On the geometric simple connectivity of open manifolds

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Abstract

We prove that there exists an obstruction to an open simply connected n-manifold of dimension $n \geq 5$ being geometrically simply connected. In particular there exist uncountably many simply connected n-manifolds which are not w.g.s.c. We also proves that for $n \neq 4$ an n-manifold proper homotopy equivalent to a w.g.s.c. polyhedron is w.g.s.c. (for $n = 4$ it is only end compressible). We analyze further the case $n = 4$ and Poénaru's conjecture.

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Contents

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1 Introduction

The problem we address in this paper is whether 1-handles are necessary in a handle decomposition of a simply connected manifold. Moreover we investigate when it is possible to kill 1-handles within the proper homotopy type of a given open manifold.

The relation between algebraic connectivity and geometric connectivity (in various forms) was explored first by E.C.Zeeman (see [28]) in connection with the Poincaré conjecture. Zeeman's definition of the geometric k-connectivity of a manifold amounts to asking that any k-dimensional compact can be engulfed in a ball. His main result was the equivalence of algebraic k-connectivity and geometric k-connectivity for n-manifolds, under the condition $k \leq n-3$. Notice that it makes no difference whether one considers open and compact manifolds. Later C.T.C.Wall ([26]) introduced another concept of geometric connectivity using handle theory which was further developed by V.Poénaru in his work around the Poincaré conjecture. A similar equivalence between the geometric and algebraic connectivities holds but this time one has to replace the previous codimension condition by $k \leq n-4$. In this respect all results in low codimension are hard results. There is also a non-compact version of this definition which we can state precisely as follows:

Definition 1.1. A (non-compact) manifold, possibly with boundary, is geometrically k-connected (abbreviated. g.k.c.) if there exists a (proper) handlebody decomposition without j-handles, for $1 \leq j \leq k$.

Remark 1.1. Handle decomposition are known to exist for all manifolds in the topological, PL and smooth settings, except in the case of topological 4-manifolds. In the latter case the existence of a handlebody decomposition is equivalent to that of a PL (or smooth) structure. However in the open case such a smooth structure always exist (in dimension 4). Although most results below can be restated and proved for other categories, we will restrict ourselves to considering PL manifolds and handle decompositions in the sequel.

We will be mainly concerned with geometric simple connectivity (abbreviated. g.s.c.) in the sequel. A related concept, relevant only in the non-compact case is:

Definition 1.2. A (non-compact) polyhedron P is weakly geometrically simply connected (abbreviated. w.g.s.c.) if $P = \cup_{j=1}^{\infty} K_j$, where K_j are compact sub-polyhedra with $\pi_1(K_j) = 0$. Alternatively, any compact subspace is contained in a simply connected sub-polyhedron.

Remark 1.2. The w.g.s.c. spaces with which we will be concerned in the sequel are usually manifolds. Similar definitions can be given in the case of topological (respectively smooth) manifolds where we require the exhaustions to be by topological (respectively smooth) submanifolds. All results below hold true for this setting too (provided handlebodies exists) except those concerning Dehn exhaustibility, since the later is essentially a PL concept.

The first result of this paper is (see Theorem 3.1 and Proposition 3.4):

Theorem 1.1. *An open simply connected n-manifold which is w.g.s.c. is end compressible. Conversely, in dimension $n \geq 5$ an open manifold which is end compressible is g.s.c.*

Remark 1.3. A similar result holds more generally for non-compact manifolds with boundary, with the appropriate definition of end compressibility and w.g.s.c.

End compressibility (see Definition 3.6) is an algebraic condition which is defined in terms of the fundamental groups of the submanifolds which form an exhaustion. Notice that end compressibility is weaker than simple connectivity at infinity for $n > 3$.

Remark 1.4. If W^k is compact and simply-connected then the product $W^k \times D^n$ with a closed n-disk is g.s.c. if $n + k \geq 5$. However there exist non-compact n-manifolds with boundary which are simply connected but not end compressible (hence not w.g.s.c.) in any dimension n, for instance $W^3 \times M^n$ where $\pi_1^{\infty} W^3 \neq 0$. Notice that $W^k \times int(D^n)$ is g.s.c. for $n \geq 1$ since $\pi_1^{\infty}(W^k \times int(D^n)) = 0$.

We will also prove (see Theorem 4.1):

Theorem 1.2. *There exist uncountably many open contractible n-manifolds for any $n \geq 4$ which are not w.g.s.c.*

The original motivation for this paper was to try to kill 1-handles of open 3-manifolds at least 'stably' (i.e. after stabilising the 3-manifold). The meaning of the word stably in [19], where such results first arose, is to do so at the expense of taking products with some high dimensional compact ball. This was extended in [5, 6] by allowing the 3-manifold be replaced by any other one having the same proper homotopy type. The analogous result is trivially true for $n \geq 5$. We prove that for $n = 4$ a weaker statement holds true (see Theorem 5.1):

Theorem 1.3. *A non-compact n-manifold, for $n \neq 4$, (respectively $n = 4$) which is proper homotopically dominated by (in particular which is proper homotopy equivalent to) a w.g.s.c. polyhedron is w.g.s.c. (respectively end compressible).*

It is very probable that there exist examples of open 4-manifolds which are not w.g.s.c., but their products with a closed ball are w.g.s.c. Thus in some sense the previous result is sharp.

The dimension 4 deserves special attention also because one expects that the w.g.s.c. and the g.s.c. conditions are not equivalent. Specifically V.Poénaru conjectured that:

Conjecture 1.1 (Poénaru Conjecture). *If the interior of a compact contractible 4-manifold with boundary a homology sphere is g.s.c. then the compact 4-manifold is also g.s.c.*

An immediate corollary would be that the interior of a Poénaru-Mazur 4-manifold may be w.g.s.c. but not g.s.c., because some (compact) Poénaru-Mazur 4-manifolds are known to be not g.s.c. (the geometrisation conjecture implies this statement for all 4-manifolds whose boundary is not a homotopy sphere). The proof is due to Casson and it was based on partial positive solutions to the following algebraic conjecture [10, p.117] [13, p.403].

Conjecture 1.2 (Kervaire Conjecture). *Suppose one adds an equal number of generators* $\alpha_1, \ldots, \alpha_n$ *and relations* $r_1, \ldots, r_n$ *to a non-trivial group* G*, then the group* $\frac{G * < \alpha_1, \ldots, \alpha_n >}{<< r_1, \ldots, r_n >>}$ *that one obtains is also non-trivial.*

Casson showed that certain 4-manifolds $(W^4, \partial W^4)$ have no handle decompositions without 1-handles by showing that if they did, then $\pi_1(\partial W^4)$ violates the Kervaire conjecture. Our aim would be to show that most contractible 4-manifolds are not g.s.c., and the method of the proof is to reduce this statement to the compact case. However our methods permit us to obtain only a weaker result, in which one shows that the interior of such a manifold cannot have handlebody decompositions without 1-handles, if the decomposition has also some additional properties (see Theorems 7.2 and 7.3 for precise statements):

Theorem 1.4. *Assume that we have a proper handlebody decomposition without 1-handles for the interior of a Poénaru-Mazur 4-manifold. If there exists a far away intermediary level 3-manifold* M^3 *whose homology is represented by disjoint embedded surfaces and whose fundamental group projects to the trivial group on the boundary, then the compact 4-manifold is also g.s.c. There always exists a collection of immersed surfaces, which might have non-trivial intersections and self-intersections along homologically trivial curves, that fulfills the previous requirements.*

Remark 1.5. Almost all of this paper deals with geometric 1-connectivity. However the results can be reformulated for higher geometric connectivities within the same range of codimensions.

Remark 1.6. It seems plausible that for $n \geq 5$ any contractible n-manifold covering an aspherical closed n-manifold should be geometrically $(n-3)$-connected. Well-known examples of M.Davis (and results in the next section) show that the codimension 3 bound cannot be improved, in general.

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2 On the g.s.c. condition

2.1 Killing 1-handles of 3-manifolds after stabilisation

This section provides motivation for trying to kill the excess of 1-handles of compact manifolds after stabilisation.

Definition 2.1. The geometric 1-defect $\epsilon(M^n)$ of the compact manifold M^n is $\epsilon(M^n) = \mu_1(M^n) - rank\ \pi_1(M)$, where $\mu_1(M^n)$ is the minimal number of 1-handles in a handlebody decomposition and $rank\ \pi_1(M)$ is the minimal number of generators of $\pi_1(M)$.

Remark 2.1. The defect (i.e., the geometric 1-defect) is always non-negative. There exist examples (see [2]) of 3-manifolds with positive defect. It is still unknown whether there exist closed 4-manifolds with positive defect.

The defect is meaningless in high dimensions because:

Proposition 2.1. *For a compact manifold $\epsilon(M^n) = 0$ holds, if $n \geq 5$.*

Corollary 2.1. *For a closed 3-manifold M^3 one has $\epsilon(M^3 \times D^2) = 0$.*

Proof. Consider a presentation P of $\pi_1(M^n)$ with the least possible number of generators $r = rank\ \pi_1(M)$ and set K^2 for the 2-complex associated to it.

Lemma 2.1. *There exists an embedding $f : K^2 \to M^n$ inducing an isomorphism of fundamental groups.*

Proof. One chooses embedded simple loops based at $* \in M^n$ which represent the generators of $\pi_1(M^n)$ and are disjoint outside the basepoint. For any relation $w = 1$ in P there exists a 2-disk $\delta^2(w)$ mapped in M^n whose boundary circle represents the word w. By general position one can assume the interiors $int(\delta^2(w))$ are pairwise disjoint embedded disks. The obvious map sending K^2 into the union of loops and 2-disks satisfies the requirements. □

Set $\Theta^n(f(K^2))$ for the closed regular neighborhood of $f(K^2)$ in M^n.

Lemma 2.2. *$\Theta^n(f(K^2))$ has a handlebody decomposition with r 1-handles.*

Proof. $\Theta^n(f(K^2))$ collapses on K^2 and 1-handles correspond to 1-cells. □

By the first lemma the pair $(M^n - int(\Theta^n(f(K^2))), \partial\Theta^n(f(K^2)))$ is 1-connected hence ([26, 22]) it is g.s.c. for $n \geq 5$. □

Remark 2.2. As a consequence if Σ^3 is a homotopy 3-sphere then $\Sigma^3 \times D^2$ is g.s.c. Results of Mazur ([15], improved by Milnor in dimension 3) show that $\Sigma^3 \times D^3 = S^3 \times D^3$, but it is still unknown whether $\Sigma^3 \times D^2 = S^3 \times D^2$ holds. An earlier result of Poénaru states that $(\Sigma^3 - nD^3) \times D^2 = (S^3 - nD^3) \times D^2$ for some $n \geq 1$. More recently, Poénaru's program reduced the Poincaré Conjecture to the g.s.c. of $\Sigma^3 \times [0,1]$. It would be very interesting to find whether $\epsilon(M^3 \times [0,1]) = 0$ holds true for all 3-manifolds.

2.2 π_1^∞ and g.s.c.

The following result was proved in ([21], Thm. 1):

Proposition 2.2. *Let W^n be an open simply connected n-manifold of dimension $n \geq 5$. If $\pi_1^\infty(W^n) = 0$ then W^n is g.s.c.*

Remark 2.3. The converse fails as the following examples show. Namely, for any $n \geq 5$ there exist open n-manifolds W^n which are geometrically $(n-4)$-connected but $\pi_1^\infty(W^n) \neq 0$.

Proof. There exist compact contractible n-manifolds M^n with $\pi_1(\partial M^n) \neq 0$, for any $n \geq 4$ (see [14, 17, 8]). Since k-connected compact n-manifolds are geometrically k-connected if $k \leq n-4$ (see [22, 26]), these manifolds are geometrically $(n-4)$-connected. Let us consider now $W^n = int(M^n)$, which is diffeomorphic to $M^n \cup_{\partial M^n \cong \partial M^n \times \{0\}} \partial M^n \times [0,1)$. Any Morse function on M^n extends over $int(M^n)$ to a proper one which has no critical points in the open collar $\partial M^n \times [0,1)$, hence $int(M^n)$ is also geometrically $(n-4)$-connected. On the other hand $\pi_1(\partial M^n) \neq 0$ implies $\pi_1^\infty(W^n) \neq 0$. □

However the following partial converse holds:

Proposition 2.3. *Let W^n be a non-compact simply connected n-manifold which has a proper handlebody decomposition*

1. *without $1-$ or $(n-2)-$handles, or*
2. *without $(n-1)-$ or $(n-2)-$handles.*

Then $\pi_1^\infty(W^n) = 0$.

Remark 2.4. When $n = 3$ this simply says that 1-handles are necessary unless $\pi_1^\infty(W^3) = 0$.

Proof. Consider the handlebody decomposition $W^n = B^n \cup_{j=1}^\infty h_j^{i_j}$, where $h_j^{i_j}$ is an i_j-handle ($B^n = h_0^0$). Set $X_m = B^n \cup_{j=1}^m h_j^{i_j}$, for $m \geq 0$. Assume that this decomposition has no $1-$ nor $(n-2)-$handles. Since there are no 1-handles it follows that $\pi_1(X_j) = 0$ for any j (it is only here one uses the g.s.c.).

Lemma 2.3. *If X^n is a compact simply connected n-manifold having a handlebody decomposition without $(n-2)-$handles then $\pi_1(\partial X^n) = 0$.*

Proof. Reversing the handlebody decomposition of X^n one finds a decomposition from ∂X^n without 2-handles. One slides the handles to be attached in increasing order of their indices. Using Van Kampen Theorem it follows that $\pi_1(X^n) = \pi_1(\partial X^n) * \mathbf{F}(r)$, where r is the number of 1-handles, and thus $\pi_1(\partial X^n) = 0$. □

Lemma 2.4. *If the compact submanifolds ... $\subset X_m \subset X_{m+1} \subset$ ... exhausting the simply connected manifold W^n satisfy $\pi_1(\partial X_m) = 0$, (for all m) then $\pi_1^\infty(W^n) = 0$.*

Proof. For $n = 3$ this is clear. Thus we suppose $n \geq 4$. For any compact $K \subset W^n$ choose some $X_m \supset K$ such that $\partial X_m \cap K = \emptyset$. Consider a loop $l \subset W^n - X_m$. Then l bounds an immersed (for $n \geq 5$ embedded) 2-disk δ^2 in W^n. We can assume that δ^2 is transversalto ∂X_m. Thus it intersects ∂X_m along a collection of circles $l_1, ..., l_p \subset \partial X_m$. Since $\pi_1(X_m) = 0$ one is able to cap off the loops l_j by some immersed 2-disks $\delta_j \subset \partial X_m$. Excising the subsurface $\delta^2 \cap X_m$ and replacing it by the disks δ_j one obtains an immersed 2-disk bounding l in $W^n - K$. □

This proves the first claim. In order to prove the second case choose some connected compact subset $K \subset W^n$. By compactness there exists k such that $K \subset X_k$. Let r be large enough (this exists comes by the properness) such that any handle $h_p^{i_p}$ whose attaching zone touches the lateral surface of one of the handles $h_1^{i_1}, h_2^{i_2}, ..., h_k^{i_k}$ satisfies $p \leq r$. The following claim will prove the theorem:

Proposition 2.4. *Any loop l in $W^n - X_r$ is null-homotopic in $W^n - K$.*

Actually the following (more general) engulfing result holds:

Proposition 2.5. *If C^2 is a 2-dimensional polyhedron whose boundary ∂C^2 is contained in $W^n - X_r$ then there exists an isotopy of W^n (with compact support), fixing ∂C^2 and moving C^2 into $W^n - K$.*

This yields the previous claim by taking for C^2 any 2-disk parameterizing a null homotopy of l.

Proof. Suppose that $C^2 \subset X_m$. One reverses the handlebody decomposition of X_m and obtains a decomposition from ∂X_m without 1- or 2-handles. Assume that we can move C^2 such that it misses the last $j \leq r - 1$ handles. By general position there exists an isotopy (fixing the last j handles) making C^2 disjoint of the cocore ball of the $(j-1)$-th handle, since the cocore disk has dimension at most $n - 3$. The uniqueness of the regular neighborhood implies that we can move C^2 out of the $(j-1)$-th handle (see e.g. [24]), by an isotopy which is identity on the last j handles. This proves the Proposition 2.5. □

□

2.3 G.s.c. and w.g.s.c.

Proposition 2.6. *The non-compact manifold W^n $(n \neq 4)$, which one supposes to be irreducible if $n = 3$, is w.g.s.c. if and only if it is g.s.c.*

Proof. For $n = 3$ it is well-known that g.s.c. is equivalent to w.g.s.c. which is also equivalent to $\pi_1^\infty = 0$ if the manifold is irreducible. For $n \geq 5$ this is a consequence of Wall's result stating the equivalence of g.s.c. and simple connectivity in the compact case (see [26]). If W^n is w.g.s.c. then it has an exhaustion by compact simply connected sub-manifolds M_j (by taking suitable regular neighborhoods of the polyhedra). One can also refine the exhaustion such that the boundaries are disjoint. Then the pairs $(cl(M_{j+1} - M_j), \partial M_j)$ are 1-connected, hence ([26]) they have a handlebody decomposition without 1-handles. Gluing together these intermediary decompositions we obtain a proper handlebody decomposition as claimed. □

3 W.g.s.c. and end compressibility

3.1 Algebraic preliminaries

Definition 3.1. The pair $(\varphi : A \to B,\ \psi : A \to C)$ of group morphisms is strongly compressible if $\varphi(\ker \psi) = \varphi(A)$.

Remark 3.1. Strong compressibility is symmetric in the arguments (φ, ψ) i.e. $\varphi(\ker \psi) = \varphi(A)$ is equivalent to $\psi(\ker \varphi) = \psi(A)$. The proof is an elementary diagram chase.

Definition 3.2. The pair $(\varphi : A \to B,\ \psi : A \to C)$ of group morphisms, is stably compressible if there exists some free group $\mathbf{F}(r)$ on finitely many generators, and a morphism $\beta : \mathbf{F}(r) \to C$, such that the pair $(\varphi * 1_{\mathbf{F}(r)} : A * \mathbf{F}(r) \to B * \mathbf{F}(r),\ \psi * \beta : A * \mathbf{F}(r) \to C)$ is strongly compressible.

Definition 3.3. Consider a fixed pair $(\varphi : A \to B,\ \psi : A \to C)$ of group morphisms. We define inductively a subgroup $G_\alpha \subset C$ for any ordinal α. Set $G_0 = C$. If G_α is defined for every $\alpha < \beta$ (i.e. β is a limit ordinal) then set $G_\beta = \cap_{\alpha<\beta} G_\alpha$. Further set $G_{\alpha+1} = \mathcal{N}(\psi(\ker \varphi), G_\alpha) \triangleleft G_\alpha$ for any other ordinal, where $\mathcal{N}(K, G)$ is the smallest normal group containing K in G. The groups G_α form a decreasing sequence of subgroups of C. Using the Zorn lemma there exist an infimum of the lattice of groups G_α, ordered by the inclusion, which we denote by $G_\infty = \cap_\alpha G_\alpha$ (over all ordinals α). The pair $(\varphi : A \to B, \psi : A \to C)$ is said to be α-compressible if $\psi(A) \subset G_\alpha$ (where α might be ∞).

Lemma 3.1. *Given a subgroup $L \subset C$ there exists a maximal subgroup $\Gamma = \Gamma(L, C)$ of C so that $L \subset N(L, \Gamma) = \Gamma$.*

Proof. There exists at least one group Γ, for instance $\Gamma = L$. Further if Γ and Γ' verify the condition $N(L, \Gamma) = \Gamma$, then their product $\Gamma\Gamma'$ does. Thus, Zorn's Lemma says that a maximal element for the lattice of subgroups verifying this property (the order is inclusion) exists. □

Lemma 3.2. *We have $\Gamma(L, C) = G_\infty$, where $L = \psi(\ker \varphi)$.*

Proof. Firstly, G_∞ satisfies the condition $N(L, \Gamma) = \Gamma$ otherwise the minimality will be contradicted. Pick an arbitrary Γ satisfying this condition. If $\Gamma \subset G_\alpha$ it follows that $\Gamma = N(L, \Gamma) \subset N(L, G_\alpha) = G_{\alpha+1}$, hence by a transfinite induction we derive our claim. □

Definition 3.4. One says that K is full in Γ if $\mathcal{N}(K, \Gamma) = \Gamma$. If we have a pair and $\psi(K)$ is full in Γ we call Γ admissible.

Remark 3.2. If Γ is admissible then $\psi(K) \subset G_\infty$, since G_∞ is the largest group with this property.

Proposition 3.1. *The pair $(\varphi : A \to B,\ \psi : A \to C)$, where A, B and C are finitely generated and $\varphi(A)$ is finitely presented is stably compressible if and only if it is ∞-compressible and there is a subgroup $\Gamma \subset G_\infty$ which is finitely generated such that $\psi(K)$ is full in Γ.*

Proof. We establish first:

Lemma 3.3. *If the pair $(\varphi : A \to B,\ \psi : A \to C)$ is stably compressible then it is ∞-compressible.*

Proof. We will use a transfinite recurrence with the inductive steps provided by the next two lemmas. Set $\beta : \mathbf{F}(r) \to C$ for the morphism making the pair $(\varphi * 1, \psi * \beta)$ compressible.

Lemma 3.4. *If $\beta(\mathbf{F}(r)) \subset G_i$ and $\psi(A) \subset G_i$ then $\psi(A) \subset G_{i+1}$.*

Proof. By hypothesis $\varphi * 1(\ker \psi * \beta) \supset \varphi * 1(A * \mathbf{F}(r))$. Let K be the kernel of φ. Alternatively, for any $b \in \varphi(A) \subset B \subset B * \mathbf{F}(r)$ there exists some $x \in A * \mathbf{F}(r)$ such that $\varphi * 1(x) = b$ and $\psi * \beta(x) = 1$. One can write uniquely x in normal form (see [12], Thm.1.2., p.175) as $x = a_1 f_1 a_2 f_2 ... a_m f_m$ where $a_j \in A, f_j \in \mathbf{F}(r)$ are non-trivial (except maybe f_m). Then $\varphi * 1(x) = \varphi(a_1) f_1 \varphi(a_2) f_2 ... \varphi(a_m) f_m$.

Since the normal form is unique in $B * \mathbf{F}(r)$ one derives that x has the following property. There exists a sequence $p_0 = 1 < p_1 < ... < p_l \leq m$ of integers for which

$$\varphi(a_{p_j}) = b_j \neq 1 \in B, \text{ where } b_1 b_2 ... b_j = b,$$

$$\varphi(a_j) = 1, \text{ for all } j \notin \{p_0, p_1, ..., p_l\},$$

and

$$f_{p_j} f_{p_j+1} ... f_{p_{j+1}-1} = 1, \text{ (for all } j\text{, with the convention } p_{l+1} = m).$$

Furthermore $1 = \psi * \beta(x)$ implies that

$$1 \in \psi(a_1 K)\beta(f_1)\psi(K)\beta(f_2)\psi(K)...$$

$$...\beta(f_{p_1-1})\psi(a_{p_1}K)\beta(f_{p_1})\psi(K)\beta(f_{p_1+1})\psi(K)...\beta(f_m).$$

However each partial product starting at the p_j-th term and ending at the $(p_{j+1}-1)$-th term is a product of conjugates of $\psi(K)$ by elements from the image of β:

$$\beta(f_{p_j})\psi(K)\beta(f_{p_j+1})\psi(K)...\psi(K)\beta(f_{p_{j+1}-1}) =$$

$$\prod_{i=0}^{p_{j+1}-p_j-1} \left(\beta \left(\prod_{k=0}^{i} f_{p_j+k} \right) \psi(K) \beta \left(\prod_{k=0}^{i} f_{p_j+k}^{-1} \right) \right) \subset \mathcal{N}(\psi(K), G_i) = G_{i+1}.$$

We used above the inclusions $\psi(K) \subset \psi(A) \subset G_i$ and $\beta(\mathbf{F}(r) \subset G_i$. Therefore

$$1 \in \psi(a_1 K) G_{i+1} \psi(a_{p_1} K) G_{i+1} ... \psi(a_{p_l} K) G_{i+1} =$$

$$= \psi(aK) G_{i+1},$$

for any $a \in A$ such that $\varphi(a) = b$. This implies that $\psi(aK) \subset G_{i+1}$ and hence $\psi(A) \subset G_{i+1}$. □

Lemma 3.5. *If $\beta(\mathbf{F}(r)) \subset G_{i-1}$ and $\psi(A) \subset G_i$ then $\beta(\mathbf{F}(r)) \subset G_i$.*

Proof. One can use the symmetry of the algebraic compressibility and use the argument from the previous lemma. Alternatively, choose $f \in \mathbf{F}(r) \subset B * \mathbf{F}(r)$ and some $x \in A * \mathbf{F}(r)$ such that $\varphi * 1(x) = f$ and $\psi * \beta(x) = 1$. Using the normal form as above we find this time $1 \in \beta(f)\mathcal{N}(\psi(K), G_{i-1})$ hence $\beta(\mathbf{F}(r)) \subset G_i$. □

Using in an alternate way the two previous lemmas one gets the claim.

□

□

Lemma 3.6. *Let $\beta : \mathbf{F}(r) \to C$ be a homomorphism such that $(\varphi * 1,\ \psi * \beta)$ is strongly compressible. Set $\beta(\mathbf{F}(r)) = H$. Then $\psi(K)$ is full in $\psi(K)H$. In particular if A, B, C are finitely generated and $\varphi(A)$ is finitely presented then the subgroup $\Gamma = \psi(K)H$ is finitely generated.*

Proof. We already saw that that $H \subset G_\infty$. Set $W(L; X) = \{x = \prod_i g_i x_i g_i^{-1},\ \ , g_i \in X,\ \ , x_i \in L\}$ for two subgroups $L, X \subset C$. The proof we used to show that $\psi(A) \subset G_\infty$ and $H \subset G_\infty$ actually yields $\varphi(A) \subset W(\psi(K), H)$ and respectively $H \subset W(\psi(K), H)$. We remark now that $W(\psi(K), H) = \mathcal{N}(\psi(K), \psi(K)H)$. The left inclusion is obvious. The other inclusion consists in writing any element gxg^{-1} with $g \in \psi(K)H$, $x \in \psi(K)$ as a product of conjugates by elements of H. This might be done by recurrence on the length of g, by using the following trick. If $g = a_1y_1a_2y_2$, $a_i \in \psi(K), y_i \in H$ then $gxg^{-1} = y_1(a_1y_2a_1^{-1})(a_1a_2xa_2^{-1}a_1^{-1})a_1y_2a_1^{-1}y_1^{-1}$.

Consequently the fact that $H \subset W(\psi(K), H)$ implies

$$W(\psi(K), H) \subset W(\psi(K), W(\psi(K), H)) = \mathcal{N}(\psi(K), \mathcal{N}(\psi(K), \psi(K)H) \subset$$

$$\subset \mathcal{N}(\psi(K), \psi(K)H) = W(\psi(K), H),$$

hence all inclusions are equalities. Also $\psi(K)H \subset \mathcal{N}(\psi(K), \psi(K)H)$ since both components $\psi(K)$ and H are contained in $\mathcal{N}(\psi(K), \psi(K)H)$. This shows that $\mathcal{N}(\psi(K), \psi(K)H) = \psi(K)H$ hence $\psi(K)$ is full in $\psi(K)H$.

We take therefore $\Gamma = \psi(K)H$. It suffices to show now that each of the groups K and H are finitely generated now. H is finitely generated since it is the image of $\mathbf{F}(r)$. Furthermore K is finitely generated since $A/K = \varphi(A)$ is finitely presented and A is finitely generated. The theorem of Neumann ([1], p.52) shows that K must be finitely generated. This proves the claim. □

Lemma 3.7. *Assume that $\psi(K)$ is full in $\Gamma \subset G_\infty$, where Γ is finitely generated. If the pair (φ, ψ) is ∞-compressible then it is stably compressible.*

Proof. Consider r big enough and a surjective homomorphism $\beta : \mathbf{F}(r) \to \Gamma$. This implies that $\psi * \beta(A * \mathbf{F}(r) = \psi(K)\Gamma = \Gamma$. We have to show that any $x \in \Gamma$ is in $\psi * \beta(\ker \varphi * 1)$.

Recall that $\mathcal{N}(\psi(K), \Gamma) = \Gamma$. Then $x = \prod_i\ {}^{g_i}x_i$ can be written as a product of conjugates of elements $x_i \in \psi(K)$ by elements $g_i \in \Gamma$. Choose $f_i \in \mathbf{F}(r)$ so that $\beta(f_i) = g_i$ and $y_i \in K$ so that $\psi(y_i) = x_i$. Then $\psi * \beta(\prod_i f_i^{-1}y_if_i) = x$ and $\varphi * 1(\prod_i f_i^{-1}y_if_i) = \prod_i f_i^{-1}\varphi(y_i)f_i = 1$, since $y_i \in K$. □

3.2 End compressible manifolds

Definition 3.5. The pair of spaces (T', T) is (respectively strongly, stably) compressible if for each component S_j of ∂T and component V_j of $T' - int(T)$ such that $S_j \subset V_j$, the pair $(*_j\pi_1(S_j) \to \pi_1(T),\ *_j\pi_1(S_j) \to *_j\pi_1(V_j))$ is algebraically (respectively weakly, stably) compressible. The morphisms are induced by the obvious inclusions. Set also $G_\infty(T, T')$for the G_∞ group associated to the previous pair.

Remark 3.3. These morphisms are not uniquely defined and depend on the various choices of base points in each component. However the compressibility does not depend on the particular choice of the representative.

Definition 3.6. The open manifold W^n is end compressible (respectively k-compressible) if it admits an exhaustion by compact submanifolds

$$W^n = \bigcup_{i=1}^{\infty} T_i,\ T_i \subset int(T_{i+1}),$$

such that:

1. all pairs (T_{i+1}, T_i) are ∞-compressible (respectively k-compressible).
2. if $S_{i,j}$ denote the components of ∂T_i then the homomorphism $*_j \pi_1(S_{i,j}) \to \pi_1(T_i)$ induced by the inclusion is surjective.
3. any component of $T_{i+1} - int(T_i)$ intersect T_i along precisely one component.
4. For each i there exists an admissible subgroup Γ_i of $G_\infty(T_i, T_{i+1})$ which is finitely presented.

Remark 3.4. As in the case of the compressibility the condition 2. above is independent of the homomorphism we chose, which might depend on the base points in each component.

Theorem 3.1. *Any w.g.s.c. open n-manifold W^n is end compressible. Conversely, for $n \geq 5$, W^n is end compressible if and only if it is w.g.s.c.*

3.3 Proof of Theorem 3.1

The first assertion of Theorem 3.1 is immediate by considering an exhaustion by simply connected submanifolds T_i verifying the condition (3) from Definition 3.6. The pair (φ, ψ) associated to consecutive terms satisfies $K = A$, hence $\psi(K) = \psi(A)$. By the Van Kampen theorem we derive $\mathcal{N}(\psi(A), C) = C$, hence $G_\infty = C = \pi_1(T_{i+1} - int(T_i))$, which is finitely presented.

We will actually prove a stronger statement below. Let us consider an exhaustion $\{T_i^n\}_{i=1,\infty}$ of W^n by compact submanifolds, and fix some index i. One can ask that each connected component of $T_{i+1} - int(T_i)$ has exactly one boundary component from ∂T_i. By adding to an arbitrary given T_i the regular neighborhoods of arcs in $T_{i+1} - int(T_i)$ joining different connected component this condition will be fulfilled. The following result is the main tool in checking that specific manifolds are not w.g.s.c.

Proposition 3.2. *Any exhaustion as above of the w.g.s.c. manifold W^n has a refinement for which consecutive terms fulfill the conditions:*

1. *all pairs (T_{i+1}, T_i) are ∞-compressible.*
2. *if $S_{i,j}$ denote the components of ∂T_i the map $*\pi_1(S_{i,j}) \to \pi_1(T_i)$ induced by the inclusion is surjective.*
3. *For each i there is some admissible subgroup $\Gamma_i \subset G_\infty(T_i, T_{i+1})$ which is finitely presented.*

Proof. Since W^n is w.g.s.c. there exists a compact 1-connected submanifold M^n of W^n such that $T_i^n \subset M^n$. We can suppose $M^n \subset T_{i+1}^n$, without loss of generality. From now on we will focus on the pair (T_{i+1}^n, T_i^n) and suppress the index i, and denote it (T', T), for the sake of notational simplicity.

Lemma 3.8. *The pair (T', T) is stably compressible.*

Proof. Let $\varphi : \pi_1(\partial T) \to \pi_1(T)$ and $\psi : \pi_1(\partial T) \to \pi_1(T' - int(T))$ be the homomorphisms induced by the inclusions $\partial T \hookrightarrow T$, and $\partial T \hookrightarrow T' - int(T)$. If ∂T has several components then we choose base points in each component and set $\pi_1(\partial T) = *_j \pi_1(S_j)$ for notational simplicity.

Let us consider a handlebody decomposition of $M - int(T)$ (respectively a connected component) from ∂T,

$$M - int(T) = \partial T \times [0,1] \bigcup_{\lambda=1}^{n-1} \cup_j h_j^{\lambda},$$

where h_j^{λ} is a handle of index λ. We suppose the handles are attached in increasing order of their index. Since the distinct components of ∂T are not connected outside T the 1-handles which are added have the extremities in the same connected component of ∂T. Set $M_2^n \subset M^n$ (respectively M_1^n) for the submanifold obtained by attaching to T only the handles h_j^{λ} of index $\lambda \leq 2$ (respectively those of index $\lambda \leq 1$). Then $\pi_1(M_2^n) = 0$, because adding higher index handles does not affect the fundamental group and we know that $\pi_1(M^n) = 0$.

Lemma 3.9. *The pair (T', M_1^n) is strongly compressible.*

Proof. Let $\{\gamma_j\}_{j=1,p} \subset \partial M_1^n$ be the set of attaching circles for the 2-handles of M_2^n and $\{\delta_j^2\}_{j=1,p}$ be the corresponding core of the 2-handles h_j^2 $(j = 1, p)$. Since δ_j^2 is a 2-disk embedded in $M^n - int(T)$ it follows that the homotopy class $[\gamma_j]$ vanishes in $\pi_1(M^n - int(T))$.

Let $\Gamma \subset \pi_1(\partial M_1^n)$ be the normal subgroup generated by the homotopy classes of the curves $\{\gamma_j\}_{j=1,p}$ which are contained in ∂M_1^n. Notice that this amounts to picking base points which are joined to the loops. Therefore the image of Γ under the map $\pi_1(\partial M_1^n) \to \pi_1(M_2^n - int(M_1^n))$, induced by the inclusion, is zero. In particular its image in $\pi_1(T' - int(M_1^n))$ is zero.

On the other hand the images of the classes $[\gamma_j]$ in $\pi_1(M_1^n)$ normally generate all of the group $\pi_1(M_1^n)$ because $\pi_1(M_2^n) = 1$ (notice that the map $\pi_1(\partial M_1^n) \to \pi_1(M_1^n)$ is surjective).

These two properties are equivalent to the strong compressibility of the pair (M_1^n, M_2^n) which in turn implies that of (T', M_1^n). □

Rest of the proof of Lemma 4.1: Assume now that the number of 1-handles h_j^1 is r. Notice that $\pi_1(M_1^n - int(T^n)) \cong \pi_1(\partial T^n) * \mathbf{F}(r)$, because it can be obtained from $\partial T^n \times [0,1]$ by adding 1-handles and the 1-handles we added do not join distinct boundary components, so that each one contributes with a free factor. In particular the inclusion $\partial T^n \hookrightarrow M_1^n - int(T^n)$ induces a monomorphism $\pi_1(\partial T^n) \hookrightarrow \pi_1(\partial T^n) * \mathbf{F}(r)$. Observe that $M_1^n - int(T^n)$ can also be obtained from $\partial M_1^n \times [0,1]$ by adding $(n-1)$-handles hence the inclusion $\partial M_1^n \hookrightarrow M_1^n - int(T^n)$ induces the isomorphism $\pi_1(\partial M_1^n) \cong \pi_1(\partial T^n) * \mathbf{F}(r)$. The same reasoning gives the isomorphism $\pi_1(M_1^n) \cong \pi_1(T^n) * \mathbf{F}(r)$.

In particular we can view the subgroup Γ as a subgroup of $\pi_1(\partial T^n) * \mathbf{F}(r)$. The previous lemma tells us that Γ lies in the kernel of $\pi_1(\partial T^n) * \mathbf{F}(r) \to \pi_1(T' - int(T^n))$ and also projects epimorphically onto $\pi_1(T^n) * \mathbf{F}(r)$. The identification of the respective maps with the morphisms induced by inclusions yields our claim. $\square$

$\square$

It remains to show that there exists an admissible group Γ which is finitely presented. Let $\Gamma \subset \pi_1(T' - int(T))$ be the image of $\psi * \beta$. Since the number of 2-handles we added is finite we derive that $\ker(\psi * \beta)$ is normally generated by finitely many elements, so that Γ is finitely presented. The admissibility of Γ follows from the previous section.

Conversely assume that W^n has an exhaustion in which consecutive pairs are stably compressible. Then it is sufficient to show the following:

Proposition 3.3. *If (T', T) is a stably compressible pair of n-manifolds and $n \geq 5$ then $T \subset int(M^n) \subset T'$ where M^n is a compact submanifold with $\pi_1(M^n) = 0$ if there exists an admissible $\Gamma \subset G_\infty(T, T')$ which is finitely presented.*

Proof. We saw that stably compressible implies ∞-compressible. In particular if we consider some admissible group Γ and use the construction from Lemma 3.7 there exists some β which surjects on Γ so that the corresponding pair $(\varphi * 1, \psi * \beta)$ is strongly compressible. This β might be different from the original one furnished by the stable compressibility.

One can realize the homomorphism $\beta : \mathbf{F}(r) \to \pi_1(T' - int(T))$ by a disjoint union of bouquets of circles $\vee^r S^1 \to T' - int(T)$. There is one bouquet in each connected component of $T' - int(T)$. One joins each wedge point to the unique connected component of ∂T for which that is possible by an arc, and set M_1^n for the manifold obtained from T by adding a regular neighborhood of the bouquets $\vee^r S^1$ in T' (plus the extra arcs). This is equivalent to adding 1-handles with the induced framing.

Lemma 3.10. *The kernel $\ker \psi * \beta \subset \pi_1(\partial M_1^n)$ is normally generated by a finite number of elements $\gamma_1, \gamma_2, ..., \gamma_p$.*

Proof. Consider a finite presentation $\mathbf{F}(k)/H \to \pi_1(\partial T)$. We know that $\pi_1(\partial M_1^n) = \pi_1(\partial T) * \mathbf{F}(r)$. Furthermore the composition map

$$\lambda : \mathbf{F}(k) * \mathbf{F}(r) \to \pi_1(\partial T) * \mathbf{F}(r) \stackrel{\psi * \beta}{\to} \Gamma,$$

is surjective (since β is). The first map is the free product of the natural projection with the identity. Therefore $\mathbf{F}(k + r)/\ker \lambda \cong \Gamma$ is a presentation of the group Γ. The Theorem of Neumann (see [1], p.52) states that any presentation on finitely many generators of a finitely presented group has a presentation on these generators with only finitely many of the given relations. Applying this to Γ one derives that there exist finitely many elements which normally generate $\ker \lambda$ in $\mathbf{F}(k+r)$. Then the images of these elements in $\pi_1(\partial T) * \mathbf{F}(r)$ normally generate $\ker \psi * \beta$ (the projection $\mathbf{F}(k) * \mathbf{F}(r) \to \pi_1(\partial T) * \mathbf{F}(r)$ is surjective). This yields the claim. $\square$

Lemma 3.11. *The elements γ_i are also in the kernel of $\pi_1(\partial M_1^n) \to \pi_1(T' - int(M_1^n))$.*

Proof. The map $\pi_1(T'-int(M_1^n)) \to \pi_1(T'-int(T))$ induced by the inclusion is injective because $T'-int(T)$ is obtained from $T'-int(M_1^n)$ by adding $(n-1)$-handles (dual to the 1-handles from which one gets M_1^n starting from T), and $n \geq 5$. Thus the map $\pi_1(\partial M_1^n) \to \pi_1(T'-int(T))$ factors through

$$\pi_1(\partial M_1^n) \to \pi_1(T'-int(M_1^n)) \to \pi_1(T'-int(T)),$$

and any element in the kernel must be in the kernel of the first map, as stated. □

The dimension restriction $n \geq 5$ implies that we can assume γ_j are represented by embedded loops having only the base point in common. Then γ_j bound singular 2-disks $D_j^2 \subset T'-int(M_1^n)$. By a general position argument, one can arrange such that the 2-disks D_j^2 are embedded in $T'-int(M_1^n)$ and have disjoint interiors.

As a consequence the manifold M^n obtained from M_1^n by attaching 2-handles along the γ_j's (with the induced framing) can be embedded in $T'-int(T)$. Moreover M^n is a compact manifold whose fundamental group is the quotient of $\pi_1(M_1^n)$ by the subgroup normally generated by the elements $\varphi * 1(\gamma_j)$'s. The group $\varphi * 1(\ker \psi * \beta)$ is normally generated by the elements $\varphi * 1(\gamma_j)$. By hypothesis the pair $(\varphi * 1, \psi * \beta)$ is compressible hence $\varphi * 1(\ker \psi * \beta)$ contains $\varphi * 1(\pi_1(\partial M_1^n)) = \varphi * 1(\pi_1(\partial T) * \mathbf{F}(r)) = \varphi(\pi_1(\partial T)) * \mathbf{F}(r)$. Next

$$\frac{\pi_1(M_1^n)}{\varphi * 1(\ker \psi * \beta)} = \frac{\pi_1(T) * \mathbf{F}(r)}{\varphi(\pi_1(\partial T)) * \mathbf{F}(r)} \cong \frac{\pi_1(T)}{\varphi(\pi_1(\partial T))} = 1,$$

since φ has been supposed surjective. Therefore the quotient of $\pi_1(M_1^n)$ by the subgroup normally generated by the elements $\varphi * 1(\gamma_j)$ is trivial. □

3.4 End 1-compressibility is trivial for $n \geq 5$

We defined an infinite sequence of obstructions (namely k-compressibility for each k) to the w.g.s.c. However the first obstruction is trivial in dimension $n \neq 4$. In fact the main result of this section establishes the following:

Proposition 3.4. *End 1-compressibility and simple connectivity (s.c.) are equivalent for open n-manifolds of dimension $n \geq 5$.*

Proof. We first consider a simpler case:

Proposition 3.5. *The result holds in the case of a manifold W^n of dimension at least 5 with one end.*

Proof. In this case W^n has an exhaustion T_i with ∂T_i connected for all i.

Lemma 3.12. *W^n has an exhaustion such that the map $\varphi : \pi_1(\partial T_i) \to \pi_1(T_i)$ induced by inclusion is a surjection for each i.*

Proof. As W^n is simply connected, by taking a refinement we can assume that each inclusion map $\pi_1(T_i) \to \pi_1(T_{i+1})$ is the zero map. As usual we denote T_i and T_{i+1} by T and T' respectively.

Now, take a handle-decomposition of T starting with the boundary ∂T. Suppose the core of each 1-handle of this decomposition is homotopically trivial in T, then it is immediate that φ is a surjection. We will enlarge T by adding some 1-handles and 2-handles (that are embedded in T') at ∂T in order to achieve this.

Namely, let γ be the core of a 1-handle. By hypothesis, there is a disc D^2 in T' bounding the core of each of the 1-handles, which we take to be transversal to ∂T. As the dimension of W^n is at least 5, D^2 can be taken to be embedded. Notice that the 2-disks corresponding to all 1-handles can also be made disjoint, by general position. Thus D^2 intersects $T' - int(T)$ in a collection of embedded disjoint planar surfaces. The neighborhood of each disc component of this intersection can be regarded as a 2-handle (embedded in T') which we add to T at ∂T. For components of $D^2 \cap (T' - int(T)) = D^2 - int(T)$ with more than one boundary component, we take embedded arcs joining distinct boundary components. We add to T a neighborhood of each arc, which can be regarded as a 1-handle. After doing this for a finite collection of arcs, $D^2 - D^2 \cap T$ becomes a union of discs. Now we add 2-handles as before. The disc D^2 that γ bounds is now in T.

Further, the dual handles to the handles added are of dimension at least 3. In particular we can extend the previous handle-decomposition to a new one for (the new) T starting at (the new) ∂T with no new 1-handles. Thus, after performing the above operation for the core of each 1-handle of the original handle decomposition, the core of each 1-handle of the resulting handle-decomposition of T starting at ∂T bounds a disc in T. Thus ϕ is a surjection. ☐

The above exhaustion is in fact 1-compressible by the following algebraic lemma.

Lemma 3.13. *Suppose that we have a square of maps verifying the Van Kampen theorem:*

$$\begin{array}{ccc} A & \xrightarrow{\psi} & C \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\gamma} \\ B & \xrightarrow{\beta} & D \end{array}$$

Let $\xi : A \to D$. Suppose also that φ is surjective and β is the 0 map. Then $\psi(A) \subset \mathcal{N}(\psi(\ker\varphi)) = \mathcal{N}(\psi(\ker\varphi), C)$.

Proof. Observe that $\xi(\ker\varphi) = 0$, hence $\psi(\ker\varphi) \subset \ker\gamma$. Hence we can define another diagram with $A' = A/\ker\varphi$, $B' = B$, $C' = C/\mathcal{N}(\psi(\ker\varphi))$, $D' = D$:

$$\begin{array}{ccc} A' = A/\ker\varphi & \xrightarrow{\psi} & C' = C/\mathcal{N}(\psi(\ker\varphi)) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\gamma} \\ B' = B & \xrightarrow{\beta} & D' = D \end{array}$$

Notice that the map $A/\ker\varphi \to C/\mathcal{N}(\psi(\ker\varphi))$ is well-defined. Again this diagram verifies the Van Kampen theorem. For this diagram, the induced φ is an isomorphism. It is immediate then that the universal (freest) D' must be C'. In fact $D' = C' * A'/\mathcal{N}(\{\psi(a)a^{-1}, a \in A'\})$. Consider the map $C' \to C' * A'/\mathcal{N}(\{\psi(a)a^{-1}, a \in A'\}) \to C'$, where the second arrow consists in replacing any occurrence of a by the element $\psi(a)$ and taking the product in C'. This composition is the identity and the first map is a surjection, hence the map $C' \to D'$ is an isomorphism.

The map induced by ξ is 0 since β is 0. But the map $\xi : A' \to D'$ is the map $\psi : A' \to C'$ followed by an isomorphism, hence $\psi(A') = 0$. This is equivalent to $\psi(A) \subset \mathcal{N}(\psi(\ker\varphi))$. □

Note that the above lemma is purely algebraic, and in particular independent of dimension. The two lemmas immediately give us the proposition for one-ended manifolds W^n with $n \geq 5$. □

The general case. We now consider the general case of a simply-connected open manifold W^n of dimension at least 5, with possibly more than one end. We shall choose the exhaustion T_i with more care in this case.

We will make use of the following construction several times. Start with a compact submanifold A^n of codimension 0, with possibly more than one boundary component. Assume for simplicity (by enlarging A^n if necessary) that no complementary component of A^n is precompact. As W^n is simply connected, we can find a compact submanifold B^n containing A^n in its interior such that the inclusion map on fundamental groups is the zero map. Further, we can do this by thickening and then adding the neighborhood of a 2-complex, i.e., a collection of 1-handles and 2-handles. Namely, for each generator γ of $\pi_1(A^n)$, we can find a disc D^2 that γ bounds, and then add 1-handles and 2-handles as in lemma 3.12. Thus, as $n \geq 5$, the boundary components of B^n correspond to those of A^n. We repeat this with B^n in place of A^n to get another submanifold C^n.

Observe that as a consequence of this and the simple-connectivity of W^n, the inclusion map $\pi_1(A^n \cup V^n) \to \pi_1(B^n \cup V^n)$ is the zero map for any component V^n of $W^n - A^n$. More generally if $Z^n \subset V^n$, then $\ker(\pi_1(A^n \cup Z^n) \to \pi_1(B^n \cup Z^n)) = \ker(\pi_1(A^n \cup Z^n) \to \pi_1(B^n \cup Z^n \cup (W^n - V^n)))$. Similar results hold with B^n and C^n in place of A^n and B^n.

Now start with some A_1^n as above and construct B_1^n and C_1^n. Thicken C_1^n slightly to get $T = T_1$. We will eventually choose a $T_2 = T'$, but for now we merely note that it can (and so it will) be chosen in such a manner that the inclusion map $\pi_1(T) \to \pi_1(T')$ is the zero map. Let $S_j, j = 1, \ldots, n$ be the boundary components of T. Let X_j be the union of the component of $T - int(B^n)$ containing S_j and B^n, and define Y_j analogously with C^n in place of B^n. Denote the image of $\pi_1(X_j)$ in $\pi_1(Y_j)$ by $\bar{\pi}(X_j)$. We then have a natural map $\varphi : \pi_1(S_j) \to \bar{\pi}(X_j)$. Let V_j be the component of $T' - int(T)$ containing S_j.

Lemma 3.14. *By adding 1-handles and 2-handles to S_j, we can ensure that $\pi_1(S_j)$ surjects onto $\bar{\pi}(X_j)$.*

Proof. The proof is essentially the same as that of lemma 3.12. We start with a handle-decomposition for X_j starting from S_j. We shall ensure that the image in $\bar{\pi}(X_j)$ of the core of each 1-handles is trivial. Namely, for each core, we take a disc D that it bounds. By the above remarks, we can, and do, ensure that the disc lies in Y_j, and in particular does not intersect any boundary component of T except S_j. As in lemma 3.12 we may now add 1-handles and 2-handles to S_j to achieve the desired result. □

Notice that the changes made to T in the above lemma do not affect S_k, X_k and Y_k for $k \neq j$. Hence, by repeated application of the above lemma, we can ensure that all the maps

$\pi_1(S_j) \to \bar{\pi}(X_j)$ are surjections. Also notice that the preceding remarks show that $\ker\varphi = \ker(\pi_1(S_j) \to \pi_1(T))$. Now take $A = \pi_1(S_j)$, $B = \bar{\pi}(X_j)$ and $C = \pi_1(V_j)$, and let D be the image of $\pi_1(V_j \cup X_j)$ in $\pi_1(V_j \cup Y_j)$. Then, by the preceding remarks and lemma 3.14, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & C \\ \varphi\downarrow & & \downarrow\gamma \\ B & \xrightarrow{\beta} & D \end{array}$$

satisfies the hypothesis of the lemma 3.13. The 1-compressibility for the pair (T', T) follows.

Now we continue the process inductively. Suppose T_k has been defined, choose A_{k+1} so that it contains T_k and also in such a manner as to ensure that A_i's exhaust M. Then find B_{k+1}, C_{k+1} and T_{k+1} as above. The rest follows as above. □

Remark 3.5. For the case of simply-connected, one-ended (hence contractible) 3-manifolds, a theorem of Luft says that M can be exhausted by a union of homotopy handlebodies. These satisfy the conclusion of lemma 3.12, hence the proposition still holds. More generally, we can apply the sphere theorem to deduce that we have an exhaustion by connected sums of homotopy handlebodies. It follows that each pair (T, T') of this exhaustion is 1-compressible as we can decompose T and consider each component separately without affecting φ or ψ.

3.5 Open manifolds which are not w.g.s.c.

In general the tower of obstructions is not trivial as is shown below:

Proposition 3.6. *If W^3 denotes the Whitehead manifold then $W^3 \times N^k$ is not ∞-compressible for any closed simply connected k-manifold N^k.*

Proof. It is sufficient to consider the case of the Whitehead manifold since the pair of groups appearing in the product exhaustions are the same. Recall that the Whitehead manifold W^3 is an increasing union of solid tori T_i, with T_i embedded in T_{i+1} as a neighborhood of a Whitehead link. Let T and T' be as usual and let $M^3 = T' - T$. Then $C = \pi_1(M^3)$ with our usual notation. Note that $\ker(\varphi)$ is generated by the meridian of the inner torus in M^3, and hence $\pi_1(M^3)/\mathcal{N}(\ker(\varphi), C) = \pi_1(T') = \mathbb{Z}$. Thus G_1 consists of homologically trivial elements in M^3.

Consider now the cover $\widetilde{M^3}$ of M^3 with fundamental group G_1. This is $\mathbb{R}^3$ with the neighborhood of an infinite component link, say indexed by the integers, deleted. Further each pair of adjacent components has linking number 1.

In this cover, $\psi(A)$ is the image of the bounding torus T of one of the components of this link, and $\psi(\ker(\varphi))$ is generated by the meridian of this component. Thus, $G_1/\mathcal{N}(\psi(ker(\varphi), G_1)$ is the fundamental group of $\widetilde{M^3} \cup_T D^2 \times S^1$, i.e., of $\widetilde{M^3}$ with a solid torus glued along T to kill the meridian. But, because of the linking, the longitude $\lambda \subset T$ is not trivial in this group, i.e. $\lambda \notin G_2$. Since $\lambda \in \psi(A)$, we see that the Whitehead link is not 2-compressible.

A similar argument shows that any refinement of this exhaustion is not n-compressible for some n. Thus the manifold is not stably compressible. □

Remark 3.6. Observe that this proof works for uncountably many similar manifolds - namely we may embed T_i in T_{i+1} as a link similar to the Whitehead link that winds around the solid torus several times.

Remark 3.7. It follows that $W^3 \times D^k$ is not w.g.s.c. using the criterion from [5, 6]. However the previous theorem is more precise regarding the failure of g.s.c. for these product manifolds.

4 Uncountably many Whitehead-type manifolds

Recall the following definition from [27]:

Definition 4.1. A Whitehead link $T_0^n \subset T_1^n$ is a null-homotopic embedding of the solid torus T_0^n in the (interior of the) unknotted solid torus T_1^n lying in S^n such that the pair (T_1^n, T_0^n) is (boundary) incompressible.

The solid n-torus is $T^n = D^2 \times S^1 \times S^1 ... \times S^1$. By iterating the ambient homeomorphism which sends T_0^n onto T_1^n one obtains an ascending sequence $T_0^n \subset T_1^n \subset T_2^n \subset ...$ whose union is called a Whitehead-type n-manifold. A Whitehead-type manifold is open contractible and not s.c.i. Wright ([27]) gave a recurrent procedure to construct many Whitehead links in dimensions $n \geq 3$. One shows below that this construction provides uncountably many distinct contractible manifolds.

We introduce an invariant for pairs of solid tori which generalizes the wrapping number in dimension 3. Moreover this provides invariants for open manifolds of Whitehead-type answering a question raised in [27].

Definition 4.2. A spine of the solid torus T^n is an embedded $t^{n-2} = \{*\} \times S^1 \times S^1 \times ... \times S^1 \subset T^n$ having a trivial normal bundle in T^n. This gives T^n the structure of a trivial 2-disk bundle over t^{n-2}.

Remark 4.1. Although the spine is not uniquely defined, its isotopy class within the solid torus is.

Consider a pair of solid tori $T_0^n \subset T^n$. We fix some spine t^{n-2} for T^n. To specify the embedding of T_0^n is the same as giving the embedding of a spine t_0^{n-2} of T_0^n in T^n. The isotopy class of the embedding $t_0^{n-2} \hookrightarrow T^n$ is therefore uniquely defined by the pair. Let us pick-up a Riemannian metric g on the torus T^n such that T^n is identified with the regular neighborhood of radius r around t^{n-2}. We denote this by $T^n = t^{n-2}[r]$, and suppose for simplicity that $r = 1$. Then $t^{n-2}[\lambda]$ for $\lambda \leq 1$ will denote the radius λ tube around t^{n-2} in this metric.

Definition 4.3. The wrapping number of the Whitehead link $T_0^n \subset T^n$ is defined as follows:

$$w(T^n, T_0^n) = \lim_{\varepsilon \to 0} \inf_{t_0^n \in \mathcal{I}(t_0^n \subset t^{n-2}[\varepsilon])} \frac{vol(t_0^{n-2})}{vol(t^{n-2})},$$

where $\mathcal{I}(t_0^n \subset T^n)$ is the set of all embeddings of the spine t_0^{n-2} of T_0^n in the given isotopy class, and vol is the $(n-2)$-dimensional volume.

Remark 4.2. Notice that a priori this definition might depend on the particular choice of the spine t^{n-2} and on the metric g.

Proposition 4.1. *The wrapping number is a topological invariant of the pair* (T^n, T_0^n).

Proof. There is a natural projection map on the spine $\pi : T^n \to t^{n-2}$, which is the fiber bundle projection of T^n (with fiber a 2-disk). When both T^n and t^{n-2} are fixed then such a projection map is also defined only up to isotopy. Set therefore

$$l(T^n, T_0^n) = \inf_{t_0^n \in \mathcal{I}(t_0^n \subset T^n)} \inf_{x \in t^{n-2}} \sharp \left\{ \pi^{-1}(x) \cap t_0^{n-2} \right\}.$$

Since $\inf_{x \in t^{n-2}} \sharp \left\{ \pi^{-1}(x) \cap t_0^{n-2} \right\}$ does not depend on the particular projection map (in the fixed isotopy class) this number represent a topological invariant of the pair (T^n, T_0^n). Hence the claim follows from the following result:

Proposition 4.2. $w(T^n, T_0^n) = l(T^n, T_0^n)$.

Proof. Consider a position of t_0^{n-2} for which the minimum value $l(T^n, T_0^n)$ is attained. A small isotopy make t_0^{n-2} transversal to π. Then, for this precise position of t_0^{n-2} there exists some number M such that

$$\sharp \left\{ \pi^{-1}(x) \cap t_0^{n-2} \right\} \leq M, \text{ for any } x \in t^{n-2}.$$

Denote by μ the Lebesgue measure on t^{n-2}.

Lemma 4.1. *For any* $\varepsilon > 0$ *one can move* t_0^{n-2} *in* T^n *by an ambient isotopy such that the following conditions are fulfilled:*

$$\sharp \left\{ \pi^{-1}(x) \cap t_0^{n-2} \right\} \leq M, \textit{ for any } x \in t^{n-2}.$$

$$\mu \left(\left\{ x \in t^{n-2} | \ \sharp \left\{ \pi^{-1}(x) \cap t_0^{n-2} \right\} > l(T^n, T_0^n) \right\} \right) < \varepsilon.$$

Proof. The set $U = \left\{ x \in t^{n-2} | \ \sharp \left\{ \pi^{-1}(x) \cap t_0^{n-2} \right\} = l(T^n, T_0^n) \right\}$ is an open subset of positive measure. Consider then a flow φ_t on the torus t^{n-2} which expands a small ball contained in U into the complement of a measure ε set (e.g. a small tubular neighborhood of a spine of the 1-holed torus). Extend this flow as $1_{D^2} \times \varphi_t$ all over T^n and consider its action on t_0^{n-2}. □

Lemma 4.2. $w(T^n, T_0^n) \leq l(T^n, T_0^n)$.

Proof. The map π is the projection of the metric tube around t^{n-2} on its spine, hence the Jacobian $Jac(\pi|_{t_0^{n-2}})$ has bounded norm $|Jac(\pi|_{t_0^{n-2}})| \leq 1$. It follows that

$$\frac{vol(t_0^{n-2})}{vol(t^{n-2})} = \frac{\int \pi^* d\mu}{\int d\mu} \leq \frac{\int_{\pi^{-1}(U)} |Jac(\pi|_{t_0^{n-2}})| d\mu}{\int_U d\mu} + M\varepsilon \leq$$

$$\leq l(T^n, T_0^n)(1 - \varepsilon) + M\varepsilon,$$

for any $\varepsilon > 0$, hence the claim. □

Lemma 4.3. $w(T^n, T_0^n) \geq l(T^n, T_0^n)$.

Proof. Set $\lambda_t : t^{n-2}[\delta] \to t^{n-2}[t\delta]$ for the map given in coordinates by $\lambda_t(p, x) = (tp, x)$, $p \in D^2, x \in t^{n-2}$. Here the projection π provides a global trivialisation of $t^{n-2}[\delta] \subset T^n$. Then

$$\lim_{t\to 0} |Jac(\pi|_{t_0^{n-2}} \circ \lambda_t)| = 1.$$

Therefore for t close enough to 0 one derives

$$\lim_{t\to 0} \frac{vol(\lambda_t(t_0^{n-2}))}{vol(t^{n-2})} = \lim_{t\to 0} \frac{\int_{\lambda_t(t_0^{n-2})} |Jac(\pi|_{t_0^{n-2}})| d\mu}{\int_{t^{n-2}} d\mu} \geq l(T^n, T_0^n).$$

Since the position of t_0^{n-2} was chosen arbitrary, this inequality survives after passing to the infimum and the claim follows. □

□

□

Theorem 4.1. *There exist uncountably many Whitehead-type manifolds for $n \geq 5$.*

Proof. The proof here follows the same pattern as that given by McMillan ([16]) for the 3-dimensional case. Let us establish first the following useful property of the wrapping number:

Proposition 4.3. *If $T_0^n \subset T_1^n \subset T_2^n$ then $w(T_2^n, T_0^n) = w(T_2^n, T_1^n)w(T_1^n, T_0^n)$.*

Proof. This is a consequence of the two lemmas below:

Lemma 4.4. $l(T_2^n, T_0^n) \leq l(T_2^n, T_1^n)l(T_1^n, T_0^n)$.

Proof. Consider $t_0^{n-2} \subset T_1^n \subset T_2^n$, where T_1^n is a very thin tube around t_1^{n-2}, and the two projections to $\pi_2 : t_1^{n-2} \to t_2^{n-2}$ and $\pi_1 : t_0^{n-2} \to t_1^{n-2}$ respectively. Using Lemma 4.1 one can assume that the conditions

$$\mu\left(\left\{x \in t_1^{n-2} |\ \sharp\left\{\pi_1^{-1}(x)\right\} = l(T_1^n, T_0^n)\right\}\right) > 1 - \varepsilon,$$

$$\mu\left(\left\{x \in t_2^{n-2} |\ \sharp\left\{\pi_2^{-1}(x)\right\} = l(T_2^n, T_1^n)\right\}\right) > 1 - \varepsilon,$$

hold. For small enough ϵ one derives that

$$\mu\left(\left\{x \in t_2^{n-2} |\ \sharp\left\{(\pi_2 \circ \pi_1)^{-1}(x)\right\} = l(T_2^n, T_1^n)l(T_1^n, T_0^n)\right\}\right) > 0.$$

This proves that the minimal cardinal of the $(\pi_2 \circ \pi_1)^{-1}(x)$ is not greater than $l(T_2^n, T_1^n)l(T_1^n, T_0^n)$, hence the claim. □

Lemma 4.5. $w(T_2^n, T_0^n) \geq w(T_2^n, T_1^n)w(T_1^n, T_0^n)$.

Proof. We can assume that $w(T_2^n, T_1^n) \neq 0$. Consider an embedding of the $(n-2)$-torus $s_1^{n-2} \subset T_2 = t_2^{n-2}[\varepsilon]$ for which the value of $\frac{vol(t_1^{n-2})}{vol(t_2^{n-2})}$ (as function of t_1^{n-2}) is closed to the infimum in the isotopy class. We will assume that in all formulas below the tori lay in their respective isotopy classes. Then

$$\left(\inf_{t_0^{n-2}\subset s_1^{n-2}[2\varepsilon]} \frac{vol(t_0^{n-2})}{vol(s_1^{n-2})}\right)\left(\inf_{t_1^{n-2}\subset t_2^{n-2}[\varepsilon]} \frac{vol(t_1^{n-2})}{vol(t_2^{n-2})}\right) \leq$$

$$\leq \left(\inf_{t_0^{n-2}\subset s_1^{n-2}[2\varepsilon]} \frac{vol(t_0^{n-2})}{vol(t_1^{n-2})}\right)\frac{vol(s_1^{n-2})}{vol(t_2^{n-2})} =$$

$$= \inf_{t_0^{n-2}\subset s_1^{n-2}[2\varepsilon]} \frac{vol(t_0^{n-2})}{vol(t_2^{n-2})} \leq \inf_{t_0^{n-2}\subset t_2^{n-2}[\varepsilon]} \frac{vol(t_0^{n-2})}{vol(t_2^{n-2})}.$$

The last inequality follows from the fact that $s_1^{n-2}[2\varepsilon] \subset t_2^{n-2}[\varepsilon]$. In fact $w(T_2^n, T_1^n) \neq 0$ implies that s_1^{n-2} intersects any 2-disk $D^2 \times \{*\}$ (i.e. any fiber of the projection $\pi_2 : T_2^n \to t_2^{n-2}$) of $T_2^n = t_2^{n-2}[\varepsilon]$ in at least one point. Then the transversal disk $D^2 \times \{*\}$ of radius ε is therefore contained in the tube $s_1^{n-2}[2\varepsilon]$ of radius 2ε around s_1^{n-2}, establishing the claimed inclusion.

On the other hand the following holds

$$\lim_{\varepsilon\to 0} \inf_{t_0^{n-2}\subset s_1^{n-2}[2\varepsilon]} \frac{vol(t_0^{n-2})}{vol(s_1^{n-2})} = w(T_1^n, T_0^n),$$

due to the topological invariance of the wrapping number. Letting ε go to 0 in the previous inequality yields the claim. □

□

Proposition 4.4. *There exist Whitehead links whose wrapping number has the form $2^{n-2}p$ for any natural number p.*

Proof. The claim is well-known for $n = 3$. One uses Wright's construction ([27]) of Whitehead links by induction on the dimension. If $T_0^n \subset T^n$ is a Whitehead link then set $T^{n+1} = T^n \times S^1$. Consider the projection q of the solid torus $T_0^n \times S^1 \cong D^2 \times S^1 \times ... \times S^1$ onto $D^2 \times S^1$ (the first and the last factors). Choose some Whitehead link $Q^3 \subset D^2 \times S^1$, and set then $T_0^{n+1} = q^{-1}(Q^3)$. The pair $T_0^{n+1} \subset T^{n+1}$ is a Whitehead link of dimension $n+1$. The Proposition then is an immediate consequence of:

Lemma 4.6. $w(T^{n+1}, T_0^{n+1}) = w(T^n, T_0^n)w(D^2 \times S^1, Q^3)$.

Proof. From the multiplicativity of w and the triviality of the projection q it is sufficient to prove that $w(T^n \times S^1, T_0^n \times S^1) = w(T^n, T_0^n)$. This formula can be checked directly using l instead of w. □

□

Proposition 4.5. *For any sequence* $\mathbf{p} = p_0, p_1, \ldots$ *of positive integers consider a Whitehead-type manifold* $W^n(\mathbf{p}) = \cup_{k=1}^{\infty} T_k^n$*, where* $w(T_{k+1}^n, T_k^n) = 2^{n-2}p_k$*. If the sequences* $\mathbf{p}$ *and* $\mathbf{q}$ *have infinitely many non-overlapping prime factors then the manifolds* $W^n(\mathbf{p})$ *and* $W^n(\mathbf{q})$ *are not PL homeomorphic.*

Proof. The proof is similar to that of ([16], p.375). Set $W^n(\mathbf{p}) = \cup_{k=1}^{\infty} T_k^n$, $W^n(\mathbf{q}) = \cup_{k=1}^{\infty} s_k^n$, where T_k^n,$\widetilde{T}_k^n$, are tori, as above. If $h : W^n(\mathbf{q}) \to W^n(\mathbf{p})$ is a PL homeomorphism, there exist integers j, k such that $T_0^n \subset int(h(\widetilde{T}_j^n))$, q_k has a prime factor which occurs in $\mathbf{q}$ but not in $\mathbf{p}$, $k > j+1$ and $h(\widetilde{T}_k^n) \subset int(T_m^n)$. We have therefore

$$w(T_m^n, T_0^n) = w((T_m^n, h(\widetilde{T}_k^n))w(h(\widetilde{T}_k^n), h(\widetilde{T}_j^n))w(h(\widetilde{T}_j^n, T_0^n)).$$

We have obtained a contradiction because q_k divides $w(h(\widetilde{T}_k^n), h(\widetilde{T}_j^n))$ but not the left hand side (which is non-zero also). □

□

5 The proper homotopy invariance of the w.g.s.c.

5.1 Dehn exhaustibility

We study in this section to what extent the w.g.s.c. is a proper homotopy invariant.

Definition 5.1. A polyhedron M is (proper) homotopically dominated by the polyhedron X if there exists a map $f : M \to X$ such that the mapping cylinder Z_f (properly) retracts on M.

Remark 5.1. A proper homotopy equivalence is the simplest example of a proper homotopically domination.

The main result of this section is:

Theorem 5.1. *For* $n \neq 4$ *a non-compact n-manifold is w.g.s.c. if and only if it is proper homotopically dominated by a w.g.s.c. polyhedron.*

Remark 5.2. It seems that the result does not hold, as stated, for $n = 4$ (see also the next section).

Proof. The main ingredient of the proof is the following notion, weaker than the w.g.s.c., introduced by Poénaru:

Definition 5.2. The simply-connected non-compact PL space W is Dehn exhaustible if, for any compact $K \subset W$ there exists some simply connected compact PL space L and a commutative diagram

$$\begin{array}{ccc} K & \stackrel{f}{\to} & L \\ & i \searrow & \downarrow g \\ & & W \end{array}$$

where i is the inclusion, f is an embedding, g is a generic immersion and $f(K) \cap M_2(g) = \emptyset$. Here $M_2(g)$ is the set of double points, namely $M_2(g) = \{x \in L; \sharp g^{-1}(g(x)) \geq 2\} \subset L$.

The first step is to establish:

Proposition 5.1. *An open simply-connected manifold which is proper homotopically dominated by a w.g.s.c. polyhedron is Dehn exhaustible.*

Proof. The proof given in [5] for the 3-dimensional statement extends without any essential modification, and we skip the details. □

Remark 5.3. Poénaru proved a Dehn-type lemma (see [19], p.333-339) which states that a Dehn exhaustible 3-manifold is w.g.s.c. This settles the dimension 3 case.

Lemma 5.1. *If the open simply-connected n-manifold W^n is Dehn exhaustible and $n \geq 5$ then it is w.g.s.c.*

Proof. Consider a connected compact submanifold $K \subset W^n$. Assume that there exists a compact polyhedron M^n with $\pi_1(M^n) = 0$ and a generic immersion F

$$\begin{array}{ccc} K & \stackrel{f}{\to} & M \\ & i \searrow & \downarrow F \\ & & W^n \end{array}$$

such that $M_2(F) \cap K = \emptyset$.

Lemma 5.2. *One can suppose that M^n is a manifold.*

Proof. The polyhedron M^n is endowed with an immersion F into the manifold W^n. Among all abstract regular neighborhoods (i.e. thickenings) of M^n there is a n-dimensional one $U(M^n, F)$, which is called the regular neighborhood determined by the immersion, such that the following conditions are fulfilled:

1. $F : M^n \to W^n$ extends to a generic immersion $\widetilde{F} : U(M^n, F) \to W^n$.
2. The image of $\widetilde{F}(U(M^n, F)) \subset W^n$ is the regular neighborhood of the polyhedron $F(M^n)$ in W^n.

The construction of the PL regular neighborhood determined by an immersion of polyhedra is given in [11]. The authors were building on the case of an immersion of manifolds, considered previously in [9]. Moreover, if one replaces M^n by the manifold $U(M^n, F)$ and F by $\widetilde{F}$ we are in the conditions required by the Dehn-type lemma. □

Consider now a handlebody decomposition of $M^n - K$ and let N_2^n be the union of K with handles of index 1 and 2. Then $\pi_1(N_2^n) = 0$. Let δ_j^2, δ_j^1 be the cores of these extra 1-handles. Since F is a generic immersion, up to a small homotopy, $F(\delta_j^2) \subset W^n$ can be replaced by some embedded 2-disks with the same boundary. This follows from the assumption $n \geq 5$. Also by general position these 2-disks can be chosen to have disjoint interiors. Now the condition $M_2(F) \cap K = \emptyset$ implies that $F(int(\delta_j^2)) \cap K = \emptyset$. This shows that there is a small homotopy such that the restriction of F to δ_j^2 (and δ_j^1) is an embedding into $W^n - K$. Using the uniqueness of the regular neighborhood it follows that F can be chosen to be an embedding on N_2^n. In particular K is engulfed in the 1-connected compact $F(N_2^n)$. □

□

5.2 Dehn exhaustibility and end compressibility in dimension 4

Proposition 5.2. *An open 4-manifold is end compressible if and only if it is Dehn exhaustible.*

Proof. We have to reconsider the proof of Proposition 3.3. Everything works as above except that the disks δ_j^2 cannot be anymore embedded, but only (generically) immersed. They may have finitely many double points in their interior. Then the manifold M^4 obtained by adding 2-handles along the γ_j has a generic immersion $F : M \to T'$, whose double points $M_2(F)$ are outside of T. This implies that W^4 is Dehn exhaustible.

Conversely assume that W^4 is Dehn exhaustible. Let K^4 be a compact submanifold of W^4 and M^4 be the immersible simply connected polyhedron provided by the Dehn exhaustibility property. Lemma 5.2 allow us to assume that M^4 and $F(M^4)$ are 4-manifolds. Consider now to the proof of the first claim from Theorem 5.1. It is sufficient to consider the case when $M^4 = M_2^4$ i.e. M^4 is obtained from K^4 by adding 1- and 2-handles. If $\Gamma \subset \pi_1(\partial M_1^4)$ is the normal subgroup generated by the attaching curves of the 2-handles of M^4 then the same argument yields:

$$\Gamma \subset \ker(\pi_1(\partial M_1^4) \to \pi_1(M^4 - int(K^4))).$$

Since F is a generic immersion we can suppose that F is an embedding of the cores of the 1-handles and so $F|_{M_1^4}$ is an embedding.

We have $F(M^4 - int(K^4)) \subset F(M^4) - int(K^4)$ because the double points of F are outside K^4. Now the homomorphism induced by F on the left side of the diagram

$$\begin{array}{ccc} \pi_1(\partial M_1^4) & \to & \pi_1(M^4 - int(K^4)) \\ F\downarrow & & \downarrow F \\ \pi_1(\partial F(M_1^4)) & \to & \pi_1(F(M^4) - int(K^4)) \end{array}$$

is an isomorphism and we derive that

$$F(\Gamma) \subset \ker(\pi_1(\partial F(M_1^4)) \to \pi_1(F(M^4) - int(K^4))).$$

Meantime $F(\Gamma)$ surjects onto $\pi_1(F(M_1^4))$ under the map $\pi_1(\partial F(M_1^4)) \to \pi_1(F(M_1^4))$. But $F(M_1^4)$ is homeomorphic to M_1^4, hence it is obtained from K^4 by adding 1-handles. This shows that the pair $(F(M^4), K^4)$ is stably compressible, from which one obtains the end compressibility of W^4 as in the proof of Theorem 3.1. □

Remark 5.4. If the open 4-manifold W^4 is Dehn exhaustible then $W^4 \times [0, 1]$ is also Dehn exhaustible hence w.g.s.c. Therefore an example of an open 4-manifold W^4 which is end compressible but which is not w.g.s.c. will show that the result of Theorem 5.1 cannot be extended to dimension 4, as stated. Such examples are very likely to exist, as the Dehn lemma is known to fail in dimension 4 (by S.Akbulut's examples).

6 G.s.c. for 4-manifolds

6.1 W.g.s.c. versus g.s.c.

Definition 6.1. A geometric Poénaru-Mazur-type manifold M^4 is a compact simply connected 4-manifold satisfying the following conditions:

1. $H_2(M^4) = 0$.

2. the boundary ∂M^4 is connected and π_1-dominates a virtually geometric 3-manifold group, i.e. there exists a surjective homomorphism

$$\pi_1(\partial M^4) \to \pi_1(N^3),$$

onto the (non-trivial) fundamental group of a virtually geometric 3-manifold N^3.

Proposition 6.1. *The interior $int(M^4)$ of a geometric Poénaru-Mazur-type manifold M^4 does not have a proper handlebody decomposition without 1-handles with the boundary of a cofinal subset of the intermediate manifolds obtained on a finite number of handle additions being homology spheres.*

6.2 Casson's proof of Proposition 6.1

The main ingredient is the following proposition extending an unpublished result of Casson:

Proposition 6.2. *Consider the 4-dimensional (compact) cobordism (W^4, M^3, N^3) such that (W^4, M^3) is 1-connected. Assume moreover that the following conditions are satisfied:*

1. *$H_2(W^4, M^3; \mathbf{Q}) = 0$, both M^3 and N^3 are connected.*

2. *$\pi_1(N^3)$ is a group which π_1-dominates a virtually geometric non-trivial 3-manifold group. Let K be the kernel of this epimorphism.*

3. *$b_1(W^4) \leq b_1(N^3)$, where b_1 denotes the first Betti number.*

4. *The map $\pi_1(N^3) \to \pi_1(W^4)$ induced by the inclusion $N^3 \hookrightarrow W^4$ has kernel strictly bigger than the subgroup K. In particular this is true if this map is trivial.*

Then any handlebody decomposition of W^4 from M^3 has 1-handles i.e. the pair (W^4, M^3) is not g.s.c.

Remark 6.1. A necessary condition for the g.s.c. of (W^4, M^3) is that the map $\pi_1(M^3) \to \pi_1(W^4)$, induced by the inclusion $M^3 \hookrightarrow W^4$, be onto. In fact adding 2-handles amounts to introducing new relations to the fundamental group of the boundary, whereas the latter is not affected by higher dimensional handle additions.

Casson's result was based on partial positive solutions to the Kervaire Conjecture 1.2. One proves that certain 4-manifolds $(N, \partial N)$ have no handle decompositions without 1-handles by showing that if they did, then $\pi_1(\partial N)$ violates the Kervaire conjecture. Casson's argument works to the extent that the Kervaire conjecture is known to be true. Casson originally applied it using a theorem of Gerstenhaber and Rothaus [7], which said that the Kervaire conjecture holds for subgroups of a compact Lie group. Subsequently, Rothaus [23] showed that the conjecture in fact holds for residually finite groups. Since residual finiteness for all 3-manifold groups is implied by the geometrisation conjecture, Casson's argument works in particular for all manifolds satisfying the geometrisation conjecture. A simple argument (Remark 6.3 below) extends the class of groups for which the Kervaire conjecture is known further.

Proposition 6.3. *If some non-trivial quotient Q of a group G satisfies the Kervaire conjecture, then so does G. In particular if a finitely generated group G has a proper finite-index subgroup, then G satisfies the Kervaire conjecture (since finite groups satisfy the Kervaire conjecture by [7]).*

Proof. Let $\phi : G \twoheadrightarrow Q$ be the quotient map. Assume that Q satisfies the Kervaire conjecture. Suppose that G violates the Kervaire conjecture. Then we have generators $\alpha_1, \ldots, \alpha_n$ and relations such that $\frac{G*<\alpha_1,\ldots,\alpha_n>}{<<r_1,\ldots,r_n>>}$ is the trivial group. Let $\tilde{\phi} : G* < \alpha_1, \ldots, \alpha_n > \to Q* < \bar{\alpha}_1, \ldots, \bar{\alpha}_n >$ be the map extending ϕ by mapping α_i to $\bar{\alpha}_i$. This is clearly a surjection, and induces a surjective map $\bar{\phi} : \frac{G*<\alpha_1,\ldots,\alpha_n>}{<<r_1,\ldots,r_n>>} \twoheadrightarrow \frac{Q*<\bar{\alpha}_1,\ldots,\bar{\alpha}_n>}{<<\tilde{\phi}(r_1),\ldots,\tilde{\phi}(r_n)>>}$. But since the domain of the surjection $\bar{\phi}$ is trivial, so is the codomain. But this means that $\frac{Q*<\bar{\alpha}_1,\ldots,\bar{\alpha}_n>}{<<\tilde{\phi}(r_1),\ldots,\tilde{\phi}(r_n)>>}$ is trivial, and so Q violates the Kervaire conjecture, a contradiction. ☐

Proof of Proposition 6.2. Suppose that

$$W^4 = M^3 \times [0,1] \sharp_k 2\text{-handles} \sharp_r 3\text{-handles},$$

(with some 0-handle or 4-handle added if one boundary component is empty). It is well-known (see [24]) that the homology groups $H_*(W^4, M^3)$ are the same as those of a differential complex C_*, whose component C_j is the free module generated by the j-handles. Therefore this complex has the form:

$$0 \to \mathbf{Z}^r \to \mathbf{Z}^k \to 0.$$

Thus $H_2(W^4, M^3; \mathbf{Q}) = 0$ implies that $k \leq r$ holds.

Consider now the handlebody decomposition is turned up-side down:

$$W^4 = N^3 \times [0,1] \sharp_r 1\text{-handles } \sharp_k 2\text{-handles},$$

(plus possibly one 0-handle or 4-handle if the respective boundary component is empty).

By the van Kampen theorem it follows that $\pi_1(W^4)$ is obtained from $\pi_1(N^3) = \pi_1(N^3 \times [0,1])$ by adding one generator for each 1-handle and one relation for each 2-handle. Therefore

$$\pi_1(W^4) = \pi_1(N^3) * \mathbf{F}(r)/W(k),$$

where $\mathbf{F}(r)$ is the free group on r generators $x_1, ..., x_r$ and $W(k)$ is a normal subgroup of the free product generated also by k words $Y_1, ..., Y_k$.

Consider a virtually geometric 3-manifold L^3 such that $\pi_1(M^3) \to \pi_1(L^3)$ is surjective. If L^3 is a geometric 3-manifold then its fundamental group is residually finite (see e.g. [25], Thm.3.3, p.364). Let d_{ij} be the degree of the letter x_j in the word representing Y_i. The result of Rotthaus ([23], Thm. 18, p.611) states that for any locally residually finite group G and choice of words Y_i such that $\mathbf{d} = (d_{ij})_{i,j}$ is of (maximal) rank k, the natural morphism $G \to G * \mathbf{F}(r)/W(k)$ is an injection. We have therefore a commutative diagram

$$\begin{array}{ccc} \pi_1(M^3) & \to & \pi_1(W^4) \\ \downarrow & & \downarrow \\ \pi_1(L^3) & \hookrightarrow & \pi_1(L^3) * \mathbf{F}(r)/W(k) \end{array}$$

whose vertical arrows are surjections. The kernel of the map induced by inclusion, $\pi_1(M^3) \to \pi_1(W^4)$ is contained in K. This contradicts our hypothesis.

On the other hand if the rank of $\mathbf{d}$ is not maximal then by considering the abelianisations one derives $H_1(G*\mathbf{F}(r)/W(k)) \subset H_1(G) \oplus \mathbf{Z}$, hence $b_1(W^4) \geq b_1(N^3)+1$, which is also false. □

Corollary 6.1. *Consider a 4-manifold W^4 which is compact connected simply-connected with non-simply connected boundary ∂M. If the boundary is (virtually) geometric and $H_2(W^4) = 0$ then W^4 is not g.s.c.*

Proof of Proposition 6.1. Assume now that $int(M^4)$ admitted a proper handlebody decomposition without 1-handles. One identifies $int(M^4)$ with $M^4 \cup_{\partial M \cong \partial M \times \{0\}} \partial M \times [0,1)$. We can truncate the handle decomposition at a finite stage in order to obtain a manifold Q^4 such that $\partial Q^4 \subset \partial M^4 \times (0,1)$, because the decomposition is proper. We can suppose that ∂Q^4 is connected since $int(M^4)$ has one end. Then Q^4 is g.s.c. hence $\pi_1(Q^4) = 0$.

By hypothesis, we can choose ∂Q^4 to be a homology sphere. Then ∂Q^4 separates the cylinder $\partial M^4 \times [0,1]$ into two manifolds with boundary which, by Mayer-Vietoris, have the homology of S^3. This implies that $H_2(Q^4) = 0$ (again by Mayer-Vietoris).

Let us consider now the map $f : \partial Q^4 \hookrightarrow \partial M^4 \times [0,1] \to \partial M^4$, the composition of the inclusion with the obvious projection.

Lemma 6.1. *The map f has degree one hence induces a surjection on the fundamental groups.*

Proof. The 3-manifold ∂Q^4 separates the two components of the boundary. In particular the generic arc joining $\partial M^4 \times \{0\}$ to $\partial M^4 \times \{1\}$ intersects transversally ∂Q^4 in a number of points, which counted with the sign sum up to 1 (or -1). If properly interpreted this is the same as claiming the degree of f is one.

It is well-known that a degree one map between orientable 3-manifolds induce a surjective map on the fundamental group (more generally, the image of the homomorphism induced by a degree d is a subgroup whose index is bounded by d). □

This shows that $\pi_1(\partial Q^4) \to \pi_1(\partial M^4)$ is surjective. On the other hand $\pi_1(\partial M^4)$ surjects onto a non-trivial residually finite group. Since $\pi_1(\partial Q^4) \to \pi_1(Q^4) = 1$ is the trivial map, the argument we used previously (from Rothaus' theorem) gives us a contradiction. This settles our claim. □

7 Handle decompositions without 1-handles in dimension 4

7.1 Open tame 4-manifolds

Definition 7.1. An exhaustion is g.s.c. if it corresponds to a proper sequence of handle additions with no 1-handles. Alternatively one has a proper Morse function, which we will refer to as *time*, with words like *past* and *future* having obvious meanings, with no critical points of index one. The inverse images of regular points are 3-manifolds, which we refer to as the *manifold at that time.*

We assume henceforth that we have a g.s.c. handle decomposition of the interior $int(W^4)$ of $(W^4, \partial W^4)$, a compact four manifold with boundary a homology 3-sphere and $H_2(W^4) = 0$.

Now let $(K_i, \partial K_i), i \in \mathbb{N}$ denote the 4-manifolds obtained by successively attaching handles to the zero handle (B^4, S^3), that is if $t : (W^4, \partial W^4) \to \mathbb{R}$ is the Morse function time, then $(K_i, \partial K_i) = t^{-1}((\infty, a_i])$, with a_i being points lying between pairs of critical values of the Morse function.

Lemma 7.1. *∂K_{i+1} is obtained from ∂K_i by one of the following:*

- *A 0-frame surgery about a homologically trivial knot in ∂K_i.*
- *Cutting along a non-separating 2-sphere in ∂K_i and capping off the result by attaching a 3-ball.*

These correspond respectively to attaching 2-handles and 3-handles to $(K_i, \partial K_i)$.

Proof. Since attaching 2-handles and 3-handles correspond to surgery and cutting along 2-spheres respectively, we merely have to show that the surgery is 0-frame about a homologically trivial curve and the spheres along which one cuts are non-separating.

First note that the absence of 1-handles implies $H_1(K_i) = 0 = \pi_1(K_i)$, for all i. Further, each ∂K_i is connected because $int(W^4)$ has one end. Thus the 2-spheres along which any ∂K_i is split have to be non-separating.

Using Mayer-Vietoris, the fact that $H_2(W^4) = 0$, and the long exact sequence in homology we derive that $H_2(K_i) = H_2(\partial K_i)$. Also adding a 3-handle decreases the rank of $H_2(\partial K_i)$ by one hence every surgery increases the rank of $H_2(\partial K_i)$ by one unit. But this means that the surgery must be a zero-frame surgery about a homologically trivial curve. □

For i large enough, ∂K_i lies in a collar $\partial W^4 \times [0, \infty)$ hence we have a map $f_i : \partial K_i \to \partial W^4$ which is the composition of the inclusion with the projection. By the Lemma 5.1 the maps f_i are of degree one and induce surjections $\phi_i : \pi_1(\partial K_i) \to \pi_1(\partial W^4)$. Here and henceforth we always assume that the index i is large enough so that f_i is defined.

Lemma 7.2. *The homotopy class of a curve along which surgery is performed is in the kernel of $\phi_i : \pi_1(\partial K_i) \to \pi_1(\partial W^4)$.*

Proof. If a surgery is performed along a curve γ, this means that a 2-handle is attached along the curve in the 4-manifold W^4. Hence γ bounds a disk in $\partial W^4 \times [0, \infty)$, which projects to a disk bounded by $f_i(\gamma)$ in ∂W^4. □

Remark 7.1. The maps ϕ_i and ϕ_{i+1} are related in a natural way. To define the map ϕ_{i+1}, take a generic curve γ representing any given element of $\pi_1(\partial K_{i+1})$. If ∂K_{i+1} is obtained from ∂K_i by splitting along a sphere, then γ is a curve in ∂K_i, and so we can simply take its image. On the other hand, if a surgery was performed, then we may assume that γ lies off the solid torus that has been attached, and hence lies in ∂K_i, so we can take its image as before. This map is well-defined by lemma 7.2.

Definition 7.2. A curve $\gamma' \subset \partial K_i$ is a *descendant* of the surgery curve $\gamma \subset \partial K_i$ if it is homotopic to it in ∂K_i (though not in general homotopic to γ after the surgery). A curve $\gamma \subset \partial K_i$ is said to *persist till* ∂K_{i+n} if some descendant of γ persists, i.e., we can homotope γ in ∂K_i so that it is disjoint from all the future 2-spheres on which 3-handles are attached while passing from $\partial K_i = M_i^3$ to $\partial K_{i+n} = M_{i+n}^3$.

Definition 7.3. A curve $\gamma \subset \partial K_i$ is said to *die by* ∂K_{i+n} if it is homotopically trivial in the 4-manifold obtained by attaching 2-handles to K_i along the curves in ∂K_i where surgeries are performed in the process of passing to ∂K_{i+n}, or equivalently, γ is trivial in the group obtained by adding relations to $\pi_1(M_i^3)$ corresponding to curves along which the surgery is performed.

We prove now a key property of the sequence ∂K_i.

Lemma 7.3. *For each i, there is a uniform $n = n(i)$ such that any curve $\gamma \subset \partial K_i$, $\gamma \in \ker \phi_i$ that persists till ∂K_{i+n} dies by ∂K_{i+n}.*

Proof. We can find $x \in [0, \infty)$ so that $\partial W^4 \times \{x\}$ is entirely after ∂K_i, and n_1 so that $\partial W^4 \times \{x\} \subset K_{i+n_1}$, because the handlebody decomposition is proper. We then define n by repeating this process once, i.e. $\partial W^4 \times \{x_1 + \varepsilon\} \subset K_{i+n}$, for some $x_1 + \varepsilon > x_1 > x$ for which $\partial W^4 \times \{x_1\}$ is entirely after ∂K_{i+n_1}. Consider $\gamma \in \ker \phi_i$ which persists till ∂K_{i+n_1}. This means that there is an annulus properly embedded in $K_{i+n} - int(K_i)$, whose boundary curves are γ and $\tilde{\gamma} \in \partial K_{i+n} \subset \partial W^4 \times [x_1, x_1 + \varepsilon)$. Since $\gamma \in \ker \phi_i$ it bounds a disc in $\partial W^4 \times [x_1, \infty)$. This disc together with the above annulus ensure that γ dies by ∂K_{i+n}, as they bound together a disc entirely in $K_{i+n} - int(K_i)$, and 3-handles do not affect the fundamental group. □

7.2 The structure theorem

Suppose henceforth that we have a sequence of connected 3-manifolds $M_i^3 \subset \partial W^4 \times [0, \infty)$ and associated maps onto $f_i : M_i^3 \to \partial W^4$ that satisfies the properties of ∂K_i stated above. Specifically one asks that:

- The maps $f_i : M_i^3 \to \partial W^4$ are of degree one, hence inducing surjection $\phi_i : \pi_1(M_i^3) \to \pi_1(\partial W^4)$.
- M_{i+1} is obtained from M_i^3 either by a 0-frame surgery along a homologically trivial knot in M_i^3, or else by cutting along a non-separating 2-sphere in M_i^3.
- The surgery curves in M_i^3 belong to $\ker \phi_i$.
- The maps ϕ_i and ϕ_{i+1} are related as in Remark 7.1.
- For any i there exists some $n = n(i)$ such that any curve in M_i^3 which persists till M_{i+n}^3 dies by M_{i+n}^3.

We show in this section that, after possibly changing the order of attaching handles, any handle decomposition without 1-handles is of a particular form.

We first describe a procedure for attempting to construct a handle decomposition for $int(W^4)$ starting with a partial handle decomposition, with boundary M_i^3. In general, M_i^3 has non-trivial homology. It follows readily from the proof of lemma 7.1 that $H_1(M_i^3)$ is a torsion free abelian group. The only way we can remove homology is by splitting along spheres. To this end, we take a collection of surfaces representing the homology, perform surgeries along curves in these surfaces so that they compress down to spheres, and then split along these spheres. By doing the surgeries, we have created new homology, and hence have to take new surfaces representing this homology and continue this procedure. In addition to this, we may need to perform other surgeries to get rid of the *homologically trivial portion* of the kernel of $\phi_i : \pi_1(M_i^3) \to \pi_1(\partial W^4)$.

The above construction may meet obstructions, since the surgeries have to be performed about curves that are homologically trivial as well as lie in the kernel of ϕ_i, hence it may not be always possible to perform enough of them to compress the surfaces to spheres. The construction terminates at some finite stage if at that stage all the homology is represented by spheres and no surgery off these surfaces is necessary.

Theorem 7.1. *After possibly changing the order of attaching handles, any handle decomposition without* 1*-handles may be described as follows. We have a collection of surfaces* $F_j(i)$*, with disjoint simple closed curves* $l_{j,k} \subset F_j(i)$ *and a generic immersion* $\psi_i : \cup_j F_j(i) \to M_i^3$ *such that:*

- *The surfaces represent the homology of* M_i^3*, i.e.* ψ_i *induces a surjection* $\psi_i : H_2(\cup_j F_j(i)) \to H_2(M_i^3)$.
- *The immersion* ψ_i *has only ordinary double points and the restriction to each individual surface* $F_j(i)$ *is an embedding. The double curves of* ψ_i *are among the curves* $l_{j,k}$*. Their images* $\psi_i(l_{j,k})$ *are called seams.*
- *When compressed along the seams (i.e. by adding 2-handles along them) the surfaces* $\psi_i(F_j(i))$ *become unions of spheres.*
- *The seams are homologically trivial curves in* M_i^3 *and lie in the kernel of* ϕ_i.
- *The pull backs (see the definition below) of the surfaces* $\psi_m(F_k(m)) \subset M_m^3$ *for* $m > i$*, which are surfaces with boundary in* M_i^3*, can only intersect the* $F_j(i)$*'s either transversely at the seams or by having some boundary components along the seams.*

We attach 2*-handles along all the seams of* M_i^3*, and possibly also along some curves that are completely off the surfaces* $F_j(i)$ *in* M_i^3 *and have no intersection with any future surface* $F_k(m)$*,* $m > i$*. We then attach* 3*-handles along the 2-spheres obtained by compressing the surfaces* $\psi_i(F_j(i))$*. Iterating this procedure gives us the handle decomposition.*

We will see that once we construct the surfaces, all of the properties follow automatically.

Let $F \subset M_{i+n}^3$ be an embedded surface. We let $M_i^3 = \partial K_i^4$, where K_i^4 is the bounded component in $int(W^4)$. Then $K_{i+n} - int(K_i) = M_i^3 \times [0,\varepsilon] \cup h_j^2 \cup h_k^3$, where h_j^m are the attached m-handles.

Lemma 7.4. *There exists an isotopy of* $K_{i+n} - int(K_i)$ *such that* $F \subset M_i^3 \times \{\varepsilon\} \cup h_j^2$ *and* $F \cap h_j^2 = \cup_k \delta_{j,k}^2$*, where* $\delta_{j,k}^2$ *are disjoint 2-disks properly embedded in the pair* $(h_j^2, \partial_a h_j^2)$ *(here* $\partial_a h_j^2$ *denotes the attachment zone of the handle, which is a solid torus), which are parallel to the core of the handle. Moreover* $\partial \delta_{j,k}^2 \subset \partial(\partial_a h_j^2))$ *are concentric circles on the torus, parallel to the 0-framing of the attaching circle.*

Proof. It follows from a transversality argument that the image of F intersects only the 2-handles, along 2-disks. Further it is sufficient to see that the circles $\partial\delta^2_{j,k}$ are homotopic to the 0-framing since in $K_{i+n} - int(K_i)$ homotopy implies isotopy for circles. If one circle is null-homotopic then it can be removed by means of an ambient isotopy. If a circle turns p-times around the longitude, then it cannot bound a disk in h^2_j unless $p = 1$. $\square$

Definition 7.4. Consider a parallel copy in $M^3_i = M^3_i \times \{0\}$ of the surface with boundary $F' = F - \cup_{j,k}\delta^2_{j,k} \subset M^3_i \times \{\varepsilon\} - \cup_j \partial_a h^2_j$, and use standardly embedded annuli in the torus $\partial_a h^2_j$, which join the parallel circles to the central knot in order to get a surface with boundary on the surgery loci. We calls this a *pull back* of the surface $F \subset M_{i+n}$.

Lemma 7.5. *Let $\alpha : \pi_1(M^3_i) \to H_1(M^3_i)$ be the Hurewicz map. Then $\phi_i(\ker(\alpha)) = \pi_1(\partial W^4)$, i.e. the pair (α, ϕ_i) is algebraically compressible.*

Proof. Consider the diagram $\begin{array}{ccc} \Gamma & \xrightarrow{\phi} & G \\ \pi\downarrow & & \downarrow \\ \Gamma_{ab} & \xrightarrow{\phi_{ab}} & G_{ab} \end{array}$ where ϕ is surjective, and the subscript ab means abelianisation. Then it is automatically that $\pi(\ker\phi) = \ker\phi_{ab}$. Since $H_1(\partial W^4) = 0$, and the algebraic compressibility is symmetric, the result follows. $\square$

Now, let $n = n(i)$ be as in the conclusion of lemma 7.3. We consider a maximal set of disjoint non-parallel essential 2-spheres (which is uniquely defined up to isotopy) and pull back these spheres up to time i to get a collection of planar surfaces, whose union is a 2-dimensional polyhedron $\Sigma_i \subset M_i$.

Lemma 7.6. *If ι denotes the map induced by the inclusion $\pi_1(M_i - \Sigma_i) \to \pi_1(M_i)$ then the restriction*

$$\phi_i : \iota(\pi_1(M_i - \Sigma_i)) \cap \ker(\alpha) \to \pi_1(\partial W^4)$$

is surjective.

Proof. The pull-backs in M^3_i of spheres $S^2_m \subset M^3_{i+j}$ are planar surfaces with boundary components being the loci of future surgeries. Further, after compressing the spheres S^2_m of M_{i+j} (hence arriving into M^3_{i+j+k}) we have a surjection ϕ_{i+j+k}, thus the map $\pi_1(M^3_{i+j} - \cup S^2_m) \to \pi_1(\partial W^4)$ is also surjective. This means that there exist curves in the complement of the planar surfaces in M^3_i mapping to every element of $\pi_1(\partial W^4)$. Moreover, by the above lemma, we have such curves that are homologically trivial in M^3_{i+j}, and hence in M^3_i as all surgery curves are null-homologous. $\square$

Lemma 7.7. $i_* : H_1(M^3_i - \Sigma_i) \to H_1(M^3_i)$ *is the zero map.*

Proof. If not then there exists a curve $\gamma \subset M^3_i - \Sigma_i$ that represents a non-trivial element of $H_1(M^3_i)$. Modifying by a homologically trivial element if necessary, we may assume that $\gamma \in ker(\phi_i)$. By the previous lemma γ persists. The group $\pi_1(K_{i+n} - int(K_i))$ is the quotient of $\pi_1(M^3_i)$ by the relations generated by the surgery curves, which are homologically trivial. In particular $H_1(K_{i+n} - int(K_i)) = H_1(M^3_i)$. Then the class of $\gamma \in H_1(K_{i+n} - int(K_i))$ is non-zero since its image in $H_1(M^3_i)$ is non-zero by hypothesis. This gives the required contradiction. $\square$

We are now in a position to prove the structure theorem. The images of the immersion ψ_i is obtained from the polyhedron Σ_i by *stitching together* several planar surfaces along the boundary knots. These knots will be the seams of the surfaces. It is clear by construction that we have all the desired properties as soon as we show that there are enough planar surfaces to be stitched together to represent all the homology.

To see this, we consider the reduced homology exact-sequence of the pair $(M_i^3, M_i^3 - \Sigma_i)$, and use the fact that $M_i^3 - \Sigma$ is connected, since M_{i+n}^3 is, as well as lemma 7.7. Thus, we have the exact sequence

$$\cdots \to H_1(M_i^3 - \Sigma_i) \to H_1(M_i^3) \to H_1(M_i^3, M_i^3 - \Sigma_i) \to \tilde{H}_0(M_i^3 - \Sigma_i)$$

which gives the exact sequence

$$0 \to H_1(M_i^3) \to H_1(M_i^3, M_i^3 - \Sigma_i) \to 0$$

which together with an application of Alexander duality gives $H_1(M_i^3) \cong H_1(M_i^3, M_i^3 - \Sigma_i) \cong H^2(\Sigma_i)$. Further, as the isomorphisms $H_1(M_i^3) \cong H^2(M_i^3)$ and $H_1(M_i^3, M_i^3 - \Sigma_i) \cong H^2(\Sigma_i)$, given respectively by Poincaré and Alexander duality, are obtained by taking cup products with the fundamental class, the diagram

$$\begin{array}{ccc} H^2(M_i^3) & \longrightarrow & H^2(\Sigma_i) \\ \downarrow & & \downarrow \\ H_1(M_i^3) & \longrightarrow & H_1(M_i^3, M_i^3 - \Sigma_i) \end{array}$$

commutes.

Thus the inclusion of Σ_i in M_i^3 gives an isomorphism $H^2(M_i^3) \cong H^2(\Sigma_i)$. Since $H_2(M_i^3)$ and $H_2(\Sigma_i)$ have no torsion, the cap product induces perfect pairings $H_2(M_i^3) \times H^2(M_i^3) \to \mathbb{Z}$ and $H_2(\Sigma_i) \times H^2(\Sigma_i) \to \mathbb{Z}$. Therefore, by duality, the map $H_2(\Sigma_i) \to H_2(M_i^3)$ induced by inclusion is also an isomorphism.

Now take a basis for $H^2(\Sigma_i)$. Each element of this basis can be looked at as an integral linear combination of the planar surfaces (as in cellular homology), with trivial boundary. We obtain a surface corresponding to each such homology class by taking copies of the planar surfaces, with the number and orientation determined by the coefficient. Since the homology classes are cycles, these planar surfaces can be glued together at the boundaries to form closed, oriented, immersed surfaces. Without loss of generality, we can assume these to be connected.

Remark 7.2. By doing surgeries on the seams of $\Sigma_i \subset M_i^3$ some new homology is created (the homology of M_{i+1}) One constructs naturally surfaces representing the homology of M_{i+1}^3, as follows. One considers *generalized* Seifert surfaces in M_i of the loci of the surgeries, which are surfaces which might have boundary components along other seams. Then one caps-off the boundaries by using the cores of the 2-handles which are added and push the closed surfaces into M_{i+1}^3. Notice that we can consider also some Seifert surface whose boundary components are seams in some M_{i+n}^3 for $n > 1$.

7.3 On Casson finiteness

Suppose we do have a 4-manifold $(W^4, \partial W^4)$ with a g.s.c. handle decomposition of its interior. Since there may be infinitely many handles, we cannot use Casson's argument. However, we note that we can use Casson's argument if we can show that

- $(W^4, \partial W^4)$ has a (finite) handle decomposition without 1-handles.
- Some $(Z^4, \partial Z^4)$ has a handle decomposition without 1-handles, where Z^4 is compact, contractible with $\pi_1(\partial Z^4) = \pi_1(\partial W^4)$.
- Some $(Z^4, \partial Z^4)$ has a handle decomposition without 1-handles, where Z^4 is compact, contractible and there is a surjection $\pi_1(\partial Z^4) \twoheadrightarrow \pi_1(\partial W^4)$ (by Proposition 6.3).

Thus, we can apply Casson's argument if we show finiteness, or some weak form of finiteness such as the latter statements above.

We now assume that the handle decomposition is as in the conclusion of Theorem 7.1. We will change our measures of time so that passing from M_i^3 to M_{i+1}^3 consists of performing all the surgeries required to compress the surfaces, splitting along the 2-spheres, and also performing the necessary surgeries off the surface.

In M_i^3, we have a collection of embedded surfaces representing all the homology of M_i^3. We see that we have Casson finiteness in a special case.

Theorem 7.2. *If there exists i so that the immersion $\psi_i : \cup_j F_j(i) \to M_i^3$ is actually an embedding and $\phi_i(\psi_i(F_j(i)) = \{1\} \subset \pi_1(\partial W^4)$ then $\pi_1(\partial W^4)$ violates the Kervaire conjecture.*

Proof. Let k be the rank of $H_1(M_i^3)$ and $P_j, 1 \leq j \leq k$ be the fundamental groups of the surfaces. Since the surfaces are disjoint, $\pi_1(M_i^3)$ is obtained by HNN extensions from the fundamental group G of the complement of the surfaces. Thus, if ψ_j are the gluing maps, we have

$$\pi_1(M_i^3) = \left\langle G, t_1, \ldots, t_k; t_j x t_j^{-1} = \psi_j(x) \forall x \in P_j \right\rangle$$

Now, since $\phi_i(P_j) = 1$ and $\phi_i(G) = \pi_1(\partial W^4)$, $\pi_1(M_i^3)$ surjects onto $\langle \pi_1(\partial W^4), t_1, \ldots, t_n \rangle$, the group obtained by adding k generators to $\pi_1(\partial W^4)$. But, M_i^3 is obtained by using n 2-handles and $n-k$ 3-handles. Thus, as in Casson's theorem, $\pi_1(M_i^3)$ is killed by adding $n-k$ generators and n relations. This implies that $\pi_1(\partial W^4)$ is killed by adding n generators and n relations. □

Theorem 7.3. *There exists always an immersion as in the structure theorem with the additional property that $\phi_i(\psi_i(F_j(i)) = \{1\} \subset \pi_1(\partial W^4)$ holds true.*

Proof. By construction the images of the seams (which are roughly speaking half the generators of the fundamental group) are null-homotopic. If the fundamental groups of the generalized Seifert surfaces from the previous remark map to the trivial group, then after doing surgery on the seams we obtain surfaces $F_j(i+1)$ representing homology with trivial π_1 images by ϕ_{i+1}. Thus, it suffices to show that we obtain this condition for a choice of Seifert surfaces for all seams, at some time in the future.

Fix i large enough so that M_i^3 is in the collar $\partial W^4 \times [0, \infty)$.

Lemma 7.8. *There exists some $n' = n'(i)$ such that, whenever a 2-sphere immersed in $int(W^4) - K_{i+n'}$ bound a 3-ball immersed in the collar, it actually bounds a 3-ball immersed in $int(W^4) - K_i$.*

Proof. Choose n' large enough so that a small collar $\partial W^4 \times [x, y] \subset K_{i+n'} - int(K_i)$. Then use the horizontal flow to send $\partial W^4 \times [0, y]$ into $\partial W^4 \times [x, y]$ by preserving the right side boundary. This yields a ball in the complement of K_i. □

We need the following analogue, for π_2 instead of π_1, of Lemma 7.3.

Lemma 7.9. *There exists $k = k(i)$ such that any immersed 2-sphere in K_i (respectively in the intersection of K_i with the collar) which bounds a 3-ball in int(W^4) (respectively in the collar) does so in K_{i+k} (respectively in the collar).*

Proof. The same trick we used in the proof of the previous lemma applies. □

Choose now n large enough such that $K_{i+n} - K_{i+n'}$ contains a non-trivial collar $\partial W^4 \times [x, y]$, and $n > k(i + n')$ provided by the Lemma 7.3. Consider a surgery curve $\gamma \subset M_j$ for some $i + n' \leq j < i + n$.

Lemma 7.10. *There exists a generalized Seifert surface for γ in M_i, so that the other boundary components are surgery curves from M_m, with $m \leq i + n$, and whose fundamental group maps to the trivial group under ϕ_i.*

Proof. The curve γ bounds a disc in the 2-handle attached to it. Further, as it can be pulled back, say along an annulus, to time $M_{i+n'}$, and then dies by M_{i+n}, it bounds another disc consisting of the annulus and the disc by which it dies. These discs together form an immersed 2-sphere. Consider the class $\nu \in \pi_2(\partial W^4)$ of this 2-sphere by using the projection of the collar on ∂W^4. We can realize the element ν by an immersed 2-sphere in a small collar $\partial W^4 \times [x, y] \subset K_{i+n} - K_i$. Therefore by modifying the initial 2-sphere by this sphere (which is far from γ) in the small collar one finds an immersed 2-sphere whose image in $\pi_2(\partial W^4)$ is trivial. Since i was large enough $K_{i+n} - K_i$ is a subset of a larger collar $\partial W^4 \times [0, z]$. Then the 2-sphere we constructed bounds a 3-ball in $\partial W^4 \times [0, z]$, and so by Lemma 7.8 it also does so in $X = K_{i+n} - K_i$. Let $\mu : S^2 \to X$ denote this immersion realizing a trivial element of $\pi_2(X)$. Then μ lifts to a map $\tilde{\mu} : S^2 \to \tilde{X}$, where $\tilde{X}$ is the universal covering space of X. Since μ is null-homotopic the homology class of $[\tilde{\mu}] = 0 \in H_2(\tilde{X})$ is trivial, when interpreting $\tilde{\mu}$ as a 2-cycle in $\tilde{X}$.

The homology of $\tilde{X}$ is computed from the $\pi_1(X)$-equivariant complex associated to the handle decomposition, whose generators in degree d are the d-handles attached to K_i in order to get K_{i+n}. Therefore one has then the following relation in this differential complex:

$$[\tilde{\mu}] = \sum_j c_j \, \partial \, [h_j^3], \quad c_j \in \mathbb{Z}.$$

The action of the algebraic boundary operator ∂ on the element $[h_j^3]$ can be described geometrically as the class of the 2-cycle which represents the attachment 2-sphere $\partial^+ h_j^3$ of the 3-handle h_j^3. Consequently the previous formula can be rewritten as

$$\tilde{\mu} = \sum_j c_j \, \partial^+ h_j^3 + \sum_k d_k \, L_k, \quad c_j, d_k \in \mathbb{Z}$$

where L_k are closed surfaces (actually these are closed 2-cycles, but they can be represented by surfaces by the well-known results of R.Thom) with the property that

$$[L_k] \cdot [\delta_m^2] = 0, \forall k, m$$

(δ^2_m denotes the core of the 2-handle h^2_m). Let us compute explicitly the boundary operator on the 3-handles, in terms of the surfaces we have in the 2-complex Σ_i. Set

$$\partial\,[h^3_j] = \sum_k m_{jk}\,[h^2_k].$$

Then the coefficient m_{jk} is the number of times the boundary $\partial^+ h^3_j$ runs over the core of h^2_k. But the 2-sphere $\partial^+ h^3_j$, when pulled back in M^3_i, is a planar surface in M^3_i whose boundary circles (i.e. at seams) are capped-off by the core disks δ^2_k of the 2-handles h^2_k. Therefore the number m_{jk} is the number of times the seam $\partial^+ h^2_k$ appears in the planar surface which is a pull-back of $\partial^+ h^3_j$. In particular the coefficient of a 2-handle vanishes in a 3-cycle only if the boundaries of the planar surface glue together to close up at the corresponding surgery locus.

Thus the pull-backs of the surfaces $\sum_j c_j \partial^+ h^3_j$ give a surface F in M^3_i with boundary the curve γ with which we started, plus some other curves along which surgery is performed by time $i+n$. As this is in fact a closed cycle in the universal cover, the surface F lifts to a surface in $\tilde{X}$, with a single boundary component, corresponding to a curve which is not surgered by the time $i+n$. Therefore the map $\pi_1(F) \to \pi_1(X) \to \pi_1(\partial W^4)$ factors through $\pi_1(\tilde{X}) = 1$, hence the image of $\pi_1(F)$ in $\pi_1(\partial W^4)$ is trivial. □

Thus, after surgering along the curves up to the M^3_{i+n} we do have the required Seifert surfaces to compress to get embedded surfaces with trivial $\pi_1(\partial W^4)$ image. □

Proposition 7.1. *If $\gamma_{i,k} \subset M_i$ are the surgery curves (i.e. the seams) then $\gamma_{i,k} \in LCS_\infty(\pi_1(M^3_i))$, where $LCS_k(G)$ is the lower central series of the group G, $LCS_1(G) = G, LCS_{k+1}(G) = [G, LCS_k(G)]$, and $LCS_\infty(G) = \cap_{k=1}^\infty LCS_k(G)$.*

Proof. We will express each surgery locus γ as a product of commutators of the form $[\alpha_i, \beta_i]$, with each α_i being conjugate to a surgery locus (possibly γ itself). It then follows readily that $\gamma \in LCS_\infty$, as now if each $\gamma_{i,k} \in LCS_k$, then each $\gamma_{i,k} \in LCS_{k+1}$

Suppose now $\gamma = \gamma_{i,k}$ is a surgery locus. Then the 0-frame surgery along γ creates homology,in M^3_{i+1} which by our structure theorem is represented by a surface $S = \psi_{i+1}(F_j(i+1))$. The pullback of S to i gives a surface with boundary along seams, and being compressed to a sphere by the seams, so that the algebraic multiplicity of γ is 1 while that of all other seams is 0. In terms of the fundamental group, this translates to the relation that was claimed. □

Since, we have immersed surfaces of the required form, the obstruction we encounter is in making these surfaces disjoint at some finite stage. Note that for a finite decomposition, we do indeed have disjoint surfaces representing the homology after finitely many surgeries, since we in fact have a family of such spheres.

Proposition 7.2. *There exists a 2-complex $\Sigma = \cup_{i=1}^\infty \Sigma_i$ with intersections along double-curves, coming from a handle-decomposition as above, where all the seams are trivial in homology, but which does not carry disjoint, embedded surfaces representing all of the homology.*

Proof. For the first stage, take two surfaces of genus 2, and let them intersect transversely along two curves (which we call seams) that are disjoint and homologically independent in each surface.

Next, take as Seifert surfaces for these curves once punctured surfaces of genus 2 intersecting in a similar manner, and glue their boundary to the abovementioned curves of intersection. Repeat this process to obtain the complex.

At the first stage, we cannot have embedded, disjoint surfaces representing the homology as the cup product of the surfaces is non-trivial. As the surfaces are compact, we must terminate at some finite stage. We will prove that if we can have disjoint surfaces at the stage $k+1$, then we do at stage k. This will suffice to give the contradiction.

Now, we know the complex cannot be embedded in the first stage. Suppose we did have disjoint embedded surface S_1 and S_2 at stage $k+1$. Since these form a basis for the homology, they contain curves on them that are the seams at the first stage with algebraically non-zero multiplicity, i.e., the collection of curves representing the seam is not homologically trivial in the intersection of the first stage with the surface. Further, some copy of the first seam must bound a subsurface S_i' in each of the surfaces, for otherwise the surface contains a curve dual to the seam. For, the cup product of such a dual curve with the homology class of the other surface is non-trivial, hence it must intersect the other surface, contradicting the hypothesis that the surfaces are disjoint. Similarly, at the other seam we get surfaces S_i''.

By deleting the first stage surfaces and capping off the first stage seams by attaching discs, we get a complex exactly as before with the $(j+1)$th stage having become the jth stage. Further, the S_1' and S_2'' now give disjoint, embedded surfaces representing the homology that are supported by stages up to k. This suffices as above to complete the induction argument.

It is easy to construct a handle-decomposition corresponding to this complex. Figure 1 shows a construction of tori with one curve of intersection. Here we have used the notation of Kirby calculus, with the thickened curves being an unlink along each component of which 0-frame surgery has been performed. It is easy to see that the same construction can give surfaces of genus 2 intersecting in 2 curves. On attaching the first two 2-handles, the boundary is $(S^2 \times S^1)\#(S^2 \times S^1)$. Since the curves of intersection are unknots, after surgery they bound spheres. Further, it is easy to see by cutting along these that the boundary is $(S^2 \times S^1)\#(S^2 \times S^1)$ after attaching the 2-handles and 3-handles as well. Repeating this process, we obtain our embedding.

Thus we have an infinite handle-decomposition satisfying our hypothesis for which this 2-complex is Σ. □

7.4 A wild example

We will construct an example of an open, contractible 4-manifold that is not tame, and that has a handle-decomposition without 1-handles.

Theorem 7.4. *There is a proper handle-decomposition of an open, contractible 4-manifold W^4 such that W^4 is not the interior of a compact 4-manifold. In particular W^4 does not have a finite handle-decomposition.*

Proof. We will take a variant of the example in the last section. Namely, we construct an explicit handle-decomposition according to a canonical form.

Start with a 0-handle and attach to its boundary 3 2-handles along an unlink. the resulting manifold has boundary $(S^2 \times S^1)\#(S^2 \times S^1)\#(S^2 \times S^1)$ obtained by 0-frame-surgery about

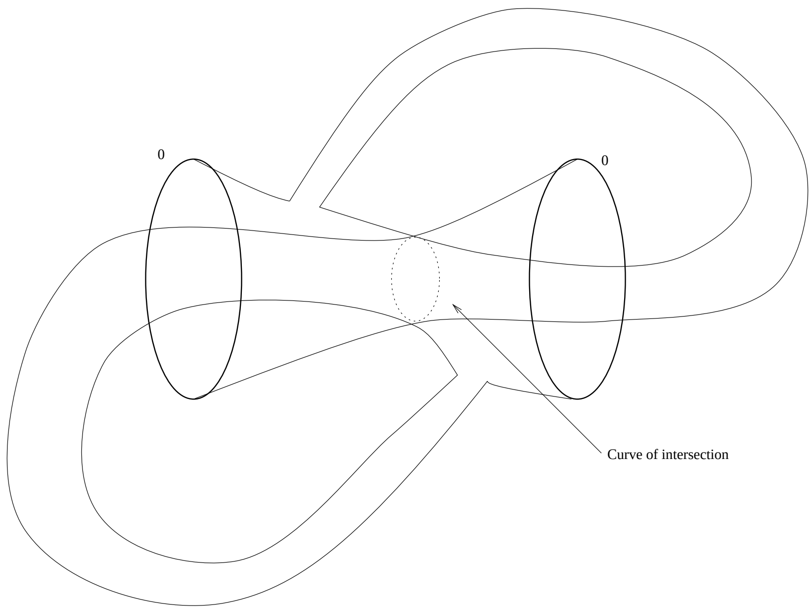

Figure 1: Tori with one intersection curve

each component of an unlink with 3 components. We now take as Seifert surfaces for these components surfaces of genus 2, so that each pair intersects in a single curve, so that the curves of intersection form an unlink and are unlinked with the original curves.

Now, attach 2-handles along the curves of intersections, and then 3-handles along the Seifert surfaces compressed to spheres by adding discs in the 2-handles just attached. It is easy to see that the resulting manifold once more has boundary $(S^2 \times S^1)\#(S^2 \times S^1)\#(S^2 \times S^1)$. Thus, we may iterate this process. Further, the generators of the fundamental group at any stage are the commutators of the generators at the previous stage.

Suppose W^4 is in fact tame. Then, we may use the results of the previous sections. Now, by construction no curve dies as only trivial relations have been added. Thus every element in kernel(ϕ_i) must fail to persist by some uniform time. In particular, the image of the group after that time in the present (curves that persist beyond that time) must inject under ϕ_i. But we know that it also surjects. Thus, we must have an isomorphism.

Thus, there is a unique element mapping onto each element of $\pi_1(\partial W^4)$. Hence this element must persist till infinity as we have a surjection at all times. On the other hand, since the limit of the lower central series of the free group is trivial, no non-trivial element persists. This gives a contradiction unless $\pi_1(\partial W^4)$ is trivial.

But there are non-trivial elements that do persist beyond any give time. As no element dies, we again get a contradiction. □

7.5 Further obstructions from Gauge theory

To further explore some of the subtleties that one might encounter in trying to construct a handle decomposition without 1-handles for a contractible manifold, given one for its interior, we consider a more general situation. We will consider sequences of 3-manifolds M_i that begin with S^3. As before, we require that each manifold comes from the previous one by 0-frame surgery about a homologically trivial curve or by splitting along a non-separating S^2 and capping off. Also, we require degree-one maps f_i to a common manifold N^3, related as before. We will say that the sequence limits to N^3 if 'any curve that persists dies' as in lemma 7.3.

In this situation, our main question generalizes to a 'relative version,' namely, given any such sequence $\{M_i\}$, with M_k an element in the sequence, is there a finite sequence that agrees up to M_k with the old sequence and whose final term is N^3?

We will show that there is an obstruction to completing certain sequences to finite sequences when $N^3 = S^3$. We do not know whether there are infinite sequences limiting to N^3 in this case.

Let $\mathfrak{P}$ denote the Poincaré homology sphere. Observe that we cannot pass from this to S^3 by 0-frame surgery about homologically trivial curves and capping-off non-separating spheres. For, if we could, $\mathfrak{P}$ would bound a manifold with $H^2 = \oplus_k [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]$, which is impossible as $\mathfrak{P}$ has Rochlin invariant 1. On the other hand, for the same reason, $\mathfrak{P}$ cannot be part of any sequence of the above form.

Using Donaldson's theorem [3], we have a similar result for the connected sum $\mathfrak{P}\#\mathfrak{P}$ of $\mathfrak{P}$ with itself. The main part of the proof of this lemma was communicated to us by R. Gompf.

Lemma 7.11. *One cannot pass from $\mathfrak{P}\#\mathfrak{P}$ to S^3 by 0-frame surgery along homologically trivial curves and capping off non-separating S^2's.*

Proof. If we did have such a sequence of surgeries, then $\mathfrak{P}\#\mathfrak{P}$ bounds a 4-manifold M_1 with $H^2 = \oplus_k [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]$, with a half-basis formed by embedded spheres. Now glue this to a manifold with form $E_8 \oplus E_8$ which is bounded by $\mathfrak{P}\#\mathfrak{P}$ to get M.

We can surger out the disjoint family of S^2's from M to get a 4-manifold with form $E_8 \oplus E_8$ and trivial H_1. This contradicts Donaldson's theorem. ∎

We still do not have a sequence as claimed. For, Cassson's argument shows that $\mathfrak{P}\#\mathfrak{P}$ cannot be part of a sequence. To obtain such a sequence, we will construct a manifold M that can be obtained by 0-frame surgery on algebraically unlinked 2-handles from each of S^3 and $\mathfrak{P}\#\mathfrak{P}$. Thus, M is part of a sequence. On the other hand, if we had a sequence starting at M that terminated at S^3, then we would have one starting at $\mathfrak{P}\#\mathfrak{P}$, which contradicts the above lemma.

To construct M, take a contractible 4-manifold K that bounds $\mathfrak{P}\#\mathfrak{P}$. By Freedman's theorem [4], this exists, and can moreover be smoothed after taking connected sums with sufficiently many copies of $S^2 \times S^2$. Take a handle decomposition of M. This may include 1-handles, but these must be boundaries of 2-handles. Hence, by handle-slides, we can ensure that each 1-handle is, at the homological level, a boundary of a 2-handle and is not part of the boundary of any other 2-handle. Replacing the 1-handle by a 2-handle does not change the boundary, and changes $H^2(M)$ to $H^2(M) \oplus (\oplus_k [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}])$. We do this dually with 3-handles too. Sliding 2-handles over the new ones, we can ensure that the attaching maps of the 2-handles having the same algebraic linking (and framing) structure as a disjoint union of Hopf links.

Now take M obtained from S^3 by attaching half the links, so that these are pairwise algebraically unlinked. The manifold M has the required properties.

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