INVARIANTS ASSOCIATED TO GROUP ACTIONS: AN ALGEBRAIC VIEWPOINT

MIHAI HALIC

Prépublication de l'Institut Fourier n° 491 (2000)

http://www-fourier.ujf-grenoble.fr/prepublications.html

Introduction

In the recent preprint [1] the authors define a certain invariant associated to Hamiltonian group actions on (non-necessarily closed) symplectic manifolds. They also state (conjecture 3.6) that their invariant should coincide (at least in some "good" cases) with the Gromov-Witten invariants of the Marsden-Weinstein quotient of the symplectic manifold. Their idea for proving the conjecture is based on an "adiabatic limit" argument.

The purpose of my note is to put the construction given in [1] into an algebro-geometric perspective and to relate it to the problem studied in [2]. In particular I shall answer in positive the conjecture in the case studied in my thesis (see corollary 4.6). More precisely, the invariant Φ introduced in [1] and whose definition is recalled in section 4 turns out to be related to the Gromov-Witten invariant as follows:

Theorem *Let X be a complex projective variety on which a complex torus $G^c := (\mathbb{C}^*)^r$ acts. Assume that this action is linearized on a very ample line bundle on X and denote $\hat{X}$ the invariant quotient of X. Consider $\alpha \in H^*_{G^c}(X)^{\otimes k}$ and a class $\hat{B} \in H_2(\hat{X};\mathbb{Z})$ which is defined by a map $C \to \hat{X}$ with C a smooth, irreducible algebraic curve. Denote $P \to C$ the $\mathscr{C}^\infty$ principal bundle obtained pulling back $X^{ss} \to \hat{X}$ to C and fix a connection A_0 in P. Denote $\mathcal{P}^c_r(A_0) \to (Pic^0 C)^r \times C$ a universal $(\mathbb{C}^*)^r$-bundle and let $\bar{X} := \mathcal{P}^c_r(A_0) \times_{G^c} X$. Denote $\bar{B} \in H_2(\bar{X};\mathbb{Z})$ the class induced by $\hat{B}$.*
Assume that:

(A$_1$) *the action is free on the semi-stable locus of X and*

(A$_2$) *the spaces $\overline{M}_{C,k}(\bar{X};\bar{B})//G^c$ and $\overline{M}_{C,k}(\hat{X};\hat{B})$ have both expected dimension.*

Then invariant Φ coincides with the Gromov-Witten invariant of $\hat{X}$.

The structure of the present note is as follows: in the first section I describe the context of the problem in more detail. I introduce the relevant definitions

Math. classification: 14L30 14C22 14H10 14D25.
Keywords: Actions des tores, groupe de Picard, invariants de Gromov-Witten.

and I study the action of the gauge group of a fixed principal bundle on the space of connections. It turns out that the (pointed) complex gauge equivalence classes of principal torus bundles over an algebraic curve are parameterized by (a power of) the Picard variety of the curve. The purpose of section 2 is to describe explicitly the universal $(\mathbb{C}^*)^r$-bundle over $(Pic^0 C)^r \times C$, denoted $\mathcal{P}^c_r(A_0)$ in the theorem above. In section 3, I introduce the relevant spaces of maps for the definition of the invariants and I concentrate on the description of the link between the analytic and the algebraic point of view. Finally, in section 4, I am in position to prove the main result.

1. **Preliminaries**

I shall describe briefly the construction given in [1] in the following particular context: X is a projective manifold and G^c is a connected, linearly reductive complex group which acts holomorphically on X. Assume further that this action linearizes a (very) ample line bundle $\mathcal{O}_X(1) \to X$. Then the maximal compact subgroup G of G^c preserves a Kähler form ω on X which represents the first Chern class of $\mathcal{O}_X(1)$. In symplectic terms, we get a Hamiltonian action on the symplectic manifold (X, ω).

The G-equivariant homology of X is defined as $H^G_*(X) := H_*(\mathrm{EG} \times_G X)$, where $\mathrm{EG} \to \mathrm{BG}$ is the universal G-bundle. The space EG is uniquely determined (up to homotopy) by the condition that G acts freely on it. Notice that one may take $\mathrm{EG}^c = \mathrm{EG} =: E$ since G, being a subgroup of G^c, acts freely on EG^c.

Elements in $H^G_k(X)$ can be constructed as follows: you start with a closed $\mathscr{C}^\infty$ manifold M of real dimension k together with a G-equivariant map $U : P \to X$. The k-dimensional equivariant homology class defined by this data is the image of the fundamental class of M under

$$H^G_k(M) \xleftarrow{\cong} H^G_k(P) \longrightarrow H^G_k(X).$$

It is proved in [1], proposition 2.1, that for every G-equivariant 2-homology class $B \in H^G_2(X; \mathbb{Z})$ there is a closed (not-necessarily connected) Riemann surface Σ and a G-principal bundle $P \to \Sigma$ together with a G-equivariant map $U : P \to X$ which represents the class B. Moreover, if Σ is connected and $P, P' \to \Sigma$ represent the same class, then P and P' are isomorphic as G-bundles. In other words, the choice of an equivariant homology class determines uniquely the topological type of the principal bundles, over a *fixed* Riemann surface, which can represent this homology class.

Remark 1.1. (i) Notice that if $P \to \Sigma$ is a principal G-bundle coming with a G-equivariant map $U : P \to X$, there is an induced commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\mathrm{id}_P \times U} & P \times X \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\bar{u}} & P \times_G X. \end{array}$$

(ii) Suppose that $U, U' : P \to X$ are two maps representing the same class in $B \in H_2^G(X)$. Then the induced maps $\bar{u}, \bar{u}' : \Sigma \to P \times_G X$ represent the same homology class in $H_2(P \times_G X)$. This homology class can be visualized as follows: the topological fibration $E \times_G X \to BG$ with fiber X gives an isomorphism $H_2^G(X;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \oplus H_2(BG;\mathbb{Z})$. If B_X is the $H_2(X;\mathbb{Z})$-component of B, the 2-homology class represented in $P \times_G X$ is $B_X + [C]$. $\diamond$

We now turn to another ingredient used in [1] for defining the invariant. Given a principal G-bundle $P \to \Sigma$, denote $\mathscr{A}(P)$ the (affine) space of connections in P. The underlying vector space is $\Omega^1(\Sigma, \mathrm{ad}P)$, where $\mathrm{ad}P := P \times_G \mathfrak{g}$ and G acts on its Lie algebra by the adjoint action. The gauge group of the principal bundle $P \to \Sigma$ is defined as

$$\mathscr{G}(P) := \{f : P \to G \mid f(R_g p) = \mathrm{Ad}(g) f(p)\},$$

where R_g denotes the right action of G on P. It can be identified with the group of automorphisms of P (as a G-bundle) via the map

$$\mathscr{G}(P) \ni f \longmapsto \bar{f} \in Aut_G(P), \quad p \overset{\bar{f}}{\longmapsto} R_{f(p)} p.$$

Using a fixed monomorphism $j : G \to G^c$, consider the complexification $P^c := P \times_G G^c$ of P which is a principal bundle over Σ with structure group G^c. Its gauge group $\mathscr{G}(P^c)$ is defined similarly as $\mathscr{G}(P)$ and can be thought off as the complexification $\mathscr{G}^c(P)$.

Let us consider the normal subgroup $\mathscr{G}_0^c(P)$ of pointed gauge transformations of P^c which are defined by the property that they are the identity on the fiber P_{ζ_0} of P over a fixed point $\zeta_0 \in \Sigma$. This group fits into the exact sequence

$$1 \to \mathscr{G}_0^c(P) \to \mathscr{G}^c(P) \to G^c \to 1$$

which is split when $G = \underbrace{S^1 \times \cdots \times S^1}_{\text{r times}}$ is a torus. In this case, the real (resp. complex) gauges are

$$\mathscr{G}(P) = \{f : \Sigma \to (S^1)^r\} \quad \text{resp.} \quad \mathscr{G}^c(P) = \{f : \Sigma \to (\mathbb{C}^*)^r\}.$$

Let now fix a metric and an orientation on Σ. This data induces naturally a complex structure on Σ, turning it into a smooth, irreducible, projective curve C.

Definition 1.2 (cf. [1] section 2.4)**.** (i) For a connection $A \in \mathscr{A}(P)$ and a G-equivariant map $U : P \to X$, define the operator $\mathrm{d}_A U$ acting as

$$T_p P \ni w \longmapsto \mathrm{d}U_p(w) + \xi(A(w))_{U(p)},$$

where $\xi(a)_x$ is the tangent vector at $x \in X$ determined by $a \in Lie(G)$.

(ii) A G-equivariant map $U : (P, A) \to X$ is called *A-holomorphic* if $\bar{\partial}_A U = 0$, where

$$\bar{\partial}_A U := \frac{1}{2}(\mathrm{d}_A U + J_X \circ \mathrm{d}_A U \circ J_C).$$

The notation J_C stands for the complex structure induced on the A-horizontal spaces of P by the complex structure of C. ◇

In more down to earth terms, a G-equivariant map $U : (P, A) \to X$ is A-holomorphic if and only if

$$\mathrm{d}U_p(\widetilde{Jv}) = J\mathrm{d}U_p(\widetilde{v}),$$

where $\widetilde{v}$ and $\widetilde{Jv}$ denote respectively the A-horizontal liftings in $p \in P$ of the vectors $v, Jv \in TC$.

Definition 1.3 (cf. [1] section 3.2)**.** Denote

$$\mathfrak{X}_B := \big\{(U, A) \in C^\infty_G(P, X; B) \times \mathscr{A}(P) \mid \bar{\partial}_A U = 0\big\}$$

the space of G-equivariant, A-holomorphic smooth maps which represent the class $B \in H^G_2(X)$. ◇

Remark 1.4. It was already mentioned in remark 1.1 that U induces a map $\bar{u} : C \to P \times_G X$. It should be noticed is that U is A-holomorphic if and only if $\bar{u}$ is holomorphic. One has to be careful with the (integrable) complex structure on $P\times_G X$: this one is induced by the connection A. So, for vectors tangent to the fibers of $P \times_G X \to C$, the complex structure agrees with that of X, while for $v \in TC$ (in a local trivialization $P \cong C \times G$),

$$J_{P\times_G X}(v) = Jv + \xi(A(Jv)) - J\xi(A(v)).$$ ◇

Obviously the gauges of P act on $\mathfrak{X}_B$ but it turns out that the complex gauges act also. One obtains a particularly nice formula when G is a torus.

Assumption From now on, I shall assume that G^c is a complex, r-dimensional torus.

Lemma 1.5. *When the structure group of P is a torus, the complex gauges $\mathscr{G}^c(P)$ act on $\mathfrak{X}_B$ as follows: $f \times (U, A) \longmapsto (f \cdot U, f \cdot A)$, where*

$$(f \cdot U)(p) := f^{-1}(p)U(p) \tag{1.1}$$

and

$$f \cdot A := A + (f^{-1}\mathrm{d}f)_{\mathfrak{g}} + *(f^{-1}\mathrm{d}f)_{i\mathfrak{g}}. \tag{1.2}$$

Some explanation is in order: any $a \in Lie(G^c) = Lie(G) \oplus iLie(G)$ can be uniquely written $a = a_{\mathfrak{g}} + ia_{i\mathfrak{g}}$, with $a_{\mathfrak{g}}, a_{i\mathfrak{g}} \in Lie(G)$. The $*$ in the formula represents the Hodge star operator on C.

Proof. It is clear that formula (1.1) just extends the action of the real gauges by composition on the right. We shall prove the formula for the action of f on A searching a connection A' on P which makes the map $U' := f \cdot U$ A'-holomorphic. For doing computations I use a local trivialization of P, so I may assume that P itself is trivial (as long as the objects found in the end are globally defined).

In what follows, ζ will denote a point on C. By assumption $P = C \times G$ and $U(\zeta, g^{-1}) = gu(\zeta)$, for some $u : C \to X$. I want to find a connection A'

on P such that $\bar\partial_{A'}U' = 0$. Since U' is G-equivariant, it is enough to check this condition at points $(\zeta, 1) \in P$. Because U is A-holomorphic,

$$\mathrm{d}U_{(\zeta,1)}(Jv - A(Jv)) = J\mathrm{d}U_{(\zeta,1)}(v - A(v))$$

for $v \in T_\zeta C$, or equivalently

$$\mathrm{d}u_\zeta(Jv) + \xi(A(Jv))_{u(\zeta)} = J\mathrm{d}u_\zeta(v) + J\xi(A(v))_{u(\zeta)}. \tag{1.3}$$

Using formula (1.1), an immediate computation shows that

$$\begin{aligned}\mathrm{d}U'_{(\zeta,1)}(Jv - A'(Jv)) &= f^{-1}(\zeta)\mathrm{d}u_\zeta(Jv) - f^{-1}(\zeta)\xi((f^{-1}\mathrm{d}f)(Jv))_{u(\zeta)}\\ &\quad + f^{-1}(\zeta)\xi(A'(Jv))_{u(\zeta)}\end{aligned}$$

and

$$\begin{aligned}J\mathrm{d}U'_{(\zeta,1)}(v - A'(v)) &= f^{-1}(\zeta)J\mathrm{d}u_\zeta(v) - f^{-1}(\zeta)J\xi((f^{-1}\mathrm{d}f)(v))_{u(\zeta)}\\ &\quad + f^{-1}(\zeta)J\xi(A'(v))_{u(\zeta)}.\end{aligned}$$

For U' to be A'-holomorphic it is necessary and sufficient that the difference of the two quantities above is zero. Imposing this condition we obtain

$$\begin{aligned}0 &= J\mathrm{d}u_\zeta(v) - \mathrm{d}u_\zeta(Jv) - J\xi((f^{-1}\mathrm{d}f)(v))_{u(\zeta)} + \xi((f^{-1}\mathrm{d}f)(Jv))_{u(\zeta)}\\ &\quad + J\xi(A'(v))_{u(\zeta)} - \xi(A'(Jv))_{u(\zeta)}\\ &\overset{(1.3)}{=} \xi(A(Jv))_{u(\zeta)} - J\xi(A(v))_{u(\zeta)} - \xi(A'(Jv))_{u(\zeta)} + J\xi(A'(v))_{u(\zeta)}\\ &\quad + \xi((f^{-1}\mathrm{d}f)(Jv))_{u(\zeta)} - J\xi((f^{-1}\mathrm{d}f)(v))_{u(\zeta)}\\ &= \xi\big(A(Jv) - A'(Jv)\big)_{u(\zeta)} - J\xi\big(A(v) - A'(v)\big)_{u(\zeta)}\\ &\quad + \xi\big((f^{-1}\mathrm{d}f)(Jv)\big)_{u(\zeta)} - J\xi\big((f^{-1}\mathrm{d}f)(v)\big)_{u(\zeta)}.\end{aligned}$$

It remains to separate the $Lie(G)$ and $iLie(G)$ components of the last line.

$$\begin{aligned}&\xi\big((f^{-1}\mathrm{d}f)(Jv)\big) - J\xi\big((f^{-1}\mathrm{d}f)(v)\big)\\ &= \xi\big((f^{-1}\mathrm{d}f)_{\mathfrak g}(Jv) + i(f^{-1}\mathrm{d}f)_{i\mathfrak g}(Jv)\big) - \xi\big(i(f^{-1}\mathrm{d}f)_{\mathfrak g}(v) - (f^{-1}\mathrm{d}f)_{i\mathfrak g}(v)\big)\\ &= \xi\big((f^{-1}\mathrm{d}f)_{\mathfrak g}(Jv) + (f^{-1}\mathrm{d}f)_{i\mathfrak g}(v)\big) - J\xi\big((f^{-1}\mathrm{d}f)_{\mathfrak g}(v) - (f^{-1}\mathrm{d}f)_{i\mathfrak g}(Jv)\big).\end{aligned}$$

Inserting this into the previous relation, we obtain

$$\begin{aligned}0 &= \xi\big(A(Jv) - A'(Jv) + (f^{-1}\mathrm{d}f)_{\mathfrak g}(Jv) + (f^{-1}\mathrm{d}f)_{i\mathfrak g}(v)\big)\\ &\quad - J\xi\big(A(v) - A'(v) + (f^{-1}\mathrm{d}f)_{\mathfrak g}(v) - (f^{-1}\mathrm{d}f)_{i\mathfrak g}(Jv)\big).\end{aligned}$$

For A' defined by

$$A' = A + (f^{-1}\mathrm{d}f)_{\mathfrak g} - (f^{-1}\mathrm{d}f)_{i\mathfrak g} \circ J,$$

the last equality is satisfied. Notice that in general this is the only possible choice for A' since the vectors ξ and $J\xi$ are in most cases linearly independent.

Using local normal coordinates on C, it follows immediately that for any 1-form $\alpha \in \Omega^1_C$, $\alpha \circ J = -(*\alpha)$. □

Remark 1.6. (i) It follows from formula (1.2) that the $\mathscr{G}_0^c(P)$-action on $\mathfrak{X}_B$ is free.

(ii) Any $f : C \to (\mathbb{C}^*)^r$ is of the form $f(\zeta) = R(\zeta)\varphi(\zeta)$, with $R : C \to \mathbb{R}^r$ and $\varphi : C \to (S^1)^r$. Formula (1.2) becomes

$$f \cdot A = A + \varphi^{-1}\mathrm{d}\varphi - i * \mathrm{d}(\log R). \tag{1.4}$$

The form $\varphi^{-1}\mathrm{d}\varphi$ is closed, but not necessarily exact; it is exact if and only if $\varphi_* : \pi_1(C) \to (\pi_1(S^1))^r$ is the zero homomorphism, or equivalently $\varphi = \exp(i\theta)$ for some $\theta : C \to \mathbb{R}^r$. However it always defines an integral 1-cohomology class and conversely, any integral 1-cohomology class can be represented in this form.

Using the Hodge decomposition of Ω^1_C, this discussion implies that the pointed complex gauge equivalence classes of connections in P are parameterized by

$$\mathbb{H}^1(C, \mathbb{R}^r)/\mathbb{H}^1(C, \mathbb{Z}^r),$$

where $\mathbb{H}^1(C, \mathbb{R}^r)$ denotes the space of harmonic $\setminus^r$-valued 1-forms on C. Let us notice that this quotient is just the r^{th} power of the familiar Picard variety of C, when $\mathbb{H}^1(C, \mathbb{R}^r)$ is given the complex structure defined by the Hodge-star of C.

(iii) In the genus zero case, i.e. $C \cong \mathbb{P}^1$, all gauges admit a globally defined logarithm. Therefore all connections are gauge equivalent, which is the same saying that in a given topological principal bundle $P \to \mathbb{P}^1$ there is only one equivalence class of holomorphic structures. $\diamond$

2. **Short digression on the Picard variety**

In this section I want to recall some basic facts about the Picard variety. The construction in proposition 2.4 is certainly well known, but I recall it because I do not know any appropriate reference for it. As mentioned already in remark 1.6, the quotient

$$\mathbb{H}^1(C, \mathbb{R}^r)/\mathbb{H}^1(C, \mathbb{Z}^r) = \big(\mathbb{H}^1(C, \mathbb{R})/\mathbb{H}^1(C, \mathbb{Z})\big)^r =: (Pic^0 C)^r$$

is the r^{th} power of the Picard variety of C, when $\mathbb{H}^1(C, \mathbb{R}^r)$ is regarded as a complex vector space with complex structure given by the Hodge-star of C. It is a projective torus which parameterizes topologically trivial, holomorphic principal $(\mathbb{C}^*)^r$-bundles over C.

It is classical that there is a universal $(\mathbb{C}^*)^r$-bundle

$$\mathcal{P}^c_r \longrightarrow (Pic^0 C)^r \times C$$

such that for any point $\tau \in (Pic^0 C)^r$, the restriction $(\mathcal{P}^c_r)_\tau \longrightarrow C_\tau$ represents the point τ. Denoting $\mathcal{P}^c \longrightarrow (Pic^0 C) \times C$ the (usual) universal $\mathbb{C}^*$-bundle, the $(\mathbb{C}^*)^r$-bundle $\mathcal{P}_r^c$ is obtained from the following diagram

$$\begin{array}{ccc} (\mathrm{id}_{(Pic^0C)^r}\times\Delta_C)^*(\mathcal{P}^c)^r =: \mathcal{P}_r^c & & (\mathcal{P}^c)^r \\ \downarrow & & \downarrow \\ (Pic^0C)^r\times C & \xrightarrow{\mathrm{id}_{(Pic^0C)^r}\times\Delta_C} & (Pic^0C)^r\times C^r. \end{array}$$

The shortcoming of this description is that it does not give any information on the connection in $\mathcal{P}_r^c$ defining its holomorphic structure. For this reason I shall indicate an alternative down-to-earth construction of $\mathcal{P}_r^c$. More precisely, I shall construct a universal principal $(\mathbb{C}^*)^r$-bundle for holomorphic $(\mathbb{C}^*)^r$-bundles over C having a fixed topological type (recall that $\mathcal{P}_r^c$ gives the topologically trivial ones).

I start with the following

Lemma 2.1. *For any $\alpha\in\mathbb{H}^1(C;\mathbb{Z}^r)$, there is a unique $\varphi_\alpha : C\to (S^1)^r$ such that $\varphi_\alpha(\zeta_0)=1$ and $\varphi_\alpha^{-1}\mathrm{d}\varphi_\alpha=\alpha$. (Recall that a point $\zeta_0\in C$ was fixed from the begining).*

Proof. Clearly, we may assume that $r=1$. The uniqueness part is immediate. For the existence part, notice that if α is exact, i.e. $\alpha=\mathrm{d}\theta$ for $\theta : C\to\mathbb{R}$, then $\varphi := \exp(i(\theta-\theta(\zeta_0)))$ does the job. Homotopy classes of maps $C\to\mathbb{R}$ are parametrized by $\mathrm{Hom}_{\mathbb{Z}}(H^1(C;\mathbb{Z});\mathbb{Z})\cong H^1(C;\mathbb{Z})$, so for $\alpha\in\mathbb{H}^1(C;\mathbb{Z})$ there exists $\varphi' : C\to S^1$ such $\varphi'(\zeta_0)=1$ and $[(\varphi')^{-1}\mathrm{d}\varphi']=[\alpha]\in H^1(C;\mathbb{Z})$. By the discussion above, there exists $\varphi_0 : C\to S^1$ such that $\varphi_0(\zeta_0)=1$ and $\varphi_0^{-1}\mathrm{d}\varphi_0=\alpha-(\varphi')^{-1}\mathrm{d}\varphi'$. Now $\varphi := \varphi_0\varphi'$ will be convenient. □

Remark 2.2. The map

$$\mathbb{H}^1(C;\mathbb{Z}^r)\ni\alpha\longmapsto\varphi_\alpha\in\mathscr{C}^\infty\big(C,(S^1)^r\big)$$

is a morphism of groups, i.e. $\varphi_\alpha\varphi_\beta=\varphi_{\alpha+\beta}$. ◇

I fix a real connection A_0 in the smooth $(\mathbb{C}^*)^r$-bundle $P^c\to C$, i.e. one coming from a connection in the real $(S^1)^r$-bundle.

Lemma 2.3. *(i) On $\mathbb{H}^1(C;\mathbb{R}^r)\times C$, there is a natural, closed $\mathbb{R}^r$-valued 1-form χ which is defined by*

$$\chi_{(A,\zeta)}(a,v) := A(v)\quad\text{for}\quad (a,v)\in T_{(A,\zeta)}\mathbb{H}^1(C;\mathbb{R}^r)\times C.$$

(ii) The (real) connection $\mathcal{A} := A_0+\chi$ defines a holomorphic structure on the bundle $\mathrm{pr}_C^*P^c\to\mathbb{H}^1(C;\mathbb{R}^r)\times C$.

Proof. The curvature of $\mathcal{A}$ is $F_{\mathcal{A}}=\mathrm{pr}_C^*F_{A_0}$. This one, being a $(1,1)$-form on $\mathbb{H}^1(C;\mathbb{R}^r)\times C$, defines a holomorphic structure. □

Proposition 2.4. *(i) The group $\mathbb{H}^1(C;\mathbb{Z}^r)$ acts holomorphically, by real gauges, on $\mathrm{pr}_C^*P^c$ by*

$$\mathbb{H}^1(C;\mathbb{Z}^r)\times\big(\mathbb{H}^1(C;\mathbb{R}^r)\times P^c\big)\longrightarrow\mathbb{H}^1(C;\mathbb{R}^r)\times P^c,$$
$$\alpha\times(A,p) := (A+\alpha, R_{\varphi_\alpha}p).$$

(ii) The holomorphic principal bundle

$$\mathcal{P}^c_r(A_0) := \mathrm{pr}_C^* P^c/\mathbb{H}^1(C;\mathbb{Z}^r) \longrightarrow (Pic^0 C)^r \times C$$

is a universal principal $(\mathbb{C}^*)^r$-bundle which parameterizes holomorphic bundles over C having fixed topological type defined by A_0. It also comes with the connection induced by $\mathcal{A}$.

Proof. (i)Remark 2.2 implies that the formula above is indeed an action. It is also holomorphic because $\mathbb{H}^1(C;\mathbb{Z}^r)$ preserves the connection $\mathcal{A}$; indeed, $A + \alpha = \varphi_\alpha A$.

(ii) The statement is a direct consequence of 1.6. □

Remark 2.5. (i) I should explain that I have worked with *complex* principal bundles throughout this section because I wanted to put emphasis on their *holomorphic* structure. But $\mathcal{P}^c_r(A_0)$ is the complexification of a real $(S^1)^r$-bundle $\mathcal{P}_r(A_0) \to (Pic^0 C)^r \times C$ and the connection $\mathcal{A}$ comes from a connection in $\mathcal{P}_r(A_0)$, because (as I stressed several times) the connection A_0 (fixed from the beginning) is *real* and the action on $\mathrm{pr}_C^* P^c$ is done by *real* gauges.

(ii) One may ask what happens if a different connection A_0' is considered in 2.3 instead of A_0, so that proposition 2.4 provides the principal bundle $\mathcal{P}^c_r(A_0')$ having a different connection $\mathcal{A}'$. I claim that in this case there is a G^c-equivariant isomorphism $E : \mathcal{P}^c_r(A_0) \to \mathcal{P}^c_r(A_0')$ covering a translation on the base and sending $\mathcal{A}$ into $\mathcal{A}'$. Indeed, there exists a (unique) gauge $f \in \mathscr{G}^c_0(P)$ such that $fA_0' = A_0 + \alpha_0'$, with $\alpha_0' \in \mathbb{H}^1(C;\mathbb{Z}^r)$ the harmonic part of $A_0' - A_0$ in the Hodge decomposition of Ω^1_C. Therefore I may assume as well that $\alpha_0 := A_0' - A_0$ is a $\mathbb{R}^r$-valued harmonic form. The map

$$\mathrm{pr}_C^* P^c = \mathbb{H}^1(C;\mathbb{R}^r) \times P^c \longrightarrow \mathbb{H}^1(C;\mathbb{R}^r) \times P^c$$

$$(A,p) \longmapsto (A + \alpha_0, p)$$

commutes with the $\mathbb{H}^1(C;\mathbb{Z}^r)$-action and sends $\mathcal{A}_0$ to $\mathcal{A}_0'$. ◇

3. Moduli spaces

In this section I shall show how the spaces introduced in [1] for defining invariants of Hamiltonian group actions have nice algebraic interpretation.

I should remind the context I place myself: X is a projective, irreducible, complex variety on which the complex torus $G^c := (\mathbb{C}^*)^r$ acts and this action is linearized in the ample line bundle $\mathcal{O}_X(1) \to X$. There exists a positive $(1,1)$-form ω on X representing $c_1(\mathcal{O}_X(1))$ for which the action of the compact torus $(S^1)^r$ on (X,ω) becomes Hamiltonian.

For defining the invariants, the authors in [1] introduce (a perturbation of) the spaces

$$\tilde{S}_{C,k}(X;B) := \left\{(U,A) \in \mathfrak{X}_B \mid *F_A + \mu \circ U = 0\right\} \times (P^k)_o \Big/ \mathscr{G}(P) \tag{3.1}$$

and

$$S_{C,k}(X;B) := \tilde{S}_{C,k}(X;B)/G^k \tag{3.2}$$

In the definition above, $\mu : X \to \mathbb{R}^r$ is the moment map corresponding to the $(S^1)^r$-action and $(P^k)_o$ is the complementary in P^k of the diagonals. The group actions are as follows:

$$f \times \big((U,A) \times (p_1,\dots,p_k)\big) := (f\cdot U, f\cdot A) \times (R_{f(p_1)}p_1,\dots,R_{f(p_k)}p_k)$$

and

$$(g_1,\dots,g_k) \times \big[(U,A)\times(p_1,\dots,p_k)\big] := \big[(U,A)\times(R_{g_1}p_1,\dots,R_{g_k}p_k)\big],$$

for $f \in \mathscr{G}(P)$ and $g_1,\dots,g_k \in G = (S^1)^r$. The expected dimension of this space is

$$2D := \text{exp.dim}_{\mathbb{R}} S_{C,k}(X;B) = 2(1-g)(n-r) + 2c_1^G(X)\cdot B + 2k, \tag{3.3}$$

where $c_1^G(X)$ denote the G-equivariant first Chern class of X.

It is stated in [1], section 3.2, that the case when X is Kähler was studied in [3] where it is proved that

$$S_{C,k}(X,B) = \Big(\mathfrak{X}_B^s \times (P^c)^k_o/\mathscr{G}^c(P)\Big)\Big/\,(G^c)^k. \tag{3.4}$$

The notation $\mathfrak{X}_B^s \subset \mathfrak{X}_B$ stands for the set of so-called stable pairs, while $\mathscr{G}^c(P)$ acts on $\mathfrak{X}_B$ as described in lemma 1.5.

For my purposes I find more convenient to use this second description of $S_{C,k}(X;B)$. I mentioned that when G^c is a torus $\mathscr{G}^c(P) = \mathscr{G}^c_0(P)\times G^c$ is a product. Because the actions of $\mathscr{G}^c(P)$ and $(G^c)^k$ on $\mathfrak{X}_B$ commute, I may construct the space $S_{C,k}(X;B)$ differently: first I take the quotient $\mathfrak{X}_B \times (P^c)^k_o/\mathscr{G}^c_0(P)\times(G^c)^k$ (which is finite dimensional) and after take the invariant quotient for the remaining G^c-action. I shall use interchangeably G-equivariant maps $U : P \to X$ and G^c-equivariant maps $U^c : P^c \to X$. That there is no harm in doing so follows from the fact that any G-equivariant U defines (in the obvious way) a G^c-equivariant U^c; conversely, any such U^c defines a corresponding U composing with the inclusion $j : P \hookrightarrow P^c$.

Lemma 3.1. *(i) Using the notations of 2.5, the variety $\bar{X}$ defined by*

$$\bar{X} := \mathcal{P}^c_r(A_0)\times_{G^c} X = \mathcal{P}_r(A_0)\times_G X,$$

carries a natural complex projective structure. Its complex dimension is $\dim \bar{X} = gr + \dim X$, where g is the genus of C and $r = \dim G^c$.

(ii) Any G-equivariant, A-holomorphic map $U : P \to X$, $A \in \mathscr{A}(P)$, which represents an equivariant 2-homology class $B \in H_2^G(X;\mathbb{Z})$ defines a holomorphic map $\bar{u} : C \to \bar{X}$ which represents a class $B \in H_2(\bar{X};\mathbb{Z})$ depending only on B. For $\pi : \bar{X} \to (Pic^0 C)^r \times C$ the natural projection, $c_1^G(X)\cdot B = c_1(T^{\text{rel}}_{\bar{X}})\cdot \bar{B}$, where $T^{\text{rel}}_{\bar{X}}$ denotes the π-relative tangent bundle.

(iii) Consider an A-holomorphic, G-equivariant map $U : P \to X$ and $g \in \mathscr{G}^c_0(P)$. Then U and gU define the same map $C \to \bar{X}$.

Proof. (i) $\bar{X}$ has a holomorphic structure because $\mathcal{P}_r^c(A_0) \to (Pic^0 C)^r \times C$ is a holomorphic bundle, according to lemma 2.3. That it is also a projective variety, follows from the fact that the Picard torus is projective.

(ii) Remark 1.6 implies that given $U : (P, A) \to X$ there is a unique $f_A \in \mathscr{G}_0^c(P)$ (depending on A) such that

$$f_A \cdot A = \text{harmonic part of } A - A_0 =: h(A - A_0) \in \mathbb{H}^1(C; \mathbb{R}^r).$$

The composed map

$$P \xrightarrow{(\{h(A-A_0)\}\times \mathrm{id}_P)\times f_A U} \underbrace{\mathbb{H}^1(C;\mathbb{R}^r) \times P}_{=\mathrm{pr}_C^* P} \times X \longrightarrow \mathcal{P}_r(A_0) \times X \tag{3.5}$$

is G-equivariant and therefore it defines

$$\bar{u} : C \longrightarrow \mathcal{P}_r(A_0) \times_G X = \bar{X}.$$

Since $f_A U : P \to X$ is $f_A A$-holomorphic, it follows that this map is holomorphic. If $p_\zeta \in P$ (or in P^c) denotes a point lying over $\zeta \in C$, the explicit formula for $\bar{u}$ is

$$\zeta \overset{\bar{u}}{\longmapsto} \big[[h(A - A_0), p_\zeta], (f_A U)(p_\zeta)\big], \tag{3.6}$$

where the square brackets denote obvious equivalence classes.

The reason why $\bar{u}$ defines a class depending only on B was already explained in remark 1.1. The difference is that this time there is a fibration $\bar{X} \to (Pic^0 C)^r \times C$ and not just $\bar{X} \to C$. If $B_X \in H_2(X; \mathbb{Z})$ is the $H_2(X;\mathbb{Z})$-component of B, then $\bar{B} = 0 \oplus [C] \oplus B_X$.

If $\rho : P \to E$ is the G-equivariant map which defines P topologically, the last statement follows from the diagram

$$\begin{array}{ccccc} T^{\mathrm{rel}}_{P\times_G X} = & (\rho, \mathrm{id}_X)^*(E \times_G T_X) & \longrightarrow & E \times_G T_X \\ & \downarrow & & \downarrow \\ C \quad \xrightarrow{\bar{u}} & P \times_G X & \xrightarrow{(\rho,\mathrm{id}_X)} & E \times_G X. \end{array}$$

(iii) Consider now $U : P \to X$ and $g \in \mathscr{G}_0^c(P)$. Then g is of the form $g = R \cdot \varphi$, with $\varphi : C \to (S^1)^r$ and $R : C \to \mathbb{R}^r$. According to (1.4), $gA = A + \varphi^{-1}\mathrm{d}\varphi - i * \mathrm{d}(\log R)$, so

$$h(gA - A_0) = h(A - A_0) + h(\varphi^{-1}\mathrm{d}\varphi).$$

Notice that $\alpha_\varphi := h(\varphi^{-1}\mathrm{d}\varphi)$ is actually an *integral* $\mathbb{R}^r$-valued harmonic form and according to lemma 2.1 it exists a unique $\psi_\varphi \in \mathscr{G}_0(P)$ such that $\psi_\varphi^{-1}\mathrm{d}\psi_\varphi = \alpha_\varphi$. I claim that $f_{gA} = \psi_\varphi f_A g^{-1}$, i.e. that $(\psi_\varphi f_A g^{-1})(gA) = A_0 + h(gA - A_0)$. Indeed,

$$\begin{aligned} (\psi_\varphi f_A g^{-1})(gA) &= \psi_\varphi(f_A A) = \psi_\varphi(A_0 + h(A - A_0)) \\ &= A_0 + h(A - A_0) + \psi_\varphi^{-1}\mathrm{d}\psi_\varphi \\ &= A_0 + h(A - A_0) + \alpha_\varphi = A_0 + h(gA - A_0). \end{aligned}$$

I am going to check now that U and gU induce the same (holomorphic) map $C \to \bar{X}$. Using formula (3.6), I deduce that gU induces the map

$$\begin{aligned}
\zeta &\longmapsto \big[[h(A-A_0)+\alpha_\varphi, p_\zeta], \psi_\varphi^{-1}(p_\zeta)\cdot(f_A U)(p_\zeta)\big] \\
&= \big[[h(A-A_0)+\alpha_\varphi, R_{\psi_\varphi} p_\zeta], (f_A U)(p_\zeta)\big] \\
&= \big[[h(A-A_0), p_\zeta], (f_A U)(p_\zeta)\big].
\end{aligned}$$

This finishes the proof. □

Proposition 3.2. *There is a one-to-one map*

$$\mathfrak{X}_B \times (P^c)^k_o/\mathscr{G}^c_0(P) \times (G^c)^k \xleftrightarrow{1:1} M_{C,k}(\bar{X};\bar{B}),$$

where $M_{C,k}(\bar{X};\bar{B})$ *denotes the space of stable maps* $(C,\underline{x}) \to \bar{X}$ *with* k *marked points* $\underline{x} \in (C^k)_o$ *and representing the* 2*-homology class* $\bar{B}$. *As usual,* $(C^k)_o$ *denotes the complement in* C^k *of the diagonals.*

Proof. Consider an A-holomorphic, G-equivariant map $U: P \to X$ together with k marked points $\underline{p} \in (P^c)^k_o$. Denote $\underline{x} := \pi_C(\underline{p}) \in (C^k)_o$. Lemma 3.1 says that this data induces a morphism $C \to \bar{X}$ and moreover, it does not depend on the $\mathscr{G}^c_0(P)$-orbit of (U,A). So we get a map

$$\begin{aligned}
F: \mathfrak{X}_B \times (P^c)^k_o/\mathscr{G}^c_0(P) &\longrightarrow M_{C,k}(\bar{X};\bar{B}) \\
[(U,A),\underline{p}] &\longmapsto (\bar{u},\underline{x}).
\end{aligned}$$

This map is clearly $(G^c)^k$-invariant and therefore descends to the quotient

$$F: \mathfrak{X}_B \times (P^c)^k_o/\mathscr{G}^c_0(P) \times (G^c)^k \longrightarrow M_{C,k}(\bar{X};\bar{B}).$$

Since the composition $\pi \circ \bar{u} = \{\tau\} \times \mathrm{id}_C$, $U \overset{F}{\longmapsto} \bar{u}$ defines a stable map instead of just a morphism i.e. $\bar{u}$ is actually a representative for a whole equivalence class of morphisms.

The map F is clearly surjective: for a point $(\bar{u},\underline{x}) \in M_{C,k}(\bar{X};\bar{B})$, consider the diagram

$$\begin{array}{ccccc}
\bar{u}^*(\mathcal{P}_r(A_0)\times X) = P & \xrightarrow{\bar{U}} & \mathcal{P}_r(A_0)\times X & \xrightarrow{\mathrm{pr}_X} & X \\
\downarrow & & \downarrow & & \\
C & \xrightarrow{\bar{u}} & \bar{X}. & &
\end{array}$$

The composed map $U := \mathrm{pr}_X \circ \bar{U}$ will be a G-equivariant, A-holomorphic map, for $A := \bar{u}^*\mathcal{A}$ (see 2.3 for the definition of $\mathcal{A}$). As marked points in P, one may take any $\underline{p}$ lying over $\underline{x}$.

I have to prove that F is injective. Assume that $U : (P, A, \underline{p}) \to X$ and $U' : (P, A', \underline{p}') \to X$ induce the same morphism $\bar{u} : C \to \bar{X}$. It follows that necessarily $h(f_A A - A_0) \equiv h(f_{A'}A' - A_0) \bmod \mathbb{H}^1(C;\mathbb{Z}^r)$ and therefore $h(f_A A - f_{A'}A') \equiv 0 \bmod \mathbb{H}^1(C;\mathbb{Z}^r)$. I deduce from 1.6 that $f_A A$ and $f_{A'}A'$ are in the same $\mathscr{G}^c_0(P)$-orbit, so A and A' are in the same $\mathscr{G}^c_0(P)$-orbit also and, since $\bar{u}$ is gauge invariant, I may assume that $A = A'$ and even that $A - A_0$ is a harmonic form.

The problem is reduced to the following: two maps $U : (P, A, \underline{p}) \to X$ and $U' : (P, A', \underline{p}') \to X$ which define the same $(\bar{u}, \bar{x})$ must be equal. Formula (3.6) says that

$$\big[[A - A_0, p_\zeta], U(p_\zeta), [\underline{p}]\big] = \big[[A' - A_0, p_\zeta], U'(p_\zeta), [\underline{p}']\big] \quad \forall p_\zeta \in P.$$

One sees immediately that this imply $U = U'$ and $[\underline{p}] = [\underline{p}']$. □

The advantage of working with the space $M_{C,k}(\bar{X}; \bar{B})$ is that it is a quasi-projective scheme and has nice compactification in terms of stable maps, which I shall denote $\overline{M}_{C,k}(\bar{X}; \bar{B})$.

Recall that for obtaining the space $S_{C,k}(X; B)$ it remains to take a $G^c = (\mathbb{C}^*)^r$-quotient. There is a natural G^c-action on $\bar{X}$:

$$g \times \big[[A, p], x\big] := \big[[A, p], gx\big] \quad \text{for} \quad \big[[A, p], x\big] \in \bar{X} = \mathcal{P}^c_r(A_0) \times_{G^c} X.$$

I have to stress at this point that this action is well-defined precisely because G^c is a torus, namely because it is commutative.

The action on $\bar{X}$ induces one on $\overline{M}_{C,k}(\bar{X}; \bar{B})$,

$$G^c \times \overline{M}_{C,k}(\bar{X}; \bar{B}) \longrightarrow \overline{M}_{C,k}(\bar{X}; \bar{B}),$$

$$g\underline{x} := \underline{x} \quad \text{and} \quad (g\bar{u})(\zeta) := gu(\zeta)$$

and is immediate to check that this is exactly the G^c-action induced, *via* proposition 3.2, on $\overline{M}_{C,k}(\bar{X}; \bar{B})$.

Remark 3.3. (i) It follows from (3.4) and 3.2 that

$$S_{C,k}(X; B) = M_{C,k}(\bar{X}; \bar{B}) /\!/ G^c.$$

I must emphasize that the linearization of the G^c-action on $\overline{M}_{C,k}(\bar{X}; \bar{B})$ used in my thesis may not coincide with that used to obtain the space $\mathfrak{X}^s_B \subset \mathfrak{X}_B$ of stable pairs. However, two invariant quotients of an irreducible variety are birational if both non-empty. Consequently the evaluation of a form of maximal degree on their fundamental cycle is the same. Of course, it is open the question when $M_{C,k}(\bar{X}; \bar{B})$ is irreducible.

(ii) One more major problem is that in algebraic context the moduli spaces are rarely of expected dimension. It should be possible to remediate this introducing virtual cycles which are obtained from obstruction theory *relative to* $\pi : \bar{X} \to (Pic^0 C)^r \times C$. From symplectic point of view, one looks at pseudo-holomorphic curves in $\bar{X}$ for generic almost complex structure in the "vertical" X-direction. ◇

First I shall linearize the G^c-action on $\bar{X}$. By hypothesis, the G^c-action on X is linearized on a very ample line bundle $\mathcal{O}_X(1) \to X$. This one determines

$$L := \mathcal{P}^c_r(A_0) \times_{G^c} \mathcal{O}_X(1) \longrightarrow \bar{X},$$

which is π-ample, for the projection $\pi : \bar{X} \to (Pic^0 C)^r \times C$. It is immediate to see that G^c still acts on $\bar{L}$ and covers the action on $\bar{X}$. Again, for obtaining this linearization I use in essential way that G^c is a torus. For $\ell \to (Pic^0C)^r \times C$ sufficiently ample line bundle,

$$\bar{L} := \pi^*\ell \otimes L \longrightarrow \bar{X}$$

is ample and the G^c-action can be linearized on it.

I claim that the G^c-semi-stable points of $\bar{X}$ are

$$\bar{X}^{ss}(\bar{L}) = \mathcal{P}^c_r(A_0) \times_{G^c} X^{ss}(\mathcal{O}_X(1)).$$

The inclusion "$\subset$" follows from the fact that the restriction of $\bar{L}$ to the fibers is precisely $\mathcal{O}_X(1)$. For the other inclusion, take $x \in X^{ss}(\mathcal{O}_X(1))$ and a point $* \in \mathcal{P}^c_r(A_0)$. There is a section $s \in \Gamma(X, \mathcal{O}_X(m))^{G^c}$ with $s(x) \neq 0$, for some $m > 0$. This one defines a section $\bar{s} \in \Gamma(\bar{X}, \bar{L})^{G^c}$ which still does not vanish at $[*, x] \in \bar{X}$. For an arbitrary section $\sigma \in \Gamma((Pic^0C)^r \times C, \ell^m)$ not vanishing at $\pi([*, x])$, $\bar{s}' := \pi^*\sigma \otimes \bar{s}$ is G^c-invariant and non-zero at $[*, x]$.

Notice that if G^c acts freely on $X^{ss}(\mathcal{O}_X(1))$ then it does the same on $\bar{X}^{ss}(\bar{L})$. Moreover

$$\bar{X}/\!/G^c = (Pic^0C)^r \times C \times \hat{X} \quad \text{where} \quad \hat{X} := X/\!/G^c.$$

I have mentioned already that for $\bar{u} \in M_{C,k}(\bar{X}; \bar{B})$, the composition $\pi \circ \bar{u} = \{\tau\} \times \mathrm{id}_C$, for a certain point $\tau \in (Pic^0C)^r$. This means that in fact $\mathrm{Image}(\bar{u}) \subset \mathcal{P}^c_r(A_0)_\tau \times_{G^c} X$, where $\mathcal{P}^c_r(A_0)_\tau \to C$ denotes the obvious restriction. The next lemma is useful to "visualize" better the space $M_{C,k}(\bar{X}; \bar{B})$.

Lemma 3.4. *Assume that G^c acts freely on X^{ss} and let $q : X^{ss} \to \hat{X}$ be the quotient map. Consider $\bar{u} \in M_{C,k}(\bar{X}; \bar{B})$ with* $\mathrm{Image}(\bar{u}) \subset \mathcal{P}^c_r(A_0)_\tau \times_{G^c} X^{ss}$. *Then $(q \circ \bar{u})^* X^{ss} \to C$ represents the point $\tau \in (Pic^0C)^r$.*

Proof. Notice that in the diagram

$$\begin{array}{ccccc} & & \mathcal{P}^c_r(A_0)_\tau \times X^{ss} & & X^{ss} \\ & & \downarrow & & \downarrow \\ C & \xrightarrow{\bar{u}} & \mathcal{P}^c_r(A_0)_\tau \times_{G^c} X^{ss} & \xrightarrow{q} & \hat{X}, \end{array}$$

$\mathcal{P}^c_r(A_0)_\tau \times X^{ss} = q^* X^{ss}$. Indeed, for $\big([p, x], x'\big) \in q^* X^{ss}$ there is a unique p' such that $[p', x'] = [p, x]$, so we may identify $\big([p, x], x'\big) = (p', x')$. Consequently, $(q \circ \bar{u})^* X^{ss} = \bar{u}^*(\mathcal{P}^c_r(A_0)_\tau \times X^{ss})$ and I obtain a G^c-equivariant map $(q \circ \bar{u})^* X^{ss} \to \mathcal{P}^c_r(A_0)_\tau$ which covers the identity of C. This one must necessarily be an isomorphism. □

4. Definition of the invariants

First I introduce the evaluation maps

$$\begin{aligned} \widetilde{\mathrm{EV}}_k : \mathfrak{X}_B \times (P^c)^k_o / \mathscr{G}^c_0(P) &\longrightarrow X^k, \\ \big[(U, A), (p_1, \dots, p_k)\big] &\longmapsto \big(U(p_1), \dots, U(p_k)\big) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \mathrm{EV}_k : \mathfrak{X}^s_B \times (P^c)^k_o/\mathscr{G}^c(P) &\longrightarrow X^k, \\ \big[(U,A),(p_1,\dots,p_k)\big] &\longmapsto \big(U(p_1),\dots,U(p_k)\big) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \overline{\mathrm{EV}}_k : \mathfrak{X}_B \times (P^c)^k_o/\mathscr{G}^c_0(P) &\longrightarrow (\mathcal{P}^c_r(A_0)\times X)^k, \\ \big[(U,A),\underline{p}\big] &\longmapsto \big([h(A-A_0),R_{f_A}\underline{p}],U(\underline{p})\big). \end{aligned} \tag{4.3}$$

All of the are $(G^c)^k$-equivariant and the last map induces on $M_{C,k}(\bar{X};\bar{B})$ the usual evaluation map

$$ev_k : M_{C,k}(\bar{X};\bar{B}) = \mathfrak{X}_B \times (P^c)^k_o/\mathscr{G}^c_0(P) \times (G^c)^k \longrightarrow \bar{X}^k. \tag{4.4}$$

The key for understanding the relationship between the analytic point of view and the algebraic one developed in the present paper is the following diagram

$$\begin{array}{ccc} \mathfrak{X}^s \times (P^c)^k_o/\mathscr{G}^c_0(P) & \subset & \mathfrak{X}_B \times (P^c)^k_o/\mathscr{G}^c_0(P) \\ \Big\downarrow {\scriptstyle \text{quot out the free (for } k\geq 1) \; G^c\text{-action}} & & \Big\downarrow {\scriptstyle \text{quot out the free } (G^c)^k\text{-action}} \\ \mathfrak{X}^s \times (P^c)^k_o/\mathscr{G}^c(P) & & \mathfrak{X}_B \times (P^c)^k_o/\mathscr{G}^c_0(P) \times (G^c)^k \\ \Big\downarrow {\scriptstyle \text{quot out the (not necessarily free) } (G^c)^k\text{-action}} & & \Big\| {\scriptstyle \text{proposition 3.2}} \\ S_{C,k}(X;B) = M_{C,k}(\bar{X};\bar{B})/\!/G^c & \dashleftarrow & M_{C,k}(\bar{X};\bar{B}). \end{array} \tag{4.5}$$

Recall that $2D$ was defined as the expected dimension of $S_{C,k}(X;B)$, the exact formula being given by (3.3). Since $G^c/G(=\mathbb{R}^r_{>0})$ is contractible, the G and G^c-equivariant cohomology of X coincides; I shall prefer working with G^c-equivariant classes.

Definition 4.1. The invariant for Hamiltonian group actions introduced in [1] is defined in the following way: consider an equivariant cohomology class $\alpha \in H^*_{G^c}(X)^{\otimes k}$ with $\deg\alpha = 2D$. Under the assumption that $(G^c)^k$ acts freely on $\mathfrak{X}^s_B \times (\mathcal{P}^c)^k_o/\mathscr{G}^c(P)$, the pull-back defines a cohomology class on $S_{C,k}(X;B)$ denoted the same. The invariant is

$$\Phi^{X,B}_{C,k}(\alpha) := \int_{S_{C,k}(X;B)} (\mathrm{EV}_k)^*\alpha. \tag{4.6}$$

I have to say that Φ is defined this way only when $S_{C,k}(X;B)$ has the correct dimension. This is the reason why the authors in [1] work with perturbations of $S_{C,k}(X;B)$. In algebraic context, one has to integrate on a π-relative virtual cycle, as mentioned in remark 3.3. ◇

The question of compactifying $S_{C,k}(X;B)$, needed to make well-defined the integral above is left open in [1]. The conjecture in [1] is

Conjecture *Consider* $\alpha \in H^*_{G^c}(X)^{\otimes k}$ *and* $\hat{B} \in H_2(\hat{X};\mathbb{Z})$*, with* $\hat{X} := X/\!/G^c$*. Denote* $\hat{\alpha} \in H^*\big(\hat{X}^k\big)$ *and* $B \in H_2^{G^c}(X;\mathbb{Z})$ *respectively the classes defined by*

$$\hat{X} \xleftarrow{\sim} E \times_{G^c} X^{ss} \longrightarrow E \times_{G^c} X.$$

Then $\Phi^{X,B}_{C,k}(\alpha) = GW^{\hat{X},\hat{B}}_{C,k}(\hat{\alpha})$*.*

Assumption In what follows I shall always assume that $\overline{M}_{C,k}(\bar{X};\bar{B})$ has the expected dimension.

As $S_{C,k}(X;B) = M_{C,k}(\bar{X};\bar{B})/\!/G^c$, it is natural to define the invariant Φ using the evaluation map ev_k on $M_{C,k}(\bar{X};\bar{B})$. Roughly, it means passing from the left hand side of diagram (4.5) to the right hand side.

Remark 4.2. For making this passage, one needs to understand the relationship between the various group actions and evaluation maps which appear in the context.

On $\mathfrak{X}^s_B \times (P^c)^k_o/\mathscr{G}^c_0(P)$ there are two commuting actions. First, $G^c \subset \mathscr{G}^c(P)$ acts by constant complex gauges

$$g \times \big[(U,A),\underline{p}\big] = \big[(gU,A),R_g\underline{p}\big].$$

The map $\widetilde{\mathrm{EV}}_k$ is G^c-*invariant* for this action, while $\overline{\mathrm{EV}}_k$ is G^c-*equivariant*, for the diagonal right action of G^c on $(\mathcal{P}^c_r(A_0) \times X)^k$ on the $\mathcal{P}^c_r(A_0)$-factor.

Secondly, we have the $(G^c)^k$-action on $\mathfrak{X}^s_B \times (P^c)^k_o/\mathscr{G}^c_0(P)$ (with quotient $M_{C,k}(\bar{X};\bar{B})^{ss}$),

$$(g_1,\dots,g_k) \times \big[(U,A),(p_1,\dots,p_k)\big] = \big[(U,A),(R_{g_1}p_1,\dots,R_{g_k}p_k)\big].$$

The evaluation map $\overline{\mathrm{EV}}_k$ is $(G^c)^k$-equivariant for the G^c-action on $\mathcal{P}^c_r(A_0)\times X$ on both terms. ◇

Convention In what follows, the symbol "$\sim$" will denote homotopy equivalence and the letter "$\jmath$" obvious inclusions.

$$\begin{aligned} M_{C,k}(\bar{X};\bar{B})/\!/G^c &\sim E^k \times_{(G^c)^k} \Big(\mathfrak{X}^s_B \times (P^c)^k_o/\mathscr{G}^c(P)\Big) \\ &\sim E^k \times_{(G^c)^k} \Big(E \times_{G^c} \Big(\mathfrak{X}^s_B \times (P^c)^k_o/\mathscr{G}^c_0(P)\Big)\Big) \\ &= E \times_{G^c} \Big(E^k \times_{(G^c)^k} \Big(\mathfrak{X}^s_B \times (P^c)^k_o/\mathscr{G}^c_0(P)\Big)\Big). \end{aligned}$$

The map

$$\overline{\mathrm{EV}}_k : \mathfrak{X}^s_B \times (P^c)^k_o/\mathscr{G}^c_0(P) \longrightarrow (\mathcal{P}^c_r(A_0) \times X)^k$$

being $(G^c)^k$-equivariant and G^c-invariant, induces

(4.7)
$$\begin{array}{ccc}
E\times_{G^c}\Big(E^k\times_{(G^c)^k}\Big(\mathfrak{X}^s_B\times(P^c)^k_o/\mathscr{G}^c_0(P)\Big)\Big) & \xrightarrow{\ \sim\ } & E\times_{G^c} M_{C,k}(\bar X;\bar B)\\
\Big\downarrow \overline{\mathrm{EV}}_k & & \Big\downarrow ev_k\\
E\times_{G^c}\Big(E^k\times_{(G^c)^k}(\mathcal{P}^c_r(A_0)\times X)^k\Big) & \longrightarrow & E\times_{G^c}\bar X^k.
\end{array}$$

On the other hand,

$$\begin{array}{ccc}
 & E\times_{G^c}\Big(E^k\times_{(G^c)^k}(\mathcal{P}^c_r(A_0)\times X)^k\Big) & \\
{}^{ev_k}\swarrow\sim & & \searrow^{\epsilon}\\
E\times_{G^c}\bar X^k & & E\times_{G^c}(E^k\times_{(G^c)^k}X^k)=\mathrm{BG}^c\times(E^k\times_{(G^c)^k}X^k).
\end{array}$$

The reason for the last equality is that, as mentioned in remark 4.2, G^c acts only on $\mathcal{P}^c_r(A_0)$ and therefore the G^c-action on $E^k\times_{(G^c)^k}X^k$ is trivial.

The class α we start with lives in in $H^*_{G^c}(X)^{\otimes k}$. Pulling it back using ϵ, I get a class $\bar\alpha\in H^*_{G^c}(\bar X^k)$.

Following carefully the map EV_k within (4.7), we see that it becomes ev_k, so the invariant Φ can be defined as $\Phi=\int_{\overline{M}_{C,k}(\bar X;\bar B)/\!/G^c} ev_k^*\bar\alpha$, where the relevant maps fit in the diagram

(4.8)
$$\begin{array}{ccc}
E\times_{G^c}\overline{M}_{C,k}(\bar X;\bar B) & \xrightarrow{\ ev_k\ } & E\times_{G^c}\bar X^k\\
\Big\uparrow \jmath_M & & \Big\uparrow \jmath_{\bar X}\\
E\times_{G^c}\overline{M}_{C,k}(\bar X;\bar B)^{ss} & & E\times_{G^c}(\bar X^k)^{ss}\\
\Big\downarrow \sim & & \Big\downarrow\\
\overline{M}_{C,k}(\bar X;\bar B)/\!/G^c & \overset{\widehat{ev}_k}{\dashrightarrow} & \bar X^k/\!/G^c.
\end{array}$$

Let's go on. In [2] I have proved that if the image of a map $\bar u: C\to\bar X$ is contained in the stable locus of $\bar X$, then it defines a stable point in $\overline{M}_{C,k}(\bar X;\bar B)$; conversely (as we are dealing with torus actions), if $\bar u\in\overline{M}_{C,k}(\bar X;\bar B)$ is a semi-stable point, then $\bar u(C)$ is allowed to intersect the unstable locus of $\bar X$ in only finitely many points. Notice that the semi-stable locus of $\overline{M}_{C,k}(\bar X;\bar B)^{ss}\neq\emptyset$ for the following reason: let $\hat u: C\to\hat X$ be a morphism representing the class $\hat B$. The pull-back $\hat u^*X^{ss}\to C$ is a holomorphic principal G^c-bundle and therefore isomorphic to $\mathcal{P}^c_r(A_0)_\tau$ for some $\tau\in(Pic^0C)^r$. Then the induced morphism $\bar u: C\to\hat u^*X^{ss}\times_{G^c}X^{ss}\subset\bar X$ represents the class $\bar B$, by the very definition of $\bar B$.

An immediate consequence is that $\overline{M}_{C,k}(\bar X;\bar B)^{ss}_o\subset\overline{M}_{C,k}(\bar X;\bar B)^{ss}$ defined by the property that the image under the evaluation map of the marked points are all contained in $\bar X^{ss}$ is (Zariski) open and dense. Therefore the invariant $\Phi = \int_{\overline{M}_{C,k}(\bar{X};\bar{B})_o^{ss}/G^c} ev_k^* \jmath_{\bar{X}^{ss}}^* \bar{\alpha}$, for $\jmath_{\bar{X}^{ss}} : E \times_{G^c} (\bar{X}^{ss})^k \to E \times_{G^c} \bar{X}^k$ the inclusion. The reason for doing this artifice is that I am still working with equivariant cohomology classes and I want to pass to "usual" ones.

>From the commutative diagram

$$\begin{array}{ccccc} & & & & E^k \times_{(G^c)^k} X^k \\ & & & & \Big\uparrow \jmath_{X^{ss}} \\ E \times_{G^c} \left(E^k \times_{(G^c)^k} (\mathcal{P}_r^c(A_0) \times X^{ss})^k\right) & & \xrightarrow{\epsilon} & & E^k \times_{(G^c)^k} (X^{ss})^k \\ \Big\downarrow \sim & & & & \Big\downarrow \sim \\ E \times_{G^c} (\bar{X}^{ss})^k & \xrightarrow{\sim} & (\bar{X}^{ss})^k / G^c & \xrightarrow{q} & \hat{X}^k , \end{array}$$

I deduce that $\jmath_{X^{ss}}^* \bar{\alpha}$ is actually the pull-back from $\hat{X}^k$ of the cohomology class $\alpha \in H_{G^c}^*(X)^{\otimes k}$ i.e. $\jmath_{\bar{X}^{ss}}^* \epsilon^* \alpha = q^* \hat{\alpha}$, with $\hat{\alpha}$ the cohomology class on $\hat{X}^k$ determined by the equivariant cohomology class α. Therefore $\Phi = \int_{\overline{M}_{C,k}(\bar{X};\bar{B})_o^{ss}/G^c} (\widehat{ev}_k)^* q^* \hat{\alpha}$.

Let me remind that the goal in [2] was to compare the invariant quotient of the moduli space of stable maps into a projective variety with the moduli space of stable maps into the invariant quotient of the variety. This eventually leaded to some relations between the two sets of Gromov-Witten invariants.

The quotient $\overline{M}_{C,k}(\bar{X};\bar{B})/\!/G^c$ appeared already. I must compare it with $\overline{M}_{C,k}(\bar{X}/\!/G^c;[C]+\hat{B})$; indeed, the 2-homology class $\bar{B} \in H_2(\bar{X};\mathbb{Z})$ determines the class $[C]+\hat{B} \in H_2(\bar{X}/\!/G^c;\mathbb{Z}) = H_2((Pic^0C)^r \times C \times \hat{X};\mathbb{Z})$. Since any map into $\bar{X}/\!/G^c = (Pic^0C)^r \times C \times \hat{X}$ is constant on the first component, I obtain a morphism

$$\overline{M}_{C,k}(\bar{X}/\!/G^c,[C]+\hat{B}) \longrightarrow \overline{M}_{C,k}(C \times \hat{X};[C]+\hat{B}).$$

Proposition 4.3. *The composed map*

$$T : \overline{M}_{C,k}(\bar{X};\bar{B})/\!/G^c \dashrightarrow \overline{M}_{C,k}(C \times \hat{X};[C]+\hat{B})$$

is birational.

Proof. I show first that T is dominant. Consider

$$\begin{array}{ccccc} \hat{u}^* X^{ss} & \longrightarrow & C \times X^{ss} & \longrightarrow & X^{ss} \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ C & \xrightarrow{\mathrm{id}_C \times \hat{u}} & C \times \hat{X} & \longrightarrow & \hat{X}. \end{array}$$

As $\hat{u}_*[C] = \hat{B} \in H_2(\hat{X};\mathbb{Z})$, the G^c-equivariant map $\hat{u}^* X^{ss} \to X^{ss} \subset X$ defines the class $B \in H_2^{G^c}(X;\mathbb{Z})$. Moreover, $\hat{u}^* X^{ss} \cong \mathcal{P}_r^c(A_0)_\tau$, for a certain $\tau \in (Pic^0C)^r$, so I obtain a map $C \xrightarrow{\bar{u}} \mathcal{P}_r^c(A_0)_\tau \times_{G^c} X^{ss} \subset \mathcal{P}_r^c(A_0) \times_{G^c} X =$

$\bar{X}$ representing $\bar{B} \in H_2(\bar{X};\mathbb{Z})$. Since Image$(\bar{u}) \subset \bar{X}^{ss}$, it follows that $\bar{u}$ is a semi-stable point in $\overline{M}_{C,k}(\bar{X};\bar{B})$. It is obvious that $\bar{u}$ is mapped to $\hat{u}$ by T.

I shall prove now that T is generically injective. Suppose that

$$\begin{array}{ccccc} C & \xrightarrow{\bar{u}} & \mathcal{P}^c_r(A_0)_\tau \times_{G^c} X^{ss} & & \\ & & & \searrow^{q} & \\ & & & & C \times \hat{X} \\ & & & \nearrow_{q} & \\ C & \xrightarrow{\bar{u}'} & \mathcal{P}^c_r(A_0)_{\tau'} \times_{G^c} X^{ss} & & \end{array}$$

are such that $q \circ \bar{u} = q \circ \bar{u}'$. Then $(q\circ\bar{u})^*(X^{ss} \to \hat{X}) = (q \circ \bar{u}')^*(X^{ss} \to \hat{X})$ and lemma 3.4 implies that $\tau = \tau'$. Since

$$C \xrightarrow{\bar{u},\bar{u}'} \mathcal{P}^c_r(A_0)_\tau \times_{G^c} X^{ss}$$

induce the same map to $\hat{X}$, for each $\zeta \in C$ there is a unique $g(\zeta) \in G^c$ such that $\bar{u}'(\zeta) = g(\zeta)u(\zeta)$, because G^c was assumed to act freely on X^{ss}. This defines a morphism $C \to G^c$ which has to be constant as G^c is affine. Therefore $\bar{u}' = g\bar{u}$ and they define the same point in $\overline{M}_{C,k}(\bar{X};\bar{B})/\!/G^c$. □

Remark 4.4. For curves of genus $g \geq 2$,

$$\overline{M}_{C,k}(C\times\hat{X};[C]+\hat{B}) \xrightarrow{\#\mathrm{Aut}(C):1} \overline{M}_{C,k}(\hat{X};\hat{B}),$$

while for elliptic curves and for $\mathbb{P}^1$, we have respectively

$$\overline{M}_{C_1,k}(C_1\times\hat{X};[C_1]+\hat{B}) \xleftrightarrow{1:1} \overline{M}_{C_1,k+1}(\hat{X};\hat{B})$$

and

$$\overline{M}_{\mathbb{P}^1,k}(\mathbb{P}^1\times\hat{X};[\mathbb{P}^1]+\hat{B}) \xleftrightarrow{1:1} \overline{M}_{\mathbb{P}^1,k+3}(\hat{X};\hat{B}). \qquad \diamond$$

I am finally in position to state the main result.

Theorem 4.5. *Under the assumption that the spaces $\overline{M}_{C,k}(\bar{X};\bar{B})/\!/G^c$ and $\overline{M}_{C,k}(\hat{X};\hat{B})$ have both expected dimension D, the invariant Φ coincides with the Gromov-Witten invariant of $\hat{X}$.*

More precisely, for $\alpha \in H^*_{G^c}(X)^{\otimes k}$ of degree $2D$, ,

(i) $\Phi^{X,B}_{C,k}(\alpha) = \#\mathrm{Aut}(C)\cdot GW^{\hat{X},\hat{B}}_{C,k}(\hat{\alpha})$ for a curve C of genus $g(C) \geq 2$;

(ii) $\Phi^{X,B}_{C_1,k}(\alpha) = GW^{\hat{X},\hat{B}}_{C_1,k+1}(\hat{\alpha})$ for an elliptic curve C_1;

(iii) $\Phi^{X,B}_{\mathbb{P}^1,k}(\alpha) = GW^{\hat{X},\hat{B}}_{\mathbb{P}^1,k+3}(\hat{\alpha})$.

Proof. The result follows from the commutative diagram

$$\begin{array}{ccccc} \overline{M}_{C,k}(\bar{X};\bar{B})^{ss}_o & \xrightarrow{\hat{ev}_k} & (\bar{X}^{ss})^k/G^c & & \\ & & & \searrow^{q} & \\ \Big\downarrow \text{1:1} & & \Big\downarrow & & \hat{X}^k \\ & & & \nearrow_{\mathrm{pr}_{\hat{X}}} & \\ \overline{M}_{C,k}(C\times\hat{X};[C]+\hat{B}) & \xrightarrow{ev^{\hat{X}}_k} & (C\times\hat{X})^k & & \end{array}$$

Indeed, it allows me to deduce that

$$\Phi^{X,B}_{C,k}(\alpha) = \int_{\overline{M}_{C,k}(\bar{X};\bar{B})^{ss}_{o}/G^c} \widehat{ev}^*_k q^* \hat{\alpha} = \int_{\overline{M}_{C,k}(C\times\hat{X};[C]+\hat{B})} \big(ev^{\hat{X}}_k\big)^* \mathrm{pr}^*_{\hat{X}} \hat{\alpha}.$$

The fact that the right hand side of the expression above is precisely a Gromov-Witten invariant of $\hat{X}$ follows from remark 4.4. □

A (very) particular case when all the assumptions are satisfied

Corollary 4.6. *Consider an action of the complex torus G^c on the homogeneous variety X which is linearized on an ample line bundle over it. Assume that the action is free on the semi-stable locus (or more generally, the stabilizer of the semi-stable points is a certain finite (normal) subgroup of G^c) and that the invariant quotient $\hat{X}$ is still homogeneous. Assume further that $\hat{B} \in H_2(\hat{X};\mathbb{Z})$ is a homology class induced by the composed map $\mathbb{P}^1 \xrightarrow{u} X^{ss} \xrightarrow{q} \hat{X}$ and denote $B \in H_2^{G^c}(X)$ the induced equivariant homology class. Consider now an equivariant cohomology class $\alpha \in H^*_{G^c}(X)^{\otimes k}$ which induces on $\hat{X}^k$ the class $\hat{\alpha} \in H^*(\hat{X}^k;\mathbb{Z})$. Then*

$$\Phi^{X,B}_{\mathbb{P}^1,k}(\alpha) = GW^{\hat{X},\hat{B}}_{\mathbb{P}^1,k+3}(\hat{\alpha}).$$

Proof. The situation is rather trivial under these assumptions: the morphism $u : \mathbb{P}^1 \to X^{ss}$ can be thought off as a section in the principal bundle $\hat{u}^* X^{ss} \to \mathbb{P}^1$, so we are dealing with the trivial $(\mathbb{C}^*)^r$-bundle over $\mathbb{P}^1$ (any topologically trivial torus bundle over $\mathbb{P}^1$ is holomorphically trivial). Therefore $\bar{X} = \mathbb{P}^1 \times X$ in this case, which is homogeneous again. The conclusion follows from the fact that the spaces of rational maps into homogeneous varieties are irreducible (see [4]) and have expected dimension (as they are convex varieties). □

5. Concluding remarks

Of course, the most unpleasant assumption in theorem 4.5 is that on the dimension of the spaces of curves. For proving this result in general using only algebraic methods, one has to clarify the construction of a π-relative virtual cycle on $\overline{M}_{C,k}(\bar{X};\bar{B})$, as explained in remark 3.3. Then, one should compare the two virtual cycles: that on $\overline{M}_{C,k}(\bar{X};\bar{B})$ with that on $\overline{M}_{C,k}(\hat{X};\hat{B})$. This approach seems rather difficult.

Another possibility is to consider a generic G-invariant almost complex structure J on X and to work with $\bar{X} = \mathcal{P}_r(A_0) \times_G X$ endowed with the almost complex structure determined by J and the *holomorphic* structure on the universal bundle $\mathcal{P}^c_r(A_0)$ (this is the meaning of the term π-relative). The shortcoming of this approach might be that one looses the powerful tools of the geometric invariant theory.

Finally, in [1] the authors work with arbitrary compact groups G whose algebraic counterpart is to consider arbitrary complex, linearly reductive groups G^c. The situation can be much more involved than for torus actions.

A good reason is that there does not exist anymore a universal principal bundle which gives all the principal G^c-bundles over a fixed curve and having a given topological type. A guess is that the object which enters into the game is the moduli space of stable principal bundles of fixed topological type, which was constructed by Ramanathan.

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Address: Institut Fourier
UMR5582 CNRS-UJF
38402 St Martin d'hères Cedex (France

Fakultät für Mathematik NA,
Ruhr-Universität Bochum, D-44780 Bochum.

Email address: halic@ag.ruhr-uni-bochum.de