LOCALIZATION OF THE RIEMANN-ROCH CHARACTER

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Abstract. We present a K-theoritic approach to the Guillemin-Sternberg conjecture [15], about the commutativity of geometric quantization and symplectic reduction, which was proved by Meinrenken [22, 23] and Tian-Zhang [27]. Besides providing a new proof of this conjecture for the full non-abelian group action case, our methods lead to a generalisation for compact Lie group actions on manifolds that are not symplectic. Instead, these manifolds carry an invariant almost complex structure and an abstract moment map.

Contents

1. Introduction

Consider a compact manifold M on which a compact Lie group G acts. If M carries a G-invariant almost complex structure J, we have a quantization map

$$RR^{G,J}(M,-) : K_G(M) \to R(G) \ ,$$

from the equivariant K-theory of complex vector bundles over M to the character ring of G. Let $\mathfrak{g}$ be the Lie algebra of G.

Let $L \to M$ be a G-equivariant Hermitian line bundle over M. The choice of an Hermitian connection ∇^L on L defines a map $f_L : M \to \mathfrak{g}^*$ such that

$$\mathcal{L}^L(X) - \nabla^L_{X_M} = \imath\langle f_L, X\rangle, \quad X \in \mathfrak{g} \ ,$$

where $\mathcal{L}^L(X)$ is the infinitesimal action of X on the section of $L \to M$ (in [8][section 7.1] they call f_L the 'moment'), and X_M is the vector field on M generated by $X \in \mathfrak{g}$.

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We will work under the following assumptions.
Assumption 1: 0 *is a regular value of* f_L.

Under this first assumption, $\mathcal{Z} := f_L^{-1}(0)$ is a smooth submanifold of M which carries a locally free action of G, and we consider the orbifold reduced space $\mathcal{M}_{red} = f_L^{-1}(0)/G$.
Assumption 2 *We have the following decomposition of the tangent space* $\mathbf{T}M$ *under* $\mathcal{Z}$*:* $\mathbf{T}M|_{\mathcal{Z}} = \mathbf{T}\mathcal{Z} \oplus J(\mathfrak{g}_{\mathcal{Z}})$*, where* $\mathfrak{g}_{\mathcal{Z}} = \{X_{\mathcal{Z}},\, X \in \mathfrak{g}\}$.

Under this second assumption, the almost complex structure J induces an almost complex structure J_{red} on $\mathcal{M}_{red}$. We have then a quantization map $RR^{J_{red}}(\mathcal{M}_{red}, -) : K(\mathcal{M}_{red}) \to \mathbb{Z}$. Let $L_{red} \to \mathcal{M}_{red}$ be the orbifold line bundle induced by L.

Under these two assumptions we prove in this paper the following

Theorem A *We have the equality*[1]

$$\left[RR^{G,J}(M, L^{\overset{k}{\otimes}})\right]^G = RR^{J_{red}}\left(\mathcal{M}_{red}, L_{red}^{\overset{k}{\otimes}}\right), \quad k \in \mathbb{N},$$

if any of the following hold:
(i) $G = T$ *is a torus; or*
(ii) $k \in \mathbb{N}$ *is large enough , so that the connected component containing* 0 *of the set of regular values of* f_L *contains* $\frac{1}{k}(w\rho - \rho)$ *for all* w *in the Weyl group* W *of* G*, where* $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ *is half the sum of the positive roots of* G.

A similar result was proved by Jeffrey-Kirwan [17] when (M, ω) is a symplectic manifold, L is the prequantum line bundle, f_L is the moment map associated to an Hamiltonian action of G on M, but where one relax the condition of positivity of J with respect to the symplectic form ω.

In this paper, we start from a (abstract) moment map[2] $f_G : M \to \mathfrak{g}$ (see Definition 6.1). An equivariant vector bundle E is called f_G-positive (see [27]) if the following hold: for any $m \in M$, such that $f_G(m) = \beta$ with $\beta_M(m) = 0$, we have

$$\langle \xi, \beta \rangle \geq 0 \tag{1.1}$$

for any weights ξ of the $\mathbb{T}_\beta$-action on E_m ($\mathbb{T}_\beta$ is the torus of G generated by $\exp_G(t.\beta)$, $t \in \mathbb{R}$). An equivariant vector bundle E is called f_G-*strictly* positive when furthermore the inequality (1.1) is strict for any $\beta \neq 0$. Note that any Hermitian line bundle L is strictly positive for its 'moment' f_L.

Theorem B *Let* f_G *be an abstract moment map satisfying the Assumptions 1) and 2). For any* f_G-strictly *positive* G*-complex vector* $E \to M$ *we have the equality*

$$\left[RR^{G,J}(M, E^{\overset{k}{\otimes}})\right]^G = RR^{J_{red}}\left(\mathcal{M}_{red}, E_{red}^{\overset{k}{\otimes}}\right), \quad k \in \mathbb{N},$$

if any of the following hold:
(i) $G = T$ *is a torus; or*
(ii) k *is large enough.*

[1] $[V]^G$ means the G-invariants of V.
[2] We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ with a G-invariant scalar product.

In this paper we look also to the Hamiltonian case where the moment map f_G and the almost complex structure J are related by means of a G-invariant symplectic 2-form ω :

i) f_G is the moment map of an Hamiltonian of G over (M,ω) : $d\langle f_G, X\rangle = \omega(X_M, -)$, for $X \in \mathfrak{g}$, and

ii) The data (ω, J) are compatible : $(v,w) \to \omega(Jv,w)$ is a Riemannian metric on M.

Note that in this case, Assumption 2 is automatically fulfilled if 0 is a regular value of f_G. More precisely, the compatible data (ω, J) induces compatible data (ω_{red}, J_{red}) on $\mathcal{M}_{red}$.

In this situation, we recover the results of Meinrenken [23] and Tian-Zhang [27].

Theorem C *Let f_G the moment map of an Hamiltonian action of G over (M,ω), and suppose that (ω, J) are compatible. We suppose furthermore that 0 is a regular value of f_G. Let E be a G-complex vector bundle over M.*

1. *If $0 \notin f_G(M)$ and E is f_G-strictly positive, we have*
$$\left[RR^{G,J}(M,E)\right]^G = 0.$$
2. *If $0 \in f_G(M)$ and E is f_G-positive, we have*
$$\left[RR^{G,J}(M,E)\right]^G = RR^{J_{red}}(\mathcal{M}_{red}, E_{red}).$$

We now turn to an introduction to our method. We associate to the abstract moment map f_G the vector field
$$\mathcal{H}_m = [f_G(m)]_M.m, \quad m \in M \ ,$$
and we denote C^{f_G} the set where $\mathcal{H}$ vanishes. There are two important cases. First, when the map f_G is constant equal to an element γ in the centre of $\mathfrak{g}$, the set C^{f_G} corresponds to the submanifold M^γ of fixed points for the infinitesimal action of γ on M. Witten [30] introduces, in the Hamiltonian case, the vector field $\mathcal{H}$ to realise, in the context of equivariant cohomology, a localisation on the set $\mathrm{Cr}(\| f_G \|^2)$ of critical points of the function $\| f_G \|^2$. Here $\mathcal{H}$ is the Hamiltonian vector field of $\| f_G \|^2$, hence $C^{f_G} = \mathrm{Cr}(\| f_G \|^2)$.

Using a deformation argument in the context of transversally elliptic operator introduced by Atiyah [1] and Vergne [29], we proved in section 4 that the map[3] RR^G can be localised near C^{f_G}. More precisely, we have the finite decomposition[4] $C^{f_G} = \cup_{\beta\in\mathcal{B}_G} C^G_\beta$ with $C^G_\beta = G(M^\beta \cap f_G^{-1}(\beta))$, and
$$RR^G(M,E) = \sum_{\beta\in\mathcal{B}_G} RR^G_\beta(M,E) \ ,$$
where each term $RR^G_\beta(M,E)$ is a generalised character of G which only depends of the behaviour of the data M, E, J, f_G, near the subset C^G_β. In fact, $RR^G_\beta(M,E)$ is defined as the index of transversally elliptic operator defined in an open neighbourhood of C^G_β.

[3]We fix one for once a G-invariant almost complex structure J and denote RR^G the quantization map.

[4]$\mathcal{B}_G$ is a finite set in the Lie algebra of a maximal torus of G.

The major work of this paper is the analysis of the localised Riemann-Roch character $RR^G_\beta(M,-) : K_G(M) \to R^{-\infty}(G)$ for $\beta \in \mathcal{B}_G$. We consider three different situations[5] :
Situation 1 : $\beta = 0$,
Situation 2 : $\beta \neq 0$ and $G_\beta = G$,
Situation 3 : $G_\beta \neq G$.

We work out *Situation 1* in subsection 6.1. Here the generalised character $RR^G_0(M,E)$ is localised near $C^G_0 = f_G^{-1}(0)$, and we compute it under Assumptions 1) and 2). We proved in particular that the multiplicity of the trivial representation in $RR^G_0(M,E)$ is $RR^{J_{red}}(\mathcal{M}_{red}, E_{red})$.

Situation 2 is studied in section 5 for the particular case $f_G = \beta$ and in subsection 6.2 for the general case. We proved then a localisation formula on the (G-invariant) submanifold M^β which relates the map $RR^G_\beta(M,-)$ with the map $RR^G_\beta(M^\beta,-)$. With this localisation formula in hand we show that $\left[RR^G_\beta(M,E)\right]^G = 0$ if the vector bundle is f_G-strictly positive.

The subsection 6.3 is devoted to *Situation 3*. The most important result is the induction formula proved in Theorem 6.11 and Corollary 6.12, between $RR^G_\beta(M,E)$ and the generalised character $RR^{G_\beta}_\beta(M,E)$, defined for G_β, which is localised near $M^\beta \cap f^{-1}_{G_\beta}(\beta)$. As β is a central element in G_β, the induction formula reduces the analysis of *Situation 3* to the one of *Situation 2*. But when we look at the multiplicities we loose some information. If the vector bundle E is f_G-strictly positive, we see that $E|_{M^\beta}$ is f_{G_β}-strictly positive, so from the result proved in *Situation 2*, we see that $\left[RR^{G_\beta}_\beta(M^\beta,E)\right]^{G_\beta} = 0$. But this does not implies in general that $\left[RR^G_\beta(M,E)\right]^G = 0$. We have to take the tensor product of E (so that $E^{\overset{k}{\otimes}}$ becomes more and more f_{G_β}-strictly positive) to see that $\left[RR^G_\beta(M,E^{\overset{k}{\otimes}})\right]^G = 0$, when k is large enough. In the Hamiltonian situation considered in section 7, we refine this induction formula by using the symplectic slice at β, and prove that $\left[RR^G_\beta(M,E)\right]^G = 0$ holds for any f_G-positive complex vector bundle E if $0 \in f_G(M)$.

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2. Quantization of compact manifolds

Let M be a smooth compact manifold provided with an action of a compact connected Lie group G. A G-invariant almost complex structure J on M defines a map $RR^{G,J}(M,-) : K_G(M) \to R(G)$ from the equivariant K-theory of complex vector bundles over M to the character ring of G.

[5] G_β is the stabiliser of β in G.

Let us recall the definition of this map. The almost complex structure on M gives the decomposition $\Lambda^\bullet T^*M \otimes \mathbb{C} = \oplus_{i,j}\Lambda^{i,j}T^*M$ of the bundle of differential forms. Using hermitian structure in the tangent bundle $\mathbf{T}M$ of M, and in the fibres of E, we define a twisted Dirac operator

$$\mathcal{D}_E^+ : \mathcal{A}^{0,even}(M,E) \to \mathcal{A}^{0,odd}(M,E)$$

where $\mathcal{A}^{i,j}(M,E) := \Gamma(M, \Lambda^{i,j}T^*M \otimes_{\mathbb{C}} E)$ is the space of E-valued forms of type (i,j). The Riemann-Roch character $RR^{^{G,J}}(M,E)$ is defined as the index of the elliptic operator $\mathcal{D}_E^+$:

$$RR^{^{G,J}}(M,E) = [Ker\mathcal{D}_E^+] - [Coker\mathcal{D}_E^+].$$

In fact the virtual character $RR^{^{G,J}}(M,E)$ is independent of the choice of the hermitian metrics on the vector bundles $\mathbf{T}M$ and E.

If M is a compact complex analytic manifold, and E is an holomorphic complex vector bundle, we have

$$RR^{^{G,J}}(M,E) = \sum_{q=0}^{q=dim M} (-1)^q [\mathcal{H}^q(M, \mathcal{O}(E))],$$

where $\mathcal{H}^q(M,\mathcal{O}(E))$ is the q-th cohomology group of the sheaf $\mathcal{O}(E)$ of the holomorphic sections of E over M.

In this paper, we will use an equivalent definition of the map $RR^{^{G,J}}$. We associate to an invariant almost complex structure J on M the symbol $\mathrm{Thom}_G(M,J) \in K_G(\mathbf{T}M)$ defined as follow. Consider a Riemannian structure q on M such that the endomorphism J is orthogonal relatively to q, and let h be the following hermitian structure on $\mathbf{T}_xM$: $h(v,w) = q(v,w) - \imath q(Jv,w)$ for $v,w \in \mathbf{T}_xM$. Let $p : \mathbf{T}M \to M$ be the canonical projection. The symbol $\mathrm{Thom}_G(M,J) : p^*(\wedge_{\mathbb{C}}^{even}\mathbf{T}M) \to p^*(\wedge_{\mathbb{C}}^{odd}\mathbf{T}M)$ is equal, at $(x,v) \in \mathbf{T}M$, to the Clifford map

$$Cl_x(v) \ : \ p^*(\wedge_{\mathbb{C}}^{even}\mathbf{T}M)|_{(x,v)} \longrightarrow p^*(\wedge_{\mathbb{C}}^{odd}\mathbf{T}M)|_{(x,v)}, \tag{2.2}$$

where $Cl_x(v).w = v \wedge w - c_h(v).w$ for $w \in \wedge_{\mathbb{C}}^\bullet \mathbf{T}_xM$. Here $c_h(v) : \wedge_{\mathbb{C}}^\bullet \mathbf{T}_xM \to \wedge^{\bullet-1}\mathbf{T}_xM$ denotes the contraction map relatively to h : for $w \in \mathbf{T}_xM$ we have $c_h(v).w = h(w,v)$. Note that $(\mathbf{T}M, J)$ is considered as a complex vector bundle over M.

The symbol $\mathrm{Thom}_G(M,J)$ determines the Thom isomorphism Thom_J : $K_G(M) \longrightarrow K_G(\mathbf{T}M)$ by $\mathrm{Thom}_J(E) := \mathrm{Thom}_G(M,J) \otimes p^*(E)$, $E \in K_G(M)$. To make the notation clearer, $\mathrm{Thom}_J(E)$ is the symbol $\sigma^E : p^*(\wedge_{\mathbb{C}}^{even}\mathbf{T}M \otimes E) \to p^*(\wedge_{\mathbb{C}}^{odd}\mathbf{T}M \otimes E)$ with

$$\sigma^E(x,v) := Cl_x(v) \otimes Id_{E_x} \ , \quad (x,v) \in \mathbf{T}M, \tag{2.3}$$

where E_x is the fibre of E at $x \in M$.

Consider the index map $\mathrm{Index}_M^G : K_G(\mathbf{T}^*M) \to R(G)$ where $\mathbf{T}^*M$ is the cotangent bundle of M. Using a G-invariant auxiliary metric on $\mathbf{T}M$, we can identify the vector bundle $\mathbf{T}^*M$ and $\mathbf{T}M$, and produce an 'index' map $\mathrm{Index}_M^G : K_G(\mathbf{T}M) \to R(G)$. We verify easily that this map is independent of the choice of the metric on $\mathbf{T}M$.

Lemma 2.1. *We have the following commutative diagram*

$$(2.4) \qquad \begin{array}{ccc} K_G(M) & \xrightarrow{\mathrm{Thom}_J} & K_G(\mathbf{T}M) \\ & \searrow^{RR^{G,J}} & \big\downarrow \mathrm{Index}_M^G \\ & & R(G)\ . \end{array}$$

Proof: If we use the natural identification $(\mathbf{T}^{0,1}M, \imath) \cong (\mathbf{T}M, J)$ of complex vector bundles over M, we see that the principal symbol of the operator $\mathcal{D}_E^+$ is equal to σ^E modulo some constant (see [11]).

We will conclude with the following Lemma. Let J^0, J^1 be two G-invariant almost complex structures on M, and let RR^{G,J^0}, RR^{G,J^1} be the respective quantization maps.

Lemma 2.2. *The maps RR^{G,J^0} and RR^{G,J^1} are identical in the following cases:*
i) There exists a G-invariant section $A \in \Gamma(M, \mathrm{End}(\mathbf{T}M))$, homotopic to the identity *in $\Gamma(M, \mathrm{End}(\mathbf{T}M))^G$ such that A_x is invertible, and $A_x.J_x^0 = J_x^1.A_x$ for every $x \in M$.*
ii) There exists an homotopy J^t, $t \in [0,1]$ of G-invariant almost complex structures between J^0 and J^1.

Proof of i) : Take a riemannian structure q^1 on M such that $J^1 \in O(q^1)$ and define another riemannian structure q^0 by $q^0(v,w) = q^1(Av, Aw)$ so that $J^0 \in O(q^0)$. Hence the section A defines a bundle unitary map $\underline{A} : (\mathbf{T}M, J^0, h^0) \to (\mathbf{T}M, J^1, h^1)$, $(x,v) \to (x, A_x.v)$, where $h^l(.,.) := q^l(.,.) - \imath q^l(J^l.,.)$, $l = 0,1$. This gives an isomorphism $A_x^\wedge : \wedge_{J^0}\mathbf{T}_xM \to \wedge_{J^1}\mathbf{T}_xM$ such that the following diagram is commutative

$$\begin{array}{ccc} \wedge_{J^0}\mathbf{T}_xM & \xrightarrow{Cl_x(v)} & \wedge_{J^0}\mathbf{T}_xM \\ A_x^\wedge \big\downarrow & & \big\downarrow A_x^\wedge \\ \wedge_{J^1}\mathbf{T}_xM & \xrightarrow{Cl_x(A_x.v)} & \wedge_{J^1}\mathbf{T}_xM\ . \end{array}$$

Then $A^\wedge$ induces an isomorphism between the symbols $\mathrm{Thom}_G(M, J^0)$ and $\underline{A}^*(\mathrm{Thom}_G(M, J^1))$: $(x,v) \to \mathrm{Thom}_G(M, J^1)(x, A_x.v)$. Here $\underline{A}^* : K_G(\mathbf{T}M) \to K_G(\mathbf{T}M)$ is the map induced by the isomorphism $\underline{A}$. Thus the complexes $\mathrm{Thom}_G(M, J^0)$ and $\underline{A}^*(\mathrm{Thom}_G(M, J^1))$ defines the same class in $K_G(\mathbf{T}M)$. We have supposed that A is homotopic to the identity, thus $\underline{A}^* = $ Identity. We have proved that $\mathrm{Thom}_G(M, J^0) = \mathrm{Thom}_G(M, J^1)$ in $K_G(\mathbf{T}M)$, and by Lemma 2.1 this shows that $RR^{G,J^0} = RR^{G,J^1}$.

Proof of ii) : We construct A as in $i)$. Take first $A^{1,0} := Id - J^1J^0$ and remark that $A^{1,0}.J^0 = J^1.A^{1,0}$. Here we consider the homotopy $A_u^{1,0} := Id - uJ^1J^0$, $u \in [0,1]$. If $-J^1J^0$ is close to Id, for example $|Id + J^1J^0| \leq 1/2$, the bundle map $A_u^{1,0}$ will be invertible for every $u \in [0,1]$. Then we can conclude with Point $i)$. In general we use the homotopy J^t, $t \in [0,1]$. First we decompose the interval $[0,1]$ in $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$ and consider the maps $A^{t_{l+1},t_l} := Id - J^{t_{l+1}}J^{t_l}$, with the corresponding homotopy for $A_u^{t_{l+1},t_l}$, $u \in [0,1]$, for $l = 0, \ldots, k-1$.

Because $-J^{t_{l+1}}J^{t_l} \to Id$ when $t \to t'$, the bundle maps $A_u^{t_{l+1},t_l}$ are invertible for all $u \in [0,1]$ if $t_{l+1} - t_l$ is small enough. Then we take the G-equivariant bundle map $A := \Pi_{l=0}^{k-1} A^{t_{l+1},t_l}$ with the homotopy $A_u := \Pi_{l=0}^{k-1} A_u^{t_{l+1},t_l}$, $u \in [0,1]$. We have $A.J^0 = J^1.A$ and A_u is invertible for every $u \in [0,1]$, hence we conclude with the point i). □

3. Transversally elliptic symbols

We give here a brief review of the material we need in the next sections. The references are [1, 9, 10, 29].

Let M be a smooth manifold provided with an action of a compact connected Lie group G, with Lie algebra $\mathfrak{g}$.

Assumption 3.1. *In this section we assume that M is compact, or is an open subset of a compact manifold.*

Like in the previous section, we identify the tangent bundle $\mathbf{T}M$ and the cotangent bundle $\mathbf{T}^*M$ via a G-invariant metric $(.,.)_M$ on $\mathbf{T}M$. For any $X \in \mathfrak{g}$, we denote X_M the following vector field : for $m \in M$, $X_M(m) := \frac{d}{dt}\exp(-tX).m|_{t=0}$.

If E^0, E^1 are G-equivariant vector bundles over M, a morphism $\sigma \in \Gamma(\mathbf{T}M, \hom(p^*E^0, p^*E^1))$ of G-equivariant complex vector bundles will be called a symbol. The subset of all $(x,v) \in \mathbf{T}M$ where $\sigma(x,v) : E^0_x \to E^1_x$ is not invertible will be called the characteristic set of σ, and will be denoted $\mathrm{Char}(\sigma)$.

We denote $\mathbf{T}_G M$ the following subset of $\mathbf{T}M$:

$$\mathbf{T}_G M = \{(x,v) \in \mathbf{T}M,\ (v, X_M(m))_M = 0 \quad \text{for all } X \in \mathfrak{g}\}.$$

A symbol σ will be called *elliptic* if σ is invertible outside a compact subset of $\mathbf{T}M$ ($\mathrm{Char}(\sigma)$ is compact), and it will be called *transversally elliptic* if the restriction of σ to $\mathbf{T}_G M$ is invertible outside a compact subset of $\mathbf{T}_G M$ ($\mathrm{Char}(\sigma) \cap \mathbf{T}_G M$ is compact). An elliptic symbol σ defines an element of $K_G(\mathbf{T}M)$, and the index of σ is a virtual finite dimensional representation of G [2, 3, 4, 5]. A transversally elliptic symbol σ defines an element of $K_G(\mathbf{T}_G M)$, and the index of σ is defined (see [1] for the analytic index and [9, 10] for the cohomological one) and is a trace class virtual representation of G. Remark that any elliptic symbol of $\mathbf{T}M$ is transversally elliptic, hence we have a restriction map $K_G(\mathbf{T}M) \to K_G(\mathbf{T}_G M)$.

Let $R(G)$ be the representation ring of G, and let $R^{-\infty}(G)$ be the set of generalised characters of G. Let H be a maximal torus of G with Lie algebra $\mathfrak{h}$, and $\Lambda = \ker\{\exp_H : \mathfrak{h} \to H\} \subset \mathfrak{t}$ the integral lattice. By the choice of a positive Weyl chamber $\mathfrak{h}^*_+$, we label the irreducible representations of G by the set of dominant weights $\Lambda^*_+ = \Lambda \cap \mathfrak{t}^*_+$.

An element $h \in R^{-\infty}(G)$ (resp. $h \in R(G)$) is of the form

$$h = \sum_{\lambda \in \Lambda^*_+} m_\lambda \chi^G_\lambda, \tag{3.5}$$

where the map $\lambda \mapsto m_\lambda, \Lambda^*_+ \to \mathbb{Z}$, has at most polynomial growth (resp. the map is zero almost everywhere). We have a natural embedding of $R(G)$ in $R^{-\infty}(G)$, and of $R^{-\infty}(G)$ in the set $\mathcal{C}^{-\infty}(G)^G$ of generalised functions over G, invariant by conjugation.

We have the following commutative diagram

$$\begin{array}{ccc} K_G(\mathbf{T}M) & \longrightarrow & K_G(\mathbf{T}_G M) \\ \Big\downarrow \mathrm{Index}^G_M & & \Big\downarrow \mathrm{Index}^G_M \\ R(G) & \longrightarrow & R^{-\infty}(G)\ . \end{array} \tag{3.6}$$

3.1. **Excision lemma.** Let $i : U \hookrightarrow M$ be the inclusion map of a G-invariant open subset, and denote $i_* : K_G(\mathbf{T}_G U) \to K_G(\mathbf{T}_G M)$ the direct image map. We have two index maps $\mathrm{Index}^G_M : K_G(\mathbf{T}_G M) \to R^{-\infty}(G)$, and $\mathrm{Index}^G_U : K_G(\mathbf{T}_G U) \to R^{-\infty}(G)$ such that $\mathrm{Index}^G_M \circ i_* = \mathrm{Index}^G_U$. Suppose that σ is a transversally elliptic symbol on $\mathbf{T}M$ with characteristic set contained in $\mathbf{T}M|_U$. Then, the restriction $\sigma|_U$ of σ to $\mathbf{T}U$ is a transversally elliptic symbol on $\mathbf{T}U$, and

$$i_*(\sigma|_U) = \sigma \quad \text{in} \quad K_G(\mathbf{T}_G M). \tag{3.7}$$

In particular, this gives $\mathrm{Index}^G_M(\sigma) = \mathrm{Index}^G_U(\sigma|_U)$.

3.2. **Free action case.** Let G and H compact Lie groups and let M be a *compact* $G \times H$ manifold where H acts freely. Consider the principal bundle $\pi : M \to M/H$, then the map π is G-equivariant. In this situation we have $\mathbf{T}_{G\times H} M \cong \pi^*(\mathbf{T}_G(M/H))$, and thus a morphism

$$\pi^* \ : \ K_G(\mathbf{T}_G(M/H)) \longrightarrow K_{G\times H}(\mathbf{T}_{G\times H} M) \tag{3.8}$$

We rephrase now Theorem 3.1 of Atiyah in [1]. Let $\{W_a, a \in \hat{H}\}$ be a completed set of inequivalent irreducible representations of H. To each W_a, we associate the complex vector bundle $\underline{W}_a := M \times_H W_a$ on M/H and denote $\underline{W}^*_a$ its dual. The group G acts trivially on W_a, this makes $\underline{W}^*_a$ a G-vector bundle.

Theorem 3.2. *If $\sigma \in K_G(\mathbf{T}_G(M/H))$, then we have the following equality in $R^{-\infty}(G \times H)$*

$$\mathrm{Index}^{G\times H}_M(\pi^*\sigma) \ = \ \sum_{a\in\hat{H}} \mathrm{Index}^G_{M/H}(\sigma \otimes \underline{W}^*_a).W_a \quad . \tag{3.9}$$

In particular the H-invariant part of $\mathrm{Index}^{G\times H}_M(\pi^*\sigma)$ *is* $\mathrm{Index}^G_{M/H}(\sigma)$.

An interesting example is when $M = H$, $G = H_r$ acts by right multiplications on H, and $H = H_l$ acts by left multiplications on H. Then the zero map $\sigma_0 : H \times \mathbb{C} \to H \times \{0\}$ define a $H_r \times H_l$-transversally elliptic symbol associated to the zero differential operator $\mathcal{C}^\infty(H) \to 0$. This symbol is equal to the pullback of $\mathbb{C} \in K_{H_r}(\mathbf{T}_{H_r}\{\text{point}\}) \cong R(H_r)$. In this case $\mathrm{Index}^{H_r\times H_l}_H(\sigma_0)$ is equal to $L^2(H)$, the L^2-index of the zero operator on $\mathcal{C}^\infty(H)$. The H_r-vector bundle $\underline{W}^*_a \to \{\text{point}\}$ is just the vector space W^*_a with the canonical action of H_r. Finally, the equality (3.9) is the Peter-Weyl decomposition of $L^2(H)$ in $R^{-\infty}(H_r \times H_l)$: $L^2(H) = \sum_{a\in\hat{H}} W^*_a \otimes W_a$.

3.3. **Induction.** We will now introduced the induction map. Let $i : H \hookrightarrow G$ be a closed subgroup with Lie algebra $\mathfrak{h}$, and $\mathcal{Y}$ be a smooth H-manifold (satisfying Assumption 3.1). We will now define, for $\mathcal{X} := G \times_H \mathcal{Y}$, a map

$$i_* : K_H(\mathbf{T}_H\mathcal{Y}) \to K_G(\mathbf{T}_G\mathcal{X}) \ , \tag{3.10}$$

which is an *isomorphism.*

First we notice that $\mathbf{T}(G \times_H \mathcal{Y}) \cong G \times_H (\mathfrak{g}/\mathfrak{h} \oplus \mathbf{T}\mathcal{Y})$. This identity comes from the following $G \times H$-equivariant isomorphism of vector bundle over $G \times \mathcal{Y}$:

$$\begin{array}{rcl} \mathbf{T}_H(G \times \mathcal{Y}) & \longrightarrow & G \times (\mathfrak{g}/\mathfrak{h} \oplus \mathbf{T}\mathcal{Y}) \\ \left(g, m; \frac{d}{dt}_{|t=0}(g.e^{tX}) + v_m\right) & \longmapsto & \left(g, m; pr_{\mathfrak{g}/\mathfrak{h}}(X) + v_m\right) \ , \end{array} \tag{3.11}$$

where $pr_{\mathfrak{g}/\mathfrak{h}} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the orthogonal projection. Starting from a H invariant metric on $\mathbf{T}\mathcal{Y}$, and a H-invariant scalar product on $\mathfrak{g}/\mathfrak{h}$, we construct a G-invariant metric on $\mathbf{T}(G \times_H \mathcal{Y})$ that makes the bundles $G \times_H (\mathfrak{g}/\mathfrak{h})$ and $G \times_H \mathbf{T}Y$ *orthogonal.* Then, we have

$$\mathbf{T}_G(G \times_H \mathcal{Y}) \cong G \times_H (\mathbf{T}_H\mathcal{Y}).$$

The map $i_* : K_H(\mathbf{T}_H\mathcal{Y}) \to K_G(G \times_H (\mathbf{T}_H Y))$ is canonically defined as follow. At the level of vector bundles, it associates a (continuous) H-vector bundle E over $\mathbf{T}_H\mathcal{Y}$ to the (continuous) G-vector bundle $G \times_H E$ over $G \times_H \mathbf{T}_H\mathcal{Y}$. For an H-equivariant smooth symbol $\sigma \in \Gamma(\mathbf{T}Y, \hom(E^0, E^1))$, where E^0, E^1 are smooth H-equivariant vector bundles over $\mathbf{T}\mathcal{Y}$, and σ is H-transversally elliptic, the map i_* is defined similarly. First we extend trivially σ to $\mathfrak{g}/\mathfrak{h} \oplus \mathbf{T}\mathcal{Y}$, and we define $i_*(\sigma) \in \Gamma(G \times_H (\mathfrak{g}/\mathfrak{h} \oplus \mathbf{T}\mathcal{Y}), \hom(G \times_H E^0, G \times_H E^1))$ by $i_*(\sigma)([g; \xi, x, v]) := \sigma(x, v)$ for $g \in G$, $\xi \in \mathfrak{g}/\mathfrak{h}$ and $(x, v) \in \mathbf{T}\mathcal{Y}$.

To express the G-index of $i_*(\sigma)$ in terms of the H-index of σ, we need the induction map

$$\mathrm{Ind}_H^G : \mathcal{C}^{-\infty}(H)^H \longrightarrow \mathcal{C}^{-\infty}(G)^G \ , \tag{3.12}$$

where $\mathcal{C}^{-\infty}(H)$ is the set of generalised functions on H, and the H and G invariants are taken with the conjugation action. The map Ind_H^G is defined as follow : for $\phi \in \mathcal{C}^{-\infty}(H)^H$, we have

$$\int_G \mathrm{Ind}_H^G(\phi)(g) f(g) dg = \frac{\mathrm{vol}(G, dg)}{\mathrm{vol}(H, dh)} \int_H \phi(h) f|_H(h) dh,$$

for every $f \in \mathcal{C}^{\infty}(G)^G$.

We can now recall Theorem 4.1 of Atiyah in [1].

Theorem 3.3. *Let $i : H \to G$ the inclusion of a closed subgroup, let $\mathcal{Y}$ be a H-manifold satisfying Assumption 3.1, and set $\mathcal{X} = G \times_H \mathcal{Y}$. Then we have the commutative diagram*

$$\begin{array}{ccc} K_H(\mathbf{T}_H\mathcal{Y}) & \xrightarrow{i_*} & K_G(\mathbf{T}_G\mathcal{X}) \\ \mathrm{Index}_{\mathcal{Y}}^H \big\downarrow & & \big\downarrow \mathrm{Index}_{\mathcal{X}}^G \\ \mathcal{C}^{-\infty}(H)^H & \xrightarrow[\mathrm{Ind}_H^G]{} & \mathcal{C}^{-\infty}(G)^G \ . \end{array}$$

3.4. **Reduction.** Let us recall a multiplicative property of the index for the product of manifold. Let a compact Lie group G acts smoothly on two manifolds $\mathcal{X}$ and $\mathcal{Y}$, and assume that another compact Lie group H acts smoothly on $\mathcal{Y}$ commuting with the action of G. The external product of complexes on $\mathbf{T}\mathcal{X}$ and $\mathbf{T}\mathcal{Y}$ induces a multiplication (see [1] and [29], section 2):

$$\begin{array}{rcl} K_G(\mathbf{T}\mathcal{X}) \times K_{G\times H}(\mathbf{T}\mathcal{Y}) & \longrightarrow & K_{G\times H}(\mathbf{T}(\mathcal{X}\times\mathcal{Y})) \\ (\sigma_1,\sigma_2) & \longmapsto & \sigma_1 \odot \sigma_2 \ . \end{array} \tag{3.13}$$

Let us recall the definition of this external product. Let $E^{\pm}, F^{\pm}$ be G-equivariant Hermitian vector bundles over $\mathcal{X}$ and $\mathcal{Y}$ respectively, and let $\sigma_1 : E^+ \to E^-$, $\sigma_2 : F^+ \to F^-$ be G-equivariant symbols. We consider the G-equivariant symbol

$$\sigma_1 \odot \sigma_2 : E^+ \otimes F^+ \oplus E^- \otimes F^- \longrightarrow E^- \otimes F^+ \oplus E^+ \otimes F^-$$

defined by

$$\sigma_1 \odot \sigma_2 = \begin{pmatrix} \sigma_1 \otimes I & -I \otimes \sigma_2^* \\ I \otimes \sigma_2 & \sigma_1^* \otimes I \end{pmatrix} . \tag{3.14}$$

We see that the set $\mathrm{Char}(\sigma_1 \odot \sigma_2) \subset \mathbf{T}\mathcal{X} \times \mathbf{T}\mathcal{Y}$ is equal to $\mathrm{Char}(\sigma_1) \times \mathrm{Char}(\sigma_2)$. This exterior product defines the $R(G)$-module structure on $K_G(\mathbf{T}\mathcal{X})$, by taking $\mathcal{Y} = point$ and $H = \{e\}$. If we take $\mathcal{X} = \mathcal{Y}$ and $H = \{e\}$, the product on $K_G(\mathbf{T}\mathcal{X})$ is defined by

$$\sigma_1 \tilde{\odot} \sigma_2 := s_{\mathcal{X}}^* (\sigma_1 \odot \sigma_2) \ , \tag{3.15}$$

where $s_{\mathcal{X}} : \mathbf{T}\mathcal{X} \to \mathbf{T}\mathcal{X} \times \mathbf{T}\mathcal{X}$ is the diagonal map.

In the transversally elliptic case we need to be careful in the definition of the exterior product, because $\mathbf{T}_{G\times H}(\mathcal{X}\times\mathcal{Y}) \neq \mathbf{T}_G\mathcal{X} \times \mathbf{T}_H\mathcal{Y}$.

Definition 3.4. *Let σ be a H-transversally elliptic symbol on $\mathbf{T}\mathcal{Y}$. This symbol is call H-*transversally-good *if the characteristic set of σ intersects $\mathbf{T}_H\mathcal{Y}$ in a compact subset of $\mathcal{Y}$.*

Recall Lemma 3.4 and Theorem 3.5 of Atiyah in [1]. Let σ_1 be a G-transversally elliptic symbol on $\mathbf{T}\mathcal{X}$, and σ_2 be a H-transversally elliptic symbol on $\mathbf{T}\mathcal{Y}$ that is G-equivariant. Suppose furthermore that σ_2 is *H-transversally-good*, then the product $\sigma_1 \odot \sigma_2$ is $G \times H$-transversally elliptic. Because every class of $K_{G\times H}(\mathbf{T}_H\mathcal{Y})$ can be represented by an *H-transversally-good* elliptic symbol, we have a multiplication

$$\begin{array}{rcl} K_G(\mathbf{T}_G\mathcal{X}) \times K_{G\times H}(\mathbf{T}_H\mathcal{Y}) & \longrightarrow & K_{G\times H}(\mathbf{T}_{G\times H}(\mathcal{X}\times\mathcal{Y})) \\ (\sigma_1,\sigma_2) & \longmapsto & \sigma_1 \odot \sigma_2 \ . \end{array} \tag{3.16}$$

Suppose now that the manifolds $\mathcal{X}$ and $\mathcal{Y}$ satisfy Assumption 3.1: the index maps $\mathrm{Index}_{\mathcal{X}}^G : K_G(\mathbf{T}_G\mathcal{X}) \to R^{-\infty}(G)$, $\mathrm{Index}_{\mathcal{Y}}^{G\times H} : K_{G\times H}(\mathbf{T}_H\mathcal{Y}) \to R^{-\infty}(G\times H)$, and $\mathrm{Index}_{\mathcal{X}\times\mathcal{Y}}^{G\times H} : K_{G\times H}(\mathbf{T}_{G\times H}(\mathcal{X}\times\mathcal{Y})) \to R^{-\infty}(G\times H)$ are well defined. After Theorem 3.5 of [1], we know that

$$\mathrm{Index}_{\mathcal{X}\times\mathcal{Y}}^{G\times H}(\sigma_1 \odot \sigma_2) = \mathrm{Index}_{\mathcal{X}}^G(\sigma_1) \cdot \mathrm{Index}_{\mathcal{Y}}^{G\times H}(\sigma_2) \quad \text{in} \quad R^{-\infty}(G\times H) \ , \tag{3.17}$$

for $\sigma_1 \in K_G(\mathbf{T}_G\mathcal{X})$ and $\sigma_2 \in K_{G\times H}(\mathbf{T}_H(\mathcal{X}\times H))$.

In the rest of this subsection we suppose that the subgroup $H \subset G$ is the centralizer of an element $\gamma \in \mathfrak{g}$.

We define now a map $r^{\gamma}_{G,H} : K_G(\mathbf{T}_G\mathcal{X}) \to K_H(\mathbf{T}_H\mathcal{X})$ for every G-manifold $\mathcal{X}$. We consider the manifold $\mathcal{X} \times G$ with two actions of $G \times H$: for $(g,h) \in G \times H$ and $(x,a) \in \mathcal{X} \times G$

- we have $(g,h).(x,a) := (g.x, gah^{-1})$ on $\mathcal{X} \overset{1}{\times} G$, and
- we have $(g,h).(x,a) := (h.x, gah^{-1})$ on $\mathcal{X} \overset{2}{\times} G$.

The map $\Theta : \mathcal{X} \overset{2}{\times} G \to \mathcal{X} \overset{1}{\times} G$, $(x,a) \mapsto (a.x, a)$ is $G \times H$-equivariant, and induces $\Theta^* : K_{G\times H}(\mathbf{T}_{G\times H}(\mathcal{X} \overset{1}{\times} G)) \to K_{G\times H}(\mathbf{T}_{G\times H}(\mathcal{X} \overset{2}{\times} G))$. The action of G is free on $\mathcal{X} \overset{2}{\times} G$, then the quotient map $\pi : \mathcal{X} \overset{2}{\times} G \to \mathcal{X}$ induces a isomorphism $\pi^* : K_H(\mathbf{T}_H\mathcal{X}) \to K_{G\times H}(\mathbf{T}_{G\times H}(\mathcal{X} \overset{2}{\times} G))$.

We consider the manifold G/H with the G-invariant complex structure J_γ defined by the element γ. At $e \in G/H$, the map $J_\gamma(e)$ equals $ad(\gamma).\sqrt{-ad(\gamma)^2}$ on $\mathbf{T}_e(G/H) = \mathfrak{g}/\mathfrak{h}$. We denote $\sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}} \in K_{G\times H}(\mathbf{T}_H G)$ the pullback of the Thom class $\mathrm{Thom}_G(G/H, J_\gamma) \in K_G(\mathbf{T}(G/H))$, via the quotient map $G \to G/H$.

Consider the manifold $\mathcal{Y} = G$ with the action of $G \times H$ defined by $(g,h).a = gah^{-1}$ for $a \in G$, $(g,h) \in G \times H$. As the symbol $\sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}$ is H-transversally good on $\mathbf{T}G$, the product by $\sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}$ induces, by Equation 3.16, the map

$$\begin{array}{ccc} K_G(\mathbf{T}_G\mathcal{X}) & \longrightarrow & K_{G\times H}(T_{G\times H}(\mathcal{X} \overset{1}{\times} G)) \\ \sigma & \longmapsto & \sigma \odot \sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}} \ . \end{array}$$

Definition 3.5. *The map $r^{\gamma}_{G,H} : K_G(\mathbf{T}_G\mathcal{X}) \to K_H(\mathbf{T}_H\mathcal{X})$ is defined for every $\sigma \in K_G(\mathbf{T}_G\mathcal{X})$ by*

$$r^{\gamma}_{G,H}(\sigma) := (\pi^*)^{-1} \circ \Theta^*(\sigma \odot \sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}) \ .$$

Theorem 4.2 in [1] tells us that the following diagram is commutative

$$\begin{array}{ccc} K_G(\mathbf{T}_G\mathcal{X}) & \xrightarrow{\ r^{\gamma}_{G,H}\ } & K_H(\mathbf{T}_H\mathcal{X}) \\ \Big\downarrow \mathrm{Index}^G_{\mathcal{X}} & & \Big\downarrow \mathrm{Index}^H_{\mathcal{X}} \\ \mathcal{C}^{-\infty}(G)^G & \xleftarrow{\ \mathrm{Ind}^G_H\ } & \mathcal{C}^{-\infty}(H)^H . \end{array} \tag{3.18}$$

We show now a more explicit description of the map $r^{\gamma}_{G,H}$. Consider the moment map

$$\mu_G : \mathbf{T}^*\mathcal{X} \to \mathfrak{g}^*$$

for the (canonical) Hamiltonian action of G on the symplectic manifold $\mathbf{T}^*\mathcal{X}$. If we identify the tangent bundle $\mathbf{T}\mathcal{X}$ with the cotangent bundle $\mathbf{T}^*\mathcal{X}$ via a G-invariant metric, and $\mathfrak{g}$ with $\mathfrak{g}^*$ via a G-invariant scalar product the 'moment map' is a map $\mu_G : \mathbf{T}\mathcal{X} \to \mathfrak{g}$ defined as follow. If $E^1, \cdots, E^l$ is an orthonormal basis of $\mathfrak{g}$, we have $\mu_G(x,v) = \sum_i (E^i_M(x), v)_M E^i$ for $(x,v) \in \mathbf{T}\mathcal{X}$. We have for the moment map the decomposition $\mu_G = \mu_H + \mu_{G/H}$, relative to the H-invariant orthogonal decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. It is important to note that $\mathbf{T}_G\mathcal{X} = \mu_G^{-1}(0)$, $\mathbf{T}_H\mathcal{X} = \mu_H^{-1}(0)$, and $\mathbf{T}_G\mathcal{X} = \mathbf{T}_H\mathcal{X} \cap \mu_{G/H}^{-1}(0)$.

The real vector space $\mathfrak{g}/\mathfrak{h}$ is endowed with the complex structure defined by γ. Consider over $\mathbf{T}\mathcal{X}$ the H-equivariant symbol

$$\begin{aligned} \sigma^{\mathcal{X}}_{G,H} : \mathbf{T}\mathcal{X} \times \wedge^{even}_{\mathbb{C}}\mathfrak{g}/\mathfrak{h} &\longrightarrow \mathbf{T}\mathcal{X} \times \wedge^{odd}_{\mathbb{C}}\mathfrak{g}/\mathfrak{h} \\ (x, v; w) &\longrightarrow (x, v; w')\ , \end{aligned}$$

with $w' = Cl(\mu_{G/H}(x, v)).w$. Here $\mathfrak{h}^\perp \simeq \mathfrak{g}/\mathfrak{h}$, and $Cl(X) : \wedge_{\mathbb{C}}\mathfrak{g}/\mathfrak{h} \to \wedge_{\mathbb{C}}\mathfrak{g}/\mathfrak{h}$, $X \in \mathfrak{g}/\mathfrak{h}$, denotes the Clifford action. This symbol has $\mu_{G/H}^{-1}(0)$ for characteristic set. For any symbol σ over $\mathbf{T}\mathcal{X}$, with characteristic set $\mathrm{Char}(\sigma)$, the product $\sigma \,\tilde{\odot}\, \sigma^{\mathcal{X}}_{G,H}$, defined at Equation (3.15), is a symbol over $\mathbf{T}\mathcal{X}$ with characteristic set $\mathrm{Char}(\sigma \,\tilde{\odot}\, \sigma^{\mathcal{X}}_{G,H}) = \mathrm{Char}(\sigma) \cap \mu_{G/H}^{-1}(0)$. Then if σ is a G-transversally elliptic symbol over $\mathbf{T}\mathcal{X}$, the product $\sigma \,\tilde{\odot}\, \sigma^{\mathcal{X}}_{G,H}$ is a H-transversally elliptic symbol.

Proposition 3.6. *The restriction $r^{\gamma}_{G,H} : K_G(\mathbf{T}_G\mathcal{X}) \to K_H(\mathbf{T}_H\mathcal{X})$ has the following equivalent definition: for every $\sigma \in K_G(\mathbf{T}_G\mathcal{X})$*

$$r^{\gamma}_{G,H}(\sigma) = \sigma \,\tilde{\odot}\, \sigma^{\mathcal{X}}_{G,H} \quad \text{in} \quad K_H(\mathbf{T}_H\mathcal{X}).$$

Proof : We have to show that for every $\sigma \in K_G(\mathbf{T}_G\mathcal{X})$, $\sigma\tilde{\odot}\sigma^{\mathcal{X}}_{G,H} = (\pi^*)^{-1} \circ \Theta^*(\sigma \odot \sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}})$ in $K_H(\mathbf{T}_H\mathcal{X})$. Recall first that $\sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}} : p_G^*(G \times \wedge^{even}_{\mathbb{C}}\mathfrak{g}/\mathfrak{h}) \to p_G^*(G \times \wedge^{odd}_{\mathbb{C}}\mathfrak{g}/\mathfrak{h})$, with $p_G : \mathbf{T}G \to G$ the canonical projection, and $\sigma_{\mathfrak{g}/\mathfrak{h}}(a, Z) = Cl(Z_{\mathfrak{g}/\mathfrak{h}})$ for $(a, Z) \in \mathbf{T}G \simeq G \times \mathfrak{g}$, where $Z_{\mathfrak{g}/\mathfrak{h}}$ is the $\mathfrak{g}/\mathfrak{h}$- component of $Z \in \mathfrak{g}$.

Consider $\sigma : p_{\mathcal{X}}^*E_0 \to p_{\mathcal{X}}^*E_1$, a G-transversally elliptic symbol on $\mathbf{T}\mathcal{X}$, where E_0, E_1 are G-complex vector bundles over $\mathcal{X}$, and $p_{\mathcal{X}} : \mathbf{T}\mathcal{X} \to \mathcal{X}$ is the canonical projection. The product $\sigma \odot \sigma^{\gamma}_{\mathfrak{g}/\mathfrak{h}}$ acts on the bundles $p_{\mathcal{X}}^*E_\bullet \otimes p_G^*(G \times \wedge^\bullet_{\mathbb{C}}\mathfrak{g}/\mathfrak{h})$ at $(x, v; a, Z) \in \mathbf{T}(\mathcal{X} \times G)$ by

$$\sigma(x, v) \odot Cl(Z_{\mathfrak{g}/\mathfrak{h}}).$$

The pullback $\sigma_o := \Theta^*(\sigma \odot \sigma_{\mathfrak{g}/\mathfrak{h}})$ acts on the bundle $G \times (p_{\mathcal{X}}^*E_\bullet \otimes \wedge^\bullet_{\mathbb{C}}\mathfrak{g}/\mathfrak{h})$ (here we identify $\mathbf{T}(\mathcal{X} \times G)$ with $G \times (\mathfrak{g} \oplus \mathbf{T}\mathcal{X})$). At $(x, v; a, Z) \in \mathbf{T}(\mathcal{X} \times G)$ we have

$$\sigma_o(x, v; a, Z) = \sigma \odot \sigma_{\mathfrak{g}/\mathfrak{h}}(a.x, v'; a, Z'), \quad \text{with}$$

$(v', Z') = \left([\mathbf{T}_{(x,a)}\Theta]^*\right)^{-1}(v, Z)$. Here $\mathbf{T}_{(x,a)}\Theta : \mathbf{T}_{(x,a)}(\mathcal{X} \times G) \to \mathbf{T}_{(a.x,a)}(\mathcal{X} \times G)$ is the tangent map of Θ at (x, a), and $[\mathbf{T}_{(x,a)}\Theta]^* : \mathbf{T}_{(a.x,a)}(\mathcal{X} \times G) \to \mathbf{T}_{(x,a)}(\mathcal{X} \times G)$ its transpose. A small computation shows that $Z' = Z + \mu_G(v)$ and $v' = a.v$. Finally, we get

$$\sigma_o(x, v; a, Z) = \sigma(a.x, a.v) \odot Cl(Z_{\mathfrak{g}/\mathfrak{h}} + \mu_{G/H}(v)).$$

Hence, the symbol $(\pi^*)^{-1}(\sigma_o)$ acts on the bundle $p_{\mathcal{X}}^*E_\bullet \otimes \wedge^\bullet_{\mathbb{C}}\mathfrak{g}/\mathfrak{h}$ by

$$(\pi^*)^{-1}(\sigma_o)(x, v) = \sigma(x, v) \odot Cl(\mu_{G/H}(v)).$$

□

For any G-invariant function $\phi \in \mathcal{C}^\infty(G)^G$, we denoted $\phi_{|H} \in \mathcal{C}^\infty(H)^H$, the restriction to $H = G_\gamma$. The Weyl integration formula can be written in the following way

$$\phi = \mathrm{Ind}_H^G\left(\phi_{|H}.\det{}^{\mathbb{C}}_{\mathfrak{g}/\mathfrak{h}}(1 - h)\right) \text{ in } \mathcal{C}^{-\infty}(G)^G\ . \tag{3.19}$$

Equation (3.19) remains true for any $\phi \in \mathcal{C}^{-\infty}(G)^G$ that admits a restriction to H.

Lemma 3.7. *Let σ be a G-transversally elliptic symbol. Suppose furthermore that σ is H-transversally elliptic. This symbol defines two classes $\sigma \in K_G(\mathbf{T}_G\mathcal{X})$ and $\sigma_{|H} \in K_H(\mathbf{T}_H\mathcal{X})$ with the relation*[6]

$$r^{\gamma}_{G,H}(\sigma) = \sigma_{|H} \otimes \wedge^{\bullet}_{\mathbb{C}}\mathfrak{g}/\mathfrak{h}.$$

Hence for the generalised character $\mathrm{Index}^G_{\mathcal{X}}(\sigma) \in R^{-\infty}(G)$ *we have a 'Weyl integration' formula*

$$\mathrm{Index}^G_{\mathcal{X}}(\sigma) = \mathrm{Ind}^{G}_{H}\left(\mathrm{Index}^H_{\mathcal{X}}(\sigma_{|H}) \det{}^{\mathbb{C}}_{\mathfrak{g}/\mathfrak{h}}(1-h)\right) . \tag{3.20}$$

Proof : If σ is H-transversally elliptic, the symbol $(x,v) \to \sigma(x,v) \odot Cl(\mu_{G/H}(v))$ is homotopic to $(x,v) \to \sigma(x,v) \odot Cl(0)$ in $K_H(\mathbf{T}_H\mathcal{X})$. Hence $\sigma_{|H} \odot \sigma^{\mathcal{X}}_{G,H} = \sigma_{|H} \otimes \wedge^{\bullet}_{\mathbb{C}}\mathfrak{g}/\mathfrak{h}$ in $K_H(\mathbf{T}_H\mathcal{X})$. Equation (3.20) follows from the diagram (3.18). □

Corollary 3.8. *Let σ be a G-transversally elliptic symbol which furthermore is H-transversally elliptic, and let $\phi \in \mathcal{C}^{-\infty}(G)^G$ which admits a restriction to H. We have*

$$\phi = \mathrm{Index}^G_{\mathcal{X}}(\sigma) \Longleftrightarrow \phi_{|H} = \mathrm{Index}^H_{\mathcal{X}}(\sigma_{|H}) \ .$$

In fact, if we come back to the definition of the analytic index given by Atiyah [1], one can show the following stronger result. Let σ be a G-transversally elliptic symbol, and suppose that σ is H-transversally elliptic. Then $\mathrm{Index}^G_{\mathcal{X}}(\sigma) \in \mathcal{C}^{-\infty}(G)^G$ admits a *restriction* to H equal to $\mathrm{Index}^H_{\mathcal{X}}(\sigma_{|H}) \in \mathcal{C}^{-\infty}(H)^H$.

4. Localisation - The general procedure

We recall briefly the notations. Let (M, J, G) be a compact G-manifold provided with a G-invariant almost complex structure. We denote $RR^{G,J} : K_G(M) \to R(G)$ (or simply RR^G), the corresponding quantization map. We choose an G-invariant Riemannian metric $(.,.)_M$ on M.

We define in this section a general procedure to localise the quantization map $RR^G : K_G(M) \to R(G)$ through the use of a G-equivariant vector field λ. This idea of localisation goes back, when G is a circle group, to Atiyah [1] (see Lecture 6) and Vergne [29] (see part II).

We denote by $\Phi_\lambda : M \to \mathfrak{g}^*$ the map defined by $\langle \Phi_\lambda(m), X\rangle := (\lambda_m, X_M|_m)_M$ for $X \in \mathfrak{g}$. We denote by $\sigma^E(m,v)$, $(m,v) \in \mathbf{T}M$ the elliptic symbol associated to $\mathrm{Thom}_G(M) \otimes p^*(E)$ for $E \in K_G(M)$ (see section 2).

Let σ^E_1 be the following G-invariant elliptic symbol

$$\sigma^E_1(m,v) := \sigma^E(m, v - \lambda_m), \quad (m,v) \in \mathbf{T}M. \tag{4.21}$$

The symbol σ^E_1 is obviously homotopic to σ^E and then defines the same class in $K_G(\mathbf{T}M)$. The characteristic set $\mathrm{Char}(\sigma^E)$ is $M \subset \mathbf{T}M$, but we see easily that $\mathrm{Char}(\sigma^E_1)$ is equal to the graph of the vector field λ, and

$$\mathrm{Char}(\sigma^E_1) \cap \mathbf{T}_G M = \{(m, \lambda_m) \in \mathbf{T}M,\ \ m \in \{\Phi_\lambda = 0\}\}.$$

We will now decompose the elliptic symbol σ^E_1 in $K_G(\mathbf{T}_G M)$ near

$$C_\lambda := \{\Phi_\lambda = 0\} \ .$$

[6] Here we note $\sigma_{|H} \otimes \wedge^{\bullet}_{\mathbb{C}}\mathfrak{g}/\mathfrak{h}$ for the difference $\sigma_{|H} \otimes \wedge^{even}_{\mathbb{C}}\mathfrak{g}/\mathfrak{h} - \sigma_{|H} \otimes \wedge^{odd}_{\mathbb{C}}\mathfrak{g}/\mathfrak{h}$.

If a G-invariant subset C is a union of *connected components* of C_λ there exists a G-invariant open neighbourhood $\mathcal{U}^c \subset M$ of C such that $\mathcal{U}^c \cap C_\lambda = C$ and $\partial\mathcal{U}^c \cap C_\lambda = \emptyset$. We associated to the subset C the symbol $\sigma^E_C := \sigma^E_1|_{\mathcal{U}^c} \in K_G(\mathbf{T}_G\mathcal{U}^c)$ which is the restriction of σ^E_1 to $\mathbf{T}\mathcal{U}^c$. It is well defined because $\mathrm{Char}(\sigma^E_1|_{\mathcal{U}^c}) \cap \mathbf{T}_G\mathcal{U}^c = \{(m, \lambda_m) \in \mathbf{T}M,\ m \in C\}$ is compact.

Proposition 4.1. *Let $C^a, a \in A$, be a finite collection of disjoint G-invariant subsets of C_λ, each of them being a union of connected components of C_λ, and let $\sigma^E_{C^a} \in K_G(\mathbf{T}_G\mathcal{U}^a)$ be the localised symbols. If $C_\lambda = \cup_a C^a$, we have*

$$\sigma^E = \sum_{a\in A} i^a_*(\sigma^E_{C^a}) \quad \text{in} \quad K_G(\mathbf{T}_G M),$$

where $i^a : \mathcal{U}^a \hookrightarrow M$ is the inclusion and $i^a_* : K_G(\mathbf{T}_G\mathcal{U}^a) \to K_G(\mathbf{T}_G M)$ is the corresponding direct image.

Proof : This a consequence of the property of excision. We consider disjoint neighbourhoods $\mathcal{U}^a$ of C^a, and take $i : \mathcal{U} = \cup_a\mathcal{U}^a \hookrightarrow M$. Let $\chi_a \in \mathcal{C}^\infty(M)^G$ be a test function (i.e. $0 \leq \chi_a \leq 1$) with compact support on $\mathcal{U}^a$ such that $\chi_a(m) \neq 0$ if $m \in C^a$. Then the function $\chi := \sum_a \chi_a$ is a G-invariant test function with support in $\mathcal{U}$ such that χ never vanishes on C_λ.

We consider the G-equivariant symbol on M

$$\sigma^E_\chi(m, v) := \sigma^E(m, \chi(m)v - \lambda_m),$$

for $(m, v) \in \mathbf{T}M$.

We will prove the following :

i) the symbol σ^E_χ is G-transversally elliptic and $\mathrm{Char}(\sigma^E_\chi) \subset \mathbf{T}M|_{\mathcal{U}}$,

ii) the symbols σ^E_χ and σ^E_1 are equal in $K_G(\mathbf{T}_G M)$, and

iii) the restrictions $\sigma^E_\chi|_{\mathcal{U}}$ and $\sigma^E_1|_{\mathcal{U}}$ are equal in $K_G(\mathbf{T}_G\mathcal{U})$.

With Point i) we can apply the excision property to σ^E_χ, hence $\sigma^E_\chi = i_*(\sigma^E_\chi|_{\mathcal{U}})$. By ii) and iii), the last equality gives $\sigma^E_1 = i_*(\sigma^E_1|_{\mathcal{U}}) = \sum_a i^a_*(\sigma^E_{C^a})$.

Proof of i). The point (m, v) belongs to $\mathrm{Char}(\sigma^E_\chi)$ if and only if $\chi(m)v = \lambda_m (*)$. If m is not included in $\mathcal{U}$, we have $\chi(m) = 0$ and the equality $(*)$ becomes $\lambda_m = 0$. But $\{\lambda = 0\} \subset C_\lambda \subset \mathcal{U}$, thus $\mathrm{Char}(\sigma^E_\chi) \subset \mathbf{T}M|_{\mathcal{U}}$. The point (m, v) belongs to $\mathrm{Char}(\sigma^E_\chi) \cap \mathbf{T}_G M$ if and only if $\chi(m)v = \lambda_m$ and v is orthogonal to the G-orbit in m. This imposes $m \in C_\lambda$, and finally we see that $\mathrm{Char}(\sigma^E_\chi) \cap \mathbf{T}_G M \simeq C_\lambda$ is compact because the function χ never vanishes on C_λ.

Proof of ii). We use the homotopy σ^E_t, $t \in [0, 1]$, defined by

$$\sigma^E_t(m, v) = \sigma^E(m, (t + (1 - t)\chi(m))v - \lambda_m).$$

We see as before that the symbols σ^E_t, $t \in [0, 1]$, are G-transversally elliptic on $\mathbf{T}M$.

Proof of iii). Here we use the homotopy $\sigma^E_t|_{\mathcal{U}},\ t \in [0, 1]$.

□

Because $RR^G(M, E) = \mathrm{Index}^G_M(\sigma^E) \in R(G)$, we obtain from Proposition 4.1 the following decomposition

$$RR^G(M, E) = \sum_{a\in A} \mathrm{Index}^G_{\mathcal{U}^a}(\sigma^E_{C^a}) \quad \text{in} \quad R^{-\infty}(G). \tag{4.22}$$

The rest of this article is devoted to the description, in some particular cases, of the localised Riemann-Roch character near C^a:

$$(4.23) \qquad \begin{array}{rcl} RR^G_{C^a}(M,-) \ : \ K_G(M) & \longrightarrow & R^{-\infty}(G) \\ E & \longmapsto & \mathrm{Index}^G_{\mathcal{U}^a}(\sigma^E_{C^a}). \end{array}$$

5. Localisation on M^β

Let (M, J, G) be a compact G-manifold provided with a G-invariant almost complex structure. We denote $RR^G : K_G(M) \to R(G)$ the quantization map. Let β be an element in the *centre* of the Lie algebra of G, and consider the G-invariant vector field $\lambda := \beta_M$ generated by the infinitesimal action of β. In this case we have obviously

$$\{\Phi_{\beta_M} = 0\} = \{\beta_M = 0\} = M^\beta \ .$$

In this section, we compute the localisation of the quantization map on the submanifold M^β following the technique explained in the section 4. We need first to understand the case of a vector space. Most of the ideas are taken from Vergne [29][Part II], where the same computation was carried out in the Spin case with an action of the circle group.

5.1. **Action on a vector space.** Let (V, q, J) a real vector space equipped with a complex structure J and an euclidean metric q such that $J \in O(q)$. Suppose that a compact Lie group G acts on (V, q, J) in a unitary way, and that there exists β in the centre of $\mathfrak{g} := Lie(G)$ such that

$$V^\beta = \{0\}.$$

We denote $\mathbb{T}_\beta$ the subtorus generated by $\exp(t.\beta), t \in \mathbb{R}$, and $\mathfrak{t}_\beta$ its Lie algebra.

The complex $\mathrm{Thom}_G(V, J)$ does not define an element in $K_G(\mathbf{T}V)$ because its characteristic set is V.

Definition 5.1. *Let* $\mathrm{Thom}^\beta_G(V) \in K_G(\mathbf{T}_G V)$ *be the* G*-transversally elliptic complex defined by*

$$\mathrm{Thom}^\beta_G(V)(x,v) := \mathrm{Thom}_G(V)(x, v - \beta_V(x)) \quad \text{for} \quad (x,v) \in \mathbf{T}V.$$

We see easily that $\mathrm{Char}(\mathrm{Thom}^\beta_G(V)) \cap \mathbf{T}_G V = \{(0,0)\}$.

The aim of this section is the computation of the index of $\mathrm{Thom}^\beta_G(V)$. We denote by ρ the action of G in the unitary group of (V, q, J). This G-action and the complex structure J are extended canonically on the complexified vector space $V \otimes \mathbb{C}$. We denote $z \overset{J}{\cdot} v := x.v + y.J(v)$, $z = x + \imath y \in \mathbb{C}$, the action of $\mathbb{C}$ on the complex vector space (V, J) or $(V \otimes \mathbb{C}, J)$. For $\alpha \in \mathfrak{t}^*_\beta$, we denote $V(\alpha)$ (resp. $(V \otimes \mathbb{C})(\alpha)$) the following subspace of V (resp. $V \otimes \mathbb{C}$)

$$V(\alpha) := \{v \in V,\ \rho(\exp X)(v) = e^{\imath\langle\alpha,X\rangle} \overset{J}{\cdot} v,\ \forall X \in \mathfrak{t}_\beta\}$$

(resp. $(V \otimes \mathbb{C})(\alpha) := \{v \in V \otimes \mathbb{C},\ \rho(\exp X)(v) = e^{\imath\langle\alpha,X\rangle} \overset{J}{\cdot} v,\ \forall X \in \mathfrak{t}_\beta\}$). The subspaces $V(\alpha)$ and $(V \otimes \mathbb{C})(\alpha)$ inherit the action of G and the complex structure J.

An element $\alpha \in \mathfrak{t}^*_\beta$, is called a weight for the action of $\mathbb{T}_\beta$ on (V, J) (resp. on $(V \otimes \mathbb{C}, J)$) if $V(\alpha) \neq 0$ (resp. $(V \otimes \mathbb{C})(\alpha) \neq 0$). We denote $\Delta(\mathbb{T}_\beta, V)$ (resp. $\Delta(\mathbb{T}_\beta, V \otimes \mathbb{C})$) the set of weights for the action of $\mathbb{T}_\beta$ on V (resp. $V \otimes \mathbb{C}$).

Definition 5.2. *We denote $V^{\beta,+}$ the following G-stable subspace of V*

$$V^{\beta,+} := \sum_{\alpha \in \Delta_+(\mathbb{T}_\beta, V)} V(\alpha) \ ,$$

where $\Delta_+(\mathbb{T}_\beta, V) = \{\alpha \in \Delta(\mathbb{T}_\beta, V),\ \langle \alpha, \beta \rangle \geq 0\}$. In the same way, we denote $(V \otimes \mathbb{C})^{\beta,+}$ the following G-stable subspace of $V \otimes \mathbb{C}$: $(V \otimes \mathbb{C})^{\beta,+} := \sum_{\alpha \in \Delta_{\beta,+}(V \otimes \mathbb{C})} (V \otimes \mathbb{C})(\alpha)$, where $\Delta_+(\mathbb{T}_\beta, V \otimes \mathbb{C}) = \{\alpha \in \Delta(\mathbb{T}_\beta, V \otimes \mathbb{C}),\ \langle \alpha, \beta \rangle \geq 0\}$.

Remark 5.3. *The vector space $V^{\beta,+}$ can be either equal to $\{0\}$ or to V, but $(V \otimes \mathbb{C})^{\beta,+}$ satisfies $V \otimes \mathbb{C} = (V \otimes \mathbb{C})^{\beta,+} \oplus \overline{(V \otimes \mathbb{C})^{\beta,+}}$.*

For any representation W of G, we denote $\det W$ the representation $\wedge^{max}_{\mathbb{C}} W$. In the same way, if $W \to M$ is a G complex vector bundle we denote $\det W$ the corresponding line bundle.

Proposition 5.4. *We have the following equality in $R^{-\infty}(G)$:*

$$\mathrm{Index}^G_V(\mathrm{Thom}^\beta_G(V)) = (-1)^{\dim_{\mathbb{C}} V^{\beta,+}} \ \det V^{\beta,+} \otimes \sum_{k \in \mathbb{N}} S^k((V \otimes \mathbb{C})^{\beta,+}) \ ,$$

where $S^k((V \otimes \mathbb{C})^{\beta,+})$ is the k-th symmetric product over $\mathbb{C}$ of $(V \otimes \mathbb{C})^{\beta,+}$.

The generalised function $\chi := \mathrm{Index}^V_G(\mathrm{Thom}^\beta_G(V))$ is an inverse, in $R^{-\infty}(G)$ of the function $g \in G \to \det_{V,\mathbb{C}}(1 - g^{-1})$.

The rest of this subsection is devoted to the proof of Proposition 5.4. The case $V^{\beta,+} = V$ or $V^{\beta,+} = \{0\}$ is considered by Atiyah [1] (see Lecture 6) and Vergne [29] (see Lemma 6, Part II).

Let H be a maximal torus of G containing $\mathbb{T}_\beta$. The symbol $\mathrm{Thom}^\beta_G(V)$ is also H-transversally elliptic and let $\mathrm{Thom}^\beta_H(V)$ be the corresponding class in $K_H(\mathbf{T}_H V)$. Following Corollary 3.8, we can reduce the proof of Proposition 5.4 to the case where the group G is equal to the torus H.

Proof of Th. 5.4 for a torus action.

We first recall the index theorem of Atiyah. Let $\mathbb{T}_m$ the circle group act on $\mathbb{C}$ with the representation t^m, $m > 0$. We have two classes $\mathrm{Thom}^\pm_{\mathbb{T}_m}(\mathbb{C}) \in K_{\mathbb{T}_m}(\mathbf{T}_{\mathbb{T}_m}(\mathbb{C}))$ that correspond respectively to $\beta = \pm\imath \in Lie(S^1)$. Atiyah denotes these elements $\overline{\partial}^\pm$.

Lemma 5.5 (Atiyah). *We have, for $m > 0$, the following equalities in $R^{-\infty}(\mathbb{T}_m)$:*

$$\mathrm{Index}^{\mathbb{T}_m}_{\mathbb{C}}(\mathrm{Thom}^+_{\mathbb{T}_m}(\mathbb{C})) = \left[\frac{1}{1-t^{-m}}\right]^+ = -t^m. \sum_{k \in \mathbb{N}} (t^m)^k$$

$$\mathrm{Index}^{\mathbb{T}_m}_{\mathbb{C}}(\mathrm{Thom}^-_{\mathbb{T}_m}(\mathbb{C})) = \left[\frac{1}{1-t^{-m}}\right]^- = \sum_{k \in \mathbb{N}} (t^{-m})^k$$

Here we follow the notation of Atiyah: $[\frac{1}{1-t^{-m}}]^+$ and $[\frac{1}{1-t^{-m}}]^-$ are the Laurent expansions of the meromorphic function $t \in \mathbb{C} \to \frac{1}{1-t^{-m}}$ around $t = 0$ and $t = \infty$ respectively.

From this Lemma we can compute the index of $\mathrm{Thom}^{\pm}_{\mathbb{T}_m}(\mathbb{C})$ when $m<0$. Suppose $m<0$ and consider the morphism $\kappa:\mathbb{T}_m\to\mathbb{T}_{|m|}, t\to t^{-1}$. Using the induced morphism $\kappa^*: K_{\mathbb{T}_{|m|}}(\mathbf{T}_{\mathbb{T}_{|m|}}(\mathbb{C}))\to K_{\mathbb{T}_m}(\mathbf{T}_{\mathbb{T}_m}(\mathbb{C}))$, we see that $\kappa^*(\mathrm{Thom}^{\pm}_{\mathbb{T}_{|m|}}(\mathbb{C}))=\mathrm{Thom}^{\mp}_{\mathbb{T}_m}(\mathbb{C})$. This gives $\mathrm{Index}^{\mathbb{T}_m}_{\mathbb{C}}(\mathrm{Thom}^{+}_{\mathbb{T}_m}(\mathbb{C}))=\kappa^*(\sum_{k\in\mathbb{N}}(t^{-|m|})^k)=\sum_{k\in\mathbb{N}}(t^{-m})^k$ and $\mathrm{Index}^{\mathbb{T}_m}_{\mathbb{C}}(\mathrm{Thom}^{-}_{\mathbb{T}_m}(\mathbb{C}))=\kappa^*(-t^{|m|}.\sum_{k\in\mathbb{N}}(t^{|m|})^k)=-t^m\sum_{k\in\mathbb{N}}(t^m)^k$.

We can summarize these different cases as follow.

Lemma 5.6. *Let $\mathbb{T}_\alpha$ be the circle group act on $\mathbb{C}$ with the representation $t\to t^\alpha$ for $\alpha\in\mathbb{Z}\setminus\{0\}$. Let $\beta\in Lie(\mathbb{T}_\alpha)\simeq\mathbb{R}$ a non-zero element. We have the following equalities in $R^{-\infty}(\mathbb{T}_\alpha)$:*

$$\mathrm{Index}^{\mathbb{T}_\alpha}_{\mathbb{C}}\left(\mathrm{Thom}^{\beta}_{\mathbb{T}_\alpha}(\mathbb{C})\right)(t)=\left[\frac{1}{1-u^{-1}}\right]^{\varepsilon}_{u=t^\alpha},$$

where ε is the sign of $\langle\alpha,\beta\rangle$.

We decompose now the vector space V in an orthogonal sum $V=\oplus_{i\in I}\mathbb{C}_{\alpha_i}$, where $\mathbb{C}_{\alpha_i}$ is a H-stable subspace of dimension 1 over $\mathbb{C}$ equipped with the representation $t\in H\to t^{\alpha_i}\in\mathbb{C}$. Here the set I parametrizes the weights for the action of H on V, counted with their multiplicities. Consider the circle group $\mathbb{T}_i$ with the trivial action on $\oplus_{k\neq i}\mathbb{C}_{\alpha_k}$ and with the canonical action on $\mathbb{C}_{\alpha_i}$. We consider V equipped with the action of $H\times\Pi_k\mathbb{T}_k$. The symbol $\mathrm{Thom}^{\beta}_H(V)$ is $H\times\Pi_k\mathbb{T}_k$-equivariant and is either H-transversally elliptic, $H\times\Pi_k\mathbb{T}_k$-transversally elliptic (we denote σ_B the corresponding class), and $\Pi_k\mathbb{T}_k$-transversally elliptic (we denote σ_A the corresponding class). We have the following canonical morphisms :

$$(5.24)\qquad \begin{array}{ccccc} K_H(\mathbf{T}_HV) & \longleftarrow & K_{H\times\Pi_k\mathbb{T}_k}(\mathbf{T}_HV) & \longrightarrow & K_{H\times\Pi_k\mathbb{T}_k}(\mathbf{T}_{H\times\Pi_k\mathbb{T}_k}V)\\ \mathrm{Thom}^{\beta}_H(V) & \longleftarrow & \sigma_{B_1} & \longrightarrow & \sigma_B\ , \end{array}$$

$$\begin{array}{ccccc} K_{H\times\Pi_k\mathbb{T}_k}(\mathbf{T}_{H\times\Pi_k\mathbb{T}_k}V) & \leftarrow & K_{H\times\Pi_k\mathbb{T}_k}(\mathbf{T}_{\Pi_k\mathbb{T}_k}V) & \to & K_{\Pi_k\mathbb{T}_k}(\mathbf{T}_{\Pi_k\mathbb{T}_k}V)\\ \sigma_B & \leftarrow & \sigma_{B_2} & \to & \sigma_A\ . \end{array}$$

We consider the following characters:
- $\phi(t)\in R^{-\infty}(H)$ the H-index of $\mathrm{Thom}^{\beta}_H(V)$,
- $\phi_B(t,t_1,\cdots,t_l)\in R^{-\infty}(H\times\Pi_k\mathbb{T}_k)$ the $H\times\Pi_k\mathbb{T}_k$-index of σ_B (the same for σ_{B_1} and σ_{B_2}).
- $\phi_A(t_1,\cdots,t_l)\in R^{-\infty}(\Pi_k\mathbb{T}_k)$ the $\Pi_k\mathbb{T}_k$-index of σ_A.

They satisfy the relations

i) $\phi(t)=\phi_B(t,1,\cdots,1)$ and $\phi_B(1,t_1,\cdots,t_l)=\phi_A(t_1,\cdots,t_l)$.

ii) $\phi_B(tu,t_1u^{-\alpha_1},\cdots,t_lu^{-\alpha_1})=\phi_B(t,t_1,\cdots,t_l)$, for all $u\in H$.

Point i) is a consequence of the morphisms (5.24). Point ii) follows from the fact that the elements $(u,u^{-\alpha_1},\cdots,u^{-\alpha_l})$, $u\in H$ act trivially on V.

The symbol σ_A can be expressed through the map

$$\begin{array}{ccc} K_{\mathbb{T}_1}(\mathbf{T}_{\mathbb{T}_1}\mathbb{C}_{\alpha_1})\times K_{\mathbb{T}_2}(\mathbf{T}_{\mathbb{T}_2}\mathbb{C}_{\alpha_2})\times\cdots\times K_{\mathbb{T}_l}(\mathbf{T}_{\mathbb{T}_l}\mathbb{C}_{\alpha_l}) & \longrightarrow & K_{\Pi_k\mathbb{T}_k}(\mathbf{T}_{\Pi_k\mathbb{T}_k}V)\\ (\sigma_1,\sigma_2,\cdots,\sigma_l)\longmapsto\sigma_1\odot\sigma_2\odot\cdots\odot\sigma_l\ . & & \end{array}$$

Here we have $\sigma_A = \odot_{k=1}^{l} \mathrm{Thom}^{\varepsilon_k}_{\mathbb{T}_k}(\mathbb{C}_{\alpha_k})$ in $K_{\Pi_k \mathbb{T}_k}(\mathbf{T}_{\Pi_k\mathbb{T}_k} V)$, where ε_k is the sign of $\langle \alpha_k, \beta\rangle$. Finally, we get

$$\begin{aligned}\phi(u) &= \phi_B(u,1,\cdots,1) = \phi_B(1,u^{\alpha_1},\cdots,u^{\alpha_1})\\ &= \phi_A(u^{\alpha_1},\cdots,u^{\alpha_1}) = \Pi_k \left[\frac{1}{1-t^{-1}}\right]^{\varepsilon_k}_{t=u^{\alpha_k}} .\end{aligned}$$

To finish the proof, it suffices to note that we have the following identification of H-vector spaces : $V^{\beta,+} \simeq \oplus_{\varepsilon_k>0}\mathbb{C}_{\alpha_k}$ and $(V\otimes\mathbb{C})^{\beta,+} \simeq \oplus_k \mathbb{C}_{\varepsilon_k\alpha_k}$. □

5.2. **Localisation of the quantization map on M^β.** We decompose the fixed point set M^β in connected components $P_a,\ a\in\mathcal{F}$. The almost complex structure J on M induces an almost complex structure J_a on each submanifold P_a. We have then the quantization maps $RR^{^G}(P_a,-) : K_G(P_a)\to R(G)$ for each $a\in\mathcal{F}$.

Let $\mathcal{N}_a$ be the normal bundle of P_a in M. For $m\in P_a$, we have the decomposition $\mathbf{T}_m M = \mathbf{T}_m P_a \oplus \mathcal{N}_a|_m$. The linear action of β on T_mM precises this decomposition. The map $\mathcal{L}^M(\beta) : \mathbf{T}_mM\to\mathbf{T}_mM$ commutes with the map J and satisfies $\mathbf{T}_mP_a = \ker(\mathcal{L}^M(\beta))$. Here we take $\mathcal{N}_a|_m := \mathrm{Image}(\mathcal{L}^M(\beta))$, then the almost complex structure J induces a G-invariant complex structure $J_{\mathcal{N}_a}$ on the fibre of $\mathcal{N}_a\to P_a$. The subgroup $\mathbb{T}_\beta$ generated by $\exp(t.\beta),\ t\in\mathbb{R}$ acts linearly on the fibre of the complex vector bundle $\mathcal{N}_a$. Thus we associate as in the previous section the polarized complex G-vector bundles $\mathcal{N}_a^{\beta,+}$ and $(\mathcal{N}_a\otimes\mathbb{C})^{\beta,+}$.

Theorem 5.7. *For every $E\in K_G(M)$, we have the following equality in $R^{-\infty}(G)$:*

$$RR^{^G}(M,E) = \sum_{a\in\mathcal{F}}(-1)^{n_a(\beta)}\sum_{k\in\mathbb{N}} RR^{^G}(P_a, E|_{P_a}\otimes\wedge^{max}_{\mathbb{C}}\mathcal{N}_a^{\beta,+}\otimes S^k((\mathcal{N}_a\otimes\mathbb{C})^{\beta,+})\ ,$$

where $n_a(\beta)$ is the complex rank of $\mathcal{N}_a^{\beta,+}$.

Note that Theorem 5.7 gives a proof of some rigidity properties (see [6, 24]). Let T be a maximal torus of G. Following Meinrenken and Sjamaar, a G-equivariant complex vector bundle $E\to M$ is called *rigid* if the action of T on $E|_{M^T}$ is trivial. Take $\beta\in\mathfrak{t}$ such that $M^\beta = M^T$, and apply Theorem 5.7, with β and $-\beta$, to $RR^{^T}(M,E)$, with E rigid.

If we take $+\beta$, Theorem 5.7 shows that $t\in T\to RR^{^T}(M,E)(t)$ is of the form $t\in T\to\sum_{a\in\hat{T}} n_a t^a$ with

$$n_a\neq 0 \Longrightarrow \langle a,\beta\rangle \geq 0\ .$$

(see Lemma 9.4). If we take $-\beta$, we find $RR^{^T}(M,E)(t) = \sum_{a\in\hat{T}} n_a t^a$, with $n_a\neq 0\Longrightarrow -\langle a,\beta\rangle\geq 0$. Comparing the two results, and using the genericity of β, we see that $RR^{^T}(M,E)$ is a *constant* function on T ($RR^{^G}(M,E)$ is then a constant function on G). We can now rewrite the equation of Theorem 5.7, where we keep on the right hand side the *constant* terms:

$$RR^{^G}(M,E) = \sum_{F\subset M^{\beta,+}} RR(F,E|_F)\ . \tag{5.25}$$

Here the summation is over all connected components F of M^T such that $\mathcal{N}_F^{\beta,+}=0$ (i.e. we have $\langle\xi,\beta\rangle>0$ for all weights ξ of the T-action on the normal bundle $\mathcal{N}_F$ of F).

Proof of Theorem 5.7 :

We know from section 4 that we have to study the modified symbol of $(m, v) \to \mathrm{Thom}_G(M) \otimes p^*(E)(m, v - \beta_M|_m)$ in the neighbourhood of each submanifold P_a. Here a G-invariant neighbourhood $\mathcal{U}_a$ of P_a in M is diffeomorphic to a G-invariant neighbourhood $\mathcal{V}_a$ of P_a in the bundle $\mathcal{N}_a$. We study here the G-transversally elliptic symbol

$$\mathrm{Thom}_G^\beta(\mathcal{V}_a, J)(n, w) := \mathrm{Thom}_G(\mathcal{V}_a, J)(n, w - \beta_{\mathcal{N}_a}(n)), \qquad (n, w) \in \mathbf{T}\mathcal{V}_a,$$

where we still denote J the almost complex structure transported on $\mathcal{V}_a$ via the diffeomorphism $\mathcal{U}_a \simeq \mathcal{V}_a$

If we note $p_a : \mathcal{N}_a \to P_a$ the canonical projection, we have an isomorphism of G-vector bundles over $\mathcal{N}_a$:

$$\begin{array}{rcl} \mathbf{T}\mathcal{N}_a & \xrightarrow{\sim} & p_a^*(\mathbf{T}P_a \oplus \mathcal{N}_a) \\ w & \longmapsto & \mathbf{T}p_a(w) \oplus (w)^V \end{array} \tag{5.26}$$

Here $w \to (w)^V$, $\mathbf{T}\mathcal{N}_a \to p_a^*\mathcal{N}_a$ is the projection which associates to a tangent vector its *vertical* part (see [8][section 7] or [25][section 4.1]). The map $\widetilde{J} := p_a^*(J_a \oplus J_{\mathcal{N}_a})$ defines an almost complex structure on the manifold $\mathcal{N}_a$ which is constant over the fibre of p_a. With this new almost complex structure $\widetilde{J}$ we construct the G-transversally elliptic symbol over $\mathcal{N}_a$

$$\mathrm{Thom}_G^\beta(\mathcal{N}_a)(n, w) = \mathrm{Thom}_G(\mathcal{N}_a, \widetilde{J})(n, w - \beta_{\mathcal{N}_a}(n)), \qquad (m, w) \in \mathbf{T}\mathcal{N}_a.$$

We denote $i : \mathcal{V}_a \to \mathcal{N}_a$ the inclusion map, and $i_* : K_G(\mathbf{T}_G\mathcal{V}_a) \to K_G(\mathbf{T}_G\mathcal{N}_a)$ the induced map.

Lemma 5.8. *For any G-complex vector bundle E over $\mathcal{V}_a$, we have*

$$i_*(\mathrm{Thom}_G^\beta(\mathcal{V}_a, J) \otimes E) = \mathrm{Thom}_G^\beta(\mathcal{N}_a) \otimes p_a^*(E|_{P_a}) \quad \text{in} \quad K_G(\mathbf{T}\mathcal{N}_a).$$

Proof : We proceed like in Lemma 2.2. The complex structure J_n, $n \in \mathcal{V}_a$ and $\widetilde{J}_n$, $m \in \mathcal{N}_a$ are equal when $n \in P_a$, and are related by the homotopy $J^t_{(x,v)} := J_{(x,t.v)}$, $u \in [0, 1]$ for $n = (x, v) \in \mathcal{V}_a$. Then, as in Lemma 2.2, we can construct an invertible bundle map $A \in \Gamma(\mathcal{V}_a, \mathrm{End}(\mathbf{T}\mathcal{V}_a)^G$, which is homotopic to the identity and such that $A.J = \widetilde{J}.A$ on $\mathcal{V}_a$. We conclude as in Lemma 2.2 that the symbols $\mathrm{Thom}_G^\beta(\mathcal{V}_a, J) \otimes E$ and $\mathrm{Thom}_G^\beta(\mathcal{N}_a) \otimes p_a^*(E|_{P_a})|\mathcal{V}_a$ are equal in $K_G(\mathbf{T}\mathcal{V}_a)$. □

We consider now the Hermitian vector bundle $\mathcal{N}_a \to P_a$ with the action of $G \times \mathbb{T}_\beta$. First we use the decomposition $\mathcal{N}_a = \oplus_\alpha \mathcal{N}_a^\alpha$ relatively to the unitary action of $\mathbb{T}_\beta$ on the fibres of $\mathcal{N}_a$. Let N_a^α be an Hermitian vector space of dimension equal to the rank of $\mathcal{N}_a^\alpha$, equipped with the representation $t \to t^\alpha$ of $\mathbb{T}_\beta$. Let U_a be the group of $\mathbb{T}_\beta$-*equivariant unitary maps* of vector space $N_a := \oplus_\alpha N_a^\alpha$, and let R_a be the $\mathbb{T}_\beta$-equivariant unitary frame of $(\mathcal{N}_a, J_{\mathcal{N}_a})$ framed on N_a. Note that R_a is provided with a $U_a \times G$-action and a trivial action of $\mathbb{T}_\beta$: for $x \in P_a$, any element of $R_a|_x$ is a $\mathbb{T}_\alpha$-equivariant unitary map from N_a^α to $\mathcal{N}_a|_x$. The manifold $\mathcal{N}_a$ is isomorphic to $R_a \times_{U_a} N_a$, where G acts on R_a and $\mathbb{T}_\beta$ acts on N_a.

Here the symbol $\mathrm{Thom}_G^\beta(\mathcal{N}_a)$ is $G \times \mathbb{T}_\beta$-equivariant, and it can be considered as a G, $G \times \mathbb{T}_\beta$, or $\mathbb{T}_\beta$-transversally elliptic symbol. Now we consider $\mathrm{Thom}_G^\beta(\mathcal{N}_a)$ as an element of $K_{G\times\mathbb{T}_\beta}(\mathbf{T}_{\mathbb{T}_\beta}(R_a \times_{U_a} N_a))$. Recall that we have two isomorphisms

$$\pi_N^* : K_{G\times\mathbb{T}_\beta}(\mathbf{T}_{\mathbb{T}_\beta}(R_a \times_{U_a} N_a)) \xrightarrow{\sim} K_{G\times\mathbb{T}_\beta\times U_a}(\mathbf{T}_{\mathbb{T}_\beta\times U_a}(R_a \times N_a)), \tag{5.27}$$

$$\pi^* : K_G(\mathbf{T}P_a) \xrightarrow{\sim} K_{G\times U_a}(\mathbf{T}_{U_a}R_a), \tag{5.28}$$

where $\pi_N : R_a \times N_a \to R_a \times_{U_a} N_a$ and $\pi : R_a \to R_a/U_a \simeq P_a$ are the quotient maps relative to the free U_a action. We have also an operation

(5.29)
$$\kappa : K_{G\times U_a}(\mathbf{T}_{U_a}R_a) \times K_{\mathbb{T}_\beta\times U_a}(\mathbf{T}_{\mathbb{T}_\beta}N_a) \longrightarrow K_{G\times\mathbb{T}_\beta\times U_a}(\mathbf{T}_{\mathbb{T}_\beta\times U_a}(R_a\times N_a))$$

We have three different Thom classes :
- $\mathrm{Thom}^\beta_G(\mathcal{N}_a) \in K_{G\times\mathbb{T}_\beta}(\mathbf{T}_{\mathbb{T}_\beta}(R_a \times_{U_a} N_a))$,
- $\mathrm{Thom}^\beta_{\mathbb{T}_\beta\times U_a}(N_a) \in K_{\mathbb{T}_\beta\times U_a}(\mathbf{T}_{\mathbb{T}_\beta}N_a)$, and
- $\mathrm{Thom}_G(P_a) \in K_G(\mathbf{T}P_a)$.

These Thom classes are related by the following equality in $K_{G\times\mathbb{T}_\beta\times U_a}(\mathbf{T}_{\mathbb{T}_\beta\times U_a}(R_a\times N_a))$:

$$\pi_N^*\left(\mathrm{Thom}^\beta_G(\mathcal{N}_a)\right) = \kappa\left(\pi^*(\mathrm{Thom}_G(P_a)), \mathrm{Thom}^\beta_{\mathbb{T}_\beta\times U_a}(N_a)\right). \tag{5.30}$$

We will justify the equality (5.30) later.

Following Theorem 3.5 of Atiyah [1], equality (5.30) gives, after taking the index, the following equality in $R^{-\infty}(G\times\mathbb{T}_\beta\times U_a)$:

$$\mathrm{Index}^{G\times\mathbb{T}_\beta\times U_a}\left(\pi_N^*\mathrm{Thom}^\beta_G(\mathcal{N}_a)\right) =$$
$$\mathrm{Index}^{G\times U_a}\left(\pi^*\mathrm{Thom}_G(P_a)\right).\mathrm{Index}^{\mathbb{T}_\beta\times U_a}\left(\mathrm{Thom}^\beta_{\mathbb{T}_\beta\times U_a}(N_a)\right). \tag{5.31}$$

The equalities (5.30) and (5.31) are still true if we replace, for any $E \in K_G(M)$, $\mathrm{Thom}^\beta_G(\mathcal{N}_a)$ by $\mathrm{Thom}^\beta_G(\mathcal{N}_a)\otimes p_a^*(E|_{P_a})$ and $\mathrm{Thom}_G(P_a)$ by $\mathrm{Thom}_G(P_a)\otimes E|_{P_a}$.

Now we conclude using Theorem 3.1 of Atiyah (see also subsection 3.2) and the computation of $\mathrm{Index}^{\mathbb{T}_\beta\times U_a}(\mathrm{Thom}^\beta_{\mathbb{T}_\beta\times U_a}(N_a))$ given in Proposition 5.4.

Using Theorem 3.2, the index of $\mathrm{Thom}^\beta_G(\mathcal{N}_a)\otimes p_a^*(E|_{P_a})$ is equal to the U_a-invariant part of $\mathrm{Index}^{G\times\mathbb{T}_\beta\times U_a}(\pi_N^*(\mathrm{Thom}^\beta_G(\mathcal{N}_a)\otimes p_a^*E|_{P_a}))$, and the index of $\pi^*(\mathrm{Thom}_G(P_a)\otimes p_a^*E|_{P_a})$ is equal to

$$\sum_{i\in\widehat{U_a}} RR^G(P_a, E|_{P_a}\otimes \underline{W_i^*}).W_i\ ,$$

where $\{W_i\}_i$ is a complete set of inequivalent irreducible representations of U_a. In the last equality, $RR^G(P_a, E|_{P_a}\otimes\underline{W_i^*})$ belongs to $R(G)$. It suffices now to observe that for any $L\in R(U_a)$, the U_a-invariant part of

$$\sum_{i\in\widehat{U_a}} RR^G(P_a, E|_{P_a}\otimes \underline{W_i^*}).W_i\otimes L$$

is equal to $RR^G(P_a, E|_{P_a}\otimes\underline{L})$ where $\underline{L} = R_a\times_{U_a} L$.

We give now an explanation for equation (5.30), which is a direct consequence of the fact that the almost complex structure $\widetilde{J}$ admits the decomposition $\widetilde{J} = p_a^*(J_a\oplus J_{\mathcal{N}_a})$. Hence $\wedge_{\mathbb{C}}\mathbf{T}_n\mathcal{N}_a$ equipped with the map $Cl_n(v-\beta_{\mathcal{N}_a}(n))$, $v\in\mathbf{T}_n\mathcal{N}_a$ is isomorphic to $\wedge_{\mathbb{C}}\mathbf{T}_xP_a\otimes\wedge_{\mathbb{C}}\mathcal{N}_a|_x$ equipped with $Cl_x(v_1)\otimes Cl_x(v_2-\beta_{\mathcal{N}_a}(n))$ where $x = p_a(n)$, and the vector $v\in\mathbf{T}_n\mathcal{N}_a$ is decomposed, following the isomorphism (5.26), in $= v_1+v_2$ with $v_1\in\mathbf{T}_xP_a$ and $v_2\in\mathcal{N}_a|_x$. Note that the vector $w = \beta_{\mathcal{N}_a}(n)\in\mathbf{T}_n\mathcal{N}_a$ is vertical, that is $w = (w)^V$. □

6. Localisation via a moment map

Let (M, J, G) be a compact G-manifold provided with a G-invariant almost complex structure. We denote $RR^{G} : K_G(M) \to R(G)$ the quantization map. Here we suppose that the G-manifold is equipped with a *a moment map* $f_{G} : M \to \mathfrak{g}^*$ in the following sense (see [12, 13, 18]) :

Definition 6.1. *A smooth map $f_{G} : M \to \mathfrak{g}^*$ is called a* moment map *if*
- the map f_{G} is equivariant for the action of the group G, and
- for every Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$, the induced map $f_{H} : M \to \mathfrak{h}^*$ is locally constant on the submanifold M^H of fixed points for the action of H (the map f_{H} is the composition of f_{G} with the projection $\mathfrak{g}^* \to \mathfrak{h}^*$).

The terminology "moment map" is usually used when we work in the case of an Hamiltonian action. More precisely, when the manifold is equipped with a symplectic 2-form ω that is invariant for the G-action, a *moment map* $\mu : M \to \mathfrak{g}^*$ relative to ω is a G-equivariant map satisfying $d\langle\mu, X\rangle = \omega(X_M, -)$, $X \in \mathfrak{g}$. We note that (when it exists) a moment map is uniquely defined up to a constant $\xi \in (\mathfrak{g}^*)^G$.
In [13], Ginzburg, Guillemin and Karshon study G-manifolds with the additional structure of an (abstract) moment map. When G is a torus they give a necessary and sufficient condition for a G-manifold to admit a (abstract) moment map.

For the rest of this paper we make the choice of a G-invariant scalar product over $\mathfrak{g}^*$. This defines an identification $\mathfrak{g}^* \simeq \mathfrak{g}$, and we work with a given moment map $f_{G} : M \to \mathfrak{g}$.

Definition 6.2. *Let $\mathcal{H}^{G}$ the G-invariant vector field over M defined by*

$$\mathcal{H}^{G}_m := (f_{G}(m)_M)_m, \quad \forall\, m \in M.$$

The aim of this section is to compute the localisation, as in section 4, with the G-invariant vector field $\mathcal{H}^{G}$. We know that the Riemann-Roch character is localised near the set $\{\Phi_{\mathcal{H}^G} = 0\}$, but we see that $\{\Phi_{\mathcal{H}^G} = 0\} = \{\mathcal{H}^{G} = 0\}$. We will denote C^{f_G} this set. Let H be a maximal torus of G, with Lie algebra $\mathfrak{h}$, and let $\mathfrak{h}_+$ be a Weyl chamber in $\mathfrak{h}$.

Lemma 6.3. *There exist a finite subset $\mathcal{B}_{G} \subset \mathfrak{h}_+$, such that*

$$C^{f_G} = \bigcup_{\beta\in\mathcal{B}_{G}} C^{G}_\beta, \quad \text{with} \quad C^{G}_\beta = G.(M^\beta \cap f_{G}^{-1}(\beta)).$$

Proof : We first observe that $\mathcal{H}^{G}_m = 0$ if and only if $f_{G}(m) = \beta'$ and $\beta'_M|_m = 0$, that is $m \in M^{\beta'} \cap f_{G}^{-1}(\beta')$, for some $\beta' \in \mathfrak{g}$. For every $\beta' \in \mathfrak{g}$, there exists $\beta \in \mathfrak{h}_+$, with $\beta' = g.\beta$ for some $g \in G$. Hence $M^{\beta'} \cap f_{G}^{-1}(\beta') = g.(M^\beta \cap f_{G}^{-1}(\beta))$. We have shown that $C^{f_G} = \bigcup_{\beta\in\mathfrak{h}_+} C^{G}_\beta$, and we need to prove that the set $\mathcal{B}_{G} := \{\beta \in \mathfrak{h}_+,\ M^\beta \cap f_{G}^{-1}(\beta) \neq \emptyset\}$ is finite. Consider the set $\{H_1, \cdots, H_l\}$ of stabilisers for the action of the torus H on the compact manifold M. For each $\beta \in \mathfrak{h}$ we denote $\mathbb{T}_\beta$ the subtorus of H generated by $\exp(t.\beta)$, $t \in \mathbb{R}$, and we observe that

$$\begin{aligned} M^\beta \cap f_{G}^{-1}(\beta) \neq \emptyset \quad &\Longleftrightarrow \quad \exists H_i \text{ such that } \mathbb{T}_\beta \subset H_i \text{ and } M^{H_i} \cap f_{G}^{-1}(\beta) \neq \emptyset \\ &\Longleftrightarrow \quad \exists H_i \text{ such that } \beta \in f_{G}(M^{H_i}) \cap Lie(H_i). \end{aligned}$$

But $f_{G}(M^{H_i}) \cap Lie(H_i) \subset f_{H_i}(M^{H_i})$ is a finite set after Definition 6.1. The proof is now completed. □

Definition 6.4. *Let* $\mathrm{Thom}^f_{G,[\beta]}(M) \in K_G(\mathbf{T}_G\mathcal{U}^{G,\beta})$ *defined by*

$$\mathrm{Thom}^f_{G,[\beta]}(M)(x,v) := \mathrm{Thom}_G(M)(x, v - \mathcal{H}^G_x), \quad \text{for} \quad (x,v) \in \mathbf{T}\mathcal{U}^{G,\beta} \ .$$

Here $i^{G,\beta} : \mathcal{U}^{G,\beta} \hookrightarrow M$ *is any* G*-invariant neighbourhood of* C^G_β *such that* $\overline{\mathcal{U}^{G,\beta}} \cap C^{f_G} = C^G_\beta$.

Definition 6.5. *For every* $\beta \in \mathcal{B}_G$, *we denote* $RR^G_\beta(M,-) : K_G(M) \to R^{-\infty}(G)$ *the localised Riemann-Roch character near* C^G_β, *defined as in equation (4.23), by*

$$RR^G_\beta(M,E) = \mathrm{Index}^G_{\mathcal{U}^{G,\beta}}\left(\mathrm{Thom}^f_{G,[\beta]}(M) \otimes E_{|\mathcal{U}^{G,\beta}}\right) \ ,$$

for $E \in K_G(M)$.

After Proposition 4.1, we have the partition $RR^G(M,-) = \sum_{\beta\in\mathcal{B}_G} RR^G_\beta(M,-)$ and the rest of this article is devoted to the analysis of the maps $RR^G_\beta(M,-)$, $\beta \in \mathcal{B}_G$.

In the next section, we compute the map $RR^G_0(M,-) : K_G(M) \to R^{-\infty}(G)$ when 0 is a regular value of the moment map f_G.

6.1. **The map** RR^G_0**.** Recall that the map $RR^G_0(M,-) : K_G(M) \to R^{-\infty}(G)$ is the localisation of the Riemann-Roch character near $C^G_0 = f_G^{-1}(0)$ (see Definition 6.5). In particular, $RR^G_0(M,-)$ is the zero map if 0 does not belong to $f_G(M)$.

We assume in this subsection that 0 is a regular value of f_G. Then the submanifold $\mathcal{Z} := f_G^{-1}(0)$ carries a locally free action of G (see [18][Lemma 7.1]). Let $\mathcal{M}_{red} := \mathcal{Z}/G$ be the corresponding 'reduced' space, and we denote $\pi : \mathcal{Z} \to \mathcal{M}_{red}$ the projection map.

Let $\{W_a,\ a \in \hat{G}\}$ be a completed set of inequivalent irreducible representations of G.

Proposition 6.6. *There exists an elliptic symbol* $\sigma^{red} \in K(\mathbf{T}\mathcal{M}_{red})$ *such that*

$$RR^G_0(M,E) = \sum_{a\in\hat{G}} \mathrm{Index}_{\mathcal{M}_{red}}(\sigma^{red} \otimes E_{red} \otimes \underline{W_a}^*).W_a \quad \text{in} \quad R^{-\infty}(G) \ ,$$

for every $E \in K_G(M)$. *Here* $E_{red} \in K(\mathcal{M}_{red})$ *is the reduced vector bundle on* $\mathcal{M}_{red}$ *induced by* E, *and* $\underline{W_a} = \mathcal{Z} \times_G W_a$. *In particular, the* G*-invariant part of* $RR^G_0(M,E)$ *is equal to* $\mathrm{Index}_{\mathcal{M}_{red}}(\sigma^{red} \otimes E_{red}) \in \mathbb{Z}$.

Proof: The map $RR^G_0(M,-) : K_G(M) \to R^{-\infty}(G)$ is defined by $\mathrm{Thom}^f_{G,[0]}(M) \in K_G(\mathbf{T}_G\mathcal{U}^{G,0})$ (see Definition 6.5), where $\mathcal{U}^{G,0}$ is a (small) neighbourhood of $\mathcal{Z}$ in M. As 0 is a regular value of f_G, $\mathcal{U}^{G,0}$ is diffeomorphic to $\mathcal{Z} \times \mathfrak{g}$. The moment map f_G, the vector field $\mathcal{H}^G$, and $\mathrm{Thom}^f_{G,[0]}(M)$ are transported by this diffeomorphism to $\mathcal{Z}\times\mathfrak{g}$. In a neighbourhood of $\mathcal{Z}$ in $\mathcal{Z}\times\mathfrak{g}$, the moment map is equal to the projection $f : \mathcal{Z} \times \mathfrak{g} \to \mathfrak{g}$, and we denote $\sigma_{\mathcal{Z}} \in K_G(\mathbf{T}_G(\mathcal{Z} \times \mathfrak{g}))$ the symbol corresponding to $\mathrm{Thom}^f_{G,[0]}(M)$ through the diffeomorphism $\mathcal{U}^{G,0} \cong \mathcal{Z} \times \mathfrak{g}$.

Let $\mathrm{Index}^G_{\mathcal{Z}\times\mathfrak{g}} : K_G(\mathbf{T}_G(\mathcal{Z} \times \mathfrak{g})) \to R^{-\infty}(G)$ be the index map on $\mathcal{Z} \times \mathfrak{g}$. The map RR^G_0 is defined by $RR^G_0(M,E) = \mathrm{Index}^G_{\mathcal{Z}\times\mathfrak{g}}(\sigma_{\mathcal{Z}} \otimes f^*(E_{|\mathcal{Z}}))$.

Following Atiyah, the inclusion map $j : \mathcal{Z} \hookrightarrow \mathcal{Z} \times \mathfrak{g}$ induces an $R(G)$-module morphism $j_! : K_G(\mathbf{T}_G\mathcal{Z}) \to K_G(\mathbf{T}_G(\mathcal{Z} \times \mathfrak{g}))$, with the commutative diagram

$$(6.32) \qquad \begin{array}{ccc} K_G(\mathbf{T}_G\mathcal{Z}) & \xrightarrow{\ j_!\ } & K_G(\mathbf{T}_G(\mathcal{Z} \times \mathfrak{g})) \\ & \searrow^{\mathrm{Index}^G_{\mathcal{Z}}} & \downarrow{\scriptstyle \mathrm{Index}^G_{\mathcal{Z}\times\mathfrak{g}}} \\ & & R^{-\infty}(G) \end{array} \quad .$$

(see [1][Theorem 4.3]).

Note that the map $i_! : K_G(\mathbf{T}_G\mathcal{Z}) \to K_G(\mathbf{T}_G\mathcal{Y})$ is defined by Atiyah for any embedding $i : \mathcal{Z} \hookrightarrow \mathcal{Y}$ of G-manifolds with $\mathcal{Z}$ compact. Consider now the case where i is the zero-section of a G-vector bundle $p_{\mathcal{E}} : \mathcal{E} \to \mathcal{Z}$. In general the map $i_!$ is *not* an isomorphism.

If the G-action is *locally free* over $\mathcal{Z}$, then $\mathbf{T}_G\mathcal{Z} \to \mathcal{Z}$ (resp. $\mathbf{T}_G\mathcal{E} \to \mathcal{E}$) is a subbundle of $\mathbf{T}\mathcal{Z} \to \mathcal{Z}$ (resp. $\mathbf{T}\mathcal{E} \to \mathcal{E}$), and the projection $\mathbf{T}_G\mathcal{E} \to \mathbf{T}_G\mathcal{Z}$ is a vector bundle isomorphic to $s^*(\mathbf{T}\mathcal{E})$ (where $s : \mathbf{T}_G\mathcal{Z} \hookrightarrow \mathbf{T}\mathcal{Z}$ is the inclusion). Hence the vector bundle $\mathbf{T}_G\mathcal{E} \to \mathbf{T}_G\mathcal{Z}$ inherits a complex structure over the fibres (coming from the complex vector bundle $\mathbf{T}\mathcal{E} \to \mathbf{T}\mathcal{Z}$). In this situation, the map $i_! : K_G(\mathbf{T}_G\mathcal{Z}) \to K_G(\mathbf{T}_G\mathcal{E})$ is the Thom isomorphism.

In the case of the (trivial) vector bundle $f : \mathcal{Z}\times\mathfrak{g} \to \mathcal{Z}$, the map $j_! : K_G(\mathbf{T}_G\mathcal{Z}) \to K_G(\mathbf{T}_G(\mathcal{Z} \times \mathfrak{g}))$ is then an *isomorphism.* Take $\tilde{\sigma}_{\mathcal{Z}} = (j_!)^{-1}(\sigma_{\mathcal{Z}})$, and from the commutative diagram (6.32) we have $RR_0^G(M, E) = \mathrm{Index}^G_{\mathcal{Z}}(\tilde{\sigma}_{\mathcal{Z}} \otimes E|_{\mathcal{Z}})$. From Theorem 3.2 we get

$$\mathrm{Index}^G_{\mathcal{Z}}(\tilde{\sigma_{\mathcal{Z}}} \otimes E|_{\mathcal{Z}}) = \sum_{a\in\hat{G}} \mathrm{Index}_{\mathcal{M}_{red}}(\sigma^{red} \otimes E_{red} \otimes \underline{W_a}^*).W_a \ ,$$

where $\sigma^{red} \in K(\mathbf{T}\mathcal{M}_{red})$ correspond to $\tilde{\sigma}_{\mathcal{Z}}$ through the isomorphism $\pi^* : K(\mathbf{T}\mathcal{M}_{red}) \to K_G(\mathbf{T}_G\mathcal{Z})$. □

Proposition 6.7. *Suppose that Assumption 2 of the Introduction is satisfied. Then,* $\mathcal{M}_{red}$ *inherits an almost complex structure* J_{red}*, and the elliptic symbol* σ^{red} *of Proposition 6.6 is equal to* Thom$(\mathcal{M}_{red}, J_{red})$ *in* $K(\mathbf{T}\mathcal{M}_{red})$*. We have*

$$(6.33) \qquad RR_0^G(M, E) = \sum_{a\in\hat{G}} RR(\mathcal{M}_{red}, E_{red} \otimes \underline{W_a}^*).W_a \quad \text{in} \quad R^{-\infty}(G) \ ,$$

for every $E \in K_G(M)$*. In particular* $[RR_0^G(M, E)]^G$ *is equal to* $RR(\mathcal{M}_{red}, E_{red})$*.*

The equality (6.33) has been obtain by Vergne [29][Part II] in the case of an Hamiltonian action of the circle group on a compact symplectic manifold.

Proof of Proposition 6.7 : The action of G is locally free over $\mathcal{Z}$ then $\mathbf{T}\mathcal{Z} = \mathbf{T}_G\mathcal{Z} \oplus \mathfrak{g}_{\mathcal{Z}}$, where $\mathfrak{g}_{\mathcal{Z}} := \{X_{\mathcal{Z}}, X \in \mathfrak{g}\}$ is the tangent space of the G-orbits in $\mathbb{Z}$. With the Assumption 2 one get the following decomposition of $\mathbf{T}M|_{\mathcal{Z}}$:

$$(6.34) \qquad \mathbf{T}M|_{\mathcal{Z}} = \mathbf{T}_G\mathcal{Z} \oplus \mathfrak{g}_{\mathcal{Z}} \oplus J(\mathfrak{g}_{\mathcal{Z}}) \ .$$

The subspace $\mathfrak{g}_{\mathcal{Z}} \oplus J(\mathfrak{g}_{\mathcal{Z}})$ is J-stable, hence the subspace $\mathbf{T}_G\mathcal{Z}$ is equipped with the G-invariant almost complex structure $J_{red} = pr \circ J$ where $pr : \mathbf{T}M|_{\mathcal{Z}} \to \mathbf{T}_G\mathcal{Z}$ is the projection relative to the decomposition (6.34). In this context we can define the symbol $\mathrm{Thom}(\mathcal{M}_{red}, J_{red}) \in K(\mathbf{T}\mathcal{M}_{red}) \cong K(\mathbf{T}_G\mathcal{Z})$ (see Eq. (2.2)), and the quantization map

$$RR(\mathcal{M}_{red}, -) : K(\mathcal{M}_{red}) \to \mathbb{Z} ,$$

by $RR(\mathcal{M}_{red}, \mathcal{E}) = \mathrm{Index}_{\mathcal{M}_{red}}(\mathrm{Thom}(\mathcal{M}_{red}, J_{red}) \otimes \mathcal{E})$.

Proposition 6.7 follows immediately from

Lemma 6.8. *We have*

$$j_! \circ (\pi)^* \left(\mathrm{Thom}(\mathcal{M}_{red}, J_{red})\right) = \sigma_{\mathcal{Z}}$$

in $K_G(\mathbf{T}_G(\mathcal{Z} \times \mathfrak{g}))$.

Proof of the Lemma : We still denotes J the almost complex structure transported on $\mathcal{Z}\times\mathfrak{g}$. Recall that $\pi^*(\mathbf{T}\mathcal{M}_{red})$ is identified with $\mathbf{T}_G\mathcal{Z}$. The decomposition of Equation (6.34) can be rewritten

$$\mathbf{T}(\mathcal{Z} \times \mathfrak{g})|_{\mathcal{Z}} = \mathbf{T}_G\mathcal{Z} \oplus \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z} ,$$

with the isomorphism $\mathfrak{g}_{\mathbb{C}} \times \mathcal{Z} \cong \mathfrak{g}_{\mathcal{Z}} \oplus J(\mathfrak{g}_{\mathcal{Z}})$, $(X + \imath Y; z) \mapsto J_z(X_{\mathcal{Z}}|_z) - Y_{\mathcal{Z}}|_z$. On $\mathbf{T}(\mathcal{Z}\times\mathfrak{g})|_{\mathcal{Z}}$, the almost almost complex structure J is equal to $J_{red} \times (\imath)$ (we denote $(\imath)$ the multiplication by $\imath$ on $\mathfrak{g}_{\mathbb{C}}$). We extend $J_{red} \times (\imath)$ to a complex structure $\widetilde{J}$ on $\mathcal{Z} \times \mathfrak{g}$ which is constant on the fibres of the map $f : \mathcal{Z} \times \mathfrak{g} \to \mathcal{Z}$. The almost complex structures J and $\widetilde{J}$ are then *homotopic* near $\mathcal{Z}$. Hence the complex $\sigma_{\mathcal{Z}}$ can be defined on $\mathcal{Z} \times \mathfrak{g}$ with $\widetilde{J}$, but

$$\wedge^{\bullet}_{\widetilde{J}}\mathbf{T}(Z \times \mathfrak{g}) = \left(\wedge^{\bullet}_{J_{red}}\mathbf{T}_G\mathcal{Z} \otimes (\wedge^{\bullet}\mathfrak{g}_{\mathbb{C}} \times \mathcal{Z})\right) \times \mathfrak{g} .$$

Hence for $v_1 \in \mathbf{T}_G\mathcal{Z}|_z$, $(X + \imath Y; z, \xi) \in \mathfrak{g}_{\mathbb{C}} \times \mathcal{Z} \times \mathfrak{g}$, the map $\sigma_{\mathcal{Z}}(z, \xi; v_1 + X + \imath Y)$ acts on $(\pi)^* \left(\wedge^{\bullet}\mathbf{T}\mathcal{M}_{red}\right)|_z \otimes \wedge^{\bullet}\mathfrak{g}_{\mathbb{C}}$ as the product

$$Cl_z(v_1) \odot Cl_\xi(X + \imath(Y + \xi)) .$$

Note that the vector field $\mathcal{H}^{G}$ satisfies $\mathcal{H}^{G}_{(z,\xi)} = -\imath\xi$ for any $(z, \xi) \in \mathcal{Z} \times \mathfrak{g}$. Now we see that the map $Cl_z(v_1) \odot Cl_\xi(X + \imath(Y + \xi))$ is homotopic, as a G-transversally elliptic symbol, to $Cl_z(v_1) \odot Cl_\xi(X + \imath\xi)$ which is the symbol map of $j_! \circ (\pi)^* \left(\mathrm{Thom}(\mathcal{M}_{red})\right)$ (see the construction of the map $j_!$ in [1][Lecture 4]). We have shown that $j_! \circ (\pi)^* \left(\mathrm{Thom}(\mathcal{M}_{red})\right) = \sigma_{\mathcal{Z}}$ in $K_G(\mathbf{T}_G(\mathcal{Z} \times \mathfrak{g}))$. □

6.2. **The map RR^{G}_{β} with $G_\beta = G$.** When $\beta \neq 0$ is in the centre of $\mathfrak{g}$, the map $RR^{G}_{\beta}(M, -) : K_G(M) \to R^{-\infty}(G)$ is the Riemann-Roch character localised near $M^\beta \cap f_{G}^{-1}(\beta)$.

On the manifold M^β, the almost complex structure J and the moment map $f_{G} : M \to \mathfrak{g}$ restrict to an almost complex structure J_β and a moment map $f_{G}|_{M^\beta} : M^\beta \to \mathfrak{g}$. Here the set $M^\beta \cap f_{G}^{-1}(\beta) = (f_{G}|_{M^\beta})^{-1}(\beta)$ is a component of the critical set of $C^{f_{G}|_{M^\beta}}$, and we denote $RR^{G}_{\beta}(M^\beta, -) : K_G(M^\beta) \to R^{-\infty}(G)$ the localisation of the Riemann-Roch character $RR^{G}(M^\beta, -) : K_G(M^\beta) \to R(G)$ near the component $(f_{G}|_{M^\beta})^{-1}(\beta)$ (see Definition 6.5).

Here we proceed like in the section 5. Let $\mathcal{N}$ be the normal bundle of M^β in M. The subgroup $\mathbb{T}_\beta \hookrightarrow G$ generated by $\exp(t.\beta)$, $t \in \mathbb{R}$ acts linearly on the fibre of the complex vector bundle $\mathcal{N}$. Thus we associate, like in Theorem 5.7, the polarized complex G-vector bundles $\mathcal{N}^{\beta,+}$ and $(\mathcal{N} \otimes \mathbb{C})^{\beta,+}$.

Proposition 6.9. *For every $E \in K_G(M)$, we have the following equality in $R^{-\infty}(G)$:*

$$RR^G_\beta(M,E) = (-1)^{r_{\mathcal{N}}} \sum_{k\in\mathbb{N}} RR^G_\beta(M^\beta, E|_{M^\beta} \otimes \det\mathcal{N}^{\beta,+} \otimes S^k((\mathcal{N}\otimes\mathbb{C})^{\beta,+})\ ,$$

where $r_{\mathcal{N}}$ is the locally constant function on M^β equal to the complex rank of $\mathcal{N}^{\beta,+}$.

Consider the decomposition of $RR^G_\beta(M,E)$ in irreducible character $\chi^G_\lambda,\ \lambda\in\Lambda^*_+$,

$$RR^G_\beta(M,E) = \sum_\lambda m_{\beta,\lambda}(E).\chi^G_\lambda, \quad m_{\beta,\lambda}(E)\in\mathbb{Z}\ . \tag{6.35}$$

If E is a f_G-strictly positive complex vector bundle over M, the vector bundle $E|_{M^\beta}$ is then $f_G|_{M^\beta}$-strictly positive. If $\mathcal{Z}$ is a connected component of M^β which intersects $f_G^{-1}(\beta)$, every weight a of the $\mathbb{T}_\beta$-action on the fibres of the complex vector bundle $E|_\mathcal{Z}\otimes\det\mathcal{N}^{\beta,+}\otimes S^k((\mathcal{N}\otimes\mathbb{C})^{\beta,+}$ satisfy $\langle a,\beta\rangle>0$.

Lemma 9.4 and Corollary 9.5, applied to this situation, show that

$$m_{\beta,\lambda}(E)\neq 0 \implies \langle\lambda,\beta\rangle > 0\ ,$$

for any f_G-strictly positive complex vector bundle E. Moreover, if we consider $\eta_{E,\beta} = \inf_a\langle a,\beta\rangle$, where a runs over the set of weights for the $\mathbb{T}_\beta$-action on the fibres of the complex vector bundles $E|_\mathcal{Z}$ with $\mathcal{Z}\cap f_G^{-1}(\beta)\neq\emptyset$, we get

$$m_{\beta,\lambda}(E^{\overset{k}{\otimes}})\neq 0 \implies \langle\lambda,\beta\rangle \geq k.\eta_{E,\beta}\ . \tag{6.36}$$

Note that $\eta_{E,\beta} > 0$, for every $\beta\in\mathcal{B}_G - \{0\}$, when E is f_G-strictly positive. Finally, we obtain

Corollary 6.10. *Let E be a f_G-strictly positive complex vector bundle over M. For any $\beta\in\mathcal{B}^G, \beta\neq 0$, with $G_\beta = G$, the G-invariant part of $RR^G_\beta(M,E)$ is equal to* 0.

Proof of Proposition 6.9 : Here we proceed as in the proof of Theorem 5.7. The almost complex structure J induces an almost complex structure J_β on M^β and a complex structure $J_\mathcal{N}$ on the fibres of vector bundle $p:\mathcal{N}\to M^\beta$. The $G\times\mathbb{T}_\beta$-vector bundle $p:\mathcal{N}\to M^\beta$ is isomorphic to $R\times_U N\to M^\beta = R/U$, where R is the $\mathbb{T}_\beta$-equivariant unitary frame of $(\mathcal{N},J_\mathcal{N})$ frame on N.

Let $\mathcal{U}^{G,\beta}$ be a neighbourhood of C^G_β in M, and consider the G-transversally elliptic symbol $\mathrm{Thom}^f_{G,[\beta]}(M)\in K_G(\mathbf{T}_G\mathcal{U}^{G,\beta})$ introduced in Definition 6.4. Here we choose $\mathcal{U}^{G,\beta}$ diffeomorphic to an open subset of $\mathcal{N}$ of the form $\mathcal{V}:=\{n=(x,v)\in\mathcal{N},\ x\in\mathcal{U}$ and $|v|<\varepsilon\}$, where $\mathcal{U}$ is a neighbourhood of $(f_G|_{M^\beta})^{-1}(\beta)$ in M^β. The moment map f_G, the vector field $\mathcal{H}^G$, and $\mathrm{Thom}^f_{G,[\beta]}(M)$ are transported by this diffeomorphism to $\mathcal{V}$ (we keep the same symbol for these elements).

We defined now the homogeneous vector field $\widetilde{\mathcal{H}}^G$ on $\mathcal{N}$ by

$$\widetilde{\mathcal{H}}^G_n := \Big(f_G(p(n))\Big)_\mathcal{N}(n),\ n\ \in\ \mathcal{N}\ . \tag{6.37}$$

Using the isomorphism $\mathbf{T}\mathcal{N}\tilde{\to}p^*(\mathbf{T}M^\beta\oplus\mathcal{N})$ (see Eq. (5.26)) we endowed the manifold $\mathcal{N}$ with the almost complex structure $\widetilde{J}:=p^*(J_\beta\oplus J_\mathcal{N})$. With the data $(\widetilde{J},\ \widetilde{\mathcal{H}}^G)$, we construct the following G-transversally elliptic symbol over $\mathcal{N}$:

$$\mathrm{Thom}^f_{G,[\beta]}(\mathcal{N})(n,w) := \mathrm{Thom}_G(\mathcal{N},\widetilde{J})(n,w-\widetilde{\mathcal{H}}^G_n), \quad \text{for} \quad (n,w)\ \in\ \mathbf{T}\mathcal{N}\ . \tag{6.38}$$

Let us now verify that

$$\mathrm{Thom}^f_{G,[\beta]}(M) = \mathrm{Thom}^f_{G,[\beta]}(\mathcal{N}) \quad \text{in } K_G(\mathbf{T}_G\mathcal{V}) \ .$$

The invariance of the Thom class after the modification of the almost complex structure is carried out in Lemma 5.8 : the class of $\mathrm{Thom}^f_{G,[\beta]}(M)$ is equal in $K_G(\mathbf{T}_G\mathcal{V})$ to the class of the symbol

$$\sigma_1(n,w) := \mathrm{Thom}_G(\mathcal{N},\widetilde{J})(n, w - \mathcal{H}^{G}_n), \quad (n,w) \ \in \ \mathbf{T}\mathcal{V} \ .$$

Using now the family of vectors field $\mathcal{H}^{G}_t(n) := \Big(f_{G}(x,t.v)\Big)_{\mathcal{V}}(n)$, $t \in [0,1]$, $n=(x,v)\in\mathcal{V}$, we construct the homotopy

$$\sigma_t(n,w) := \mathrm{Thom}_H(\mathcal{N},\widetilde{J})(n, w - \mathcal{H}^{G}_t(n)), \quad (n,w) \ \in \ \mathbf{T}\mathcal{V}$$

of G-transversally elliptic symbol between σ_1 and $\mathrm{Thom}^f_{G,[\beta]}(\mathcal{N})$ (one easily verifies that $\mathrm{Char}(\sigma_t)\cap \mathbf{T}_G\mathcal{V} = C^{G}_\beta$ for every $t\in[0,1]$). Finally, we have shown that $\mathrm{Thom}^f_{G,[\beta]}(\mathcal{N}) = \mathrm{Thom}^f_{G,[\beta]}(M)$ in $K_G(\mathbf{T}_G\mathcal{V})$, thus

$$RR^{G}_\beta(E) = \mathrm{Index}^G_{\mathcal{N}}\Big(\mathrm{Thom}^f_{G,[\beta]}(\mathcal{N})\otimes p^*(E_{|M^\beta})\Big)$$

for every $E\in K_G(M)$ (here $p:\mathcal{N}\to M^\beta$ denotes the projection).

Now we proceed as follows. For every $(n,w)\in\mathbf{T}\mathcal{V}$, the Clifford action $\mathrm{Thom}^f_{G,[\beta]}(\mathcal{N})(n,w) = Cl_n(w-\widetilde{\mathcal{H}}^{G}_n)$ on $\wedge_{\mathbb{C}}\mathbf{T}_n\mathcal{V}$ is equal to the exterior product

$$Cl_x(w_1 - [\widetilde{\mathcal{H}}^{G}_n]_1)\odot Cl_x(w_2 - [\widetilde{\mathcal{H}}^{G}_n]_2) \tag{6.39}$$

acting on $\wedge_{\mathbb{C}}\mathbf{T}_xM^\beta\otimes\wedge_{\mathbb{C}}\mathcal{N}|_x$, where $x=p(n)$. Here $w\to w_1$, $\mathbf{T}_n\mathcal{V}\to\mathbf{T}_xM^\beta$ is the tangent map $\mathbf{T}p|_n$, and $w\to w_2=[w]^V$, $T_n\mathcal{V}\to\mathcal{N}|_x$ is the 'vertical' map. Here we see that $[\widetilde{\mathcal{H}}^{G}_n]_1 = \mathcal{H}^{G}_x$ is the vector field on M^β generated by the moment map $f_{G}|_{M^\beta}$ (see Definition 6.2).

Suppose that the exterior product (6.39) can be modified in

$$Cl_x(w_1-\mathcal{H}^{H}_x)\odot Cl_x(w_2-\beta_{\mathcal{N}}|_n), \tag{6.40}$$

without changing the K-theoric class. This will prove a modified version of the equality (5.30) in $K_{G\times\mathbb{T}_\beta\times U}(\mathbf{T}_{G\times\mathbb{T}_\beta\times U}(R\times N))$:

$$\pi_N^*\Big(\mathrm{Thom}^f_{G,[\beta]}(\mathcal{N})\Big) = \kappa\Big(\pi^*(\mathrm{Thom}^f_{G,[\beta]}(M^\beta)), \mathrm{Thom}^\beta_{\mathbb{T}_\beta\times U}(N)\Big) \ . \tag{6.41}$$

where $\pi_N: R\times N\to R_a\times_U N=\mathcal{N}$ and $\pi: R\to R/U = M^\beta$ are the quotient maps relative to the free U-action. The symbols $\mathrm{Thom}^f_{G,[\beta]}(\mathcal{N})$, $\mathrm{Thom}^f_{G,[\beta]}(M^\beta)$ and $\mathrm{Thom}^\beta_{\mathbb{T}_\beta\times U}(N)$ belong respectively to $K_{G\times\mathbb{T}_\beta}(\mathbf{T}_{G\times\mathbb{T}_\beta}(R\times_U N))$, $K_G(\mathbf{T}_G(R/U))$, and $K_{\mathbb{T}_\beta\times U}(\mathbf{T}_{\mathbb{T}_\beta\times U}N)$.

The Proposition 6.9 follows after taking the index, and the U-invariants, in the equality (6.41).

Finally we explain why the change of $[\widetilde{\mathcal{H}}^{G}_n]_2$ in $\beta_{\mathcal{N}}|_n$ can be done in the tensor product (6.39) without changing the class of $\mathrm{Thom}^f_{G,[\beta]}(\mathcal{N})$.

Let $\mu^{\mathcal{N}} : \mathfrak{g} \to \Gamma(M^\beta, \mathrm{End}(\mathcal{N}))$ be the 'moment' relative to the choice of a connection on $\mathcal{N} \to M^\beta$ (see Definition 7.5 in [8]). Then, for every $X \in \mathfrak{g}$ we have

$$[X_{\mathcal{N}}(x,v)]^V = -\mu^{\mathcal{N}}(X)|_x.v, \quad (x,v) \in \mathcal{N}$$

(see Proposition 7.6 in [8]). When $X = \beta$, the vector field $\beta_{\mathcal{N}}$ is vertical, hence we have $\mu^{\mathcal{N}}(\beta)|_x.v = \mathcal{L}^{\mathcal{N}}(\beta)|_x.v = -\beta_{\mathcal{N}}(x,v)$, where $\mathcal{L}^{\mathcal{N}}(\beta)$ is the infinitesimal action of β on the fibre of $\mathcal{N} \to M^\beta$. We have also $[\widetilde{\mathcal{H}}^{G}_n]_2 = -\mu^{\mathcal{N}}(f_G(x))|_x.v$, for every $n = (x,v) \in \mathcal{N}$.

Note that the quadratic form $v \in \mathcal{N}_x \to |\mathcal{L}^{\mathcal{N}}(\beta)|_x.v|^2$ is positive definite for $x \in M^\beta$. Hence, for every $X \in \mathfrak{g}$ close enough to β, the quadratic form $v \in \mathcal{N}_x \to (\mu^{\mathcal{N}}(\beta)|_x.v, \mu^{\mathcal{N}}(X)|_x.v)$ is positive definite for $x \in M^\beta$.

Consider now the homotopy

$$\sigma^t(n,w) := Cl_x(w_1 - \mathcal{H}^{G}_x) \odot Cl_x(w_2 - t.[\widetilde{\mathcal{H}}^{G}_n]_2 - (1-t).\beta_{\mathcal{N}}|_n), \quad (n,v) \in \mathcal{V} \quad t \in [0,1].$$

We see that $(n,w) \in \mathrm{Char}(\sigma^t) \cap \mathbf{T}_G\mathcal{V}$ if and only if

i) $w_1 = \mathcal{H}^{G}_x$,

ii) $w_2 = t[\widetilde{\mathcal{H}}^{G}_n]_2 + (1-t)\beta_{\mathcal{N}}(n)$, and

iii) $(w_1, X_{M^\beta}(x)) + (w_2, [X_{\mathcal{N}}(x,v)]^V) = 0$ for all $X \in \mathfrak{g}$.

Take now $X = f_G(x)$ in iii). With i) and ii), we get

$$\left|\mathcal{H}^{G}_x\right|^2 + t.\left|\mu^{\mathcal{N}}(f_G(x))|_x.v\right|^2 + (1-t).\Sigma(x,v) = 0 , \tag{6.42}$$

with $\Sigma(x,v) := (\mu^{\mathcal{N}}(\beta)|_x.v, \mu^{\mathcal{N}}(f_G(x))|_x.v)$.

If $x \in M^\beta$ is sufficiently close to $(f_G|_{M^\beta})^{-1}(\beta)$, the term $\Sigma(x,v)$ is positive for all $v \in \mathcal{N}_x$. In this case, Equality (6.42) gives $\mathcal{H}^{G}_x = 0$ and $\Sigma(x,v) = 0$, which insures that $x \in C^{G}_\beta$ and $v = 0$.

We have proved that $\mathrm{Char}(\sigma^t) \cap \mathbf{T}_G\mathcal{V} = C^{G}_\beta$ for every $t \in [0,1]$ if $\mathcal{V}$ is 'small' enough. Hence σ^t is an homotopy of G-transversally elliptic symbols over $\mathbf{T}\mathcal{V}$ between the exterior products of Equations 6.39 and 6.40. □

6.3. **Induction formula.** We prove in this section an induction formula which compare the map $RR^{G}_\beta(M,-)$ with the similar localised Riemann-Roch character defined for the maximal torus. The idea of this induction comes from a previous paper of the author [26] where we prove a similar induction formula in the context of equivariant cohomology.

Consider the restriction $f_H : M \to \mathfrak{h}$ of the moment map f_G to the maximal torus H with Lie algebra $\mathfrak{h}$. In this situation we use the vector field $\mathcal{H}^{H}|_m = f_H(m)_M|_m, m \in M$ to decompose the map $RR^{H}(M,-) : K_H(M) \to R(H)$ near the set $C^{f_H} = \{\mathcal{H}^{H} = 0\}$. From Lemma 6.3 there exists a finite subset $\mathcal{B}_H \subset \mathfrak{h}$, such that

$$C^{f_H} = \bigcup_{\beta \in \mathcal{B}_H} C^{H}_\beta, \quad \text{with} \quad C^{H}_\beta = M^\beta \cap f_H^{-1}(\beta).$$

Like in Definition 6.5, we define for every $\beta \in \mathcal{B}_H$, the map $RR^{H}_\beta(M,-) : K_H(M) \to R^{-\infty}(H)$ which is the localised Riemann-Roch character near C^{H}_β.

Let W be the Weyl group of (G,H). Note that $\mathcal{B}_H$ is a W-stable subset of $\mathfrak{h}$, and that $\mathcal{B}_G \subset \mathcal{B}_H \cap \mathfrak{h}_+$.

Theorem 6.11. *We have, for every $\beta \in \mathcal{B}_G$, the following induction formula between $RR^G_\beta(M,-)$ and $RR^H_\beta(M,-)$. For every $E \in K_G(M)$, we have*[7]

$$RR^G_\beta(M,E) = \frac{1}{|W_\beta|}\mathrm{Ind}^G_H\left(RR^H_\beta(M,E).\,\det{}^{\mathbb{R}}_{\mathfrak{g}/\mathfrak{h}}(1-h)\right) \quad \text{in} \quad \mathcal{C}^{-\infty}(G)^G,$$

where W_β is the stabilizer of β in W.

We can use the previous induction formula between G and H index maps to produce an induction formula between G and G_β index maps. Consider the restriction $f_{G_\beta} : M \to \mathfrak{g}_\beta$ of the moment map to the stabiliser G_β of β in G. Let $RR^{G_\beta}_\beta(M,-) : K_{G_\beta}(M) \to R^{-\infty}(G_\beta)$ be the localised Riemann-Roch character near $C^{G_\beta}_\beta = M^\beta \cap f^{-1}_{G_\beta}(\beta)$[8].

Corollary 6.12. *For every $\beta \in \mathcal{B}_G$ and every $E \in K_G(M)$, we have*

$$RR^G_\beta(M,E) = \mathrm{Ind}^G_{G_\beta}\left(RR^{G_\beta}_\beta(M,E).\,\det{}^{\mathbb{R}}_{\mathfrak{g}/\mathfrak{g}_\beta}(1-h)\right) \quad \text{in} \quad \mathcal{C}^{-\infty}(G)^G,$$

Proof of the Corollary : It comes immediately by applying the induction formula of Theorem 6.11 to the couples (G,H) and (G_β,H).

Corollary 6.13. *For every complex vector bundle $E \to M$, we have*

$$\left[RR^G_\beta(M,E^{\overset{k}{\otimes}})\right]^G = 0,$$

if $k \in \mathbb{N}$ is large enough, and E is f_G-strictly positive.

Corollary 6.14. *Let $L \to M$ be an Hermitian line bundle on M, such that f_G is equal to its moment f_L (see the introduction). We have*

$$\left[RR^G_\beta(M,L^{\overset{k}{\otimes}})\right]^G = 0,$$

if $k \in \mathbb{N}$ is large enough, so that $k.\parallel \beta \parallel > \parallel w\rho - \rho \parallel$ for all w in the Weyl group W of G, where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is half the sum of the positive roots of G.

Proof of Corollary 6.13 :

Using the holomorphic induction map $\mathrm{Hol}^G_{G_\beta}$ (see equation 9.54 in Appendix B), the equality of Corollary 6.12 can be rewritten

$$RR^G_\beta(M,E^{\overset{k}{\otimes}}) = \mathrm{Hol}^G_{G_\beta}\left(RR^{G_\beta}_\beta(M,E^{\overset{k}{\otimes}}).\overline{\det{}^{\mathbb{C}}_{\mathfrak{g}/\mathfrak{g}_\beta}(1-h)}\right) .$$

First write the decomposition[9] $RR^{G_\beta}_\beta(M,E^{\overset{k}{\otimes}}) = \sum_{\lambda\in\Lambda^+_\beta} m_{\lambda,\beta}(E^{\overset{k}{\otimes}})\chi^{G_\beta}_\lambda$, $m_{\lambda,\beta}(E^{\overset{k}{\otimes}}) \in \mathbb{Z}$, in irreducible character of G_β. We know from equation (6.36) that there exists $\eta > 0$ such that

$$m_{\lambda,\beta}(E^{\overset{k}{\otimes}}) \neq 0 \implies \langle\lambda,\beta\rangle \geq k.\eta \quad k \in \mathbb{N} .$$

[7]See Eq. (3.12) for the definition of the induction map $\mathrm{Ind}^G_H : \mathcal{C}^{-\infty}(H)^H \longrightarrow \mathcal{C}^{-\infty}(G)^G$

[8]Note that $M^\beta \cap f^{-1}_{G_\beta}(\beta) = M^\beta \cap f^{-1}_G(\beta)$ because $f_{G_\beta} = f_G$ on M^β.

[9]We choose a set $\Lambda^*_{\beta,+}$ of dominant weight for G_β that contains the set Λ^*_+ of dominant weight for G.

Each irreducible character $\chi_\lambda^{G_\beta}$ is equal to $\mathrm{Hol}_H^{G_\beta}(h^\lambda)$, then $RR_\beta^G(M, E^{\overset{k}{\otimes}}) = \mathrm{Hol}_H^G\left((\sum_\lambda m_{\lambda,\beta}(E^{\overset{k}{\otimes}})\, h^\lambda)\Pi_{\alpha\in\Delta(\mathfrak{g}/\mathfrak{g}_\beta)}(1-h^{-\alpha})\right)$ where $\Delta(\mathfrak{g}/\mathfrak{g}_\beta)$ is the set of H-weight on $\mathfrak{g}/\mathfrak{g}_\beta$[10]. We know from Appendix B that $\mathrm{Hol}_H^G(h^{\lambda'})$ is either 0 or the character of an irreducible representation; in particular $\mathrm{Hol}_H^G(h^{\lambda'})$ is equal to ± 1 (1 is the character of the trivial representation) only if $\langle \lambda', X\rangle \leq 0$ for every $X \in \mathfrak{h}_+$ in the Weyl chamber. The generalised character $RR_\beta^G(M, E^{\overset{k}{\otimes}})$ is the sum of terms of the form $m_{\lambda,\beta}(E^{\overset{k}{\otimes}})\,\mathrm{Hol}_H^G(h^{\lambda-\alpha_I})$ with $\alpha_I = \sum_{\alpha\in I}\alpha$ where I is a subset of $\Delta(\mathfrak{g}/\mathfrak{g}_\beta)$. Let $k_o \in \mathbb{N}$ such that

$$k_o.\eta \;>\; \sum_{\alpha\in\Delta(\mathfrak{g}/\mathfrak{g}_\beta)} \langle \alpha, \beta\rangle .$$

Then for every $k \geq k_o$,

$$\begin{aligned} m_{\lambda,\beta}(E^{\overset{k}{\otimes}}) \neq 0 \;&\Longrightarrow\; \langle \lambda, \beta\rangle \geq k.\eta \geq k_o.\eta \\ &\Longrightarrow\; \langle \lambda - \alpha_I, \beta\rangle > 0 \quad \text{for all} \quad I \subset \Delta(\mathfrak{g}/\mathfrak{g}_\beta) \\ &\Longrightarrow\; \mathrm{Hol}_H^G(h^{\lambda-\alpha_I}) \neq \pm 1 \quad \text{for all} \quad I \subset \Delta(\mathfrak{g}/\mathfrak{g}_\beta)\ . \end{aligned}$$

We have proved that $\left[RR_\beta^G(M, E^{\overset{k}{\otimes}})\right]^G = 0$ if $k \geq k_o$. □

Proof of Corollary 6.14 :

Using the holomorphic induction map Hol_H^G (see equation 9.54 in Appendix B), the equality of Corollary 6.11 can be rewritten

$$RR_\beta^G(M, L^{\overset{k}{\otimes}}) = \frac{1}{|W_\beta|}\sum_{w\in W} \mathrm{Hol}_H^G\left(w.RR_\beta^H(M, L^{\overset{k}{\otimes}})\right) .$$

First write the decomposition $RR_\beta^H(M, L^{\overset{k}{\otimes}}) = \sum_{\lambda\in\Lambda} m_{\lambda,\beta}^H(L^{\overset{k}{\otimes}})\, h^\lambda$, $m_{\lambda,\beta}^H(L^{\overset{k}{\otimes}}) \in \mathbb{Z}$, in irreducible character of H. We know from equation (6.36) that

$$\begin{aligned} m_{\lambda,\beta}^H(L^{\overset{k}{\otimes}}) \neq 0 \quad &\Longrightarrow\quad \langle \lambda, \beta\rangle \;\geq k.\parallel \beta \parallel^2 \\ &\Longrightarrow\quad \parallel \lambda \parallel \geq k.\parallel \beta \parallel \end{aligned}$$

(Here $\eta_{L,\beta} = \parallel \beta \parallel^2$). We know from Lemma 9.1 that $\mathrm{Hol}_H^G(h^{\lambda'})$ is equal to ± 1 if only if $w.(\lambda'+\rho)-\rho = 0$ for some $w \in W$: this implies in particular $\parallel \lambda' \parallel = \parallel w.\rho - \rho \parallel$.

The generalised character $RR_\beta^H(M, L^{\overset{k}{\otimes}})$ is the sum of terms of the form $m_{\lambda,\beta}^H(L^{\overset{k}{\otimes}})\,\mathrm{Hol}_H^G(h^{w.\lambda})$ with $w \in W$. Let $k_o \in \mathbb{N}$ such that

$$k_o.\parallel \beta \parallel \;>\; \parallel w'.\rho - \rho \parallel\ ,$$

for all $w' \in W$. Then for every $k \geq k_o$,

$$\begin{aligned} m_{\lambda,\beta}(E^{\overset{k}{\otimes}}) \neq 0 \;&\Longrightarrow\; \parallel \lambda \parallel \geq k.\parallel \beta \parallel \geq k_o.\parallel \beta \parallel \\ &\Longrightarrow\; \parallel w.\lambda \parallel > \parallel w'.\rho - \rho \parallel \quad \text{for all} \quad w, w' \in W \\ &\Longrightarrow\; \mathrm{Hol}_H^G(h^{w.\lambda}) \neq \pm 1 \quad \text{for all} \quad w \in W\ . \end{aligned}$$

[10] The complex structure on $\mathfrak{g}/\mathfrak{g}_\beta$ is defined by β, so that $\langle\alpha,\beta\rangle > 0$ for all $\alpha \in \Delta(\mathfrak{g}/\mathfrak{g}_\beta)$.

We have proved that $\left[RR^{G}_{\beta}(M, L^{\overset{k}{\otimes}})\right]^{G} = 0$ if $k \geq k_o$. □

The rest of this section is devoted to the proof of Theorem 6.11. Consider the map $r^{\gamma}_{G,H} : K_G(\mathbf{T}_G M) \to K_H(\mathbf{T}_H M)$ defined with $\gamma \in \mathfrak{h}$ in the interior of the Weyl chamber, so that $G_\gamma = H$ (see subsection 3.4).

The map $RR^{G}_{\beta}(M, -)$ is defined through the symbol $\mathrm{Thom}^f_{G,[\beta]}(M) \in K_G(\mathbf{T}_G \mathcal{U}^{G,\beta})$ where $i^{G,\beta} : \mathcal{U}^{G,\beta} \to M$ is any G-invariant neighbourhood of C^{G}_{β} such that $\overline{\mathcal{U}^{G,\beta}} \cap C^{f_G} = C^{G}_{\beta}$ (see Definition 6.4). We define in the same way the localised Thom complex $\mathrm{Thom}^f_{H,[\beta]}(M) \in K_H(\mathbf{T}_H \mathcal{U}^{H,\beta})$.

For notational convenience, we will note in the same way the direct image of $\mathrm{Thom}^f_{G,[\beta]}(M)$ (resp. $\mathrm{Thom}^f_{H,[\beta]}(M)$) in $K_G(\mathbf{T}_G M)$ (resp. $K_H(\mathbf{T}_H M)$) via $i^{G,\beta}_* : K_G(\mathbf{T}_G \mathcal{U}^{G,\beta}) \to K_G(\mathbf{T}_G M)$ (resp. $i^{H,\beta}_* : K_H(\mathbf{T}_H \mathcal{U}^{H,\beta}) \to K_H(\mathbf{T}_H M)$).

Then we have $RR^{G}_{\beta}(M, E) = \mathrm{Index}^G_M(\mathrm{Thom}^f_{G,[\beta]}(M) \otimes E)$ for $E \in K_G(M)$. The Weyl group acts on $K_H(\mathbf{T}_H M)$ and we remark that $w.\mathrm{Thom}^f_{H,[\beta]}(M) = \mathrm{Thom}^f_{H,[w.\beta]}(M)$ for every $\beta \in \mathcal{B}_H$, and $w \in W$.

Lemma 6.15. *We have the following equality*

$$r^{\gamma}_{G,H}\left(\mathrm{Thom}^f_{G,[\beta]}(M)\right) = \sum_{\beta' \in W.\beta} \mathrm{Thom}^f_{H,[\beta']}(M) \otimes \wedge^{\bullet}_{\mathbb{C}} \mathfrak{g}/\mathfrak{h} \quad \text{in} \quad K_H(\mathbf{T}_H M) \ .$$

This Lemma implies that

$$\begin{aligned} RR^{G}_{\beta}(M, E) &= \mathrm{Ind}^{G}_{H}\left((\sum_{\beta' \in W.\beta} RR^{H}_{\beta'}(M, E)).\det{}^{\mathbb{C}}_{\mathfrak{g}/\mathfrak{h}}(1-h)\right) \\ &= \frac{1}{|W|}\mathrm{Ind}^{G}_{H}\left((\sum_{\beta' \in W.\beta} RR^{H}_{\beta'}(M, E)).\det{}^{\mathbb{R}}_{\mathfrak{g}/\mathfrak{h}}(1-h)\right) \\ &= \frac{1}{|W_\beta|}.\mathrm{Ind}^{G}_{H}\left(RR^{H}_{\beta}(M, E).\det{}^{\mathbb{R}}_{\mathfrak{g}/\mathfrak{h}}(1-h)\right). \end{aligned}$$

The second equality is due to the fact that $\sum_{\beta' \in W.\beta} RR^{H}_{\beta'}(M, E)$ is a W-invariant element of $R^{-\infty}(H)$ and $\sum_{\varepsilon \in W} \varepsilon.\det^{\mathbb{C}}_{\mathfrak{g}/\mathfrak{h}}(1-h) = \left|\det{}^{\mathbb{C}}_{\mathfrak{g}/\mathfrak{h}}(1-h)\right|^2 = \det{}^{\mathbb{R}}_{\mathfrak{g}/\mathfrak{h}}(1-h)$.

The last equality follows from the fact that $w.\mathrm{Thom}^f_{H,[\beta]}(M) = \mathrm{Thom}^f_{H,[w.\beta]}(M)$, and $h \to \det{}^{\mathbb{R}}_{\mathfrak{g}/\mathfrak{h}}(1-h)$ is W-invariant. So, we have proved that Lemma 6.15 implies Theorem 6.11.

Proof of Lemma 6.15 : Consider a G-invariant open neighbourhood $\mathcal{U}^{G,\beta}$ of C^{G}_{β} such that $\overline{\mathcal{U}^{G,\beta}} \cap C^{f_G} = C^{G}_{\beta}$. We know from Proposition 3.6 that the symbol of $\sigma := r_{G,H}\left(\mathrm{Thom}^f_{G,[\beta]}(M)\right)$ is the restriction on $\mathbf{T}\mathcal{U}^{G,\beta}$ of the symbol

$$\sigma_I(m, v) = Cl_x(v - \mathcal{H}^{G}_{m}) \odot Cl(\mu_{G/H}(v)), \qquad (m, v) \in \mathbf{T}M \ .$$

Let $f_{G/H} : M \to \mathfrak{g}/\mathfrak{h}$ (resp. $f_H : M \to \mathfrak{h}$) the $\mathfrak{g}/\mathfrak{h}$-part (resp. the $\mathfrak{h}$-part) of the moment map f. Note that $(\mu_{G/H}(\mathcal{H}^{G}), f_{G/H})_{\mathfrak{g}} = |\mathcal{H}^{G}|^2_M - (\mathcal{H}^{G}, \mathcal{H}^{H})_M$ (★), where the equality $\mathcal{H}^{G} = \mathcal{H}^{H} + \mathcal{H}^{G/H}$ comes from the decomposition $f_G = f_H + f_{G/H}$. Note that the vector field $\mathcal{H}^{H}$ belongs to the H-orbits and consider the family of H-equivariant symbol σ_θ, $\theta \in [0,1]$, defined on $\mathbf{T}M$ by

$$\sigma_\theta(m,v) = Cl_x(v - \mathcal{H}^{G}_m) \odot Cl\left(\theta\mu_{G/H}(v) + (1-\theta)f_{G/H}(m)\right), \qquad (m,v) \in \mathbf{T}M .$$

We see that $(m,v) \in \mathrm{Char}(\sigma_\theta) \iff v = \mathcal{H}^{G}_m$ and $\theta\mu_{G/H}(\mathcal{H}^{G}_m) + (1-\theta)f_{G/H}(m) = 0$. The equality ($\bigstar$) shows that $\mathrm{Char}(\sigma_\theta) \cap \mathbf{T}_H M \subset \{\mathcal{H}^{G} = 0\}$, for every $\theta \in [0,1]$. By this way we prove have proved that $\sigma_I|_{\mathcal{U}^{G,\beta}}$ is homotopic to the H-transversally elliptic symbol $\sigma_{II}|_{\mathcal{U}^{G,\beta}}$ where

$$\sigma_{II}(m,v) = Cl_x(v - \mathcal{H}^{G}_m) \odot Cl(f_{G/H}(m)), \qquad (m,v) \in \mathbf{T}M .$$

We transform now σ_{II} via the following homotopy of H-transversally elliptic symbols

$$\sigma^u(m,v) := Cl_x(v - \mathcal{H}^{H}_m - u.\mathcal{H}^{G/H}_m) \odot Cl(f_{G/H}(m)), \quad (m,v) \in \mathbf{T}M ,$$

for $u \in [0,1]$. We see here that $\mathrm{Char}(\sigma^u) \cap \mathbf{T}_H M = \{\mathcal{H}^{G} = 0\} \cap \{f_{G/H} = 0\}$ for all $u \in [0,1]$, hence $\sigma_{II}|_{\mathcal{U}^{G,\beta}}$ is homotopic to the H-transversally elliptic symbol $\sigma_{III}|_{\mathcal{U}^{G,\beta}}$ where

$$\sigma_{III}(m,v) = Cl_x(v - \mathcal{H}^{H}_m) \odot Cl(f_{G/H}(m)), \qquad (m,v) \in \mathbf{T}M .$$

At this stage we have proved that $\sigma_I|_{\mathcal{U}^{G,\beta}} = \sigma_{III}|_{\mathcal{U}^{G,\beta}}$ in $K_H(\mathbf{T}_H\mathcal{U}^{G,\beta})$.

Note that we have

$$\begin{aligned}\mathrm{Char}(\sigma_{III}|_{\mathcal{U}^{G,\beta}}) \cap \mathbf{T}_H\mathcal{U}^{G,\beta} &= G.(M^\beta \cap f_G^{-1}(\beta)) \bigcap \{f_{G/H} = 0\} \\ &= \bigcup_{\beta' \in W.\beta} M^{\beta'} \cap f_H^{-1}(\beta') ,\end{aligned}$$

because $G.\beta \cap \mathfrak{h} = W.\beta$. Let $i : \mathcal{U} \hookrightarrow \mathcal{U}^{G,\beta}$ be a H-invariant neighbourhood of $\cup_{\beta' \in W.\beta} M^{\beta'} \cap f_H^{-1}(\beta')$ such that $\overline{\mathcal{U}} \cap \{\mathcal{H}^{H} = 0\} = \cup_{\beta' \in W.\beta} M^{\beta'} \cap f_H^{-1}(\beta')$. The symbol $\sigma_{III}|_{\mathcal{U}}$ is H-transversally elliptic and

$$i_*(\sigma_{III}|_{\mathcal{U}}) = \sigma_{III}|_{\mathcal{U}^{G,\beta}} \quad \text{in} \quad K_H(\mathbf{T}_H\mathcal{U}^{G,\beta}) . \tag{6.43}$$

As in the proof of Proposition 4.1, Equality (6.43) is an immediate consequence of the excision property.

The symbol $(m,v) \to Cl_x(v - \mathcal{H}^{H}_m)$ is H-transversally elliptic on $\mathbf{T}\mathcal{U}$, and equal (by definition) to $\sum_{\beta' \in W.\beta} \mathrm{Thom}^f_{H,[\beta']}(M)$. Hence $\sigma_{III}|_{\mathcal{U}}$ is homotopic, in $K_H(\mathbf{T}_H\mathcal{U})$, to $(m,v) \to Cl_x(v - \mathcal{H}^{H}_m) \otimes 0_{\mathfrak{g}/\mathfrak{h}}$, where $0_{\mathfrak{g}/\mathfrak{h}}$ is the zero map from $\wedge^{even}_{\mathbb{C}}\mathfrak{g}/\mathfrak{h}$ to $\wedge^{odd}_{\mathbb{C}}\mathfrak{g}/\mathfrak{h}$. Finally we have shown that $\sigma_{III}|_{\mathcal{U}} = \sum_{\beta' \in W.\beta} \mathrm{Thom}^f_{H,[\beta']}(M) \otimes \wedge^\bullet_{\mathbb{C}}\mathfrak{g}/\mathfrak{h}$ in $K_H(\mathbf{T}_H\mathcal{U})$, and Equality (6.43) finish the proof. □

7. The Hamiltonian case

In this section, we assume that (M,ω) is a compact symplectic manifold with a Hamiltonian action of a compact Lie group G. The corresponding moment map $\mu_G : M \to \mathfrak{g}^* \cong \mathfrak{g}$ is defined by

$$d\langle \mu_G, X\rangle = \omega(X_M, -), \quad \forall\, X \in \mathfrak{g}, \tag{7.44}$$

In this situation, the manifold always carries an almost complex structure J *compatible* with ω, that is :

$$\omega_x(J_x v, J_x w) = \omega_x(v, w) \quad x \in M \ ,$$

for every $v, w \in \mathbf{T}_x M$, and the symmetric bilinear form $\omega_x(J_x ., .)$ is *definite positive* on $\mathbf{T}_x M$. Moreover two *compatible* almost complex structures on (M, ω) are homotopic.

We fix once for all a *compatible* almost complex structure J, and we denote $(.,.)_M := \omega(J.,.)$ the corresponding Riemannian metric. Let $RR^G(M, -) : K_G(M) \to R(G)$ be the quantization map defined with the *compatible* almost complex structure J (note that this map does not depend of the choice of J).

Here the vector field $\mathcal{H}^G$ is the hamiltonian vector field of the function[11] $\frac{1}{2}|\mu_G|^2 : M \to \mathbb{R}$, and $\{\mathcal{H}^G = 0\}$ is the set of critical points of $|\mu_G|^2$. We know from the beginning of section 6 that we have the decomposition $RR^G(M, -) = \sum_{\beta \in \mathcal{B}_G} RR^G_\beta(M, -)$, where $RR^G_\beta(M, -) : K_G(M) \to R^{-\infty}(G)$ is the localisation of the Riemann-Roch character near the critical set $C^G_\beta = G(M^\beta \cap \mu_G^{-1}(\beta))$. In this section we will prove in particular the following

Theorem 7.1. *Let* $E \to M$ *a* G*-equivariant vector bundle over* M*. Suppose that* 0 *is a regular value of* μ_G*. The* G*-invariant part of* $RR^G_0(M, E)$ *is equal to* $RR(\mathcal{M}_{red}, E_{red})$*. If* E *is* μ_G-positive *and* $\mu_G^{-1}(0) \neq \emptyset$*, the* G*-invariant part of* $RR^G_\beta(M, E)$ *is equal to* 0 *for* $\beta \in \mathcal{B}_G, \beta \neq 0$.

<u>The map RR^G_0</u>

We assume here that 0 is a regular value of μ_G. In the Hamiltonian case the compatibility of J with ω insures that

$$(\mathbf{T}\mu_G(J(X_M)), X) = \omega(X_M, J(X_M)) = - \parallel X_M \parallel^2 \ .$$

In particular $\mathbf{T}\mu_G(J(X_M)) \neq 0$ on $\mathcal{Z} = \mu_G^{-1}(0)$ if $X \neq 0$, hence $\mathbf{T}\mathcal{Z} \cap J(\mathfrak{g}_\mathcal{Z}) = \{0\}$. So the Assumption 2 is fulfilled, and the map RR^G_0 is determined by the Proposition 6.7: for any $E \in K_G(M)$,

$$RR^G_0(M, E) = \sum_{a \in \hat{G}} RR(\mathcal{M}_{red}, E_{red} \otimes \underline{W_a}^*).W_a \quad \text{in} \quad R^{-\infty}(G) \ .$$

In particular, the G-invariant part of $RR^G_0(M, E)$ is equal to $RR(\mathcal{M}_{red}, E_{red}) \in \mathbb{Z}$ (see subsection 6.1 for the notations).

<u>The map RR^G_β with $G_\beta = G$</u>

When $\beta \neq 0$ is in the centre of $\mathfrak{g}$, recall the localisation formula on M^β obtain in Proposition 6.9. For every $E \in K_G(M)$, we have the following equality in $R^{-\infty}(G)$

$$RR^G_\beta(M, E) = (-1)^{r_\mathcal{N}} \sum_{k \in \mathbb{N}} RR^G_\beta(M^\beta, E|_{M^\beta} \otimes \det \mathcal{N}^{\beta,+} \otimes S^k((\mathcal{N} \otimes \mathbb{C})^{\beta,+}) \ ,$$

where $r_\mathcal{N}$ is the locally constant function on M^β equal to the complex rank of $\mathcal{N}^{\beta,+}$.

In the Hamiltonian case we can refined Corollary 6.10.

[11] Equality 7.44 gives $\frac{1}{2} d|\mu_G|^2 = \omega(\mathcal{H}^G, -)$

Lemma 7.2. *Let E be a complex vector bundle over M and let $\beta \in \mathcal{B}^{G}, \beta \neq 0$ be a* central *element in $\mathfrak{g}$.*

1) Suppose that $\langle \mu_G, \beta\rangle^{-1}(0)$ is not empty. Then the G-invariant part of $RR^{G}_{\beta}(M,E)$ is equal to 0 if E is μ_G-positive.

*2) In general, the G-invariant part of $RR^{G}_{\beta}(M,E)$ is equal to 0 if E is μ_G-*strictly *positive.*

The point 2) is proved in Corollary 6.10. For the first point we need just the following fact. If $\langle \mu_G, \beta\rangle^{-1}(0)$ is not empty, the vector bundle $\mathcal{N}^{\beta,+} \to Z$ is not trivial on each connected component Z of M^β which intersects $\mu_G^{-1}(\beta)$. The reason is the following. Consider the set of weights $\{\alpha_i,\ i \in I\}$ for the action of $\mathbb{T}_\beta$ on the fibres of the vector bundle $\mathcal{N} \to Z$. We have then the following description of the function $\langle \mu_G, \beta\rangle$ in the neighbourhood of Z. For $v \in \mathcal{N}_x$, with the decomposition $v = \oplus_i v_i$, we have if $|v|$ is small enough

$$\langle \mu_G, \beta\rangle(x,v) = |\beta|^2 - \frac{1}{2}\sum_{i\in I}\langle \alpha_i, \beta\rangle |v_i|^2 \ .$$

We know from Lemma 5.1 of [14] that the map $\langle \mu_G, \beta\rangle : M \to \mathbb{R}$ admits a *unique* local minimum. But if $\langle \alpha_i, \beta\rangle < 0$ for every $i \in I$, we have $\langle \mu_G, \beta\rangle \geq |\beta|^2$ in a neighbourhood of Z, so $|\beta|^2$ will be the unique local minimum of $\langle \mu_G, \beta\rangle$ on M; and it contradicts the fact that $\langle \mu_G, \beta\rangle$ vanishes on M. Hence $\langle \alpha_i, \beta\rangle > 0$ for some $i \in I$. □

The map RR^{G}_{β} with $G_\beta \neq G$

We consider the situation of $\beta \in \mathcal{B}_G$ with stabiliser $G_\beta \neq G$. Here the Riemann-Roch character is localised near $C^G_\beta = G(M^\beta \cap \mu_G^{-1}(\beta))$.

The symplectic slice at the point β is a symplectic (locally closed) submanifold $\mathcal{Y}_\beta$ of M with an induced G_β Hamiltonian action (see [16][Theorem 26.7], [21][Definition 3.1]). Here we take

$$\mathcal{Y}_\beta := \mu_G^{-1}\left(\{\xi \in \mathfrak{g}_\beta, |\xi - \beta| < \varepsilon\}\right) \ ,$$

with $\varepsilon > 0$ small enough, and

$$\mathcal{Z}_\beta := \left[\mathcal{Y}_\beta\right]^\beta \ .$$

Recall that a G-invariant open neighbourhood of C^{G}_β in M is isomorphic to $G\times_{G_\beta}\mathcal{Y}_\beta$ and $\mathcal{Z}_\beta = (G \times_{G_\beta} \mathcal{Y}_\beta)^\beta$ is an open neighbourhood of $M^\beta \cap \mu_G^{-1}(\beta)$ in M^β.

The open subset $\mathcal{Z}_\beta$ of M^β inherits a symplectic structure ω_β and an Hamiltonian action of the group G_β. The moment map $\mu_{G_\beta} : \mathcal{Z}_\beta \to \mathfrak{g}_\beta$ is the restriction of μ_G to $\mathcal{Z}_\beta$. A ω_β-compatible almost complex structure on $\mathcal{Z}_\beta$ (says J_β) defines a quantization map $RR^{G_\beta}(\mathcal{Z}_\beta, -) : K_{G_\beta}(\mathcal{Z}_\beta) \to R(G_\beta)$. With the moment map μ_{G_β} we define

$$RR^{G_\beta}_\beta(\mathcal{Z}_\beta, -) : K_{G_\beta}(\mathcal{Z}_\beta) \to R^{-\infty}(G_\beta)$$

which is the Riemann-Roch character localised near $\mu_{G_\beta}^{-1}(\beta) = M^\beta \cap \mu_G^{-1}(\beta)$ (see Definition 6.5).

Let $\widetilde{\mathcal{N}}$ be the normal bundle of $\mathcal{Z}_\beta$ in $\mathcal{Y}_\beta$. The subgroup $\mathbb{T}_\beta \hookrightarrow G_\beta$ generated by $\exp(t.\beta)$, $t \in \mathbb{R}$ acts linearly on the fibre of the complex vector bundle $\mathcal{N}$. Thus we associate the polarized complex G_β-vector bundles $\widetilde{\mathcal{N}}^{\beta,+}$ and $(\widetilde{\mathcal{N}} \otimes \mathbb{C})^{\beta,+}$. (see subsection 5).

The next theorem determines $RR_\beta^{G}(M,-)$ in terms of $RR_\beta^{G_\beta}(\mathcal{Z}_\beta,-)$.

Theorem 7.3. *For every $E\in K_G(M)$, we have*

$$RR_\beta^{G}(M,E)=\mathrm{Ind}_{G_\beta}^{G}\left(\widetilde{\Theta}(E)(g)\det{}_{\mathfrak{g}/\mathfrak{g}_\beta}^{\mathbb{C}}(1-g)\right)\quad\text{in}\quad \mathcal{C}^{-\infty}(G)^G\ ,$$

where $\widetilde{\Theta}(E)\in R^{-\infty}(G_\beta)$ is determined by

$$\widetilde{\Theta}(E)=(-1)^{r_{\widetilde{\mathcal{N}}}}\sum_{k\in\mathbb{N}}RR_\beta^{G_\beta}\left(\mathcal{Z}_\beta,E|_{\mathcal{Z}_\beta}\otimes\det\widetilde{\mathcal{N}}^{\beta,+}\otimes S^k((\widetilde{\mathcal{N}}\otimes\mathbb{C})^{\beta,+}\right)\ .$$

Here $r_{\mathcal{N}}\in\mathbb{N}$ is the locally constant function on $\mathcal{Z}_\beta$ equal to the complex rank of $\widetilde{\mathcal{N}}^{\beta,+}$.

Corollary 7.4. *If the vector bundle $E\to M$ is μ_G-positive we have*

$$\left[RR_\beta^{G}(M,E)\right]^G=0\ ,$$

for every $\beta\in\mathcal{B}_G$ with $G_\beta\neq G$.

Proof of Corollary : Using the holomorphic induction map $\mathrm{Hol}_{G_\beta}^{G}$ (see equation 9.54 in Appendix B), the first equality of Theorem 7.3 can be rewritten

$$RR_\beta^{G}(M,E)=\mathrm{Hol}_{G_\beta}^{G}\left(\widetilde{\Theta}(E)\right)\ .$$

The second equality of Theorem 7.3 and Corollary 9.5 shows that $\widetilde{\Theta}(E)$ $\sum_{\lambda\in\lambda_\beta^+}m_{\lambda,\beta}(E)\chi_\lambda^{G_\beta}$, $m_{\lambda,\beta}(E)\in\mathbb{Z}$, with

$$m_{\lambda,\beta}(E)\neq 0\implies\langle\lambda,\beta\rangle>0.$$

Like in the proof of Lemma 7.2, we use here the fact that $\widetilde{\mathcal{N}}^{\beta,+}$ is not equal to 0.

Finally Lemma 9.3 of Appendix B tells us that $\mathrm{Hol}_{G_\beta}^{G}(\widetilde{\Theta}(E))$ does not contains the trivial representation. □

Proof of Theorem 7.3 : We know from Proposition 6.9 and Corollary 6.12 that

$$RR_\beta^{G}(M,E)=\mathrm{Ind}_{G_\beta}^{G}\left(RR_\beta^{G_\beta}(M,E)(g)\det{}_{\mathfrak{g}/\mathfrak{g}_\beta}^{\mathbb{R}}(1-g)\right)\quad\text{in}\quad\mathcal{C}^{-\infty}(G)^G\ ,$$

and

$$RR_\beta^{G_\beta}(M,E)=(-1)^{r_{\mathcal{N}}}\sum_{k\in\mathbb{N}}RR_\beta^{G_\beta}\left(M^\beta,E|_{M^\beta}\otimes\det\mathcal{N}^{\beta,+}\otimes S^k((\mathcal{N}\otimes\mathbb{C})^{\beta,+}\right)\ ,$$

where $\mathcal{N}\to M^\beta$ is the normal bundle of M^β in M. The proof will be completed if we show that[12]

$$\begin{aligned}&\det{}_{\mathfrak{g}/\mathfrak{g}_\beta}^{\mathbb{C}}(1-g^{-1}).(-1)^{r_{\mathcal{N}}}\sum_{k\in\mathbb{N}}RR_\beta^{G_\beta}\left(M^\beta,E|_{M^\beta}\otimes\det\mathcal{N}^{\beta,+}\otimes S^k((\mathcal{N}\otimes\mathbb{C})^{\beta,+})\right)\\ =\ &(-1)^{\widetilde{\mathcal{N}}}\sum_{i\in\mathbb{N}}RR_\beta^{G_\beta}\left(\mathcal{Z}_\beta,E|_{\mathcal{Z}_\beta}\otimes\det\widetilde{\mathcal{N}}^{\beta,+}\otimes S^i((\widetilde{\mathcal{N}}\otimes\mathbb{C})^{\beta,+})\right).\end{aligned}$$

[12]Note that $\det{}_{\mathfrak{g}/\mathfrak{g}_\beta}^{\mathbb{R}}(1-g)=|\det{}_{\mathfrak{g}/\mathfrak{g}_\beta}^{\mathbb{C}}(1-g)|^2=\det{}_{\mathfrak{g}/\mathfrak{g}_\beta}^{\mathbb{C}}(1-g).\det{}_{\mathfrak{g}/\mathfrak{g}_\beta}^{\mathbb{C}}(1-g^{-1})$.

For any G_β-complex vector $W \to M^\beta$ the generalised character $RR_\beta^{G_\beta}(M^\beta, W)$ can be computed as the index of a G_β-transversally elliptic symbol with support in $\mathcal{Z}_\beta$. The excision Lemma (see subsection 3.1) tells us that

$$RR_\beta^{G_\beta}(M^\beta, W) = RR_\beta^{G_\beta}(\mathcal{Z}_\beta, W|_{\mathcal{Z}_\beta}) \ .$$

The symplectic slice $\mathcal{Y}_\beta$ has an induced symplectic two form $\omega_{\mathcal{Y}_\beta}$, whit a moment map $\mu_{\mathcal{Y}_\beta} : \mathcal{Y}_\beta \to \mathfrak{g}_\beta$ equal to the restriction of $\mu_{G_\beta} : M \to \mathfrak{g}_\beta$ to $\mathcal{Y}_\beta$. Let $J_{\mathcal{Y}_\beta}$ be a $\omega_{\mathcal{Y}_\beta}$-compatible almost complex structure on $\mathcal{Y}_\beta$.

The complex structure on the fibres of the vector bundle $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are induces respectively by the compatible almost complex structure J and $J_{\mathcal{Y}_\beta}$ on the symplectic manifold $G \times_{G_\beta} \mathcal{Y}_\beta$ and $\mathcal{Y}_\beta$.

The symplectic form ω, when restricted to on $G \times_{G_\beta} \mathcal{Y}_\beta$, can be written in terms of the moment map $\mu_{\mathcal{Y}_\beta}$ and the symplectic form $\omega_{\mathcal{Y}_\beta}$:

$$\omega_{[g,y]}(X + v, Y + w) = -(\mu_{\mathcal{Y}_\beta}(y), [X, Y]) + \omega_{\mathcal{Y}_\beta}|_y (v, w) \ , \tag{7.45}$$

where $X, Y \in \mathfrak{g}/\mathfrak{g}_\beta$, and $v, w \in \mathbf{T}_y\mathcal{Y}_\beta$ [13]. With the complex structure J_β on G/G_β determined by β, we form the almost complex structure $\widetilde{J} := J_\beta \times J_{\mathcal{Y}_\beta}$ on $G \times_{G_\beta} \mathcal{Y}_\beta$. Equation (7.45) shows that $\widetilde{J}$ is compatible with ω on $G \times_{G_\beta} \mathcal{Y}_\beta$, hence $\widetilde{J}$ is homotopic to J on $G \times_{G_\beta} \mathcal{Y}_\beta$. We see then that the computation of the localised Riemann-Roch character $RR_\beta^G(M, E)$ can be carried out with $\widetilde{J}$, and that we can take on the fibres of bundles $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ the complex structures induces respectively by the compatible almost complex structure $\widetilde{J}$ and $J_{\mathcal{Y}_\beta}$

Under this modification of the complex structures, we have on $\mathcal{Z}_\beta$ the following decomposition of the normal bundle $\mathcal{N} \to M^\beta$ in sum of two complex vector bundles:

$$\mathcal{N} = [\mathfrak{g}/\mathfrak{g}_\beta] \oplus \widetilde{\mathcal{N}} \ , \tag{7.46}$$

Here $[\mathfrak{g}/\mathfrak{g}_\beta] \to \mathcal{Z}_\beta$ is the trivial complex vector bundle isomorphic to $\mathfrak{g}/\mathfrak{g}_\beta \times \mathcal{Z}_\beta$: for any $m \in \mathcal{Z}_\beta$, $[\mathfrak{g}/\mathfrak{g}_\beta]_m = \{X_{\mathcal{Z}_\beta}|_m,\ m \in \mathfrak{g}/\mathfrak{g}_\beta\}$. The complex structures on the fibres of $[\mathfrak{g}/\mathfrak{g}_\beta]$ is defined by J_β, and the $\mathbb{T}_\beta$-weights on $[\mathfrak{g}/\mathfrak{g}_\beta]$ are all positive for β, that is $[\mathfrak{g}/\mathfrak{g}_\beta]^{\beta,+} = [\mathfrak{g}/\mathfrak{g}_\beta]$ and $([\mathfrak{g}/\mathfrak{g}_\beta] \otimes \mathbb{C})^{\beta,+} \cong [\mathfrak{g}/\mathfrak{g}_\beta]$.

Equality 7.46 gives $\mathcal{N}^{\beta,+} = [\mathfrak{g}/\mathfrak{g}_\beta] \oplus \widetilde{\mathcal{N}}^{\beta,+}$ and $(\mathcal{N}^{\beta,+} \otimes \mathbb{C}) = [\mathfrak{g}/\mathfrak{g}_\beta] \oplus (\widetilde{\mathcal{N}} \otimes \mathbb{C})^{\beta,+}$. We have now the following decomposition

$$\begin{aligned}
(-1)^{r_{\mathcal{N}}} \sum_{k \in \mathbb{N}} RR_\beta^{G_\beta}\left(\mathcal{Z}_\beta, E|_{\mathcal{Z}_\beta} \otimes \det \mathcal{N}^{\beta,+} \otimes S^k((\mathcal{N} \otimes \mathbb{C})^{\beta,+})\right) &= (-1)^{r_{\mathcal{N}}} \times \\
\sum_{i,j \in \mathbb{N}} RR_\beta^{G_\beta}\Big(\mathcal{Z}_\beta, E|_{\mathcal{Z}_\beta} \otimes \det \widetilde{\mathcal{N}}^{\beta,+} \otimes S^i((\widetilde{\mathcal{N}} \otimes \mathbb{C})^{\beta,+}) &\otimes \det[\mathfrak{g}/\mathfrak{g}_\beta] \otimes S^j([\mathfrak{g}/\mathfrak{g}_\beta])\Big) \\
= \quad A_{[\mathfrak{g}/\mathfrak{g}_\beta]}.(-1)^{\widetilde{\mathcal{N}}} \sum_{i \in \mathbb{N}} RR_\beta^{G_\beta}\Big(\mathcal{Z}_\beta, E|_{\mathcal{Z}_\beta} \otimes \det \widetilde{\mathcal{N}}^{\beta,+} &\otimes S^i((\widetilde{\mathcal{N}} \otimes \mathbb{C})^{\beta,+})\Big) \ ,
\end{aligned}$$

[13]We use here the identification $\mathbf{T}(G \times_{G_\beta} \mathcal{Y}_\beta) \cong G \times_{G_\beta} (\mathfrak{g}/\mathfrak{g}_\beta \oplus \mathbf{T}\mathcal{Y}_\beta)$ (see Eq. 3.11).

with $A_{[\mathfrak{g}/\mathfrak{g}_\beta]}(g) = (-1)^{dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{g}_\beta)} \det^{\mathbb{C}}_{\mathfrak{g}/\mathfrak{g}_\beta}(g). \sum_{j\in\mathbb{N}} \mathrm{Tr}_{S^j(\mathfrak{g}/\mathfrak{g}_\beta)}(g)$. But $A_{[\mathfrak{g}/\mathfrak{g}_\beta]}(g). \det^{\mathbb{C}}_{\mathfrak{g}/\mathfrak{g}_\beta}(1-g^{-1}) = 1$, hence

$$\begin{aligned} &\det{}^{\mathbb{C}}_{\mathfrak{g}/\mathfrak{g}_\beta}(1-g^{-1}).(-1)^{r_{\mathcal{N}}} \sum_{k\in\mathbb{N}} RR^{G_\beta}_\beta\left(\mathcal{Z}_\beta, E|_{\mathcal{Z}_\beta} \otimes \det\mathcal{N}^{\beta,+} \otimes S^k((\mathcal{N}\otimes\mathbb{C})^{\beta,+})\right) \\ &= (-1)^{\widetilde{\mathcal{N}}} \sum_{i\in\mathbb{N}} RR^{G_\beta}_\beta\left(\mathcal{Z}_\beta, E|_{\mathcal{Z}_\beta} \otimes \det\widetilde{\mathcal{N}}^{\beta,+} \otimes S^i((\widetilde{\mathcal{N}}\otimes\mathbb{C})^{\beta,+})\right). \end{aligned}$$

□

8. Appendix A: G=SU(2)

We restrict our attention to an action of $G = SU(2)$ on a compact manifold M. We suppose that M is endowed with a G-invariant almost complex structure J and a moment map $f : M \to \mathfrak{g}$. In this situation, the decomposition $RR^{G}(M,-) = \sum_{\beta\in\mathcal{B}_G} RR^{G}_\beta(M,-)$ become simple.

Let S^1 be the maximal torus of $SU(2)$, and $f_{S^1} : M \to \mathbb{R}$ the induced moment map for the S^1-action. The critical set $\{\mathcal{H}^{G} = 0\}$ has a particularly simple expression

$$\{\mathcal{H}^{G} = 0\} = f^{-1}(0) \cup \bigcup_{\substack{F\subset M^{S^1} \\ f_{S^1}(F)>0}} G.F \ .$$

Hence the set $\mathcal{B}_G$ is $\{f_{S^1}(F) > 0,\ F \in M^{S^1}\} \cup \{0\}$. Let $M^{S^1}_+$ be the union of the connected components $F \subset M^{S^1}$ with $f_{S^1}(F) > 0$

The non-symplectic case

Note that the critical set $\{\mathcal{H}^{S^1} = 0\}$ is equal to $f^{-1}_{S^1}(0) \cup M^{S^1}$, then the set $\mathcal{B}_{S^1}$ is $\{f(F), F \in M^{S^1}\} \cup \{0\}$. Here the induction formula of Theorem 6.11, and Proposition 6.9 gives

$$RR^{G}(M,E) = RR^{G}_0(M,E) + \mathrm{Ind}^{G}_{S^1}\left(\Theta(E)(t).\left|1-t^2\right|^2\right) \tag{8.47}$$

where $\Theta(E) \in R^{-\infty}(S^1)$ is determined by

$$\Theta_1(E) = (-1)^{r_{\mathcal{N}_1}} \sum_{k\in\mathbb{N}} RR^{S^1}(M^{S^1}_+, E|_{M^{S^1}_+} \otimes \det\mathcal{N}^+ \otimes S^k((\mathcal{N}\otimes\mathbb{C})^+)) \tag{8.48}$$

where $\mathcal{N} \to M^{S^1}_+$ is the normal bundle of $M^{S^1}_+$ in M.

The Hamiltonian case

Here we suppose that (M,ω) is a symplectic manifold, with moment map μ and ω-compatible almost complex structure J. Let $\mathcal{Y} = \mu^{-1}(\mathbb{R}_{>0})$ be the symplectic slice associated to the interior of the Weyl chamber $\mathbb{R}_{>0} \subset Lie(S^1)$.

The induction formula of Theorem 7.3 gives

$$RR^{G}(M,E) = RR^{G}_0(M,E) + \mathrm{Ind}^{G}_{S^1}\left(\widetilde{\Theta}(E)(t).(1-t^2)\right) \tag{8.49}$$

where $\widetilde{\Theta}(E) \in R^{-\infty}(S^1)$ is determined by

$$\widetilde{\Theta}(E) = (-1)^{r_{\widetilde{\mathcal{N}}}} \sum_{k\in\mathbb{N}} RR^{S^1}(M^{S^1}_+, E|_{M^{S^1}_+} \otimes \det\widetilde{\mathcal{N}}^+ \otimes S^k((\widetilde{\mathcal{N}}\otimes\mathbb{C})^+))\ , \tag{8.50}$$

where $\tilde{\mathcal{N}} \to M_+^{S^1}$ is the normal of $M_+^{S^1}$ in $\mathcal{Y}$.

Recall that the irreducible characters ϕ_n of $G = SU(2)$ are labelled by $\mathbb{Z}_{\geq 0}$, and are completely determined by the relation

$$\phi_n = \mathrm{Ind}_{S^1}^{G}\left(t^n.(1-t^2)\right) \quad \text{in} \quad R^{-\infty}(G)$$

(See Lemma 9.1). This explains the important differences between Equation 8.47 where the term $(1-t^2)(1-t^{-2})$ appears, and Equation 8.49 where we have 'only' $(1-t^2)$.

Hence the component $\mathrm{Ind}_{S^1}^{G}\left(\Theta(E)(t).\left|1-t^2\right|^2\right)$ of Equation (8.47) does not contains the trivial character ϕ_0 if $\Theta(E) \in R^{-\infty}(S^1)$ is of the form $\Theta(E) = \sum_{n\in\mathbb{Z}} a_n t^n$ with

$$a_n \neq 0 \Longrightarrow n \geq 3 \ . \tag{8.51}$$

Now, we look at Equation (8.47), and we notice that Equation (8.51) is satisfied if the weights for the action of S^1 in the fibre of the complex vector bundle $E|_{M_+^{S^1}} \otimes \det \mathcal{N}^+ \to M_+^{S^1}$ are all bigger than 3.

The conditions are weaker in the 'Hamiltonian' situation. The term $\mathrm{Ind}_{S^1}^{G}\left(\tilde{\Theta}(E)(t).(1-t^2)\right)$ of Equation (8.49) does not contains the trivial character ϕ_0 if $\Theta(E) \in R^{-\infty}(S^1)$ is of the form $\Theta(E) = \sum_{n\in\mathbb{Z}} a_n t^n$ with

$$a_n \neq 0 \Longrightarrow n \geq 1 \ , \tag{8.52}$$

and this condition is fulfilled if the weights for the action of S^1 in the fibre of the complex vector bundle $E|_{M_+^{S^1}} \otimes \det \tilde{\mathcal{N}}^+ \to M_+^{S^1}$ are all bigger than 1. But we have here another important difference (noticed in the Proof of Lemma 7.2): the vector bundle $\tilde{\mathcal{N}}^+ \to M_+^{S^1}$ is not equal to the zero bundle if $0 \in \mu(M)$.

We see finally that, in the Hamiltonian case, the condition '*E is μ-positive*' implies

$$0 \in \mu(M) \Longrightarrow \left[RR^{G}(M,E)\right]^G = \left[RR_0^{G}(M,E)\right]^G \ .$$

9. Appendix B: Induction map and multiplicities

Let G be a compact connected Lie group, with maximal torus H, and $\mathfrak{h}_+^* \subset \mathfrak{h}^* = (\mathfrak{g}^*)^H$ some choice of the positive Weyl chamber. We denote $\mathfrak{R}_+$ the associated system of positive roots, and we label the irreducible representations of G by the set $\Lambda_+^* = \Lambda^* \cap \mathfrak{h}_+^*$ of dominants weights. For any weights $\alpha \in \Lambda^*$ we denote $H \to \mathbb{C}^*$, $h \mapsto h^\alpha$ the corresponding character[14].

Let W be the Weyl group of (G,H), and $\mathrm{L}^2(H)$ be the vector space of square integrable complex functions on H. For $f \in \mathrm{L}^2(H)$, we consider following [7][Section 7.4]

$$J(f) = \sum_{w\in W} (-1)^w \ w.f \ ,$$

where $W \to \{1,-1\}$, $w \to (-1)^w$, is the signature operator and $w.f \in \mathrm{L}^2(H)$ is defined by $w.f(h) = f(w^{-1}.h)$, $h \in H$. The map $\frac{1}{|W|}J$ is the orthogonal projection from $\mathrm{L}^2(H)$ to the space of W-anti-invariant elements of $\mathrm{L}^2(H)$.

[14] For any $X \in \mathfrak{h}$, $(\exp(X))^\alpha = e^{\imath\langle\alpha,X\rangle}$.

Let $\rho \in \mathfrak{h}^*$ be the half sum of the positive roots. The function $H \to \mathbb{C}^*$, $h \mapsto h^\rho$ is well defined as an element of $L^2(H)$ (even if ρ is not a weight). The Weyl's character formula can be written in the following way. For any dominant weight $\lambda \in \Lambda^*_+$, the corresponding irreducible character χ_λ satisfies the relation[15]

$$J(h^\rho).\chi_\lambda|_H = J(h^{\lambda+\rho}) \quad \text{in} \quad L^2(H)\ . \tag{9.53}$$

For our purpose we give an expression of the character χ_λ through the induction map $\mathrm{Ind}_H^G : \mathcal{C}^{-\infty}(H) \to \mathcal{C}^{-\infty}(G)^G$ (see equation 3.12). Following [24] [Section 2], we consider the affine action of the weyl group on the set of weights : $w\circ\lambda = w.(\lambda+\rho)-\rho$ for $w \in W$ and $\lambda \in \Lambda^*$.

Lemma 9.1. *1) For any dominant weight $\lambda \in \Lambda^*_+$, the character χ_λ is determined by the relation*

$$\chi_\lambda = \mathrm{Ind}_H^G\Big(h^\lambda \prod_{\alpha\in\mathfrak{R}_+}(1-h^\alpha)\Big) \quad \text{in} \quad \mathcal{C}^{-\infty}(G)^G\ .$$

2) For $\lambda \in \Lambda^*$ and $w \in W$, we have $\mathrm{Ind}_H^G(h^{w\circ\lambda}\Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha)) = (-1)^w\mathrm{Ind}_H^G(t^\lambda\Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha))$.
3) For any weight λ, the following statements are equivalent :
a) $\mathrm{Ind}_H^G(h^\lambda\Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha)) = 0$,
b) $W\circ\lambda \cap \Lambda^*_+ = \emptyset$,
c) The element $\lambda+\rho$ is not a regular element of $\mathfrak{h}^*$.

Proof of 1) : This relation was already given in the Corollaire 4, section 7.4 of [7]. To prove it, we need the following relations in $L^2(H)$:
i) $\overline{J(h^\rho)} = h^{-\rho}\prod_{\alpha\in\mathfrak{R}_+}(1-h^\alpha)$, ii) $J(h^\rho).\overline{J(h^\rho)} = \prod_{\alpha\in\mathfrak{R}}(1-h^\alpha)$.

Let dg and dt be the normalized Haar measures on G and H respectively. For any $f \in \mathcal{C}^\infty(G)^G$ we have

$$\begin{aligned}
\int_G \chi_\lambda(g)\, f(g)\, dg &= \frac{1}{|W|}\int_H \chi_\lambda|_H(h)\, \Pi_{\alpha\in\mathfrak{R}}(1-h^\alpha)\, f|_H(h)\, dh && [1]\\
&= \frac{1}{|W|}\int_H J(h^{\lambda+\rho})\,\overline{J(h^\rho)}\, f|_H(h)\, dh && [2]\\
&= \int_H h^{\lambda+\rho}\,\overline{J(h^\rho)}\, f|_H(h)\, dh && [3]\\
&= \int_H h^\lambda\, \Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha)\, f|_H(h)\, dh\ . && [4]
\end{aligned}$$

The first equality is the Weyl integration formula. The equality [2] comes from ii) and (9.53). The fact that the map $\frac{1}{|W|}J$ is the orthogonal projection on $L^2(H)^{W-anti-invariant}$ implies the third equality, because the map $h \mapsto \overline{J(h^\rho)}\, f|_H(h)$ is W-anti-invariant. The equality [4] comes from i).

Proof of 2) : From $i)$, wee see that $h^{w\circ\lambda}\Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha) = h^{w(\lambda+\rho)}\overline{J(h^\rho)} = (-1)^w\, w^{-1}.(h^{\lambda+\rho}\overline{J(h^\rho)}) = (-1)^w\, w^{-1}.(h^\lambda\Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha))$, hence the relation 2) is proved.

Proof of 3) : The implication $a) \Longrightarrow b)$ is an immediate consequence of 1) and 2). Proposition 3, section 7.4 of [7] tells us that $\{J(h^{\lambda'+\rho}),\ \lambda' \in \Lambda^*_+\}$ is an orthogonal basis of the Hilbert space $L^2(H)^{W-anti-invariant}$. For $\lambda \in \Lambda^*$

[15] $\chi_\lambda|_H$ is the restriction of $\chi_\lambda \in \mathcal{C}^\infty(G)$ to H.

and $\lambda' \in \Lambda^*_+$ we have $< J(h^{\lambda+\rho}), J(h^{\lambda'+\rho}) >_{L^2} = |W| < J(h^{\lambda+\rho}), h^{\lambda'+\rho} >_{L^2} = |W| \sum_{w\in W}(-1)^w \int_T t^{w\circ\lambda-\lambda'} dt$. Thus, the condition $W\circ\lambda \cap \Lambda^*_+ = \emptyset$ is equivalent to $J(h^{\lambda+\rho}) = 0$. But the point 2) gives $\mathrm{Ind}^G_H(h^\lambda \Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha)) = \frac{1}{|W|}\mathrm{Ind}^G_H(J(h^{\lambda+\rho})h^{-\rho}\Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha))$, hence $J(h^{\lambda+\rho}) = 0$ implies the point $a)$. We have proved that $b) \Longrightarrow a)$. Finally we see that $J(h^{\lambda+\rho}) = 0 \Longleftrightarrow \exists w \in W, w.(\lambda+\rho) = \lambda+\rho \Longleftrightarrow \lambda+\rho$ is not a regular value of $\mathfrak{h}^*$. We have proved that $b) \Longleftrightarrow c)$. □

From the previous Lemma, we see that $v \mapsto \mathrm{Ind}^G_H(v(h)\Pi_{\alpha\in\mathfrak{R}_+}(1-h^\alpha))$ is the holomorphic induction functor $\mathrm{Hol}^G_H : R(H) \to R(G)$ (which is denote Ind^G_H in [24][Section 2]).

We arrive now to the principal result of this appendix. Let $\beta \in \mathfrak{h}$, $\beta \neq 0$, such that $\langle\alpha,\beta\rangle \geq 0$ for every $\alpha \in \mathfrak{R}_+$.

Lemma 9.2. *Let $A(h) = \sum_{\lambda\in\Lambda^*} m^A_\lambda h^\lambda$ be an element of $R^{-\infty}(H)$, and consider $B = \mathrm{Hol}^G_H(A)$ the induced element of $R^{-\infty}(G)$. Suppose that $m^A_\lambda \neq 0 \Longrightarrow \langle\lambda,\beta\rangle \geq \eta > 0$, then $B = \sum_{\lambda\in\Lambda^*_+} m^B_\lambda \chi_\lambda$ with $m^B_\lambda \neq 0 \Longrightarrow \| \lambda \| \geq \eta' > 0$, where the constant η' is equal $\frac{\eta}{\|\beta\|}$. In particular, B does not contains the trivial representation.*

Proof : Lemma 9.1 shows that $m^B_\lambda = \sum_{w\in W}(-1)^w m^A_{w\circ\lambda}$ for every $\lambda \in \Lambda^*_+$, hence $m^B_\lambda \neq 0$ only if $m^A_{w\circ\lambda} \neq 0$ for some $w \in W$. The condition $\langle w\circ\lambda,\beta\rangle \geq \eta$ can be rewritten $\langle w.\lambda,\beta\rangle \geq \eta + \langle\rho - w.\rho,\beta\rangle$. A small computation shows that $\rho - w.\rho = \sum_{\alpha>0, w^{-1}.\alpha<0}\alpha$, hence $\langle\rho - w.\rho,\beta\rangle \geq 0$. Finally the condition $m^B_\lambda \neq 0$ implies that $\langle w.\lambda,\beta\rangle \geq \eta$ for some $w \in W$, but $\| \lambda \| . \| \beta \| \geq \langle w.\lambda,\beta\rangle$, so the proof is completed. □

Consider now the stabiliser G_β of the non-zero element $\beta \in \mathfrak{h}_+$. The subgroup H is also a maximal torus of G_β. The Weyl group W_β of (G_β, H) is identified with $\{w \in W,\ w.\beta = \beta\}$. We consider a Weyl chamber $\mathfrak{h}^*_{\beta,+} \subset \mathfrak{h}^*$ for G_β that contains the Weyl chamber $\mathfrak{h}^*_+$ of G. The irreducible representations $\chi^{G_\beta}_\lambda$, $\lambda \in \Lambda^*_{\beta,+}$ of G_β are labelled by the set $\Lambda^*_{\beta,+} = \Lambda^* \cap \mathfrak{h}^*_{\beta,+}$ of dominant weights.

We have a unique 'holomorphic' induction map $\mathrm{Hol}^G_{G_\beta} : R(G_\beta) \to R(G)$ such that $\mathrm{Hol}^G_H = \mathrm{Hol}^G_{G_\beta} \circ \mathrm{Hol}^{G_\beta}_H$. This map is defined precisely by the equation[16]

$$\mathrm{Hol}^G_{G_\beta}(v) = \mathrm{Ind}^G_{G_\beta}\left(v(g)\det{}^{\mathbb{C}}_{\mathfrak{g}/\mathfrak{g}_\beta}(1-g)\right) \quad \text{in} \quad \mathcal{C}^{-\infty}(G)^G \ , \tag{9.54}$$

for every $v \in R(G_\beta)$.

Lemma 9.2 can rewritten in the case of the (extended) map $\mathrm{Hol}^G_{G_\beta} : R^{-\infty}(G_\beta) \to R^{-\infty}(G)$.

Lemma 9.3. *Let $A = \sum_{\lambda\in\Lambda^*_{\beta,+}} m^A_\lambda \chi^{G_\beta}_\lambda$ be an element of $R^{-\infty}(G_\beta)$, and consider $B = \mathrm{Hol}^G_{G_\beta}(A)$ the induced element of $R^{-\infty}(G)$. Suppose that $m^A_\lambda \neq 0 \Longrightarrow \langle\lambda,\beta\rangle \geq \eta > 0$, then B has a decomposition $B = \sum_{\lambda\in\Lambda^*_+} m^B_\lambda \chi_\lambda$ with $m^B_\lambda \neq 0 \Longrightarrow \| \lambda \| \geq \eta' > 0$, where η' is equal $\frac{\eta}{\|\beta\|}$. In particular, B does not contains the trivial representation.*

[16] We take on $\mathfrak{g}/\mathfrak{g}_\beta$ the complex structure defined by β.

Proof : Lemma 9.1 says that $\chi_\lambda^{G_\beta} = \mathrm{Hol}_H^{G_\beta}(h^\lambda)$ for every $\lambda \in \Lambda^*_{\beta,+}$, then $B = \mathrm{Hol}_{G_\beta}^{G}(A)$ is equal to $\mathrm{Hol}_H^G(\sum_{\lambda\in\Lambda^*_{\beta,+}} m^A_\lambda\ h^\lambda)$ and we conclude with Lemma 9.2. □

We finish now this appendix with some general remark about P-transversally elliptic on a compact manifold M, when a subgroup $\mathbb{T}$ in the centre of P acts trivially on M.

More precisely, let H be a compact maximal torus in P, $\mathfrak{h}_+$ be a choice of a positive Weyl chamber in the Lie algebra $\mathfrak{h}$ of H, and let $\beta \in \mathfrak{h}_+$ be a non-zero element in the centre of Lie algebra $\mathfrak{p}$[17]. We suppose here that the subtorus $\mathbb{T} \subset H$, which is equal to the closure of $\{\exp(t.\beta),\ t \in \mathbb{R}\}$, acts trivially on M.

Every P-equivariant complex vector bundle $E \to M$ can be decomposed relatively to the $\mathbb{T}$-action: $E = \oplus_{a\in\hat{\mathbb{T}}} E^a \otimes \mathbb{C}_a$, where $E^a := \hom_{\mathbb{T}}(E, \mathbb{C}^*_a)$[18] is a P-complex vector bundle with a trivial action of $\mathbb{T}$. Then, each P-equivariant symbol $\sigma : p^*(E_1) \to p^*(E_2)$ where E_1, E_2 are P-equivariant complex vector bundles over M, and where $p : \mathbf{T}M \to M$ is the canonical projection, admits the finite $P \times \mathbb{T}$-equivariant decomposition

$$\sigma = \sum_{a\in\hat{\mathbb{T}}} \sigma^a \otimes \mathbb{C}_a . \tag{9.55}$$

Here $\sigma^a : p^*(E^a_1) \to p^*(E^a_2)$ is a P-equivariant symbol, trivial for the $\mathbb{T}$-action.

Let us consider the inclusion map $i : \mathbb{T} \hookrightarrow H$, with the induced maps $i : Lie(\mathbb{T}) \to \mathfrak{h}$ at the level of Lie algebra and $i^* : \mathfrak{h}^* \to Lie(\mathbb{T})^*$. Note that $i^*(\lambda)$ is a weight for $\mathbb{T}$ if λ is a weight for H.

Lemma 9.4. *Let M be a P-manifold with the same properties as above. Let $\sigma : p^*(E_1) \to p^*(E_2)$ be a P-equivariant transversally elliptic symbol over M and denote $m_\lambda(\sigma)$, $\lambda \in \Lambda^*_{P,+}$, the multiplicities of its index :* $\mathrm{Index}^P_M(\sigma) = \sum_{\lambda\in\Lambda^*_{P,+}} m_\lambda(\sigma)\chi^P_\lambda$. *Then, if $m_\lambda(\sigma) \neq 0$, the weight $a = i^*(\lambda)$ occurs in the decomposition (9.55).*

Corollary 9.5. *Suppose that the weights $a \in \hat{\mathbb{T}}$ which occur in the decomposition (9.55) satisfy $\langle a, \beta\rangle \geq \eta > 0$. Then, for the multiplicities, we get*

$$m_\lambda(\sigma) \neq 0 \Longrightarrow \langle\lambda, \beta\rangle \geq \eta\ .$$

In particular, $\mathrm{Index}^P_M(\sigma)$ *does not contains the trivial representation.*

Remark 9.6. *The previous Lemma and Corollary remain true if M is a P-invariant open subset of a compact P-manifold.*

For the Corollary, we have just to notice that [19] $\langle\lambda, \beta\rangle = \langle a, \beta\rangle$ for $a = i^*(\lambda)$. Then, if we have $\langle a, \beta\rangle \geq \eta > 0$ for all $\mathbb{T}$-weights occurring in σ, we get $\langle\lambda, \beta\rangle \geq \eta$ for every λ such that $m_\lambda(\sigma) \neq 0$.

Proof of the Lemma: Let P' be a Lie subgroup of P such that $r : \mathbb{T} \times P' \to P$, $r(t, g) = t.g$, is a finite cover of P. The map r induces $r^* : K_P(\mathbf{T}_P M) \to K_{\mathbb{T}\times P'}(\mathbf{T}_{P'} M)$[20] and an injective map $r^* : R^{-\infty}(P) \to R^{-\infty}(\mathbb{T} \times P')$, such that $\mathrm{Index}^{\mathbb{T}\times P'}_M(r^*\sigma) = r^*(\mathrm{Index}^P_M(\sigma))$.

[17]The Lie group P is supposed connected then $\beta \in (\mathfrak{p})^P$.

[18]The torus $\mathbb{T}$ acts on the complex line $\mathbb{C}_a$ with the representation $t \to t^a$.

[19]We use the same notations for $\beta \in Lie(\mathbb{T})$ and $i(\beta) \in \mathfrak{h}$.

[20]Note that $\mathbf{T}_{P'} M = \mathbf{T}_P M$ because $\mathbb{T}$ acts trivially on M.

The decomposition (9.55) can be read through the identification $K_{\mathbb{T}\times P'}(\mathbf{T}_{P'}M) = K_{P'}(\mathbf{T}_{P'}M)\otimes R(\mathbb{T})$: we have $r^*\sigma = \sum_{a\in\hat{\mathbb{T}}}\sigma^a\otimes\mathbb{C}_a$ with $\sigma^a\in K_{P'}(\mathbf{T}_{P'}M)$. Hence

$$\text{(9.56)}\qquad \mathrm{Index}_M^{\mathbb{T}\times P'}(r^*\sigma)(t,g) = \sum_{a\in\hat{\mathbb{T}}}\mathrm{Index}_M^{P'}(\sigma^a)(g).\,t^a\ ,\quad (t,g)\in\mathbb{T}\times P'\ .$$

The irreducible characters χ_λ^P satisfy $r^*\chi_\lambda^P(t,g) = \chi_\lambda^P|_{P'}(g).\,t^{i^*(\lambda)}$. If we start from the decomposition $\mathrm{Index}_M^P(\sigma) = \sum_{\lambda\in\Lambda^*_{P,+}} m_\lambda(\sigma)\chi_\lambda^P$ relative to the irreducible characters of P, we get

(9.57)

$$r^*\left(\mathrm{Index}_M^{\mathbb{T}\times P'}(\sigma)\right)(t,g) = \sum_{a\in\hat{\mathbb{T}}}\left(\sum_{i^*(\lambda)=a} m_\lambda(\sigma)\chi_\lambda^P|_{P'}(g)\right).\,t^a\ ,\quad (t,g)\in\mathbb{T}\times P'\ .$$

If we compare Equations 9.56 and 9.57, we get $\mathrm{Index}_M^{P'}(\sigma^a) = \sum_{i^*(\lambda)=a} m_\lambda(\sigma)\chi_\lambda^P|_{P'}$. The map $r^* : R^{-\infty}(P)\to R^{-\infty}(\mathbb{T}\times P')$ is injective, so $\sum_{i^*(\lambda)=a} m_\lambda(\sigma)\chi_\lambda^P|_{P'} = 0$ if and only if $m_\lambda(\sigma) = 0$ for every λ satisfying $i^*(\lambda) = a$. Hence if the multiplicity $m_\lambda(\sigma)$ is non zero, the element $a = i^*(\lambda)$ is a weight for the action of $\mathbb{T}$ on $\sigma : p^*(E_1)\to p^*(E_2)$. $\square$

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