Index growth of hypersurfaces with constant mean curvature

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Abstract. In this paper, we give estimates for the index growth of complete noncompact nonminimal constant mean curvature (cmc) hypersurfaces. In particular, we show that complete, properly embedded finite topology surfaces with constant mean curvature in Euclidean 3-space have linear index growth, and we find an explicit asymptotic growth rate in terms of geometric properties of the ends. We give similar results for cmc hypersurfaces in $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$ as well.

Résumé. Dans cet article, nous donnons des estimées de la croissance de l'indice des hypersurfaces complètes, non compactes, à courbure moyenne constante (cmc), non minimales. Nous montrons en particulier que l'indice des surfaces complètes, proprement plongées, de topologie finie, de l'espace euclidien, croît linéairement. Le comportement asymptotique de l'indice dépend de certaines caractéristiques des bouts. Nous donnons des résultats analogues pour les hypersurfaces cmc dans $\mathbb{R}^{n+1}$ et dans $\mathbb{H}^{n+1}$.

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1 Introduction

In this paper, we investigate the index of complete constant mean curvature (cmc) nonminimal hypersurfaces, *i.e.* the number of negative eigenvalues of the stability operator, with respect to Dirichlet boundary conditions (see (2.2), Section 2). When the hypersurface is noncompact, the index is defined as the supremum of the indexes of all bounded regions in the hypersurface.

A complete constant mean curvature nonminimal surface without boundary in $\mathbb{R}^3$ has finite index if and only if it is compact [LR], [S]. If the surface is noncompact, the index is infinite, so it is natural to ask at what rate the index grows to infinity on an exhaustion of the surface by bounded regions. In this paper, we prove, under some natural geometric conditions, that complete non-compact cmc embedded hypersurfaces have linear index growth. With $B(R)$ defined to be the ball of radius R in $\mathbb{R}^3$ centered at the origin, a typical statement is

Theorem 1.1 *Let $M \subset \mathbb{R}^3$ be a complete properly embedded finite-topology cmc-1 surface. Let $E_j, j = 1 \dots N$, be the ends of M. They are asymptotic to Delaunay surfaces $\mathcal{D}(\mu_j)$, with weight parameter $\mu_j > 0$. If $T(\mu_j)$ denotes the period of the Delaunay surface $\mathcal{D}(\mu_j)$, then*

$$\lim_{R\to\infty} \frac{Ind(M \cap B(R))}{R} = \sum_{j=1}^{N} \frac{2}{T(\mu_j)}. \tag{1.1}$$

Using the fact that $2 \le T(\mu_j) \le \pi$ (see Section 3.1), we can conclude from the preceding theorem that the index growth provides upper and lower bounds on the number of ends of the surface.

There are many known surfaces to which this theorem can by applied. Complete noncompact properly embedded finite-topology cmc-1 surfaces have been constructed by N. Kapouleas [K], R. Mazzeo et al. [MP] [MPP], and K. Grosse-Brauckmann et al. [GKS]. And there are presently other works in progress for constructing such surfaces (*e.g.* that of J. Dorfmeister, N. Schmitt and M. Kilian). Furthermore, the structure of such surfaces is well understood [KK], [KKS].

The rough idea of the proofs of our results is to decompose the hypersurfaces into components, one which is the central part and the others which are compact pieces of ends and are close to Delaunay surfaces, and then to apply Dirichlet-Neumann bracketing. The fact that the ends are asymptotic to Delaunay hypersurfaces follows from [KKS], [KKMS], and we will see that the indexes of these end pieces are close to the indexes of the actual Delaunay pieces. To complete the proof, we must understand the indexes of the Delaunay pieces (with both Dirichlet and Neumann boundary conditions), and we do this by the following procedure: we find bounds for the potential function of the stability operator on Delaunay hypersurfaces; we then show that the spectrum of the stability operator is the disjoint union of the spectra of a sequence of operators defined on functions of only one real variable (since the domains of the functions are 1-dimensional, these operators are easier to handle); and then finally we study the possible range of those spectra, using the fact that cmc graphs are stable. This leads us to a knowledge of the indexes of the Delaunay pieces.

An interesting byproduct of these proofs in an exact knowledge of the index of any Delaunay piece with boundary in two hyperplanes perpendicular to the axis of the Delaunay piece. We will see in Section 4.3 that the index of such a Delaunay piece is essentially equal to the number of bulges and necks it contains.

Remark. One may ask whether the assumptions in the above theorem are necessary in order to insure that the index grows linearly. As a matter of fact, we also prove quadratic growth for the index of some embedded Kapouleas examples with infinite topology.

The paper is organized as follows. In Section 2, we recall the definition of the (Morse) index of a cmc hypersurface. In Section 3 we recall the basic facts on Delaunay hypersurfaces (in Euclidean or in hyperbolic spaces) and we introduce some special domains on Delaunay hypersurfaces. Section 4 is devoted to estimating the index of these special domains. Section 5 is the core of this paper. The main results are stated in Subsection 5.4. The other subsections contain technical results needed in the proofs.

2 Definition of Index

Here we consider nonminimal cmc-H hypersurfaces M^n of an $(n+1)$-dimensional space form $\overline{M}^{n+1}$ with constant sectional curvature $c \in \{-1, 0\}$. We consider both cases with boundary and without boundary. Let $x : M^n \to \overline{M}^{n+1}$ be a cmc-H immersion of a compact manifold M^n. Since the immersion is of nonzero cmc, the mean curvature vector provides an orientation, so we let N denote an associated unit normal vector on M. Consider a smooth variation M_t of $M (= M_0)$ such that the boundary is fixed and call W the variation vector field, $W = \frac{\partial}{\partial t} M_t\big|_{t=0}$. The first variation formula for the area is given by

$$A'(0) = -\, nH \int_M \langle W, N \rangle \, dv_M \ ,$$

where H is the mean curvature in the direction N (it is normalized in such a way that the mean curvature of a standard sphere is 1) and where dv_M is the measure for the induced metric on M. Changing the variation M_t if necessary, we may assume that W is of the form $W = u\,N$ for some $u \in C^\infty(M)$ with $u|_{\partial M} = 0$. Indeed, for every such u, there exists a smooth variation M_t with variation vector field $u\,N$ ([BdCE], Lemma 2.2).

For volume-preserving variations (as defined in [BdCE]), *i.e.* when the function u satisfies the additional condition $\int_M u \, dv_M = 0$, the second variation formula is

$$A''(0) = \int_M u \, Lu \, dv_M \, ,$$

where

(2.2) $$L := \Delta - nc - ||B||^2$$

is the *stability operator* of the immersion (here, $||B||$ is the norm of the second fundamental form and Δ is the Laplace-Beltrami operator determined by the metric). Note that for any u such that $\int_M u \, dv_M = 0$, there exists a volume preserving variation with variation vector field $u\,N$ [BdCE].
In the case that $\overline{M} = \mathbb{R}^3$, the stability operator can also be written

(2.3) $$L = \Delta + 2K - 4H^2 \, ,$$

where K is the Gauss curvature.

Remark. The sign of Δ is chosen so that its eigenvalues are nonnegative. (For example, $\Delta = -\frac{\partial^2}{\partial x^2}$ on Euclidean $\mathbb{R}^1$.)

Definition 2.1 *For M compact, the* index *Ind(M) is the number of negative eigenvalues of L, with respect to Dirichlet boundary conditions,* i.e. *the maximum possible dimension of a subspace V of $C_0^\infty(M)$ on which $\int_M u \, Lu \, dv_M < 0$. For M non-compact, Ind(M) is the supremum of Ind(Ω) over all compact regions $\Omega \subset M$.*

This index is usually called the *strong index* in the literature. There is also a notion of *weak index*, which is more geometrically natural. Loosely speaking, the weak index measures the maximum possible dimension of a "vector space" of variations that are *volume-preserving* and area-decreasing. Thus it measures the degree to which the hypersurface is unstable in the cmc sense [BdCE].

Definition 2.2 *For M compact, the* weak index $Ind_w(M)$ *is the maximum possible dimension of a subspace V of $C_0^\infty(M) \cap \{u \mid \int_M u\, dv_M = 0\}$ on which $\int_M u\, Lu\, dv_M < 0$ (the variations under consideration have variation vector field $u\,N$, they are volume-preserving and strictly area decreasing). For M non-compact, $Ind_w(M)$ is the supremum of $Ind_w(\Omega)$ over all compact regions $\Omega \subset M$.*

As discussed in [BB] and [LiRo], it is known that either $\mathrm{Ind}(M) = \mathrm{Ind}_w(M)$ or $\mathrm{Ind}(M) = \mathrm{Ind}_w(M) + 1$. (Examples in [BB] show that both cases are possible.) As the more geometrically natural index, $\mathrm{Ind}_w(M)$ is the index we wish to estimate. However, $\mathrm{Ind}(M)$ is easier to compute, and since the two indices differ by at most 1, estimating the asymptotic behavior of one is the same as estimating the other.

In order to estimate the index growth, we will have to consider the number of negative eigenvalues of the stability operator L on compact domains with Dirichlet, Neumann, or with mixed boundary conditions.

Definition 2.3 *For M compact, the* Dirichlet index *(*resp. Neumann index*) of M is the number of negative eigenvalues of the stability operator L on M, with Dirichlet (*resp. *with Neumann) boundary conditions on ∂M.* Mixed index *will refer to the number of negative eigenvalues of L with Dirichlet condition on part of ∂M and Neumann condition on the complementary part of ∂M.*

3 Delaunay hypersurfaces

In this section, we give some description of Delaunay hypersurfaces with constant nonzero mean curvature in Euclidean and hyperbolic spaces.

3.1 Delaunay hypersurfaces in Euclidean space, with constant mean curvature $H > 0$

Let us consider a rotation hypersurface M in $\mathbb{R}^{n+1}$ parametrized by

$$\mathbb{R} \times S^{n-1} \ni (x,\omega) \to F(x,\omega) = \big(x, f(x)\,\omega\big) \tag{3.4}$$

where f is assumed to be positive and defined over $(-\infty,\infty)$. Notice that $f(x)$ measures the Euclidean distance from the point $F(x,\omega)$ on M to the axis of revolution. We choose the unit normal vector to be

$$N(x,\omega) = \big(1 + f'^2(x)\big)^{-1/2}\,\big(f'(x), -\omega\big). \tag{3.5}$$

The Riemannian metric is given in the above parametrization by the matrix

$$G(x,\omega) = \begin{pmatrix} 1 + f'^2(x) & 0 \\ 0 & f^2(x)\,g_S \end{pmatrix} \tag{3.6}$$

where g_S is the canonical metric on the unit sphere S^{n-1}.

Provided that $f' \not\equiv 0$, the profile curves of Delaunay hypersurfaces are given by the following differential equation

$$(3.7) \qquad \frac{m}{\omega_{n-1}} = \frac{f^{n-1}(x)}{(1+f'^2(x))^{1/2}} - Hf^n(x)\,,$$

where H is the normalized mean curvature (nonzero, constant), ω_{n-1} is the volume of S^{n-1} and m is a real parameter. The constant m can be interpreted as the flux of the unit vector field parallel to the axis of revolution, as in dimension 2 [KKS]. (We remark that the definition of H in [KKS] is n times our definition for H, hence the term to the right in our formula differs by a factor of n from the right-most term in the formula of [KKS].)

Let us, for notational convenience, fix $H = 1$ and introduce the weight parameter $\mu := m/\omega_{n-1}$. Equation (3.7) becomes

$$(3.8) \qquad \mu = \frac{f^{n-1}(x)}{(1+f'^2(x))^{1/2}} - f^n(x)\,.$$

It can be shown that the curves corresponding to equation (3.8) are embedded if and only if $\mu > 0$. In that case, we must have $\mu \in (0, \frac{1}{n}(\frac{n-1}{n})^{n-1}]$ and the extreme values correspond to a chain of spherical beads of radii 1 (when $\mu = 0$), and to a cylinder with radius $\frac{n-1}{n}$ (when $\mu = \frac{1}{n}(\frac{n-1}{n})^{n-1}$).

Given $\mu \in (0, \frac{1}{n}(\frac{n-1}{n})^{n-1}]$, let $a_\pm(\mu)$ be the two positive roots of the equation $X^n - X^{n-1} + \mu = 0$ with $a_-(\mu) \le a_+(\mu)$.

Let $\mathcal{D}(\mu)$ be the Delaunay hypersurface with constant mean curvature 1 and weight parameter $\mu \in (0, \frac{1}{n}(\frac{n-1}{n})^{n-1}]$, whose profile curve f satisfies Equation (3.8). One can show that the function f is defined over $\mathbb{R}$, pinched between the two positive values $a_\pm(\mu)$,

$$(3.9) \qquad a_-(\mu) \le f(x) \le a_+(\mu),$$

and $T(\mu)$-periodic, where $T(\mu)$ is the distance between two consecutive values of x at which f achieves its least value $a_-(\mu)$. (For a true cylinder, $\mu = \frac{1}{n}(\frac{n-1}{n})^{n-1}$ and $f(x) = a_-(\mu) = a_+(\mu) = \frac{n-1}{n}$ is constant. In that case, $T(\mu)$ has some finite positive limiting value as μ increases up to $\frac{1}{n}(\frac{n-1}{n})^{n-1}$.)

In dimension 2, it is a well known fact [E] that the profile curves of embedded Delaunay surfaces are traces of one of the focuses of an ellipse as it is rolled along a line in $\mathbb{R}^2$. These ellipses are congruent to

$$(\frac{1}{2}\cos(s), \sqrt{\mu}\sin(s))\,, \quad s \in [0, 2\pi)$$

and they have eccentricity $e(\mu) = \sqrt{1-4\mu}$. The length of this ellipse is equal to $T(\mu)$, hence we have the formula

$$T(\mu) = \int_0^{2\pi} \sqrt{\frac{1}{4}\sin^2(s) + \mu\cos^2(s)}\, ds \;\in [2, \pi]$$

in the 2-dimensional case.

The stability operator of the n-dimensional Euclidean Delaunay hypersurface $\mathcal{D}(\mu)$ is given by

$$(3.10) \qquad \begin{cases} L = \Delta - V, \\ V = \|B\|^2 = n\left(1 + (n-1)\mu^2 f^{-2n}\right)\,. \end{cases}$$

where $\|B\|$ is the norm of the second fundamental form. The following lemma will be needed to estimate the index of certain Delaunay pieces.

Lemma 3.1 *For $n \geq 2$ and for any $x \in \mathbb{R}$ the function V in equation (3.10) satisfies $V(x)\, f^2(x) \leq n^2$.*

Proof. We have already seen that the weight parameter μ of $\mathcal{D}(\mu)$ satisfies $0 < \mu \leq \frac{1}{n}(\frac{n-1}{n})^{n-1}$. Consider the polynomial $P(t) = t^n - t^{n-1} + \mu$, whose positive roots are the numbers $a_\pm(\mu)$. The function $P(t)$, considered on the domain $\mathbb{R}_+$, achieves its non-positive minimum $\mu - \frac{1}{n}(\frac{n-1}{n})^{n-1}$ at $t = \frac{n-1}{n}$. Since $P(\mu^{1/(n-1)})$ and $P(1)$ are both positive, it follows that

$$\mu^{1/(n-1)} \leq a_-(\mu) \leq \frac{n-1}{n} \leq a_+(\mu) \leq 1\,.$$

Consider the function $Q(t) = nt^2(1+(n-1)\mu^2 t^{-2n})$, for $t > 0$. When t varies from 0 to ∞, Q decreases from ∞ to its minimum $Q((n-1)^{\frac{1}{n}}\mu^{\frac{1}{n}}) \geq 0$ and then increases to ∞. It follows immediately that, for all $x \in \mathbb{R}$,

$$(Vf^2)(x) \leq \max\{Q(a_-), Q(a_+)\} \leq \max\{Q(\mu^{1/(n-1)}), Q(1)\}\,.$$

Using the fact that $\mu \leq \frac{1}{n}(\frac{n-1}{n})^{n-1}$, it follows that $Vf^2 \leq n^2$ on $\mathbb{R}$ as claimed. ∎

3.2 Special parts of Euclidean Delaunay hypersurfaces

Take a Delaunay surface $\mathcal{D}(\mu)$. Without loss of generality, we may assume that the function f defining the profile curve satisfies $f(0) = a_-(\mu)$. It follows easily that $f(T(\mu)) = a_-(\mu)$, $f(T(\mu)/2) = a_+(\mu)$ and that f is symmetric with respect to the values $kT(\mu)/2$, $k \in \mathbb{Z}$.

Let the *basic Dirichlet block* for the Euclidean Delaunay hypersurface $\mathcal{D}(\mu)$ be the compact domain

(3.11) $$\mathcal{B}(\mu) := F([0, \frac{T(\mu)}{2}] \times S^{n-1}) \;\text{ or }\; F([\frac{T(\mu)}{2}, T(\mu)] \times S^{n-1})\,,$$

where F is the parametrization (3.4), see Figure 1.

We also introduce the pieces $\mathcal{B}_\ell(\mu)$ obtained by glueing ℓ basic Dirichlet blocks,

(3.12) $$\mathcal{B}_\ell(\mu) := F([0, \ell\,\frac{T(\mu)}{2}] \times S^{n-1}) \;\text{ or }\; F([\frac{T(\mu)}{2}, \frac{T(\mu)}{2} + \ell\,\frac{T(\mu)}{2}] \times S^{n-1})\,.$$

Let a be the function

(3.13) $$a(x) = (1 + f'^2(x))^{-1/2}\, f'(x)\,,$$

where f is as above.

Lemma 3.2 *The function a vanishes precisely at the half-integer multiples of $T(\mu)$. Furthermore, a' has exactly two zeroes $\zeta_1(\mu), \zeta_2(\mu)$ in the interval $[0, T(\mu)]$, with*

$$0 < \zeta_1(\mu) < \frac{T(\mu)}{2} < \zeta_2(\mu) < T(\mu).$$

(In the case of a true cylinder, $a(x)$ is constant, so then the values $\zeta_1(\frac{1}{n}(\frac{n-1}{n})^{n-1})$ and $\zeta_2(\frac{1}{n}(\frac{n-1}{n})^{n-1})$ must be determined by the limits of $\zeta_1(\mu)$ and $\zeta_2(\mu)$ as μ increases to $\frac{1}{n}(\frac{n-1}{n})^{n-1}$.)

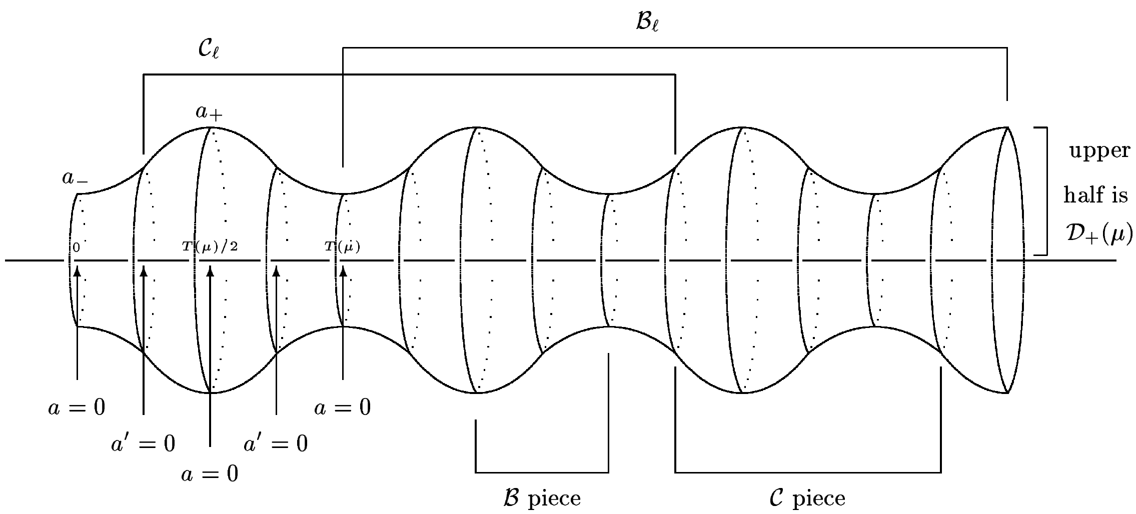

Figure 1: A portion of $\mathcal{D}(\mu) \subset \mathbb{R}^3$, $\mu > 0$.

The proof of this Lemma will be given in Section 4.3.

Let the *basic Neumann block* for the Euclidean Delaunay hypersurface $\mathcal{D}(\mu)$ be the compact domain

$$\mathcal{C}(\mu) := F([\zeta_1(\mu), T(\mu) + \zeta_1(\mu)] \times S^{n-1}) . \tag{3.14}$$

We also introduce the pieces $\mathcal{C}_\ell(\mu)$ obtained by glueing ℓ basic Neumann blocks, see Figure 1,

$$\mathcal{C}_\ell(\mu) := F([\zeta_1(\mu), \ell T(\mu) + \zeta_1(\mu)] \times S^{n-1}) . \tag{3.15}$$

3.3 Delaunay hypersurfaces in hyperbolic space, with constant mean curvature $H > 1$

We choose the half-space model $\{(x_1, \dots, x_n, y) \in \mathbb{R}^{n+1} \mid y > 0\}$ for hyperbolic space $\mathbb{H}^{n+1}$ (with the hyperbolic space metric), and we fix the geodesic $\gamma(t) = (0, \dots, 0, e^t)$.

The profile curve of a hyperbolic Delaunay hypersurface is described, say in the vertical 2-dimensional plane $\{x_1, y\}$, as a geodesic graph. The point $m(t)$ on the profile curve is at geodesic distance $\rho(t)$ from the point $\gamma(t)$. Let $\varphi(t)$ be the angle $\angle(\gamma(t)\,\vec{0}\,m(t))$, see Figure 2. Then, $\sinh \rho(t) = \tan \varphi(t)$.

With these notations, the profile curve is given by $\big(e^t \sin \varphi(t), e^t \cos \varphi(t)\big)$, where φ satisfies the differential equation ([KKMS], Equation (6.3) page 34)

$$\mu := \frac{m}{\omega_{n-1}} = (\tan \varphi)^{n-1} \frac{1}{\cos \varphi \sqrt{1 + \varphi'^2}} - H(\tan \varphi)^n . \tag{3.16}$$

Here, $\mu > 0$ (is the weight parameter, m is the flux) and the (normalized) mean curvature H satisfies $H > 1$. Note that the mean curvature is not normalized in [KKMS].

The hyperbolic Delaunay hypersurfaces $\mathcal{D}(\mu)$ are given by

$$\mathbb{R} \times S^{n-1} \ni (t, \omega) \overset{\Phi}{\to} (e^t \sin \varphi(t)\, \omega, e^t \cos \varphi(t)) =: (f(t)\, \omega, g(t)) \in \mathbb{H}^{n+1} , \tag{3.17}$$

where φ satisfies the differential equation (3.16).

As in the case of the Euclidean Delaunay hypersurfaces, it can be shown that the function φ (or equivalently ρ) is pinched between two values $0 < \alpha_-(\mu) \le \varphi(t) \le \alpha_+(\mu)$ and periodic with period $\tau(\mu)$. The Delaunay hypersurfaces obtained in this way with $\mu > 0$ are embedded.

A unit normal vector to the hypersurface $\mathcal{D}(\mu)$ is given (with the above notations) by

$$N(t, \omega) = \frac{\cos \varphi}{\sqrt{1 + \varphi'^2}} (g'\, \omega, -f') . \tag{3.18}$$

The metric on $\mathcal{D}(\mu)$ is given by

$$G(t, \omega) = \begin{pmatrix} (1 + \varphi'^2)(1 + \tan^2 \varphi) & 0 \\ 0 & \tan^2 \varphi\, g_S \end{pmatrix} \tag{3.19}$$

where g_S is the canonical metric on the unit $(n-1)$-sphere as above.

Finally, the stability operator L is of the form

$$\begin{cases} L = \Delta - V, \\ V = -n + \|B\|^2, \end{cases} \tag{3.20}$$

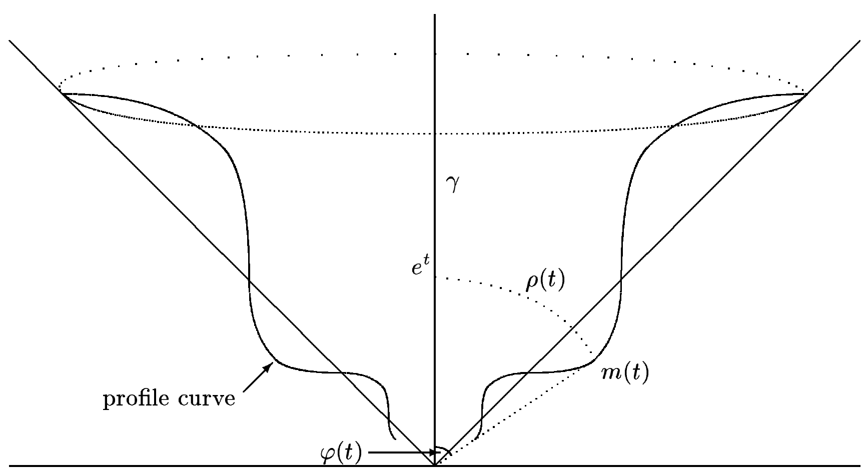

Figure 2: A portion of $\mathcal{D}_H(\mu) \subset \mathbb{H}^3$, $\mu > 0$, $H > 1$.

where $\|B\|$ is the norm of the second fundamental form of the immersion. The important point here is that V is periodic and hence bounded on $\mathbb{R}$, as in the Euclidean case. It turns out that one can find a nice expression for the function V, as in the Euclidean case.

Lemma 3.3 *With the notations as in Equations (3.17) and (3.20), we have*

$$V(\Phi(t,\omega)) = n(H^2-1) + n(n-1)\mu^2(\tan\varphi)^{-2n}$$

for the n-dimensional hyperbolic Delaunay hypersurface.

Note. In dimension 2, this formula already appears in [C].

Proof. In order to prove this formula, we need to explain how one establishes Delaunay's differential equation following [Hs]. In the upper half-plane model of $\mathbb{H}^2$ we fix the vertical geodesic $\gamma(t) = (0, e^t)$ parametrized by arc-length. A point $p = (\xi,\eta) \in \mathbb{H}^2$, with $\xi,\eta > 0$ is uniquely characterized by $(t,\rho) \in \mathbb{R}_+ \times \mathbb{R}_+$, where $\rho = d(p,\gamma)$ is the hyperbolic distance from the point p to the geodesic γ, which is achieved at some $q = (0,e^t) \in \gamma$. Let φ denote the angle $\angle p\vec{0}q$, where $\vec{0} = (0,0)$ in the upper half-plane model, and so $\sinh\rho = \tan\varphi$.
The hyperbolic metric is given, in the coordinates (ξ,η), respectively (t,ρ), by

$$\eta^{-2}\,(d\xi^2 + d\eta^2) = \cosh^2\rho\, dt^2 + d\rho^2\,.$$

In the $\{t,\rho\}$ coordinates, we have the orthonormal frame $\{e_t = (\cosh\rho)^{-1}\frac{\partial}{\partial t}, e_\rho = \frac{\partial}{\partial\rho}\}$ and the covariant derivative of the hyperbolic metric is given by the relations $D_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = -\sinh\rho\,\cosh\rho\,\frac{\partial}{\partial\rho}$, $D_{\frac{\partial}{\partial\rho}}\frac{\partial}{\partial\rho} = 0$, $D_{\frac{\partial}{\partial t}}\frac{\partial}{\partial\rho} = D_{\frac{\partial}{\partial\rho}}\frac{\partial}{\partial t} = \sinh\rho\,(\cosh\rho)^{-1}\,\frac{\partial}{\partial t}$.
Let $c(s) = (t(s),\rho(s))$ be a curved parametrized by (hyperbolic) arc-length. Write

$$\left\{\begin{array}{lcl} \tau(s) & = & \dot{c}(s) = \sin\alpha\, e_t + \cos\alpha\, e_\rho \\ \nu(s) & = & \cos\alpha\, e_t - \sin\alpha\, e_\rho \end{array}\right. \tag{3.21}$$

On the other-hand, we can write

$$\tau(s) = \cosh\rho\,\frac{dt}{ds}e_t + \frac{d\rho}{ds}\,e_\rho$$

and hence

$$\left\{\begin{array}{lcl} \cosh\rho\,\frac{dt}{ds} & = & \sin\alpha \\ \frac{d\rho}{ds} & = & \cos\alpha \end{array}\right. \tag{3.22}$$

Computing $D_{\dot{c}}\dot{c}$ using the above formulas, it follows that the geodesic curvature of the curve c in the direction ν is given by

$$\kappa = \frac{d\alpha}{ds} + \sinh\rho\,(\cosh\rho)^{-1}\,\sin\alpha = \frac{d\alpha}{ds} + \sinh\rho\,\frac{dt}{ds}\,. \tag{3.23}$$

We now consider the cmc hypersurface $M \subset \mathbb{H}^{n+1}$ generated by the rotation about the geodesic γ (this is an $O(n)$-action). The orbit of the point $p = (t,\rho) \in c$ is an $(n-1)$-sphere of Euclidean radius $e^t\sin\varphi$ (where $\tan\varphi = \sinh\rho$) which is contained in the horosphere $\{\eta = e^t\cos\varphi\}$ of $\mathbb{H}^{n+1} = \{(\xi_1,\ldots,\xi_n,\eta) \mid \eta > 0\}$. The volume of this sphere, for the metric induced by the hyperbolic metric, is $v(\rho) = \omega_{n-1}(\sinh\rho)^{n-1}$, where ω_{n-1} is the volume of the unit Euclidean sphere $S^{n-1} \subset \mathbb{R}^n$. One can then show ([Hs], page 487, notice his notations differs from ours) that the normalized mean curvature H of $M^n \subset \mathbb{H}^{n+1}$ satisfies

$$nH = \kappa - \frac{\partial}{\partial\nu}\ln v\,,$$

where ν is given in (3.21) and where v is the volume of the orbit. It follows that

$$nH = \kappa + \sin\alpha \, \frac{v'(\rho)}{v(\rho)} = \kappa + (n-1)k\,, \tag{3.24}$$

where $k = \sin\alpha \cosh\rho \,(\sinh\rho)^{-1}$ is the common value of the principal curvatures of the $O(n)$-orbit.

Let us set, for the sake of simplicity, $f(\rho) = \cosh\rho$ and $u(\rho) = \frac{\omega_{n-1}}{n}(\sinh\rho)^n$. Then $u'(\rho) = v(\rho)f(\rho)$ and it follows from (3.23) and (3.24) that

$$nH = \frac{d\alpha}{ds} + \frac{f'}{f}\sin\alpha + \frac{v'}{v}\sin\alpha\,. \tag{3.25}$$

One can now view ρ as a function of t and check that $f(\rho)\big(f^2(\rho) + (\frac{d\rho}{dt})^2\big)^{-1/2} = \sin\alpha$. It follows that

$$\begin{aligned}
&\frac{d}{dt}\{vf^2\big(f^2 + (\frac{d\rho}{dt})^2\big)^{-1/2} - nHu\} \\
&\quad = \frac{d}{dt}\{vf\sin\alpha - nHu\} \\
&\quad = \frac{d\rho}{dt}\frac{d}{d\rho}\{vf\sin\alpha - nHu\} \\
&\quad = vf\frac{d\rho}{dt}\{\frac{v'}{v}\sin\alpha + \frac{f'}{f}\sin\alpha + \frac{d\alpha}{ds} - nH\} \equiv 0\,,
\end{aligned}$$

i.e.

$$\left\{\begin{array}{l} vf^2\big(f^2 + (\frac{d\rho}{dt})^2\big)^{-1/2} - nHu \equiv m \\ \\ vf\sin\alpha - nHu \equiv m \end{array}\right. \tag{3.26}$$

where m is a constant. Writing $\sinh\rho = \tan\varphi$, a straightforward computation gives that equation (3.26) is equivalent to equation (3.16), so that m is the flux of M.

The second fundamental form B of M can be written as $B = B^0 - Hg$, where B^0 is traceless and where g is the induced metric on M. It follows that

$$||B||^2 = nH^2 + ||B^0||^2 = nH^2 + \frac{n-1}{n}(\kappa - k)^2\,. \tag{3.27}$$

We now turn to the computation of $k - \kappa$. According to (3.23) and (3.24) and (3.25), we have

$$k - \kappa = \frac{\sin\alpha}{\sinh\rho\,\cosh\rho} - \frac{d\alpha}{ds}\,.$$

Using (3.25) and (3.26) with $\mu = \frac{m}{\omega_{n-1}}$, we find that

$$k - \kappa = \frac{n\mu}{(\sinh\rho)^n}\,.$$

Finally, we find that

$$||B||^2 = nH^2 + n(n-1)\mu^2(\sinh\rho)^{-2n}$$

and the formula for V follows immediately (recall that $\sinh\rho = \tan\varphi$). ∎

In order to estimate the index of certain pieces of Delaunay hypersurfaces, we will need the following lemma.

Lemma 3.4 *With the notations as in Equations (3.20) and (3.16), there exists a constant $c(n, H)$ which only depends on n and H such that*

$$V \tan^2 \varphi \leq c(n, H)$$

on the n-dimensional hyperbolic Delaunay hypersurface $\mathcal{D}(\mu, H)$ with weight parameter μ and mean curvature $H > 1$.

Proof. We give two arguments. One in dimension 2 (which shows that the constant does not depend on H) and one in arbitrary dimension.
Case 1, dimension 2. Recall that $\mu > 0$ and that $H > 1$. We have the differential equation

$$\begin{aligned}\mu = \frac{m}{2\pi} &= \frac{\tan\varphi\sqrt{1+\tan^2\varphi}}{\sqrt{1+\varphi'^2}} - H\tan^2\varphi \\ &\leq \sup_{r>0}\{r\sqrt{1+r^2} - Hr^2\} = \frac{H - \sqrt{H^2-1}}{2} = \mu_+ \,.\end{aligned}$$

We must also have

$$H\tan^2\varphi - \tan\varphi\sqrt{1+\tan^2\varphi} + \mu \leq 0\,,$$

thus

$$(H^2-1)X^4 + (2H\mu - 1)X^2 + \mu^2 \leq 0\,,$$

where $X = \tan\varphi$. This second order polynomial in X^2 has positive discriminant when $\mu \in (0, \mu_+)$, which shows that its two positive roots are numbers $a_- \leq a_+$ such that

$$a_- \leq \tan^2\varphi \leq a_+\,.$$

By Lemma 3.3, we have

$$V\tan^2\varphi = 2\{(H^2-1)\tan^2\varphi + \frac{\mu^2}{\tan^2\varphi}\}\,,$$

and looking at the function $r \to P(r) = (H^2-1)r^2 + \mu^2 r^{-2}$, we see that

$$V\tan^2\varphi \leq 2\max\{P(a_-), P(a_+)\} = 2(1 - 2H\mu) \leq 2\,,$$

where the middle equality follows from the above second order polynomial in X^2. ∎
Case 2, dimension n. Looking at the differential equation which describes the profile curve of the hyperbolic Delaunay hypersurfaces, with $\mu > 0$ and $H > 1$, we see that there exists some number $\mu_+(n, H)$ such that

$$0 < \mu \leq \max_{r>0}\{r^{n-1}\sqrt{1+r^2} - Hr^n\} = \mu_+(n, H)\,.$$

For the Delaunay hypersurface $\mathcal{D}(\mu, H) \subset \mathbb{H}^{n+1}$, the function φ satisfies

$$a_-(\mu, H) \leq \tan\varphi \leq a_+(\mu, H)$$

where $a_\pm(\mu, H)$ are the two positive roots of the equation

$$Hr^n - r^{n-1}\sqrt{1+r^2} + \mu = 0$$

i.e. of the equation

$$\psi(r) := Hr^n - r^{n-1}\sqrt{1+r^2} = -\mu\,.$$

Looking at the graph of ψ, it follows immediately that $a_-(\mu, H)$ is an increasing function of μ and $a_+(\mu, H)$ is a decreasing function of μ, for fixed $H > 1$. It follows that

$$a_+(\mu, H) \leq 1/\sqrt{H^2-1}\,.$$

Using Lemma 3.3, we can write

$$V \tan^2\varphi = n(H^2-1)\tan^2\varphi + n(n-1)\mu^2\tan^{2-2n}\varphi\,.$$

Define $\lambda(r) = n(H^2-1)r^2 + n(n-1)\mu^2 r^{2-2n}$ and observe that $\lambda(r)$ decreases and then increases when r varies from 0 to ∞. It follows that

$$V\tan^2\varphi \leq \max\{\lambda(a_-), \lambda(a_+)\}\,.$$

Let $s = a_\pm$. Then, $\mu = s^{n-1}\sqrt{1+s^2} - Hs^n$ which implies that

$$\lambda(s) = n(H^2-1)s^2 + n(n-1)(\sqrt{1+s^2} - Hs)^2$$

and hence

$$\lambda(s) \leq n(H^2-1)s^2 + n(n-1)(\sqrt{1+s^2} + Hs)^2\,.$$

Call $c(n, H)$ the number obtained by evaluating the right hand side of the preceding equation at $s = 1/\sqrt{H^2-1}$ and get the estimate stated in the lemma. ∎

3.4 Special parts of hyperbolic Delaunay hypersurfaces

Take a hyperbolic Delaunay surface $\mathcal{D}(\mu)$. Without loss of generality, we may assume that the function φ defining the profile curve satisfies $\varphi(0) = \alpha_-(\mu)$. It follows easily that $\varphi(\tau(\mu)) = \alpha_-(\mu), \varphi(\tau(\mu)/2) = \alpha_+(\mu)$ and that φ is symmetric with respect to the values $k\,\tau(\mu)/2$, $k \in \mathbb{Z}$.

Let the *basic Dirichlet block* for the hyperbolic Delaunay hypersurface $\mathcal{D}(\mu)$ be the compact domain

$$\mathcal{B}(\mu) := \Phi([0, \frac{\tau(\mu)}{2}] \times S^{n-1}) \;\text{ or }\; \Phi([\frac{\tau(\mu)}{2}, \tau(\mu)] \times S^{n-1})\,, \tag{3.28}$$

where Φ is the parametrization (3.17).

We also introduce the pieces $\mathcal{B}_\ell(\mu)$ obtained by glueing ℓ basic Dirichlet blocks,

$$\mathcal{B}_\ell(\mu) := \Phi([0, \ell\,\frac{\tau(\mu)}{2}] \times S^{n-1}) \;\text{ or }\; \Phi([\frac{\tau(\mu)}{2}, \frac{\tau(\mu)}{2} + \ell\,\frac{\tau(\mu)}{2}] \times S^{n-1})\,. \tag{3.29}$$

Let a be the function

$$a(x) = \frac{\varphi'(x)}{\cos\varphi(x)\,\sqrt{1+\varphi'^2(x)}} \tag{3.30}$$

where φ is as above.

Lemma 3.5 *The function a vanishes precisely at the half-integer multiples of $\tau(\mu)$. Furthermore, a' has exactly two zeroes $\zeta_1(\mu), \zeta_2(\mu)$ in the interval $[0, \tau(\mu)]$, with*

$$0 < \zeta_1(\mu) < \frac{\tau(\mu)}{2} < \zeta_2(\mu) < \tau(\mu).$$

(Again, the values $\zeta_j(\mu)$ and $\tau(\mu)$ for the true hyperbolic cylinder are determined as limiting values of the $\zeta_j(\mu)$ and $\tau(\mu)$ for noncylindrical hyperbolic Delaunay hypersurfaces.)

The proof of this Lemma will be given in Section 4.3.

Let the *basic Neumann block* for the Delaunay hypersurface $\mathcal{D}(\mu)$ be the compact domain

$$\mathcal{C}(\mu) := \Phi([\zeta_1(\mu), \tau(\mu) + \zeta_1(\mu)] \times S^{n-1})\,. \tag{3.31}$$

We also introduce the pieces $\mathcal{C}_\ell(\mu)$ obtained by glueing ℓ basic Neumann blocks,

$$\mathcal{C}_\ell(\mu) := \Phi([\zeta_1(\mu), \ell\,\tau(\mu) + \zeta_1(\mu)] \times S^{n-1})\,. \tag{3.32}$$

4 Index estimates for certain pieces of Delaunay hypersurfaces

4.1 Preparatory lemmas

In this section, we gather classical results which we will need later on.

Lemma 4.1 *Let P, Q, R, be smooth functions on $I\!R$, with $P, R > 0$. Consider the eigenvalue problem*

$$-\frac{d}{dx}\big(P(x)\,\frac{d}{dx}y(x)\big) + Q(x)\,y(x) = \lambda R(x)\,y(x) \tag{4.33}$$

in the interval $]a, b[$, with any one of the following boundary conditions

$$\begin{array}{ll} \text{[DD]} & y(a) = y(b) = 0\,, \\ \text{[ND]} & y'(a) = y(b) = 0 \;\; \text{or} \;\; y(a) = y'(b) = 0\,, \\ \text{[NN]} & y'(a) = y'(b) = 0\,. \end{array} \tag{4.34}$$

Write the eigenvalues $\lambda_1 < \lambda_2 < \ldots$ in increasing order, starting with the index 1. (Note that these eigenvalues are always simple.)

Then any eigenfunction of (4.33), with one of the boundary conditions in (4.34), associated with the n-th eigenvalue λ_n has exactly *n nodal domains.*

In particular, if u is an eigenfunction of (4.33) for one of the boundary conditions (4.34), associated with the eigenvalue λ, and if u has n nodal domains, then $\lambda = \lambda_n$.

Proof. This is an easy consequence of Sturm's comparison argument. ∎

Remark. The preceding lemma means that Courant's nodal domain theorem actually gives a way of determining the rank of an eigenvalue in the case of one-dimensional eigenvalue problems.

Lemma 4.2 *Let M be a cmc hypersurface in $\overline{M} = I\!R^{n+1}$ or $I\!H^{n+1}$, with unit normal field N. Let X be a Killing vector field in $\overline{M}$. Then the function $a_X := \overline{g}(X, N)$ is a Jacobi function on M,* i.e. *it satisfies $L\,a_X = 0$. Here, $\overline{g}$ is the metric of the ambient space, and $L := \Delta_M - V$ is the stability operator of the immersion M,* i.e. *Δ_M is the Laplacian for the induced metric and $V = n\,c + \|B\|^2$, where B is the second fundamental form of M and c is the sectional curvature of the ambient space.*

Remark. This lemma is stated in [Ch], Lemma 1 page 196; it also follows from Theorem 2.7 and its proof in [BGS]. In the case of Delaunay hypersurfaces, the lemma also follows from simpler computations. In the Euclidean case, taking as Killing field the translations parallel to the axis of revolution of the Delaunay hypersurface, we get as a_X the function given by (3.13). In the hyperbolic case, taking as Killing field the field corresponding to the group of isometries which translates the vertical geodesic axis, we get as a_X the function given by (3.30). In both cases the fact that the function satisfies the Jacobi equation $(\Delta_M - V)a_X = 0$ follows by making use of the Delaunay differential equations.

Lemma 4.3 *Let M be a Riemannian manifold and let q be a function on M. Assume that there is a positive solution u of the equation $(\Delta + q)\, u = 0$ in the interior of M. Then the operator $\Delta + q$ has positive Dirichlet spectrum in any proper subset $\Omega \subset M$ such that $\overline{\Omega}$ is compact in $\overline{M}$.*

Proof. Indeed, let $D \subset M$ be any relatively compact domain and let f be a smooth function with compact support in D. We can write

$$\int_D |df|^2 + qf^2 = \int_D |df|^2 - \frac{\Delta u}{u} f^2.$$

The right hand side can be written as

$$\int_D |df|^2 - \Delta(\ln u)\, f^2 + \frac{|du|^2}{u^2} f^2 = \int_D |df - f\, d\ln(u)|^2 - \int_{\partial D} f^2 \partial_\nu \ln(u)$$

where ν is the unit inward normal to ∂D. The lemma follows immediately. ∎

Lemma 4.4 *Let A, B, V be smooth functions on $I\!R$. Assume that A, B are bounded from below by a positive constant and that V is bounded on $I\!R$. Let P be the manifold $[a, b] \times S^{n-1}$ equipped with the metric $g := A^2(x)dx^2 + B^2(x)g_S$, where g_S is the canonical metric on S^{n-1}. We are interested in the eigenvalue problem $(\Delta_g - V)\, y(x,\omega) = \lambda\, y(x,\omega)$, with Dirichlet or Neumann conditions on $\{a\} \times S^{n-1}$ and on $\{b\} \times S^{n-1}$. Let $\Lambda_k = k(k+n-2)$, $k \geq 0$, denote the eigenvalues of the Laplacian on S^{n-1} and let $m(\Lambda_k)$ denote the multiplicity of Λ_k (this is a polynomial in k, of degree $(n-2)$). Let $L := \Delta_g - V$ and define the operators $L_k, k \geq 0$, by*

$$L_k\, u = -\frac{d}{dx}\left(A^{-1} B^{n-1} \frac{du}{dx}\right) + AB^{n-3}\, \left(\Lambda_k - B^2\, V\right)\, u\,.$$

Let us denote by $\sigma(L)$ the set of eigenvalues of L, counted with multiplicities and by $\sigma(L_k)$ the eigenvalues of the problem $L_k\, u(x) = \lambda\, AB^{n-1}\, u(x)$. Then

$$\sigma(L) = \bigsqcup_{k=0}^{\infty} m(\Lambda_k)\, \sigma(L_k)$$

where the expression in the right-hand side means that each eigenvalue of L_k appears with multiplicity $m(\Lambda_k)$ in $\sigma(L)$ (summing up multiplicities if the same number λ appears in several $\sigma(L_k)$). In particular, the index (number of negative eigenvalues) of L is given by

$$\mathrm{Index}(L) = \sum_{k=0}^{\infty} \mathrm{Index}(L_k)$$

and the sum in the right-hand side only involves finitely many terms.

Proof. If $y(x,\omega)$ satisfies $Ly = \lambda y$, then

$$-\frac{\partial}{\partial x}\left(A^{-1} B^{n-1} \frac{\partial y}{\partial x}\right) + AB^{n-3}\, \left(\Delta_S\, y - B^2\, Vy\right) = \lambda\, AB^{n-1} y\,.$$

In order to prove the lemma, it suffices to decompose the function $y(x,\omega)$ into a series of spherical harmonics. The generic term in this series will be of the form $u(x)Y(\omega)$ where Y is a k-spherical harmonic and the preceding equation becomes

$$-\frac{d}{dx}\left(A^{-1} B^{n-1} \frac{du}{dx}\right) + AB^{n-3}\, \left(\Lambda_k - B^2\, V\right)\, u = \lambda\, AB^{n-1} u\,.$$

The first assertion of the lemma follows easily. For the second assertion we only have to remark that Λ_k tends to infinity with k and hence that the operators L_k are positive for k large enough (this is because A, B are bounded from below by positive constants and V is bounded). ∎

4.2 A stability result for half-Delaunay hypersurfaces

Let $\mathcal{D}$ be a Delaunay hypersurface in $\mathbb{R}^{n+1}$ (with $H > 0, \mu > 0$) or in $\mathbb{H}^{n+1}$ (with $H > 1, \mu > 0$), where H is the mean curvature and μ the weight parameter. Let $\mathbb{R}^{n+1}_+$ and $\mathbb{H}^{n+1}_+$ denote one of the closed half-spaces defined by a geodesic hyperplane containing the axis of $\mathcal{D}$.

With these notations, we have

Proposition 4.1 *The stability operator $\Delta - V$ of the Delaunay hypersurface $\mathcal{D}$ is positive in any Ω contained in $\mathcal{D}_+ := \mathcal{D} \bigcap \mathbb{R}^{n+1}_+$ or $\mathcal{D}_+ := \mathcal{D} \bigcap \mathbb{H}^{n+1}_+$, with respect to Dirichlet boundary conditions. In particular, the half-Delaunay hypersurfaces $\mathcal{D}_+$ are (strongly) stable.*

Proof. By Lemma 4.3, in order to prove the proposition, it suffices to find a positive solution of the equation $(\Delta - V)\, y = 0$ on $\mathcal{D}_+$. Such a solution will be given by the normal component of some well chosen Killing field, using Lemma 4.2.

In the Euclidean case, given any $\theta \in S^{n-1}$ we consider the Killing vector-field $Y_\theta(x,\omega) \equiv (0, \theta)$. It follows from Lemma 4.2 that the function

$$a_\theta(x,\omega) := -\overline{g}(N, Y_\theta)(x,\omega) = (1 + f'^2(x))^{-1/2} \langle \omega, \theta \rangle_{\mathbb{R}^n}$$

is a positive Jacobi function on $\mathcal{D}_+ = \mathcal{D} \cap \{\langle \omega, \theta \rangle_{\mathbb{R}^n} > 0\}$ and hence the proposition follows, by Lemma 4.3.

In the hyperbolic case, given any $\theta \in S^{n-1}$, we consider the Killing field $Y_\theta(\omega, t) = (\theta, 0)$ in $\mathbb{H}^{n+1}$. With the notations of Section 3.3, the function

$$a_\theta(t,\omega) := \overline{g}(N, Y_\theta)(\omega, t)$$

is equal to $g'(t)\, \langle \theta, \omega \rangle$ up to a positive factor (recall that $g(t) = e^t \cos \varphi(t)$). In order to prove that $a_\theta > 0$, it suffices to look at the sign of $g'(t) = e^t \big(\cos \varphi(t) - \varphi'(t) \sin \varphi(t) \big)$. Assume there is a point t_0 at which g' vanishes, then

$$\frac{1}{\varphi'(t)} = \tan \varphi(t) > 0$$

at $t = t_0$, and the differential equation (3.16) implies

$$0 < \mu = \frac{1}{(\varphi'(t_0))^{n-1}} \left(\sqrt{\frac{1 + (\varphi'(t_0))^{-2}}{1 + (\varphi'(t_0))^2}} - \frac{H}{\varphi'(t_0)} \right) = \left(\frac{1}{(\varphi'(t_0))^n}\right)(1 - H) < 0\,,$$

a contradiction. ∎

4.3 Proofs of Lemmas 3.2, 3.5

The function a given by equation (3.13) (*resp.* by equation (3.30)) is up to sign the scalar product of the unit normal to the Delaunay hypersurface with the Killing field corresponding to translations along the axis of revolution. According to Lemma 4.2, this function satisfies $(\Delta - V)a = 0$, where V is the *positive* potential given by equation (3.10) (*resp.* by equation (3.20) and Lemma 3.3). The metric on the surface of revolution is of the form

$$G = \begin{pmatrix} A^2 & 0 \\ 0 & B^2\, g_S \end{pmatrix}$$

where A, B are positive functions of one variable x (the variable along the axis of revolution) and g_S the metric of the sphere. Since the function a only depends on x, it follows that it satisfies the equation

$$\frac{d}{dx}\big(A^{-1}\, B^{n-1}\, \frac{da}{dx}\big) + A\, B^{n-1}\, V\, a = 0\,.$$

It follows that the function $A^{-1}\, B^{n-1}\, a'$ is monotonic between two consecutive zeroes of a (because the functions A, B, V are positive) and hence that a' can at most vanish once between two consecutive zeroes of a. On the other-hand, it has to vanish at least once by the mean value theorem so finally we conclude that a' has exactly one zero between two consecutive zeroes of a. ∎

4.4 Index estimates for certain Delaunay pieces

With the notations as in Definition 2.3 and in Sections 3.2 and 3.4, we have the following estimates for the indexes of the Delaunay pieces $\mathcal{B}_\ell$ and $\mathcal{C}_\ell$.

Proposition 4.2 *The Dirichlet index of the Delaunay piece $\mathcal{B}_\ell(\mu)$ is exactly $\ell - 1$.*

Proposition 4.3 *There is a constant $c_1(n, H)$, which only depends on the dimension n and the mean curvature H such that the Neumann index of the Delaunay piece $\mathcal{C}_\ell(\mu)$ satisfies*

$$2\ell \leq \textit{Neumann Index}(\mathcal{C}_\ell(\mu)) \leq 2\ell + c_1(n, H)\,.$$

Proofs. The proofs of these two propositions are quite similar.
Step 1. The induced metric on the (Euclidean or hyperbolic) pieces $\mathcal{B}_\ell(\mu)$ or $\mathcal{C}_\ell(\mu)$ is of the type described in Lemma 4.4. It follows that in order to estimate the index we only need to look at the index of the corresponding operators $A^{-1}B^{1-n}L_k$, where $A = \sqrt{1+(f')^2}$ and $B = f$ in the Euclidean case, and $A = \sqrt{(1+(\varphi')^2)(1+\tan^2\varphi)}$ and $B = \tan\varphi$ in the hyperbolic case, and we already know that for k large enough the operator $A^{-1}B^{1-n}L_k$ is positive, implying that its index is zero. Looking at the precise form of the potential V, one can actually show that that there exists a constant $c(n, H)$ such that $A^{-1}B^{1-n}L_k$ is positive whenever $k \geq c(n, H)$, see Lemma 3.1 for the Euclidean case (and the constant does not depend on H) and Lemma 3.4 for the hyperbolic case (where the constant does not depend on H at least when $n = 2$).

Proof of 4.2, Step 2. Consider the function $a = \overline{g}(Y, N)$ where Y is the Killing field generated by the one-parameter group of isometries which translates the geodesic axis of the Delaunay hypersurface along itself. According to Lemma 4.2, this function satisfies $(\Delta - V)a = 0$ and $a\,|\partial\mathcal{B}_\ell = 0$. This function has precisely ℓ nodal domains in $\mathcal{B}_\ell$. According to Lemma 4.1, 0 is the ℓ-th eigenvalue of the operator $A^{-1}B^{1-n}L_0$ (see Step 1). It follows that the Dirichlet index of $\mathcal{B}_\ell$ is at least $(\ell - 1)$ and that it is bigger than $(\ell - 1)$ if and only if some of the operators $A^{-1}B^{1-n}L_k, k \geq 1$, have negative eigenvalues. Assume this is the case and that (u, λ) satisfies $A^{-1}B^{1-n}L_k\, u = \lambda\, u$, for some $\lambda < 0$. This implies that

$$(\Delta - V)\, u\, Y = \lambda\, u\, Y$$

for any spherical harmonic Y of degree k. Choosing, for example, a radial spherical harmonic Y, we can always find a domain $D_k \subset \mathcal{D}_+$ such that the function $u\, Y$ is positive in D_k and satisfies

$$\begin{cases} (\Delta - V)\, u\, Y = \lambda\, u\, Y & \text{in } D_k\,, \\ u\, Y = 0 & \text{on } \partial D_k\,. \end{cases}$$

This contradicts Proposition 4.1.

Proof of 4.3, Step 2. We use the same function a as before, $(\Delta - V)a = 0$. The domain $\mathcal{C}_\ell$ was designed so that $a' = \frac{\partial a}{\partial \nu} = 0$ on $\partial \mathcal{C}_\ell$ and so that a has exactly $(2\ell + 1)$ nodal domains in $\mathcal{C}_\ell$. It follows, as in the preceding argument, that 0 is the $(2\ell+1)$-st eigenvalue of L_0 and hence that the Neumann index of $\mathcal{C}_\ell$ is at least 2ℓ. In order to obtain the upper bound, we remark that the Neumann index of $A^{-1}B^{1-n}L_k, k \geq 1$, is at most 2. Indeed, assume it is at least 3. This means, using Lemma 4.1, that there is an eigenfunction u of $A^{-1}B^{1-n}L_k, k \geq 1$, with at least three nodal domains and hence with an interior nodal domain. We can then repeat the argument in Step 2, proof of Proposition 4.2, and arrive at a contradiction with Proposition 4.1. An eigenvalue of $A^{-1}B^{1-n}L_k$ gives an eigenvalue of L with multiplicity a polynomial of degree $(n-2)$ in k. Since $A^{-1}B^{1-n}L_k$ is positive for $k \geq c(n, H)$, the result follows. ∎

5 Index growth results

5.1 Preparatory lemma

Lemma 5.1 *Let M be a compact manifold with boundary. Assume that it can be decomposed as $M = \bigsqcup_{i=0}^{m} M_i$ into m pieces with smooth boundaries. Let L be an operator of the form $L = \Delta_M - V$. We consider the counting function $\mathcal{N}(M, \lambda)$ of the L-eigenvalues less than λ on M, with either Dirichlet or Neumann boundary condition on ∂M. We let $\mathcal{N}(M_i, D, \lambda)$,* resp. *$\mathcal{N}(M_i, N, \lambda)$, denote the counting functions of the L-eigenvalues less than λ on M_i with the given boundary condition on the parts of ∂M contained in ∂M_i and with Dirichlet,* resp. *Neumann, condition on the remaining parts of ∂M_i. Then (Dirichlet – Neumann bracketing),*

$$\sum_{i=0}^{m} \mathcal{N}(M_i, D, \lambda) \leq \mathcal{N}(M, \lambda) \leq \sum_{i=0}^{m} \mathcal{N}(M_i, N, \lambda). \tag{5.35}$$

Proof. The proof uses the min-max principle, see [RS]. ∎

5.2 Eigenvalue estimates for almost Delaunay pieces

Fix a Delaunay hypersurface $\mathcal{D}(\mu)$ and a piece $\mathcal{E} \subset \mathcal{D}$ which is bounded by two "parallel spheres" in geodesic hyperplanes orthogonal to the axis of revolution. We call $\widetilde{\mathcal{E}}$ an *almost Delaunay piece* if it is a cylindrical graph over $\mathcal{E}$.

Lemma 5.2 *There exists a constant $c_2(n, H)$, depending only on the dimension n and mean curvature H, such that if $\widetilde{\mathcal{E}}$ is close enough to $\mathcal{E}$ in the C^2-sense, then*

$$\mathit{Ind}(\mathcal{E}) \leq \mathit{Ind}(\widetilde{\mathcal{E}}) \leq \mathit{Ind}(\mathcal{E}) + c_2(n, H),$$

where Ind denotes the index for either Dirichlet or Neumann conditions on the corresponding boundary components of $\partial \mathcal{E}, \partial \widetilde{\mathcal{E}}$.

Proof. Indeed, once the piece $\mathcal{E}$ is *fixed*, we can write the eigenvalues of the operator L on $\mathcal{E}$ (with respect to some Dirichlet or Neumann conditions on the boundary components) as

$$\lambda_1(\mathcal{E}) < \lambda_2(\mathcal{E}) \leq \ldots \leq \lambda_k(\mathcal{E}) < 0 \leq \lambda_{k+1}(\mathcal{E}) \leq \ldots$$

where $k = \mathrm{Ind}(\mathcal{E})$. If $\widetilde{\mathcal{E}}$ is close enough to $\mathcal{E}$ in the C^2-sense, the negative eigenvalues of the operator $\widetilde{L}$ corresponding to L are close to the corresponding eigenvalues of L. It follows that $\mathrm{Ind}(\mathcal{E}) \leq \mathrm{Ind}(\widetilde{\mathcal{E}})$ because $\lambda_k(\widetilde{\mathcal{E}}) < 0$, with $\mathrm{Ind}(\mathcal{E}) = \mathrm{Ind}(\widetilde{\mathcal{E}})$ unless $\lambda_{k+1}(\mathcal{E}) = 0$, in which case we may have $\lambda_{k+1}(\widetilde{\mathcal{E}}) < 0$ and the constant $c_2(n, H)$ takes the possible multiplicity of $\lambda_{k+1}(\mathcal{E})$ into account. This multiplicity can be bounded as indicated in the proof of Proposition 4.3. ∎

Note. The operator L_p introduced earlier cannot have eigenvalue 0 when $p \geq 2$ in case of Dirichlet boundary condition (use Proposition 4.1, $p = 2$ and the monotonicity of Dirichlet eigenvalues).

5.3 Asymptotically Delaunay hypersurfaces

Let $M \subset \mathbb{R}^{n+1}$ be an embedded hypersurface such that

1. The hypersurface M can be decomposed as

$$M = M_0 \bigsqcup \bigsqcup_{j=1}^{k} E_j$$

   where M_0 is compact with boundary and where each E_j is an end of M.

2. Each end E_j is a cylindrical graph over half an embedded Delaunay hypersurface $\mathcal{D}_j$ with positive flux m_j, with semi-axis $a_j + \mathbb{R}_+ \, d_j$ for some $a_j, d_j \in \mathbb{R}^{n+1}$. The boundary of E_j is the intersection of E_j with the hyperplane through a_j orthogonal to d_j and $\partial M_0 = \bigsqcup \partial E_j$.

3. The graph E_j above $\mathcal{D}_j$ is given by a parametrization of the form

$$\mathbb{R} \times S^{n-1} \ni (x, \omega) \to F_j(t, \omega) = \big(x, (f(x) + w_j(x, \omega))\, \omega\big) \qquad (5.36)$$

   with some function $w_j(x, \omega)$, for $(x, \omega) \in \mathbb{R}_+ \times S^{n-1}$, where $f(x)$ satisfies equation (3.7). We assume that w_j tends to zero in C^2-norm on $[r, \infty[\times S^{n-1}$ when r tends to infinity.

Definition 5.1 *We will say that a hypersurface which satisfies the preceding conditions (1) to (3) is an asymptotically Delaunay hypersurface.*

This preceding definition extends mutatis mutandis to the case of hypersurfaces in $\mathbb{H}^{n+1}$ (in this case, the axis is a geodesic ray parametrized by arc-length).

If M is an asymptotically Delaunay hypersurface, we can introduce the operator $L := \Delta - nc - \|B\|^2$, where B is the second fundamental form and c the curvature of the ambient space $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$.

With the above notations, we also introduce the following subsets of M.

- For $R > 0$, we let

$$M^R = M_0 \bigsqcup \bigsqcup_{j=1}^{k} E_j^R$$

where E_j^R is the part of E_j which lies above $a_j + [0, R]\, d_j$.

- For $R > S > 0$, we let

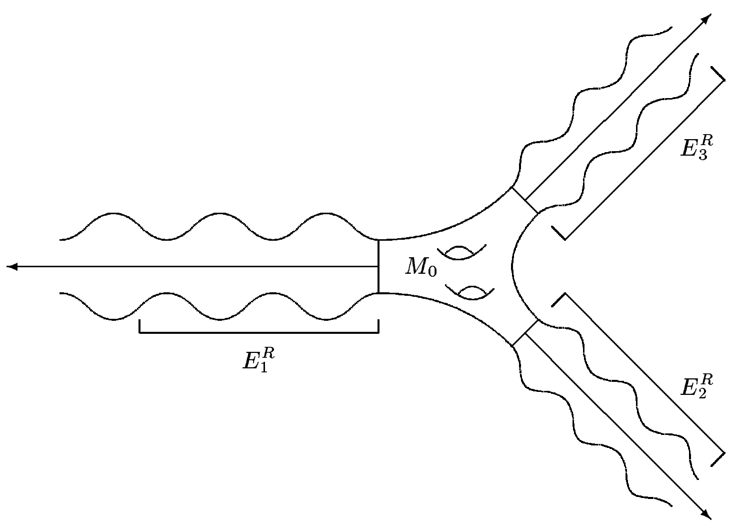

Figure 3: An asymptotically Delaunay surface $M \subset \mathbb{R}^3$.

$$M^{S,R} = M^R \setminus M^S = \bigsqcup_{j=1}^{k} E_j^{S,R}$$

where $E_j^{S,R}$ is the part of E_j which lies above $a_j + [S,R]\, d_j$.

- For $R > 0$, we let Σ^R denote the part of the boundary of E_j^R which lies above $\{a_j + R\, d_j\}$.

We can use similar notations for hypersurfaces M in $\mathbb{H}^{n+1}$.

5.4 Statements of the main results

With the notations of Section 5.3, we have the following results.

Theorem 5.1 *Let $M \subset \mathbb{R}^3$ be a complete properly embedded finite-topology cmc surface. Then, M is an asymptotically Delaunay surface in the sense of Definition 5.1. Let $E_j, j = 1 \ldots N$, be the ends of M. They are asymptotic to Delaunay surfaces $\mathcal{D}(\mu_j)$, with weight parameter $\mu_j > 0$. Denote by $T(\mu_j)$ the period of the Delaunay surface $\mathcal{D}(\mu_j)$, as in Section 3.1. Then*

$$\lim_{R\to\infty} \frac{Ind(M \cap B(R))}{R} = 2 \sum_{j=1}^{N} \frac{1}{T(\mu_j)}\,, \tag{5.37}$$

where $B(R)$ is the Euclidean ball in $\mathbb{R}^3$.

Note. In dimension 2, $T(\mu_j) \in [2, \pi]$ and we deduce from the preceding theorem that

$$\frac{2N}{\pi} \le \lim_{R\to\infty} \frac{Ind(M \cap B(R))}{R} \le N\,,$$

where N is the number of ends.

Theorem 5.2 *Let $M \subset \mathbb{R}^{n+1}$ be a complete embedded cmc hypersurface.* Assume *it is an asymptotically Delaunay hypersurface in the sense of Definition 5.1. Let $E_j, j = 1 \ldots N$, be the ends of M. They are asymptotic to Delaunay surfaces $\mathcal{D}(\mu_j)$, with weight parameter $\mu_j > 0$. Denote by $T(\mu_j)$ half the period of the Delaunay surface $\mathcal{D}(\mu_j)$, as in Section 3.1. Then*

$$\lim_{R\to\infty} \frac{Ind(M \cap B(R))}{R} = 2 \sum_{j=1}^{N} \frac{1}{T(\mu_j)}, \tag{5.38}$$

where $B(R)$ is the Euclidean ball in $\mathbb{R}^{n+1}$.

Theorem 5.3 *Let $M \subset \mathbb{H}^{n+1}(-1)$ be a complete properly embedded hypersurface, with constant mean curvature $H > 1$ and finite topology. Then M is an asymptotically Delaunay hypersurface in the sense of Definition 5.1. Let $E_j, j = 1 \ldots N$, be the ends of M. They are asymptotic to Delaunay surfaces $\mathcal{D}(\mu_j)$, with weight parameter $\mu_j > 0$. Denote by $\tau(\mu_j)$ the period of the Delaunay surface $\mathcal{D}(\mu_j)$. Then*

$$\lim_{R\to\infty} \frac{Ind(M \cap B(R))}{R} = 2 \sum_{j=1}^{N} \frac{1}{\tau(\mu_j)}, \tag{5.39}$$

where $B(R)$ is the hyperbolic ball in $\mathbb{H}^{n+1}$.

Remark. The general philosophy of the proofs of these theorems is to apply the *Dirichlet–Neumann bracketing* to the decomposition of M^R as $M^R = M^S \sqcup M^{S,R}$ (see Section 5.3) and to use the fact that each component of $M^{S,R}$ is asymptotic to a Delaunay piece for which we can estimate the index. The assumptions in Theorems 5.1 and 5.3 imply, using [KKS, KKMS], that M is actually an asymptotically Delaunay hypersurface. No such result is known for $M^n \subset \mathbb{R}^{n+1}$, $n \geq 3$ and we therefore have to make this additional assumption in Theorem 5.2.

Proofs, main argument. Assume M is an asymptotically Delaunay hypersurface in the sense of Definition 5.1. We want to estimate the limits

$$\liminf \frac{\mathrm{Ind}(M \cap B(R))}{R} \ , \quad \limsup \frac{\mathrm{Ind}(M \cap B(R))}{R}$$

when $R \to \infty$. It is easy to see that this is equivalent to estimating

$$\liminf \frac{\mathrm{Ind}(M^R)}{R} \ , \quad \limsup \frac{\mathrm{Ind}(M^R)}{R}$$

for some decomposition of M.

Fix some $\varepsilon > 0$ and fix some $\ell \in \mathbb{N}$ such that $\ell\,\varepsilon \geq 1$.

Step 1, Estimating the index from below.

Recall that $M^R = M_0 \bigsqcup \bigsqcup_{j=1}^N E_j^R$, where M_0 is compact and where the N ends E_j of M, $j = 1 \ldots N$, are cylindrical graphs over Delaunay hypersurfaces $\mathcal{D}(\mu_j)$, with weight parameter μ_j, as in Definition 5.1.

For $R > S$, we again decompose M^R into pieces $M^R = M^S \bigsqcup \bigsqcup_{j=1}^N E_j^{S,R}$, having in mind to decompose $E_j^{S,R}$ into $\widetilde{\mathcal{B}}_\ell$-pieces.

We choose S so that the components of ∂M^S lie above boundaries of $\mathcal{B}$-type blocks of the corresponding Delaunay hypersurfaces. Taking $R > S$ we can decompose E_j^R as $E_j^R = E_j^S \bigsqcup E_j^{S,R}$, where $E_j^S = E_j^R \cap M^S$. Each piece $E_j^{S,R}$ can again be decomposed into pieces that are graphs above $\mathcal{B}_l(\mu_j)$-type pieces of the Delaunay hypersurface $\mathcal{D}(\mu_j)$, plus a remainder part. We write such a decomposition as

$$E_j^{S,R} = \bigsqcup_{p=1}^{m_j} \widetilde{\mathcal{B}}_{\ell,p}(\mu_j) \bigsqcup \widetilde{\mathcal{R}}_j$$

where $\widetilde{\mathcal{R}}_j \subset \widetilde{\mathcal{B}}_{\ell,m_j+1}(\mu_j)$.

Using Lemma 5.1, we can write

$$\mathrm{Ind}(M^R) \geq \text{D-Ind}(M^S) + \sum_{j=1}^N \left\{ \sum_{p=1}^{m_j} \text{D-Ind}(\widetilde{\mathcal{B}}_{\ell,p}(\mu_j)) \right\} ,$$

where D-Ind means Dirichlet-index (see Section 2).

The number ℓ being fixed, we can choose S (and $R > S$) so large that each piece $\widetilde{\mathcal{B}}_{\ell,p}(\mu_j)$ is close enough to a $\mathcal{B}_\ell(\mu_j)$-piece so that $\text{D-Ind}(\widetilde{\mathcal{B}}_{\ell,p}(\mu_j)) \geq \ell - 1$, by Lemma 5.2 and Proposition 4.2.

We can now look at the extrinsic length R and write, for each end E_j,

$$S + m_j\,\ell\,\frac{T(\mu_j)}{2} \leq R \leq S + (m_j+1)\,\ell\,\frac{T(\mu_j)}{2} \ .$$

It follows that

$$\mathrm{Ind}(M^R) \geq \text{D-Ind}(M^S) + \sum_{j=1}^{N} (1 - \frac{1}{\ell}) \left(\frac{2(R-S)}{T(\mu_j)} - \ell \right) .$$

Dividing the preceding inequality by R and letting R tend to infinity, we find that

$$\liminf_{R\to\infty} \frac{\mathrm{Ind}(M^R)}{R} \geq 2\,(1 - \frac{1}{\ell}) \sum_{j=1}^{N} \frac{1}{T(\mu_j)}.$$

Since ℓ is an arbitrary positive integer, we conclude

$$\liminf_{R\to\infty} \frac{\mathrm{Ind}(M^R)}{R} \geq 2 \sum_{j=1}^{N} \frac{1}{T(\mu_j)}.$$

Step 2, Estimating the index from above.

Again recall $M^R = M_0 \bigsqcup \bigsqcup_{j=1}^{N} E_j^R$, where M_0 is compact and where the ends $E_j, j = 1 \ldots N$, are cylindrical graphs over Delaunay hypersurfaces $\mathcal{D}(\mu_j)$, with weight parameter μ_j, as in Definition 5.1.

For $R > S$, we again decompose M^R into pieces $M^R = M^S \bigsqcup \bigsqcup_{j=1}^{N} E_j^{S,R}$, having in mind to decompose $E_j^{S,R}$ into $\widetilde{\mathcal{C}}_\ell$-pieces instead of $\widetilde{\mathcal{B}}_\ell$-pieces.

We choose S so that the components of ∂M^S lie above boundaries of $\mathcal{C}$-type blocks of the corresponding Delaunay hypersurfaces. Taking $R > S$ we can decompose E_j^R as $E_j^R = E_j^S \bigsqcup E_j^{S,R}$, where $E_j^S = E_j^R \cap M^S$. Each piece $E_j^{S,R}$ can again be decomposed into almost Delaunay pieces above $\mathcal{C}_l(\mu_j)$-type pieces of the Delaunay hypersurfaces $\mathcal{D}(\mu_j)$ plus a remainder part. We write such a decomposition

$$E_j^{S,R} = \bigsqcup_{p=1}^{m_j} \widetilde{\mathcal{C}}_{\ell,p}(\mu_j) \bigsqcup \widetilde{\mathcal{R}}_j$$

where $\mathcal{R}_j \subset \widetilde{\mathcal{C}}_{\ell,m_j+1}(\mu_j)$.

Using Lemma 5.1, we can write

$$\mathrm{Ind}(M^R) \leq \text{N-Ind}(M^S) + \sum_{j=1}^{N} \left\{ \sum_{p=1}^{m_j} \text{N-Ind}(\widetilde{\mathcal{C}}_{\ell,p}(\mu_j)) + \text{ND-Ind}(\widetilde{\mathcal{R}}_j) \right\} ,$$

where N-Ind stands for Neumann-index and where ND-Ind stands for a mixed Neumann–Dirichlet index.

The number ℓ being fixed, we can choose S (and $R > S$) so large that each piece $\widetilde{\mathcal{C}}_{\ell,p}(\mu_j)$ is close enough to a $\mathcal{C}_\ell(\mu_j)$-piece so that $\text{N-Ind}(\widetilde{\mathcal{C}}_{\ell,p}(\mu_j)) \leq 2\ell + c(n, \mu_j, H)$, by Lemma 5.2 and Proposition 4.3.

We can now look at the extrinsic length R and write, for each end E_j,

$$S + m_j\, \ell\, T(\mu_j) \leq R \leq S + (m_j + 1)\, \ell\, T(\mu_j) .$$

It follows that

$$\mathrm{Ind}(M^R) \le \text{N-Ind}(M^S) + \sum_{j=1}^{N} (1 + \frac{c(n,\mu_j,H)}{2\ell}) \frac{2(R-S)}{T(\mu_j)} + \text{ND-Ind}(\widetilde{\mathcal{C}}_\ell(\mu_j)) \ .$$

$$\le \text{N-Ind}(M^S) + \sum_{j=1}^{N} (1 + \frac{c(n,\mu_j,H)}{2\ell}) \frac{2(R-S)}{T(\mu_j)} + \text{ND-Ind}(\mathcal{C}_\ell(\mu_j)) + c(n,\mu_j,H) \ .$$

Dividing the preceding inequality by R and letting R tend to infinity, we find that

$$\limsup_{R\to\infty} \frac{\mathrm{Ind}(M^R)}{R} \le 2 \sum_{j=1}^{N} (1 + \frac{c(n,\mu_j,H)}{2\ell}) \frac{1}{T(\mu_j)}.$$

Since ℓ is an arbitrary positive integer, we have that

$$\limsup_{R\to\infty} \frac{\mathrm{Ind}(M^R)}{R} \le 2 \sum_{j=1}^{N} \frac{1}{T(\mu_j)}.$$

Proof of Theorem 5.1.
Under the assumptions of Theorem 5.1, it follows from a result of Meeks ([KKS], Corollary 1.12) that M has cylindrically-bounded ends. It follows from [KKS], Theorem 5.18, that each end E_j of M is a cylindrical graph over half a Delaunay surface $\mathcal{D}(\mu_j)$ and that E_j tends to $\mathcal{D}(\mu_j)$ uniformly in the C^2-topology at infinity ([KKS], page 510). By Theorem 4.6 of [KKS], the curvature of M remains bounded and hence, using the C^2 convergence of the ends, it follows that the Delaunay surfaces $\mathcal{D}(\mu_j)$ have strictly positive flux and hence strictly positive weight parameter μ_j. This proves the first assertion of Theorem 5.1.

In order to prove the second assertion, we apply the preceding argument.

Proof of Theorem 5.2.
This is the main argument given above.

Proof of Theorem 5.3.
In order to prove the first assertion in Theorem 5.3, we apply the results of [KKMS]. The second assertion follows from the main argument given above in the hyperbolic framework, in particular, change $T(\mu)$ to $\tau(\mu)$.

5.5 Other growth results

Kapouleas [K] has constructed examples of complete constant mean curvature surfaces in $\mathbb{R}^3$ which are periodic with respect to some 2-dimensional (*resp.* to some 3-dimensional) lattice. Using Lemma 5.1 of this paper, it is not difficult to establish that, for each of the doubly periodic (*resp.* triply periodic) surfaces M in [K], there exist finite positive constants c_1 and c_2 such that $c_1 R^2 \le \mathrm{Index}(M \cap B(R)) \le c_2 R^2$ (*resp.* $c_1 R^3 \le \mathrm{Index}(M \cap B(R)) \le c_2 R^3$) for large R.

In fact, in some cases, one can estimate c_1 explicitly. For example, consider the surface shown in Figure 1.1 of [K]. This surface is contained between two parallel planes P_1 and P_2, and is symmetric with respect to reflection through some plane P parallel to P_1 and P_2. There exists a line ℓ perpendicular to P which does not intersect the surface and is a symmetry line of the surface, in the sense that the surface in invariant with respect to rotation by $\frac{\pi}{2}$ about ℓ. (Actually there are infinitely many choices for the line ℓ.) With respect to the Killing vector field determined by rotation about ℓ, one can determine the nodal domains, and then use the methods of step 1 of the proof of Theorems 5.1, 5.2, and 5.3 to determine an explicit value for c_1.

References

[BB] L. Barbosa, P. Bérard. — A "twisted" eigenvalue problem and applications to geometry, Prébublication 466, Institut Fourier Grenoble 1999 (http://www-fourier.ujf-grenoble.fr/PREP/prep.html).

[BdCE] L. Barbosa, M. do Carmo, J. Eschenburg. — Stability of hypersurfaces of constant mean curvature in Riemannian manifolds, *Math. Z.* 197 (1988), 123–138.

[BGS] L. Barbosa, J.M. Gomes, A.M. Silveira. — Foliation of 3-dimensional space form by surfaces with constant mean curvature, Bol. Soc. Bras. Mat. 18 (1987), 1–12.

[C] Ph. Castillon. — Sur les sous-variétés à courbure moyenne constante dans l'espace hyperbolique, Thèse de Doctorat, Université Joseph Fourier, Grenoble 1997.

[Ch] Jaigyoung Choe. — Index, vision number and stability of complete minimal surfaces, Arch. Rat. Mech. Anal. 109 (1990), 195–212.

[E] J. Eells. — The surfaces of Delaunay, *Math. Intelligencer* 9(1) (1987), 53–57.

[GKS] K. Grosse-Brauckmann, R.B. Kusner, J.M. Sullivan. — Constant mean curvature surfaces with three ends, to appear in Proc. Nat. Acad. Sciences.

[Hs] W.Y. Hsiang. — On generalization of theorems of A.D. Alexandrov and C. Delaunay on hypersurfaces of constant mean curvature, Duke Math. J. 49 (1982), 485–496.

[K] N. Kapouleas. — Constant mean curvature surfaces in Euclidean three-space, *Annals of Math.* 131 (1990), 239–330.

[KK] N. Korevaar, R. Kusner. — The global stucture of constant mean curvature surfaces, *Invent. Math.* 114(2) (1993), 311–332.

[KKS] N. Korevar, R. Kusner, and B. Solomon. — The stucture of complete embedded surfaces with constant mean curvature, *J. Diff. Geom.* 30(2) (1989), 465–503.

[KKMS] N. Korevar, R. Kusner, W. Meeks, and B. Solomon. — Constant mean curvature surfaces in hyperbolic space, *American J. Math.* 114 (1992) 1–43.

[LiRo] L.L. de Lima, W. Rossman. — Index of constant mean curvature 1-surfaces in $\mathbb{H}^3$, *Indiana Univ. Math. J.* 47 (1998) 685–723.

[LR] F.J. Lopez, A. Ros. — Complete minimal surfaces with index one and stable constant mean curvature surfaces, *Commentarii Math. Helv.* 64 (1989), 34–43.

[MP] R. Mazzeo, F. Pacard. — Constant mean curvature surfaces with Delaunay ends, preprint.

[MPP] R. Mazzeo, F. Pacard, D. Pollack. — Connected sums of constant mean curvature surfaces in Euclidean 3 space, preprint.

[RS] M. Reed, B. Simon. — Methods of Modern Mathematical Physics (Vol. I to IV), Academic Press 1979.

[S] A. da Silveira. — Stability of complete noncompact surfaces with constant mean curvature, *Math. Ann.* 277 (1987), 629–638.

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