ON PERIODIC AND APERIODIC TILINGS

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Prépublication de l'Institut Fourier 489 (1999)

http://www-fourier.ujf-grenoble.fr/prepublications.html

Abstract. In this work we briefly discuss the *substitution process*, the most common tool to study aperiodicity, and some well-known results which derive from this concept. Using *k-coronas*, we prove then some theorems about the link between aperiodicity and periodicity. Finally we propose a new approach to the "aperiodicity problem".

Résumé. Dans cet article nous traitons brièvement le *processus de substitution*, l'outil le plus connu pour étudier l'apériodicité, et nous présentons quelques résultats bien connus qui découlent de ce concept. Grâce aux *k-couronnes*, nous démontrons quelques théorèmes reliant la périodicité et l'apériodicité. Enfin nous proposons une nouvelle approche au "problème de l'apériodicité".

Math. Subject Classification 1991 : 05B45, 52C20.
Keywords: Tiling, Substitution Tiling.

1. Introduction

The existence of a polygon tiling the plane only non-periodically is an old and famous open problem of two dimensional geometry. So far the most interesting and well-known result is the set of *kites and darts* and the analogous set of "arrowed rhombuses" of Penrose [17]. The discovery of *quasi-crystals* gave great importance to the study of *aperiodic* tilings and their properties: the physical and logical implications of the aspects of tilings brought so much activity of research that, today, the original question (the existence of an *aperiodic* polygon) appears as one among several questions about tilings, for which mathematical instruments are still "rough".

Inflation, *substitution* and *composition* ([1], [13], [15], [25]) are usual terms in the study of aperiodicity: they denote the same concept, namely, the existence of a "substitution process" (see section 3). In this work we start from some well-known results following from this approach, then, with the use of *k-coronas*, we show how the substitution process is a fundamental instrument to understand and analyze aperiodicity and its links with periodicity.

The paper is outlined as follows: in section 2 we give basic notions, in section 3 and 4 we present the concept of substitution and its connections with periodicity and aperiodicity; in section 5 we prove the main Theorems, and in section 6 we propose a new approach (*generalized matching rules*) to the "aperiodicity problem" Also, some physical consequences and possible applications are considered paying specific attention to the problem of partial order in globally disordered real structures. Finally, in the light of the proved results, new problems and research paths in the study of tilings are proposed.

2. Basic definitions and notations

Definition 1 *A ball with center x and radius R (we say $B(x,R)$) in a Euclidean space $\mathbb{E}^d$ is the set $\{p \in \mathbb{E}^d \mid d(p,x) < R\}$: a closed ball is the set $\{p \in \mathbb{E}^d \mid d(p,x) \leq R\}$.*

Definition 2 *A tiling T of the space $\mathbb{E}^d$ is a representation of $\mathbb{E}^d$ as countable union of "tiles" $\{F_1, F_2, \dots\}$ where:*

1. *there is a finite set of polygons, called protoset, $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ of* prototiles, *not congruent to each other;*
2. *each tile F_i is a congruent copy (obtained by an orientation-preserving isometry) of some prototile;*
3. *the tiles cover $\mathbb{E}^d$ without holes or overlappings, i.e. $intF_i \cap intF_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{\infty} F_i = \mathbb{E}^d$;*
4. *the tiles match edge to edge;*

we say that $\mathcal{P}$ admits T. If $\mathcal{P}$ is composed of a single element, then T is called monohedral.

Definition 3 *The* symmetry group *of a tiling T is the set of the isometries of $\mathbb{E}^d$ which preserve T.*

Definition 4 *A tiling of $\mathbb{E}^d$ is called periodic if its symmetry group contains a subgroup of translations isomorphic to $\mathbb{Z}^d$.*

Definition 5 *A tiling T which admits k independent translations, where $1 \leq k < d$, is sub-periodic. A tiling T is called non-periodic if its symmetry group does not contain translations.*

Definition 6 *A protoset is called* aperiodic *if it admits* ***only*** *non-periodic tilings: in turn, tilings admitted by aperiodic protosets are called aperiodic.*

In the literature, given a finite protoset $\mathcal{P} = (P_1, P_2, \ldots, P_n)$, the "matching constraints" to arrange together the prototiles P_i are often called **matching rules** ($\mathcal{R}$). We can also say that the matching rules "reduce" the tilings admitted by a given protoset. The constraints can be given via deformation of the tiles, via "admitted atlases" and so on: for example, in figure 1 a), the matching rules (the arrows) are such that the only admitted tilings are non-periodic. With an abuse of notation we call protoset the pair ($\mathcal{P}$, $\mathcal{R}$). $\mathcal{T}(\mathcal{P}, \mathcal{R})$ will denote the family of all (congruence classes of) tilings admitted by ($\mathcal{P}$, $\mathcal{R}$).

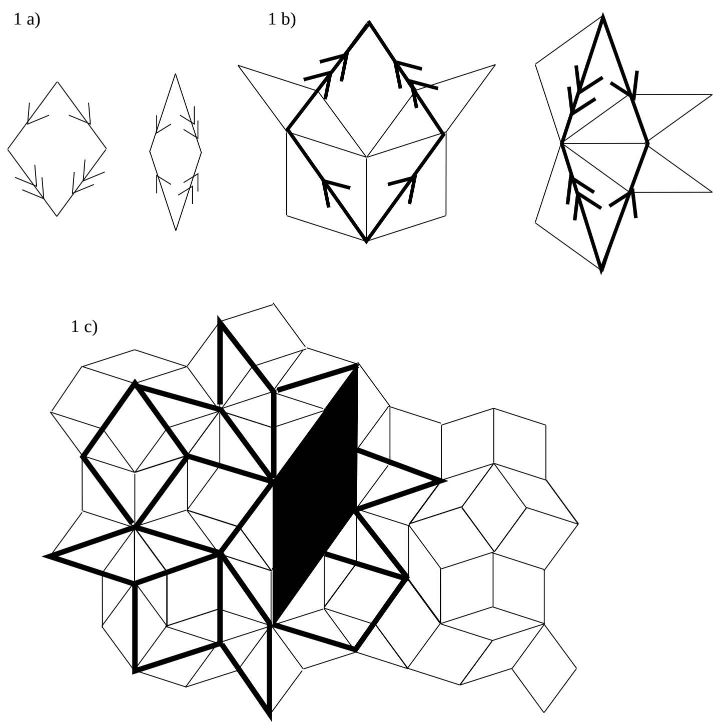

Figure 1: Matching rules for Penrose's tilings, the substitution process for arrowed rhombuses and a partial tiling where the substitution process is put in evidence

Definition 7 *Let ($\mathcal{P}$, $\mathcal{R}$) be a finite protoset and T_1, T_2 two tilings in $\mathcal{T}(\mathcal{P}, \mathcal{R})$: T_1 and T_2 are* ***locally isomorphic*** *if every bounded configuration which appears* *in one of them appears also in the other. If all the tilings in $\mathcal{T}(\mathcal{P},\mathcal{R})$ are locally isomorphic, we say that $\mathcal{T}(\mathcal{P},\mathcal{R})$ has the property of local isomorphism.*

Definition 8 *A configuration in a tiling T is a finite set of tiles in T; a configuration $\mathcal{C}$ of radius r in T is a configuration $\mathcal{C}$ that covers a ball of radius r and such that every configuration $\mathcal{D} \subset \mathcal{C}$ does not cover this ball.*

Definition 9 *A tiling $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$ is* ***repetitive*** *if every bounded configuration is relatively dense in the tiling, i.e., if for every configuration $\mathcal{C}$ of radius r in T there exists a radius R (depending on $\mathcal{C}$) such that for any point x in the plane there exists $\mathcal{C}' \subset B(x,R)$ with $\mathcal{C}'$ congruent to $\mathcal{C}$.*

A tiling T_1 is obtained by a *composition process* from a tiling T_2, if each tile of T_1 is a union of tiles of T_2. If every tile of T_1 is a union of k-tiles of T_2, then T_1 is called *k-composition* of T_2.

In particular, a tiling T_1 is called a *similarity* tiling if it is "self-composed", i.e. if there exists a similitude that moves T_1 onto T_2, where T_2 is a k-composition of T_1.

A monohedral tiling T is a *k-similarity* tiling if T is equal to a k-composition of itself, and no smaller value of $k > 1$ has this property. To obtain a k-similarity tiling, for a fixed k, a useful method is to study k rep-tiles ([11], [12]). A *k rep-tile* is a tile F which it can be dissected into k congruent parts F', each of which is similar to F (there exists an analogous for polyhedral tilings, the *croc of tiles* [5]).

Definition 10 *Given $k \in \mathbb{N}$ and a tile F of a tiling T, the* ***k-corona*** *of F is the set*

$$C^k(F) = \{F' \mid \|F', F\| \leq k\}$$

where $\|F, F'\|$ indicates the combinatorial length of the shortest chain connecting F to F'; otherwise stated $C^k(F)$ consists in all tiles which have a common vertex with a tile of $C^{k-1}(F)$ ($C^0(F) = F$ by convention).

From now on, for simplicity, we will refer particularly to the bidimensional case.

3. The substitution process and aperiodicity

It is useful to recall the basic theorem in the study of aperiodicity: even if it is proved in [13] and [25], we provide a detailed proof, because it is useful for the continuation.

Theorem 1 *Let T be a monohedral k-similarity tiling with unique k-composition process. Then T is non-periodic.*

Proof: By hypothesis, T determines a unique $k-$composition tiling, say T^1. If τ is an element of the symmetry group of T, i.e. $\tau \in \Gamma(T)$, then both T^1 and $\tau(T^1)$ are two k-composed tilings of T. Hence $\tau(T^1) = T^1$, i.e. $\tau \in \Gamma(T^1)$. Suppose that there exists a translation $\overrightarrow{u} \in \Gamma(T)$ that moves the tile $F \in T$ onto F'; it follows that $\overrightarrow{u} \in \Gamma(T^1)$. We can iterate the composition process to obtain a sequence $T^1, T^2, \ldots$

where T^j is the unique k-composition tiling determined by T^{j-1}; $\overrightarrow{u} \in \Gamma(T^p)$ for every $p \in \mathbb{N}$. This process leads to tilings of arbitrarily large tiles; then there exists a value l such that the translation vector $\overrightarrow{u}$ lies entirely inside a tile "of level l": this is absurd, because $\overrightarrow{u} \in \Gamma(T^l)$. □

Notice that in the proof of the theorem we use the fact, implicit in the definition of k-similarity, that the composition process can be iterated arbitrarily. Such a property does not necessarily hold if the k-composition tiling of the original tiling T is not similar to T.

Remark: For details and proofs of the following statements see [13] and [25]. Suppose we have a tiling T built by arrowed rhombuses $\{r_1, r_2\}$ (figure 1 a)); we can mark locally the tiling as in figure 1 b). One can show that

- we obtain a well defined marking on the tiling
- if we delete the original tiles and we look at the marking as edges, vertices and arrows of a new tiling, we obtain a tiling T' built by arrowed rhombuses $\{R_1, R_2\}$ (figure 1 c)), similar to the original tiles (the ratio is $\tau : 1$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the famous *golden number*);
- this process can be iterated arbritarely;
- the tiling T' that we have obtained is the unique tiling built by $\{R_1, R_2\}$ that has the set of vertices contained in the set of vertices of T.

As in the proof of Theorem 1, one can show that T is non periodic; since this is true for all tilings admitted by $\{r_1, r_2\}$, this protoset is aperiodic.

Generally, take a tiling T admitted by a certain finite protoset $(\mathcal{P}, \mathcal{R})$ with $\mathcal{P} = \{P_1, \ldots, P_n\}$. Suppose that we can mark the tiles in such a way that the following properties hold:

1. if we delete the original tiles and we look at the marking as edges and vertices of a new tiling, we obtain a tiling T' admitted by $(\mathcal{P}', \mathcal{R}')$ where $\mathcal{P}' = \{P'_1, \ldots, P'_n\}$, P'_i similar to P_i with a fixed similarity ratio and $\mathcal{R} = \mathcal{R}'$;
2. the above process can be iterated arbritarely.

Then T is a *substitution tiling.* The tiles of the original tiling are *tiles of level 0*, the tiles obtained after the first composition are *tiles of level 1* and so on. If we have also the property that:

3. the tiling T' that we have obtained is the unique tiling admitted by $(\mathcal{P}', \mathcal{R}')$ that has the set of vertices contained in the set of vertices of T,

then T is non periodic. Moreover, if all the tilings allowed by $(\mathcal{P}, \mathcal{R})$ can be marked as T and they are substitution tilings in respect of this marking, then we say that $(\mathcal{P}, \mathcal{R})$ admits a *substitution process.* If all the tilings have property 3., the substitution process is unique and $(\mathcal{P}, \mathcal{R})$ is aperiodic.

This technique is frequently used to show that a given protoset is aperiodic: indeed, to our knowledge, whenever protosets containing a small number of tiles are adopted, the above procedure was used to show aperiodicity ([1], [13], [25], [2], [3], [7]).

Remark: All the known aperiodic protosets have the property that the associated substitution process selects one particular way to build the tiles of level $(i+1)$ from tiles of level (i) independently of i. In the sequel we require that the substitution process has always this property.

4. The substitution process and periodicity

We introduce *substitution matrices* [25]. Suppose that T is a substitution tiling. Let $P_1^0, P_2^0, \ldots, P_k^0$ be the prototiles of T (in other words, the congruence classes of the tiles in T) and let $P_1^i, P_2^i, \ldots, P_k^i$ be the corresponding prototiles at level i. By the last assumption of section 3, each tile, congruent to some P_j^i, in T (as composition of tiles of T) can be decomposed into tiles of level $i-1$ and the decomposition is independent of i. Then, a *substitution matrix* $\mathcal{U} = (u_{ij})$ can be associated to the composition process (with a hopefully comprehensive abuse of notation), such that:

$$\begin{aligned} P_1^1 &= u_{1\,1}P_1 + u_{2\,1}P_2 + \cdots + u_{k\,1}P_k \\ P_2^1 &= u_{1\,2}P_1 + u_{2\,2}P_2 + \cdots + u_{k\,2}P_k \\ &\vdots \\ P_k^1 &= u_{1\,k}P_1 + u_{2\,k}P_2 + \cdots + u_{k\,k}P_k \end{aligned}$$

where the coefficients $u_{i\,j}$ are nonnegative integers describing the "multiplicity" of the tile P_i^0 in P_j^1.

The matrix associated to a substitution tiling T is called the *substitution matrix* of T. Let $(\mathcal{P}, \mathcal{R})$ be a protoset that admits a substitution process: then the matrix associated to the substitution process of $(\mathcal{P}, \mathcal{R})$ is called *substitution matrix of* $(\mathcal{P}, \mathcal{R})$.

Since now on we require that the matrix $\mathcal{U} = (u_{i\,j})$, which represents the substitution tiling, is *primitive*, i.e. all entries of $\mathcal{U}$ are nonnegative integers and, for some $i \in N$, all entries of $\mathcal{U}^i$ are positive: this last property means that, in each tile of level i, at least one copy of each original prototile must be present.

The requirement of the previous remark that the substitution process selects one particular way to build the tiles of level $(i+1)$ from tiles of level (i) independently of i does not mean that the composition is unique: for example, the usual "chessboard tiling" T of congruent squares admits four possible tilings (all isomorphic) of congruent squares that are 4-composition of T but T is the 4-composition of only one tiling of congruent squares. Thus, given a protoset $(\mathcal{P}, \mathcal{R})$ and the pertinent substitution process, to each tiling of level $i+1$ (i.e. composed by tiles of level $i+1$) can be associated only one tiling of level i. If such a relation is biunivocal, i.e. if we can associate to each tiling of level i only one tiling of level $i+1$, then we have shown that $(\mathcal{P}, \mathcal{R})$ is aperiodic.

However, the very existence of a substitution process (and then of a substitution matrix) leads to the following lemmas (for a proof, see [25]):

Lemma 1 *Given a protoset* $(\mathcal{P}, \mathcal{R})$ *and its substitution matrix* $\mathcal{U}$*, all tilings admitted by* $(\mathcal{P}, \mathcal{R})$ *and* $\mathcal{U}$ *are locally isomorphic.*

Lemma 2 *Any tiling admitted by* $(\mathcal{P}, \mathcal{R})$ *and* $\mathcal{U}$ *is repetitive.*

Notice that also periodic tilings can belong to a substitution process. In figure 2 we show a rhombus; if its sides are marked as in figure it belongs to a substitution process. Thus only one tiling is obtained, which is periodic.

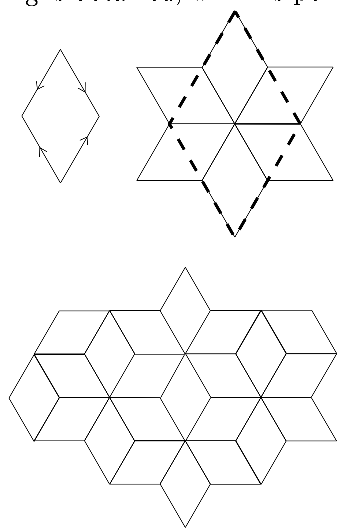

Figure 2: Substitution process for a rhombus with acute angle of $\pi/3$, corresponding marking of rhombus sides and a portion of the obtained periodic tiling

Given a substitution matrix associated to a substitution process, it can be difficult [20] or even impossible (*Fibonacci strings*, [13], [25]) to find a protoset and a set of matching rules such that all admitted tilings belong to the substitution process.

5. The main Theorems

We have shown some properties of a substitution process and that periodic as well as non-periodic tilings can belong to a substitution process. The following material allows to establish a clear link between periodicity and aperiodicity. We will use some techniques and notations introduced by Dolbilin [8].

Given a finite protoset $(\mathcal{P}, \mathcal{R})$, let T be a tiling in $\mathcal{T}(\mathcal{P}, \mathcal{R})$. Let us consider pairs $(F, C^k(F))$, where $C^k(F)$ is the k-corona surrounding a tile $F \in T$. Two pairs $(F, C^k(F))$ and $(F', C^k(F'))$ are *equivalent* if a suitable isometry of the plane moves F onto F' and $C^k(F)$ onto $C^k(F')$. Two different tiles F and F' may have the same k-corona without being equivalent. However, from now on, we say that "$C^k(F)$ and $C^k(F')$ are equivalent" ($C^k(F) \sim C^k(F')$) when the corresponding pairs, $(F, C^k(F))$ and $(F', C^k(F'))$ are equivalent.

Given a tiling $T \in \mathcal{T}(\mathcal{P}, \mathcal{R})$ and $k \in \mathbb{N}$, let

$$\mathcal{C}^k(T) = \{C^k(F) \,|\, F \in T\}/\sim$$

denote the set of all equivalence classes of k-coronas C^k in T. This set is clearly finite for all k; with a little abuse of notation we denote $\mathcal{C}^k$ an element of $\mathcal{C}^k(T)$.

Similarly, consider the set of equivalence classes of k-coronas belonging to any of the tilings in the family $\mathcal{T}(\mathcal{P},\mathcal{R})$. Denote this (finite) set by

$$\mathcal{C}^k_{\mathcal{T}} = \bigcup_{T\in\mathcal{T}(\mathcal{P},\mathcal{R})} \mathcal{C}^k(T)\,.$$

Definition 11 *Given a tiling $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$, an equivalence class $\mathcal{C}^k$ is* stable *(with respect to T) if $C^k(F) \sim C^k(F') \in \mathcal{C}^k$ implies $C^r(F) \sim C^r(F')$ (namely, they belong to the same class of r-coronas) for all $r > k$.*

For the proof of the following Lemmas see [8].

Lemma 3 *Let $F, F' \in T$, be tiles of a tiling $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$ such that $C^k(F) \sim C^k(F') \in \mathcal{C}^k$, where $\mathcal{C}^k$ is a stable class with respect to T. Then there exists an isometry τ of the plane such that $\tau(F) = F'$ and $\tau(T) = T$, i.e. $\tau \in \Gamma_T$ (the symmetry group of T) .*

Remark: Note that the equivalence class of a given k-corona may be stable in some tilings and unstable in others. One can hence only speak of stability with respect to a given tiling.

For a fixed $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$ and two natural numbers k, l with $k < l$ we write $\mathcal{C}^k \prec \mathcal{C}^l$ if $F \in T$ exists such that $C^k(F) \in \mathcal{C}^k$ and $C^l(F) \in \mathcal{C}^l$. The relation $\prec$ introduces a partial order on the set of all classes $\bigcup_{k=0}^{\infty} \mathcal{C}^k(T)$. Then we can introduce an infinite graph $\mathcal{G}_T$. The vertices are $\bigcup_{k=0}^{\infty} \mathcal{C}^k(T)$. Two vertices $\mathcal{C}^k$, $\mathcal{C}^l$ define an *edge* iff $l = k+1$ and $\mathcal{C}^k \prec \mathcal{C}^l$. For every $\mathcal{C}^l$, there exists exactly one predecessor $\mathcal{C}^{l-1}$ such that $\mathcal{C}^{l-1} \prec \mathcal{C}^l$. In particular $\mathcal{G}_T$ is a finite forest consisting in a finite number of infinite trees, each tree having its root in the protoset $\mathcal{P}$ (0-coronas).

Remark: The tree below any stable class is an infinite chain without branchings.

Lemma 4 *Suppose that $\mathcal{T}(\mathcal{P},\mathcal{R})$ is at most countable (namely, there is only a countable number of congruence classes of tilings). For every $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$ and for every class $\mathcal{C}^k \in \mathcal{C}^k(T)$ there exists a stable class $\mathcal{C}^s \in \mathcal{C}^s(T)$ such that $\mathcal{C}^k \prec \mathcal{C}^s$.*

Remark: By Bieberbach's theorem (for a proof, see [26]), the periodicity of a tiling T means that the group Γ_T of isomorphisms of T partitions T into finitely many Γ_T-equivalence classes (we say that Γ_T possesses a *compact fundamental domain*).

Lemma 5 *Let $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$ be a periodic tiling such that the action of Γ_T has m orbits. Then there exists $k \leq m-1$ such that the k-corona $C^k(F) \subset T$, of any tile F in T, contains a fundamental domain of Γ_T.*

Lemma 6 *Given a stable class $\mathcal{C}^s$ of a non-periodic tiling $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$ there exists a tiling $T' \in \mathcal{T}(\mathcal{P},\mathcal{R})$, such that:*

1. $\mathcal{C}^s \notin \mathcal{C}^s(T')$

2. if $\mathcal{C}^k \in \mathcal{C}^k(T')$, then $\mathcal{C}^k \in \mathcal{C}^k(T)$.

Remark: These results show that k-coronas are useful for studying tilings: we remark that the property of local isomorphism for $\mathcal{T}(\mathcal{P},\mathcal{R})$ is equivalent to the requirement that all the tilings in $\mathcal{T}(\mathcal{P},\mathcal{R})$ are associated to the same graph, i.e. $\mathcal{G}_T = \mathcal{G}_{T'}$, for every couple $T, T' \in \mathcal{T}(\mathcal{P},\mathcal{R})$. We can also rewrite Definition 8 with respect to k-coronas; a tiling $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$ is **repetitive** if for any couple of tiles $F, F' \in T$ and for any integer $l \in \mathbb{N}$ there exists a number $m \in \mathbb{N}$ such that $C^{l+m}(F') \supset C^l(F'')$, where $C^l(F'')$ is a l-corona equivalent to $C^l(F)$.

Now we can state and prove the Main Theorems of this section.

Theorem 2 ("Periodicity" Theorem) *Let $\mathcal{T}(\mathcal{P},\mathcal{R})$ have the property of local isomorphism. If $\mathcal{T}(\mathcal{P},\mathcal{R})$ contains a periodic tiling T then all tilings of $\mathcal{T}(\mathcal{P},\mathcal{R})$ are periodic and isomorphic to T.*

Proof: Since T is periodic, there exists only a finite number m of Γ_T-equivalence classes. The graph $\mathcal{G}_T$, at the vertices of level-0 (i.e., the 0-coronas) consists of n_0 elements: if $n_0 = m$ then all the 0-coronas (the tiles) are stable. Otherwise, we examine the 1-coronas: if $n_1 = m$ all the 1-coronas are stable (some tiles, corresponding to vertices of level-0, are perhaps already stable, but this does not matter). The process can be iterated: by Lemma 5, for any tile F, in $C^{m-1}(F)$ there are all the m equivalence classes, thereby a finite level, say h, of $\mathcal{G}_T$ exists, such that $n_h = m$, i.e., all the h-coronas of T are stable. The finite number of possible chains $\mathcal{W}$ implies that **every** $\mathcal{W}$ represents T: since $\mathcal{G}_T$ is the same for all tilings in $\mathcal{T}(\mathcal{P},\mathcal{R})$ (see the precedent remark), it follows that T is the only element in $\mathcal{T}(\mathcal{P},\mathcal{R})$, modulo congruence. □

Theorem 3 ("Aperiodicity" Theorem) *Let $\mathcal{T}(\mathcal{P},\mathcal{R})$ have the property of local isomorphism. If all tilings in $\mathcal{T}(\mathcal{P},\mathcal{R})$ are non periodic then $\mathcal{T}(\mathcal{P},\mathcal{R})$ is uncountable.*

Proof: If $\mathcal{T}(\mathcal{P},\mathcal{R})$ is only countable, we can apply Lemma 4 to a tiling $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$ and conclude that there is some stable class $\mathcal{C}^s \in \mathcal{C}^s(T)$. By Lemma 6, a tiling $T' \in \mathcal{T}(\mathcal{P},\mathcal{R})$ exists, such that $\mathcal{C}^s \notin \mathcal{C}^s(T')$, but this result contradicts the assumption that $\mathcal{G}_T = \mathcal{G}_{T'}$. □

In the bidimensional case we have an interesting result [13]. [1]

Theorem 4 *Let be $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$ sub-periodic; then there is at least one periodic tiling $T' \in \mathcal{T}(\mathcal{P},\mathcal{R})$.*

Thus, in the bidimensional case, we can give a "stronger form" of Theorem 3:

[1] Indeed the statement in [13] is quite different, namely: *Let T be a tiling sub-periodic with protoset $P = (P_1, P_2, \ldots, P_n)$, where the P_i are polygons; then there is also a periodic tiling with protoset $P = (P_1, P_2, \ldots, P_n)$.* This is no problem, because we can suppose that the matching rules are always given via "deformation" [25]; then we have a bigger protoset $\mathcal{F}$, that we can suppose to be composed of polygons, without matching rules, equivalent to the original protoset $\mathcal{P}$ with matching rules $\mathcal{R}$. Namely, $\mathcal{T}(\mathcal{P},\mathcal{R}) = \mathcal{T}(\mathcal{F})$.

Theorem 5 *If $\mathcal{T}(\mathcal{P},\mathcal{R})$ has the property of local isomorphism and there is a non periodic tiling $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$ then $\mathcal{T}(\mathcal{P},\mathcal{R})$ is aperiodic and uncountable.*

Proof: Suppose that $T' \in \mathcal{T}(\mathcal{P},\mathcal{R})$ is a sub-periodic tiling. By Theorem 4 the existence of a sub-periodic tiling in $\mathcal{T}(\mathcal{P},\mathcal{R})$ implies the existence of a periodic tiling $T'' \in \mathcal{T}(\mathcal{P},\mathcal{R})$. By Theorem 2, the tiling T is periodic, which is absurd. $\mathcal{T}(\mathcal{P},\mathcal{R})$ is hence aperiodic and we can apply Theorem 3. □

Remark: The Theorems 2, 3, 5 and the interpretations of local isomorphism and repetitivity in terms of k-coronas are original. Indeed the previous Lemmas are used in [8] to show that:

Theorem 6 *If $\mathcal{T}(\mathcal{P},\mathcal{R})$ is at most countable (and non empty), then there exists at least one periodic tiling $T \in \mathcal{T}(\mathcal{P},\mathcal{R})$.*

6. Generalized matching rules

Given the above results, a new definition of matching rules can be formulated.

Definition 12 *Given a finite protoset*

$$P = (P_1, P_2, \dots, P_n)\,,$$

we define ***generalized matching rules*** *($\mathcal{RG}$) a set of matching rules for the P_i such that $(\mathcal{P},\mathcal{RG})$ admits a substitution process (and then a substitution matrix with the properties enumerated in section 4).*

Now, in the bidimensional case, given a protoset $(\mathcal{P},\mathcal{RG})$, Lemma 1 and Theorems 2 and 5 show that only two cases are possible:

- $\mathcal{T}(\mathcal{P},\mathcal{RG})$ contains a unique periodic tiling;
- $\mathcal{T}(\mathcal{P},\mathcal{RG})$ is uncountable, aperiodic and all the admitted tilings are locally isomorphic.

Thus periodicity is a "special" case of aperiodicity: in the periodic case local equivalences (local isomorphism) imply global equivalence (Theorem 2).

The substitution process appears to have interesting physical implications.

- The error is intrinsic to aperiodic [9]: otherwise, given a protoset which can tile the plane (or the space), we can always build configurations, complying with the matching rules, that cannot be extended. Penrose [18] used the substitution process to show that, for kites and darts, the inability to tile the plane increases (exponentially) with the size of the region covered by tiles.
- It has been shown ([10] and [19]) that the way to generate medium range order (MRO) and long range order (LRO) starting from the **same** set of structural elements which are used to define the short range order (SRO) of a condensed system, is determined by the constraints imposed when combining together such elements (e.g., in the matching scheme). If such constraints,

which correspond to the matching rules defined above, are rigid, a single periodic structure with intrinsic LRO, i.e. a crystal, is generated. If the same constraints admit variability up to a certain extent, a non-periodic structure with a degree of MRO strongly reminiscent of some features of crystalline LRO is obtained. Such a hierarchy has many analogies with the behavior of tilings like those in figure 3 a), where there is a portion of a monohedral tiling, formed by trapeziums generated by the usual tiling with hexagons, cut along a principal diagonal. Such a tiling can be 4-composed in a unique way to give another tiling, in turn monohedral, with trapeziums similar to the initial one. However this last tiling cannot be 4-composed to obtain a monohedral tiling of trapeziums. In general any tiling of this kind admits infinite *successors* (trapeziums can be decomposed into similar trapeziums, and so on, *ad infinitum*), but it is not possible *a priori* to assess the existence of a substitution process. Matching constraints in figure 3 are a *1-coronas atlas* (i.e., the 1-coronas and their reflections are the only 1-coronas admitted): while they do not allow for obtaining exclusively a substitution tiling, corresponding to LRO, yet they are compatible with a composition process, blocked at a given stage (in the example, the first), corresponding to MRO; a local order reminiscent of the global order is generated. Such problems are connected to the concept of *locality* of the matching constraints ([4], [14], [25]).

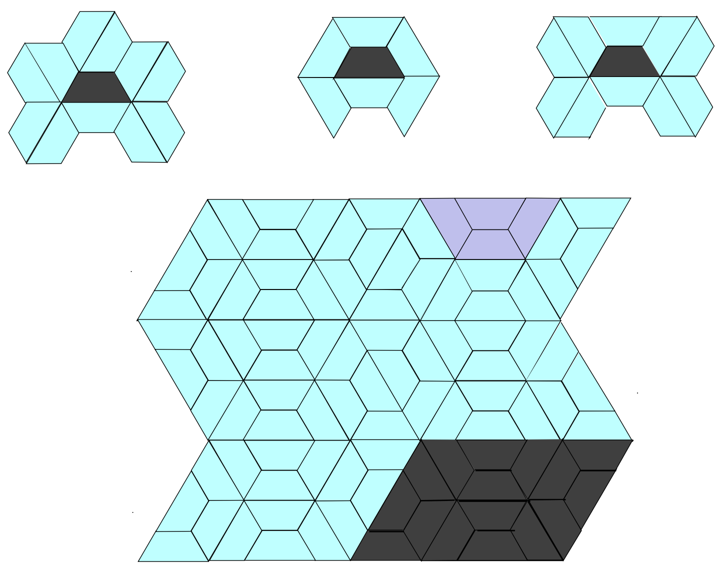

Figure 3: An example of finite substitution process: the dark region is the fundamental domain

7. Concluding remarks

In conclusion, periodicity can be interpreted as a special case of aperiodicity. This formulation upsets the classical statement: indeed, so far, aperiodic sets were considered “pathologic”, or so peculiar that they do not allow any periodic tiling. The present formulation opens new questions:

- are there protosets which admit more "disordered" tilings than those we presently known?
- How is it possible to classify tilings and protosets with respect to their degree of order ([21], [22], [23])? In this respect we mention also the concept of *mutual local derivability* introduced by Baake [2] for a possible classification of quasi-crystals.

It appears interesting to study if the uniqueness of the substitution process is a necessary condition for the non periodicity of all admitted tilings: in other words, is a substitution process leading to non-periodic tilings necessarily unique?

We could also like to mention the *Wang tilings* and their connections with decidability problems and dynamical systems ([13], [6], [16], [24], [27]): a systematic approach to them could give new insights in the analysis of aperiodicity.

ACKNOWLEDGEMENTS: The authors are grateful to Prof. R. Kenyon, Laboratoire de Topologie et de Dynamique, Université de Paris Sud, for a critical reading of a first version of this manuscript.

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