SETS OF SEMISIMPLICITY

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ABSTRACT. We introduce the notion of a set of semisimplicity, or S_3-set, as a set Λ such that if T is a representation of a LCA group G with $Sp(T) \subset \Lambda$, then T generates a semisimple Banach algebra. We prove that the union of S_3-sets is a S_3-set, provided their intersection is countable. In particular, the union of a countable set and a Helson S-set is a S_3-set. Using the construction of the limit isometric representations, we obtain conditions for semisimplicity of Banach algebras generated by representations of semigroups, and results on stability of such representations. We apply the approach to polynomially bounded and power-bounded operators and show, in particular, that if T is a polynomially bounded (power-bounded) operator whose spectrum on the unit circle has measure zero (resp., is a Helson S-set), then $T^n x \to 0$ for all $x \in X$ if and only if T^* does not have an invariant subspace on which it is similar to an invertible isometry.

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1. Introduction.

According to the classical Gelfand-Naimark theorem, the Banach algebra generated by a normal (in particular, self-adjoint or unitary) operator on a Hilbert space is isometric to a Banach algebra of continuous functions on a compact and, in particular, is semisimple. Sinclair [**S**] proved that the Banach algebra generated by a Hermitian operator T on a Banach space is semisimple, if the spectrum of T is countable. The analogous result for isometric operators was obtained by Feldman [**F**]. In [**M-V**], the authors proved that the Banach algebra $\mathcal{A}(T)$ generated by a strongly continuous isometric representation T of an arbitrary locally compact abelian group is semisimple, if the spectrum $Sp(T)$ of the representation T is scattered (in particular, if $Sp(T)$ is countable). It follows from Malliavin's theorem on the existence of sets which are not of spectral synthesis (non-S-sets) on any non-discrete groups (i.e. the failure of spectral synthesis on $L^1(G)$ for every

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noncompact locally compact abelian group G), that without the spectral condition the algebra $\mathcal{A}(T)$ is not, in general, semisimple.

In this paper, we introduce the notion of sets of semisimplicity, or S_3-sets, and investigate their properties. A closed subset Λ of the dual group $\Gamma := \widehat{G}$ is called S_3-set, if for every representation T of G by bounded linear operators on a Banach space such that $Sp(T) \subset \Lambda$, the Banach algebra $\mathcal{A}(T)$, generated by "functions" of T, is semisimple. Following an argument in **[F]**,**[S]**, it is not difficult to see that any S_3-set is a set of spectral synthesis (or S-set), and that any Helson set of spectral synthesis is a S_3-set. The above result **[F]**,**[S]**,**[M-V]** implies that any scattered set is a S_3-set. Moreover, every S_3-set is a set of spectral resolution in the sense of Malliavin (see $[\mathbf{B}]_1$, p.174) and, therefore, not every S-set is a S_3-set. We introduce the notion of archipelago of closed sets, and show that any archipelago of S_3-sets is a S_3-set; in particular, this implies that the union of a S_3-set and a scattered set is a S_3-set. Moreover, we prove that the union of two S_3-sets is a S_3-set provided that their intersection is scattered (answering a question of G.M. Feldman). Using a standard construction of the limit isometric representation of bounded representations of locally compact abelian semigroups, we obtain similar results on semisimplicity of the Banach algebra generated by a representation T of a locally compact semigroup S, and we establish some relationship between (non-) semisimplicity and the stability of the corresponding semigroup representations, resp. the existence of hyperinvariant subspaces. In particular, we show, that if T is a polynomially bounded (power-bounded) operator whose spectrum on the unit circle has measure zero (resp., is a Helson S-set), then $T^n x \to 0$ for all $x \in X$ if and only if T^* does not have an invariant subspace on which it is similar to an invertible isometry. Analogously, if T is a polynomially bounded (power-bounded) operator of class $C_{1\cdot}$ whose spectrum on the unit circle has measure zero (is a Helson S-set), then T has a nontrivial hyperinvariant subspace.

Throughout the paper, we denote by $\mathbf{D}$ the open unit disk, by $\partial\mathbf{D}$ the unit circle. If $\mathcal{A}$ is a commutative Banach algebra, then by $\mathcal{R}(\mathcal{A})$ we denote its radical, i.e. the set of all topological nilpotent elements of $\mathcal{A}$. If X is a Banach space, then by $L(X)$ we denote the set of all bounded linear operators on X. As usual, $\mathbf{Z}$ is the set of integers, $\mathbf{Z}_+ := \{n \in \mathbf{Z} : n \geq 0\}$.

2. S_3-sets.

Let G be a Hausdorff locally compact abelian group, written additively, with Haar measure m and dual group Γ. By $L^1(G)$ we denote the usual group algebra, and by $A(\Gamma)$ the corresponding algebra of Fourier transforms of elements of $L^1(G)$. The results in this paper can be presented in several ways, e.g., using the language of Banach modules as given in **[D-M]**. Here we will use the framework of representation theory for simplicity of notations.

Let T be a bounded strongly continuous representation of G by bounded linear operators on a Banach space X ($X \neq \{0\}$), i.e. $\{T(t) : t \in G\}$ is a family of bounded linear operators on X satisfying the following conditions:

(i) $T(e) = I$, where e is the unit in G;
(ii) $T(t_1 + t_2) = T(t_1)T(t_2)$ for all t_1, t_2 in G;
(iii) The mapping $t \mapsto T(t)x$ is continuous for every $x \in X$;
(iv) $sup_{t \in G}\|T(t)\| < \infty$.

By introducing an equivalent norm on X

$$|||x||| := \sup_{t \in G} \|T(t)x\|, \ \forall x \in X,$$

one can assume that T is an isometric representation. For each function $f \in L^1(G)$, let

$$\hat{f}(\chi) = \int_G f(t)\chi(t)dt,$$

and

$$\hat{f}(T) = \int_G f(t)T(t)dt.$$

The spectrum of the representation T is defined by

$$Sp(T) = \{\chi \in \Gamma : \hat{f}(\chi) = 0 \text{ whenever } \ \hat{f}(T) = 0\}.$$

Let $\mathcal{A}(T)$ be the Banach algebra generated by $\hat{f}(T), f \in L^1(G)$. The spectrum of T, $Sp(T)$, can be identified with the Gelfand space of $\mathcal{A}_0(T)$ via the formula

$$\phi_\chi(\hat{f}(T)) = \hat{f}(\chi) \text{ (see } \textbf{[A]}, \textbf{[L-M-F]}, \textbf{[B-V]}).$$

Definition. A closed subset $\Lambda \subset \Gamma$ is called a *set of semisimplicity* or S_3*-set*, if for every isometric representation $T : G \to L(X)$ such that $Sp(T) \subset \Lambda$, the algebra $\mathcal{A}(T)$ is semisimple.

As mentioned above, every scattered set as well as every Helson S-set is a S_3-set. Recall that for every closed subset E of Γ, there associate two closed ideals, $I(E)$, consisting of functions $\varphi \in A(\Gamma)$ such that $\varphi|E = 0$, and $J(E)$, consisting of functions which can be approximated by functions vanishing on a neighborhood of E. Clearly, $J(E) \subset I(E)$. A set E is called a *set of spectral synthesis*, or S_3*-set*, if $I(E) = J(E)$. A compact subset $E \subset \Gamma$ is called a *Helson set*, if every continuous function on E is the restriction of a function from $A(\Gamma)$. It is well known that there are Helson sets which are not S-sets (conditions for Helson sets to be S-sets are given in $[\textbf{Be}]_2$). There are countable sets (scattered sets) which are not Helson sets, as well as Helson S-sets which are not scattered (see $[\textbf{Be}]_1$, **[H-R]**).

Proposition 1. *Every* S_3*-set is a* S*-set.*

Proof. Assume that E is a closed subset of Γ which is not a S-set. Consider the quotient algebra $A(\Gamma)/J(E)$. If $\varphi \in A(\Gamma)$, then the image of φ under this homomorphism is denoted by $\hat{\varphi}$. Since E is not a S-set, the quotient algebra $A(\Gamma)/J(E)$

is not semisimple: indeed, any element $\varphi \in I(E) \setminus J(E)$ under the natural homomorphism $A(\Gamma) \to A(\Gamma)/J(E)$ will be mapped into a non-zero topological nilpotent element. Consider the representation $V : G \to L(A(\Gamma))$ defined by

$$(V(t)\varphi)(\chi) = \chi(t)\varphi(\chi),$$

and let $T : G \to L(A(\Gamma)/J(E))$ be defined by $T(g)\widehat{\varphi} = \widehat{(V(g)\varphi)}$. Then the algebra $\mathcal{A}(T)$ is isometrically isomorphic to $A(\Gamma)/J(E)$, hence is not semisimple. □

Definition. A family of closed subsets of Γ, $\{E_\alpha\}_{\alpha \in F}$ is called an *archipelago* if:
(i) for every $\alpha_1, \alpha_2 \in F, \alpha_1 \neq, \alpha_2$, we have $E_{\alpha_1} \cap E_{\alpha_2} = \emptyset$, and
(ii) for every $F_0 \subset F$ there exists an open set $V \subset \Gamma$ and there exists $\alpha_0 \in F_0$ such that $E_{\alpha_0} \subset V$ and $V \cap E_{\alpha_j} = \emptyset$ for all $\alpha_j \in F_0, \alpha_j \neq \alpha_0$.

Proposition 2. *If $Sp(T) = \Lambda_1 \cup \Lambda_2$, where Λ_1, Λ_2 are nonempty closed subsets such that one of them is compact and $\Lambda_1 \cap \Lambda_2 = \emptyset$, then there is a projection $P \in \mathcal{A}_0(T)$ such that $Sp(T|PX) = \Lambda_1, Sp(T|(I-P)X) = \Lambda_2$.*

Proof. Assume, for definiteness, that Λ_1 is compact. Let $\Lambda_\alpha \subset \Lambda_2$ be a compact set, $Q_\alpha := \Lambda_1 \cup \Lambda_\alpha$, and consider the spectral subspace $X_\alpha := X(Q_\alpha)$. Let $T_\alpha(g) := T(g)|X_\alpha$. Then $Sp(T_\alpha)$ is compact, hence T_α is uniformly continuous and the algebra $\mathcal{A}_0(T_\alpha)$ has unit. By Silov's Idempotent Theorem, there is an idempotent element $P_\alpha \in \mathcal{A}_0(T_\alpha)$ such that $Sp(T_\alpha|P_\alpha X_\alpha) = \Lambda_1, Sp(T_\alpha|(I-P_\alpha)X_\alpha) = \Lambda_\alpha$. It is easy to see that the family of projections P_α is uniformly bounded is can be extended to a projection P on X such that $Sp(T|PX) = \Lambda_1, Sp(T|(I-P)X) = \Lambda_2$. □

It follows from Proposition 1 that if E_1 and E_2 are two compact S_3-sets such that $A_1 \cap A_2 = \emptyset$, then $E_1 \cup E_2$ is a S_3-set. A more general fact is proved in the next theorem.

Theorem 1. *Let $\{E_\alpha\}_{\alpha \in F}$ be an archipelago of compact S_3-sets of Γ, then $E := \cup_{\alpha \in F} E_\alpha$ is a S_3-set.*

Proof. Let T be a representation of G on $L(X)$ such that $Sp(T) \subset E$. Let $a \in R(\mathcal{A}(T))$. Define

$$F_a := \{\alpha \in F : E_\alpha \cap Sp(a) \neq \emptyset\}.$$

There exists an open set V in Γ and an α_i such that $E_{\alpha_i} \subset V$ and $V \cap E_{\alpha_j} = \emptyset$ for all $\alpha_j \in F_a, \alpha_j \neq \alpha_i$. Take an element $f \in L^1(G)$ such that $\hat{f}(\gamma) = 1$ for all $\gamma \in A_{\alpha_i}$ and $\hat{f}(\gamma) = 0$ for all $\gamma \notin V$.

Let $X_1 := \overline{\{\hat{f}(T)ax : x \in X\}}$ and $\tilde{T}(t) := T(t)|X_1$. Since $Sp(\hat{f}(T)a) \subset E_{\alpha_i} \cap Sp(a)$, and since $\hat{f}(T)a$ is in the radical of $\mathcal{A}((\tilde{T}))$, it follows $\hat{f}(T)a = 0$, Hence, $Sp(a) \cap E_{\alpha_i} = \emptyset$, which is a contradiction. □

Proposition 3. *If $F = \{E_i : i = 1, 2, ...\}$ is a sequence of compact sets such that $E_i \cap E_j = \emptyset$ for $i \neq j$, and assume that $E = \cup_{i=1}^{\infty} E_i$ is compact, then F is an archipelago.*

Proof. Let $F_0 \subset F$. If F_0 is finite, say $F_0 = \{E_{k_1}, ..., E_{k_m}\}$, then one can take $V = \Gamma \backslash [E_{k_2} \cup E_{k_3} \cup ... \cup E_{k_m}]$. Hence $E_{k_1} \subset V$ and $E_j \cap V = \emptyset$ for all $j = k_2, k_3, ..., k_m$.
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Proposition 4. *If E is a closed set and B is scattered, then $F := \{E, x \in B \setminus E\}$ is an archipelago.*

Proof. Let $F_0 \subset F$. There are three possibilities:

(i) $F_0 = \{E\}$. Then we can take as V any open set containing E.

(ii) $F_0 \subset \{x : x \in B \setminus E\}$. Since B is scattered, F_0 contains an isolated point $x_0 \in B \setminus E$, i.e. there is an open set V, $x_0 \in V$, $[V \setminus \{x_0\}] \cap B \setminus E = \emptyset$, so that the definition is fulfilled.

(iii) F_0 contains E and elements in $B \setminus E$. Since $F_0 \setminus \{E\}$ contains an isolated point, say x_0, there exists an open set V, such that $x_0 \in V$ and $V \cap [F_0 \setminus E] = \emptyset$. Choose $W = V \cap E^c$ (where $E^c = \Gamma \setminus E$), then $x_0 \in W$ and $W_0 \cap E = \emptyset$, hence the definition is fulfilled. □

Proposition 4 and Theorem 1 imply the following corollary.

Corollary 1. *If E is a S_3-set and B is scattered, then $E \cup B$ is S_3-set. In particular, the union of a Helson S-set and a scattered set is a S_3-set.*

Now we consider the general question of when is the union of S_3-sets a S_3-set. Let a be an element in $\mathcal{A}(T)$. We define

$$I_a := \{f \in L^1(G) : \hat{f}(T)a = 0\},$$

and let

$$Sp(a) := \{\chi \in \Gamma : \hat{f}(\chi) = 0 \ \forall f \in I_a\}.$$

It is not difficult to see that $Sp(a) = Sp(T|\overline{aX})$.

Lemma 1. *Assume that Λ_1, Λ_2 are S_3-sets, $T : G \to L(X)$ is an isometric representation such that $Sp(T) \subset \Lambda_1 \cup \Lambda_2$. If $a \in \mathcal{R}(\mathcal{A}(T))$, then $Sp(a) \subset \Lambda_1 \cap \Lambda_2$.*

Proof. We show that $Sp(a) \subset \Lambda_2$. Let U_2 be an open set, $\Lambda_2 \subset U_2$. We show that $Sp(a) \subset \overline{U_2}$. Assume, on the contrary, that there exists $\chi \in Sp(a)$, such that $\chi \notin \overline{U_2}$. Take an element $f \in L^1(G)$ such that $\hat{f}|U_2 = 0, \hat{f}(\chi) = 1$.

Since $Sp(\hat{f}(T)a) \subset supp(\hat{f}) \cap Sp(a) \subset [\Gamma \setminus U] \cap Sp(a) \subset \Lambda_1$, and since Λ_1 is a S_3-set and $\hat{f}(T)a$ is a topological nilpotent element, it follows that $\hat{f}(T)a = 0$, i.e. $f \in I_a$. Therefore, $\hat{f}(\chi) = 0$, a contradiction. □

We also need the following lemma (see **[M-V]**, Proposition 6).

Lemma 2. *If $a \in \mathcal{R}(\mathcal{A}(T))$ and $a \neq 0$, then $Sp(a)$ has no isolated point.*

Theorem 2. *If Λ_1 and Λ_2 are S_3-sets and $\Lambda_1 \cap \Lambda_2$ is scattered, then $\Lambda_1 \cup \Lambda_2$ is a S_3-set.*

Proof. Let T be an isometric representation of G on $L(X)$ such that $Sp(T) \subset \Lambda_1 \cup \Lambda_2$. Assume that there exists an element $a \in \mathcal{R}(\mathcal{A}(T))$ such that $a \neq 0$. By Lemma 1, $Sp(a) \subset \Lambda_1 \cap \Lambda_2$, hence $Sp(a)$ contains an isolated point, which is impossible by Lemma 2. □

Theorems 1, 2 and Corollary 1 are, of course, analogous to the corresponding results concerning S-sets (see [**B**]$_1$, p. 172, 187). It is not known whether finite unions of S_3-sets are always S_3-sets. If Λ_1 and Λ_2 are S_3-sets, $Sp(T) \subset \Lambda_1 \cup \Lambda_2$ and $a \in \mathcal{R}(\mathcal{A}(T))$, then Lemma 1 implies only that $a^2 = 0$.

3. Semisimplicity of representations of semigroups.

Now we consider the question of semisimplicity for Banach algebras generated by representations of semigroups. Let S be a measurable subsemigroup of a locally compact abelian group G such that $S - S = G$. Let $\widehat{S}$ denote the semigroup of continuous semi-unitary characters on S, i.e. the set of all complex homomorphisms from S into the unit disk $\overline{\mathbf{D}}$.

A *representation* T *of the semigroup* S on a Banach space X is by definition a homomorphism from S to $L(X)$ such that the mapping $s \mapsto T(s)x$ is continuous for every $x \in X$.

If T is a representation of S such that $\sup_{s\in S} \|T(s)\| < \infty$, then T is called a *bounded representation.* If T is a bounded representation of S, then there is an equivalent norm on X in which $T(s)$ are contractions. As examples of the semigroup S we can take $S = \mathbf{Z}_+^m := \{\mathbf{n} = (n_1, ..., n_m) : n_i \in \mathbf{Z}_+\}$. Then every representation T of $\mathbf{Z}_+^m$ has form: $T(\mathbf{n})x = T_1^{n_1} T_2^{n_2} ... T_m^{n_m} x$, $\forall x \in X$, where $T_1, ..., T_m$ are commuting operators. Similarly, every representation of the semigroup $\mathbf{R}_+^m := \{\mathbf{t} = (t_1, ..., t_m) : t_i \in \mathbf{R}_+\}$ has form $T(\mathbf{t}) = T_1(t_1)...T_m(t_m)$, where $T_i(t)$ are commuting C_0-semigroups. Such multi-parameter semigroups were considered by Hille-Phillips [**H-P**].

Let $f \in L^1(S)$. Define

$$\hat{f}(T) := \int_S f(s)T(s)ds,$$

where ds is the restriction of the Haar measure from G to S. The spectrum of T is defined by the following formula

$$Sp(T) := \{\chi \in \widehat{S} : |\hat{f}(\chi)| \leq \|\hat{f}(T)\| \text{ for all } f \in L^1(S)\} \text{ (see [\mathbf{B\text{-}V}]).}$$

We denote by $Sp_u(T)$ the unitary part of the spectrum of T, i.e. $Sp_u(T) = Sp(T) \cap \Gamma$. If T is a single power-boubded operator (i.e., $\sup_{n\geq 0} \|T^n\| < \infty$), then $Sp(T)$ coincides with the union of $\sigma(T)$ and all bounded components of the resolvent set.

We will also need the following construction of the *limit isometric representation* for bounded representations T. For this, we regard S as a directed set with the quasi-order on S defined by:

$$s < t \ (s,t \in S) \text{ if there exists } u \in S \text{ such that } t = u + s.$$

Define a seminorm on X by

$$l(x) := \lim_S \|T(s)x\|, \ x \in X.$$

Let $L = ker(l) = \{x \in X : l(x) = 0\}$, and consider the quotient space $\widehat{X} = X/L$, equipped with the norm $\hat{l}(\hat{x}) := l(x), \ x \in X$. The operators $T(s)$ generate the corresponding operators $\widehat{T}(s)$ on $\widehat{X}$ in the natural way, namely $\widehat{T}(s)\hat{x} := \widehat{T(s)x}$. Clearly, $\widehat{T}(s)$ is a (strongly continuous) representation (since $\hat{l}(\hat{x}) \leq \|x\|, \ \forall x \in X$) of S by isometric operators on $\widehat{X}$. We denote by Z the completion of $\widehat{X}$ in the norm $\hat{l}$, and by $V(s)$ the continuous extention of $\widehat{T}(s)$ from $widehat X$ to Z.

It is known that $Sp(V) \subset Sp(T)$. If $T(s)$ has a dense range for each $s \in S$, then $V(s)$ also has a dense range, hence $V(s)$ are invertible isometrices, which implies that V can be extended to an isometric representation of the group G on Z. In general, there is an isometric representation of the group G on a Banach space $E \supset Z$ such that $U(s)|Z = V(s)$ and $Sp(U) \subset Sp_u(V)$ (see **[B-V], [D]**). Moreover, the construction in **[B-V], [D]** has the property that if the algebra $\mathcal{A}(U)$ is semisimple, then the algebra $\mathcal{A}(V)$ also is semisimple.

If A is an operator on X which commutes with $T(s)$, then L is invariant with respect to A and one can define $\widehat{A}$ on $\widehat{X}$ by $\widehat{A}\hat{x} := \widehat{Ax}$. Since

$$\|\widehat{A}\hat{x}\|_{\widehat{X}} = \lim_S \|T(s)Ax\| \leq \|A\| \lim_S \|T(s)x\| = \|A\|\|\hat{x}\|_{\widehat{X}},$$

it follows that $\varphi : A \mapsto \widehat{A}$ is a continuous homomorphism from the algebra $\mathcal{A}(T)$ to the algebra $\mathcal{A}(V)$. In particular,

$$(*) \qquad \varphi(\mathcal{R}(\mathcal{A}(T))) \subset \mathcal{R}(\mathcal{A}(V)).$$

Theorem 3. *Assume that $T : S \to L(X)$ is a bounded representation such that:*
(i) for every $x \in X, x \neq 0$, $T(s)x$ does not converge to 0;
(ii) $Sp_u(T)$ is a S_3-set.
Then the algebra $\mathcal{A}(T)$ is semisimple.

Proof. Since $Sp(U) \subset Sp_u(T)$, which is a S_3-set, we conclude that the algebra $\mathcal{A}(U)$, and hence $\mathcal{A}(V)$, is semisimple. Assume, on the contrary, that $\mathcal{A}(T)$ is not semisimple. Then there exists $a \in \mathcal{R}(\mathcal{A}(T)), a \neq 0$. By $(*)$, the operator $\hat{a}$ is in the radical of the algebra $\mathcal{A}(V)$, hence $\hat{a} = 0$, which implies that $\lim_S \|T(s)ax\| = 0$ for all $x \in X$, a contradiction to (i). $\square$

Corollary 2. *Assume that $T : S \to L(X)$ is a bounded representation such that $Sp_u(T)$ is the union of a scattered set and a Helson S-set. If $T(s)x$ does not converge strongly to 0, for all $x \neq 0$, then $\mathcal{A}(T)$ is semisimple.*

Corollary 3. *Assume that T is a power-bounded operator, i.e. $\sup_{n\geq 0} \|T^n\| < \infty$, and $\sigma(T) \cap \partial\mathbf{D}$ is a union of a countable set and a Helson S-set. If $T^n x$ does not converge strongly to 0, for all $x \neq 0$, then $\mathcal{A}(T)$ is semisimple.*

Corollary 4. *Assume that $T(t), t \geq 0$, is a bounded C_0-semigroup, i.e. $\sup_{t\geq 0} \|T(t)\| < \infty$, with the generator A, and assume that $\sigma(A) \cap i\mathbf{R}$ is a union of a countable set and a Helson S-set. If $T(t)x$ does not converge strongly to 0, for all $x \neq 0$, then $\mathcal{A}(T)$, the Banach algebra generated by $T(t)$ and $(\lambda I - A)^{-1}, \lambda \in \rho(A)$, is semisimple.*

For semigroups of operators on Hilbert space Theorem 3 and its corollaries hold without the spectral assumption, since isometric representations on Hilbert space always generate semisimple algebras. It should be noted that the norm $|||x||| = \sup_{s\in S} \|T(s)x\|$ is not, in general, a Hilbert space norm. Therefore, when considering bounded representations T on a Hilbert space, we can not assume that the operators $T(s)$ are contractions.

In order to formulate and prove the corresponding result, we show that if the space $X = H$ is a Hilbert space, then the construction of $\widehat{H}$ can be modified to be also a Hilbert space (such construction was contained in **[K]** for single operators). For this purpose, let $glim_S$ be a fixed Banach limit on $l^\infty(S)$, and consider the following bilinear form on H:

$$l(x, y) := glim_S \langle T(s)x, T(s)y \rangle, \ x, y \in H.$$

It is easy to see that $l(x, y)$ is a bilinear form, i.e., it is linear in x, anti-linear in y, and $l(x, x) \geq 0$. We put

$$L = \{x : l(x, x) = 0\} = \{x : l(x, y) = 0 \ \ \forall y \in H\}.$$

Clearly, L is a closed subspace of H which is invariant for $T(s)$ as well as for every operator commuting with $T(s), s \in S$. Note that, since $T(s)$ are uniformly bounded, $\inf_S \|T(s)x\| = 0$ if and only if $\lim_S \|T(s)x\| = 0$. Hence, $glim_S \|T(s)x\| = 0$ if and only if $\lim_S \|T(s)x\| = 0$.

Let $\widehat{H} := H/L$, and $\hat{l}$ be a bilinear form on $\widehat{H}$ defined by

$$\hat{l}(\hat{x}, \hat{y}) := l(x, y), \ x, y \in H.$$

Then $\hat{l}$ is a scalar product, so that $\widehat{H}$ is a pre-Hilbert space. Let K be the completion of H. Let $\widehat{T}(s) : \widehat{H} \to \widehat{H}$ be defined as before, namely by $\widehat{T}(s)\hat{x} := \widehat{T(s)x}, x \in H$, and let $V(s)$ be the continuous extension of $\widehat{T}(s)$ from $\widehat{H}$ to K. As before, V is a strongly continuous representation of S by isometric operators on the Hilbert space

K. If $T(s)$ have dense range, then $V(s)$ also have dense range, so that $V(s)$ are invertible isometries, i.e. unitary operators, on K. In general, the construction in **[D]** shows that there exists a unitary representation U of G on a Hilbert space $\mathcal{K} \supset K$ such that $V(s) = U(s)|K, s \in S$. Moreover, each operator $\mathbf{a}$ in the commutant of $V(s)$ has an unique extension to an operator $\mathbf{b}$ in the commutant of $U(s)$ such that $\|\mathbf{a}\| = \|\mathbf{b}\|$. This implies that, since the algebra $\mathcal{A}(U)$ is semisimple (by the Gelfand-Naimark Theorem), $\mathcal{A}(V)$ also is semisimple.

Every operator A on H, which commutes with $\widehat{T}(s)$, generates a corresponding operator $\widehat{A}$ on $\widehat{H}$ by $\widehat{A}\hat{x} := \widehat{Ax}$. Since

$$\|\widehat{A}\hat{x}\|^2_{\widehat{H}} = l(Ax, Ax) = glim_S\|T(s)Ax\|^2 \leq \|A\|glim_S\|T(s)x\| = \|A\|\|\hat{x}\|_{\widehat{H}},$$

it follows that

$$(**) \qquad \text{if } a \in \mathcal{R}(\mathcal{A}(T)) \text{ then } \hat{a} \in \mathcal{R}(\mathcal{A}(V)).$$

Theorem 4. *Assume that $T : S \to L(H)$ is a bounded representation such that $T(s)x$ does not converge strongly to 0, for all $x \neq 0$. Then the algebra $\mathcal{A}(T)$ is semisimple.*

Proof. Assuming the contrary, there exists an element $a \in \mathcal{R}(\mathcal{A}(T)), a \neq 0$. By (**), $\hat{a} \in \mathcal{R}(\mathcal{A}(V))$, hence $\hat{a} = 0$, i.e. $glim_S\|T(s)ax\| = 0$. This implies that $\lim_S \|T(s)ax\| = 0$, which is a contradiction. □

Theorems 3 and 4 imply the following corollaries.

Corollary 5. *Assume that $T : S \to L(X)$ is a bounded representation on a Banach space X such that $Sp_u(T)$ is a S_3-set and assume that there exists $x \in X$ such that $T(s)x$ does not converge to 0. If the Banach algebra $\mathcal{A}(T)$ is not semisimple, then T has a nontrivial hyperinvariant subspace.*

A power-bounded operator T is called an *operator of class $C_{1\cdot}$, if there exists $x \in X$ such that $T^n x$ does not converge to 0*

Corollary 8. *If T is a power-bounded operator of class $C_{1\cdot}$ on a Banach space X, such that $\sigma(T) \cap \partial\mathbf{D}$ is a S_3-set and assume that the Banach algebra generated by T is not semisimple, then T has a nontrivial hyperinvariant subspace.*

Corollary 7. *Assume that $T : S \to L(X)$ is a bounded representation on Hilbert space H and there exists $x \in X$ such that $T(s)x$ does not converge to 0. If the Banach algebra $\mathcal{A}(T)$ is not semisimple, then T has a nontrivial hyperinvariant subspace.*

Corollary 8. *If T is a power-bounded operator on H of class $C_{1\cdot}$ such that the Banach algebra generated by T is not semisimple, then T has a nontrivial hyperinvariant subspace.*

We remark that the question of whether every contraction of class $C_{1\cdot}$ on Hilbert space has a nontrivial invariant subspace is still open (see **[B]** for partial results concerning this problem). Corollary 8 indicates that in the study of this problem we can assume that the Banach algebra generated by the operator is semisimple.

4. Semisimplicity, stability.

A representation $T : S \to L(X)$ of the abelian semigroup S is called *stable*, if $\lim_S \|T(s)x\| = 0, \forall x \in S$. Such representations were studied in **[L-V]**$_1$, **[B-V]**. For a bounded representation $T : S \to L(X)$, we let

$$M_0 = \bigcup\{ran(\hat{f}(T)) : f \in L^1(S), \hat{f}(\chi) = 0 \quad \forall \chi \in Sp_u(T)\},$$

$$M_1 = \bigcup\{ran(A) : A \in \mathcal{R}(\mathcal{A}(T))\}$$

and $M = \overline{span}(M_0, M_1)$ (here $ran(A)$ denotes the range of A).

Theorem 5. *Assume that $Sp_u(T)$ is a S_3-set. Then $\lim_S \|T(s)x\| = 0$ for every $x \in M$. In particular, if $M = X$ then the representation $T(s)$ is stable.*

Proof. Assume that $x = \hat{f}(T)y$ where $f \in L^1(S), \hat{f}(\chi) = 0 \ \forall \chi \in Sp_u(T)$. Let V be the corresponding limit isometric representation, and U the corresponding representation of the group G on E, as constructed in Section 3. Then $Sp(U) \subset Sp_u(T)$. Hence $\mathcal{A}(U)$ is semisimple. By the Spectral Mapping Theorem (see **[L-M-F]**), $\sigma(\hat{f}(U)) = \{\hat{f}(\chi) : \chi \in Sp(U)\}$, which implies that $\hat{f}(U) = 0$. Hence $\hat{f}(V) = 0$, which implies that $\lim_S \|T(s)\hat{f}(T)y\| = 0, \forall y \in X$. Thus, $\lim_S \|T(s)x\| = 0 \ \forall x \in M_0$.

Next, assume that $x = Ay$, where $A \in \mathcal{R}(\mathcal{A}(T))$. Since $\widehat{A} \in \mathcal{R}(\mathcal{A}(V))$ and $\mathcal{A}(V)$ is semisimple, we have $\widehat{A} = 0$, which implies $\lim_S \|T(s)Ay\| = 0$. Hence, $\lim_S \|T(s)x\| = 0, \forall x \in M$. □

For bounded representations on a Hilbert space the spectral assumption in Theorem 5 is not needed.

Theorem 6. *Let $T(s)$ be a bounded representation of the semigroup S on a Hilbert space H. Then $\lim_S \|T(s)x\| = 0$ for all $x \in M$. In particular, if $M = X$ then T is stable.*

Proof. Let V be the limit isometric representation of S on the Hilbert space K. As noted before, the algebra $\mathcal{A}(V)$ is semisimple. The same argument as in the proof of Theorem 5 shows that $glim_S \|T(s)x\| = 0$, hence $\lim_S \|T(s)x\| = 0, \forall x \in M$. □

5. Polynomially bounded operators.

A bounded linear operator T on a Banach space X is called *polynomially bounded*, if there exists a constant $M > 0$ such that

$$\|p(T)\| \leq M \sup_{|z|\leq 1} |p(z)|,$$

for all polynomials p. It is well known that contractions on Hilbert space are polynomially bounded (von Neumann's Inequality), contractions on Banach spaces are, in general, not polynomially bounded, and that there exist polynomially bounded operators on Hilbert space which are not similar to contractions (Pisier **[P]**).

A power-bounded operator T on a Banach space X is said to be an *operator of class* C_1 , if $\|T^n x\|$ does not converge to 0, for all $x \in X, x \neq 0$. It is known **[N-F]** that if T is contraction of class C_1, such that $\sigma(T) \cap \partial\mathbf{D}$ has measure zero, then T is unitary. A closely related result of Nagy and Foias is the stability theorem for contractions on Hilbert space, which states that if T is a completely nonunitary contraction such that $\sigma(T) \cap \partial\mathbf{D}$ has measure zero, then $\|T^n x\| \to 0$ as $n \to \infty$, for all $x \in H$ (see **[N-F]**,**[B]**).

Recently, Kerchy and van Neerven **[K-vN]** have shown that if T is a polynomially bounded operator of class C_1, on a Banach space X, then T is similar to an invertible isometry. However, the stability question was left open in **[K-vN]**. Here, we extend the above mentioned stability theorem of Nagy-Foias to polynomially bounded operators whose spectrum on the unit circle has measure zero, as well as to power-bounded operators whose spectrum on the unit circle is a Helson S-set.

Lemma 3. *If V is a polynomially bounded invertible isometry on a Banach space E, then the algebra $\mathcal{A}(V)$ is isomorphic to $C(\sigma(V))$.*

Proof. It was shown in **[K-vN]** that there is a homomorphism $\varphi : C(\partial\mathbf{D}) \to L(E)$ such that $\|\varphi\| \leq M$, i.e. there is a functional calculus on $C(\partial D)$ which satisfies: $\|f(T)\| \leq M\|f\|_\infty$. Moreover, $f(T)$ is completely determined by its values on $\sigma(V)$, and the spectral mapping theorem holds: $\sigma(f(V)) = f(\sigma(V))$. Therefore, the functional calulus can be defined for $C(\sigma(V))$, and we have

$$\sup_{\lambda\in\sigma(V)} |f(\lambda)| \leq \|f(V)\| \leq M \sup_{\lambda\in\sigma(V)} |f(\lambda)|,$$

i.e. the homomorphism is in fact an isomorphism. □

Now let T be a polynomially bounded operator on a Banach space X. Assume that there exists $x \in X$ such that $\|T^n x\|$ does not converge to 0. Let V be the limit isometric operator, acting on E, and assume that V is invertible (which holds, e.g., if T has a dense range or $\sigma(T)$ does not contain the whole unit circle). Since V also is polynomially bounded, Lemma 3 implies that $\mathcal{A}(V)$ is isomorphic to $C(\sigma(V))$. Then, for each $z \in E, z^* \in E^*$, the mapping $f \mapsto \langle f(V)z, z^*\rangle$ is a continuous linear functional on $C(\sigma(V))$. Hence, by Riesz's Theorem, for every $z \in E, z \in E^*$, there exists a measure μ_{z,z^*} on $\sigma(V)$ such that

$$(***) \qquad \langle f(V)z, z^*\rangle = \int_{\sigma(V)} f(\lambda) d\mu_{z,z^*}(\lambda), \ \forall f \in C(\sigma(V)).$$

Note that, in general, V does not have a spectral measure, i.e., it is not a spectral operator of scalar type in the sense of Dunford **[D-S]**. But formula (***), which resembles the functional calculus for spectral operators of scalar type and holds in our case only for continuous functions f on the spectrum of V, will be one of the main ingredients in the proof of Lemma 4.

Lemma 4. *Assume that T is polynomially bounded, the operator T^* does not have an invariant subspace K such that $T^*|K$ is invertible and $\sup_{n\in\mathbf{Z}} \|(T^*|K)^n\| < \infty$. Then the measures μ_{z,z^*} are absolutely continuous with respect to the Lebesgue measure.*

The proof of Lemma 4 uses the main ideas in the proof of Proposition 2.1 in [**B**], Chapter XII, (in which it was attributed to A. Atzmon). However, there are some essential modifications, and, therefore, we give the proof below.

Proof. Assume that there exist $z \in E, z^* \in E^*$ such that μ_{z,z^*} is not absolutely continuous with respect to the Lebesgue measure m, i.e. there exists a compact set K with $m(K) = 0$ and $\mu_{z,z^*}(K) \neq 0$. For convenience, we denote the duality in (X, X^*) by (x, x^*), and the duality in (E, E^*) by $\langle z, z^*\rangle$.

By Fatou's Theorem, there exists $h \in A(\mathbf{D})$ such that $h|K = 1$ and $|h||\overline{\mathbf{D}}\backslash K < 1$. Let $\tilde{h}(z) := \overline{h(\bar{z})}$. Then $\tilde{h} \in A(\mathbf{D}), \|\tilde{h}\| = 1$. Since V^* also is polynomially bounded, $\tilde{h}^n(V^*)$ is defined and

$$\sup_{n\geq 0} \|\tilde{h}^n(V^*)\| \leq M < \infty.$$

From weak* compactness of the unit ball in E^*, there exists a subsequence n_k such that $\tilde{h}^{n_k}(V^*)z^* \to z_0^*$ in the (E^*, E)-topology. Let $x^*, x_0^* \in X^*$ be defined by: $(x, x^*) := \langle \hat{x}, z^*\rangle$, and $(x, x_0^*) := \langle \hat{x}, z_0^*\rangle$, $x \in X$. Then, for every $x \in X$,

$$(x, \tilde{h}^{n_k}(T^*)x^*) = (h^{n_k}(T)x, x^*) = \langle (\widehat{h^{n_k}(T)}x), z^*\rangle = \langle h^{n_k}(V)\hat{x}, z^*\rangle = \langle \hat{x}, \tilde{h}^{n_k}(V^*)z^*\rangle.$$

Therefore,

$$\lim_{k\to\infty} (x, \tilde{h}^{n_k}(T^*)x^*) = \lim_{k\to\infty} \langle \hat{x}, \tilde{h}^{n_k}(V^*)z^*\rangle = \langle \hat{x}, z_0^*\rangle = (x, x_0^*),$$

i.e. $\tilde{h}^{n_k}(T^*)x^*$ converges to x_0^* in the (X, X^*)-topology. Now we have, by adopting (***) and the Dominated Convergence Theorem,,

$$(y, x_0^*) = \lim_{k\to\infty} (y, \tilde{h}^{n_k}(T^*)x^*) = \lim_{k\to\infty} (h^{n_k}(T)y, x_0^*) = \lim_{k\to\infty} \langle h^{n_k}(V)\hat{y}, z^*\rangle = \lim_{k\to\infty} \int_{-\pi}^{\pi} h^{n_k}(e^{i\lambda}) d\mu_{\hat{y},z^*}(\lambda) = \mu_{\hat{y},z^*}(K).$$

Since $\mu_{z,z^*}(K) \neq 0$, and $\widehat{X}$ is dense in E, there exists $\hat{y}$ such that $\mu_{\hat{y},z^*}(K) \neq 0$, so that $x_0^* \neq 0$.

By Rudin-Carleson Theorem, there exists a fucntion $\phi \in A(\mathbf{D})$ such that

$$\phi(e^{i\lambda}) = e^{-i\lambda} \text{ for } \lambda \in K \text{ and } \|\phi\|_\infty = 1.$$

We show that

$$(****) \qquad T^*\phi(T^*)x_0^* = x_0^*.$$

Indeed, we have,

$$\begin{aligned}&(y,[I-T^*\phi(T^*)]x_0^*) = ([I-T\phi(T)]y,x_0^*) =\\ &\langle [I-V\phi(V)]\hat{y},z_0^*\rangle = \lim_{k\to\infty}\langle [I-V\phi(V)]\hat{y},\tilde{h}^{n_k}(V^*)z^*\rangle =\\ &\lim_{k\to\infty}\langle h^{n_k}(V)[I-V\phi(V)]\hat{y},z^*\rangle\\ &= \lim_{k\to\infty}\int_{-\pi}^{\pi} h^{n_k}(e^{i\lambda})(1-e^{i\lambda}\phi(e^{i\lambda}))d\mu_{\hat{y},z^*}(\lambda)\\ &= \int_K (1-e^{i\lambda}\phi(e^{i\lambda}))d\mu_{\hat{y},z^*}(\lambda) = 0,\ \forall y\in X,\end{aligned}$$

which implies that (****) holds. Let $W := \phi(T^*)$. Then $\sup_{n\geq 0}\|W^n\| \leq M$, and $(WT^*)^n(T^*)^k x_0^* = (T^*)^k x_0^*, k = 0,1,2,\ldots.$ Let $K := \overline{span}\{(T^*)^k x_0^* : k\geq 0\}$. Then K is invariant subspace for T^*, $T^*|K$ is invertible and $\sup_{n\in\mathbf{Z}}\|(T^*|K)^n\| \leq M$, which is a contradiction. □

¿From Lemma 4 we obtain the following result which is a generalization of the Nagy-Foias Theorem.

Theorem 7. *Let T be a polynomially bounded operator on a Banach space X such that $\sigma(T)\cap\partial\mathbf{D}$ has measure 0. Then the following are equivalent:*

(i) $T^n x\to 0$ for every $x\in X$;

(ii) T^* does not have an invariant subspace K on which $T^*|K$ is invertible and $\sup_{n\in\mathbf{Z}}\|(T^*|K)^n\| < \infty$.

Proof. The condition implies that, if (i) holds, then the limit isometry V is invertible. The implication $(ii)\Rightarrow(i)$ follows from Lemma 4. To show the converse, assume that $K\subset X^*$ is a nonzero subspace which is invariant under T^* and such that $T^*|K$ is invertible and $\sup_{n\in\mathbf{Z}}\|(T^*|K)^n\| < \infty$, and assume that (i) holds. Let $V := T^*|K$. Then, for every $x^*\in K$, $\{V^{-n}x^* : n\geq 0\}$ are uniformly bounded, hence $\langle x,x^*\rangle = \langle x,V^n(V^{-n})x^*\rangle = \langle x,(T^*)^nV^{-n}x^*\rangle == \langle T^nx,V^{-n}x^*\rangle \to 0,\ \forall x\in X$, which is a contradiction. □

In the next result, we only require the operator T to be power-bounded, but we need a stronger condition on the peripheral spectrum of T. We will need the following lemma.

Lemma 5. *Assume that V is an invertible isometry on a Banach space such that $\sigma(V)$ is a Helson S-set. Then the algebra $\mathcal{A}(V)$ is isomorphic to $C(\sigma(V))$.*

Proof. Let W be the Wiener algebra

$$W = \{f(z) = \sum_{n=-\infty}^{\infty} c_n z^n : \sum_{n=-\infty}^{\infty}|c_n| < \infty\}.$$

Define $\varphi : W\to\mathcal{A}(V)$ by

$$\varphi(f) = f(T).$$

Then φ is continuous. Since $\sigma(V)$ is a Helson S-set, the algebra $\mathcal{A}(V)$ is semisimple, so that elements of $\mathcal{A}(V)$ can be regarded as functions on $\sigma(V)$. It follow that the functional calculus is defined for all functions in $C(\sigma(V))$, and the spectral mapping theorem holds, so that $\mathcal{A}(V)$ is isomorphic to $C(\sigma(V))$. □

The following lemma has the same proof as Lemma 4.

Lemma 4. *Assume that T is power-bounded, $\sigma(T) \cap \partial \mathbf{D}$ is a Helson S-set and the operator T^* does not have an invariant subspace K such that $T^*|K$ is invertible and $\sup_{n\in\mathbf{Z}} \|(T^*|K)^n\| < \infty$. Then the measures μ_{z,z^*} are absolutely continuous with respect to the Lebesgue measure.*

Theorem 8. *Assume that T is a power-bounded operator on a Banach space X such that $\sigma(T) \cap \partial \mathbf{D}$ is a Helson S-set. Then $T^n x \to 0$ for all $x \in X$ if and only if T^* does not have an invariant subspace K such that $T^*|K$ is invertible and $\sup_{n\in\mathbf{Z}} \|(T^*|K)^n\| < \infty$.*

Proof. The proof follows from Lemmas 4, 5, the fact that any Helson set has measure 0, and that any continuous function f on a Helson set, which is analytic inside D, is the restriction of a function in W_+, for which $f(T)$ is defined. □

Note that Theorems 7 and 8 are in the same spirit as the stability theorems in **[At]**, **[A-B]**, **[L-V]**$_2$, where the condition that $\sigma(T) \cap \partial \mathbf{D}$ is a Helson S-set is replaced by its countability, and condition (ii) is replaced by the absence of eigenvalues of T^* on the unit circle. Theorems 7 and 8 also give a partial solution to problems 1 and 2 in **[V]**$_2$. As an immediate corollary of Theorems 7 and 8 we obtain the following results on invariant subspaces for operators of class $C_{1\cdot}$.

Corollary 9. *Assume that T is a polynomially bounded operator of class $C_{1\cdot}$, on a Banach space X, such that the spectrum of T on the unit circle has measure zero. Then T has a nontrivial hyperinvariant subspace.*

Corollary 10. *Assume that T is a power-bounded operator of class $C_{1\cdot}$, on a Banach space X, such that the spectrum of T on the unit circle is a Helson S-set. Then T has a nontrivial hyperinvariant subspace.*

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