ON ORBIT CLOSURES OF BOREL SUBGROUPS IN SPHERICAL VARIETIES

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Abstract

Let $\mathscr{F}$ be the flag variety of a complex semi-simple group G, let H be an algebraic subgroup of G acting on $\mathscr{F}$ with finitely many orbits, and let V be an H-orbit closure in $\mathscr{F}$. Expanding the cohomology class of V in the basis of Schubert classes defines a union V_0 of Schubert varieties in $\mathscr{F}$ with positive multiplicities. If G is simply-laced, we show that these multiplicites are equal to the same power of 2. For arbitrary G, we show that V_0 is connected in codimension 1. If moreover all multiplicities are 1, we show that the singularities of V are rational, and we construct a flat degeneration of V to V_0. Thus, for any effective line bundle L on $\mathscr{F}$, the restriction map $H^0(\mathscr{F}, L) \to H^0(V, L)$ is surjective, and $H^i(V, L) = 0$ for $i \geqslant 1$.

Introduction

Let X be a spherical variety, that is, X is a normal algebraic variety endowed with an action of a connected reductive group G such that the set of orbits of a Borel subgroup B in X is finite. These B-orbits play an important role in the geometry and topology of X: they define a stratification by products of affine spaces with tori, and the Chow group of X is generated by the classes of their closures. Moreover, the B-orbits in a spherical homogeneous space G/H, viewed as H-orbits in the flag variety G/B, are of importance in representation theory.

The set $\mathscr{B}(X)$ of B-orbit closures in X is partially ordered by inclusion. A weaker order $\preceq$ of $\mathscr{B}(X)$ is defined by: $Y \preceq Y'$ if there exists a sequence $(P_1, \ldots, P_n)$ of subgroups containing B such that $Y' = P_1 \cdots P_n Y$. In this paper, we establish some properties of this weak order and its associated graph, with applications to the geometry of B-orbit closures.

Both orders are well known in the case where X is the flag variety of G. Then $\mathscr{B}(X)$ identifies to the Weyl group W, and the inclusion (resp. weak) order is the Bruhat-Chevalley (resp. left) order, see e.g. [14] 5.8. The B-orbit closures are the Schubert varieties; their singularities are rational, in particular, they are normal and Cohen-Macaulay.

Other important examples of homogeneous spherical varieties are symmetric spaces. In that case, the inclusion and weak orders have been studied in detail by Richardson and Springer

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[24], [25], [27]. But the geometry of B-orbit closures is far from being fully understood; some of them are non-normal, see [1].

Returning to the general setting of spherical varieties, examples of B-orbit closures of arbitrary dimension and depth 1 are given at the beginning of Section 3. On the other hand, the singularities of all G-orbit closures in a spherical G-variety are rational, see e.g. [6]. A criterion for B-orbit closures to have rational singularities will be formulated below, in terms of the oriented graph $\Gamma(X)$ associated with the weak order.

For this, we endow $\Gamma(X)$ with additional data, as in [24]: each edge from Y to Y' is labeled by a simple root of G corresponding to a minimal parabolic subgroup P such that $PY = Y'$. The degree of the associated morphism $P \times^B Y \to Y'$ being 1 or 2, this defines simple and double edges. There may be several labeled edges with the same endpoints, but they are simultaneously simple or double (Proposition 1).

For a spherical homogeneous space G/H, the cohomology classes of H-orbit closures in G/B can be read off the graph $\Gamma(G/H)$: each H-orbit closure V in G/B corresponds to a B-orbit closure Y in X. Consider an oriented path γ in $\Gamma(X)$, joining Y to X. Denote by $D(\gamma)$ its number of double edges, and by $w(\gamma)$ the product in W of the simple reflections associated with its labels. It turns out that $D(\gamma)$ depends only of Y and $w(\gamma)$ (Lemma 6) and that we have in the cohomology ring of G/B:

$$[V] = \sum_{w=w(\gamma)} 2^{D(\gamma)} \; [\overline{Bw_0wB}/B],$$

the sum over the $w(\gamma)$ associated with all oriented paths from Y to X. Here w_0 denotes the longest element of W.

Thus, we are led to study oriented paths in $\Gamma(X)$ and their associated Weyl group elements; this is the topic of Section 1. The main tool is a notion of neighbor paths that reduces several questions to the case where G has rank two. Using this, we show that the union of Schubert varieties

$$V_0 = \bigcup_{w=w(\gamma)} \overline{Bw_0wB}/B$$

is connected in codimension 1 (Corollary 5). If moreover G is simply-laced, then $D(\gamma)$ depends only on the endpoints of γ (Proposition 5). As a consequence, all coefficients of $[V]$ in the basis of Schubert classes are equal. For symmetric spaces, the latter result is due to Richardson and Springer [28]. It does not extend to multiply-laced groups, see Example 4 in Section 1.

In Section 2, we analyze the intersections of B-orbit closures with G-orbit closures in an important class of spherical varieties, the (complete) regular G-varieties in the sense of Bifet, De Concini and Procesi [2]. This generalizes results of [7] §1 where the intersections with closed G-orbits were described. Here a new ingredient is the construction of a "slice" $S_{Y,w}$ associated with a B-orbit closure Y in complete regular X, and with the Weyl group element w defined by an oriented path from Y to X. The $S_{Y,w}$ are toric varieties; each oriented path γ in $\Gamma(X)$ defines a finite surjective morphism of degree $2^{D(\gamma)}$ between "slices" of its endpoints. If the target of γ is X, then the intersection multiplicities of Y with those G-orbit closures that meet $S_{Y,w}$ are divisors of $2^{D(\gamma)}$. And given a G-orbit closure X' and an irreducible component Y' of $Y \cap X'$, there exists a "slice" meeting Y' (Theorem 1.)

This distinguishes the B-orbit closures Y such that all oriented paths in $\Gamma(X)$ with source Y contain simple edges only; we call them multiplicity-free. In a regular variety, any irreducible component of the intersection of multiplicity-free Y with a G-orbit closure is multiplicity-free as well, and the corresponding intersection multiplicity equals 1 (Corollary 3.)

Section 3 contains our main result, Theorem 3: the singularities of any multiplicity-free B-orbit closure Y in a spherical variety X are rational, if X contains no fixed points of simple normal subgroups of G of type G_2, F_4 and E_8. This technical assumption is used in one of the reduction steps of the proof, but the statement should hold in full generality. The argument goes by decreasing induction on Y, like Seshadri's proof of normality of Schubert varieties [26]. This result applies, e.g., to all regular G-varieties; for them, we show that the scheme-theoretical intersection of Y with any G-orbit closure is reduced.

For a H-orbit closure V in G/B, the corresponding B-orbit closure Y is multiplicity-free if and only if $[V] = [V_0]$. In that case, we construct a flat degeneration of V to V_0, where the latter is viewed as a reduced subscheme of G/B (Corollary 5). Thus, the equality $[V] = [V_0]$ holds in the Grothendieck group of G/B as well. As another consequence, the restriction map $H^0(G/B, L) \to H^0(V, L)$ is surjective for any effective line bundle L on G/B; moreover, the higher cohomology groups $H^i(V, L)$ vanish for $i \geqslant 1$ (Corollary 6.) Applied to symmetric spaces and combined with Theorem B of [1], this result implies a version of the Parthasaraty-Ranga Rao-Varadarajan conjecture, see [1] §6. It extends to certain smooth H-orbit closures, but not to all of them, see the example in [5] 4.3. In fact, surjectivity of all restriction maps for spherical G/H is equivalent to multiplicity-freeness of all H-orbit closures in G/B (Proposition 8.)

In Section 4, we relate our approach to work of Knop [18], [19]. He defined an action of W on $\mathscr{B}(X)$ such that the W-conjugates of the maximal element X are the orbit closures of maximal rank (in the sense of [19].) Moreover, the isotropy group $W_{(X)}$ of this maximal element is closely related to the "Weyl group of X", as defined in [18]. It is easy to see that all orbit closures of maximal rank are multiplicity-free, and hence their singularities are rational if X is regular. In that case, we describe the intersections of B-orbit closures of maximal rank with G-orbit closures, in terms of W and $W_{(X)}$ (Proposition 10.)

This implies two results on the position of $W_{(X)}$ in W: firstly, all elements of W of minimal length in a given $W_{(X)}$-coset have the same length. Secondly, $W_{(X)}$ is generated by reflections or products of two commuting reflections of W. This gives a simple proof of the fact that the Weyl group of X is generated by reflections [18].

A remarkable example of a spherical homogeneous space where all orbit closures of a Borel subgroup have maximal rank is the group G viewed as a homogeneous space under $G \times G$. If moreover G is adjoint, then it has a canonical $G \times G$-equivariant completion $\mathbf{X}$. It is proved in [9] that the $B \times B$-orbit closures in $\mathbf{X}$ are normal, and that their intersections are reduced. This follows from the fact that $\mathbf{X}$ is Frobenius split compatibly with all $B \times B$-orbit closures.

It is tempting to generalize this to any spherical variety X. By [6], X is Frobenius split compatibly with all G-orbit closures. But this does not extend to B-orbit closures, since their intersections may be not reduced. This happens, e.g., for the space of all symmetric $n \times n$ matrices of rank n, that is, the symmetric space $\mathrm{GL}(n)/\mathrm{O}(n)$: consider the subvarieties $(a_{11} = 0)$ and $(a_{11}a_{22} - a_{12}^2 = 0)$. On the other hand, many B-orbit closures in that space are not normal for $n \geqslant 5$, see [23].

So the present paper generalizes part of the results of [9] to all spherical varieties, by other methods. It raises many further questions, e.g., is it true that the normalization of any B-orbit closure in a spherical variety has rational singularities ? And do our results extend to positive characteristics (the proof of Theorem 3 uses an equivariant resolution of singularities) ?

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Notation. Let G be a complex connected reductive algebraic group. Let B, B^- be opposite Borel subgroups of G, with unipotent radicals U, U^- and common torus T, a maximal torus of G. Let $\mathscr{X}$ be the character group of B; we identify $\mathscr{X}$ with the character group of T and we choose a scalar product on $\mathscr{X}$, invariant under the Weyl group W. Let Φ be the root system of (G,T), with the subset Φ^+ of positive roots, i.e., of roots of (B,T), and its subset Δ of simple roots.

For $\alpha \in \Delta$, let $s_\alpha \in W$ be the corresponding simple reflection, and let $P_\alpha = B \cup Bs_\alpha B$ be the corresponding minimal parabolic subgroup. For any subset I of Δ, let P_I be the subgroup of G generated by the P_α, $\alpha \in I$. The map $I \mapsto P_I$ is a bijection from subsets of Δ to subgroups of G containing B, that is, to standard parabolic subgroups of G.

Let L_I be the Levi subgroup of P_I that contains T; let Φ_I be the root system of (L_I, T), with Weyl group W_I. We denote by ℓ the length function on W and by W^I the set of all $w \in W$ such that $\ell(ws_\alpha) = \ell(w) + 1$ for all $\alpha \in I$ (this amounts to: $w(I) \subseteq \Phi^+$). Then W^I is a system of representatives of the set of right cosets W/W_I.

1. The weak order and its graph

In the sequel, we denote by X a complex spherical G-variety and by $\mathscr{B}(X)$ the set of B-orbit closures in X. One associates to a given $Y \in \mathscr{B}(X)$ several combinatorial invariants, see [19]: The *character group* $\mathscr{X}(Y)$ is the set of all characters of B that arise as weights of eigenvectors of B in the function field $\mathbb{C}(Y)$. Then $\mathscr{X}(Y)$ is a free abelian group of finite rank $r(Y)$, the *rank* of Y.

Let Y^0 be the open B-orbit in Y and let $P(Y)$ be the set of all $g \in G$ such that $gY^0 = Y^0$; then $P(Y)$ is a standard parabolic subgroup of G. Let $L(Y)$ be its Levi subgroup that contains T and let $\Delta(Y)$ be the corresponding subset of Δ: the set of *simple roots of* Y.

We note some easy properties of these invariants.

Lemma 1. — *(i)* $\mathscr{X}(Y)$ *is isomorphic to the quotient of the group of invertible regular functions on* Y^0*, by the subgroup of constant non-zero functions.*

(ii) The derived subgroup $[L(Y), L(Y)]$ *fixes a point of* Y^0.

(iii) The group $W_{\Delta(Y)}$ *fixes pointwise* $\mathscr{X}(Y)$*. Equivalently, any simple root of* Y *is orthogonal to* $\mathscr{X}(Y)$.

Proof. — (i) Let f be an eigenvector of B in $\mathbb{C}(Y)$ with weight $\chi(f)$. Then f restricts to an invertible regular function on Y^0, and is uniquely determined by $\chi(f)$ up to a constant. Conversely, let f be an invertible regular function on the B-orbit Y^0. Then f pulls back to an invertible regular function on B, that is, to a scalar multiple of a character of B. Thus, f is an eigenvector of B in $\mathbb{C}(Y)$.

(ii) Choose $y \in Y^0$. Let B_y (resp. $P(Y)_y$) be the isotropy group of y in B (resp. $P(Y)$). Since $Y^0 = By = P(Y)y$, we have $P(Y) = BP(Y)_y$. Thus, $P(Y)_y$ acts transitively on $P(Y)/B$, the flag variety of $P(Y)$. Using e.g. [10], it follows that $P(Y)_y$ contains a maximal connected semisimple subgroup of $P(Y)$, that is, a conjugate of $[L(Y), L(Y)]$.

(iii) follows from [19] Lemma 3.2; it can be deduced from (ii) as well. □

Let $\mathscr{D}(X)$ be the subset of $\mathscr{B}(X)$ consisting of all irreducible B-stable divisors that are not G-stable. The elements of $\mathscr{D}(X)$ are called *colors*; they play an important role in the classification of spherical embeddings, see [16]. They also allow to describe the parabolic subgroups associated with G-orbit closures:

LEMMA 2. — *Let Y be the closure of a G-orbit in X and let $\mathscr{D}_Y(X)$ be the set of all colors that contain Y. Then $P(Y)$ is the set of all $g \in G$ such that $gD = D$ for any $D \in \mathscr{D}(X) - \mathscr{D}_Y(X)$. Moreover, there exists $y \in Y^0$ fixed by $[L(Y), L(Y)]$, such that the map $R_u(P(Y)) \times Ty \to Y^0$, $(g, x) \mapsto gx$ is an isomorphism. Then the dimension of Ty equals the rank of Y.*

Proof. — Let X_0 be the complement in X of the union of those colors that do not contain Y. Then X_0 is an open affine B-stable subset of X, and $X_0 \cap Y$ equals Y^0; see [16] Theorem 3.1. Let Q be the stabilizer of X_0 in G, then Q consists of all $g \in G$ such that $gD = D$ for all $D \in \mathscr{D}(X) - \mathscr{D}_Y(X)$. Clearly, Q is a standard parabolic subgroup, contained in $P(Y)$. It follows that $R_u(P(Y)) \subseteq R_u(Q)$.

Let M be the standard Levi subgroup of Q. By [18] 2.3 and 2.4, there exists a closed M-stable subvariety S of X_0 such that the product map $R_u(Q) \times S \to X_0$ is an isomorphism; moreover, $[M, M]$ acts trivially on $S \cap Y^0$. In particular, for any $y \in S \cap Y^0$, the product map $R_u(Q) \times Ty \to Y^0$ is an isomorphism. Since $R_u(Q) = R_u(P(Y))(R_u(Q) \cap [L(Y), L(Y)])$ and since $[L(Y), L(Y)]$ fixes points of Y^0, it follows that $R_u(Q) = R_u(P(Y))$, whence $Q = P(Y)$. Moreover, the character group of Y is isomorphic to that of the torus $Ty \cong T/T_y$, whence $r(Y) = \dim(Ty)$. □

This description of Y^0 as a product of a unipotent group with a torus will be generalized in Section 4 to all B-orbits of maximal rank.

Returning to arbitrary B-orbit closures, let $Y, Y' \in \mathscr{B}(X)$ and let $\alpha \in \Delta$. We say that α *raises Y to Y'* if $Y' = P_\alpha Y \neq Y$. Let then

$$f_{Y,\alpha} : P_\alpha \times^B Y \to P_\alpha/B$$

be the homogeneous bundle with fiber the B-variety Y and basis P_α/B (isomorphic to projective line.) The map $P_\alpha \times Y \to X$, $(p, y) \mapsto py$ factors through a proper morphism

$$\pi_{Y,\alpha} : P_\alpha \times^B Y \to Y' = P_\alpha Y$$

that restricts to a finite morphism $P_\alpha \times^B Y^0 \to P_\alpha Y^0$. In particular, $\dim(Y') = \dim(Y) + 1$.

By [24] or [19] Lemma 3.2, one of the following three cases occurs.

- Type U: $P_\alpha Y^0 = Y'^0 \cup Y^0$ and $\pi_{Y,\alpha}$ is birational. Then $\mathscr{X}(Y') = s_\alpha \mathscr{X}(Y)$; thus, $r(Y') = r(Y)$.
- Type T: $P_\alpha Y^0 = Y'^0 \cup Y^0 \cup Y_-^0$ for some $Y_- \in \mathscr{B}(X)$ of the same dimension as Y, and $\pi_{Y,\alpha}$ is birational. Then $r(Y) = r(Y_-) = r(Y') - 1$.
- Type N: $P_\alpha Y^0 = Y'^0 \cup Y^0$ and $\pi_{Y,\alpha}$ has degree 2. Then $r(Y) = r(Y') - 1$.

In particular, $r(Y) \leqslant r(P_\alpha Y)$ with equality if and only if α has type U.

Our notation for types differs from that in [24] and [19]; it can be explained as follows. Choose $y \in Y^0$ with isotropy group $(P_\alpha)_y$ in P_α. Then $(P_\alpha)_y$ acts on $P_\alpha/B \cong \mathbb{P}^1$ with finitely many orbits, for B acts on $P_\alpha Y^0 \cong P_\alpha/(P_\alpha)_y$ with finitely many orbits. By [24] or [19], the image of $(P_\alpha)_y$ in $\mathrm{Aut}(P_\alpha/B) \cong \mathrm{PGL}(2)$ is a torus (resp. the normalizer of a torus) in type T (resp. N); in type U, this image contains a non-trivial unipotent normal subgroup.

Definition. Let $\Gamma(X)$ be the oriented graph with vertices the elements of $\mathscr{B}(X)$ and edges labeled by Δ, where Y is joined to Y' by an edge of label α if that simple root raises Y to Y'. This edge is simple (resp. double) if $\pi_{Y,\alpha}$ has degree 1 (resp. 2.) The partial order $\preceq$ on $\mathscr{B}(X)$ with oriented graph $\Gamma(X)$ will be called the *weak order*.

Observe that the dimension and rank functions are compatible with $\preceq$. We shall see that $Y, Y' \in \mathscr{B}(X)$ satisfy $Y \preceq Y'$ if and only if there exists $w \in W$ such that Y' equals the closure $\overline{BwY}$ (Corollary 1.)

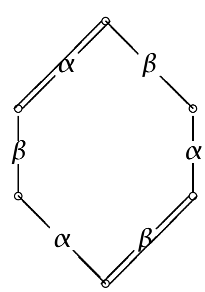
In the case where $X = G/P$ where P is a parabolic subgroup of G, the rank function is zero. Thus, all edges are of type U; in particular, they are simple.

Here is another example, where double edges occur.

Example 1. Let $G = \mathrm{GL}(3)$ with simple roots α and β. Let H be the subgroup of G consisting of matrices of the form

$$\begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \text{ or } \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ 0 & 0 & * \end{pmatrix}$$

and let $X = G/H$. It is easy to see that X is spherical of rank one and that $\Gamma(X)$ is as follows:



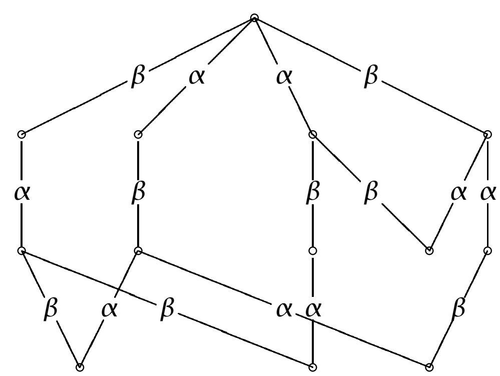
Observe that $\Gamma(X)$ is the same as $\Gamma(G/B)$, except for double edges. But the geometry of B-orbit closures is very different in both cases: all of them are smooth in G/B (the flag variety of $\mathbb{P}^2$), whereas X contains a B-stable divisor that is singular in codimension 1.

Specifically, let Z be the closed B-orbit in G/H. We claim that $Y = P_\beta P_\alpha Z$ is singular along $P_\beta Z$. Indeed, the morphism $\pi : P_\beta \times^B P_\alpha Z \to Y$ is birational, and $\pi^{-1}(P_\beta Z)$ equals $P_\beta \times^B Z$. But the restriction $P_\beta \times^B Z \to P_\beta Z$ has degree two. Now our claim follows from Zariski's main theorem.

One checks that $r(P_\beta Z) = 1$, whereas $r(Y) = 0$. Thus, the rank function is not compatible with the inclusion order.

Obviously, all closed B-orbits in a spherical homogeneous space X are minimal elements for the weak order. In fact, these closed B-orbits are isomorphic, and their codimension is the maximal length of all oriented paths in $\Gamma(X)$, see e.g. [8] 2.2. If moreover X is symmetric, then all minimal elements of $\Gamma(X)$ are closed orbits, see [24] Theorem 4.6; equivalently, all maximal oriented paths in $\Gamma(X)$ have the same length. But this does not extend to all spherical homogeneous spaces, as shown by the following

Example 2. We represent $\Gamma(X)$ for $X = \mathrm{GL}(3)/H$ where H consists of all matrices of the form $\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$:



Returning to the general situation, observe that GY is the closure of a G-orbit for any $Y \in \mathscr{B}(X)$. Moreover, Y is the source of an oriented path in $\Gamma(X)$ with target GY (for the group G is generated by the P_α, $\alpha \in \Delta$), and the length of any such path equals $\dim(GY) - \dim(Y)$. By [19] Corollary 2.4, we have $r(GY) \leqslant r(X)$, so that $r(Y) \leqslant r(X)$. It also follows that each connected component of $\Gamma(X)$ contains a unique G-orbit closure.

The simple roots of Y are determined by $\Gamma(X)$: indeed, $\alpha \in \Delta$ is not in $\Delta(Y)$ if and only if α is the label of an edge with endpoint Y. Similarly, if α raises Y then its type is determined by $\Gamma(X)$: it is U (resp. N) if there is a unique edge of label α and target $P_\alpha Y$ and this edge is simple (resp. double); and it is T if there are two such edges. It follows that the ranks of B-orbit closures are determined by $\Gamma(X)$ and the ranks of G-orbit closures.

There is no restriction on the number of edges in $\Gamma(X)$ with prescribed endpoints, as shown by the example below suggested by D. Luna. But we shall see that all such edges have the same type.

Example 3. Let n be a positive integer. Let $G = \mathrm{SL}(2) \times \cdots \times \mathrm{SL}(2)$ (n terms) and let H be the subgroup of G consisting of those n-tuples $\begin{pmatrix} t & u_1 \\ 0 & t^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} t & u_n \\ 0 & t^{-1} \end{pmatrix}$ where $t \in \mathbb{C}^*$, $u_1, \ldots, u_n \in \mathbb{C}$ and $u_1 + \cdots + u_n = 0$. One checks that G/H is spherical; the open H-orbit in $G/B \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (n terms) consists of those $(z_1, \ldots, z_n)$ such that $z_i \neq \infty$ for all i, and that $z_1 + \cdots + z_n \neq 0$. Let Y be the B-stable hypersurface in G/H corresponding to the H-stable hypersurface $(z_1 + \cdots + z_n = 0)$ in G/B. One checks that Y is irreducible and raised to G/H by all simple roots of G (there are n of them). Thus, Y is joined to G/H by n edges of type U.

PROPOSITION 1. — *Let $Y, Y' \in \mathscr{B}(X)$ and let α, β be distinct simple roots raising Y to Y'. Then either α, β are orthogonal and both of type U, or they are both of type T.*

Proof. — We begin with two lemmas that reduce the "local" study of $\Gamma(X)$ to simpler situations.

Let $Y \in \mathscr{B}(X)$ and let $P = P_I$ be a standard parabolic subgroup of G, with radical $R(P)$. Let $\mathscr{B}(P,Y)$ be the set of all closures in X of B-orbits in PY^0; in other words, $\mathscr{B}(P,Y)$ is the set of all $Z \in \mathscr{B}(X)$ such that $PZ = PY$. Let $\Gamma(P,Y)$ be the oriented graph with set of vertices $\mathscr{B}(P,Y)$, and with edges those edges of $\Gamma(X)$ that have both endpoints in $\mathscr{B}(P,Y)$ and labels in I.

LEMMA 3. — *The quotient $PY^0/R(P)$ is a $P/R(P)$-homogeneous spherical variety with graph $\Gamma(P,Y)$.*

Proof. — Since PY^0 is a unique P-orbit and $R(P)$ is a normal subgroup of P contained in B, the quotient $PY^0/R(P)$ exists and is homogeneous under $P/R(P)$; moreover, any $B/R(P)$-orbit in $PY^0/R(P)$ pulls back to a unique B-orbit in PY^0. Let $\mathscr{O}$ be a B-orbit in PY^0 and let $\alpha \in I$. Then $R(P_\alpha)$ contains $R(P)$, the square

$$\begin{array}{ccc} P_\alpha \times^B \mathscr{O} & \to & P_\alpha \mathscr{O} \\ \downarrow & & \downarrow \\ P_\alpha \times^B \mathscr{O}/R(P) & \to & P_\alpha \mathscr{O}/R(P) \end{array}$$

is cartesian, and the map $P_\alpha \times^B \mathscr{O}/R(P) \to P_\alpha/R(P) \times^{B/R(P)} \mathscr{O}/R(P)$ is an isomorphism. Thus, the type is preserved under pull back. □

Assume now that X is homogeneous under G; write then $X = G/H$. Let H' be a closed subgroup of the normalizer $N_G(H)$ such that H' contains H, and that the quotient H'/H is connected. Let $Z(G)$ be the center of G. Let $X' = G/H'Z(G)$, a homogeneous spherical variety under the adjoint group $G/Z(G)$. The natural G-equivariant map $p : X \to X'$ is the quotient by the right action of $H'Z(G)$ on G/H.

LEMMA 4. — *The pull-back under p of any B-orbit in X' is a unique B-orbit in X. This defines an isomorphism of $\Gamma(X')$ onto $\Gamma(X)$.*

Proof. — The first assertion follows from [8] Proposition 2.2 (iii). The second assertion is checked as in the proof of Lemma 3. □

LEMMA 5. — *Let $Y \in \mathscr{B}(X)$, $Y \neq X$, and let $\alpha \in \Delta$. If $P_\alpha Y^0 = X$ then α is orthogonal to $\Delta - \{\alpha\}$, and the derived subgroup of $L_{\Delta-\{\alpha\}}$ fixes pointwise X.*

Proof. — Let H be the isotropy group in G of a point of Y^0. Since $P_\alpha Y^0 = X$, we have $P_\alpha H = G$. Equivalently, the map $H/P_\alpha \cap H \to G/P_\alpha$ is an isomorphism. But since $Y \neq X$, we have $Y^0 \neq P_\alpha Y^0$, so that the image of $P_\alpha \cap H$ in $P_\alpha/R(P_\alpha) \cong \mathrm{PGL}(2)$ is a proper subgroup. It follows that $(P_\alpha \cap H)^0$ is solvable. Thus, $H/P_\alpha \cap H$ is the flag variety of H^0. Now the connected automorphism group of this flag variety is the quotient of $H^0/R(H^0)$ by its center. On the other hand, the connected automorphism group of G/P_α is $G/Z(G)$ if α is not orthogonal to $\Delta - \{\alpha\}$ (this follows e.g. from [10].) In this case, we have $G = Z(G)H^0$ so that G/H is a unique B-orbit, a contradiction. Thus, $G/Z(G)$ is the product of $L_\alpha/Z(L_\alpha)$ with $L_{\Delta-\{\alpha\}}/Z(L_{\Delta-\{\alpha\}})$, and the map $L_{\Delta-\{\alpha\}}/B \cap L_{\Delta-\{\alpha\}} \to G/P_\alpha$ is an isomorphism. It follows that the derived subgroup of $L_{\Delta-\{\alpha\}}$ is contained in H. □

We now prove Proposition 1. Applying Lemma 3 to Y' and $P_{\alpha,\beta}$, we may assume that $Y' = X = G/H$ for some subgroup H of G and that $\Delta = \{\alpha, \beta\}$.

If α has type U, then $r(Y) = r(X)$ whence β has type U as well. We claim that $\mathscr{B}(X)$ consists of Y and X. Indeed, if $Z \in \mathscr{B}(X)$ and $Z \neq X$, then Z is connected to X by an oriented path in $\Gamma(X)$. Let Z' be the source of the top edge of this path. That edge cannot have Y as its target (otherwise Y would be stable under P_α or P_β); thus, it raises Z' to X. Since α and β have type U, it follows that $Z' = Y$, whence $Z = Y$. Thus, $P_\alpha Y^0 = X$; then α and β are orthogonal by Lemma 5.

If α has type N, then $r(Y) = r(X) - 1$, whence β has type N or T. In the former case, we see as above that $X = P_\alpha Y^0 = P_\beta Y^0$. Thus, α and β are orthogonal by Lemma 5. Using Lemma 4, we may assume that $G = \mathrm{PGL}(2) \times \mathrm{PGL}(2)$ and that H contains a copy of $\mathrm{PGL}(2)$. Then H is conjugate to $\mathrm{PGL}(2)$ embedded diagonally in G. But then both α and β have type T, a contradiction.

If α has type N and β has type T, then there exists $y \in Y^0$ such that $(P_\beta)_y$ is contained in $R(P_\beta)T$. Since the homogeneous spaces $P_\beta/R(P_\beta)T$ and $R(P_\beta)T/(P_\beta)_y$ are affine, the same holds for $P_\beta/(P_\beta)_y \cong P_\beta Y^0$. It follows that $X - P_\beta Y^0$ is pure of codimension 1 in X. But $P_\beta Y^0$ meets both B-orbits of codimension 1 in X, so that $P_\beta Y^0 = X$. This case is excluded as above. Thus, type N does not occur. □

We next study oriented paths in $\Gamma(X)$. Let γ be such a path, with source Y and target Y'. Let $(\alpha_1, \alpha_2, \dots, \alpha_\ell)$ be the sequence of labels of edges of γ, where $\ell = \ell(\gamma)$ is the length of the path. Let $\ell_U(\gamma)$ (resp. $\ell_T(\gamma)$, $\ell_N(\gamma)$) be the number of edges of type U (resp. T, N) in γ. Then

$$\ell_U(\gamma) + \ell_T(\gamma) + \ell_N(\gamma) = \ell(\gamma) = \dim(Y') - \dim(Y).$$

Define an element $w(\gamma)$ of W by $w(\gamma) = s_{\alpha_\ell} \cdots s_{\alpha_2} s_{\alpha_1}$.

Lemma 6. — *(i) $(s_{\alpha_\ell}, \dots, s_{\alpha_2}, s_{\alpha_1})$ is a reduced decomposition of $w(\gamma)$; equivalently, $\ell(w(\gamma)) = \ell$.*

(ii) $\ell_T(\gamma) + \ell_N(\gamma) = r(Y') - r(Y)$. In particular, $\ell_T(\gamma) + \ell_N(\gamma)$ and $\ell_U(\gamma)$ depend only on the endpoints of γ.

(iii) The morphism $G \times^B Y \to X : (g, y)B \to gy$ restricts to a morphism $\overline{Bw(\gamma)B} \times^B Y \to Y'$ that is surjective and generically finite of degree $2^{\ell_N(\gamma)}$. In particular, $\ell_T(\gamma)$ and $\ell_N(\gamma)$ depend only on the endpoints of γ and on $w(\gamma)$. Moreover, $w(\gamma)$ is in $W^{\Delta(Y)}$, and $w(\gamma)^{-1}$ is in $W^{\Delta(Y')}$.

(iv) If the stabilizer in G of a point of Y^0 is contained in a Borel subgroup of G (e.g., if $X = G/H$ where H is connected and solvable), then $\ell_N(\gamma) = 0$ so that $\ell_T(\gamma)$ depends only on the endpoints of γ.

Proof. — (i) Observe that $Bs_{\alpha_1}Y$ is dense in $P_{\alpha_1}Y$, as P_{α_1} raises Y. By induction, it follows that $Bs_{\alpha_\ell}B \cdots s_{\alpha_2}Bs_{\alpha_1}Y$ is dense in Y'. Because $\dim(Y') = \dim(Y) + \ell$, we must have $\dim(\overline{Bs_{\alpha_\ell}B \cdots s_{\alpha_2}Bs_{\alpha_1}B}/B) = \ell$, whence $\ell(s_{\alpha_\ell} \cdots s_{\alpha_2}s_{\alpha_1}) = \ell$.

(ii) follows from the fact that $r(Y') = r(Y)$ (resp. $r(Y)+1$) if Y is the source of an edge with target Y' and type U (resp. T, N).

(iii) By (i), the product maps

$$P_{\alpha_i} \times^B \cdots \times^B P_{\alpha_2} \times^B P_{\alpha_1} \to \overline{Bs_{\alpha_i} \cdots s_{\alpha_2} s_{\alpha_1} B}$$

are birational for $1 \leqslant i \leqslant \ell$. It follows that the morphism $\overline{Bw(y)B} \times^B Y \to X$ has image Y'; moreover, its degree is the product of the degrees of the

$$\pi_i : P_{\alpha_i} \times^B (P_{\alpha_{i-1}} \cdots P_{\alpha_1} Y) \to P_{\alpha_i} P_{\alpha_{i-1}} \cdots P_{\alpha_1} Y,$$

that is, $2^{\ell_N(y)}$.

Let $w = w(y)$. We show that $w^{-1} \in W^{\Delta(Y')}$. Otherwise, there exists $\alpha \in \Delta(Y')$ such that $\ell(s_\alpha w) = \ell(w)-1$. Thus, $BwB = Bs_\alpha Bs_\alpha wB$, and $Y' = \overline{BwY} = \overline{Bs_\alpha Bs_\alpha wY}$. Let $Y'' = \overline{Bs_\alpha wY}$, then α raises Y'' to Y'. This contradicts the assumption that $\alpha \in \Delta(Y')$. A similar argument shows that $w \in W^{\Delta(Y)}$.

(iv) If $\ell_N(y) > 0$, then there exists a point $x \in GY^0$, a simple root α and a surjective group homomorphism $(P_\alpha)_x \to N$ where N is the normalizer of a torus in $\mathrm{PGL}(2)$. Since N consists of semisimple elements, it is a quotient of $(P_\alpha)_x / R_u(P_\alpha)_x$. By assumption, the latter is isomorphic to a subgroup of $B/U = T$. Thus, N is abelian, a contradiction. □

Corollary 1. — *Let $Y, Y' \in \mathscr{B}(X)$, then $Y \preceq Y'$ if and only if there exists $w \in W$ such that $Y' = \overline{BwY}$.*

Proof. — Recall that $\overline{BwB}$ (closure in G) is a product of minimal parabolic subgroups. Thus, $Y \preceq \overline{BwB}Y = \overline{BwY}$. The converse has just been proved. □

For later use, we study the behavior of $\Gamma(X)$ under parabolic induction in the following sense (see [7] 1.2.) Let $P = P_I$ be a standard parabolic subgroup with Levi subgroup $L = L_I$ and let X' be a spherical L-variety, then the induced variety is $X = G \times^P X'$ where P acts on X' through its quotient $P/R_u(P)$, isomorphic to L. In other words, X is the total space of the homogeneous bundle over G/P with fiber X'. By [*loc. cit.*], each $Y \in \mathscr{B}(X)$ can be written uniquely as $\overline{BwY'}$ for $w \in W^I$ and $Y' \in \mathscr{B}(X')$; then $r(Y) = r(Y')$. We thus identify $\mathscr{B}(X)$ to $W^I \times \mathscr{B}(X')$. The next result describes the edges of $\Gamma(X)$ in terms of those of $\Gamma(X')$.

Lemma 7. — *Let $\alpha \in \Delta$, $w \in W^I$ and $Y' \in \mathscr{B}(X')$; let $\beta = w^{-1}(\alpha)$. Then the edges of $\Gamma(X)$ with source (w, Y') and label α are as follows:*

(i) If $\beta \in \Phi^+ - I$, join (w, Y') to $(s_\alpha w, Y')$ by an edge of type U.

(ii) If $\beta \in I$ and $P_\beta \cap L$ raises Y', join (w, Y') to $(w, (P_\beta \cap L)Y')$ by an edge of the same type as the edge from Y' to $(P_\beta \cap L)Y'$.

Proof. — Since $w \in W^I$, we have $s_\alpha w \in W^I$ if and only if $\beta \notin I$. In that case, P_α raises Y if and only if $\ell(s_\alpha w) = \ell(w) + 1$, that is, $\beta \in \Phi^+$. Then $P_\alpha Y = \overline{Bs_\alpha wY'}$ and the map $\pi_{Y,\alpha}$ is the pull-back of $\pi_{\overline{BwP}/P,\alpha}$ under the map $\overline{BwY'} \to \overline{BwP}/P$. This yields case (i).

But if $\beta \in I$, then $s_\alpha w = ws_\beta$ has length $\ell(w) + 1$, so that

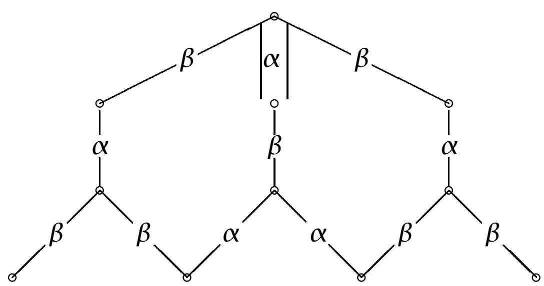
$$P_\alpha Y = \overline{Bs_\alpha BwY'} = \overline{Bs_\alpha wY'} = \overline{Bws_\beta Y'} = \overline{BwBs_\beta Y'} = \overline{Bw(P_\beta \cap L)Y'}.$$

Thus, P_α raises Y if and only if $P_\beta \cap L$ raises Y'. Then, as $s_\alpha w = ws_\beta$, we can join Y' to $P_\alpha Y$ by two paths: one beginning with $\ell(w)$ edges of type U followed by an edge from Y to $P_\alpha Y$, and another one beginning with an edge from Y' to $(P_\beta \cap L)Y'$ followed by $\ell(w)$ edges of type U. Using Lemma 6, this yields case (ii). ☐

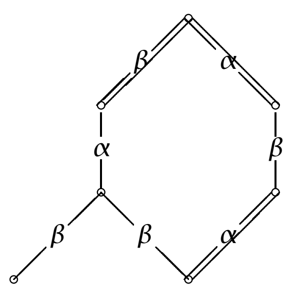
For instance, Example 1 is obtained from $\mathrm{SL}(2)/N$ by parabolic induction.

Returning to the case where X is an arbitrary spherical G-variety, we shall see that the numbers $\ell_T(\gamma)$ and $\ell_N(\gamma)$ depend only on the endpoints of the oriented path γ in $\Gamma(X)$, if G is simply-laced (that is, if all roots have the same length for an appropriate choice of the W-invariant scalar product on $\mathscr{X}$; equivalently, Φ is a product of simple root systems of type A, D or E.) This assumption cannot be omitted, as shown by

Example 4. Let $G = \mathrm{SP}(4)$ be the subgroup of $\mathrm{GL}(4)$ preserving a non-degenerate symplectic form, and let $H = \mathrm{GL}(2)$ be the subgroup of G preserving two complementary lagrangian planes. The normalizer $N_G(H)$ contains H as a subgroup of index 2. The graph $\Gamma(G/H)$ is as follows:



And here is $\Gamma(G/N_G(H))$:



Using parabolic induction, one constructs similar examples for Φ of type B, C or F.

To proceed, we need the following definition taken from [7]:

Definition. For $Y \in \mathscr{B}(X)$, let $W(Y)$ be the set of all $w \in W$ such that the morphism $\pi_{Y,w}$: $\overline{BwB} \times^B Y \to GY$ is surjective and generically finite. For $w \in W(Y)$, let $d(Y,w)$ be the degree of $\pi_{Y,w}$.

In other words, $W(Y)$ consists of all $w(\gamma)$ where γ is an oriented path from Y to GY; moreover, $d(Y, w(\gamma)) = 2^{\ell_N(\gamma)}$. By Lemma 6, $w^{-1} \in W^{\Delta(X)}$ for all $w \in W(Y)$.

We now introduce a notion of neighbors in $W(Y)$, and we show that any two elements of that set are connected by a chain of neighbors. Let α, β be distinct simple roots and let m be a positive integer. Let

$$(s_\alpha s_\beta)^{(m)} = \cdots s_\beta s_\alpha s_\beta s_\alpha \qquad (m \text{ terms.})$$

Then we have the braid relation $(s_\alpha s_\beta)^{(m(\alpha,\beta))} = (s_\beta s_\alpha)^{(m(\alpha,\beta))}$, where $m(\alpha,\beta)$ denotes the order of $s_\alpha s_\beta$ in W.

Definition. Two elements u and v of W are *neighbors* if there exist x, y in W together with distinct α, β in Δ and a positive integer $m < m(\alpha,\beta)$ such that

$$u = x(s_\alpha s_\beta)^{(m)} y,\ v = x(s_\beta s_\alpha)^{(m)} y, \text{ and } \ell(u) = \ell(x) + m + \ell(y) = \ell(v).$$

For example, any two simple reflections are neighbors.

Proposition 2. — *Let $Y \in \mathscr{B}(X)$ and let u, v be distinct elements of $W(Y)$. Then there exists a sequence $(u = u_0, u_1, \ldots, u_n = v)$ in $W(Y)$ such that each u_{i+1} is a neighbor of u_i.*

Proof. — By induction on $\ell(u) = \ell(v) = \ell$, the case where $\ell = 1$ being evident.

If there exists $\alpha \in \Delta$ such that $\ell(us_\alpha) = \ell(vs_\alpha) = \ell - 1$, then P_α raises Y, and us_α, vs_α are in $W(P_\alpha Y)$. Now the induction assumption for $P_\alpha Y$ concludes the proof in this case.

Otherwise, we can find distinct $\alpha, \beta \in \Delta$ such that $\ell(us_\alpha) = \ell(vs_\beta) = \ell - 1$. Then P_α and P_β raise Y to subvarieties of $P_{\alpha,\beta}Y$. Let m be the common codimension of $P_\alpha Y$ and $P_\beta Y$ in $P_{\alpha,\beta}Y$, then we have

$$P_{\alpha,\beta}Y = \cdots P_\alpha P_\beta P_\alpha Y = \overline{B \cdots s_\alpha s_\beta s_\alpha Y} \qquad (m \text{ terms})$$

Choose $x \in W(P_{\alpha,\beta}Y)$, then $W(Y)$ contains $x(s_\alpha s_\beta)^{(m)}$ and, similarly, $x(s_\beta s_\alpha)^{(m)}$, as neighbors. Moreover, $W(P_\alpha Y)$ contains us_α and $x(s_\beta s_\alpha)^{(m-1)}$, whereas $W(P_\beta Y)$ contains $x(s_\beta s_\alpha)^{(m-1)}$ and vs_β. Now we conclude by the induction assumption for $P_\alpha Y$ and $P_\beta Y$. □

Neighbors in $W(Y)$ are also close to each other for the Bruhat-Chevalley order $\leqslant$ on W:

Proposition 3. — *Let $Y \in \mathscr{B}(X)$. For any neighbors $u, v \in W(Y)$, there exists $w \in W$ such that $u \leqslant w$, $v \leqslant w$, $w^{-1} \in W^{\Delta(X)}$ and $\ell(w) = \ell(u) + 1 = \ell(v) + 1$.*

Proof. — Write $u = x(s_\alpha s_\beta)^{(m)} y$ and $v = x(s_\beta s_\alpha)^{(m)} y$. Let

$$w = x(s_\alpha s_\beta)^{(m)} s_\beta y.$$

We claim that $\ell(w)$ equals $\ell(x) + m + 1 + \ell(y) = \ell(u) + 1 = \ell(v) + 1$. Otherwise, $\ell(w) \leqslant \ell(x) + \ell(y) + m - 1 < l(u)$ and $w = uy^{-1}s_\beta y = us_{y^{-1}(\beta)}$. By the strong exchange condition ([14] Theorem 5.8 applied to u), one of the following cases occurs:

(i) $w = x'(s_\alpha s_\beta)^{(m)} y$ where $\ell(x') = \ell(x) - 1$. Comparing both expressions for w, we obtain $x'(s_\alpha s_\beta)^{(m)} = x(s_\alpha s_\beta)^{(m)} s_\beta$. Thus, there exists $\gamma \in \Phi^+_{\alpha,\beta}$ such that $x' = xs_\gamma$. But $\ell(xs_\alpha) = \ell(xs_\beta) = \ell(x) + 1$, for $\ell(x(s_\alpha s_\beta)^{(m)} y) = \ell(x(s_\beta s_\alpha)^{(m)} y) = \ell(x) + m + \ell(y)$. It follows that $x(\alpha)$ and $x(\beta)$ are in Φ^+. Thus, $x \in W^{\alpha,\beta}$. Since $s_\gamma \in W_{\alpha,\beta}$, we have $\ell(x') = \ell(x) + \ell(s_\gamma) \geqslant \ell(x)$, a contradiction.

(ii) $w = xzy$ where z is obtained from $(s_\alpha s_\beta)^{(m)}$ by deleting a simple reflection. Then the equality $z = (s_\alpha s_\beta)^{(m)} s_\beta$ leads to a braid relation of length at most $m < m(\alpha,\beta)$, a contradiction.

(iii) $w = x(s_\alpha s_\beta)^{(m)} y'$ where $\ell(y') = \ell(y) - 1$. Then $y' = s_\beta y$. But $\ell(s_\beta y) = \ell(y) + 1$, for $\ell(v) = \ell(x) + m + \ell(y)$; a contradiction.

By the claim and [14] Theorem 5.10, we have $u \leqslant w$ and $v \leqslant w$. Write $w = w'' w'$ where $w'' \in W_{\Delta(X)}$ and $(w')^{-1} \in W^{\Delta(X)}$; then $\ell(w) = \ell(w') + \ell(w'')$. Since $u^{-1} \leqslant w^{-1}$ and $u^{-1} \in W^{\Delta(X)}$, it follows that $u^{-1} \leqslant (w')^{-1}$ by [11] Lemma 3.5. Thus, $u \leqslant w'$ and $v \leqslant w'$. Since $u \neq v$ and $\ell(u) = \ell(v) = \ell(w) - 1 \geqslant \ell(w') - 1$, we must have $w = w'$, so that $w^{-1} \in W^{\Delta(X)}$. □

Recall that $r(Y) \leqslant r(X)$ for any $Y \in \mathscr{B}(X)$, see [19] Corollary 2.4. If equality holds, then neighbors in $W(Y)$ have a very simple form:

Proposition 4. — *Let $Y \in \mathscr{B}(X)$ such that $r(Y) = r(X)$; let $u, v \in W(Y)$ be neighbors. Then $u = x s_\alpha y$ and $v = x s_\beta y$ where $x, y \in W$ and α, β are orthogonal simple roots such that $\ell(u) = \ell(v) = \ell(x) + \ell(y) + 1$. Moreover, $\mathscr{X}(X)$ contains $x(\alpha+\beta)$.*

Proof. — Write $u = x(s_\alpha s_\beta)^{(m)} y$ and $v = x(s_\beta s_\alpha)^{(m)} y$ as in the definition of neighbors. Then $x(s_\alpha s_\beta)^{(m)}$ and $x(s_\beta s_\alpha)^{(m)}$ are neighbors in $W(\overline{ByY})$. Moreover, $r(\overline{ByY}) \geqslant r(Y)$, whence $r(\overline{ByY}) = r(X)$. Thus, we may assume that $y = 1$.

Let $Y' = \overline{B(s_\alpha s_\beta)^{(m)} Y}$ and $Y'' = \overline{B(s_\beta s_\alpha)^{(m)} Y}$, then we obtain similarly: $r(Y') = r(Y'') = r(X)$ and $x \in W(Y') \cap W(Y'')$. If $x \neq 1$, write $x = s_\gamma x'$ where $\gamma \in \Delta$ and $\ell(x) = \ell(x') + 1$. Then $\overline{Bx'Y'}$ and $\overline{Bx'Y''}$ have rank $r(X)$ and are raised to X by γ. Thus, $\overline{Bx'Y'} = \overline{Bx'Y''}$ and, by induction on $\ell(x)$, we obtain $Y' = Y''$. This subvariety is stable under $P_{\alpha,\beta}$. Applying Lemmas 3 and 4, we may assume that $Y' = X$ (i.e., $x = 1$), $\Delta = \{\alpha, \beta\}$ and $X = G/H$ where the center of G is trivial and H has finite index in its normalizer. Moreover, we have $P(X) = B$, for P_α and P_β do not stabilize X^0.

We claim that any $Z \in \mathscr{B}(X)$ can be written as

$$\overline{B(s_\alpha s_\beta)^{(n)} Y} = \cdots P_\beta P_\alpha Y \text{ or } \overline{B(s_\beta s_\alpha)^{(n)} Y} = \cdots P_\alpha P_\beta Y \qquad (n \text{ terms}),$$

where $n = \dim(Z) - \dim(Y)$ satisfies $0 \leqslant n \leqslant m$. For this, we argue by induction on the codimension of Z in X. We may assume that α raises Z. By the induction assumption, we have

$$P_\alpha Z = P_\beta P_\alpha \cdots Y \text{ or } P_\alpha Z = P_\alpha P_\beta \cdots Y \qquad (n+1 \text{ terms}).$$

In the latter case, let $Z' = P_\beta \cdots Y$ (n terms). Since $P_\alpha Z = P_\alpha Z'$ and $r(Z) = r(Z') = r(P_\alpha Z) = r(Y)$, it follows that $Z = Z'$. In the former case, $P_\alpha Z$ is stable under G and hence equal to X; in particular, Z has codimension 1 in X. Now $X = P_\alpha P_\beta \cdots Y$ (m terms), so that we are in the previous case.

By the claim, all B-orbit closures in X have the same rank, and Y^0 is the unique closed B-orbit. Let $y \in Y^0$; we may assume that $H = G_y$. Since the H-orbit in G/B corresponding to the B-orbit Y^0 in G/H is closed, the connected isotropy group B_y^0 is a Borel subgroup of H^0. It follows that $r(Y) = r(B) - r(B_y) = 2 - r(H)$. On the other hand, $r(Y) = r(G/H)$ by assumption. Thus, $r(G/H) = 2 - r(H)$.

If $r(G/H) = 0$ then H is a parabolic subgroup of G (in fact, a Borel subgroup as $P(G/H) = B$.) Moreover, Y is the B-fixed point in G/H. But then $W(Y)$ consists of a unique element (of maximal length in W), a contradiction.

If $r(G/H) = 1$ then $r(H) = 1$ as well. Using the classification of homogeneous spaces of rank 1 under semi-simple groups of rank 2 (see e.g. Table 1 of [30]), this forces $G = \mathrm{PGL}(2) \times \mathrm{PGL}(2)$ and $H = \mathrm{PGL}(2)$ embedded diagonally in G. As a consequence, the simple roots α and β are orthogonal, and $\mathscr{X}(G/H)$ is generated by $\alpha + \beta$.

If $r(G/H) = 2$ then $r(H) = 0$, that is, H^0 is unipotent. Since G/H is spherical, H^0 is a maximal unipotent subgroup of G. This contradicts the assumption that H has finite index in its normalizer. ☐

Proposition 5. — *If G is simply-laced, then*

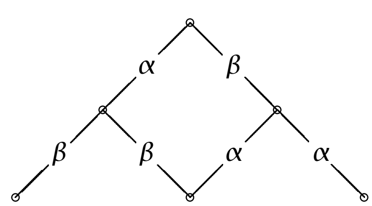
(i) for any oriented path γ in $\Gamma(X)$, both $\ell_T(\gamma)$ and $\ell_N(\gamma)$ depend only on the endpoints of γ.

(ii) for any $Y \in \mathscr{B}(X)$, there exists an oriented path γ joining Y to X through a sequence of simple edges followed by a sequence of double edges.

Proof. — (i) Let Y (resp. Y') be the source (resp. target) of γ, and let δ be another oriented path from Y to Y'. By Lemma 6, it suffices to show that $\ell_N(\gamma) = \ell_N(\delta)$. Joining Y' to X by an oriented path, we reduce to the case where $Y' = X$; then $w(\gamma)$ and $w(\delta)$ are in $W(Y)$. By Proposition 2, we may assume moreover that $w(\gamma)$ and $w(\delta)$ are neighbors. Using Lemmas 3 and 4, we reduce to the case where the center of G is trivial, $\Delta = \{\alpha, \beta\}$, $X = G/H$ where H has finite index in its normalizer, $w(\gamma) = (s_\alpha s_\beta)^{(m)}$ and $w(\delta) = (s_\beta s_\alpha)^{(m)}$ for some $m < m(\alpha, \beta)$.

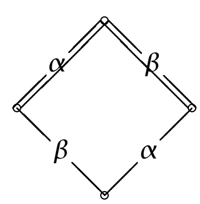
Since G is simply-laced, we have either $G = \mathrm{PGL}(2) \times \mathrm{PGL}(2)$ and $m(\alpha, \beta) = 2$, or $G = \mathrm{PGL}(3)$ and $m(\alpha, \beta) = 3$. In particular, $m \leqslant 2$. If $m = 1$ then $\ell_N(\gamma) = \ell_N(\delta) = 0$ by Proposition 1. If $m = 2$ then $G = \mathrm{PGL}(3)$. Using Lemma 6 (iv), we may assume moreover that H is not contained in any Borel subgroup. Then we see by inspection that H is conjugate to $\mathrm{PO}(3)$ or to $\mathrm{GL}(2)$.

In the latter case, here is $\Gamma(G/H)$:



Thus, $\ell_N(\gamma) = \ell_N(\delta) = 0$.

In the former case, we have $\ell_N(\gamma) = \ell_N(\delta) = 1$, since $\Gamma(G/H)$ is as follows:



(ii) Let γ be an oriented path joining Y to X. We may assume that γ contains double edges. Consider the lowest maximal subpath δ of γ that consists of double edges only; we may assume that the endpoint of δ is not X. Let Y' be the source of the top edge of δ, and let α (resp. β) be the label of that edge (resp. of the next edge of γ, a simple edge by assumption.) We claim that there exists an oriented path γ' joining Y' to X and beginning with a simple edge; then assertion (ii) will follow by induction on $\ell(\delta) + \mathrm{codim}_X(Y')$.

To check the claim, it suffices to join Y' to $P_{\alpha\beta}Y'$ by an oriented path γ' beginning with a simple edge. As above, we reduce to the case where G equals $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ or $\mathrm{PGL}(3)$, and H is not contained in a Borel subgroup of G; Moreover, H has finite index in its normalizer. Using the fact that $\Gamma(G/H)$ contains a double edge followed by a simple edge, one checks that H is a product of subgroups of $\mathrm{PGL}(2)$ if $G = \mathrm{PGL}(2) \times \mathrm{PGL}(2)$; and if $G = \mathrm{PGL}(3)$, then H is conjugate to the subgroup of Example 1, or to its transpose. The path γ' exists in all of these cases. □

From Proposition 5, we will deduce a criterion for the graph of a spherical variety to contain simple edges only. To formulate it, we need more notation, and a preliminary result.

Let $D \in \mathscr{D}(X)$ be a color; then D is the closure of its intersection with the open G-orbit G/H. Let $\tilde{D}$ be the preimage in G of $D \cap G/H$. Replacing G by a finite cover, we may assume that $\tilde{D}$ is the divisor of a regular function f_D on G. Then f_D is an eigenvector of B acting by left multiplication; let ω_D be its weight. Since f_D is uniquely defined up to multiplication by a regular invertible function on G, then ω_D is unique up to addition of a character of G. In particular, for any $\alpha \in \Delta$, the number $\langle \omega_D, \check{\alpha} \rangle$ is a non-negative integer depending only on D and α.

Lemma 8. — *(i) The degree $d(D,\alpha)$ of the morphism $\pi_{D,\alpha} : P_\alpha \times^B D \to X$ equals $\langle \omega_D, \check{\alpha} \rangle$ if $\pi_{D,\alpha}$ is generically finite; otherwise, $\langle \omega_D, \check{\alpha} \rangle = 0$.*

(ii) For any G-orbit closure X' in X and for any $D' \in \mathscr{D}(X')$, there exists $D \in \mathscr{D}(X)$ such that D' is an irreducible component of $D \cap X'$. Then $\langle \omega_{D'}, \check{\alpha} \rangle \leqslant \langle \omega_D, \check{\alpha} \rangle$ for all $\alpha \in \Delta$.

Proof. — (i) Note that D is P_α-stable if and only if f_D is an eigenvector of P_α, that is, ω_D extends to a character of that group. This amounts to: $\langle \omega_D, \check{\alpha} \rangle = 0$.

Let V be the H-stable divisor in G/B corresponding to the B-stable divisor $D \cap G/H$. Then V is the zero scheme of a section of the homogeneous line bundle on G/B associated with the character ω_D of B. Let $p : G/B \to G/P_\alpha$ be the natural map, then $d(D,\alpha)$ equals the degree of the restriction $p_V : V \to G/P_\alpha$. The latter degree is the intersection number of V with a fiber of p, that is, $\langle \omega_D, \check{\alpha} \rangle$.

(ii) For the first assertion, it suffices to show existence of $D \in \mathscr{D}(X)$ containing D' and not containing X'; but this follows from [16] Theorem 3.1. For the second assertion, note that P_α stabilizes D' if it stabilizes D. Thus, $\langle \omega_{D'}, \check{\alpha} \rangle = 0$ if $\langle \omega_D, \check{\alpha} \rangle = 0$. On the other hand, if $\langle \omega_D, \check{\alpha} \rangle = 1$ then $\pi_{D,\alpha}$ is birational. Restricting to $P_\alpha \times^B D'$, it follows that $\pi_{D',\alpha}$ is birational if generically finite. □

A direct consequence of Lemma 8 and Proposition 5 is

Corollary 2. — *If G is simply-laced, then the following conditions are equivalent:*

(i) Each edge of $\Gamma(X)$ is simple.

(ii) For any $D \in \mathscr{D}(X)$ and $\alpha \in \Delta$, we have $\langle \omega_D, \check{\alpha} \rangle \leqslant 1$.

This criterion applies, e.g., to all embeddings of the following symmetric spaces: $\mathrm{GL}(p+q)/\mathrm{GL}(p)\times\mathrm{GL}(q)$, $\mathrm{SL}(2n)/\mathrm{SP}(2n)$, $\mathrm{SO}(2n)/\mathrm{GL}(n)$ and E_6/F_4. For this, one uses the explicit description of colors of symmetric spaces given in [29]. Further applications will be given after Theorem 3 below.

Note that Corollary 2 does not extend to multiply-laced groups G. Consider, for example, $G = \mathrm{SO}(2n+1)$ and its subgroup $H = \mathrm{O}(2n)$, the stabilizer of a non-degenerate line in $\mathbb{C}^{2n+1}$. Then the homogeneous space G/H is spherical of rank 1 and its graph consists of a unique oriented path: a double edge followed by $n-1$ simple edges.

2. Orbit closures in regular varieties

Recall from [2] that a variety X with an action of G is called *regular* if it satisfies the following three conditions:

(i) X is smooth and contains a dense G-orbit whose complement is a union of irreducible smooth divisors (the *boundary divisors*) with normal crossings.

(ii) Any G-orbit closure in X is the transversal intersection of those boundary divisors that contain it.

(iii) For any $x \in X$, the normal space to the orbit Gx contains a dense orbit of the isotropy group of x.

Any regular G-variety X contains only finitely many G-orbits. Their closures are the G-stable subvarieties of X; they are regular G-varieties as well.

Regular varieties are closely related with spherical varieties: any complete regular G-variety is spherical, and any spherical G-homogeneous space G/H admits an open equivariant embedding into a complete regular G-variety X, see [3] 2.2.

Let Z be a closed G-orbit in complete regular X, then the isotropy group of each point of Z is a parabolic subgroup of G. Thus, Z contains a unique T-fixed point z such that Bz is open in Z; we shall call z the *base point* of Z. In fact, the isotropy group $Q = G_z$ is opposed to $P(X)$, see e.g. [3] 2.2.

We next recall the local structure of complete regular varieties, see e.g. [3] 2.3. For such a variety X, set $P = P(X)$ and $L = L(X)$. Let X_0 be the set of all $x \in X$ such that Bx is open in Gx. Then X_0 is an open P-stable subset of X: the complement of the union of all colors. Moreover, there exists an L-stable subvariety S of X_0, fixed pointwise by $[L, L]$, such that the map

$$\begin{array}{ccc} R_u(P)\times S & \to & X_0 \\ (g,x) & \mapsto & gx \end{array}$$

is an isomorphism. As a consequence, S is a smooth toric variety (for a quotient of T) of dimension $r(X)$, the rank of X; moreover, S meets each G-orbit along a unique T-orbit. Let $\varphi : X_0 \cong R_u(P)\times S \to S$ be the second projection, then φ is L-equivariant; it can be seen as the quotient map by the action of $R_u(P)$.

We now turn to B-orbit closures. Let $Y \in \mathscr{B}(X)$; since GY is regular, we may assume that $GY = X$. Then, by [7] 1.4, Y meets all G-orbit closures properly; moreover, for any closed G-orbit Z, the irreducible components of $Y \cap Z$ are the Schubert varieties $\overline{Bw^{-1}z}$ where $w \in$

$W(Y)$, and the intersection multiplicity of Y and Z along $\overline{Bw^{-1}z}$ equals $d(Y,w)$. To describe the intersection of Y with arbitrary G-orbit closures, we shall study the local structure of Y along $\overline{Bw^{-1}z}$ for a fixed $w \in W(Y)$. It will be more convenient to consider the translate wY along $\overline{wBw^{-1}z}$.

Note that wY meets X_0 (because $\overline{BwY} = X$), and that the intersection $wY \cap X_0$ is stable by the group $wBw^{-1} \cap P$. The latter contains $R_u(P) \cap wUw^{-1}$ as a normal subgroup. We shall see that $R_u(P) \cap wUw^{-1}$ acts freely on $wY \cap X_0$, with section

$$S_{Y,w} = wY \cap (U \cap wU^- w^{-1})S.$$

Note that $U \cap wU^- w^{-1}$ is contained in $R_u(P)$, because $w^{-1} \in W^P$. Thus, $S_{Y,w}$ is a closed T-stable subvariety of $wY \cap X_0$. Let

$$\varphi_{Y,w} : S_{Y,w} \to S$$

be the restriction of $\varphi : X_0 \to S$, then $\varphi_{Y,w}$ is T-equivariant.

Proposition 6. — *Keep notation as above.*

(i) The map

$$\begin{array}{ccc} (R_u(P) \cap wUw^{-1}) \times S_{Y,w} & \to & wY \cap X_0 \\ (g,x) & \mapsto & gx \end{array}$$

is an isomorphism.

(ii) The variety $S_{Y,w}$ is irreducible and meets each G-orbit along a unique T-orbit. In particular, $S_{Y,w} \cap GY^0$ is a unique T-orbit, dense in $S_{Y,w}$ and contained in wY^0; and $S_{Y,w} \cap Z = \{z\}$ for any closed G-orbit Z with base point z.

(iii) The morphism $\varphi_{Y,w}$ is finite surjective of degree $d(Y,w)$.

Proof. — (i) The product map $(R_u(P) \cap wUw^{-1}) \times (R_u(P) \cap wU^- w^{-1}) \to R_u(P)$ is an isomorphism; moreover, $R_u(P) \cap wU^- w^{-1} = U \cap wU^- w^{-1}$. Therefore, the product map

$$(R_u(P) \cap wUw^{-1}) \times (U \cap wU^- w^{-1})S \to X_0$$

is an isomorphism. The assertion follows by intersecting with wY.

(ii) and (iii) The union of all G-orbits in X that contain Z in their closure is a G-stable open subset of X. Thus, we may assume that Z is the unique closed G-orbit in X. Let $D_1, \ldots, D_r$ be the boundary divisors, then $r = r(X)$. Moreover, S is isomorphic to affine space $\mathbb{A}^r$ with coordinate functions $x_1, \ldots, x_r$, equations of $D_1 \cap S, \ldots, D_r \cap S$. The compositions $f_1 = x_1 \circ \varphi, \ldots, f_r = x_r \circ \varphi$ are equations of $D_1 \cap X_0, \ldots, D_r \cap X_0$; they generate the ideal of $Z \cap X_0 = Bz$ in X_0. The map $\varphi : X_0 \to S$ identifies to $(f_1, \ldots, f_r) : X_0 \to \mathbb{A}^r$. The intersections of G-orbit closures with X_0 are the pull-backs of coordinate subspaces of $\mathbb{A}^r$.

By (i), $S_{Y,w}$ is irreducible. We check that $S_{Y,w} \cap Z = \{z\}$. For this, note that the product map

$$(R_u(P) \times wUw^{-1}) \times (S_{Y,w} \cap Z) \to wY \cap X_0 \cap Z = wY \cap Bz$$

is an isomorphism. Moreover, since Y meets Z properly, with $\overline{Bw^{-1}z}$ as an irreducible component, it follows that $wY \cap Bz$ is equidimensional, with $\overline{wBw^{-1}z} \cap Bz = (B \cap wBw^{-1})z$ as an irreducible component. The latter is isomorphic to $R_u(P) \cap wUw^{-1}$. Thus, the T-stable set

$S_{Y,w} \cap Z$ is finite, so that it consists of T-fixed points. Since z is the unique T-fixed point in Bz, our assertion follows.

The map $\varphi_{Y,w} : S_{Y,w} \to S$ identifies with $(f_1, \ldots, f_r) : S_{Y,w} \to \mathbb{A}^r$. We just saw that the set-theoretical fiber of 0 is $\{z\}$. Since 0 is the unique closed T-orbit in $\mathbb{A}^r$, all fibers of $\varphi_{Y,w}$ are finite. Thus, $S_{Y,w}$ contains a dense T-orbit. Since $S_{Y,w}$ is affine and contains a T-fixed point z, it follows that $\varphi_{Y,w}$ is finite and that the pull-back of any T-orbit in S is a unique T-orbit. This implies (ii).

Finally, we check that the degree of $\varphi_{Y,w}$ equals $d(Y,w)$, that is, the degree of the natural map $\overline{BwB} \times^B Y \to X$. For this, note that the map

$$U \cap wU^- w^{-1} \to \overline{BwB}/B, \; g \mapsto gwB/B$$

is an open immersion. Thus, $d(Y,w)$ is the degree of the product map $(U \cap wU^- w^{-1}) \times wY \to X$, or, equivalently, of its restriction

$$p : (U \cap wU^- w^{-1}) \times (wY \cap X_0) \to X_0.$$

The latter map fits into a commutative diagram

$$\begin{array}{ccc} (U \cap wU^- w^{-1}) \times (wY \cap X_0) & \to & X_0 \\ \downarrow & & \downarrow \\ S_{Y,w} & \to & S, \end{array}$$

where the bottom horizontal map is $\varphi_{Y,w}$; indeed,

$$(U \cap wU^- w^{-1}) \times (wY \cap X_0) \cong (R_u(P) \cap wU^- w^{-1}) \times (R_u(P) \cap wU w^{-1}) \times S_{Y,w}$$

by (i). Moreover, the fibers of the right (resp. left) vertical map are isomorphic to $R_u(P)$ (resp. to $(R_u(P) \cap wU^- w^{-1}) \times (R_u(P) \cap wU w^{-1}) \cong R_u(P)$.) Thus, the diagram is cartesian, and the degree of p equals the degree of $\varphi_{Y,w}$. □

Thus, we can view $S_{Y,w}$ as a "slice" in wY to $wBw^{-1}z = (R_u(P) \cap wU w^{-1})z$ at z. But $S_{Y,w}$ may be non transversal to wY at z: indeed, the intersection multiplicity of $S_{Y,w}$ and wY at z equals the intersection multiplicity of Z and Y along $\overline{Bw^{-1}z}$, and the latter equals $d(Y,w)$ by [7] 1.4 (alternatively, this can be deduced from Proposition 6 (iii).) On the other hand, it is not clear whether $S_{Y,w}$ is smooth, that is, $Y \cap w^{-1}X_0$ consists of smooth points of Y; see Corollary 3 below for a partial answer to this question.

We now relate the "slices" associated with both endpoints of an edge in $\Gamma(X)$. Let $Y \in \mathscr{B}(X)$ and let $\alpha \in \Delta$ raising Y. Choose $v \in W(P_\alpha Y)$, then $w = vs_\alpha$ is in $W(Y)$, and $\ell(w) = \ell(v) + 1$. Thus, $v(\alpha) \in \Phi^+ \cap w(\Phi^-)$. Let $U_{v(\alpha)}$ be the corresponding unipotent subgroup of dimension 1, then $U_{v(\alpha)}$ is contained in $R_u(P) \cap vUv^{-1}$.

Proposition 7. — *With notation as above, $S_{Y,w}$ is contained in $U_{v(\alpha)} S_{P_\alpha Y,v}$ and the latter is isomorphic to $U_{v(\alpha)} \times S_{P_\alpha Y,\tau}$. Denoting by*

$$\varphi_{Y,\alpha} : S_{Y,w} \to S_{P_\alpha Y,v}$$

the corresponding projection, then $\varphi_{Y,w} = \varphi_{P_\alpha Y,v} \circ \varphi_{Y,\alpha}$. Moreover, $\varphi_{Y,\alpha}$ is finite surjective of degree $d(Y,\alpha)$.

Proof. — We have

$$S_{Y,w} = wY \cap (U \cap wU^-w^{-1})S = wY \cap U_{v(\alpha)}(U \cap vU^-v^{-1})S$$
$$\subseteq vP_\alpha Y \cap U_{v(\alpha)}(U \cap vU^-v^{-1})S = U_{v(\alpha)}(vP_\alpha Y \cap (U \cap vU^-v^{-1})S) = U_{v(\alpha)}S_{P_\alpha Y,v}.$$

Moreover, since $U_{v(\alpha)} \subseteq R_u(P) \cap vUv^{-1}$, the product map $U_{v(\alpha)} \times S_{P_\alpha Y,v} \to U_{v(\alpha)}S_{P_\alpha Y,v}$ is an isomorphism. Now the equality $\varphi_{Y,w} = \varphi_{P_\alpha Y,v} \circ \varphi_{Y,\alpha}$ follows from the definitions. Together with Proposition 6 (iii), it implies that $\varphi_{Y,\alpha}$ is finite surjective of degree $d(Y,w)d(P_\alpha Y,v)^{-1} = d(Y,\alpha)$. □

Using Proposition 6, we analyze the intersection of a B-orbit closure with an arbitrary G-orbit closure, generalizing [7] Theorem 1.4.

THEOREM 1. — *Let X be a complete regular G-variety, let $Y \in \mathscr{B}(X)$ be such that $GY = X$ and let X' be a G-orbit closure in X. Then $W(Y)$ is the disjoint union of the $W(C)$ where C runs over all irreducible components of $Y \cap X'$. Moreover, for any such C and $w \in W(C)$, we have*

$$d(Y,w) = d(C,w)\ i(C,Y\cdot X';X)$$

where $i(C,Y\cdot X';X)$ denotes the intersection multiplicity of Y and X' along C in X. As a consequence, this multiplicity is a power of 2.

Proof. — By [7] Lemma 1.3, $W(Y)$ is the union of the $W(C)$. Choose C and $w \in W(C)$, then $C \cap w^{-1}X_0$ is an irreducible component of $Y \cap w^{-1}X_0 \cap X'$. The latter is isomorphic to $(U \cap w^{-1}R_u(P)) \times w^{-1}(S_{Y,w} \cap X')$, and $S_{Y,w} \cap X'$ is a unique T-orbit, by Proposition 6. It follows that $Y \cap w^{-1}X_0 \cap X' = C \cap w^{-1}X_0$ is irreducible, so that C is uniquely determined by w. Equivalently, the $W(C)$ are pairwise disjoint.

Let Z be a closed G-orbit in X', then

$$d(Y,w) = i(\overline{Bw^{-1}z},Y\cdot Z;X) = i(\overline{Bw^{-1}z} \cap w^{-1}X_0,(Y \cap w^{-1}X_0)\cdot(Z \cap w^{-1}X_0);w^{-1}X_0),$$

where the former equality follows from [7] 1.4, and the latter from [13] 8.2. Moreover, we have by Proposition 6: $\overline{Bw^{-1}z} \cap w^{-1}X_0 = Bw^{-1}z$ and $Z \cap w^{-1}X_0 = w^{-1}Bz$. Thus,

$$d(Y,w) = i(Bw^{-1}z,(Y \cap w^{-1}X_0)\cdot w^{-1}Bz,w^{-1}X_0).$$

Using the fact that $Y \cap w^{-1}X_0 \cap X' = C \cap w^{-1}X_0$ is irreducible, together with associativity of intersection multiplicities (see [13] 7.1.8), we obtain

$$d(Y,w) = i(Bw^{-1}z,(C \cap w^{-1}X_0)\cdot w^{-1}Bz;w^{-1}X_0 \cap X')\ i(C,Y\cdot X';X)$$
$$= i(\overline{Bw^{-1}z},C\cdot Z;X')\ i(C,Y\cdot X';X) = d(C,w)\ i(C,Y\cdot X';X).$$

□

These results motivate the following

Definition. A B-orbit closure Y in an arbitrary spherical variety X is *multiplicity-free* if $d(Y,w) = 1$ for all $w \in W(Y)$. Equivalently, the edges of all oriented paths in $\Gamma(X)$ with source Y are simple.

For example, Y is multiplicity-free if $r(Y) = r(GY)$, or if the isotropy group in G of a point of Y^0 is contained in a Borel subgroup of G (this follows from Lemma 6.)

Other examples of multiplicity-free orbit closures arise from parabolic induction: if $X = G \times^{P_I} X'$ is induced from X' and if $Y = \overline{BwY'}$ with $w \in W^I$ and $Y' \in \mathscr{B}(X')$, then Y is multiplicity-free if and only if Y' is (this follows from Lemma 7 or, alternatively, from [7] 1.2).

COROLLARY 3. — *Let X be a complete regular G-variety, Y a multiplicity-free B-stable subvariety such that $GY = X$, and X' a G-orbit closure in X. Then all irreducible components of $Y \cap X'$ are multiplicity-free B-orbit closures of X', and the corresponding intersection multiplicities equal* 1. *Moreover, for any $w \in W(Y)$, the map $\varphi_{Y,w} : S_{Y,w} \to S$ is an isomorphism. As a consequence, $Y \cap w^{-1}X_0$ consists of smooth points of Y.*

Proof. — The first assertion follows from Theorem 1. By Proposition 6, $\varphi_{Y,w}$ is finite surjective of degree 1, hence an isomorphism because S is smooth. □

We next characterize those B-orbit closures that are multiplicity-free, in terms of the intersection numbers $\int_X [Y] \cdot [Y']$ where $Y' \in \mathscr{B}(X)$. Here $\int_X [Y] \cdot [Y']$ denotes the degree of the product of the classes of Y, Y' in the Chow ring of X. The latter is isomorphic to the integral cohomology ring of X; it is generated as an abelian group by classes of B-stable subvarieties.

COROLLARY 4. — *Let X be a complete regular G-variety and let $Y \in \mathscr{B}(X)$ such that $GY = X$. Then the numbers $\int_X [Y] \cdot [Y']$ are powers of* 2, *for all $Y' \in \mathscr{B}(X)$. Moreover, Y is multiplicity-free if and only if $\int_X [Y] \cdot [Y']$ equals* 0 *or* 1, *for any $Y' \in \mathscr{B}(X)$.*

Proof. — Let $Y' \in \mathscr{B}(X)$. By [8] 1.4 Corollary, $\int_X [Y] \cdot [Y'] \neq 0$ if and only if: $\dim(Y) + \dim(Y') = \dim(X)$, and Y meets $w_0 Y'$. Under these hypotheses, $Y \cap w_0 Y'$ consists of a unique point y, fixed by T. Moreover, the proof of [*loc. cit.*] shows that $w_0 y \in Y'^0$. Thus, $\overline{By}$ and $\overline{B^- y} = w_0 \overline{Bw_0 y} = w_0 Y'$ meet transversally at y in $\overline{Gy} = GY'$. As a consequence, we have

$$\dim(\overline{By}) = \dim(GY') - \dim(w_0 Y') = \dim(GY') + \dim(Y) - \dim(X) = \dim(Y \cap GY').$$

It follows that $\overline{By}$ is the unique irreducible component of $Y \cap GY'$ through y.

Using the projection formula, we obtain

$$\int_X [Y] \cdot [Y'] = \int_{GY'} ([Y] \cdot [GY']) \cdot [Y']$$
$$= d(\overline{By}; Y \cdot GY'; X) \int_{GY'} \overline{By} \cdot [Y'] = d(\overline{By}; Y \cdot GY'; X).$$

Thus, by Theorem 1, $\int_X [Y] \cdot [Y']$ is a power of 2; if moreover Y is multiplicity-free, then $\int_X [Y] \cdot [Y'] = 1$.

Conversely, assume that $\int_X [Y] \cdot [Y']$ equals 0 or 1 for all $Y' \in \mathscr{B}(X)$. Let then $w \in W(Y)$; choose a closed G-orbit Z with base point z and consider $Y' = \overline{Bw_0 w^{-1} z}$. Then $\dim(Y') = \operatorname{codim}_Z(\overline{Bw^{-1}z}) = \dim(X) - \dim(Y)$, and Y meets $w_0 Y'$ at $w^{-1}z$. Thus, $\int_X [Y] \cdot [Y'] = d(Y, w)$ by the argument above. It follows that Y is multiplicity-free. □

We now show that the intersections of B-orbit closures with G-orbit closures in a complete regular G-variety satisfy Hartshorne's connectedness theorem, see [12] 18.2. That theorem is proved there for schemes of depth at least 2; but B-orbit closures may have depth 1 at some points, see Example 5 in the next section.

THEOREM 2. — *Let X be a complete regular G-variety, Y a B-orbit closure, and X' a G-orbit closure in X. Then $Y \cap X'$ is connected in codimension* 1 *(that is, the complement in $Y \cap X'$ of any closed subset of codimension at least* 2 *is connected.)*

Proof. — We may assume that $GY = X$. If $X' = Z$ is a closed G-orbit, then the assertion follows from the description of $Y \cap Z$ in terms of $W(Y)$, together with Propositions 2 and 3. Indeed, for any $w \in W$ such that $w^{-1} \in W^{\Delta(X)}$, we have $\ell(w) = \ell(w^{-1}) = \mathrm{codim}_Z(\overline{Bw^{-1}z})$, where z is the base point of Z.

For arbitrary X', let Z be a closed G-orbit in X'. Let Y_1', Y_2' be unions of irreducible components of $Y \cap X'$ such that $Y \cap X' = Y_1' \cup Y_2'$. Then $Y_1' \cap Z$ and $Y_2' \cap Z$ are unions of irreducible components of $Y' \cap Z$ (for any irreducible component C of $Y \cap X'$ meets Z properly in X'); Moreover, their intersection has codimension 1 in $Y_1' \cap Z$ and $Y_2' \cap Z$, by the first step of the proof. It follows that $Y_1' \cap Y_2'$ has codimension 1 in both Y_1' and Y_2'. □

3. Singularities of orbit closures

We begin by recalling the notion of rational singularities, see e.g. [15] p. 50.

Let Y be a variety. Choose a resolution of singularities $\varphi : Z \to Y$, that is, Z is smooth and φ is proper and birational. Then the sheaves $R^i\varphi_*\mathcal{O}_Z$ $(i \geqslant 0)$ are independent of the choice of Z. The singularities of Y are rational if $R^i\varphi_*\mathcal{O}_Z = 0$ for all $i \geqslant 1$ and $\varphi_*\mathcal{O}_Z = \mathcal{O}_Y$; the latter condition is equivalent to normality of Y. Varieties with rational singularities are Cohen-Macaulay.

Let now X be a spherical variety and Y a B-stable subvariety. If Y is G-stable, then its singularities are rational, see e.g. [6]. But this does not extend to arbitrary Y: generalizing Example 1 in Section 1, we shall construct examples of B-orbit closures of arbitrary dimension but of depth 1 at some points. In particular, such orbit closures are neither normal nor Cohen-Macaulay.

Example 5. Let X be the space of unordered pairs $\{p, q\}$ of distinct points in projective space $\mathbb{P}^n$. The group $G = \mathrm{GL}(n+1)$ acts transitively on X; one checks that X is spherical of rank 1. Let $\mathbb{P}^m$ be a proper linear subspace of $\mathbb{P}^n$ of positive dimension m. Consider the space

$$Y_m = \{\{p, q\} \in X \mid p \in \mathbb{P}^m \text{ or } q \in \mathbb{P}^m\},$$

a subvariety of X of codimension $n - m$. The stabilizer P_m of $\mathbb{P}^m$ in G, a maximal parabolic subgroup, stabilizes Y_m as well; in fact, Y_m contains an open P_m-orbit (the subset of all $\{p, q\}$ such that $p \in \mathbb{P}^m$ but $q \in \mathbb{P}^n - \mathbb{P}^m$) and its complement

$$Y_m' = \{\{p, q\} \mid p, q \in \mathbb{P}^m,\ p \neq q\}$$

is a unique P_m-orbit of codimension $n - m$ in Y_m. Thus, Y_m is the closure of a B-orbit; one checks that $r(Y_m) = 0$ and $r(Y_m') = 1$.

The map

$$\begin{array}{rccc} \nu : & \mathbb{P}^m \times \mathbb{P}^n & \to & Y_m \\ & (p, q) & \mapsto & \{p, q\} \end{array}$$

is an isomorphism over the open P_m-orbit, but has degree 2 over Y_m'. Thus, ν is the normalization of Y_m, and the latter is not normal. Moreover, Y_m' is the singular locus of Y_m.

Observe that Y_{n-1} is Cohen-Macaulay, as a divisor in X (for $n = 2$ and $m = 1$, we recover Example 1 in Section 1.) But if $m < n - 1$, then Y_m has depth 1 along Y'_m by Serre's criterion, see [12] 18.3. In particular, Y_m is not Cohen-Macaulay.

Let $\alpha_1, \dots, \alpha_n$ be the simple roots of G. Then $P_{\alpha_m} Y_m = Y_{m+1}$, and α_m is the unique simple root raising Y_m. The corresponding edge in $\Gamma(X)$ is simple, except for $m = n - 1$. Thus, Y_m is the source of a unique oriented path with target X, and the top edge of this path is double. In particular, Y_m is not multiplicity-free.

Such examples of bad singularities do not occur for multiplicity-free orbit closures:

Theorem 3. — *Let Y be a multiplicity-free B-orbit closure in a spherical G-variety X. If no simple normal subgroup of G of type G_2, F_4 or E_8 fixes points of X, then the singularities of Y are rational.*

Proof. — We begin with a reduction to the case where no simple normal subgroup of G fixes points of X. For this, we may assume that G is the direct product of a torus with a family of simple, simply connected subgroups; let Γ be one of them. If Γ is not of type G_2, F_4 or E_8, then there exists a simple, simply connected group $\tilde{\Gamma}$ together with a maximal proper parabolic subgroup $\tilde{P}$ such that a Levi subgroup $\tilde{L}$ has the same adjoint group as Γ (indeed, add an edge to the Dynkin diagram of Γ to obtain that of $\tilde{\Gamma}$.) Then $\tilde{L}$ is the quotient of $\Gamma \times \mathbb{C}^*$ by a finite central subgroup F. We may assume moreover that $\mathbb{C}^*$ maps injectively to $\tilde{L}$, that is, $\mathbb{C}^* \cap F$ is trivial. Then the first projection $p_1 : F \to \Gamma$ is injective.

We claim that the second projection $p_2 : F \to \mathbb{C}^*$ is injective as well. Indeed, as $\tilde{\Gamma}$ is simply connected, its Picard group is trivial; as some open subset of $\tilde{\Gamma}$ is the direct product of $\tilde{L}$ with an affine space, the Picard group of $\tilde{L}$ is trivial as well. But $\tilde{L} = (\Gamma \times \mathbb{C}^*)/F$ is the total space of the line bundle over $\Gamma/p_1(F)$ associated with the character p_2 of $p_1(F) \cong F$, minus the zero section. Thus, $\mathrm{Pic}(\tilde{L})$ is the quotient of $\mathrm{Pic}(\Gamma/p_1(F))$ by the class of that line bundle. Moreover, $\mathrm{Pic}(\Gamma/p_1(F))$ is isomorphic to the character group of F, as Γ is simply connected. Therefore p_2 generates the character group of F. Since F is abelian, the claim follows.

By that claim, $\Gamma \cap F$ is trivial; thus, Γ embeds into $\tilde{L}$ as its derived subgroup. We shall treat $p_2 : F \to \mathbb{C}^*$ as an inclusion, which defines an action of F on $\mathbb{C}^*$. On the other hand, F acts on X via $p_2 : F \to \Gamma$, and this action commutes with that of the remaining factors of G. Thus, $X \times^F \mathbb{C}^*$ is a variety with an action of the product $\Gamma \times^F \mathbb{C}^* \cong \tilde{L}$ with the remaining factors of G. This variety is spherical and fibers equivariantly over $\mathbb{C}^*/F \cong \mathbb{C}^*$, with fiber X. Thus, we may assume that the action of Γ on X extends to an action of $\tilde{L}$. Now the parabolically induced variety $\tilde{\Gamma} \times^{\tilde{P}} X$ contains Y as a multiplicity-free subvariety (Lemma 7) but contains no fixed point of $\tilde{\Gamma}$. Iterating this argument removes the fixed points of all simple normal subgroups of G.

We now reduce to the case where X is projective. For this, we use embedding theory of spherical homogeneous spaces, see [16]. We may assume that X contains a unique closed G-orbit Z (for X is the union of G-stable open subsets, each of which contains a unique closed G-orbit.) Together with Lemma 2, the assumption that no simple factor of G fixes points of X amounts to: $P(Z)$ contains no simple factor of G. Let $\mathcal{D}_Z(X)$ be the set of all colors D that contain Z; then we can find an equivariant projective completion $\overline{X}$ of X such that $\mathcal{D}_{Z'}(X) \subseteq \mathcal{D}_Z(X)$ for any G-orbit closure Z' in $\overline{X}$. By Lemma 2, it follows that $P(Z') \subseteq P(Z)$, and that no simple factor of G fixes points of $\overline{X}$.

We next reduce to an affine situation, in the following standard way. Choose an ample G-linearized line bundle $\mathscr{L}$ over X. Replacing $\mathscr{L}$ by a positive power, we may assume that $\mathscr{L}$ is very ample and that X is projectively normal in the corresponding projective embedding. Let $\hat{X}$ be the affine cone over X. This is a spherical variety under the group $\hat{G} = G \times \mathbb{C}^*$, and the origin 0 is the unique fixed point of any simple normal subgroup of $\hat{G}$, since $[\hat{G}, \hat{G}] = [G, G]$. Moreover, the affine cone $\hat{Y}$ over Y is stable under the Borel subgroup $B \times \mathbb{C}^*$ of $\hat{G}$, and is multiplicity-free. Thus, we may assume that X is affine with a fixed point 0, and we have to show that Y has rational singularities outside 0.

By [6], the G-variety GY is spherical, with rational singularities, so that we may assume that $GY = X$. We argue then by induction on the codimension of Y in X.

Let $N_G(Y)$ be the set of all $g \in G$ such that $gY = Y$. This is a proper standard parabolic subgroup of G, acting on Y by automorphisms. Let

$$\varphi : Z \to Y$$

be a $N_G(Y)$-equivariant resolution of singularities. Denote by $\mathbb{C}[Y]$ (resp. $\mathbb{C}[Z]$) the algebra of regular functions on Y (resp. Z). Then $\mathbb{C}[Z]$ is a finite $\mathbb{C}[Y]$-module. Moreover, we have an exact sequence of $\mathbb{C}[Y]$-modules

$$0 \to \mathbb{C}[Y] \to \mathbb{C}[Z] \to C \to 0$$

where the support of C is the non-normal locus N of Y, by Zariski's main theorem. Note that $N_G(Y)$ acts on C compatibly with its $\mathbb{C}[PY]$-module structure. We first show that C is supported at 0, that is, Y is normal outside 0.

Let α be a simple root raising Y and let $P = P_\alpha$. Let

$$f = f_{Y,\alpha} : P \times^B Y \to P/B$$

be the fiber bundle with fiber the B-variety Y; let

$$\pi = \pi_{Y,\alpha} : P \times^B Y \to PY$$

be the natural morphism. Then the map

$$\pi^* : \mathbb{C}[PY] \to \mathbb{C}[P \times^B Y]$$

is injective, and makes $\mathbb{C}[P \times^B Y]$ a finite $\mathbb{C}[PY]$-module. Since Y is multiplicity-free, π is birational and PY is multiplicity-free as well. By the induction assumption, PY is normal outside 0. Therefore, the cokernel of π^* is supported at 0, by Zariski's main theorem again.

The B-equivariant resolution $\varphi : Z \to Y$ induces a P-equivariant resolution

$$\rho : P \times^B Z \to P \times^B Y.$$

Composing with π, we obtain a P-equivariant birational morphism

$$\tilde{\pi} : P \times^B Z \to PY.$$

As above, the map

$$\tilde{\pi}^* : \mathbb{C}[PY] \to \mathbb{C}[P \times^B Z]$$

is injective and its cokernel is supported at 0. We shall treat π^* and $\tilde{\pi}^*$ as inclusions.

We have

$$\mathbb{C}[P\times^B Y] = H^0(P\times^B Y, \mathcal{O}_{P\times^B Y}) = H^0(P/B,\ f_*\mathcal{O}_{P\times^B Y}).$$

Moreover, $f_*\mathcal{O}_{P\times^B Y}$ is the P-linearized sheaf on P/B associated with the (rational, infinite-dimensional) B-module

$$H^0(f^{-1}(B/B), \mathcal{O}_{P\times^B Y}) = \mathbb{C}[Y].$$

We shall use the notation

$$f_*\mathcal{O}_{P\times^B Y} = \underline{\mathbb{C}[Y]}.$$

Then

$$\mathbb{C}[PY] \subseteq H^0(P/B, \underline{\mathbb{C}[Y]}) \subseteq H^0(P/B, \underline{\mathbb{C}[Z]}) = \mathbb{C}[P\times^B Z]$$

and these $\mathbb{C}[PY]$-modules coincide outside 0.

Consider the exact sequence of P-linearized sheaves on P/B:

$$0 \to \underline{\mathbb{C}[Y]} \to \underline{\mathbb{C}[Z]} \to \underline{C} \to 0.$$

Since the restriction map $\mathbb{C}[PY] \to \mathbb{C}[Y]$ is surjective, the B-module $\mathbb{C}[Y]$ is the quotient of a rational P-module. Since P/B is a projective line, it follows that $H^1(P/B, \underline{\mathbb{C}[Y]}) = 0$. Thus, we have an exact sequence of $\mathbb{C}[PY]$-modules

$$0 \to H^0(P/B, \underline{\mathbb{C}[Y]}) \to H^0(P/B, \underline{\mathbb{C}[Z]}) \to H^0(P/B, \underline{C}) \to 0.$$

It follows that $H^0(P/B, \underline{C})$ is supported at 0. Now normality of Y outside 0 is a consequence of the following

LEMMA 9. — *Let C be a finite $\mathbb{C}[Y]$-module with a compatible action of $N_G(Y)$, such that the $\mathbb{C}[PY]$-module $H^0(P/B, \underline{C})$ is supported at 0 for any minimal parabolic subgroup P that raises Y. Then C is supported at 0.*

Proof. — Otherwise, choose an irreducible component $Y' \neq \{0\}$ of the support of C. Let $I(Y')$ be the ideal of Y' in $\mathbb{C}[Y]$. Define a submodule C' of C by

$$C' = \{c \in C \mid I(Y')c = 0\}.$$

Observe that the support of C' is Y' (indeed, the ideal $I(Y')$ is a minimal prime of the support of C; thus, this ideal is an associated prime of C.) Note that $N_G(Y)$ stabilizes Y' and acts on C'. Moreover, $H^0(P/B, \underline{C'})$ is a $\mathbb{C}[PY']$-module supported at 0 (as a $\mathbb{C}[PY]$-submodule of $H^0(P/B, \underline{C})$.)

We claim that Y' is G-stable. Otherwise, let α be a simple root raising Y'; then α raises Y. Define as above the maps

$$f' : P\times^B Y' \to P/B \text{ and } \pi' : P\times^B Y' \to PY'.$$

The $\mathbb{C}[Y']$-module C' with a compatible B-action induces a P-linearized sheaf $\mathscr{C}'$ on $P\times^B Y'$, and we have $f'_*\mathscr{C}' = \underline{C'}$ as P-linearized sheaves on P/B. It follows that the $\mathbb{C}[PY']$-module $H^0(P\times^B Y', \mathscr{C}') = H^0(P/B, \underline{C'})$ is supported at 0. On the other hand, we have $H^0(P\times^B Y', \mathscr{C}') = H^0(PY', \pi'_*\mathscr{C}')$. Moreover, the map $\pi' : P\times^B Y' \to PY'$ is generically finite (as P raises Y'), and the support of $\mathscr{C}'$ is $P\times^B Y'$ (as the support of C' is Y'). Thus, the support of $\pi'_* \mathscr{C}'$ is PY', and the same holds for the support of $H^0(PY', \pi'_* \mathscr{C}') = H^0(P/B, \underline{C'})$. This contradicts the assumption that $Y' \neq \{0\}$. The claim is proved.

Let L be the Levi subgroup of P containing T, then $P/B = [L,L]/B \cap [L,L]$. Since Y' is G-stable, it is not fixed pointwise by $[L,L]$ (here we use the assumption that no simple normal subgroup of G fixes points of $X - \{0\}$.) Since Y' is affine, $[L,L]$ acts non trivially on $\mathbb{C}[Y']$. Thus, we can find an eigenvector f of $B \cap [L,L]$ in $\mathbb{C}[Y'] = \mathbb{C}[PY']$ of positive weight with respect to the coroot $\check{\alpha}$. Then $f(0) = 0$, so that f acts nilpotently on $H^0(P/B, \underline{C'})$. But f does not act nilpotently on C', for the support of this module is Y'. Therefore we can choose a finite-dimensional $B \cap [L,L]$-submodule M of C' such that $f^n M \neq 0$ for any large integer n. For such n, all weights of $\check{\alpha}$ in $f^n M$ are positive. It follows that $H^0([L,L]/B \cap [L,L], \underline{f^n M}) \neq 0$. But

$$H^0([L,L]/B \cap [L,L], \underline{f^n M}) \subseteq H^0(P/B, \underline{f^n C'}) = f^n H^0(P/B, \underline{C'}).$$

Since $H^0(P/B, \underline{C'})$ is supported at 0, we have $f^n H^0(P/B, \underline{C'}) = 0$ for large n, a contradiction. $\square$

Next we fix $i \geqslant 1$ and consider $R^i \varphi_* \mathscr{O}_Z$, a $N_G(Y)$-linearized coherent sheaf on Y. Since Y is affine, this sheaf is associated with the $\mathbb{C}[Y]$-module $H^i(Z, \mathscr{O}_Z)$ endowed with a compatible action of $N_G(Y)$. We claim that the $\mathbb{C}[PY]$-module $H^0(P/B, \underline{H^i(Z, \mathscr{O}_Z)})$ is supported at 0.

For this, note that the map $\tilde{\pi} : P \times^B Z \to PY$ is a resolution of singularities. By the induction assumption, PY has rational singularities outside 0; thus, the $\mathbb{C}[PY]$-modules $H^q(P \times^B Z, \mathscr{O}_{P \times^B Z})$ are supported at 0, for all $q \geqslant 1$. Moreover, $\tilde{\pi} = \pi \circ \rho$ (recall that $\rho : P \times^B Z \to P \times^B Y$ denotes the P-equivariant extension of φ.) And the fibers of $\pi : P \times^B Y \to PY$ identify to closed subsets of projective line, as the map $(\pi, f) : P \times^B Y \to PY \times P/B$ is a closed immersion. Thus, $H^p(P \times^B Y, \mathscr{F}) = 0$ for any $p \geqslant 2$ and for any coherent sheaf $\mathscr{F}$ on $P \times^B Y$. It follows that the Leray spectral sequence

$$H^p(P \times^B Y, R^q \rho_* \mathscr{O}_{P \times^B Z}) \Rightarrow H^{p+q}(P \times^B Z, \mathscr{O}_{P \times^B Z})$$

degenerates at E_2: then $H^0(P \times^B Y, R^q \rho_* \mathscr{O}_{P \times^B Z})$ is a quotient of $H^q(P \times^B Z, \mathscr{O}_{P \times^B Z})$. In particular, the $\mathbb{C}[PY]$-module $H^0(P \times^B Y, R^i \rho_* \mathscr{O}_{P \times^B Z})$ is supported at 0. Moreover, $R^i \rho_* \mathscr{O}_{P \times^B Z}$ is the P-linearized sheaf on $P \times^B Y$ associated with the B-linearized sheaf $R^i \varphi_* \mathscr{O}_Z$. Thus,

$$H^0(P \times^B Y, R^i \rho_* \mathscr{O}_{P \times^B Z}) = H^0(P/B, \underline{H^i(Z, \mathscr{O}_Z)}).$$

This proves the claim.

By Lemma 9, it follows that the $\mathbb{C}[Y]$-module $H^i(Z, \mathscr{O}_Z)$ is supported at 0. Thus, Y has rational singularities outside 0. $\square$

Combining Theorem 3 with Corollary 2, we obtain examples of spherical varieties where all B-orbit closures have rational singularities, e.g., all embeddings of the symmetric spaces listed at the end of Section 1. Here are other examples, of geometric interest.

Example 6. Let $\mathscr{F}_n$ be the variety of all complete flags in $\mathbb{C}^n$. Consider the variety $X = \mathbb{P}^{n-1} \times \mathscr{F}_n$ endowed with the diagonal action of $G = \mathrm{GL}(n)$. Then X is spherical, see e.g. [22]. Clearly, the isotropy group of any point of X is contained in a Borel subgroup of G; thus, by Lemma 6, all B-orbit closures in X are multiplicity-free. Applying Theorem 3, it follows that their singularities are rational. Therefore all $\mathrm{GL}(n)$-orbit closures in $\mathbb{P}^{n-1} \times \mathscr{F}_n \times \mathscr{F}_n$ have rational singularities as well.

Example 7. Let p, q, n be positive integers such that $p \leqslant q \leqslant n$. Let $\mathscr{G}_{n,p}$ be the Grassmanian variety of all p-dimensional linear subspaces of $\mathbb{C}^n$. Consider the variety $X = \mathscr{G}_{n,p} \times \mathscr{G}_{n,q}$ endowed with the diagonal action of $G = \mathrm{GL}(n)$. By [20], X is spherical (see also [22].)

We claim that all edges of $\Gamma(X)$ are simple. Thus, the singularities of all B-orbit closures in X are rational, and the same holds for closures of $\mathrm{GL}(n)$-orbits in $\mathscr{G}_{n,p} \times \mathscr{G}_{n,q} \times \mathscr{F}_n$.

To prove the claim, consider a point (E,F) in the open G-orbit in X. Let $r = \dim(E \cap F)$, then $r = \max(p+q-n, 0)$. We can choose a basis $(v_1, \ldots, v_n)$ of $\mathbb{C}^n$ such that $E \cap F$ (resp. E; F) is spanned by $v_1, \ldots, v_r$ (resp. $v_1, \ldots, v_p$; $v_1, \ldots, v_r, v_{p+1}, \ldots, v_{p+q-r}$). Then, in the corresponding decomposition

$$\mathbb{C}^n = \mathbb{C}^r \oplus \mathbb{C}^{p-r} \oplus \mathbb{C}^{q-r} \oplus \mathbb{C}^{n-p-q+r},$$

the isotropy group of (E,F) in G consists of the following block matrices:

$$\begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Thus, the orbit $G/G_{(E,F)}$ is induced from $\mathrm{GL}(n-r)/\mathrm{GL}(p-r) \times \mathrm{GL}(q-r)$. Now the claim follows from Lemma 7 together with Corollary 2.

Remark. The varieties $\mathbb{P}^{n-1} \times \mathscr{F}_n \times \mathscr{F}_n$ and $\mathscr{G}_{n,p} \times \mathscr{G}_{n,q} \times \mathscr{F}_n$ are examples of "multiple flag varieties of finite type" in the sense of [22]. There these varieties are classified for $G = GL(n)$. Do all orbit closures in such varieties have rational singularities ?

Example 8. Let $M_{m,n}$ be the space of all $m \times n$ matrices. This is a spherical variety for the action of $G = \mathrm{GL}(m) \times \mathrm{GL}(n)$ by left and right multiplication. Arguing as in Example 7, one checks that all B-orbit closures in $M_{m,n}$ are multiplicity-free (in fact, any $Y \in \mathscr{B}(M_{m,n})$ satisfies $r(Y) = r(GY)$). Hence they have rational singularities, by Theorem 3.

The same result holds for the natural action of $\mathrm{GL}(n)$ on the space of antisymmetric $n \times n$ matrices; but it fails in the case of symmetric $n \times n$ matrices, if $n \geqslant 3$. Indeed, the subset

$$a_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = 0$$

is irreducible, stable under the standard Borel subgroup of G, and singular along its divisor $(a_{11} = a_{12} = a_{13} = 0)$.

THEOREM 4. — *Let X be a regular G-variety, let Y be a multiplicity-free B-orbit closure in X such that $GY = X$, and let X' be a G-orbit closure in X, transversal intersection of the boundary divisors $D_1, \ldots, D_r$. Then the singularities of Y are rational, and the scheme-theoretical intersection $Y \cap X'$ is reduced. Moreover, for any $y \in Y \cap X'$, local equations of $D_1, \ldots, D_r$ at y are a regular sequence in $\mathscr{O}_{Y,y}$.*

Proof. — For rationality of singularities of Y, it is enough to check that X satisfies the assumption of Theorem 3. We may assume that G acts effectively on X. If a simple normal subgroup Γ of G fixes points of X, let X' be a component of the fixed point set. Then X' is G-stable:

it is the closure of some orbit Gx. Since X is regular, the normal space $T_x(X)/T_x(Gx)$ is a direct sum of Γ-invariant lines. Since Γ is simple and fixes pointwise Gx, it fixes pointwise $T_x(X)$ as well. It follows that Γ fixes pointwise X, a contradiction.

For the remaining assertions, observe that the local equations of $D_1,\dots,D_r$ at any point $x \in X'$ are a regular sequence in $\mathcal{O}_{X,x}$. Moreover, as noted above, the scheme-theoretical intersection $Y \cap X'$ is equidimensional of codimension r, and generically reduced. Since Y is Cohen-Macaulay, then $Y \cap X'$ is reduced, and the local equations of $D_1,\dots,D_r$ at any point $y \in Y \cap X'$ are a regular sequence in $\mathcal{O}_{Y,y}$. □

We now apply these results to orbit closures in flag varieties. For this, we recall a construction from [7] 1.5. Let G/H be a spherical homogeneous space, then H acts on the flag variety G/B with only finitely many orbits. Let V be a H-orbit closure in G/B and let $\hat{V}$ be the corresponding B-orbit closure in G/H. Choose a complete regular embedding X of G/H and let Y be the closure of $\hat{V}$ in X. Then $Y \in \mathcal{B}(X)$ and $GY = X$. Consider the natural morphism

$$\pi : G \times^B Y \to X$$

and the projection

$$f : G \times^B Y \to G/B.$$

The fibers of π identify to closed subschemes of G/B via f_*. Let x be the image in X of the base point of G/H, then $\pi^{-1}(x)$ identifies to V. On the other hand, let Z be a closed G-orbit in X with B-fixed point z, then the set $f(\pi^{-1}(z))$ equals

$$V_0 = \bigcup_{w \in W(Y)} \overline{Bw_0wB}/B$$

where w_0 denotes the longest element of W. Moreover, we have in the integral cohomology ring of G/B:

$$[V] = \sum_{w \in W(Y)} d(Y,w)[\overline{Bw_0wB}/B].$$

Now Theorem 2 and Proposition 5 imply the following

Corollary 5. — *Notation being as above, V_0 is connected in codimension 1. If moreover G is simply-laced, then $[V] = 2^{\ell_N(\gamma)}[V_0]$ where γ is any oriented path in $\Gamma(X)$ joining Y to X.*

We shall call V multiplicity-free if Y is. Equivalently, the cohomology class of V decomposes as a sum of Schubert classes with coefficients 0 or 1.

Note that any multiplicity-free H-orbit closure V is irreducible, even if H is not connected. Indeed, H acts transitively on the set of all irreducible components of V, so that any two such components have the same cohomology class; but the class of V is indivisible in the integral cohomology of G/B.

Theorem 5. — *Let G/H be a spherical homogeneous space, and V a multiplicity-free H-orbit closure in G/B. Then the singularities of V are rational.*

Moreover, let X be a complete regular embedding of G/H and let Y be the B-orbit closure in X associated with V, then the natural morphism $\pi : G \times^B Y \to X$ is flat, and its fibers are reduced.

As a consequence, the fibers of π realize a degeneration of V to the reduced subscheme V_0 of G/B.

Proof. — Note that the singularities of Y are rational by Theorem 4; thus, the same holds for $\hat{V} = Y \cap G/H$. Let $\varphi : Z \to \hat{V}$ be a resolution of singularities; consider the quotient map $q_H : G \to G/H$, the preimage $V' = q_H^{-1}(\hat{V})$ in G, and the fiber product $Z' = Z \times_{\hat{V}} V'$. Then V' is smooth, since Z and q_H are; the projection $\varphi' : Z' \to V'$ is proper and birational, since φ is; and $R^i\varphi_* \mathscr{O}_{Z'} = 0$ for $i \geqslant 1$, since cohomology commutes with flat base extension. Therefore the singularities of V' are rational. Now $V' = q_B^{-1}(V)$ and q_B is a locally trivial fibration, so that the singularities of V are rational as well.

For the second assertion, we identify Y to its image $B \times^B Y$ in $G \times^B Y$. Since π is G-equivariant, it is enough to check the statement at $y \in Y$. Let $D_1, \dots, D_r$ be the boundary divisors containing y, with local equations $f_1, \dots, f_r$ in $\mathscr{O}_{X,y}$. It follows from Theorem 4 that the pull-backs $\pi^* f_1, \dots, \pi^* f_r$ are a regular sequence in $\mathscr{O}_{G\times^B Y,y}$ and generate the ideal of $\pi^{-1}(Gy)$. Moreover, the restriction of π to $\pi^{-1}(Gy)$ is flat with reduced fibers, as π is G-equivariant. Now we conclude by a local flatness criterion, see [12] Corollary 6.9. □

A direct consequence is the following

Corollary 6. — *Consider a spherical homogeneous space G/H, a multiplicity-free H-orbit closure V in G/B and an effective line bundle L on G/B. Then the restriction map $H^0(G/B, L) \to H^0(V, L)$ is surjective, and $H^i(V, L) = 0$ for all $i \geqslant 1$.*

Indeed, this holds with V replaced by V_0, a union of Schubert varieties (see [21].) The result follows by semicontinuity of cohomology in a flat family.

We now obtain a partial converse to Corollary 6:

Proposition 8. — *Let G/H be a spherical homogeneous space, let V be a H-orbit closure in G/B and let Y be the corresponding B-orbit closure in G/H. If Y is the source of a double edge of $\Gamma(G/H)$, then there exists an effective line bundle L on G/B such that the restriction $H^0(G/B, L) \to H^0(V, L)$ is not surjective.*

Proof. — Let α be the label of a double edge with source Y. Denote by $p : G/B \to G/P_\alpha$ the natural map and by $p_V : V \to \pi(V)$ its restriction to V; then p is a projective line bundle, and p_V is generically finite of degree 2. Choose an ample line bundle L on G/P_α; then $p^* L$ is an effective line bundle on G/B. Now our assertion is a direct consequence of the following claim: the restriction map

$$r_n : H^0(p^{-1} p(V), p^*(L^{\otimes n})) \to H^0(V, p^*(L^{\otimes n}))$$

is not surjective for large n. To check this, note that

$$H^0(p^{-1} p(V), p^*(L^{\otimes n})) = H^0(p(V), L^{\otimes n}),\ H^0(V, p^*(L^{\otimes n})) = H^0(p(V), L^{\otimes n} \otimes p_{V*}\mathscr{O}_V)$$

by the projection formula. Thus, r_n identifies with the map

$$H^0(p(V), L^{\otimes n}) \to H^0(p(V), L^{\otimes n} \otimes p_{V*}\mathscr{O}_V)$$

defined by the inclusion of $\mathscr{O}_{p(V)}$ into $p_{V*}\mathscr{O}_V$. Since p_V has degree 2, the quotient $\mathscr{F} = p_{V*}\mathscr{O}_V/\mathscr{O}_{p(V)}$ has rank 1 as a sheaf of $\mathscr{O}_{p(V)}$-modules. Moreover, since L is ample, the cokernel of r_n is isomorphic to $H^0(p(V), \mathscr{F} \otimes L^{\otimes n})$ for large n. This proves the claim. □

4. Orbit closures of maximal rank

Let $\mathscr{B}(X)_{max}$ be the set of all $Y \in \mathscr{B}(X)$ such that $r(Y) = r(X)$, that is, the set of all B-orbit closures of maximal rank. Recall that all such orbit closures are multiplicity-free and meet the open G-orbit. Here is another characterization of them.

Proposition 9. — *(i) For any $Y \in \mathscr{B}(X)_{max}$ and $w \in W(Y)$, we have: $BwY^0 = X^0$ and $w^{-1} \in W^{\Delta(X)}$. Moreover, $W(Y)$ is disjoint from all $W(Y')$ where $Y' \in \mathscr{B}(X)$ and $Y' \neq Y$.*

(ii) Conversely, if $Y \in \mathscr{B}(X)$ and there exists $w \in W$ such that $BwY^0 = X^0$, then Y has maximal rank. If moreover $w^{-1} \in W^{\Delta(X)}$, then $w \in W(Y)$, and $\Delta(Y)$ consists of those $\alpha \in \Delta$ such that $w(\alpha) \in \Delta(X)$.

Proof. — (i) We prove that $BwY^0 = X^0$ by induction over $\ell(w)$, the case where $\ell(w) = 0$ being evident. If $\ell(w) \geqslant 1$, we can write $w = w's_\alpha$ for some simple root α and some $w' \in W$ such that $\ell(w') = \ell(w) - 1$; then $BwB = Bw'Bs_\alpha B$. Then $X = \overline{BwY} = \overline{Bw'P_\alpha Y}$. Since $\ell(w) = \mathrm{codim}_X(Y)$, it follows that α raises Y and that $w' \in W(P_\alpha Y)$. Because Y has maximal rank, $P_\alpha Y^0$ consists of two B-orbits, both of maximal rank. But $P_\alpha Y^0 = Y^0 \cup Bs_\alpha Y^0$ so that $Bs_\alpha Y^0$ is a unique B-orbit of maximal rank and of codimension $\ell(w')$ in X. By the induction assumption, we have $Bw'Bs_\alpha Y^0 = X^0$, that is, $BwY^0 = X^0$. If moreover $w \in W(Y')$ for some $Y' \in \mathscr{B}(X)$, then a similar induction shows that $Y' = Y$.

If $w^{-1} \notin W^{\Delta(X)}$ then there exists $\beta \in \Delta(X)$ such that $\ell(s_\beta w) = \ell(w) - 1$. Thus, $BwB = Bs_\beta Bs_\beta wB$, so that $s_\beta Bs_\beta wY^0$ is contained in X^0. But $s_\beta X^0 = X^0$; therefore, $Bs_\beta wY^0 = X^0$, and $\overline{Bs_\beta wY} = X$. It follows that $\mathrm{codim}_X(Y) \leqslant \ell(s_\beta w) = \ell(w) - 1$, a contradiction.

(ii) Let $\dot{w}$ be a representative of w in the normalizer of T. By assumption, the map

$$\begin{array}{ccc} U \times Y^0 & \to & X^0 \\ (u, y) & \mapsto & u\dot{w}y \end{array}$$

is surjective. Thus, it induces an injective homomorphism from the ring $\mathbb{C}[X^0]$ of regular functions on X^0, to $\mathbb{C}[U \times Y^0]$. The group of invertible regular functions $\mathbb{C}[X^0]^*$ is mapped into $\mathbb{C}[U \times Y^0]^* = \mathbb{C}[Y^0]^*$. Quotienting by $\mathbb{C}^*$ and taking ranks, we obtain $r(X) \leqslant r(Y)$ by Lemma 1, whence $r(Y) = r(X)$.

If moreover $w^{-1} \in W^{\Delta(X)}$, we show that $w \in W(Y)$ by induction over $\ell(w)$; we may assume that $w \neq 1$. Then we can write $w = w's_\alpha$ where $w' \in W$, $\alpha \in \Delta$ and $\ell(w) = \ell(w') + 1$. It follows that $w(\alpha) \in \Phi^-$.

We begin by checking that $s_\alpha Y^0 \neq Y^0$. Otherwise, by Lemma 1, there exists $y \in Y^0$ fixed by $[L_\alpha, L_\alpha]$. Thus, $\dot{w}y \in X^0$ is fixed by $w[L_\alpha, L_\alpha]w^{-1}$. Since the unipotent radical of $P(X)$ acts freely on X^0 by Lemma 2, it follows that $w(\alpha) \in \Phi_{\Delta(X)}$. Then $\alpha \in \Delta \cap w^{-1}(\Phi^-_{\Delta(X)})$ which contradicts the assumption that $w^{-1} \in W^{\Delta(X)}$.

As above, it follows that $Bs_\alpha Y^0$ is a B-orbit of maximal rank and of dimension $\dim(Y) + 1$; moreover, $Bw'Bs_\alpha Y^0 = X^0$. We can write $w' = uv$ where $u \in W_{\Delta(X)}$, $v^{-1} \in W^{\Delta(X)}$, and $\ell(w') = \ell(u) + \ell(v)$. Thus, $BwB = BuBvBs_\alpha B$, and $BvBs_\alpha Y^0 = X^0$ as $u^{-1}X^0 = X^0$. By the induction assumption, $v \in W(\overline{Bs_\alpha Y})$. Moreover, $\ell(vs_\alpha) = \ell(v) + 1$, for $w = uvs_\alpha$ and $\ell(w) = \ell(u) + \ell(v) + 1$. It follows that $vs_\alpha \in W(Y)$; in particular, $s_\alpha v^{-1} \in W^{\Delta(X)}$. But $w^{-1} = s_\alpha v^{-1}u^{-1}$ is in $W^{\Delta(X)}$ as well. Thus, $u = 1$ and $w^{-1} \in W(Y)$.

Let α be a simple root of Y. Then we see as above that $w(\alpha) \in \Phi_{\Delta(X)}$. We have $ws_\alpha = s_{w(\alpha)}w$ with $s_{w(\alpha)} \in W_{\Delta(X)}$ and $w^{-1} \in W^{\Delta(X)}$. Thus, $\ell(ws_\alpha) = \ell(s_{w(\alpha)}) + \ell(w)$ which forces $w(\alpha) \in \Phi^+$ (as $\ell(s_\alpha w) = \ell(w) + 1$) and $w(\alpha) \in \Delta$ (as $\ell(s_{w(\alpha)}) = 1$.) We conclude that $w(\alpha)$ is a simple root of X.

Conversely, let $\alpha \in \Delta$ such that $w(\alpha)$ is a simple root of X. Then $\ell(ws_\alpha) = \ell(w) + 1$, whence

$$BwBs_\alpha Y^0 = Bws_\alpha Y^0 = Bs_{w(\alpha)}wY^0 = Bs_{w(\alpha)}BwY^0 = Bs_{w(\alpha)}X^0 = X^0.$$

Let $\mathcal{O}$ be a B-orbit in $Bs_\alpha Y^0$. Then $Bw\mathcal{O} = X^0$. By (i), we have $\mathcal{O} = Y^0$, whence $s_\alpha Y^0 = Y^0$ and $\alpha \in \Delta(Y)$. □

This preliminary result, combined with those of Section 2, implies a structure theorem for orbits of maximal rank and their closures in regular varieties:

THEOREM 6. — *Let X be a complete regular G-variety, $Y \in \mathscr{B}(X)_{max}$ and $w \in W(Y)$. Choose a "slice" $S_{Y,w}$ as in Proposition 6, so that the product map*

$$(U \cap w^{-1}R_u(P)w) \times w^{-1}S_{Y,w} \to Y \cap w^{-1}X_0$$

is an isomorphism. Then $w^{-1}S_{Y,w}$ is fixed pointwise by $[L(Y), L(Y)]$. Moreover, $Y \cap w^{-1}X_0$ is $P(Y)$-stable and meets each G-orbit along a unique B-orbit, of maximal rank in this G-orbit. In particular, there exists $y \in Y^0$ fixed by $[L(Y), L(Y)]$ such that the product map $(U \cap w^{-1}R_u(P)w) \times Ty \to Y^0$ is an isomorphism.

As a consequence, for each G-orbit closure X' in X, all irreducible components of $Y \cap X'$ have maximal rank in X'. Moreover, a given $C \in \mathscr{B}(X')$ is an irreducible component of $Y \cap X'$ if and only if $W(C)$ is contained in $W(Y)$.

Proof. — With notation as in Section 2, recall that

$$w^{-1}S_{Y,w} = Y \cap (U^- \cap w^{-1}Uw)w^{-1}S$$

where S is fixed pointwise by $[L(X), L(X)]$. Now Proposition 9 implies that $[L(Y), L(Y)]$ fixes pointwise S and normalizes $U^- \cap w^{-1}Uw$. Thus, $[L(Y), L(Y)]$ stabilizes $w^{-1}S_{Y,w}$. Moreover, intersecting that space with those boundary divisors that contain a given closed G-orbit, we obtain $[L(Y), L(Y)]$-stable hypersurfaces meeting transversally at a fixed point. Arguing as in the proof of Theorem 4, it follows that $[L(Y), L(Y)]$ fixes pointwise $w^{-1}S_{Y,w}$.

By Proposition 6, $w^{-1}S_{Y,w}$ meets each G-orbit along a unique T-orbit. As a consequence, the intersection of $Y \cap w^{-1}X_0$ with each G-orbit is contained in a unique B-orbit. We apply this to GY^0, the open G-orbit in X. Since $Y \cap w^{-1}X_0 \cap GY^0 = Y \cap w^{-1}X^0$ equals Y^0 by Proposition 9, we see that the product map

$$(U \cap w^{-1}R_u(P)w) \times (w^{-1}S_{Y,w} \cap Y^0) \to Y^0$$

is an isomorphism. Moreover, $w^{-1}S_{Y,w} \cap Y^0$ is a unique T-orbit of dimension equal to the rank of X.

It follows that each U-orbit in Y^0 is a unique orbit of $U \cap w^{-1}R_u(P)w$. Indeed, any U-orbit is isomorphic to some affine space, and its projection to $w^{-1}S_{Y,w} \cap Y^0$ is a morphism to a torus, hence is constant.

Choose $y_0 \in Y^0$ and let $y \in Y \cap w^{-1}X_0$. Since $By_0 = Y^0$ is dense in $Y \cap w^{-1}X_0$, we have $\dim(Uy) \leqslant \dim(Uy_0)$. The latter equals $\dim(U \cap w^{-1}R_u(P)w)$ by the previous step. Because $U \cap w^{-1}R_u(P)w$ acts freely on $Y \cap w^{-1}X_0$, it follows that $(U \cap w^{-1}R_u(P)w)y$ is open in Uy. But both are affine spaces, so that they are equal. Thus, $Y \cap w^{-1}X_0$ is B-stable. It is even $P(Y)$-stable, because $P(Y) \subseteq w^{-1}Pw$ by Proposition 9.

Since $w^{-1}S_{Y,w}$ meets each G-orbit along a unique T-orbit, $Y \cap w^{-1}X_0$ meets each G-orbit along a unique B-orbit. Let $y \in Y \cap w^{-1}X_0$, then $wBy \subseteq X_0$ and, therefore, $wBy \subseteq (Gy)^0$. By Proposition 9 again, we have $r(By) = r(Gy)$.

The remaining assertions follow from Theorem 1 together with Proposition 9. □

As a consequence, we determine all B-orbit closures Y' such that $\int_X [Y] \cdot [Y'] \neq 0$; by Corollary 4, this amounts to $\int_X [Y] \cdot [Y'] = 1$.

Corollary 7. — *Let Y be a B-orbit closure of maximal rank in a complete regular G-variety X and let $Y' \in \mathscr{B}(X)$. Then the intersection number $\int_X [Y] \cdot [Y']$ equals 1 if and only if $Y' = \overline{Bw_0w^{-1}z}$ for some $w \in W(Y)$ and some closed G-orbit Z with base point z.*

Proof. — If $\int_X [Y] \cdot [Y'] = 1$, then $Y \cap w_0Y'$ consists of a unique T-fixed point $y \in w_0 Y'^0$, and $\overline{By}$ is an irreducible component of $Y \cap GY'$, by the proof of Corollary 4. Therefore, $\overline{By}$ has maximal rank in $GY' = \overline{Gy'}$. But $r(\overline{By}) = 0$ because y is fixed by T. Thus Gy, being a G-orbit of rank 0, is closed in X. Let z be its base point, then $y = w^{-1}z$ for some $w \in W(Y)$, so that $Y' = \overline{Bw_0y} = \overline{Bw_0w^{-1}z}$. The converse is clear. □

We now describe the intersections of B-orbit closures of maximal rank with G-orbit closures, in terms of Knop's action of the Weyl group W on the set $\mathscr{B}(X)$. This action can be defined as follows.

Let $\alpha \in \Delta$ and $Y \in \mathscr{B}(X)$, then s_α fixes Y, except in the following cases:

- Type U: $P_\alpha Y^0 = Y^0 \cup Z^0$ for $Z \in \mathscr{B}(X)$ with $r(Z) = r(Y)$. Then s_α exchanges Y and Z.
- Type T: $P_\alpha Y^0 = Y^0 \cup Y_-^0 \cup Z^0$ for $Z \in \mathscr{B}(X)$ with $r(Y) = r(Y_-) = r(Z) - 1$. Then s_α exchanges Y and Y_-.

By [19, §4], this defines indeed a W-action (that is, the braid relations hold); moreover, $\mathscr{X}(w(Y)) = w(\mathscr{X}(Y))$ for all $w \in W$. In particular, this action preserves the rank.

For $Y \in \mathscr{B}(X)_{max}$ and $w \in W(Y)$, we have $w(Y) = X$. Thus, $\mathscr{B}(X)_{max}$ is the W-orbit of X in $\mathscr{B}(X)$.

Let $W_{(X)}$ be the isotropy group of X; then $W_{(X)}$ acts on $\mathscr{X}(X)$. Observe that $W_{(X)}$ contains $W_{\Delta(X)}$. The latter acts trivially on $\mathscr{X}(X)$ by Lemma 1. In fact, $W_{(X)}$ stabilizes $\Phi_{\Delta(X)}$ (indeed, $\Phi_{\Delta(X)}$ consists of all roots that are orthogonal to $\mathscr{X}(X)$, if X is non-degenerate in the sense of [18]; and the general case reduces to that one, by [18] §5.)

The normalizer of $\Phi_{\Delta(X)}$ in W is the semi-direct product of $W_{\Delta(X)}$ with the normalizer of $\Delta(X)$. Therefore, $W_{(X)}$ is the semi-direct product of $W_{\Delta(X)}$ with

$$W_X = \{w \in W \mid w(X) = X \text{ and } w(\Delta(X)) = \Delta(X)\}.$$

The latter identifies to the image of $W_{(X)}$ in $\mathrm{Aut}\,\mathscr{X}(X)$, that is, to the "Weyl group of X", see [19] Theorem 6.2.

In fact, W_X is the set of all $w \in W_{(X)}$ such that $w(\rho) - \rho \in \mathscr{X}(X)$, where ρ denotes the half sum of positive roots (see [17] 6.5); we shall not need this result.

Let

$$W^{(X)} = \{w \in W \mid \ell(wu) \geqslant \ell(w) \ \forall \ u \in W_{(X)}\},$$

the set of all elements of minimal length in their right $W_{(X)}$-coset.

Proposition 10. — *Notation being as above, we have*

$$W^{(X)} = \{w \in W^{\Delta(X)} \mid \ell(wu) \geqslant \ell(w) \ \forall \ u \in W_X\},$$

and, for any $w \in W$*,*

$$W(w(X)) = \{v \in W \mid v^{-1} \in W^{(X)} \cap wW_{(X)}\}.$$

As a consequence, all elements of minimal length in a given left $W_{(X)}$*-coset have the same length and are contained in a left* W_X*-coset. Moreover, the subsets* $W(Y)$*,* $Y \in \mathscr{B}(X)_{max}$*, are exactly the subsets of all elements of minimal length in a given left* $W_{(X)}$*-coset.*

If moreover X *is regular, then we have for any* G*-orbit closure* X' *in* X*:*

$$w(X) \cap X' = \bigcup_{w' \in W^{(X)} \cap wW_{(X)}} w'(X').$$

Proof. — Clearly, $W^{(X)}$ is contained in $W^{\Delta(X)}$. And since W_X stabilizes $\Delta(X)$, the set $W^{\Delta(X)}$ is stable under right multiplication by W_X. This implies the first assertion.

Let $Y = w(X)$ and observe that $\mathrm{codim}_X(Y) \leqslant \ell(w)$ with equality if and only if $w^{-1} \in W(Y)$ (indeed, a reduced decomposition of w defines a non-oriented path in $\Gamma(X)$ with end-points Y and X).

Let $v \in W(Y)$. Since $v(Y) = X$, we have $v^{-1} \in wW_{(X)}$. Moreover, $\ell(v^{-1}) = \ell(v) = \mathrm{codim}_X(Y) \leqslant \ell(w)$. Since we can change w in its right $W_{(X)}$-coset, it follows that $v^{-1} \in W^{(X)}$.

Conversely, let $u \in W$ such that $u^{-1} \in W^{(X)} \cap wW_{(X)}$. Then $u(Y) = X$, whence $\ell(u) \geqslant \ell(v)$ and $u \in W_{(X)}v$. Since $u^{-1} \in W^{(X)}$, this forces $\ell(u) = \ell(v)$ and then $u \in W(Y)$. This proves the first assertion. Together with Theorem 6, this implies the second assertion. ☐

Example 9. Let $\mathbf{G}$ be a connected reductive group. Consider the group $G = \mathbf{G} \times \mathbf{G}$ acting on $X = \mathbf{G}$ by $(x, y) \cdot z = xzy^{-1}$. Then X is a spherical homogeneous space: consider the Borel subgroup $B = \mathbf{B} \times \mathbf{B}^-$ of G, where $\mathbf{B}$ and $\mathbf{B}^-$ are opposed Borel subgroups of $\mathbf{G}$. With evident notation, the B-orbits in X are the $\mathbf{B}w\mathbf{B}^-$, $w \in \mathbf{W}$. This identifies $\mathscr{B}(X)$ to $\mathbf{W}$. Moreover, all B-orbits have maximal rank, and the Weyl group $W = \mathbf{W} \times \mathbf{W}$ acts on $\mathbf{W}$ by $(u, v)w = uwv^{-1}$. Thus, $\Delta(X)$ is empty, $W_{(X)}$ is the diagonal in $\mathbf{W} \times \mathbf{W}$, and $\mathbf{W} \times \{1\}$ is a system of representatives of $W/W_{(X)}$. One checks that

$$W^{(X)} = \{(u, v) \in \mathbf{W} \times \mathbf{W} \mid \ell(u) + \ell(v) = \ell(uv^{-1}\}.$$

In particular, $(w, 1) \in W^{(X)}$ for all $w \in \mathbf{W}$. Moreover,

$$W^{(X)} \cap (w, 1)W_{(X)} = \{(u, v) \in \mathbf{W} \times \mathbf{W} \mid uv^{-1} = w \text{ and } \ell(u) + \ell(v) = \ell(w)\}.$$

This identifies $W^{(X)} \cap (w,1)W_{(X)}$ to the set of all $u \in \mathbf{W}$ such that $u \preceq w$ for the right order on $\mathbf{W}$.

Remark. Let X be a complete regular G-variety, Y a B-orbit closure of maximal rank, and X' a G-orbit closure in X. Then the number of irreducible components of $Y \cap X'$ is at most the order of W_X by Proposition 10. If moreover X has rank 1, then W_X is trivial or has order 2, so that $Y \cap X'$ has at most 2 components.

Returning to an arbitrary spherical variety X, we shall deduce from Proposition 4 the following

THEOREM 7. — *The group $W_{(X)}$ is generated by reflections s_α where α is a root such that $\alpha \in \Phi_{\Delta(X)}$ or that $2\alpha \in \mathscr{X}(X)$, and by products $s_\alpha s_\beta$ where α, β are orthogonal roots such that $\alpha+\beta \in \mathscr{X}(X)$.*

Proof. — Let $w \in W_{(X)}$. We choose a reduced decomposition $w = s_{\alpha_\ell} \cdots s_{\alpha_2} s_{\alpha_1}$ and we argue by induction on ℓ.

If $\alpha_1 \in \Delta(X)$ then s_{α_1} is a reflection in $W_{(X)}$, so that $s_{\alpha_\ell} \cdots s_{\alpha_2} \in W_{(X)}$. Now we conclude by the induction assumption.

If $\alpha_1 \notin \Delta(X)$ then $s_{\alpha_1}(X)$ has codimension 1 in X. Let i be the largest integer such that $\operatorname{codim}_X s_{\alpha_i} \cdots s_{\alpha_1}(X) = i$. Let $Y = s_{\alpha_i} \cdots s_{\alpha_1}(X) = i$, then $Y \in \mathscr{B}(X)_{max}$ and $s_{\alpha_1} \cdots s_{\alpha_1} \in W(Y)$.

If $P_{\alpha_{i+1}} Y = Y$ then $s_{\alpha_{i+1}}(Y) = Y$ by definition of the W-action and maximality of i. Let $\alpha = s_{\alpha_1} \cdots s_{\alpha_i}(\alpha_{i+1})$. Then s_α is a reflection of $W_{(X)}$, and $w = s_{\alpha_\ell} \cdots s_{\alpha_{i+2}} s_{\alpha_i} \cdots s_{\alpha_1} s_\alpha$. If $\alpha_{i+1} \in \Delta(Y)$, then $\alpha \in \Delta(X)$ by Proposition 9. Otherwise, $P_{\alpha_{i+1}} Y^0 / R(P_{\alpha_{i+1}})$ is isomorphic to $\mathrm{PGL}(2)/T$ or to $\mathrm{PGL}(2)/N$; it follows that $2\alpha_{i+1} \in \mathscr{X}(Y)$, and that $2\alpha \in \mathscr{X}(X)$. Now we conclude by the induction assumption.

If $P_{\alpha_{i+1}} Y \neq Y$ then α_{i+1} raises Y to (say) Y'. Choose $u \in W(Y')$, then $\ell(u) = i-1$ and $u s_{\alpha_{i+1}} \in W(Y)$. Moreover, $u s_{\alpha_{i+1}} s_{\alpha_i} \cdots s_{\alpha_1} \in W_{(X)}$. We have $w = v u s_{\alpha_{i+1}} s_{\alpha_i} \cdots s_{\alpha_1}$ for some $v \in W_{(X)}$ such that $\ell(vu) = \ell - i - 1$. Thus, $\ell(v) \leqslant \ell(vu) + \ell(u) = \ell - 2$. Therefore, we may assume that there exist $Y \in \mathscr{B}(X)_{max}$ and $w_1, w_2 \in W(Y)$ such that $w = w_2 w_1^{-1}$. By Proposition 2, we may assume moreover that w_1 and w_2 are neighbors. Then we conclude by Proposition 4. □

As a direct consequence, we recover the following result of Knop, see [18] and [19].

COROLLARY 8. — *The image of W_X in $\operatorname{Aut} \mathscr{X}(X)$ is generated by reflections.*

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