A LOCAL VERSION OF LAUFER'S THEOREM FOR EMBEDDED CR MANIFOLDS

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Using Fredholm-operator theory or some L^2-estimates, it is often more easy to obtain finiteness theorems for the natural cohomology groups associated to a complex manifold than to actually show the solvability of the $\overline{\partial}$-equation. Therefore one would like to have a criterion which permits to pass from finiteness theorems to vanishing theorems.

It was proved by Laufer that for any open subset D of a Stein manifold, the Dolbeault cohomology groups $H^{p,q}(D)$ are either zero or infinite dimensional ([La]).

The purpose of this note is to give a local version of this theorem for embedded CR manifolds. We believe that this is of interest, since very little seems to be known about the solvability of the tangential Cauchy-Riemann equations if the Levi-form of the CR manifold does not have the right amount of positive or negative eigenvalues.

Let us denote by M a smooth generic CR manifold embedded into $\mathbb{C}^n$, of real codimension k, and by $H^{p,q}(M)$, $0 \leqslant p \leqslant n$, $0 \leqslant q \leqslant n-k$, the cohomology groups of the Cauchy-Riemann complexes obtained from the $\overline{\partial}_M$-operator (see [Bo] for the definitions). We then obtain the following theorem.

THEOREM 0.1. — *Let M be a smooth generic* CR *manifold embedded into $\mathbb{C}^n$, of real codimension k, and let z be an arbitrary point of M. Then there exists an open neighborhood U_z of z in M such that for every open subset $D \subset U_z$ with* $\dim H^{p,q}(D) < +\infty$, *we have $H^{p,q}(D) = 0$, $0 \leqslant p \leqslant q$, $1 \leqslant q \leqslant n-k$.*

Our result holds only for sufficiently small open subsets of CR manifolds. We also provide an example fo a CR manifold, globally embedded into some $\mathbb{C}^n$, for which the corresponding global result is false.

For later use, let us take a closer look at some local representation of M. We fix $p \in M$. We may, see [Bo] for the details, assume $p = 0$ and that M is (in a neighborhood of 0) of the form

$$M = \{(z, w = u + iv) \in \mathbb{C}^{n-k} \times \mathbb{C}^k \mid v = h(z,u)\} \tag{1}$$

where

$$h = (h_1, \ldots, h_k) : \mathbb{C}^{n-k} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$$

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is a $\mathscr{C}^\infty$-mapping with $h(0,0) = 0$, $Dh(0,0) = 0$. Moreover, there is a basis of $T^{0,1}M$ on a neighborhood of 0 of the form

$$\overline{L}_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{\ell=1}^{k} a_{j\ell} \frac{\partial}{\partial \bar{w}_\ell}, \;\; j = 1,\dots,n-k \tag{2}$$

We note (see [Ba/Tr]) that

$$[\overline{L}_i,\overline{L}_j] = 0, \;\; i,j = 1,\dots,n-k\,. \tag{3}$$

Let us define a vector field $M_1 \in \mathbb{C} \otimes TM$ by

$$M_1 = \frac{\partial}{\partial z_1} + \sum_{\ell=1}^{k} b_\ell \frac{\partial}{\partial \bar{w}_\ell} \tag{4}$$

M_1 will be tangent to M if and only if

$$\frac{\partial h_j}{\partial z_1} + \sum_{\ell=1}^{k} \frac{b_\ell}{2}\Big(\frac{\partial h_j}{\partial u_\ell} - i\delta_{j\ell}\Big) = 0, \;\; j = 1,\dots,k\,.$$

As $Dh(0,0) = 0$, we can certainly find $(b_1,\dots,b_k)$ satisfying the above system on a neighborhood of 0.

We claim that we have

$$[M_1,\overline{L}_j] = 0, \;\; j = 1,\dots,n-k\,. \tag{5}$$

This can be seen as follows (see also [Ba/Tr]).

We have

$$[M_1,\overline{L}_j]z_\ell = [M_1,\overline{L}_j]w_m = 0, \;\; \ell = 1,\dots,n-k, \;\; m = 1,\dots,k\,.$$

As the annihilator of the forms $dz_1,\dots,dz_{n-k}$, $dw_1,\dots,dw_k$ in $\mathbb{C}\otimes TM$ is spanned by $\overline{L}_1,\dots,\overline{L}_{n-k}$, this implies that $[M_1,\overline{L}_j]$ has to be a linear combination of the $\overline{L}_\alpha$'s. But $[M_1,\overline{L}_j]$ does not involve differentiation in the direction of $\bar{z}_\alpha$, hence $[M_1,\overline{L}_j] = 0$.

Let $\bar{\omega}_1,\dots,\bar{\omega}_{n-k}$ be smooth 1-forms in a neighboorhood of 0 in M such that $\bar{\omega}_j(\overline{L}_\alpha) = \delta_{j,\alpha}$. We observe that

$$d\bar{\omega}_j\,|_{T^{0,1}M} = 0, \;\; j = 1,\dots,n-k\,. \tag{6}$$

Indeed, by the classical formula for d we have

$$\begin{aligned} d\bar{\omega}_j(\overline{L}_\alpha,\overline{L}_\beta) &= \overline{L}_\alpha(\bar{\omega}_j(\overline{L}_\beta)) - \overline{L}_\beta(\bar{\omega}_j(\overline{L}_\alpha)) - \bar{\omega}_j([\overline{L}_\alpha,\overline{L}_\beta]) \\ &= \overline{L}_\alpha(\delta_{j\beta}) - \overline{L}_\beta(\delta_{j\alpha}) - \bar{\omega}_j(0) = 0 \end{aligned}$$

because of (3).

Proof of the theorem. — We fix a point $z \in M$ and choose a small neighborhood U_z of z such that all the computations (1)–(6) hold in U_z.

Let $D \subset U_z$ be an open subset with $\dim H^{p,q}(D) < +\infty$, $q \geqslant 1$. We assume $H^{p,q}(D) \neq 0$.

It is no loss of generality to take $p = n$ (see [Tr]). The (n,q)-forms in D can be uniquely written as

$$u = \sum_{|I|=q}{}' \, u_I dz_1 \wedge \cdots \wedge dz_{n-k} \wedge dw_1 \wedge \cdots \wedge dw_k \wedge \bar{\omega}^I, \; u_I \in \mathscr{C}^\infty(D),$$

where the prime indicates summation over increasing multiindices, $I = (i_1, \ldots, i_q)$ and $\bar{\omega}^I = \bar{\omega}_{i_1} \wedge \cdots \wedge \bar{\omega}_{i_q}$.

Moreover, the $\overline{\partial}_M$-operator is nothing else but the exterior differential d, *i.e.*

$$\begin{aligned}\overline{\partial}_M u = & \sum_{|I|=q}{}' \, d(u_I) \wedge dz_1 \wedge \cdots \wedge dz_{n-k} \wedge dw_1 \wedge \cdots \wedge dw_k \wedge \bar{\omega}^I \\ & + (-1)^n \sum_{|I|=q}{}' \, u_I \wedge dz_1 \wedge \cdots \wedge dz_{n-k} \wedge dw_1 \wedge \cdots \wedge dw_k \wedge d(\bar{\omega}^I) \\ = & \sum_{j=1}^{n-k} \sum_{|I|=q}{}' \, \overline{L}_j(u_I)\bar{\omega}_j \wedge dz_1 \wedge \cdots \wedge dz_{n-k} \wedge dw_1 \wedge \cdots \wedge dw_k \wedge \bar{\omega}^I\end{aligned}$$

because of (6).

From the assumption $0 < H^{n,q}(D) < +\infty$, it follows that there exists a non constant polynomial P in one complex variable such that $P(z_1)u \in \operatorname{Im}(\overline{\partial}_M)$ for every $\overline{\partial}_M$-closed (n,q)-form u in D.

We take P to be of minimal degree N among all the polynomials having this property.

For each $\overline{\partial}_M$-closed (n,q)-form u in D we can thus find an (n,q)-form α such that

$$P(z_1)u = \overline{\partial}_M \alpha \, .$$

Applying the vector field M_1 defined by (4) coefficientwise to this equation

$$(\textit{i.e.} \;\; M_1 u = \sum_{|I|=q}{}' \, (M_1 u_I) dz_1 \wedge \cdots \wedge dz_{n-k} \wedge dw_1 \wedge \cdots \wedge dw_k \wedge \bar{\omega}^I \;)$$

yields

$$P'(z_1)u + P(z_1)M_1 u = M_1 \overline{\partial}_M \alpha$$

where P' is a polynomial of degree $< N$.

In view of (5), the operators M_1 and $\overline{\partial}_M$ commute, hence the above equations yield

$$P'(z_1)u \in \operatorname{Im}(\overline{\partial}_M)$$

for all $\overline{\partial}_M$-closed (n,q)-forms in D. This contradicts the minimality of N and completes the proof. □

Remark. — The preceding theorem does not hold globally, not even for CR manifolds globally embedded into $\mathbb{C}^n$. Indeed, it was proved in [Ca/Le] that there exist compact strictly pseudoconvex CR manifolds of hypersurface type and of any dimension, embedded into some $\mathbb{C}^N$ that admit small deformations that are also embeddable but their embeddings cannot be chosen close to the original embedding. Moreover, it was shown by Tanaka [Ta] that if we are in a situation as above with $\dim_{\mathbb{R}} M \geqslant 5$ and $H^{0,1}(M) = 0$, then small deformations of M can be embedded by mappings close to the original embedding, *i.e.* the above phenomenon does not arise. In addition, if M is compact and strictly pseudoconvex of dimension greater than five, we always have $\dim H^{0,1}(M) < +\infty$ (see [Ta]). This implies the existence of compact strictly pseudoconvex hypersurfaces of dimension greater than five in $\mathbb{C}^n$ with

$$0 \neq \dim H^{0,1}(M) < +\infty .$$

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