is a group isomorphic to $(\mathbb{Z}, +)$.

4. Linear asymptotic invariants

The arrow $g_m : \mathbb{Z}^m \to \mathbb{Z}[\beta]$ in the construction of the inductive limit $\mathscr{E}$ is a $\mathbb{Z}$ - isomorphism. Therefore, we have a bijection $g_m : \bigcup_{n=m}^{+\infty} f_m^n(\pm\mathscr{K}_n) \subset \mathbb{Z}^m \to \mathbb{Z}_\beta$. We investigate now the image of $\mathbb{Z}_\beta$ under g_m^{-1}. We will show that the inductive limit $\mathscr{K}$ is canonically associated with a cut-and-project scheme around some eigensubspaces in $\mathbb{R}^m$ and points on $\mathbb{Z}^m$ gathered around these eigensubspaces.

Lemma 4.1. — *For all integer $k \geqslant m$, $\mathscr{K}_k$ is not empty. It contains at least the elements $(r, 0, 0, \cdots, 0)$ for any $r \in \{1, 2, \cdots, c_1\}$ where $c_1 = t_1 = \lfloor\beta\rfloor$ is the first digit in the β - expansion of 1.*

Proof. — Recall that we have assumed $m \geqslant 2$. Since $\beta > 1$, $c_1 = t_1 = \lfloor\beta\rfloor$ is at least equal to 1. The condition $(r, 0, 0, \cdots, 0, \cdots) < (c_1, c_2, \cdots)$ ensures that all the elements $(r, 0, 0, \cdots, 0)$ are in $\mathscr{K}_k$. $\square$

Applying g_k to such elements $(r, 0, 0, \cdots, 0)$, we see that we are looking for the linear asymptotic behaviour of the elements $r\beta^k$ in $\mathbb{R}^m$, with $r \in \{1, 2, \cdots, c_1\}$. By linearity, it suffices to understand the linear asymptotic behaviour of β^k, that is of $f_m^k((1,0,0,\cdots,0)) = g_m^{-1} \circ g_k((1,0,0,\cdots,0)) = g_m^{-1}(\beta^k) \in \mathbb{R}^m$ when k tends to infinity. This will also be sufficient to understand the linear asymptotic behaviour of any polynomial $r\beta^k + r_{k-1}\beta^{k-1} + \cdots + r_2\beta^2 + r_1\beta + r_0$ whose dominant monomial is $r\beta^k$. The number of such polynomials is given by the number of words of length $k+1$ which are lexicographically possible.

THEOREM 4.2. — *[19] Let $0.t_1t_2t_3\cdots$ the Rényi development of 1. The number d_k of words $r_kr_{k-1}r_{k-2}\ldots r_2r_1$, $r_i \in A$ of length k $(r_k \neq 0)$ of the language $L(\beta)$ is given by the following recurrence relations:*
- if $t_1, t_2, \cdots$ is not ending with zeroes, then $d_0 = 1, \cdots, d_k = t_1d_{k-1} + t_2d_{k-2} + \cdots + t_kd_0 + 1$,
- if $t_1, t_2, \cdots$ is ending with zeroes ($t_i \neq 0$ and $t_{i+1}t_{i+2}\cdots = 00\cdots$) then:

$$d_0 = 1, \cdots, d_k = t_1d_{k-1} + t_2d_{k-2} + \cdots + t_kd_0 + 1 \ \textit{for all}\ k = 1, 2, \cdots i-1$$
$$d_k = t_1d_{k-1} + t_2d_{k-2} + \cdots + t_id_{k-i}, \quad k \geqslant i$$

We have

$$\lim_{k\to\infty} \frac{d_k}{\beta^k} = \frac{1}{\beta - 1}\left(\frac{t_1}{\beta} + \frac{2t_2}{\beta^2} + \frac{3t_3}{\beta^3} + \cdots\right) \tag{1}$$

For simple β- numbers, A.Bertrand-Mathis [24] has developed other formulas for d_k from the matrix associated with the Markov topological β- shift.

A TRIVIAL REMARK .— If the Rényi development of 1 is finite and equal to $0.a_{m-1}a_{m-2}\cdots a_1a_0$ so that β becomes a simple β-number $(t_i = a_{m-i})$, then

$$\frac{P'(\beta)}{\beta^{m-1}(\beta-1)} = \lim_{k\to+\infty} \frac{d_k}{\beta^k} \tag{2}$$

where $P(X)$ is given by (4). The limit $\lim_{k\to+\infty} \frac{d_k}{\beta^k}$ tells us how many words we have asymptotically relatively to the powers of β (it *measures in the intuitive sense* the richness of the language $L(\beta)$). In the case in which β is a β - number, we obtain *rich* languages when the ratio $\frac{P'(\beta)}{\beta^{m-1}(\beta-1)}$ is large. Obviously, when β is an arbitrarily large β-number (or Pisot number), this ratio goes to zero since all the conjugates remain in a fixed disc in the complex plane. For instance, for $L(\tau)$, with τ root of $P(X) = X^2 - X - 1$, we have $1 = 0.11$ as τ - expansion and $\lim_{k\to+\infty} \frac{d_k}{\tau^k} = \sqrt{5}$ which is comparatively large; on the contrary, the language $L(\beta)$ with β the Pisot number, root of the equation $P(X) = X^2 - aX - 1$, $a > 1$ is poorer and poorer when a is tending to infinity.

Let us go back to the general situation. For all $k \geqslant 0$, write $\beta^k = z_{m-1,k}\beta^{m-1} + z_{m-2,k}\beta^{m-2} + \cdots + z_{1,k}\beta + z_{0,k}$, where all the integers $z_{0,k}, z_{1,k}, \cdots, z_{m-1,k}$ belong to $\mathbb{Z}$. Denote

$$Z_k = \begin{pmatrix} z_{0,k} \\ z_{1,k} \\ z_{2,k} \\ \vdots \\ z_{m-1,k} \end{pmatrix} \qquad B = B^{(0)} = \begin{pmatrix} 1 \\ \beta \\ \beta^2 \\ \vdots \\ \beta^{m-1} \end{pmatrix} \qquad B^{(j)} = \begin{pmatrix} 1 \\ \beta^{(j)} \\ {\beta^{(j)}}^2 \\ \vdots \\ {\beta^{(j)}}^{m-1} \end{pmatrix}$$

where the elements $\beta^{(j)}, j \in \{1, 2, \cdots, m-1\}$ are the conjugate roots of $\beta = \beta^{(0)}$ in the minimal polynomial of β. The transposed vector of Z_k is denoted by ${}^t Z_k$. Set

$$\mathscr{B}_k = \begin{pmatrix} \beta^k \\ {\beta^{(1)}}^k \\ {\beta^{(2)}}^k \\ \vdots \\ {\beta^{(m-1)}}^k \end{pmatrix} \quad \text{and } Q = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & & \cdots & 1 \\ a_0 & a_1 & & \cdots & a_{m-1} \end{pmatrix}$$

the $m \times m$ matrix with coefficients in $\mathbb{Z}$. ${}^t Q$, denoting the transposed matrix of Q, is the companion matrix of $P(X)$. For all $p, k \in \{0, 1, \cdots, m-1\}$, we have: $z_{p,k} = \delta_{p,k}$ the Kronecker symbol.

LEMMA 4.3. — *For all $k \geqslant 0$, we have $Z_{k+1} = {}^t Q\, Z_k$.*

Proof. — This is trivial if $k < m-1$. If $k \geqslant m-1$, write

$$\beta^{k+1} = \beta \cdot \beta^k = \beta(z_{m-1,k}\beta^{m-1} + z_{m-2,k}\beta^{m-2} + \cdots + z_{1,k}\beta + z_{0,k}) =$$

$$(z_{m-2,k} + a_{m-1}z_{m-1,k})\beta^{m-1} + (z_{m-3,k} + a_{m-2}z_{m-1,k})\beta^{m-2} + \cdots + (z_{0,k} + a_1 z_{m-1,k})\beta + a_0 z_{m-1,k}$$

since $\beta^m = a_{m-1}\beta^{m-1} + a_{m-2}\beta^{m-2} + \cdots + a_1\beta + a_0$. Hence, the result in a matrix form. □

Denote

$$C = \begin{pmatrix} 1 & \beta & \beta^2 & \cdots & \beta^{m-1} \\ 1 & \beta^{(1)} & {\beta^{(1)}}^2 & \cdots & {\beta^{(1)}}^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \beta^{(m-1)} & {\beta^{(m-1)}}^2 & \cdots & {\beta^{(m-1)}}^{m-1} \end{pmatrix}$$

the Vandermonde matrix of order m. We obtain $C\, Z_k = \mathscr{B}_k$ by the real and complex embeddings of $\mathbb{Q}[\beta]$ since all the coefficients $z_{j,k}, j \in \{0, 1, \cdots, m-1\}$ are integers and remain invariant.

THEOREM 4.4. — *If β is a Perron number of degree m and minimal polynomial $P(X)$ and if v_1 denote the vector defined by the first column of C^{-1}, then*

$$\lim_{k \to +\infty} \frac{Z_k}{\|Z_k\|} \text{ exists and is equal to } u := \frac{v_1}{\|v_1\|} \tag{3}$$

Moreover, all the components of v_1 are real and belong to the $\mathbb{Z}$ - module $\dfrac{\mathbb{Z}[\beta]}{\beta^{m-1}P'(\beta)}$.

Proof. — Since $P(X)$ is minimal, all the roots of $P(X)$ are distinct. Hence, the determinant of C is $\prod_{i<j}(\beta^{(i)} - \beta^{(j)})$ and is not zero Let $C^{-1} = (\xi_{ij})$. Then $C \cdot C^{-1} = I$, that is

$$\xi_{1i} + \xi_{2i}\beta^{(j)} + \xi_{3i}{\beta^{(j)}}^2 + \cdots + \xi_{mi}{\beta^{(j)}}^{m-1} = \delta_{i,j+1} \tag{4}$$

for all $i = 1, 2, \ldots, m,$ and $j = 0, 1, \ldots, m-1$. On the other hand, the Lagrange interpolating polynomials associated with $\{\beta, \beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(m-1)}\}$ are given by

$$L_s(X) = \prod_{\substack{j=0 \\ j\neq s}}^{m-1} \frac{X - \beta^{(j)}}{\beta^{(s)} - \beta^{(j)}} \qquad s = 0, 1, \ldots, m-1$$

For m arbitrary complex numbers $y_1, y_2, \cdots, y_m$, denote

$$\sigma_r = \sigma_r(y_1, y_2, \cdots, y_m) = \sum_{1\leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r \leqslant m} \prod_{j=1}^{r} y_{i_j}$$

the r-th elementary symmetric function of the m numbers $y_1, y_2, \cdots, y_m$. The degree of $L_s(X)$ is $m-1$ and $L_s(X)$ can be expressed as

$$L_s(X) = \sum_{r=0}^{m-1} (-1)^r \sigma_r^{(s)} X^{m-r-1} / \prod_{\substack{r=0 \\ r\neq s}}^{m-1} (\beta^{(s)} - \beta^{(r)})$$

where $\sigma_r^{(s)} = \sigma_r(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, \beta^{(s+1)}, \cdots, \beta^{(m-1)})$ denotes the r-th elementary symmetric function of the $m-1$ numbers $\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, \beta^{(s+1)}, \cdots, \beta^{(m-1)}$ where $\beta^{(s)}$ is missing. Since these polynomials satisfy $L_s(\beta^{(k)}) = \delta_{s,k}$ for all $s, k = 0, 1, \cdots, m-1$, comparing with (4), we obtain, by identification of the coefficients

$$\xi_{ji} = \frac{(-1)^{m-j} \sigma_{m-j}^{(i-1)}}{\prod\limits_{\substack{r=0 \\ r\neq i-1}}^{m-1} (\beta^{(i-1)} - \beta^{(r)})} = \frac{(-1)^{m-j} \sigma_{m-j}^{(i-1)}}{P'(\beta^{(i-1)})}$$

for all $i, j = 1, 2, \cdots, m$. We have

$$L_s(X) = \sum_{j=1}^{m} \xi_{j,s+1} X^{j-1} \qquad s = 0, 1, \cdots, m-1$$

Now $C \cdot Z_k = \mathscr{B}_k$ for all $k \geqslant 0$, hence $Z_k = C^{-1} \cdot \mathscr{B}_k$. Each component $z_{i,k}$, $0 \leqslant i \leqslant m-1$, $k \geqslant 0$ of Z_k can be expressed as

$$z_{i,k} = \sum_{j=1}^{m} \xi_{i+1,j} \beta^{(j-1)^k} \tag{5}$$

Since β is a Perron number, we have $|\beta^{(j)}| < \beta$ for all j, $1 \leqslant j \leqslant m-1$. Hence, for all j, $1 \leqslant j \leqslant m-1$

$$\lim_{k\to+\infty} \left(\frac{\beta^{(j)}}{\beta} \right)^k = 0$$

and, therefore

$$\lim_{k\to+\infty} \frac{z_{i,k}}{\beta^k} = \xi_{i+1,1} \qquad i = 0, 1, \cdots, m-1$$

Moreover,

$$\lim_{k\to+\infty} \frac{\left(\sum_{i=0}^{m-1} |z_{i,k}|^2\right)^{1/2}}{\beta^k} = \lim_{k\to+\infty} \frac{\|Z_k\|}{\beta^k} = \sqrt{\sum_{i=0}^{m-1} |\xi_{i+1,1}|^2} = \|v_1\|$$

hence the result. The fact that all the components of v_1 are real and belong to the $\mathbb{Z}$ - module $\mathbb{Z}[\beta]/(\beta^{m-1}P'(\beta))$ comes from the following more accurate proposition. $\square$

Proposition 4.5. — *The components of* v_1 *are*

$$\xi_{j,1} = \frac{a_{j-1}\beta^{j-1} + a_{j-2}\beta^{j-2} + \cdots + a_1\beta + a_0}{\beta^j P'(\beta)} \qquad j = 1, 2, \cdots, m$$

In particular,

$$\xi_{m,1} = \frac{1}{P'(\beta)}$$

Proof. — First $L_0(X) = \sum_{j=1}^{m} \xi_{j,1} X^{j-1}$ and

$$P(X) = X^m - a_{m-1}X^{m-1} - a_{m-2}X^{m-2} - \cdots - a_1X - a_0 = \prod_{j=0}^{m-1} (X - \beta^{(j)}) = L_0(X)(X-\beta)P'(\beta)$$

$P(X)$ is an element of $\mathbb{Z}[X]$, $P'(X)$ also. All the coefficients of $L_0(X)$ are satisfying the following relations

$$\begin{aligned} -\beta P'(\beta)\xi_{1,1} &= -a_0 \\ -\beta P'(\beta)\xi_{2,1} + \xi_{1,1}P'(\beta) &= -a_1 \\ -\beta P'(\beta)\xi_{3,1} + \xi_{2,1}P'(\beta) &= -a_2 \\ &\vdots \\ -\beta P'(\beta)\xi_{m,1} + \xi_{m-1,1}P'(\beta) &= -a_{m-1} \\ \xi_{m,1}P'(\beta) &= 1 \end{aligned} \tag{6}$$

We deduce the result recursively from $\xi_{1,1}$. $\square$

Let $u_B := B/\|B\|$ the unit vector and π_B the orthogonal projection mapping onto the eigenspace $\mathbb{R}B$.

Theorem 4.6. — *Under the assumptions of theorem 4.4, we have:*

$$(i) \;\; u \cdot u_B = \|B\|^{-1}\|v_1\|^{-1} > 0$$

$$(ii) \;\; \lim_{k\to+\infty} \frac{\|Z_{k+1}\|}{\|Z_k\|} \text{ exists and is equal to } \beta .$$

(iii) u *is an eigenvector of* ${}^tQ$ *of eigenvalue* β. *The eigenspace of* $\mathbb{R}^m$ *associated with the eigenvalue* β *of* ${}^tQ$ *is* $\mathbb{R}u$. *(iv)* B *is an eigenvector of the adjoint matrix* $({}^tQ)^* = Q$ *associated with the eigenvalue* β *and for all* $x \in \mathbb{C}^m$

$$\lim_{k\to+\infty} \frac{({}^tQ)^k}{\beta^k}(x) = (x \cdot B)\, v_1 \tag{7}$$

Proof. — (i) and (ii) First we have $v_1 \cdot B = 1$ by definition of the inverse matrix C^{-1}. Hence $u \cdot B = \|v_1\|^{-1} > 0$. Then, for all $k \geqslant 0$,

$$ {}^tZ_k\, B \;=\; \beta^k \;=\; \|Z_k\|\,{}^t\!\left(\frac{Z_k}{\|Z_k\|} - u + u\right) B \;>0 $$

which tends to infinity when k tends to $+\infty$. Since $u - Z_k/\|Z_k\|$ tends to zero when k goes to infinity, $\|Z_k\|$ behaves at infinity like $\beta^k/\,(u \cdot B)$, hence the limit.

(iii) For all $k \geqslant 0$, we have

$$ {}^tQu \;=\; {}^tQ\Big(u - \frac{Z_k}{\|Z_k\|} + \frac{Z_k}{\|Z_k\|}\Big) \;=\; {}^tQ\Big(u - \frac{Z_k}{\|Z_k\|}\Big) + \frac{\|Z_{k+1}\|}{\|Z_k\|}\frac{Z_{k+1}}{\|Z_{k+1}\|} $$

The first term is converging to zero and the second one to βu, when k goes to infinity, from theorem 4.4. Hence, the result since all the roots of $P(X)$ are distinct and the (real) eigenspace associated with β is 1 - dimensional.

(iv) It is clear that B is an eigenvector of the adjoint matrix Q. If $h_0, h_1, \cdots, h_{m-1} \in \mathbb{C}$, $x = \sum_{j=0}^{m-1} h_j Z_j$, where $Z_0, Z_1, \cdots, Z_{m-1}$ is the canonical basis of $\mathbb{C}^m$, we have

$$ \frac{({}^tQ)^k}{\beta^k}(x) \;=\; \sum_{j=0}^{m-1} h_j \beta^{-k} Z_{k+j} \;=\; \sum_{j=0}^{m-1} h_j \beta^j \left(\frac{Z_{k+j}}{\beta^{k+j}}\right) $$

but, from the proof of theorem 4.4, $\displaystyle\lim_{k\to+\infty} \frac{Z_{k+j}}{\beta^{k+j}} \;=\; v_1$ and $\sum_{j=0}^{m-1} h_j\beta^j \;=\; x \cdot B$. □

Remarks. — (i) Note that B has strictly positive components while v_1 may have negative components according to the signs of the coefficients a_i. The equality (7) is already given by the Perron-Frobenius theory, e.g. Ruelle [10] p136, Gantmacher [11], chap XIII, or Minc [12], but, here, the matrix ${}^tQ$ has not necessarily non-negative entries. The prominent lines are nevertheless those generated by the two eigenvectors B and v_1, resp. of the matrix ${}^tQ$ and its adjoint Q.

(ii) The result (ii) provides a straightforward way of computing β by means of purely vector methods.

From proposition 4.5 and theorem 4.4, we can formulate a basis of eigenvectors in $\mathbb{C}^m$ (complexication space of $\mathbb{R}^m$) of the complexification ${}^tQ_{\mathbb{C}} : \mathbb{C}^m \to \mathbb{C}^m$ of the operator ${}^tQ$. From it, we will deduce a basis of eigenvectors of ${}^tQ$ as an operator on the real vector space $\mathbb{R}^m$, in the case in which β has non real conjugates [22]. Obviously, the complexification $Q_{\mathbb{C}}$ of the adjoint operator $({}^tQ)^* = Q$ admits $\{B, B^{(1)}, B^{(2)}, \cdots, B^{(m-1)}\}$ as a basis of eigenvectors in $\mathbb{C}^m$ of respective eigenvalues $\beta, \beta^{(1)}, \beta^{(2)}, \cdots, \beta^{(m-1)}$. Let $s \geqslant 1$, resp. t, the number of real, resp. complex (up to conjugation), embeddings of the number field $\mathbb{Q}(\beta)$. We have $m = s + 2t$. Assume the conjugates of β are (up to permutation leaving β fixed, the first s elements real and grouping by pairs the complex conjugates):

$$ \beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, \beta^{(s)}, \beta^{(s+1)}, \cdots, \beta^{(m-2)} = \beta^{(s+2t-2)}, \beta^{(m-1)} = \beta^{(s+2t-1)} $$

where $\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}$ are real and $\beta^{(s+2j)} = \overline{\beta^{(s+2j+1)}}$ are complex with non zero imaginary part, for all $j = 0, 1, \cdots, t-1$. Write $r_j = |\beta^{(s+2j)}| = |\beta^{(s+2j+1)}|$ and $\beta^{(s+2j)} = r_j(\cos(\theta_j) +$

$i\sin(\theta_j))$, for all $j = 0, 1, \cdots, t-1$. Denote by $\mathrm{Diag}(a_1, a_2, \cdots, a_q)$ a square matrix having as coefficients zero everywhere except on the diagonal where the diagonal elements are the a_j 's. These entries can be complex numbers, real numbers or real Jordan blocks.

Corollary 4.7. — *(i) A basis of eigenvectors of* ${}^tQ_{\mathbb{C}}$ *is given by the collection of* m *column vectors* $\{W_k\}_{k=1,2,\cdots,m}$ *, of respective components*

$$\xi_{j,k} = \frac{a_{j-1}\beta^{(k-1)^{j-1}} + a_{j-2}\beta^{(k-1)^{j-2}} + \cdots + a_1\beta^{(k-1)} + a_0}{\beta^{(k-1)^j} P'(\beta^{(k-1)})} \qquad j = 1, 2, \cdots, m$$

In particular,

$$\xi_{m,k} = \frac{1}{P'(\beta^{(k-1)})}$$

In this basis, the matrix of the operator ${}^tQ_{\mathbb{C}}$ *is*

$$\mathrm{Diag}(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, \beta^{(s)}, \beta^{(s+1)}, \cdots, \beta^{(s+2t-2)}, \beta^{(m-1)} = \beta^{(s+2t-1)})$$

(ii) A real Jordan form for the operator ${}^tQ$ *is given by*

$$\mathrm{Diag}(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, D_1, D_2, \cdots, D_t)$$

in the basis of eigenvectors $\{V_j\}_{j=1,\cdots,m}$ *with*

$$V_1 = W_1 = v_1, V_2 = W_2, \cdots, V_s = W_s$$

$$V_{s+2j+1} = Im(W_{s+2j+1}), V_{s+2j+2} = Re(W_{s+2j+1}), \qquad j = 0, 1, \cdots, t-1$$

and the real Jordan blocks D_j *are* 2×2 *and equal to*

$$\begin{pmatrix} r_j\cos\theta_j & -r_j\sin\theta_j \\ r_j\sin\theta_j & r_j\cos\theta_j \end{pmatrix}$$

(iii) a real Jordan form of the adjoint operator $({}^tQ)^* = Q$ *is given by the same matrix*

$$\mathrm{Diag}(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, D_1, D_2, \cdots, D_t)$$

in the basis of eigenvectors $\{X_j\}_{j=1,\cdots,m}$ *with*

$$X_1 = B, X_2 = B^{(1)}, X_3 = B^{(2)}, \cdots, X_s = B^{(s-1)}$$

$$X_{s+2j+1} = Im(B^{(s+2j)}), X_{s+2j+2} = Re(B^{(s+2j)}), \qquad j = 0, 1, \cdots, t-1$$

In particular, if K *denotes the real field*

$$K = \mathbb{Q}(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, Im(\beta^{(s)}), Re(\beta^{(s)}), \cdots, Im(\beta^{(m-2)}), Re(\beta^{(m-2)})\ )$$

Q *and* ${}^tQ$ *are equivalent over* K *: there exists a* $m \times m$ *invertible matrix* U *with entries in* K *such that*

$${}^tQ = U^{-1}QU$$

Proof. — We know that

$$({}^tQ)v_1 = \beta v_1 \tag{8}$$

where ${}^tQ$ has rational entries, and that v_1 has components in the $\mathbb{Z}$ - module $\beta^{1-m}(P'(\beta))^{-1}\mathbb{Z}[\beta]$. Hence, applying component by component to equation (8) the $\mathbb{Q}$ - automorphisms of $\mathbb{C}$ which are the real and complex embeddings of the number field $\mathbb{Q}(\beta)$, we deduce

$$({}^tQ)W_j = \beta^{(j-1)}W_j, \qquad j = 1, 2, \cdots, m$$

and the diagonal form of the complexification operator ${}^tQ_{\mathbb{C}}$. In a suitable basis of $\mathbb{R}^m$, the matrix of the operator ${}^tQ$ admits a real Jordan form (e.g. [22]); this decomposition by Jordan blocks on the diagonal is done when some conjugates of β have non zero imaginary parts. The restrictions of ${}^tQ$ to the real ${}^tQ$ - invariant subspaces of $\mathbb{R}^m$ have no nilpotent parts since all the roots of $P(X)$ are distinct, and therefore the real Jordan blocks are 2×2.

Similarly

$$QB = \beta B$$

hence

$$QB^{(j)} = \beta^{(j)}B^{(j)}, \qquad j = 0, 1, \cdots, m-1$$

Obviously Q and ${}^tQ$ have the same eigenvalues. A real Jordan form of Q is the same as for ${}^tQ$. The corresponding basis is classically given by the X_i 's (e.g. [22]). Since all the components of W_j and X_j belong to K, there exist two matrices U_1, U_2 with entries in K such that

$$\begin{aligned} {}^tQ &= U_1^{-1}\mathrm{Diag}(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, D_1, D_2, \cdots, D_t)U_1 \\ Q &= U_2^{-1}\mathrm{Diag}(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, D_1, D_2, \cdots, D_t)U_2 \end{aligned}$$

Note that in general

$${}^t\mathrm{Diag}(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, D_1, D_2, \cdots, D_t) \neq \mathrm{Diag}(\beta, \beta^{(1)}, \cdots, \beta^{(s-1)}, D_1, D_2, \cdots, D_t)$$

Hence, $U = U_2^{-1}U_1$ satisfies the equivalence relation

$${}^tQ = U^{-1}QU$$

□

If K^g is the smallest number field such that $K^g/\mathbb{Q}$ is a Galois extension containing β, hence finite, the field K is included in $\mathbb{R} \cap K^g$. If we assume $s = m$ (no complex embeddings for $\mathbb{Q}(\beta)$) and $\mathbb{Q}(\beta)/\mathbb{Q}$ is a Galois extension, then $K = \mathbb{Q}(\beta)$. In general, K is a finite real extension of the field of rationals which contains strictly $\mathbb{Q}(\beta)$.

We now construct the Delone set $\mathbb{Z}_\beta$ of β -integers on the line $\mathbb{R}u_B$ by (cut and) projection from the lattice $\mathbb{Z}^m$.

Definition 4.8. — *A cut and project scheme consists of a direct product $E \times D$ of an euclidean space E of finite dimension and a locally compact abelian group D, and a lattice L in $E \times D$ so that with respect to the natural projections $p_1 : E \times D \to E$, $p_2 : E \times D \to D$,*

(i) p_1 restricted to L is one to one,

(ii) $p_2(L)$ is dense in D.

We denote by $p_1(L) = M$ and by * the mapping $p_2 \circ (p_{1_{|_L}})^{-1}$ from M to D.

THEOREM 4.9. — *Denote by* $\mathscr{D}$ *the linear subspace of* $\mathbb{R}^m$ *orthogonal to* $E = \mathbb{R}B$, $\pi_{\mathscr{D}} = Id - \pi_B$ *, and by* $L = \mathbb{Z}^m$ *. Set*

$$\begin{aligned}\mathscr{L}_{\mathscr{K}} &= \{x_k Z_k + x_{k-1}Z_{k-1} + \cdots + x_1 Z_1 + x_0 Z_0 \mid x_i \in A,\ k \geqslant 0\text{, and}\\ &\quad (x_j, x_{j-1}, \cdots, x_1, x_0, 0, 0, \cdots) < (c_1, c_2, \cdots) \text{ for all } j,\ 0 \leqslant j \leqslant k\}\end{aligned}$$

the ^tQ *- invariant subset of* L*. Then:*

(i) the map $\displaystyle\sum_{j=0}^{k} x_j \beta^j \to \sum_{j=0}^{k} x_j Z_j \ :\quad \mathbb{Z}_\beta^+ \to \mathscr{L}_{\mathscr{K}}$ *is a bijection, and, if we denote by* F *any real* ^tQ *- invariant subspace of* $\mathbb{R}^m$ *, by* π_F *the projection to* F *along its* ^tQ *- invariant complementary space and by* $\mathbb{Z}^+_{\beta^{(F)}} = \pi_F(\mathscr{L}_{\mathscr{K}})$ *, all the diagrams are commutative:*

$$\begin{array}{ccc}\mathscr{L}(\mathscr{K}) & \xrightarrow{\ ^tQ\ } & \mathscr{L}(\mathscr{K})\\ \pi_F \downarrow\simeq & & \simeq\downarrow \pi_F\\ \mathbb{Z}^+_{\beta^{(F)}} & \xrightarrow{\times\beta^{(F)}} & \mathbb{Z}^+_{\beta^{(F)}}\end{array}$$

where $\beta^{(F)}$ *is one of the real conjugate of* β *if dim F = 1, otherwise* $\beta^{(F)}$ *is one of the* 2×2 *real Jordan blocks* D_j *of corollary 4.7,*

(ii) we have

$$\pi_B(\pm\mathscr{L}_{\mathscr{K}}) = \mathbb{Z}_\beta \frac{u_B}{\|B\|}$$

and

$$\beta\pi_B(\pm\mathscr{L}_{\mathscr{K}}) \subset \pi_B(\pm\mathscr{L}_{\mathscr{K}}), \quad \textit{hence} \quad \beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta,$$

(iii) $(E \times \mathscr{D}, L)$ *is a cut and project scheme.*

Proof. — (i) For all $k \geqslant 0$ the element β^k is uniquely associated with the equivalence class in the inductive system $(\mathscr{K}_n, f_n^p)$ whose representants are $(1, 0, 0, \cdots)$ at the rank k and Z_k at the rank m by the relation $^tZ_k\, B = \beta^k$. The image of the map $g_m^{-1} : \mathbb{Z}[\beta] \to \mathbb{Z}^m$ restricted to $\mathbb{Z}_\beta^+$ ensures the surjectivity. Let us show that it is injective. Assume there exists a non-zero element $\displaystyle\sum_{j=0}^{k} x_j \beta^j$ in $\mathbb{Z}_\beta$ such that $\displaystyle\sum_{j=0}^{k} x_j Z_j = 0$. Since $^t\left(\sum_{j=0}^{k} x_j Z_j\right) B = 0 = \sum_{j=0}^{k} x_j \beta^j$, this would mean that zero could be represented by a non-zero element. This is impossible by construction, from the fact that the greedy algorithm provides a system of numeration (Fraenkel [13]). The operator ^tQ is commuting with the projection mappings to the ^tQ - invariant subspaces. For an arbitrary ^tQ - invariant subspace F, the invariance of $\mathscr{L}_{\mathscr{K}}$ by ^tQ induces the invariance by multiplication by $\beta^{(F)}$ of its projection to F.

(ii) for all $k \geqslant 0$, we have

$$\pi_B(Z_k) = \frac{\beta^k}{\|B\|}\ u_B$$

Hence the result by $\mathbb{Z}$ - linearity. Similarly, since the operator ${}^tQ$ is leaving $\pm\mathscr{L}_{\mathscr{K}}$ invariant, we have that $p_1(\pm\mathscr{L}_{\mathscr{K}})$ is left invariant by multiplication by β. It is important to note that the discrete set $p_1(\pm\mathscr{L}_{\mathscr{K}})$ is invariant by multiplication by β but the $\mathbb{R}$ - span of it is not ${}^tQ$ - invariant, that is invariant by multiplication by β, since it is generally not an eigenspace (see proposition 4.12 below) of the operator ${}^tQ$.

(iii) The set $\{Z_0, Z_1, \cdots, Z_{m-1}\}$ is exactly the canonical basis of $\mathbb{R}^m$. Any

$$X = \sum_{j=0}^{m-1} h_j Z_j \in L = \mathbb{Z}^m, \qquad h_j \in \mathbb{Z}$$

is projected by $p_1 = \pi_B : X \to (X \cdot u_B)u_B$ to $\sum_{j=0}^{m-1} h_j \beta^j u_B$ in the $\mathbb{Z}$-module $\mathbb{Z}[\beta]u_B$. The map $p_{1_{|\mathbb{Z}^m}}$ is bijective since the family $\{\beta^j u_B \mid j = 0, 1, \cdots, m-1\}$ is free over $\mathbb{Z}$. The fact that $p_2(L) = \pi_{\mathscr{D}}(L)$ is dense in $\mathscr{D}$ arises from Kronecker's theorem ([36], Appendix B) because β is an algebraic integer of degree m. Recall that $1 = \beta^0, \beta^1, \cdots, \beta^{m-1}$ are m real numbers linearly independent over $\mathbb{Q}$. Hence, if $x_0, x_1, \cdots, x_{m-1}$ are m real numbers such that the vector X of components $\{x_j\}_{j=0,1,\ldots,m-1}$ belongs to $\mathscr{D}$, and $\epsilon > 0$, then there exist a real number w and m rational integers $u_0, u_1, \cdots, u_{m-1}$ such that $\mid x_0 - \beta^0 w - u_0 \mid \leqslant \epsilon, \cdots, \mid x_{m-1} - \beta^{m-1} w - u_{m-1} \mid \leqslant \epsilon$. In other terms, there exists a point $U \in \mathbb{Z}^m$ of components $\{u_j\}_{j=0,1,\ldots,m-1}$ such that its image by $\pi_{\mathscr{D}}$ is close to the given X up to ϵ for an arbitrary ϵ. Hence the result. □

Proposition 4.10. — *(i) For any $a_0, a_1, \cdots, a_k \in \mathbb{Z}, k \geqslant 0$, we have*

$$\pi_B\left(\sum_{i=0}^{k} a_i Z_i\right) = \frac{\sum_{i=0}^{k} a_i \beta^i}{\|B\|}\ u_B \tag{9}$$

and conversely, any polynomial in β on the line generated by $u_B/\|B\|$ can be uniquely lifted up to a $\mathbb{Z}$ -linear combination of the Z_i's with the same coefficients.

(ii) Denote by (see corollary 4.7 for the definition of X_i)

$$u_{B,i} = \begin{cases} \frac{1}{\|X_i\|} X_i, & i = 1, 2, \cdots, s \\ \frac{1}{(\|X_i\|^2 + \|X_{i+1}\|^2)^{1/2}} X_i, & i = s+1, \cdots, m, \quad \text{with } i-(s+1) \text{ even} \\ \frac{1}{(\|X_{i-1}\|^2 + \|X_i\|^2)^{1/2}} X_i, & i = s+1, \cdots, m, \quad \text{with } i-(s+1) \text{ odd} \end{cases}$$

the unit vectors (with $u_B = u_{B,1}$) and, for all $i = 1, 2, \cdots, s$, $\pi_{B,i} : \mathbb{R}^m \to \mathbb{R}u_{B,i}$ the orthogonal projections to the 1-dimensional eigenspaces of Q, resp. $i = s+1, \cdots, m$ with $i-(s+1)$ even, $\pi_{B,i} : \mathbb{R}^m \to \mathbb{R}u_{B,i} + \mathbb{R}u_{B,i+1}$ the orthogonal projections to the 2-dimensional eigenspaces of Q. For any $a_0, a_1, \cdots, a_k \in \mathbb{Z}, k \geqslant 0$, we have

$$\pi_{B,i}(\sum_{j=0}^{k} a_j Z_j) = \frac{\sum_{j=0}^{k} a_j \beta^{(i-1)^j}}{\|X_i\|}\ u_{B,i} \qquad i = 1, 2, \cdots, s$$

and, for all $i = s+1, \cdots, m$ with $i-(s+1)$ even,

$$\pi_{B,i}(\sum_{j=0}^{k} a_j Z_j) =$$

$$\frac{1}{(\|X_i\|^2+\|X_{i+1}\|^2)^{1/2}}\begin{pmatrix} Re(\sum_{j=0}^k a_j\beta^{(i-1)^j}) & Im(\sum_{j=0}^k a_j\beta^{(i-1)^j}) \\ -Im(\sum_{j=0}^k a_j\beta^{(i-1)^j}) & Re(\sum_{j=0}^k a_j\beta^{(i-1)^j}) \end{pmatrix}\begin{pmatrix} u_{B,i} \\ u_{B,i+1}\end{pmatrix}$$

Proof. — (i) is obtained by linearity from theorem 4.6 (ii) and (iii) since $p_{1_{|_{\mathbb{Z}^m}}}$ is bijective.

(ii) Applying now the real embeddings of $\mathbb{Q}(\beta)$ to the relation

$$\pi_B(\sum_{j=0}^k a_jZ_j) = ((\sum_{j=0}^k a_jZ_j)\cdot u_B)u_B = \frac{\sum_{j=0}^k a_j\beta^j}{\|B\|}u_B$$

gives, for all $i=1,2,\cdots,s$,

$$\pi_{B,i}(\sum_{j=0}^k a_jZ_j) = ((\sum_{j=0}^k a_jZ_j)\cdot u_{B,i})u_{B,i} = \frac{\sum_{j=0}^k a_j\beta^{(i-1)^j}}{\|B^{(i-1)}\|}u_{B,i}$$

and the result. Similarly the complex embeddings applied to the above relation provide as orthogonal projections, with $i=s+1,\cdots,m$ with $i-(s+1)$ even

$$\frac{1}{\|B^{(i-1)}\|^2}((\sum_{j=0}^k a_jZ_j)\cdot B^{(i-1)})B^{(i-1)}$$

from which, by means of corollary 4.7, we deduce the orthogonal projection on the real plane generated by $u_{B,i}$ and $u_{B,i+1}$. □

Definition 4.11. — *A subset Λ of a finite dimensional euclidean space E is a model set (also called a cut and project set) if there exists a cut and project scheme $(E\times D, L)$ and a relatively compact set Ω of D with non empty interior such that*

$$\Lambda = \{p_1(l) \mid l\in L, p_2(l)\in\Omega\} = \{v\in M \mid v^*\in\Omega\}$$

The set Ω is called the acceptance window.

The problem is now the following. In general, for an arbitrary Perron number β, the dominant eigenspaces $\mathbb{R}u$, resp. $\mathbb{R}u_B$, of the matrices ${}^tQ$, resp. its adjoint Q, are distinct. The vectors Z_k gather angularly about the eigenspace $\mathbb{R}u$ and not about the line $\mathbb{R}u_B$ from theorem 4.4. But $\mathbb{Z}_\beta$ is naturally formed, up to the scaling constant $\|B\|^{-1}$, by projection on $\mathbb{R}u_B$ from theorem 4.9. If we set an acceptance window Ω about the origin in the internal space $\mathscr{D}$ the number of integers k such that $p_2(Z_k)\in\Omega$ will be finite. Consequently, it becomes a priori impossible to embed $\mathbb{Z}_\beta$ in a model set on the line $\mathbb{R}u_B$ using the cut-and-project scheme $(E\times\mathscr{D}, L)$ of theorem 4.9 when $u\neq u_B$. Recall that if Λ is a relatively dense subset of $\mathbb{R}$, then Λ is a Meyer set if and only if there exists a model set which contains Λ ([37], p 431). In particular, it seems difficult to prove by this process and the above mentioned cut-and-project scheme $(E\times\mathscr{D}, L)$ that Z_β is a Meyer set when β is a Pisot number, although this result holds (Meyer, [36]). This requires new constructions. In the following, up till the end of this paragraph, let us consider only Pisot numbers.

First, consider the case of equality $u=u_B$ and show that it is rarely occuring.

PROPOSITION 4.12. — *The equality case* $u = u_B$ *is satisfied if and only if* β *is a Pisot number of degree 2, root* > 1 *of the polynomial* $X^2 - aX - 1$ *, with* $a \geqslant 1$.

Proof. — Indeed, the condition $u = u_B$ is equivalent to v_1 colinear to B , that is $\xi_{j,1}\beta^{-j+1}$ is a non zero constant independent of $j = 1, 2, \cdots, m$ (with the notations of proposition 4.5). We see that if β is such a Pisot number, such equalities hold. Conversely, equating the two terms indexed by $j = 0$ and $j = m$, we obtain $a_0\beta^{m-2} = 1$, that is necessarily $m = 2$ and $a_0 = 1$. The Perron number β is then a Pisot number of negative conjugate $-\beta^{-1}$ which satisfies $\beta^2 - a_1\beta - 1 = 0$, where $a_1 = \beta - \beta^{-1}$ is an integer greater or equal than 1. This is the only possibility of quadratic Pisot number with constant term -1 (Frougny et al, [20], Lemma 3). □

When this equality condition $u = u_B$ is satisfied, we have the following result.

PROPOSITION 4.13. — *When* β *is a Pisot number of degree 2, root* > 1 *of the polynomial* $X^2 - aX - 1$ *, with* $a \geqslant 1$ *,* $\mathbb{Z}_\beta$ *is a Meyer set, i.e.* $\mathbb{Z}_\beta$ *is relatively dense and, if* Ω_a *denotes the acceptance window* $[-c_a u^\perp; +c_a u^\perp]$ *in* $\mathscr{D}$ *, with*

$$c_a \;=\; \frac{(1 + a\beta)\lfloor\beta\rfloor}{\sqrt{2 + a\beta(\beta - 1)}} \quad \textit{and} \quad \Lambda_a \;=\; \{v \in p_1(\mathbb{Z}^2) \mid v^* \in \Omega_a\}$$

the model set, we have the following inclusion

$$\pi_B(\pm\mathscr{L}_\mathscr{K}) \;\subset\; \Lambda_a$$

Proof. — Indeed, by proposition 4.12 we have $u = u_B$. With the notations of theorem 4.9, $(E \times \mathscr{D}, L)$ is a cut-and-project scheme of $\mathbb{R}^2$, with $E = \mathbb{R}u = \mathbb{R}u_B$. We will show that the model set Λ_a in E contains $\pi_B(\pm\mathscr{L}_\mathscr{K}) = \mathbb{Z}_\beta\|B\|^{-1}u_B$. Since $\mathbb{Z}_\beta$ is isometric to $\|B\|\pi_B(\pm\mathscr{L}_\mathscr{K})$, this will be enough for showing the result. Recall that $\{Z_0, Z_1\}$ is the canonical basis of $\mathbb{R}^2$ and for all $j = 0, 1, 2, \cdots, Z_j = ({}^tQ)^j Z_0$. Hence, if g denotes an arbitrary element of $\mathscr{L}_\mathscr{K}$, it can be written $g = x_k({}^tQ)^k Z_0 + x_{k-1}({}^tQ)^{k-1}Z_0 + \cdots + x_1({}^tQ)Z_0 + x_0 Z_0$ for a certain integer $k \geqslant 0$ with $x_i \in A$ and $(x_j, x_{j-1}, \cdots, x_1, x_0, 0, 0, \cdots) < (c_1, c_2, \cdots)$ for all $j,\ 0 \leqslant j \leqslant k$. Recall that $A = \{0, 1, \cdots, \lfloor\beta\rfloor\}$. Denote by

$$u^\perp \;=\; \|B\|^{-1}\begin{pmatrix} -\beta \\ 1 \end{pmatrix}$$

the unit vector of $\mathscr{D}$. We have $Z_0 \;=\; su + s^\perp u^\perp$ with $s = \|B\|^{-1}$ and $s^\perp = -\beta\|B\|^{-1}$. We can write

$$g = \sum_{j=0}^{k} x_j({}^tQ)^j Z_0 = \sum_{j=0}^{k} x_j\left(s\beta^j u + s^\perp(-1)^j\beta^{-j}u^\perp\right)$$

Thus $\pi_\mathscr{D}(g) = s^\perp \sum_{j=0}^{k} x_j(-1)^j\beta^{-j}u^\perp$ and

$$\|\pi_\mathscr{D}(g)\| \;\leqslant\; |s^\perp|\lfloor\beta\rfloor\sum_{j=0}^{+\infty}\beta^{-j} \;=\; |s^\perp|\lfloor\beta\rfloor\frac{1}{1-\beta^{-1}}$$

which is equal to c_a since $\|B\| = \sqrt{2 + a\beta}$. This constant is independent of k. Hence we have $p_1(g)^* = \pi_B(g)^* = \pi_{\mathscr{D}}(g) \in \Omega_a$, that is the inclusion $\pi_B(\pm\mathscr{L}_{\mathscr{K}}) \subset \Lambda_a$.

Now, $\mathbb{Z}_\beta$ is relatively dense since it is a self-similar tiling of the line with a finite set of prototiles (2) whose corresponding lengths are $\{\|B\|^{-1}T^i(1) |\ i = 0, 1\}$ (Thurston, [28]) with the notations of §3. We have $T^0(1) = 1$ and $T^1(1) = \beta - a$. Now apply theorem 9.1 (i) in [37]. $\square$

Remarks .— (i) The bound c_a was already given by Burdik et al ([2], §4) when β is the golden mean $\tau = (1 + \sqrt{5})/2$. In this case, $a = 1$ and $c_1 = \tau^3/\sqrt{1 + \tau^2}$. The set $\mathbb{Z}_\tau$ is a Meyer set and there exists a finite set G_τ such that $\mathbb{Z}_\tau - \mathbb{Z}_\tau \subset \mathbb{Z}_\tau + G_\tau$. A first estimate of this set G_τ is $\{0, \pm\tau^{-1}, \pm\tau^{-2}\}$ and more detailed algebraic considerations can be found about G_β in [2] when $a \neq 1$, and also in Masakova et al [29].

(ii) The above model sets and Meyer sets built in dimension 2 from Pisot numbers can be found again from the non crystallographic root system of type A_1 as shown in Patera [49].

In the following, we will construct another cut-and-project scheme $(E' \times D', L')$ which will allow us to deal with the generic case $u \neq u_B$ and complex conjugates of β. Let $E' = \mathbb{R}u_B$ and L' the standard lattice $\mathbb{Z}^m$. We chose for D' the direct sum of all the ${}^tQ$ - invariant subspaces of $\mathbb{R}^m$ except $\mathbb{R}u$.

We will assume that β is a Pisot number from now on. The eigenspaces F corresponding to real or complex $\beta^{(j)}$ with $|\beta^{(j)}| < 1$ are such that when restricted to these subspaces, the norm obeys

$$\begin{aligned} \|({}^tQ)v\| &= \beta\|v\|, & v \in \mathbb{R}u \\ \|({}^tQ)v\|_F &= \lambda_F\|v\|_F, & v \in F,\ 0 < \lambda_F < 1 \end{aligned}$$

(we write for short λ_F instead of $|\beta^{(j)}|$). Let $\pi_F : \mathbb{R}^m \to F$ be the projection to F along the complementary direct sum. We denote by $\mathscr{S}$ the class of eigenspaces of ${}^tQ$ except $\mathbb{R}u$. Set

$$D' = \oplus_{F\in\mathscr{S}} F \quad \text{and} \quad \pi_{D'} = \oplus_{F\in\mathscr{S}} \pi_F$$

We have $D' \cap \mathbb{R}u_B = \{0\}$. Indeed, if we had $B \in D'$, we should obtain

$$\lim_{k\to\infty} \|({}^tQ)^k B\| = 0$$

but B has a non trivial component on the line $\mathbb{R}u$ (theorem 4.6) and u is an eigenvector of ${}^tQ$ with eigenvalue β strictly greater than unity. Hence, a component of $({}^tQ)^k B$ is going to infinity when k goes to $+\infty$. This is a contradiction.

Since $E' \times D' \simeq \mathbb{R}^m$ and the restriction of $p_1 : \mathbb{R}^m \to E'$ to $L' = \mathbb{Z}^m$ is obviously one-to-one, we have a $*$ - operation from $M = p_1(L')$ to D': $x \in M \to x^* = \pi_{D'} \circ (p_{1_{|\mathbb{Z}^m}})^{-1}(x)$. Let

$$\mathscr{C} = \{ \sum_{j=0}^{m-1} \alpha_j Z_j \mid \alpha_j \in [0; 1] \text{ for all } j = 0, 1, \cdots, m-1 \} \tag{10}$$

be the unit cell of $\mathbb{Z}^m$ constructed on the standard orthogonal basis $\{Z_0, Z_1, \cdots, Z_{m-1}\}$. Recall that, when dim F = 2, $\pi_F(\mathscr{C})$ is a closed polygon in the plane and that the restriction of the action of ${}^tQ$ to F is a rotation followed by a contraction (corollary 4.7). Set

$$\delta_F = \max_{x\in\mathscr{C}} \|\pi_F(x)\|, \qquad \mu_F = \min\{\sqrt{m}, m\delta_F\}, \qquad\qquad F \in \mathscr{S}$$

It is reached by one of the $\|\pi_F(Z_q)\|$ for a certain integer $q \in \{0,1,\cdots,m-1\}$ since the extremal points of the convex $\pi_F(\mathscr{C})$ arise from the extremal points of $\mathscr{C}$. It can be easily computed from the components of $Z_q, q = 0,1,\cdots,m-1$ in the basis of eigenvectors $V_i, i = 1,2,\cdots,m$ (corollary 4.7). If $g = \sum_{j=0}^{k} x_j Z_j$ is an arbitrary element of $\mathscr{L}_{\mathscr{K}}$ with $k = dm-1$, and d an integer $\geqslant 1$, then

$$g = \sum_{q=0}^{d-1}\sum_{l=0}^{m-1} x_{qm+l}({}^tQ)^{qm} Z_l = \sum_{q=0}^{d-1}({}^tQ)^{qm}\left(\sum_{l=0}^{m-1} x_{qm+l}Z_l\right)$$

Hence

$$p_1(g)^* = \pi_{D'}(g) = \sum_{F\in\mathscr{S}}\sum_{q=0}^{d-1}\left[({}^tQ_{|_F})^{qm}\pi_F\left(\sum_{l=0}^{m-1} x_{qm+l}Z_l\right)\right]$$

and

$$\|\pi_{D'}(g)\| \leqslant \sum_{F\in\mathscr{S}}\sum_{q=0}^{d-1}\lfloor\beta\rfloor\lambda_F^{qm}\|\pi_F\left(\sum_{l=0}^{m-1} Z_l\right)\|$$

$$\leqslant \sum_{F\in\mathscr{S}}\lfloor\beta\rfloor\,\mu_F\sum_{q=0}^{+\infty}\lambda_F^{qm} = \lfloor\beta\rfloor\sum_{F\in\mathscr{S}}\frac{\mu_F}{1-\lambda_F^m}$$

This constant is independent of d, hence of $k = dm-1$. It is easy to check that it is also an upper bound for $\|\pi_{D'}(g)\|$ even though k is not congruent to -1 modulo m, and also for any $g \in -\mathscr{L}_{\mathscr{K}}$. Now, $\mathbb{Z}_\beta$ is obtained, as an aperiodic tiling, by concatenation of a finite number of prototiles on the line (Thurston, [28]) and therefore $p_1(\pm\mathscr{L}_{\mathscr{K}})$ is relatively dense. For any $F \in \mathscr{S}$, let

$$c_F = \lfloor\beta\rfloor\frac{\mu_F}{1-\lambda_F^m} \tag{11}$$

$$\Omega_F = \begin{cases} \text{closed interval centred at 0 in } F \text{ of length } 2c_F & if\,\dim F = 1\\ \text{closed disc centred at 0 in } F \text{ of radius } c_F & if\,\dim F = 2\end{cases}$$

and

$$\Omega = \oplus_{F\in\mathscr{S}}\Omega_F \tag{12}$$

be the compact subset of D'. If Λ denotes the model set:

$$\Lambda = \{v \in p_1(\mathbb{Z}^m) \mid v^* \in \Omega\} \tag{13}$$

we have the inclusion

$$p_1(\pm\mathscr{L}_{\mathscr{K}}) \quad\subset\quad \Lambda$$

Let us show that $(E'\times D', L')$ with the two projection mappings π_B and $\pi_{D'}$ is a cut-and-project scheme: it remains to show that $\pi_{D'}(L')$ is dense in D'. It suffices to show that $\pi_F(L')$ is dense in F for all $F \in \mathscr{S}$. But $\pi_F(L')$ is a $\mathbb{Z}$ - module in F which is structurally the direct sum of a dense part and a discrete part (e.g. Descombes [23], theorem 2.3.7). With the algebraic numbers appearing in the components of the vectors $V_j, j = 1, 2, \cdots, m$ (corollary 4.7), it is clear that the discrete part is always trivial. Indeed, the minimal polynomial $P(X)$ of β is such that: 1) $a_0 \neq 0$ (for having a degree of β equal to m), and 2) one of the coefficients a_i, $i \in \{1, 2, \cdots, m-1\}$ is not equal to zero (to consider by assumption that β is a Perron number and not a Lind number). This forces at least one of the components of V_j to have sufficently non-zero ($\mathbb{Z}$ - linearly independent) algebraic numbers, $j = 1, m$. Therefore, applying theorem in Moody [37], and since $\mathbb{Z}_\beta$ is isometric to $\|B\| \pi_B(\pm \mathscr{L}_{\mathscr{K}})$, we have proved the following result.

THEOREM 4.14. — *(i)* $(\mathbb{R}^m \simeq E' \times D', L' = \mathbb{Z}^m)$, *with the two projection mappings* $p_1 = \pi_B : \mathbb{R}^m \to E'$ *and* $\pi_{D'} = \oplus_{F \in \mathscr{S}} \pi_F : \mathbb{R}^m \to D'$, *is a cut-and-project scheme,*
(ii) If β *is a Pisot number of degree* $m \geqslant 2$, *with* Ω *defined by equation (12), we have*

$$p_1(\pm \mathscr{L}_{\mathscr{K}}) \subset \Lambda = \{ v \in p_1(\mathbb{Z}^m) \mid v^* \in \Omega \}$$

Consequently, if β *is a Pisot number of degree* $m \geqslant 2$, *then* $\mathbb{Z}_\beta$ *is a Meyer set.*

5. Additive properties of $\mathscr{L}_{\mathscr{K}}$

In this paragraph, β is a Pisot number of degree $m \geqslant 2$. In the first part A), we shall deal with representing the elements of $\mathscr{L}_{\mathscr{K}}$ as non-negative integral combinations of a fixed set of some of them of small norm. Actually we show that this can be done up to a finite set of them which can be easily described. We will be also concerned in part B) with understanding the geometrical origin of the finite sets T and T' in the relations (1), (2) and (3) of §1.

A) Recall that when β is a Pisot number, the elements of $\pm \mathscr{L}_{\mathscr{K}}$ are within a band about the dominant eigenspace $\mathbb{R}u$ of ${}^tQ$. Set $\pi := \pi_{\mathbb{R}u}$ with the notations of theorem 4.9. Let $\pi^{\|}$ be the orthogonal projection mapping of image $\mathbb{R}u$ and $\pi^{\perp} = Id - \pi^{\|}$. For $\theta > 0$, define the cone about the dominant eigenspace $\mathbb{R}u$

$$K_\theta = \{ x \in \mathbb{R}^m \mid \theta \|\pi_{D'}(x)\| \leqslant \|\pi(x)\|, 0 \leqslant \pi(x) \cdot u \}$$

For $r, w > 0$, define

$$K_\theta(r) = \{ x \in K_\theta \mid \|\pi(x)\| \leqslant r \}$$

$$K_\theta(r, w) = \{ x \in K_\theta \mid r \leqslant \|\pi(x)\| \leqslant w \}$$

If $\mathscr{A}$ is an arbitrary subset of $\mathbb{R}^m$, denote by $sg(\mathscr{A})$ the semigroup generated by $\mathscr{A}$ (under the addition). Let ρ be the *covering radius* of the subset $\pm \mathscr{L}_{\mathscr{K}} \subset \mathbb{Z}^m$. In other terms, ρ is the smallest positive real number such that for any $z \in \mathbb{R}^m$ such that $\pi_{D'}(z) \in \Omega$ and $\pi(z) \cdot u \geqslant 0$, the closed ball $B(z, \rho)$ contains at least one element of $\pm \mathscr{L}_{\mathscr{K}}$. A lower bound for ρ is given by the covering radius of the lattice $\mathbb{Z}^m$ (Rogers, [26]). From equation (11) and decomposing any vector of $\mathbb{R}^m$ in the basis $\{B, V_2, V_3, \cdots, V_{s+2t}\}$, we see that an upper bound for ρ is given by

$$\frac{1}{2}\|B\|^{-1} \max\{1, \beta - t_1, \beta^2 - t_1\beta - t_2, \cdots, \beta^{q+k-1} - t_1\beta^{q+k-2} - t_2\beta^{q+k-3} \cdots - t_{q+k-1}\}$$

$$+\ \lfloor\beta\rfloor \sum_{F\in\mathscr{S}} \frac{\mu_F}{1-\lambda_F^m} = \frac{1}{2}\|B\|^{-1} + \ \lfloor\beta\rfloor \sum_{F\in\mathscr{S}} \frac{\mu_F}{1-\lambda_F^m}$$

for a Renyi-development of 1 which is (see §3; Bertrand [7]; Burdik et al [2], Theorem 2.3; Thurston [28])

$$d_\beta(1) = 0.t_1 t_2 \cdots t_q (t_{q+1} t_{q+2} \cdots t_{q+k})^\omega$$

Define

$$M := (2 + \frac{1}{(u_B\cdot u)}) \lfloor\beta\rfloor \sum_{F\in\mathscr{S}} \frac{\mu_F}{1-\lambda_F^m}$$

Proposition 5.1. — *(i) For each $\theta > 0$, there exists an integer $j_0 = j_0(\theta)$ such that $Z_j \in K_\theta$ for all $j \geqslant j_0$.*

(ii) If ${}^tQ$ is nonnegative, and $\min\{\xi_{j,1} \mid j = 1,2,\cdots,m\} > 2(2+\frac{1}{(u_B\cdot u)})^{-1} M$, then $j_0(\theta) = 0$ for all $0 < \theta < \theta_{min}$ where

$$\theta_{min} := -2 + (2 + \frac{1}{(u_B\cdot u)})\, M^{-1} \min\{\xi_{j,1} \mid j = 1,2,\cdots,m\}$$

Proof. — (i) Fix an arbitrary $\theta > 0$. We have just to prove that $\pi(Z_j)\cdot u$ tends to $+\infty$ and not to $-\infty$ when j goes to $+\infty$. Write

$$Z_j = \pi(Z_j) + (Z_j - \pi(Z_j)) = \pi^{||}(Z_j) + \pi^{\perp}(Z_j), \qquad j \geqslant 0$$

hence

$$\|\,\pi(Z_j) - \pi^{||}(Z_j)\,\| = \|\,\pi^{\perp}(Z_j) - (Z_j - \pi(Z_j))\,\| \leqslant \|\,\pi^{\perp}(Z_j)\,\| + \|\,Z_j - \pi(Z_j)\,\|$$

$$= \|\,\pi^{\perp}(Z_j - \pi(Z_j))\| + \|Z_j - \pi(Z_j)\,\| \leqslant 2\lfloor\beta\rfloor \sum_{F\in\mathscr{S}} \frac{\mu_F}{1-\lambda_F^m} \qquad (1)$$

On the other hand

$$\|\,\pi_B(\pi^{||}(Z_j)) - \pi_B(Z_j)\,\| = \|\,(Z_j\cdot u)(u_B\cdot u)u_B - \|B\|^{-1}\beta^j u_B\,\|$$

$$= |\,(Z_j\cdot u)(u_B\cdot u) - \|B\|^{-1}\beta^j\,| \leqslant \lfloor\beta\rfloor \sum_{F\in\mathscr{S}} \frac{\mu_F}{1-\lambda_F^m} \qquad (2)$$

Hence, since $u_B\cdot u > 0$ from theorem 4.6,

$$|\,Z_j\cdot u - (u_B\cdot u)^{-1}\|B\|^{-1}\beta^j\,| \leqslant \frac{\lfloor\beta\rfloor}{(u_B\cdot u)} \sum_{F\in\mathscr{S}} \frac{\mu_F}{1-\lambda_F^m} \qquad (3)$$

Consequently

$$\|\,\pi(Z_j) - (u_B\cdot u)^{-1}\|B\|^{-1}\beta^j u\,\| = \|\,\pi(Z_j) - \pi^{||}(Z_j) + \pi^{||}(Z_j) - (u_B\cdot u)^{-1}\|B\|^{-1}\beta^j u\,\|$$

$$\leqslant \|\,\pi(Z_j) - \pi^{||}(Z_j)\,\| + \|\,\pi^{||}(Z_j) - (u_B\cdot u)^{-1}\|B\|^{-1}\beta^j u\,\|$$

$$\leqslant (2+\frac{1}{(u_B\cdot u)})\lfloor\beta\rfloor \sum_{F\in\mathscr{S}} \frac{\mu_F}{1-\lambda_F^m} \;=\; M \tag{4}$$

We obtain that $\pi(Z_j)\cdot u$ tends to $+\infty$ as $(u_B\ \cdot\ u)^{-1}\,\|B\|^{-1}\beta^j u$ when $j\to+\infty$. Denote

$$\kappa := 1+\frac{\theta}{(2+(u_B\cdot u)^{-1})} > 1$$

There exists j_0 such that

$$Z_j\cdot u \geqslant \kappa M,\quad j\geqslant j_0$$

A a consequence

$$\pi(Z_j)\cdot u\ \geqslant\ Z_j\cdot u-(u_B\cdot u)^{-1}\|B\|^{-1}\beta^j \geqslant (\kappa-1)M>0,\qquad j\geqslant j_0$$

We claim that $Z_j\in K_\theta$ for all $j\geqslant j_0$. Indeed, since $\|\pi_{D'}(Z_j)\|\leqslant(2+(u_B\cdot u)^{-1})^{-1}M$, the inequality $\theta\|\pi_{D'}(Z_j)\|\leqslant\|\pi(Z_j)\|=\pi(Z_j)\cdot u$ is satisfied for all $j\geqslant j_0$.

(ii) Computation of j_0. — Assume all the entries of the companion matrix ${}^tQ$ of β nonnegative. Recall that all the coefficients $\{a_j\}_{j=0,1,\cdots m-1}$ arising from the minimal polynomial $P(X)$ are nonnegative, with $a_0\neq 0$ and at least one of a_k 's, $k=1,2,\cdots,m-1$ non zero since β is assumed to be a Pisot number and not a Salem number. Hence, from proposition 4.5, we have

$$\xi_{1,1}=\pi^{||}(Z_0)=Z_0\cdot u=\frac{a_0}{\beta P'(\beta)}>0$$

and $\xi_{j+1,1}=\pi^{||}(Z_j)=Z_j\cdot u,\qquad j=1,2,\cdots m-1$ with

$$\min\{\xi_{j,1}\mid j=1,2,\cdots m-1\}\geqslant\frac{a_0}{\beta^m P'(\beta)}>0$$

since $P'(\beta)>0$. Because $\{Z_0,Z_1,\cdots,Z_{m-1}\}$ is the canonical basis of $\mathbb{R}^m$, any $Z_j, j\geqslant m$, can be written as an integral combination of the elements of this basis with positive coefficients. Hence,

$$Z_j\cdot u\geqslant\min\{\,\xi_{l,1}\mid l=1,2,\cdots,m-1\,\}\qquad j\geqslant 0$$

But

$$|\pi(Z_j)\cdot u-Z_j\cdot u|\leqslant 2(2+(u_B\cdot u)^{-1})^{-1}M\qquad j\geqslant 0$$

Therefore, by assumption,

$$\pi(Z_j)\cdot u\geqslant\min\{\,\xi_{l,1}\mid l=1,2,\cdots,m-1\,\}-2(2+(u_B\cdot u)^{-1})^{-1}M>0\qquad j\geqslant 0$$

Hence, by definition of θ_{min} , for all $j\geqslant 0$

$$\|\pi(Z_j)\|\geqslant\theta_{min}(2+(u_B\cdot u)^{-1})^{-1}M\geqslant\theta\|\pi_{D'}(Z_j)\|$$

that is, for all $j\geqslant 0$, for all $\theta, 0<\theta\leqslant\theta_{min}$, we have $Z_j\in K_\theta$. □

We have more.

Proposition 5.2. — *For all* $g = \sum_{i=0}^{k} x_i Z_i \in \mathscr{L}_{\mathscr{K}}$ *, we have*

$$\|\pi(g) - \|V_1\| \left(\sum_{i=0}^{k} x_i \beta^i\right) u\| \leqslant M \tag{5}$$

Proof. — The inequalities (1), (2), (3) and (4) in the proof of proposition 5.1 (i) are valid for arbitrary elements of $\mathscr{L}_{\mathscr{K}}$. Since $u \cdot u_B = \|B\|^{-1}\|V_1\|^{-1}$ by theorem 4.6, we deduce the result.

□

We have now the following situation. Up to the scaling factor $\|B\|^{-1}$, $\mathbb{Z}_\beta$ is obtained isometrically by orthogonal projection to the line $\mathbb{R}u_B$. Recall that

$$\pi_B(g) \cdot u_B = \|B\|^{-1} \sum_{i=0}^{k} x_i \beta^i \tag{6}$$

This exact result on the dominant eigenspace $\mathbb{R}u_B$ of the adjoint operator Q is replaced, on the dominant eigenspace $\mathbb{R}u$ of the companion matrix ${}^tQ$ by an approximate linear mapping L (in the sense of Moody [37], §8): it can be formulated by

$$L : \mathscr{L}_{\mathscr{K}} \to \mathbb{R}, \qquad g \to L(g) = \pi(g) \cdot u$$

with $\mathbb{Z}[\mathscr{L}_{\mathscr{K}}] = \mathbb{Z}^m$. L is an homomorphism (defined in [37]) from $\mathscr{L}_{\mathscr{K}}$ to $\mathbb{R}$ and from $\pm\mathscr{L}_{\mathscr{K}}$ to $\mathbb{R}$ by extension. In some sense, $\mathbb{Z}_\beta$ is obtained *"folded"* from $\pm\mathscr{L}_{\mathscr{K}}$ on the line $\mathbb{R}u$ with prototiles possibly counted negatively if the matrix ${}^tQ$ has negative entries, as it can be seen from equation (5).

Remark . — (i) Even though the matrix ${}^tQ$ has negative entries, it may occur that for certain values of $\theta > 0$ we have $j_0(\theta) = 0$. Hence $j_0(\theta') = 0$ for all $\theta',\ 0 < \theta' \leqslant \theta$. In other terms, a cone K_θ may contain all the Z_j's. We shall be concerned with maximal values of θ for which $j_0(\theta)$ has minimal value.

(ii) Let $\theta > 0$. For any $g \in \mathscr{L}_{\mathscr{K}}$ and any $\lambda > 0$ such that $\lambda g \in \mathscr{L}_{\mathscr{K}}$, we have $g \in K_\theta \Longrightarrow \lambda g \in K_\theta$. This property comes from the definition of the cone K_θ .

We now turn to the question of generating the elements of $\mathscr{L}_{\mathscr{K}}$ by some of them over the positive integers. We need at first a lemma.

Lemma 5.3. — *[17] Let* $\theta > 0$ *. If* $\delta = (2\theta + 2)^{-1}$ *and* $x \in K_{2\theta}$ *with* $\|\pi(x)\| = \pi(x) \cdot u > 4$*, then*

$$[x - K_\theta(1,3)] \cap K_{2\theta} \quad \text{contains a ball of radius } \delta.$$

Proof. — [17] Take $y = 2u + 3(\pi(x) \cdot u)^{-1}\pi_{D'}(x)$. We show that the ball centred at $x - y$ and of radius δ satisfies our claim. Suppose $\|z\| < \delta$. Then $x - y + z \in K_{2\theta}$. Indeed,

$$2\theta\,\|\pi_{D'}(x - y + z)\| \leqslant 2\theta\, \left[\left(1 - \frac{3}{(\pi(x) \cdot u)}\right)\|\pi_{D'}(x)\| + \delta\right]$$

$$\leqslant \left[1 - \frac{3}{(\pi(x)\cdot u)}\right](\pi(x)\cdot u) + \frac{2\theta}{2\theta+2} = (\pi(x)\cdot u) - 2 - \frac{2}{2\theta+2}$$

but $\pi_{D'}(y) = 2$. We deduce

$$2\theta\,\|\pi_{D'}(x-y+z)\| \leqslant \pi(x-y+z)\cdot u$$

Let us show that $y - z \in K_\theta$. We have

$$2\theta\|\pi_{D'}(y)\| = 2\theta \times 3(\pi(x)\cdot u)^{-1}\|\pi_{D'}(x)\| \leqslant 3(\pi(x)\cdot u)^{-1}(\pi(x)\cdot u) = 3$$

therefore

$$\theta\|\pi_{D'}(y-z)\| \leqslant \theta(\|\pi_{D'}(y)\| + \delta) \leqslant \frac{3}{2} + \frac{\theta}{2\theta+2} = 2 - \frac{1}{2\theta+2} \leqslant \pi(y-z)\cdot u$$

Now, since $\delta < 1$, we have the inequalities $1 \leqslant \pi(y-z)\cdot u \leqslant 3$, establishing the result. □

THEOREM 5.4. — *Let $\theta > 0$. For any r such that $r > \rho(2\theta+2)$, we have*

$$K_{2\theta} \cap \mathscr{L}_{\mathscr{K}} \subset sg(K_\theta(r) \cap \mathscr{L}_{\mathscr{K}}) \tag{7}$$

Proof. — Lemma (5.3) implies the following assertion: if $x \in K_{2\theta}$ is such that $\pi(x)\cdot u > 4r$ with $r > \rho(2\theta+2)$, then $[x - K_\theta(r, 3r)] \cap K_{2\theta}$ contains a ball of radius $r\delta > \rho$. But ρ is by definition the covering radius of $\pm\mathscr{L}_{\mathscr{K}}$, hence this ball intersects $\mathscr{L}_{\mathscr{K}}$.

Let $\mathscr{A} = K_\theta(4r) \cap \mathscr{L}_{\mathscr{K}}$ the finite point set of $\mathscr{L}_{\mathscr{K}}$. We show that $K_{2\theta} \cap \mathscr{L}_{\mathscr{K}} \subset sg(\mathscr{A})$. Indeed, $K_{2\theta}(4r) \cap \mathscr{L}_{\mathscr{K}} \subset sg(\mathscr{A})$. We now proceed inductively. Suppose $K_{2\theta}(r') \cap \mathscr{L}_{\mathscr{K}} \subset sg(\mathscr{A})$ for some $r' \geqslant 4r$. We show that this implies $K_{2\theta}(r'+r) \cap \mathscr{L}_{\mathscr{K}} \subset sg(\mathscr{A})$, which suffices by induction.

Take $g \in \mathscr{L}_{\mathscr{K}} \cap [K_{2\theta}(r'+r)\ K_{2\theta}(r)]$. From the preceding lemma and the above, there exists an element, say y , in $\mathscr{L}_{\mathscr{K}}$, contained in $[g - K_\theta(r, 3r)] \cap K_{2\theta}(r')$. By assumption, $y \in sg(\mathscr{A})$ and $y = g - x$ for some $x \in K_\theta(r, 3r) \cap \mathscr{L}_{\mathscr{L}} \subset sg(\mathscr{A})$. Therefore $g = x + y \in sg(\mathscr{A}) + sg(\mathscr{A}) \subset sg(\mathscr{A})$. This concludes the induction. □

LEMMA 5.5. — *For any $\theta > 0$, the set*

$$\mathscr{L}_{\mathscr{K}}(\theta) := \{\, x \in \mathscr{L}_{\mathscr{K}} \mid \pi_{D'}(x) \in \Omega, x \notin K_\theta(\rho(2\theta+2)), x \notin K_{2\theta} \,\}$$

is finite.

Proof. — This is clear since all the elements g of $\mathscr{L}_{\mathscr{K}}$ such that $\pi(g)\cdot u > 2\rho(2\theta+2)$ belong to $K_{2\theta}$. □

We are concerned with putting possible additive structures on the whole of $(\pm)\mathscr{L}_{\mathscr{K}}$ (and therefore on $\mathbb{Z}_\beta$ by projection by π_B) and try to have cones of the type $K_{2\theta}$ for θ well-chosen, that is for values of θ small enough: this has for consequence to eliminate a minimal number of elements of $(\pm)\mathscr{L}_{\mathscr{K}}$ lying outside such cones. Unfortunately, there is no reason that this should be tractable when the coefficients a_i are negative, or even positive.

Define

$$\theta_f := \max\{\theta > 0 \mid \#(\mathscr{L}_{\mathscr{K}}(\theta)) \text{ is minimal }\}$$

From proposition 5.1, if ${}^tQ$ is nonnegative and the condition (ii) satisfied, we see that $\#(\mathscr{L}_{\mathscr{K}}(\theta)) = 0$ when θ is close to zero and precisely we have $\theta_f \geqslant \theta_{min}/2 > 0$.

THEOREM 5.6. — *Minimal decomposition. — Any element*

$$g \in \mathscr{L}_{\mathscr{K}} \setminus \mathscr{L}_{\mathscr{K}}(\theta_f)$$

can be expressed as a integral nonnegative combination of elements of the finite point set

$$K_{\theta_f}(\rho(2\theta_f + 2)) \cap \mathscr{L}_{\mathscr{K}}$$

Proof. — Applying theorem 5.4 with $\theta = \theta_f$ and $r = \rho(2\theta_f + 2)$ gives the result. □

The ideal situation would be to deal with $\mathscr{L}_{\mathscr{K}}(\theta_f)$ reduced to the empty set. By projection to $\mathbb{R}u_B$, we would obtain any element of $\mathbb{Z}_\beta$ from a finite number of elements of small norm of this set. We deduce from proposition 5.1 that this situation is more likely to occur when ${}^tQ$ is nonnegative. Theorem 5.6 implies that there exist elements $g_1, g_2, \cdots, g_\eta \in \mathbb{Z}_\beta^+$ of small norm such that the semi-group $\mathbb{N}[g_1, g_2, \cdots, g_\eta]$ contains $\mathbb{Z}_\beta^+$ except possibly a finite number of elements close to the origin.

B) Let us turn now to the geometrical counterpart of the sets T and T' of §1. Let us make at first some remarks concerning the sharpness of the band defined by Ω about the eigenspace $\mathbb{R}u$ with respect to the β- numeration. Similarly as in §4, equation (10), we denote

$$\mathscr{C}^{\pm} \;=\; \{\; \sum_{j=0}^{m-1} \alpha_j Z_j \mid \alpha_j \in [-1;1] \text{ for all } j = 0, 1, \cdots, m-1 \,\} \tag{8}$$

We have

$$\delta_F = \max_{x \in \mathscr{C}^{\pm}} \|\pi_F(x)\|, , \qquad F \in \mathscr{S}$$

Hence the band

$$\{v \in \mathbb{R}^m \mid v^* \in \Omega\}$$

could have been defined from $\mathscr{C}^{\pm}$ instead of $\mathscr{C}$ in §4. This means that the elements of $\mathscr{L}_{\mathscr{K}}$ only occupy a part of this band, that this band is not sharp for the β- integers lifted up to $\mathbb{Z}^m$. This absence of sharpness may be observed, with the τ- integers, in figure 3 in [2] where a shift occurs in the drawing close to the origin. Indeed, the band defined by $\mathscr{C}^{\pm}$ is symmetrical with respect to the origin (invariant by inversion) and this is not the case of the one represented in that contribution.

Let $R > 0$ and I be an arbitrary interval of $\mathbb{R}$. We denote by

$$\mathscr{T}_{I,R} := \{x \in \mathbb{R}^m \mid \pi_{D'}(x) \in \frac{R}{\lfloor \beta \rfloor}\Omega, \pi_B(x) \cdot u_B \in \frac{I}{\|B\|}\}$$

the slice of the band defined by $\frac{R}{\lfloor \beta \rfloor}\Omega$ and $\frac{I}{\|B\|}$ about the eigenspace $\mathbb{R}u$.

LEMMA 5.7. — *Let* $R \geqslant 0$ *and* $F_R = \{\mathrm{frac}(z) \mid z = a_k\beta^k + a_{k-1}\beta^{k-1} + \cdots + a_1\beta + a_0, a_i \in \mathbb{Z}, |a_i| \leqslant R\} \subset [0,1)$ *. Then* F_R *is a finite subset of* $\mathbb{Z}[\beta]$ *and*

$$F_R \subset \{\|B\|\pi_B(g) \cdot u_B \mid g \in \mathscr{T}_{[0,1),R+\lfloor\beta\rfloor} \cap \mathbb{Z}^m\}$$

In particular Card$(F_R) \leqslant$ *Card*$(\mathscr{T}_{[0,1),R+\lfloor\beta\rfloor} \cap \mathbb{Z}^m)$.

Proof. — This is a reformulation of lemma 2.1 in Burdik et al [2] and a consequence of lemma 5.8. We recall the proof for the sake of completeness. Let $z = \sum_{j=0}^{k} a_j\beta^j$ with $a_i \in \mathbb{Z}, |a_i| \leqslant R$. We have also $z = \sum_{j=-\infty}^{k} x_j\beta^j$ as β - expansion of z. Therefore $z - \mathrm{int}(z) = \sum_{i=0}^{k} a_j\beta^k - \sum_{j=0}^{k} x_j\beta^j$. Since $0 \leqslant x_j \leqslant \lfloor\beta\rfloor$ and $|a_i| \leqslant R$, $\mathrm{frac}(z) \in [0,1)$ is a polynomial in β, the coefficients of which are bounded by $R+\lfloor\beta\rfloor$. We deduce the result from proposition 4.10 (i) and the computations of the upper bounds c_F in the proof of theorem 4.14; indeed, these bounds are calculated under the assumption that the digits are less than $\lfloor\beta\rfloor$. Here, we have to consider that the absolute values of the coefficients are not bounded by $\lfloor\beta\rfloor$ but by $R + \lfloor\beta\rfloor$ inducing the scaling factor $(R + \lfloor\beta\rfloor)/\lfloor\beta\rfloor$ of Ω. From proposition 4.10 (i), F_R is finite and is in one-to-one correspondence with a subset of the finite point set $\mathscr{T}_{[0,1),R+\lfloor\beta\rfloor}$ establishing the last upper bound of the claim. □

LEMMA 5.8. — *The set* F_R *is a finite subset of* $\mathbb{Z}[\beta] \cap [0,1)$.

Proof. — See Solomyak ([21], Lemma 6.6). □

Assume $R > 0$ and I any relatively compact interval of the line, then the set $\mathscr{T}_{I,R}$ is relatively compact. We denote by

$$\psi_{I,R} := \max\{\|y\| \mid y \in \mathscr{T}_{I,R}\}$$

A. Bertrand [7] and K. Schmidt [8] have proved that β - expansions of elements of $\mathbb{Q}(\beta)$ are periodic (theorem 3.3). We will use the formulation of K. Schmidt to control the maximal length of the *preperiod* of the β -expansions of the elements of F_R, to provide an upper bound for the value of L in eq. (3) of §1.

The important rôle played by the set Per(β) of periodic points under T given by (see §1 and Schmidt [8]) $Tx = \{\beta x\},\ x \in [0,1)$, in the study of subshifts of finite type of the β-shift was already outlined by Parry [5] and considered by many authors. Recall that $x \in [0,1)$ is said to be periodic under T if the set $\{T^l(x) \mid l \geqslant 0\}$ is finite. For any $x \in \mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0,1)$, we put

$$r^*(x) = \begin{cases} \min\{l \geqslant 1 \mid T^l x = x\} & \text{if this set is not empty} \\ \infty & \text{otherwise} \end{cases} \tag{9}$$

$$r(x) = \min_{m \geqslant 0}\ r^*\left(T^m(x)\right) \tag{10}$$

$r(x)$ is the length of the period in the β-expansion of x. Recall that x is said to be strictly periodic under T if $r^*(x) < \infty$, in which case $r(x) = r^*(x)$.

A priori we do not know if F_R (in lemma 5.8) contains few strictly periodic elements, or, on the contrary, is only composed of such elements. For every $z \in F_R$, the β - expansion of z :

$\sum_{j=1}^{+\infty} z_{-j}\beta^{-j}$ can be written, for some minimal integers $j_0(z), r(z) \geqslant 1$

$$\sum_{j=1}^{j_0(z)-1} z_{-j}\beta^{-j} + \sum_{k=0}^{+\infty} \sum_{j=j_0(z)+kr(z)}^{j_0(z)+(k+1)r(z)-1} z_{-j}\beta^{-j} \tag{11}$$

We denote by $J_R = \max\{j_0(z) \mid z \in F_R\}$ and call it the maximal preperiod of the β - expansions of the elements of F_R .. An upper bound for J_R will be computed below.

THEOREM 5.9. — *Let $R > 0$ and I a relatively compact interval of the line. Denote by*

$$L_{I,R} = \lfloor \min \left\{ \frac{\log(\, \|X_i\| \, \psi_{I,R+\lfloor\beta\rfloor})}{\log(\, \beta^{(i-1)^{-1}}\,)} \right\} \rfloor \tag{12}$$

where the minimum is taken over the real positive embeddings of $\mathbb{Q}(\beta)$, i.e. $i = 2, \cdots, s$ such that $\beta^{(i-1)}$ is a real and positive conjugate of β .

For any $x, y \in \mathbb{Z}_\beta^+$, such that $x + y$ has a finite β-expansion, we have

$$x + y \in \frac{1}{\beta^L}\,\mathbb{Z}_\beta^+$$

with $L = \min\{L_{[0,1),2\lfloor\beta\rfloor}, J_{2\lfloor\beta\rfloor}\}$.

Proof. — Let $x = x_k\beta^k + \cdots + x_0$ and $y = y_l\beta^l + \cdots + y_0$ denote two elements of $\mathbb{Z}_\beta^+$. Then $z = x + y$ is of the form $z = a_j\beta^j + \cdots + a_0$ with $0 \leqslant a_j \leqslant 2\lfloor\beta\rfloor$. Write now the β - expansion of z as

$$z = \sum_{j=1}^{+\infty} z_{-j}\beta^{-j} + \sum_{j=0}^{e} z_j\beta^j$$

and assume it is finite. Then

$$\sum_{j=1}^{J_{2\lfloor\beta\rfloor}} z_{-j}\beta^{-j} = (a_j\beta^j + \cdots + a_0) - (\sum_{i=0}^{e} z_i\beta^i) \tag{13}$$

This means that the fractional part $\sum_{j=1}^{J_{2\lfloor\beta\rfloor}} z_{-j}\beta^{-j}$ is a polynomial of the type $\sum_{i=0}^{f} b_i\beta^i$ with $-\lfloor\beta\rfloor \leqslant b_i \leqslant 2\lfloor\beta\rfloor$ hence with $|b_i| \leqslant 2\lfloor\beta\rfloor$. By lemma 5.7, $F_{2\lfloor\beta\rfloor}$ is finite and the set of all possible fractional parts of elements of $\mathbb{Z}_\beta^+$ is exactly in one-to-one correspondence with a subset of the finite point set $\mathscr{T}_{[0,1),3\lfloor\beta\rfloor} \cap \mathbb{Z}^m$ of $\mathbb{Z}^m$. Therefore, there exists a unique $g_z = \sum_{i=0}^{f} b_i Z_i \in \mathscr{T}_{[0,1),3\lfloor\beta\rfloor} \cap \mathbb{Z}^m$ such that $\|B\|\pi_B(g_z) \cdot u_B = \sum_{i=0}^{f} b_i\beta^i = \sum_{j=1}^{J_{2\lfloor\beta\rfloor}} z_{-j}\beta^{-j} = \mathrm{frac}(z)$. Applying the real and complex embeddings of the number field $\mathbb{Q}(\beta)$ to eq. (13) gives

$$\sum_{j=1}^{J_{2\lfloor\beta\rfloor}} z_{-j}\beta^{(i-1)^{-j}} = \sum_{j=0}^{f} b_j\beta^{(i-1)^j} \qquad \text{for all } i = 2, \cdots, m$$

Therefore from proposition 4.10 we have for all $i = 1, 2, \cdots, s$

$$\pi_{B,i}(g_z) = \pi_{B,i}(\sum_{j=0}^{f} b_j Z_j) = \frac{\sum_{j=0}^{f} b_j\beta^{(i-1)^j}}{\|X_i\|}\; u_{B,i} = \frac{\sum_{j=1}^{J_{2\lfloor\beta\rfloor}} z_{-j}\beta^{(i-1)^{-j}}}{\|X_i\|}\; u_{B,i}$$

with all $z_{-j} \geqslant 0$; similarly for all $i = s+1, \cdots, m$ with $i-(s+1)$ even on the invariant planes. Hence, an eigenspace of Q is indexed by an unique conjugate $\beta^{(i-1)}$ of β . There are three categories of eigenspaces of the matrix Q : those for which the conjugate $\beta^{(i-1)}$ (i) is real and positive, (ii) is real and negative, (iii) is complex with non trivial imaginary part. Some cases may not exist. Each case is associated with a collection of eigenspaces of Q , possibly empty. We separate out the first case.

Case (i): In this case, $i \in \{2, 3, \cdots, s\}$ with s assumed to be greater than 2 . Since all these conjugates $\beta^{(i-1)}$, if any, are positive and strictly smaller than 1 and the digits z_{-j} are positive, we have necessarily

$$\frac{\left(\beta^{(i-1)^{-1}}\right)^j}{\|X_i\|} \geqslant \Psi_{[0;1),3\lfloor\beta\rfloor}$$

as soon as j is large enough. With the definition of $L_{[0;1),2\lfloor\beta\rfloor}$, we see that the sum of positive terms $\sum_{j=1}^{J_{2\lfloor\beta\rfloor}} z_{-j}\beta^{(i-1)^{-j}}$ does not contain any term indexed by $-j$ with $j > L_{[0;1),2\lfloor\beta\rfloor}$.

The other possible cases (ii) and (iii) will be treated by the computation of $J_{2\lfloor\beta\rfloor}$ which will follow below. □

Theorem 5.10. — *If β is a Pisot number of degree $m \geqslant 2$, then*

$$\text{(i)} \qquad \mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+ + T$$

$$\text{(ii)} \qquad \mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta + T'$$

with

$$T = \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathscr{T}_{[0,+1),3\lfloor\beta\rfloor} \cap \mathbb{Z}^m\},$$

$$T' = \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathscr{T}_{(-1,+1),2\lfloor\beta\rfloor} \cap \mathbb{Z}^m\}$$

Proof. — This is a reformulation of theorem 2.4 in [2]: at first, we have $F_{\lfloor\beta\rfloor} \subset F_{2\lfloor\beta\rfloor}$, second $\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+ + F_{2\lfloor\beta\rfloor}$, $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta \pm F_{\lfloor\beta\rfloor}$. From lemma 5.7, we deduce the result since $\Omega^\pm$ is invariant by inversion and $F_{\lfloor\beta\rfloor} \cup -F_{\lfloor\beta\rfloor} \subset \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathscr{T}_{[0,+1),2\lfloor\beta\rfloor} \cap \mathbb{Z}^m\} \cup \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathscr{T}_{(-1,0],2\lfloor\beta\rfloor} \cap \mathbb{Z}^m\} = \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathscr{T}_{(-1,+1),2\lfloor\beta\rfloor} \cap \mathbb{Z}^m\}$. □

Corollary 5.11. — *Let β be a Pisot number of degree $m \geqslant 2$. Denote by*

$$\mathscr{L} = \min\{L_{(-1,+1),2\lfloor\beta\rfloor}, J_{2\lfloor\beta\rfloor}\}$$

Let $x, y \in \mathbb{Z}_\beta$. If $x+y$ and $x-y$ have finite β-expansions, then

$$x+y \ \ (\text{resp. } x-y) \ \in \frac{1}{\beta^{\mathscr{L}}}\,\mathbb{Z}_\beta$$

Proof. — Indeed, $T' \subset \pm T$. Hence $\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta \pm T$. Since $\pm T = \{\|B\|\pi_B(g)\cdot u_B \mid g \in \mathscr{T}_{(-1,+1),3\lfloor\beta\rfloor} \cap \mathbb{Z}^m\}$, the relevant quantity in the definition of L of theorem 5.9 becomes $\Psi_{(-1,1),3\lfloor\beta\rfloor}$ instead of $\Psi_{[0,1),3\lfloor\beta\rfloor}$. □

We now compute an upper bound of J_R following an idea of Schmidt [8]. This will allow to determine explicitely the maximal possible value of L in theorem 5.9 and $\mathscr{L}$ in corollary 5.11. First, we need a lemma.

Let $\{Z_{-j}\}_{j\geqslant 0}$ the sequence of vectors defined by $Z_0 = ({}^tQ)^j Z_{-j}$. We denote as usual the algebraic norm of β by $N(\beta) = N_{\mathbb{Q}(\beta)/\mathbb{Q}}(\beta) = \prod_{i=0}^{m-1}\beta^{(i)}$. Recall that $a_0 = (-1)^{m-1}N(\beta)$.

LEMMA 5.12. — *We have: (i)* $\lim_{j\to+\infty}\|Z_{-j}\| = +\infty$ *; (ii) For all* $j\in\mathbb{N}$, $Z_{-j}\in\frac{1}{N(\beta)^j}\mathbb{Z}^m$ *with* $\pi_B(Z_{-j})\cdot u_B = \|B\|^{-1}\beta^{-j}$. *In particular, if* β *is a unit of the number field* $\mathbb{Q}(\beta)$ *, then all the elements* Z_{-j} *will belong to* $\mathbb{Z}^m$.

Proof. — (i) Since all the conjugates of β are within the open unit disc in the complex plane, their inverse are outside it and the inverse operator $({}^tQ)^{-1}$ has the diagonal form (see corollary 4.7)

$$Diag(\beta^{-1},\beta^{(1)^{-1}},\cdots,\beta^{(s-1)^{-1}},D_1^{-1},D_2^{-1},\cdots,D_t^{-1})$$

in the basis $\{V_i\}_{i=1,2,\cdots,m}$ with the inverse of the corresponding real 2×2 Jordan blocks. Hence, all the non zero components of a vector Z_{-j} in this basis diverge when j tends to infinity.

(ii) Solving the equation $Z_0 = ({}^tQ)^1 Z_{-1}$ shows that Z_{-1} can be written

$$Z_{-1} = -a_0^{-1}\left(a_1Z_0 + a_2Z_1 + \cdots + a_{m-1}Z_{m-2} - Z_{m-1}\right) \in \frac{1}{N(\beta)}\mathbb{Z}^m \tag{14}$$

Since by construction we have $Z_j = ({}^tQ)^{-1}(Z_{j+1})$ for all $j\in\mathbb{Z}$, applying $({}^tQ)^{-1}$ to equation (14) clearly gives $Z_{-2}\in\frac{1}{N(\beta)^2}\mathbb{Z}^m$ and, by induction $Z_{-h}\in\frac{1}{N(\beta)^h}\mathbb{Z}^m$ for all $h\geqslant 0$. Now it is classical that β is a unit of $\mathbb{Q}(\beta)$ is and only if $N(\beta)=\pm 1$ establishing the result. □

THEOREM 5.13. — *Denote by*

$$\mathscr{B}_R = \{x\in\mathbb{R}^m \mid \|\pi_{B,i}(x)\| \leqslant \frac{\Psi_{[0;1),R+\lfloor\beta\rfloor}(1-|\beta^{(i-1)}|^m)+\lfloor\beta\rfloor}{\|B^{(i-1)}\|(1-|\beta^{(i-1)}|)},\quad i=2,3,\cdots,m\}$$

the cylinder about the dominant eigenspace $\mathbb{R}u$ *and*

$$\mathscr{V}_R = \{x\in\mathscr{B}_R \mid \|B\|\pi_B(x)\cdot u_B\in[0,1)\}$$

the slice of the band $\mathscr{B}_R$ *. This slice is relatively compact and*

$$J_R \leqslant \mathrm{Card}(\mathscr{V}_R\cap\frac{1}{N(\beta)^m}\mathbb{Z}^m)$$

Proof. — Each element $\alpha\in F_R$ can be written

$$\alpha = \sum_{i=0}^{m-1} p_i\beta^i \qquad p_i\in\mathbb{Z}$$

with

$$\sum_{i=0}^{m-1} p_iZ_i \in \mathscr{T}_{[0,1),R+\lfloor\beta\rfloor}\cap\mathbb{Z}^m$$

from lemma 5.7. Thus, for all $i = 0, 1, \cdots, m-1$, $|p_i| \leqslant \Psi_{[0,1),R+\lfloor\beta\rfloor}$. Now, Schmidt [8] has proved that the n-th iterated shift of α is given by

$$T^n(\alpha) = \beta^n \cdot \left(\alpha - \sum_{k=0}^{n} \epsilon_k(\alpha)\beta^{-k}\right) = \sum_{k=1}^{m} r_k^{(n)}\beta^{-k}$$

where $(\epsilon_k(\alpha))_{k\geqslant 0}$ is the β-expansion of α and $(r_1^{(n)}, r_2^{(n)}, \cdots, r_m^{(n)}) \in \mathbb{Z}^m$. Recall that $\epsilon_0(\alpha) = \lfloor\alpha\rfloor = 0$. Moreover, he has shown that the real and complex embeddings of the number field $\mathbb{Q}(\beta)$ can be applied to $T^n(\alpha)$ to provide m equalities

$$X^n \cdot \left(\sum_{i=0}^{m-1} p_i X^i - \sum_{k=1}^{n} \epsilon_k(\alpha)X^{-k}\right) = \sum_{k=1}^{m} r_k^{(n)} X^{-k}$$

where X is put for any conjugate $\beta^{(q-1)}, q = 2, \cdots, m$ of β . From this setting, we deduce

$$\begin{aligned}\left|\sum_{k=1}^{m} r_k^{(n)}\beta^{(q-1)^{-k}}\right| &\leqslant \sum_{i=0}^{m-1} |p_i||\beta^{(q-1)}|^{n+i} + \lfloor\beta\rfloor \sum_{k=0}^{n} |\beta^{(q-1)}|^k \\ &\leqslant \frac{1}{1-|\beta^{(q-1)}|}\left[\Psi_{[0,1),R+\lfloor\beta\rfloor}(1-|\beta^{(q-1)}|^m) + \lfloor\beta\rfloor\right]\end{aligned}$$

for every $n \geqslant 0, q = 2, 3, \cdots, m$ and

$$0 \leqslant \sum_{k=1}^{m} r_k^{(n)}\beta^{-k} < 1$$

From proposition 4.10 and lemma 5.12 the element $\sum_{k=1}^{m} r_k^{(n)}\beta^{-k}$ can be uniquely lifted up to the element

$$\sum_{k=1}^{m} r_k^{(n)} Z_{-k} \in \frac{1}{N(\beta)^m}\mathbb{Z}^m$$

Its orthogonal projections by $\pi_{B,i}$, $i = 2, 3, \cdots, m$ to the ${}^tQ$-invariant subspaces of $\mathbb{R}^m$ are bounded by a constant which is independant of n . Therefore the number of points

$$\mathrm{Card}(\mathscr{V}_R \cap \frac{1}{N(\beta)^m}\mathbb{Z}^m)$$

is necessarily an upper bound of J_R . ☐

Acknowledgements.— We are indebted to Christiane Frougny for very useful and valuable comments and discussions.

References

[1] J. P. GAZEAU, Pisot-Cyclotomic Integers for Quasicrystals, in *The Mathematics of Long-Range Aperiodic Order*, Ed. By R.V. Moody, Kluwer Academic Publishers, (1997), 175 - 198.

[2] Č. BURDÍK, CH. FROUGNY, J.-P. GAZEAU AND R. KREJCAR, Beta-integers as natural counting systems for quasicrystals, J. Phys. A: Math. Gen. **31**, (1998), 6449 - 72.

[3] R.V. MOODY, Meyer sets and their duals, in *The Mathematics of Long-Range Aperiodic Order*, Ed. By R.V. Moody, Kluwer Academic Publishers, (1997), 403-41.

[4] A. RENYI, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hung., **8**, (1957), 477-93.

[5] W. PARRY, On the β - expansions of real numbers, Acta Math. Acad. Sci. Hung., **11**, (1960), 401-16.

[6] A. BRAUER, On algebraic equations with all but one root in the interior of the unit circle, Math. Nachr. **4**, (1951), 250 -257.

[7] A. BERTRAND, Développements en base de Pisot et répartition modulo 1, C. R. Acad. Sc. Paris, Série A, t. 285, (1977), 419-21.

[8] K. SCHMIDT, On periodic expansions of Pisot numbers and Salem numbers, Bull. London Math. Soc. **12**, (1980), 269 - 278.

[9] J. W. S. CASSELS, An introduction to Diophantine Approximation, Cambridge University Press, (1957).

[10] D. RUELLE, Statistical Mechanics; Rigorous results, Benjamin, New York, (1969).

[11] F. R. GANTMACHER, The Theory of Matrices, Chelsea, New York, (1960).

[12] H. MINC, Nonnegative matrices, John Wiley and Sons, New York, (1988).

[13] A.S. Fraenkel, Systems of Numeration, Amer. Math. Monthly, **92**, (1), (1985), 105 - 114.

[14] J. LAGARIAS, Geometric Models for Quasicrystals I. Delone Sets of Finite Type, Discrete Comput. Geom. **21**, no 2, (1999), 161-191.

[15] J.C. LAGARIAS, Geometrical models for quasicrystals. II. Local rules under isometry, Disc. and Comp. Geom., **21**, No 3, (1999), 345-372.

[16] J.C. LAGARIAS, Mathematical Quasicrystals, preprint (1999).

[17] D. LIND, The entropies of topological Markov shifts and a related class of algebraic integers, Ergod. Th. & Dynam. Sys. **4**, (1984), 283 - 300.

[18] Č. BURDÍK, CH. FROUGNY, J.-P. GAZEAU AND R. KREJCAR, Beta-integers as a group, preprint, (1999).

[19] A. BERTRAND - MATHIS, Comment écrire les nombres entiers dans une base qui n'est pas entière, Acta Math. Hung., **54**, (3-4), (1989), 237-241.

[20] CH. FROUGNY AND B. SOLOMYAK, Finite beta-expansions, Ergod. Theor. Dynam. Syst., **12**, (1992), 713 - 723.

[21] B. SOLOMYAK, Dynamics of Self-Similar Tilings, Ergod. Th. & Dynam. Sys. **17**, (1997), 695 - 738.

[22] M. W. HIRSCH AND S. SMALE, Differential Equations, Dynamical Systems and Linear Algebra, Academic Press, New York, (1974).

[23] R. DESCOMBES, Eléments de Théorie des Nombres, PUF, Paris, (1986).

[24] A. BERTRAND - MATHIS, Développement en base θ, Répartition modulo un de la suite $(x\theta^n)_{n\geqslant 0}$. Langages codes et θ - shift, Bull. Soc. Math. France, **114**, (1986), 271-323.

[25] A. BERTRAND - MATHIS, Nombres de Perron et questions de rationnalité, Séminaire d'Analyse, Université de Clermont II, Année 1988/89, Exposé 08; A. BERTRAND - MATHIS, Nombres de Perron et problèmes de rationnalité, Astérisque, **198-199-200**, (1991), 67 - 76.

[26] C.A. ROGERS, Packing and Covering, Cambridge University Press, (1964).

[27] D. LIND, Matrices of Perron numbers, J. of Number Theory, **40**, (1992), 211 - 217.

[28] W. THURSTON Groups, tilings and finite state automata, preprint (1989).

[29] Z. MASÁKOVÁ, J. PATERA AND E. PELANTOVÁ s-convexity, model sets and their relations, preprint (2000).

[30] A. HOF, Diffraction by aperiodic structures, in *The Mathematics of Long-Range Aperiodic Order*, Ed. by R.V. Moody, Kluwer Academic Publishers (1997), 239–268.

[31] A. HOF, On diffraction by aperiodic structures, *Comm. Math. Phys.*, **169** (1995), 25–43.

[32] C. JANOT, Quasicrystals: A Primer, Oxford University Press, Cambridge, (1994), 2nd edition.

[33] G. CHAPUIS, Molecular Geometry of Incommensurate Structures, in Aperiodic '97, Proc. of the Int. Conf. on Aperiodic Crystals, Ed. M. de Boissieu, J.-L. Verger-Gaugry, R. Currat, World Scientific, (1998), 267–276; and references therein.

[34] International Union of Crystallography. Report of the Executive Committee for 1991, *Acta Cryst. A* **48** (1992), 922–946; M. L. Senechal, A critique of the projection method, in *The Mathematics of Long-Range Aperiodic Order*, Ed. by R.V. Moody, Kluwer Academic Publishers (1997), 521–548.

[35] R.V. MOODY, Model sets, a Survey, in From Quasicrystals to More Complex Systems, Springer, Ed. F. Axel, F. Denoyer and J.-P. Gazeau, (2000), 145–166.

[36] Y. MEYER, Algebraic Numbers and Harmonic Analysis, North-Holland (1972).

[37] R.V. Moody, Meyer sets and their duals, in *The Mathematics of Long-Range Aperiodic Order*, Ed. by R.V. Moody, Kluwer Academic Publishers (1997), 403–441.

[38] J. M. Luck, C. Godrèche, A. Janner and T. Janssen, The nature of the atomic surfaces of quasiperiodic self-similar structures, J. Phys. A: Math. Gen., **26**, No 8, (1993), 1951-1999.

[39] J.-P. Allouche and M. Mendès France, Automata and Automatic Sequences, Course 11, in *Beyond Quasicrystals*, Ed. by F. Axel and D. Gratias, Les Editions de Physique, Springer (1995), 293–367.

[40] J.-L. Verger-Gaugry and J. Wolny, Generalized Meyer sets and Thue-Morse quasicrystals with toric internal spaces, J. Phys. A: Math. Gen **32**, No 36, (1999), 6445-60.

[41] J.-L. Verger-Gaugry, J. Wolny and J. Patera, Mathematical quasicrystals with toric internal spaces, diffraction and Thue-Morse sequence, prepublication 458, Institut Fourier, (1999).

[42] N. Cotfas and J.-L. Verger-Gaugry, A mathematical construction of n - dimensional quasicrystals starting from G - clusters, *J. Phys. A: Math. Gen.* **30** (1997), 4283–4291.

[43] Č. Burd´ik, Ch. Frougny, J. P. Gazeau and R. Krejcar, Beta - integers as natural counting systems for quasicrystals, J. Phys. A: Math. Gen., **31** (1998), 6449–6472.

[44] E. Pelantova, J. Patera and Z. Masakova, Self-similar Delone sets and quasicrystals, J. Phys. A: Math. Gen., **21**, (1998), 4927-46.

[45] F. Gähler and R. Klitzing, The diffraction pattern of self-similar tilings, in The Mathematics of Long-Range Aperiodic Order, Ed. by R. V. Moody, Nato series, Kluwer Academic Publ. (1997), 141–174.

[46] R. Riklund, M. Severin and Y. Liu, The Thue-Morse aperiodic crystal, a link between the Fibonacci quasicrystal and the periodic crystal, Int. J. Mod. Phys., **B 1** (1987), 121-132.

[47] Z. Cheng, R. Savit and R. Merlin, Structure and electronic properties of Thue-Morse lattices, Phys. Rev. B, **37** (1988), 4375.

[48] M. Kolář, B. Iochum and L. Raymond, Structure factor of 1D systems (Superlattices) based on two-letter substitution rules I. δ (Bragg) peaks, *J. Phys. A: Math Gen.* **26** (1993), 7343–7366.

[49] J. Patera, Non-crystallographic root systems and quasicrystals, in *The Mathematics of Long-Range Aperiodic Order*, Ed. By R.V. Moody, Kluwer Academic Publishers, (1997), 443 - 465.

[50] Shu-fang Xu An introduction to inverse algebraic eigenvalue problems, Vieweg, (1998).

[51] D. Boyd, The maximal modulus of an algebraic integer, Math. Comp. **45**, (1985), 243- 250 and S17 - S20.

[52] P.M. Gruber and C.G. Lekkerkerker, Geometry of Numbers, 2nd edition, North-Holland, Amsterdam, (1987).

[53] P. Samuel, Théorie algébrique des nombres, Hermann, Paris, (1967).