Moduli of simple holomorphic pairs and effective divisors

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ABSTRACT: In this note we identify two complex structures (one is given by algebraic geometry, the other by gauge theory) on the set of isomorphism classes of holomorphic bundles with section on a given compact complex manifold. The point is that the obvious bijection is an isomorphism of complex spaces.
In the case of *line* bundles, these complex spaces are shown to be isomorphic to a space of effective divisors on the manifold.

Introduction

Let $(X, \mathcal{O}_X)$ be a compact complex analytic space. We denote by $\mathrm{Div}^+(X)$ the space of all Cartier divisors (including the empty one) on X. This set is a Zariski open subspace in the entire Douady space $\mathcal{D}(X)$, parameterizing *all* compact subspaces of X. If X is smooth (or more generally *locally factorial*), then $\mathrm{Div}^+(X)$ is a union of connected components of $\mathcal{D}(X)$ [Fu].

We consider pairs $(\mathcal{E}, \phi)$ consisting of an invertible sheaf $\mathcal{E}$ over X coupled with a holomorphic section ϕ in $\mathcal{E}$, which is locally a non zero divisor. Two such pairs $(\mathcal{E}_i, \phi_i)$ are called equivalent if there exists an isomorphism of sheaves $\Theta : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ such that $\Theta \circ \phi_1 = \phi_2$. The set of equivalence classes is called the moduli space of simple holomorphic pairs of rank one on X. This moduli space can be given a structure of a complex analytic space using standard results of deformation theory (see [KO]).
There exists a natural bijective map from the moduli space of simple holomorphic pairs of rank one into $\mathrm{Div}^+(X)$ sending the equivalence class of a pair $(\mathcal{E}, \phi)$ into the divisor, given by the vanishing locus of the section ϕ. In the first part of the paper, we prove that this one-to-one correspondence is in fact an analytic isomorphism with respect to these natural structures on the two spaces.

If X is a *smooth* compact complex manifold, there is a second possibility of defining an analytic structure on the moduli space of simple holomorphic pairs, namely by using gauge-theoretical methods (compare [OT],[S]). In the last part we show that, in this case, these two structures are isomorphic. This is the "pair"-version of a previous result due to

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Miyajima [M] in the case of moduli spaces of simple *bundles*.

Throughout this paper we adopt the following notations:

$(sets)$	: the category of sets;
(an)	: the category of (not necessarily reduced) complex spaces;
(an/R)	: the category of relative complex spaces over a complex space R;
$(germs)$	: the category of germs of complex spaces;
h_S	: the canonical contravariant functor $h_S : \mathcal{C} \longrightarrow (sets)$, $h_S = \mathrm{Hom}(\,\cdot\,, S)$ associated to an object S belonging to a category $\mathcal{C}$.

1 Main result

The purpose of this part is to prove the following

Theorem 1.1 *Let X be a compact complex space with $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Then there exists a natural complex analytic isomorphism between the moduli space of simple holomorphic pairs of rank one and the space* $\mathrm{Div}^+(X)$.

REMARK. Let $(X, \mathcal{O}_X)$ be a compact complex space with $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Then:

i.) The Picard functor $\underline{\mathrm{Pic}}_X$ is representable (cf. [Bi1, p.337]).
ii.) There exists a Poincaré line bundle over $X \times \mathrm{Pic}(X)$.

The existence of a Poincaré bundle can be seen as follows (see [Br, p.55] in the smooth case). The Leray spectral sequence for the projection morphism $\pi : X \times S \longrightarrow S$ leads to the exact sequence

$$0 \to \mathrm{Pic}(S) \to \mathrm{Pic}(X \times S) \to H^0(\mathcal{R}^1\pi_*\mathcal{O}^*_{X\times S}) \to H^2(\pi_*\mathcal{O}_{X\times S}) \to H^2(\mathcal{O}^*_{X\times S})\ .$$

Since $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$, the last morphism is *injective*, hence one gets an exact sequence

$$0 \to \mathrm{Pic}(S) \to \mathrm{Pic}(X \times S) \to H^0(\mathcal{R}^1\pi_*\mathcal{O}^*_{X\times S}) \cong \mathrm{Hom}(S, \mathrm{Pic}(X)) \to 0\ . \tag{1}$$

For $S = \mathrm{Pic}(X)$, this sequence leads to a line bundle $\mathcal{P}$ on $X \times \mathrm{Pic}(X)$ such that $\mathcal{P}|_{X\times\{[\xi]\}} \cong \xi$ for every $[\xi] \in \mathrm{Pic}(X)$. ■

A complex structure on the space of simple holomorphic pairs is given as follows:
Let $\mathcal{P}$ be a Poincaré bundle on $X \times \mathrm{Pic}(X)$. By [Fl] there exists a linear fiber space H over $\mathrm{Pic}(X)$, which represents the functor $\mathcal{H} : (an/_{\mathrm{Pic(X)}}) \longrightarrow (sets)$ given by

$$\left(S \xrightarrow{f} \mathrm{Pic}(X)\right) \longmapsto \mathrm{Hom}\left(\mathcal{O}_{X\times S}, (\mathrm{id}_X \times f)^*\mathcal{P}\right)\ .$$

(The action of $\mathcal{H}$ on morphisms is given by "pull-back".) In particular, for every complex space S over $Pic(X)$, there is a bijection

$$\mathrm{Hom}\left(\mathcal{O}_{X\times S}, (\mathrm{id}_X \times f)^*\mathcal{P}\right) \cong \mathrm{Hom}_{Pic(X)}(S, H)\ . \tag{2}$$

Let $\tilde{O} \subset H$ be the subset consisting of zero divisors. The multiplicative group $\mathbb{C}^*$ operates on $H' := H \setminus \tilde{O}$ such that the quotient $\mathrm{P(H')} := H'/\mathbb{C}^*$ becomes an open subset of a projective fiber space over $\mathrm{Pic}(X)$. The fiber over $[\xi] \in \mathrm{Pic}(X)$ can be identified with an open subset of $PH^0(X, \xi)$. Moreover, $\mathrm{P(H')}$ coincides set-theoretically with the moduli space of simple holomorphic pairs, defining a natural analytic structure on it.

In order to prove that $\mathrm{P(H')}$ and $\mathrm{Div}^+(X)$ are isomorphic as complex spaces, it suffices to

prove that the associated functors $h_{\mathrm{P(H')}}$ and $h_{\mathrm{Div}^+(X)}$ are isomorphic. More precisely, we show that both are isomorphic to the contravariant functor
$F : (an) \longrightarrow (sets)$, defined by $S \longmapsto F(S)$, where $F(S)$ denotes the equivalence classes of pairs $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is an invertible sheaf on $X \times S$, and ϕ is a holomorphic section in $\mathcal{E}$ whose restriction to each fiber $X \times \{s\}$ is locally a non zero divisor. (Two such tuples are called equivalent if there exists an isomorphism of sheaves $\Theta : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ such that $\Theta \circ \phi_1 = \phi_2$.)

REMARK. Note that if $(\mathcal{E}, \phi) \sim (\mathcal{E}, \phi)$, then necessarily $\Theta = \mathrm{id}_{\mathcal{E}}$. Using this simple observation, it is easy to see that the functor F is of local nature (it is a sheaf), i.e. given any complex space S together with an open covering $S = \cup S_i$, the following sequence is exact:

$$F(S) \longrightarrow \prod F(S_i) \longrightarrow \prod F(S_i \cap S_j) \ .$$

Recall that a *relative Cartier divisor* in $X \times S$ over S is a Cartier divisor $Z \subset X \times S$ which is flat over S. We denote by $\mathrm{Div}^+_S(X)$ the set of all relative Cartier divisors (including the empty one) over a fixed S.

Lemma 1.2 *Let S be a fixed complex space. The map*

$$\mathcal{Z} : F(S) \longrightarrow \mathrm{Div}^+_S(X)$$

sending the class of $(\mathcal{E}, \phi)$ into the Cartier divisor given by the vanishing locus of the section ϕ is well defined and bijective.

PROOF. It is clear that the vanishing locus $Z := Z(\phi) \subset X \times S$ depends only on the isomorphism class of $(\mathcal{E}, \phi)$. We need to show that Z is flat over S. Take $(x, s) \in Z$, denote $A := \mathcal{O}_{S,s}$, $B := \mathcal{O}_{X \times S,(x,s)}$ and let $\mathfrak{m} \subset A$ be the maximal ideal. Since the section ϕ restricted to each fiber $X \times \{s\}$ is locally a non zero divisor, we have $\mathcal{O}_{Z,(x,s)} = B/uB$ for some nonzero divisor u. The flatness of the morphism $A \longrightarrow B/uB$ follows then from the general Bourbaki-Grothendieck criterion [Fi, p. 152] since

$$\mathrm{Tor}_1^A(A/\mathfrak{m}, B/uB) = \ker(B/\mathfrak{m}B \stackrel{\cdot u}{\longrightarrow} B/\mathfrak{m}B) = \{0\}.$$

We take two simple pairs $(\mathcal{E}_i, \phi_i)$ over $X \times S$, defining the same divisor $Z := Z(\phi_1) = Z(\phi_2)$. The invertible sheaf $\mathcal{O}_{X \times S}(Z)$ admits a canonical section ϕ_{can}, and both pairs $(\mathcal{E}_i, \phi_i)$ are equivalent to the pair $(\mathcal{O}_{X \times S}(Z), \phi_{can})$. This proves the injectivity.
Given $Z \in \mathrm{Div}^+_S(X)$, the associated canonical section ϕ_{can} restricted to each fiber $X \times \{s\}$ is locally a non zero divisor since Z lies flat over S. This gives the surjectivity. ■
The above lemma shows that there exists a natural isomorphism of functors between F and the functor $G : (an) \longrightarrow (sets)$,

$$G(S) := \mathrm{Div}^+_S(X) \subset \mathcal{D}(X \times S) \ .$$

It follows from the general result of Douady [D] that G is a representable functor and its representation space is exactly $\mathrm{Div}^+(X)$. In order to prove theorem 1.1 it suffices to show the following

Theorem 1.3 *The functor F is representable by the space of simple holomorphic pairs* $\mathrm{P(H')}$.

PROOF. Since F is a sheaf, it suffices to prove that F and $h_{\mathrm{P(H')}}$ are isomorphic as functors defined on the category of germs of analytic spaces. The following observation shows that $h_{\mathrm{P(H')}}$ is isomorphic to the sheafified functor associated to the quotient functor $h_{H'}/h_{\mathbb{C}^*}$:

Lemma 1.4 *Let M be a complex analytic space, and let G be a complex Lie group acting smoothly and freely on M. Suppose that the quotient M/G exists in the category of analytic spaces such that the canonical morphism $M \longrightarrow M/G$ is smooth. Then the canonical morphism of functors*

$$(h_M/h_G)^{\#} \longrightarrow h_{M/G} \,. \tag{3}$$

is an isomorphism (The superscript $^{\#}$ denotes here the associated sheafified functor.)

PROOF. Since the projection $M \longrightarrow M/G$ is smooth, one has an epimorphism of sheaves $h_M \longrightarrow h_{M/G}$, hence the morphism (3) is also an epimorphism.
Furthermore, since G acts smoothly and freely on M, the morphism $G \times M \longrightarrow M \times_{M/G} M$, $(g, m) \longmapsto (m, gm)$ is an *isomorphism* by the relativ implicit function theorem. This shows that (3) is also a monomorphism, i.e. an isomorphism. ■

Let $p : H' \longrightarrow \mathrm{Pic}(X)$ be the natural morphism and consider the corresponding tautological homomorphism

$$u : \mathcal{O}_{X \times H'} \longrightarrow (\mathrm{id}_X \times p)^* \mathcal{P}$$

given by the bijection (2).
The morphism of functors $h_{H'} \longrightarrow F$ is defined by sending $\phi : S \longrightarrow H'$ to the isomorphism class of the simple pair

$$\mathcal{O}_{X \times S} \xrightarrow{\phi^* u} (\mathrm{id}_X \times p \circ \phi)^* \mathcal{P} \,.$$

(Note that $\phi^* u|_{X \times \{s\}} = \phi(s) \in H'$ for every $s \in S$.)

Injectivity: Let ϕ_1, ϕ_2 be two morphisms from S to H' such that the associated simple pairs are isomorphic. It follows in particularly that the two sheaves $(\mathrm{id}_X \times p \circ \phi_i)^* \mathcal{P}$ are isomorphic via some Θ. The sequence (1) implies $p \circ \phi_1 = p \circ \phi_2$. The isomorphism Θ becomes an automorphism, and is given by multiplication with some element $a_\Theta \in H^0(X \times S, \mathcal{O}^*_{X \times S}) \cong H^0(S, \mathcal{O}^*_S)$ (since $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$). It follows that ϕ_1, ϕ_2 are conjugate under the action of $a_\Theta : S \longrightarrow \mathbb{C}^*$, i.e. the morphism

$$(h_{H'}/h_{\mathbb{C}^*})^{\#} \longrightarrow F$$

is injective.

Surjectivity: Consider a germ $(S, 0)$ of complex space and a simple pair $(\mathcal{O}_{X \times S} \xrightarrow{\alpha} \mathcal{E})$. The corresponding morphism $f \in \mathrm{Hom}_{Pic(X)}(S, H')$ has the property

$$\mathcal{E} \cong (\mathrm{id}_X \times p \circ f)^* \mathcal{P} \otimes \mathcal{L}$$

for some $[\mathcal{L}] \in \mathrm{Pic}(S)$. However, since we are working on the germ S, we may assume that $\mathcal{L}$ is trivial. In this way, we obtain a simple pair $(\mathcal{O}_{X \times S} \xrightarrow{\alpha'} (\mathrm{id}_X \times p \circ f)^* \mathcal{P})$, which by (2) leads to a morphism from S to H'. This proves the surjectivity, and completes the proof of theorem 1.3. ■

Corollary 1.5 *Let $(X, \mathcal{O}_X)$ be a compact, reduced, connected and locally irreducible analytic space, and let $m \in H^2(X, \mathbb{Z})$ be a fixed cohomology class. Consider the closed subspace of $\mathrm{Div}^+(X)$ given by*

$$\mathrm{Dou}(m) := \left\{ Z \in \mathrm{Div}^+(X) \,\middle|\, c_1(\mathcal{O}_X(Z)) = m \right\} .$$

Then there exists an isomorphism of complex spaces

$$\mathrm{Dou}(m) \cong \left\{ (\mathcal{E}, \phi) \,\middle|\, \mathcal{E} \text{ invertible sheaf on } X,\ c_1(\mathcal{E}) = m,\ 0 \neq \phi \in H^0(X, \mathcal{E}) \right\} \big/ \sim \,.$$

PROOF. One has a commutative diagram

$$\begin{array}{ccc} \mathrm{P(H')} & \stackrel{\cong}{\longrightarrow} & \mathrm{Div}^+(X) \\ & \searrow \quad \swarrow & \\ & \mathrm{Pic}(X) & \end{array}$$

where the vertical arrows are the natural projective morphisms. The assertion follows by taking the analytic pull-back of the component $\mathrm{Pic}^m(X) \subset \mathrm{Pic}(X)$ via these two maps. ■

REMARK. If X is a smooth manifold, then $\mathrm{Dou}(m)$ is a union of connected components of $\mathrm{Div}^+(X)$. Moreover, if X admits a Kähler metric, the spaces $\mathrm{Dou}(m)$ are always compact. This follows from Bishop's compactness theorem, since all divisors in $\mathrm{Dou}(m)$ have the same volume (with respect to any Kähler metric).
This property fails in the case of manifolds which do not allow Kähler metrics, since a non-Kählerian manifold may have (nonempty) effective divisors which are homologically trivial. (Take for instance a (elliptic) surface X with $H^2(X, \mathbb{Z}) = 0$.)

2 Gauge-theoretical point of view

When X is a smooth compact, connected complex manifold it is possible to construct a "gauge-theoretical" moduli space of simple holomorphic pairs (of any rank) on X (compare [OT],[S]).

Let E be a fixed $\mathcal{C}^\infty$ complex vector bundle of rank r on X. We recall the following basic facts from complex differential geometry:

Definition 2.1 A *semiconnection* (of type (0,1)) in E is a differential operator $\bar{\delta} : A^0(E) \longrightarrow A^{0,1}(E)$ satisfying the Leibniz rule

$$\bar{\delta}(f \cdot s) = \bar{\partial}(f) \otimes s + f \cdot \bar{\delta}(s) \qquad \forall\, f \in \mathcal{C}^\infty(X, \mathbb{C}), s \in A^0(E)\ .$$

The space of all semiconnections in E will be denoted by $\bar{\mathcal{D}}(E)$; it is an affine space over $A^{0,1}(\mathrm{End}\ E)$. Every $\bar{\delta} \in \bar{\mathcal{D}}(E)$ admits a natural extension $\bar{\delta} : A^{p,q}(E) \longrightarrow A^{p,q+1}(E)$ such that

$$\bar{\delta}(\alpha \otimes s) = \bar{\partial}(\alpha) \otimes s + (-1)^{p+q} \alpha \wedge \bar{\delta}(s) \qquad \forall\, \alpha \in A^{p,q}(X), s \in A^0(E)\ .$$

Moreover, $\bar{\delta}$ induces $\bar{D} : A^{p,q}(\mathrm{End}\ E) \longrightarrow A^{p,q+1}(\mathrm{End}\ E)$ by

$$\bar{D}(\alpha) = [\bar{\delta}, \alpha] := \bar{\delta} \circ \alpha + (-1)^{p+q+1} \alpha \circ \bar{\delta}\ .$$

Every *holomorphic* bundle $\mathcal{E}$ over X of differentiable type E induces a canonical semiconnection $\bar{\delta} := \bar{\partial}_{\mathcal{E}}$ on E such that $\bar{\delta}^2 : A^0(E) \longrightarrow A^{0,2}(E)$ vanishes identically. Conversely, by [AHS] Theorem (5.1), every semiconnection $\bar{\delta}$ with $\bar{\delta}^2 = 0$ defines a unique holomorphic bundle $\mathcal{E}$, differentiably equivalent to E such that $\bar{\partial}_{\mathcal{E}} = \bar{\delta}$.
There exists a natural right action of the gauge group $GL(E) \subset A^0(\mathrm{End}\ E)$ of differentiable automorphisms of E on the space $\bar{\mathcal{D}}(E) \times A^0(E)$ by

$$(\bar{\delta}, \phi) \cdot g := (g^{-1} \circ \bar{\delta} \circ g, g^{-1}\phi)\ .$$

Denote by $\bar{\mathcal{S}}(E)$ the set of points with trivial isotropy group. After suitable L^2_k-Sobolev completions, the space $\bar{\mathcal{B}}^{s.p.}(E) := \bar{\mathcal{S}}(E)/GL(E)$ becomes a complex analytic Hilbert manifold, and $\bar{\mathcal{S}}(E) \longrightarrow \bar{\mathcal{B}}^{s.p.}(E)$ a complex analytic $GL(E)$-Hilbert principal bundle. The map

$$\Upsilon : \bar{\mathcal{D}}(E) \times A^0(E) \longrightarrow A^{0,2}(\text{End } E) \times A^{0,1}(E)$$

given by

$$\Upsilon(\bar{\delta}, \phi) = (\bar{\delta}^2, \bar{\delta}\phi)$$

is $GL(E)$-equivariant, hence it induces a section $\hat{\Upsilon}$ in the associated Hilbert vector bundle

$$\bar{\mathcal{S}}(E) \times_{GL(E)} \left[A^{0,2}(\text{End } E) \oplus A^{0,1}(E)\right]$$

over $\bar{\mathcal{B}}^{s.p.}(E)$. This section becomes analytic for apropriate Sobolev completions.

Definition 2.2 The *gauge-theoretical* moduli space $\mathcal{M}^{s.p.}(E)$ of simple holomorphic pairs of type E is the complex analytic space given by the vanishing locus of the section $\hat{\Upsilon}$.

Set-theoretically, $\mathcal{M}^{s.p.}(E)$ can be identified with the set of isomorphism classes of pairs $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is a holomorphic bundle of type E, and ϕ is a holomorphic section in $\mathcal{E}$, such that the associated evaluation map

$$ev(\phi) : H^0(X, \mathcal{E}nd(\mathcal{E})) \longrightarrow H^0(X, \mathcal{E})$$

is injective. This is equivalent to the fact that the only automorphism of $(\mathcal{E}, \phi)$ is the identity. If $\mathcal{E}$ is a *simple bundle* (this happens always if $r = 1$) and $\phi \in H^0(X, \mathcal{E})$, then $(\mathcal{E}, \phi)$ is a *simple pair* iff ϕ is nontrivial.

Definition 2.3 Fix $(\bar{\delta}_0, \phi_0) \in \Upsilon^{-1}(0)$. A *gauge-theoretical family of deformations* of $(\bar{\partial}_0, \phi_0)$ parametrized by a germ $(T, 0)$ is a complex analytic map

$$\omega = (\omega_1, \omega_2) : (T, 0) \longrightarrow (A^{0,1}(\text{End } E), 0) \times (A^0(E), \phi_0)$$

such that the image of the map $(\bar{\delta}_0 + \omega_1, \omega_2)$ is contained in $\Upsilon^{-1}(0)$.
Two families ω and ω' over $(T, 0)$ are called *equivalent* if there exists a complex analytic map

$$g : (T, 0) \longrightarrow (\text{GL}(E), \text{id}_E)$$

such that $\omega' = \omega \cdot g$.

Note that, given such a deformation $\omega = (\omega_1, \omega_2)$, the family ω_1 induces uniquely a section in the sheaf $\left(\mathcal{A}^{0,1}(\text{End } E) \times \mathcal{A}^{0,0}(E)\right)\hat{\otimes}\mathcal{O}_T$, and conversely. In particular, if $(T, 0)$ is an *artinian* germ, then ω_1 induces a morphism of sheaves

$$\mathcal{A}^{0,i}(E)_T := \mathcal{A}^{0,i}(E) \otimes_{\mathbb{C}} \mathcal{O}_T \longrightarrow \mathcal{A}^{0,i+1}(E)_T$$

We denote by $h_{gauge} : (germs) \longrightarrow (sets)$ the functor which sends a germ $(T, 0)$ into the set of equivalence classes of gauge-theoretical families of deformations over $(T, 0)$.

Theorem 2.4 *The functor h_{gauge} has a semi-universal deformation.*

PROOF. Fix $\bar{\delta}_0 \in \bar{\mathcal{D}}(E)$ and consider the orbit map $\beta : GL(E) \longrightarrow \bar{\mathcal{D}}(E) \times A^0(E)$ given by

$$\beta(g) := (\bar{\delta}_0, 0) \cdot g = (\bar{\delta}_0 + g^{-1}\bar{D}_0(g), 0) .$$

By [BK] Theorem (12.13) and [K] Theorem (1.1), the existence of a semi-universal deformation follows if:

i.) The derivative of Υ at $(\bar{\delta}_0, 0)$ and the derivative of β at id_E are direct linear maps (for apropriate Sobolev completions);

ii.) The quotient $\ker(d\Upsilon_{(\bar{\delta}_0,0)})\big/\mathrm{im}(d\beta_{\mathrm{id}_E})$ is finite dimensional.

One has

$$\ker(d\Upsilon_{(\bar{\delta}_0,0)}) = \left\{ (\alpha, \phi) \in A^{0,1}(\mathrm{End}\ E) \times A^0(E) \;\middle|\; \bar{D}_0(\alpha) = 0, \bar{\delta}_0\phi = 0 \right\}$$

and $d\beta_{\mathrm{id}_E} : A^0(\mathrm{End}E) \longrightarrow A^{0,1}(\mathrm{End}\ E) \times A^0(E)$ is given by $u \longmapsto (\bar{D}_0(u), 0)$. Therefore *i.)* and *ii.)* follow from standard Hodge theory. ■

REMARK. The existence of a moduli space of simple holomorphic pairs for arbitrary rank can be also deduced from [KO] Theorem (2.1), (2.2) and the proof of Theorem (6.4) of loc.cit.
Indeed, one has local semi-universal deformations: Fix $(\mathcal{E}_0, \phi_0)$ and let $\mathcal{E} \longrightarrow X \times (R, 0)$ be a semi-universal family of vector bundles with $\mathcal{E}|_{X\times\{0\}} \cong \mathcal{E}_0$. Similarly as in rank one, the functor $\mathcal{H} : (an/R) \longrightarrow (sets)$ given by

$$\left(S \xrightarrow{f} R\right) \longmapsto \mathrm{Hom}\left(\mathcal{O}_{X\times S}, (\mathrm{id}_X \times f)^*\mathcal{E}\right)$$

is representable [Fl] by a linear fibre space $p : \tilde{R} \longrightarrow R$. Moreover, there exists a tautological section $u : \mathcal{O}_{X\times\tilde{R}} \longrightarrow (\mathrm{id}_X \times p)^*\mathcal{E}$ arising from the representability of $\mathcal{H}$. The pair $((\mathrm{id}_X \times p)^*\mathcal{E}, u^*\phi_0)$ is a versal deformation of $(\mathcal{E}_0, \phi_0)$. By [Bi2] there exists then also a semi-universal deformation.
Moreover, it can be shown (as in loc.cit. for the case of simple sheaves), that the isomorphy locus of two families of simple holomorphic pairs over S is a *locally closed* analytic subset of S and as a consequence, this moduli space exists by [KO] (2.1).

The aim of the remaining part is to prove the following

Theorem 2.5 *The gauge-theoretical moduli space of simple holomorphic pairs of type E is analytically isomorphic to the complex-theoretical moduli space of simple holomorphic pairs of type E.*

PROOF. The arguments we use are inspired from [M], where a similar problem is treated (bundles without section). It suffices to show that the associated deformation functors h_{gauge} resp. h_{an} are isomorphic over *artinian* bases.
Note first, that there exists a well defined morphism of functors from h_{an} to h_{gauge}. Moreover, this morphism is injective since

$$(\mathcal{E}_1, \phi_1) \sim (\mathcal{E}_2, \phi_2) \Longleftrightarrow (\bar{\partial}_{\mathcal{E}_1}, \phi_1) \sim (\bar{\partial}_{\mathcal{E}_2}, \phi_2)\ .$$

In order to prove the surjectivity, we need to show that every gauge-theoretical family of simple holomorphic pairs determines a complex-theoretical family of simple holomorphic pairs. We will prove this by induction on the length of the artinian base.
For $n = 0$, this follows from the fact that every integrable semi-connection $\bar{\delta}_0$ determines a holomorphic bundle $\mathcal{E}_0$ of type E such that one has an exact sequence of sheaves

$$0 \ \rightarrow\ \mathcal{E}_0 \ \rightarrow\ \mathcal{A}^{0,0}(E) \ \overset{\bar{\partial}_{\mathcal{E}_0}}{\rightarrow}\ \mathcal{A}^{0,1}(E) \ \overset{\bar{\partial}_{\mathcal{E}_0}}{\rightarrow}\ \mathcal{A}^{0,2}(E) \ \rightarrow \dots$$

For the induction step, let $(T, 0)$ be a small extension of an infinitesimal neighbourhood $(T', 0)$ such that $\ker(\mathcal{O}_T \longrightarrow \mathcal{O}_{T'}) = \mathbb{C}$, and let $\omega = (\omega_1, \omega_2)$ be a gauge-theoretical family of simple holomorphic pairs parametrized by $(T,0)$. By the induction assumption, we can find a holomorphic vector bundle $\mathcal{E}'$ over $X \times (T',0)$ which is induced by $\omega_1|_{T'}$. Then we have the following exact sequence of sheaves

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{E}_0 & \to & \mathcal{A}^{0,0}(E) & \to & \mathcal{A}^{0,1}(E) & \to & \mathcal{A}^{0,2}(E) & \to \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{E} & \to & \mathcal{A}^{0,0}(E)_T & \to & \mathcal{A}^{0,1}(E)_T & \to & \mathcal{A}^{0,2}(E)_T & \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{E}' & \to & \mathcal{A}^{0,0}(E)_{T'} & \to & \mathcal{A}^{0,1}(E)_{T'} & \to & \mathcal{A}^{0,2}(E)_{T'} & \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & 0 & & \end{array}$$

In particular, $\mathcal{E}$ is locally free, hence it defines a holomorphic vector bundle over $X \times (T,0)$ whose restriction to $X \times (T',0)$ gives $\mathcal{E}'$. This vector bundle $\mathcal{E}$ together with the family of sections ω_2 gives rise to a complex theoretical family of simple pairs which induces the gauge-theoretical family ω.
This completes the proof of the theorem. ■

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