

The projective geometry of Freudenthal's magic square

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Abstract

We connect the algebraic geometry and representation theory associated to Freudenthal's magic square. We give unified geometric descriptions of several classes of orbit closures, describing their hyperplane sections and desingularizations, and interpreting them in terms of composition algebras. In particular, we show how a class of invariant quartic polynomials can be viewed as generalizations of the classical discriminant of a cubic polynomial.

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1 Introduction

1.1 The magic square

Let G be a complex simple Lie group, \mathfrak{g} its Lie algebra, and G^{ad} the closed G -orbit in $\mathbb{P}\mathfrak{g}$, the *adjoint variety* of G . Adjoint varieties are of current interest in algebraic geometry [2, 15, 24], it is conjectured that they are the only complex contact manifolds with ample anticanonical bundle.

In order to better understand the geometry of the adjoint varieties, one could work infinitesimally. If one calculates the space of tangent directions to lines passing through a point x of G^{ad} , one obtains a new variety $Y \subset \mathbb{P}T_x G^{ad}$. For example, in the case of $G = G_2$, $Y = v_3(\mathbb{P}^1) \subset \mathbb{P}^3$, the twisted cubic curve. In other cases, to understand the geometry of Y better, one can repeat the procedure. In the case of the remaining exceptional groups, upon a second infinitesimalization one arrives at the *Severi varieties*, the projective planes over the composition algebras. (These observations were communicated to us by Y. Ye.) The Severi varieties have been well studied, they arise in numerous geometric contexts. In particular, Zak [29] showed that they are the unique extremal varieties for secant defects. They have the unusual property that a generic hyperplane section of a Severi variety is still homogeneous. Putting the resulting varieties into a chart we have:

$v_2(Q^1)$	$\mathbb{P}(T\mathbb{P}^2)$	$G_\omega(2, 6)$	$\mathbb{O}\mathbb{P}_0^2$	hyperplane section of Severi
$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$G(2, 6)$	$\mathbb{O}\mathbb{P}^2$	Severi
$G_\omega(3, 6)$	$G(3, 6)$	\mathbb{S}_{12}	E_7/P_7	lines through a point of G^{ad}
F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}	G^{ad}

where the notations are explained below.

These varieties are homogeneous spaces of groups whose associated Lie algebras are:

\mathfrak{so}_3	\mathfrak{sl}_3	\mathfrak{sp}_6	\mathfrak{f}_4
\mathfrak{sl}_3	$\mathfrak{sl}_3 \times \mathfrak{sl}_3$	\mathfrak{sl}_6	\mathfrak{e}_6
\mathfrak{sp}_6	\mathfrak{sl}_6	\mathfrak{so}_{12}	\mathfrak{e}_7
\mathfrak{f}_4	\mathfrak{e}_6	\mathfrak{e}_7	\mathfrak{e}_8

This chart is called *Freudenthal's magic square* of semi-simple Lie algebras.

The magic square was constructed by Freudenthal and Tits as follows: Let \mathbb{A} denote a complex composition algebra (i.e. the complexification of \mathbb{R} , \mathbb{C} , the *quaternions* \mathbb{H} or the *octonions* \mathbb{O}). For a pair (\mathbb{A}, \mathbb{B}) of such composition algebras, the corresponding Lie algebra is

$$\mathfrak{g} = \text{Der}\mathbb{A} \oplus (\mathbb{A}_0 \otimes \mathcal{J}_3(\mathbb{B})_0) \oplus \text{Der}\mathcal{J}_3(\mathbb{B}),$$

where \mathbb{A}_0 is the space of imaginary elements, $\mathcal{J}_3(\mathbb{B})$ denotes the Jordan algebra of 3×3 \mathbb{B} -Hermitian matrices, and $\mathcal{J}_3(\mathbb{B})_0$ is the subspace of $\mathcal{J}_3(\mathbb{B})$ consisting of traceless matrices. From Freudenthal's construction the symmetry in the chart appears to be as miraculous as that \mathfrak{g} is actually a Lie algebra. Vinberg gave a construction where the symmetry is built in, see [25].

The Severi varieties admit a common geometric interpretation as $\mathbb{A}\mathbb{P}^2 \subset \mathbb{P}(\mathcal{J}_3(\mathbb{A}))$ and we showed in [21] that their hyperplane sections admit common geometric interpretations as $G_Q(\mathbb{A}^1, \mathbb{A}^3) \subset \mathbb{P}(\mathcal{J}_3(\mathbb{A})_0)$, the Grassmanian of \mathbb{A}^1 's in \mathbb{A}^3 isotropic for a quadratic form. In particular, when one moves from left to right in the first two rows, the varieties are naturally nested in each other. We show in §5,6 that the same is true for the varieties above in the third and fourth rows, in particular, we give a common geometric interpretation of the varieties above in the third row as $G_w(\mathbb{A}^3, \mathbb{A}^6)$.

Moreover, as remarked above, as one moves from line to line there are also natural inclusions of varieties (after fixing a point and with the caveat that the inclusion of the first row in the second is of a different nature).

The four Severi varieties share many common geometric properties: their tangent spaces have a common geometric interpretation as $\mathbb{A} \oplus \mathbb{A}$ (see §3), and the orbit structure in $\mathbb{P}(\mathcal{J}_3(\mathbb{A}))$, the classification of hyperplane sections, and the desingularizations of the singular orbit closure are all the same (see §4,7,8). We show that these extraordinary similarities also hold for the varieties above in the third and fourth rows. Moreover, we show that the more complicated spaces of the third and fourth rows can be understood in terms of a simple object, the discriminant of a cubic polynomial, as we now explain.

1.2 The discriminant and generalizations

Consider the twisted cubic curve $v_3(\mathbb{P}^1) \subset \mathbb{P}^3 = \mathbb{P}(S^3(\mathbb{C}^2))$. It is the space of cubic polynomials having a triple root, and its tangential variety, the quartic hypersurface $\tau(v_3(\mathbb{P}^1)) \subset \mathbb{P}^3$, is the space of cubics having a multiple root. The equation Δ defining $\tau(v_3(\mathbb{P}^1))$ is the classical *discriminant* of a cubic polynomial and is as follows: if $P = p_0x^3 + p_1x^2 + p_2x + p_3$, then

$$\Delta(P) = 3(3p_0p_3 - p_1p_2)^2 + 4(p_0p_2^3 + p_1^3p_3) - 4p_1^2p_2^2.$$

The ideal of $v_3(\mathbb{P}^1)$ is generated by the second derivatives of Δ . Write $W = \mathbb{C}$ and write $\mathbb{C}^4 = V = \mathbb{C} \oplus W \oplus W^* \oplus \mathbb{C}^*$. Let $C(x) = x^3$. We rewrite the discriminant (changing scales) as follows: for $w = (\alpha, r, s^*, \beta^*) \in V$ let

$$Q(w) = (3\alpha\beta^* - \frac{1}{2}\langle r, s^* \rangle)^2 + \frac{1}{3}(\beta^*C(r^3) + \alpha C^*(s^{*3})) - \frac{1}{6}\langle C^*(s^{*2}), C(r^2) \rangle.$$

We may describe $v_3(\mathbb{P}^1)$ as the image of the rational map:

$$\begin{aligned} \phi : \mathbb{P}(\mathbb{C} \oplus W) &\dashrightarrow \mathbb{P}(V) = \mathbb{P}(\mathbb{C} \oplus W \oplus W^* \oplus \mathbb{C}^*) \\ (z : w) &\mapsto (\frac{1}{6}z^3 : z^2w : zC(w^2) : \frac{1}{3}C(w^3)). \end{aligned}$$

Letting $\mathfrak{h} = \mathfrak{sl}_2$, the adjoint variety $G_2^{ad} \subset \mathbb{P}\mathfrak{g}_2$ is the image of the rational map (see §5 and [22]):

$$\begin{aligned} \psi : \mathbb{P}(\mathbb{C} \oplus V \oplus \mathbb{C}) &\rightarrow \mathbb{P}(\mathbb{C}^* \oplus V^* \oplus (\mathbb{C} \oplus \mathfrak{h}) \oplus V \oplus \mathbb{C}) \\ (u, A, v) &\mapsto (u^4, u^3A, u^3v, u^2Q(A, A, \cdot, \cdot), u^2vA - uQ(A, \cdot, \cdot, \cdot), u^2v^2 - Q(A)). \end{aligned}$$

Now, let $W = \mathbb{C}^{3m+3}$ be the vector space associated to a Severi variety $\mathbb{A}\mathbb{P}^2 \subset \mathbb{P}W$. W is equipped with a cubic form C (the determinant, see [21]). Let $Sp_6(\mathbb{A})$ (resp. $E(\mathbb{A})$) denote the groups appearing in the third (resp fourth) row of the magic chart, let $\mathfrak{h} = \mathfrak{sp}_6(\mathbb{A})$, and continue the notation $V = \mathbb{C} \oplus W \oplus W^* \oplus \mathbb{C}^*$ etc... We prove:

The varieties $G_w(\mathbb{A}^3, \mathbb{A}^6) \subset \mathbb{P}V$ are the images of the rational mapping ϕ . The quartic Q is an $Sp_6(\mathbb{A})$ -invariant form on $V = \mathbb{C}^{6m+8}$. The hypersurface $Q = 0$ is $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$ and the ideal of $G_w(\mathbb{A}^3, \mathbb{A}^6)$ is generated by the second derivatives of Q . Moreover, the adjoint variety $E(\mathbb{A})^{ad} \subset \mathbb{P}(\mathfrak{e}(\mathbb{A}))$ is the image of the rational map ψ .

The varieties $G_w(\mathbb{A}^3, \mathbb{A}^6) \subset \mathbb{P}V$ are also Legendrian for a $Sp_6(\mathbb{A})$ -invariant symplectic form Ω that generalizes the natural symplectic form on $S^3\mathbb{C}^2$, for which $v_3(\mathbb{P}^1)$ is Legendrian, see §5.

We also show in §5 that in some sense the quartic Q as well as Cayley's hyperdeterminant are determined by the classical discriminant.

The orbit structures for $G_w(\mathbb{A}^3, \mathbb{A}^6) \subset \mathbb{P}V$ are slightly more complicated than for $v_3(\mathbb{P}^1)$, as the first derivatives of the quartic define an intermediate orbit closure which we interpret as the locus of points on a family of secant lines, see §5.3. We remark that there are three orbits in the ambient space for the second row, four in the ambient space for the third row and five, not in $\mathbb{P}(\mathfrak{e}(\mathbb{A}))$, which contains an infinite number of orbits, but in the secant variety of $E(\mathbb{A})^{ad}$.

1.3 Notation

\mathbb{A}, \mathbb{B} denote complex composition algebras, i.e. $\mathbb{A} = \mathbb{A}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ where $\mathbb{A}_{\mathbb{R}} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or \mathbb{O} (the four real division algebras). If $a \in \mathbb{A}$, \bar{a} denotes its conjugate as an element of \mathbb{A} . $\mathbb{A} \otimes \mathbb{B}$ denotes the tensor product which has the algebra structure with multiplication $(\alpha \otimes b, \alpha' \otimes b') \mapsto \alpha\alpha' \otimes bb'$ and conjugation $\overline{\alpha \otimes b} = \bar{\alpha} \otimes \bar{b}$.

$\mathcal{J}_3(\mathbb{A})$ denotes the space of \mathbb{A} -Hermitian matrices of order three, with coefficients in \mathbb{A} :

$$\mathcal{J}_3(\mathbb{A}) = \left\{ \begin{pmatrix} r_1 & \bar{x}_3 & \bar{x}_2 \\ x_3 & r_2 & \bar{x}_1 \\ x_2 & x_1 & r_3 \end{pmatrix}, r_i \in \mathbb{C}, x_j \in \mathbb{A} \right\}.$$

$\mathcal{J}_3(\mathbb{A})$ has the structure of a *Jordan algebra* with the multiplication $A \circ B = \frac{1}{2}(AB + BA)$ where AB is the usual matrix multiplication. There is a well defined cubic form which we call the *determinant* on $\mathcal{J}_3(\mathbb{A})$.

We let $SO_3(\mathbb{A}) \subset GL_{\mathbb{C}}(\mathcal{J}_3(\mathbb{A}))$ denote the group of complex linear transformations preserving the Jordan multiplication (the name is motivated because the group can also be described as group preserving the cubic form and the quadratic form $Q(A) = \text{trace}(A^2)$). We have respectively $SO_3(\mathbb{A}) = SO_3, SL_3, Sp_6, F_4$.

We let $SL_3(\mathbb{A}) \subset GL_{\mathbb{C}}(\mathcal{J}_3(\mathbb{A}))$ denote the group of complex linear transformations preserving the determinant. Respectively $SL_3(\mathbb{A}) = SL_3, SL_3 \times SL_3, SL_6, E_6$.

$\mathcal{Z}_2(\mathbb{A})$ denotes the space of *Zorn matrices*,

$$\mathcal{Z}_2(\mathbb{A}) = \left\{ \begin{pmatrix} x & A \\ B & y \end{pmatrix} \mid x, y \in \mathbb{C}, A, B \in \mathcal{J}_3(\mathbb{A}) \right\}$$

It can be given the structure of an algebra, called a *Freudenthal algebra*, see [17], [11]. $Sp_6(\mathbb{A}) \subset Gl_{\mathbb{A}}(\mathcal{Z}_2(\mathbb{A}))$ respectively denotes the groups $Sp_6, SL_6, Spin_{12}, E_7$. It is the group preserving the quartic discriminant on $\mathcal{Z}_2(\mathbb{A})$ (see proposition 5.5).

If $Y \subset \mathbb{P}T_x G/P$ is a subvariety, we let $\tilde{Y} \subset T(G/P)$ denote the corresponding distribution.

If $X \subset \mathbb{P}V$, we let $\hat{X} \subset V$ denote the cone over X .

When there is an orbit closure in $\mathbb{P}V^*$ isomorphic to $X \subset \mathbb{P}V$, we utilize $X_* \subset \mathbb{P}V^*$ to denote this orbit closure.

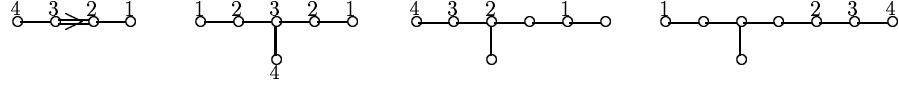
2 Freudenthal geometries

2.1 The magic square and the four geometries

Freudenthal associates to each group in the square a set of preferred homogeneous varieties (k spaces for each group in the k -th row). These spaces have the same incidence relations with the corresponding varieties for the groups in the same row. He calls the geometries associated to the groups of the rows respectively, *2-dimensional elliptic*, *2-dimensional plane projective*, *5-dimensional symplectic* and *metasymplectic*. The distinguished spaces are called respectively,

spaces of *points, lines, planes and symplecta*. To avoid confusion, we will use the terminology *F-points, F-planes* etc...

The spaces of elements are given by the following diagrams:



Here a 1 denotes the space of F-points, 2 the space of F-lines, 3 the space of F-planes, and 4 the space of F-symplecta in the metasymplectic geometries. (E.g. the space of F-points for E_6 is $E_6/P_{1,6}$, where we use the ordering of roots as in [3], and $P_{1,6}$ is the parabolic subgroup associated to the simple roots α_1 and α_6 .) Taking out the nodes numbered 4, we obtain the diagrams describing the three types of elements in the 5-dimensional symplectic geometries, and so on.

While Freudenthal was interested in the synthetic/axiomatic geometry of the spaces, we are primarily interested in the spaces as subvarieties of a projective space. We have taken embeddings of the spaces to make the geometries as uniform as possible. Below are the spaces, which are in their minimal homogeneous embeddings unless indicated. If $X \subset \mathbb{P}V$ is the minimal embedding, $v_d(X) \subset \mathbb{P}S^dV$ indicates the d -th Veronese re-embedding. We use standard nomenclature when there is one, and otherwise have continued the labelling by parabolic.

$v_4(\mathbb{P}^1)$	$\mathbb{F}_{1,2}^3$	$G_\omega(2, 6)$	$\mathbb{O}\mathbb{P}_0^2$				
$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$G(2, 6)$	$\mathbb{O}\mathbb{P}^2$	$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$G(2, 6)$	$\mathbb{O}\mathbb{P}^2$
$v_2(\mathbb{P}^5)$	$\mathbb{F}_{1,5}^6$	$G_o(2, 12)$	E_7^{ad}	$G_\omega(2, 6)$	$\mathbb{F}_{2,4}^6$	$G_o(4, 12)$	E_7/P_6
$v_2(\mathbb{O}\mathbb{P}_0^2)$	$E_6/P_{1,6}$	E_7/P_6	E_8/P_1	$v_2(F_4/P_3)$	$E_6/P_{3,5}$	E_7/P_4	E_8/P_6

F-points

F-lines

$G_\omega(3, 6)$	$G(3, 6)$	\mathbb{S}_{12}	E_7/P_7				
F_4/P_2	E_6/P_4	E_7/P_3	E_8/P_7	F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}

F-planes

F-symplecta

Here we use the following notations: $G(k, l)$ denotes the Grassmanian of \mathbb{C}^k 's in \mathbb{C}^l , $G_\omega(k, l)$ respectively $G_o(k, l)$ denotes the Grassmanian of \mathbb{C}^k 's in \mathbb{C}^l isotropic for a symplectic (resp. nondegenerate quadratic) form, $\mathbb{F}_{a,b}^c$ denotes the variety of flags $\mathbb{C}^a \subset \mathbb{C}^b$ in a fixed \mathbb{C}^c . $G^{ad} \subset \mathbb{P}\mathfrak{g}$ denotes the *adjoint variety* of G , the closed orbit in $\mathbb{P}\mathfrak{g}$.

We will use the following notations: the F-points in respectively the first, second and third rows, and columns corresponding to \mathbb{A} , will be denoted $\mathbb{A}\mathbb{P}_0^2, \mathbb{A}\mathbb{P}^2, Sp_6(\mathbb{A})^{ad} = G_\omega(\mathbb{A}^1, \mathbb{A}^6)$. (In particular, $\mathbb{O}\mathbb{P}^2$ is the Cayley plane, see [22].) The F-lines, resp. F-planes in the third row will be denoted $G_\omega(\mathbb{A}^2, \mathbb{A}^6)$, resp. $G_\omega(\mathbb{A}^3, \mathbb{A}^6)$. The notations are explained in §3.5.

2.2 The magic rectangle

If one allows the fifth algebra $\mathbb{A} = \underline{0}$, so that

$$\mathcal{J}_3(\underline{0}) = \left\{ \begin{pmatrix} r_1 & & \\ & r_2 & \\ & & r_3 \end{pmatrix} \mid r_j \in \mathbb{C} \right\}$$

one obtains a fifth column with Lie algebras $0, 0, \mathfrak{sl}_2 + \mathfrak{sl}_2 + \mathfrak{sl}_2, \mathfrak{d}_4$.

Everything works fine except that one obtains the unfortunate notations $Sl_3(\mathbb{Q}) = \mathbb{C}^* + \mathbb{C}^*$, $\mathfrak{sp}_6(\mathbb{Q}) = \mathfrak{sl}_3 + \mathfrak{sl}_3 + \mathfrak{sl}_3$, $\mathbb{Q}\mathbb{P}^2 = \mathbb{P}^0 \sqcup \mathbb{P}^0 \sqcup \mathbb{P}^0$, $G_w(\mathbb{Q}^3, \mathbb{Q}^6) = Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$.

The spaces of F-points, F-lines, F-planes and F-symplecta for \mathfrak{d}_4 are respectively the closed orbits in the projectivizations of the irreducible representations encoded in the following diagrams:



One can extend further to have a sixth column by letting $\Delta_{\mathcal{J}} \subset \mathcal{J}_3(\mathbb{Q})$ denote the homotheties. The corresponding Lie algebras are $0, 0, \mathfrak{sl}_2, \mathfrak{g}_2$.

Folding the diagram of \mathfrak{d}_4 along symmetries to obtain the diagram of \mathfrak{g}_2 , the F-planes and F-symplecta descend to give the representations $(3, 0)$ and $(0, 1)$ for \mathfrak{g}_2 . It is not clear if there are natural analogues for F-points and F-lines.

Remark. Another way to include \mathfrak{g}_2 is by adding a fifth column with $\mathbb{A} = 0$ in the naive way. Such a column also has Lie algebras $0, 0, \mathfrak{so}_3, \mathfrak{g}_2$.

Proposition 2.1 *Let $m = 0, 1, 2, 4, 8$. The dimensions of the spaces of elements are as follows:*

	F-points	F-lines	F-planes	F-symplecta
First row	$2m - 1$			
Second row	$2m$	$2m$		
Third row	$4m + 1$	$5m + 2$	$3m + 3$	
Fourth row	$9m + 6$	$11m + 9$	$9m + 11$	$6m + 9$

Freudenthal remarked that for the fourth row, the usual duality between elements of complementary dimension is lost already at the level of their dimensions. We that vestiges of this duality remain. In particular, giving the F-spaces “F-dimensions” $1, 2, 3, 4$ respectively for F-points, F-lines, F-planes, F-symplecta, if some geometric element describes a space of dimension $um + v$, the element of complementary F-dimension describes a space of dimension $vm + u$!

Remark. Deligne, Cohen and de Man [7, 8, 5] have uniform descriptions of the dimensions of the representation spaces of the exceptional Lie algebras. The dimensions are given in terms of rational functions of a parameter $\mu = \frac{2}{m+2}$. The ambient spaces for the spaces of points, lines, planes and symplecta are respectively labelled X_3, C^*, Y_2^*, X_2 .

2.3 The magic square for all n

One can define a magic square for all n , only one loses the fourth row and column:

$$\begin{array}{ccc} \mathfrak{so}_n & \mathfrak{sl}_n & \mathfrak{sp}_{2n} \\ \mathfrak{sl}_n & \mathfrak{sl}_n \times \mathfrak{sl}_n & \mathfrak{sl}_{2n} \\ \mathfrak{sp}_{2n} & \mathfrak{sl}_{2n} & \mathfrak{spin}_{4n}, \end{array}$$

where now

$$\mathfrak{g} = Der\mathbb{A} \oplus (\mathbb{A}_0 \otimes \mathcal{J}_n(\mathbb{B})_0) \oplus Der\mathcal{J}_n(\mathbb{B}).$$

We have the following chart of F-points

$$\begin{array}{ccc} v_2(Q^{n-2}) & \mathbb{F}_{1,n-1}^n & G_\omega(2, 2n) \\ v_2(\mathbb{P}^{n-1}) & Seg(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) & G(2, 2n) \\ v_2(\mathbb{P}^{2n-1}) & \mathbb{F}_{1,2n-1}^{2n} & G_o(2, 4n). \end{array}$$

The analogue of F-planes, or perhaps better to say F-hyperplanes, for the third row are the minuscule varieties

$$G_\omega(n, 2n) \quad G(n, 2n) \quad \mathbb{S}_{2n}.$$

2.4 F-Schubert varieties and F-incidence

Let P, Q be parabolic subgroups of G . Consider the diagram

$$\begin{array}{ccc} & G & \\ p \swarrow & & \searrow q \\ X = G/P & & G/Q = Y \end{array}$$

and define $\Sigma_X^{Y, y_0} = \Sigma_X^Y = p(q^{-1}(y_0)) = \{x \in X \mid x \text{ is incident to } y_0\}$. In the language of [21], Σ_X^{Y, y_0} is the (Y, X) Tits-transform of y_0 . We note that such a Schubert variety furnishes a homogeneous vector bundle over Y by taking the fiber over $y \in Y$ to be the linear span of the cone over $\Sigma_X^{Y, y}$.

The Freudenthal spaces distinguish certain Schubert varieties which we will call *F-Schubert varieties*. The F-Schubert varieties have uniform behavior as one changes \mathbb{A} and exhibit similarities as one changes the row. They play a role in understanding the geometries of the F-varieties analogous to the role of classical Schubert varieties for understanding Grassmanians.

Let $m = 1, 2, 4, 8$. In the case of the first row there is nothing to say. For the second row, the variety of F-points incident to an F-line is an $\mathbb{A}\mathbb{P}^1 = Q^m$ and of course the variety of F-lines incident to an F-point is an $\mathbb{A}\mathbb{P}^1 = Q^m$ as the two spaces are isomorphic. This symmetry is broken with the third row as $\mathbb{A}\mathbb{P}^1 = Q^m = G_o(1, m+2)$ generalizes in two different ways, to $\mathbb{A}\mathbb{P}^2$ and to $G_o(2, m+4)$.

For the third row we have

	F-points	F-lines	F-planes
F-points		$\mathbb{A}\mathbb{P}^1$	$\mathbb{A}\mathbb{P}^2$
F-lines	$G_o(2, m+4)$		$\mathbb{A}\mathbb{P}^2$
F-planes	$G_o(1, m+4)$	\mathbb{P}^1	

and for the fourth:

	F-points	F-lines	F-planes	F-symplecta
F-points		$\mathbb{A}\mathbb{P}^1$	$\mathbb{A}\mathbb{P}^2$	$G_\omega(\mathbb{A}^1, \mathbb{A}^6)$
F-lines	$G_o(3, m+6)$		$\mathbb{A}\mathbb{P}^2$	$G_\omega(\mathbb{A}^2, \mathbb{A}^6)$
F-planes	$G_o(2, m+6)$	\mathbb{P}^2		$G_\omega(\mathbb{A}^3, \mathbb{A}^6)$
F-symplecta	$G_o(1, m+6)$	\mathbb{P}^2	\mathbb{P}^1	

The \mathbb{P}^2 's corresponding to $\Sigma_{X^{F\text{-symplecta}}}^{X^{F\text{-lines}}}$ are embedded by the quadratic Veronese embedding.

One can also study the incidence relations among elements of the same space. Freudenthal ([12], pp. 169-171) describes *uniform* incidence relations for the distinguished varieties of each row, which we utilize in our study. See §6.

3 Tangent spaces

We describe the tangent bundles in each of the Freudenthal geometries G/P . Recall from [21] that, as a module over a maximal semi-simple subgroup of P , the decomposition of $T(G/P)$ into irreducible H -modules can be read off the root system of \mathfrak{g} . Indeed, up to conjugation, the

parabolic group P is determined by a set of simple roots (a single root when P is maximal), say I . Then the irreducible components of the tangent bundle are, roughly speaking, in correspondance with the possible coefficients of positive roots over the simple roots in I . We shall denote by T_k the sum of the irreducible components of T defined by coefficients over these simple roots with the coefficients summing to k .

3.1 F-points

In the case of points, the tangent space has one or two components, given by the following tables:

\mathbf{T}_1	\mathbb{C}	$\mathbb{C} \oplus \mathbb{C}$	$S^* \otimes S^\perp / S$	Δ		\mathbf{T}_2	$*$	\mathbb{C}	\mathbb{C}^3	\mathbb{C}^7
	\mathbb{C}^2	$\mathbb{C}^2 \oplus \mathbb{C}^2$	$S^* \otimes Q$	Δ_+			$*$	$*$	$*$	$*$
	\mathbb{C}^4	$\mathbb{C}^4 \oplus \mathbb{C}^4$	$S^* \otimes S^\perp / S$	Δ_+			\mathbb{C}	\mathbb{C}	\mathbb{C}	\mathbb{C}
	Δ	$\Delta \oplus \Delta$	$S^* \otimes \Delta$	Δ_+			\mathbb{C}^7	\mathbb{C}^8	\mathbb{C}^{10}	\mathbb{C}^{14}

Proposition 3.1 *Let $X = G/P$ be the space of F-points in the Freudenthal geometry associated to the pair of composition algebras (\mathbb{A}, \mathbb{B}) and let $x \in X$. Let $T_1 \subset T_x X$ denote the smallest P -invariant sub-module. Let H be a maximal semi-simple subgroup of P . Then H is a spin group (or product of spin groups) and T_1 is a spin representation (or product of such). Moreover,*

$$T_1 \simeq \mathbb{A} \otimes \mathbb{B}.$$

Proof. Let $\Delta^k = \Delta^{B_k}$, $\Delta_+^k = \Delta_+^{D_k}$, $\Delta_-^k = \Delta_-^{D_k}$. We rewrite the table for T_1 as

Δ^0	$\Delta_+^0 \oplus \Delta_-^0$	$\Delta_+^1 \otimes \Delta_+^1$	Δ^3
Δ^1	$\Delta_+^1 \oplus \Delta_-^1$	$\Delta_+^1 \otimes \Delta_+^2$	Δ_+^5
Δ^2	$\Delta_+^3 \oplus \Delta_-^3$	$\Delta_+^1 \otimes \Delta_+^4$	Δ_+^6
Δ^3	$\Delta_+^3 \oplus \Delta_-^3$	$\Delta_+^1 \otimes \Delta_+^5$	Δ_+^7

The fact that several spin representations of small dimensions have natural realizations given by composition algebras can be found in [14], from which all cases except for $\mathbb{O} \otimes \mathbb{O}$ can be deduced. The case of $\mathbb{O} \otimes \mathbb{O}$ follows from proposition 3.3 below, which gives a general relation between Clifford algebras and composition algebras. \square

Note that in the magic chart for $n > 3$, the tangent space to points in the third row does not have an analogous interpretation.

The space T_2 for points does not have a very regular behavior. Note however that for the first and last lines T_2 has the interpretation of $\mathbb{A}_0 \oplus \mathbb{B}_0$. While T_2 does not behave well for points, we have the following proposition:

Proposition 3.2 *Notations as above. For each Lie algebra \mathfrak{g} in Freudenthal's magic square, there is a parabolic subgroup \mathfrak{p} of \mathfrak{g} such that the quotient $\mathfrak{g}/\mathfrak{p}$ decomposes into the sum of*

$$T_1 \simeq \mathbb{A} \otimes \mathbb{B} \quad \text{and} \quad T_2 \simeq \mathbb{A}_0 \oplus \mathbb{B}_0,$$

where \mathfrak{g} is associated to the pair of composition algebras (\mathbb{A}, \mathbb{B}) .

H is a spin group (or product of spin groups), T_1 is a spin representation (or product of such) and T_2 is a vector representation.

For the first and last lines of the square, the corresponding G/P 's are the spaces of F-points, while for the third line they are the spaces of F-lines. For the second line they are the spaces of incident pairs of F-points and F-lines. Note in particular that the square below, formed by these G/P is perfectly symmetric, although the geometric interpretations are not.

$$\begin{array}{cccc}
v_4(\mathbb{P}^1) & \mathbb{F}_{1,2} & G_\omega(2,6) & \mathbb{O}\mathbb{P}_0^2 \\
\mathbb{F}_{1,2} & \mathbb{F}_{1,2} \times \mathbb{F}_{1,2} & \mathbb{F}_{2,4} & E_6/P_{1,6} \\
G_\omega(2,6) & \mathbb{F}_{2,4} & G_o(4,12) & E_7/P_6 \\
\mathbb{O}\mathbb{P}_0^2 & E_6/P_{1,6} & E_7/P_6 & E_8/P_1
\end{array}$$

Proposition 3.3 *Let \mathbb{A}, \mathbb{B} be complex composition algebras, other than the complexification of \mathbb{R} . Let $\mathbb{A}_0 \oplus \mathbb{B}_0$ be endowed with the quadratic form $Q(a+b) = a\bar{a} + b\bar{b} = -a^2 - b^2$. Then there is a natural diagram of maps of algebras*

$$\begin{array}{ccc}
Cl(\mathbb{A}_0 \oplus \mathbb{B}_0, Q) & \rightarrow & End(\mathbb{A} \otimes \mathbb{B} \oplus \mathbb{A} \otimes \mathbb{B}) \\
\cup & & \cup \\
Cl(\mathbb{A}_0 \oplus \mathbb{B}_0, Q)^{even} & \rightarrow & End(\mathbb{A} \otimes \mathbb{B}) \otimes End(\mathbb{A} \otimes \mathbb{B}).
\end{array}$$

The inclusion on the right is the ‘‘diagonal’’ inclusion. When $\mathbb{A} = \mathbb{B} = \mathbb{O}$ the dimensions of $Cl(\mathbb{O}_0 \oplus \mathbb{O}_0, Q)^{even}$ and $End(\mathbb{O} \otimes \mathbb{O}) \otimes End(\mathbb{O} \otimes \mathbb{O})$ coincide, showing that the two half-spin representations of $Spin_{14}$ have natural realizations on $\mathbb{O} \otimes \mathbb{O}$.

Proof. By the fundamental lemma of Clifford algebras, see [14], we have to construct a map

$$\phi : \mathbb{A}_0 \oplus \mathbb{B}_0 \rightarrow End(\mathbb{A} \otimes \mathbb{B} \oplus \mathbb{A} \otimes \mathbb{B})$$

such that $\phi(a+b)^2 = Q(a+b)Id_{\mathbb{A} \otimes \mathbb{B} \oplus \mathbb{A} \otimes \mathbb{B}}$, as then there exists a unique extension to a map $\tilde{\phi} : Cl(\mathbb{A}_0 \oplus \mathbb{B}_0, Q) \rightarrow End(\mathbb{A} \otimes \mathbb{B} \oplus \mathbb{A} \otimes \mathbb{B})$. Consider

$$\phi(a+b)(\alpha \otimes \beta, \gamma \otimes \delta) = (ia\gamma \otimes \bar{\delta} + \gamma \otimes \bar{\delta}b, ia\alpha \otimes \bar{\beta} + \alpha \otimes b\bar{\beta}).$$

A short calculation shows that ϕ has the required property. Moreover, since Cl^{even} is generated by the products of even numbers of vectors in $\mathbb{A}_0 \oplus \mathbb{B}_0$, the diagram follows. \square

3.2 F-lines

The components of the tangent spaces for F-lines are as follows:

$$\begin{array}{cccccc}
\mathbf{T}_1 & \mathbb{C}^2 & \mathbb{C}^2 \oplus \mathbb{C}^2 & \mathbb{C}^2 \otimes \mathbb{C}^4 & \Delta_+ & \mathbf{T}_3 & * & * & * & * \\
& \mathbb{C}^2 \otimes \mathbb{C}^2 & \mathbb{C}^2 \otimes \mathbb{C}^2 \oplus \mathbb{C}^2 \otimes \mathbb{C}^2 & \mathbb{C}^4 \otimes \mathbb{C}^4 & \mathbb{C}^2 \otimes \Delta_+ & & * & * & * & * \\
& \mathbb{C}^2 \otimes \mathbb{C}^3 & \mathbb{C}^2 \otimes \mathbb{C}^3 \oplus \mathbb{C}^2 \otimes \mathbb{C}^3 & \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4 & \mathbb{C}^3 \otimes \Delta_+ & & \mathbb{C}^2 & \mathbb{C}^4 & \mathbb{C}^8 & \mathbb{C}^{16} \\
\mathbf{T}_2 & * & * & * & * & \mathbf{T}_4 & * & * & * & * \\
& \mathbb{C}^3 & \mathbb{C}^4 & \mathbb{C}^6 & \mathbb{C}^{10} & & * & * & * & * \\
& \mathbb{C}^9 & \mathbb{C}^{12} & \mathbb{C}^{18} & \mathbb{C}^{30} & & \mathbb{C}^3 & \mathbb{C}^3 & \mathbb{C}^3 & \mathbb{C}^3
\end{array}$$

Let $Y_{\mathbb{A}}$ respectively denote $\emptyset, \mathbb{P}^1 \sqcup \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^3, S_5$ and let $H_{\mathbb{A}}$ respectively denote $SL_2, SL_2 \times SL_2, SL_2 \times SL_4$ and $Spin_{10}$. Note that $Y_{\mathbb{A}} \subset \mathbb{P}(\mathbb{A} \oplus \mathbb{A})$ and if we give $\mathbb{A} \oplus \mathbb{A}$ coordinates (u, v) then $I_2(Y_{\mathbb{A}}) = \{u\bar{u}, v\bar{v}, u\bar{v}\}$, having respectively 3, 4, 6 and 10 generators. Examining the spaces above, we obtain:

Proposition 3.4 *Let $X = X_{F\text{-lines}}^{p,\mathbb{A}} = G/P$ denote the space of F -lines in the p -th row whose composition algebra is \mathbb{A} . With the same notations as above,*

$$\begin{aligned} T_1 &= \mathbb{C}^{p-1} \otimes \mathbb{A}^2 \\ Y_1 &= \text{Seg}(\mathbb{P}^{p-2} \times Y_{\mathbb{A}}) \\ H &= SL_{p-1} \times H_{\mathbb{A}} \\ T_2 &= \Lambda^2 \mathbb{C}^{p-1} \otimes I_2(Y_{\mathbb{A}}). \end{aligned}$$

3.3 F-planes

Proposition 3.5 *Let $X = X_{F\text{-planes}}^{p,\mathbb{A}} = G/P$ denote the space of F -planes in the p -th row whose composition algebra is \mathbb{A} . With the same notations as above,*

$$\begin{aligned} T_1 &= \mathbb{C}^{p-2} \otimes \mathcal{J}_3(\mathbb{A}) \\ Y_1 &= \text{Seg}(\mathbb{P}^{p-3} \times \mathbb{A}\mathbb{P}^2) \\ H &= SL_{p-2} \times SL_3\mathbb{A} \end{aligned}$$

For $p = 3$, $T = T_1$ and for $p = 4$, $T_2 = \mathcal{J}_3(\mathbb{A})^$ and $T_3 = \mathbb{C}^2$.*

Unlike the case of F -points, the structure of the tangent space for F -hyperplanes has a similar interpretation for all n . As with the $n = 3$ chart, the tangent directions to lines through a point is the space of F -points of the second row. Let $\mathbb{A}\mathbb{P}^n$ respectively denote $v_2(\mathbb{P}^{n-1})$, $\text{Seg}(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$, $G(2, 2n)$ and let $SL_n(\mathbb{A})$ respectively denote SL_n , $SL_n \times SL_n$, SL_{2n} :

Proposition 3.6 *Let $X = X_{F\text{-planes}}^{\mathbb{A}} = G/P$ denote the space of F -hyperplanes in the 3-rd row of the generalized chart whose composition algebra is*

\mathbb{A} . With the same notations as above, $T_2 = 0$ and

$$\begin{aligned} T_1 &= \mathcal{J}_n(\mathbb{A}) \\ Y_1 &= \mathbb{A}\mathbb{P}^{n-1} \\ H &= SL_n(\mathbb{A}). \end{aligned}$$

3.4 F-symplecta

The spaces of F -symplecta are the adjoint varieties of the exceptional groups other than G_2 .

Proposition 3.7 *Let $X = X_{F\text{-symplecta}}^{\mathbb{A}} = G/P$ denote the space of F -symplecta whose composition algebra is \mathbb{A} . With the same notations as above,*

$$\begin{aligned} T_1 &= \mathcal{Z}_2(\mathbb{A}) \\ T_2 &= \mathbb{C} \\ Y_1 &= G_w(\mathbb{A}^3, \mathbb{A}^6) \\ H &= Sp_6(\mathbb{A}) \end{aligned}$$

3.5 Interpretations as Grassmanians

Let $X = G/P \subset \mathbb{P}V$ be a homogeneous variety with P maximal. When G is a classical group, X can be characterized as a family of k -planes in some natural representation of G . We investigate the existence of similar characterizations in the exceptional cases, in terms of composition algebras (for a different kind of such characterizations, see [22], Corollary 7.8). For the varieties of F -points in the second row of the magic square, $X = G(\mathbb{A}^1, \mathbb{A}^3) = \mathbb{A}\mathbb{P}^2$ and in [21] we gave the interpretation of the varieties of F -points in the first row as $G_Q(\mathbb{A}^1, \mathbb{A}^3) = \mathbb{A}\mathbb{P}_0^2$. (Being

null for the cubic and the trace is equivalent to being null for the cubic and quadratic forms $Q(x) = \text{tr}(x \circ x)$ where \circ is the Jordan multiplication in $\mathcal{J}_3(\mathbb{A})$.)

Rozenfeld announces ([26], theorem 7.22) a unified geometric interpretation of certain varieties in the chart, which he calls elliptic planes over $\mathbb{A} \otimes \mathbb{B}$. Also, before Rozenfeld's work was available in English, E. Weinstein [28] had conjectured that there should be a unified interpretation of varieties in the chart as some type of Grassmanian over $\mathbb{A} \otimes \mathbb{B}$.

As a first step towards Weinstein's conjecture, we felt that just as the tangent space to an ordinary Grassmanian has an interpretation as $T_E G(k, V) = E^* \otimes V/E$, for there to be a unified interpretation of the chart, there should be a unified interpretation of tangent spaces, and this infinitesimal problem is solved above. In what follows we suggest global interpretations based on our infinitesimal calculations.

We begin with the F-varieties of the third row. If $P \in \mathcal{J}_3(\mathbb{A})$, its *comatrix* is defined by

$$\text{com}(P) = P^2 - (\text{trace } P)P + \frac{1}{2}((\text{trace } P)^2 - \text{trace } P^2)I,$$

and characterized by the identity $\text{com}(P)P = \det(P)I$. Thus the linear form $P \mapsto \text{trace}(\text{com}(P)P)$ is a polarization of the determinant. The varieties of F-planes in the third row are the image of the rational map ϕ described in §1.2. On an affine open subset we may write

$$\phi(1, P) = (1, P, \text{com}(P), \det(P)).$$

Note in particular we recover the natural identification $T_x X_{F\text{-planes}}^{3, \mathbb{A}} \simeq \mathcal{J}_3(\mathbb{A})$ from the map ϕ alone. Moreover, if (I_3, P) is interpreted as a matrix of three row vectors in \mathbb{A}^6 , the map ϕ is the usual Plucker map. The condition that $P \in \mathcal{J}_3(\mathbb{A})$ can be interpreted as the fact that the three vectors defined by the matrix (I, P) are orthogonal with respect to the Hermitian symplectic two-form $w(x, y) = {}^t x A \bar{y}$, where $A = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. It is therefore natural to see $X_{F\text{-planes}}^{3, \mathbb{A}}$ as a Grassmannian of symplectic three-planes in \mathbb{A}^6 . This motivates our notation

$$X_{F\text{-planes}}^{3, \mathbb{A}} = G_\omega(\mathbb{A}^3, \mathbb{A}^6).$$

Similarly, consider a matrix of two vectors in \mathbb{A}^6 , of the form (I_2, R, S) , where R and S are two matrices of order two. These two vectors are orthogonal with respect to the Hermitian symplectic two-form w if and only if S is Hermitian. The space of F-lines can be interpreted as a Grassmannian $G_\omega(\mathbb{A}^2, \mathbb{A}^6)$ of symplectic two-planes in \mathbb{A}^6 , and its tangent space, as expected, decomposes into $T_1 = \mathbb{H} \otimes \mathbb{A}$ and $T_2 = \mathcal{J}_2(\mathbb{A})$.

The space of F-points also has an interpretation as $G_\omega(\mathbb{A}^1, \mathbb{A}^6)$, its tangent space decomposing into $T_1 = \mathbb{H} \otimes \mathbb{A}$ and $T_2 = \mathbb{C}$.

In summary:

Proposition 3.8 *The spaces of F-points, F-lines and F-planes for the third line of the magic square, have a natural interpretation as symplectic Grassmannians $G_\omega(\mathbb{A}^k, \mathbb{A}^6)$, with $k = 1, 2, 3$ respectively. Their tangent spaces are respectively $\mathbb{H} \otimes \mathbb{A} \oplus \mathbb{C}$, $\mathbb{H} \otimes \mathbb{A} \oplus \mathcal{J}_2(\mathbb{A})$ and $\mathcal{J}_3(\mathbb{A})$.*

Problem. *Find a unified interpretation of the F-varieties of the fourth row.*

Consider the quadratic form on $(\mathbb{A} \otimes \mathbb{B})^3$, with values in $\mathbb{A} \otimes \mathbb{B}$, given for $x = (x_1, x_2, x_3)$ by $Q(x) = x_1 \bar{x}_3 + x_2 \bar{x}_2$, and consider the space

$$G_Q(\mathbb{A} \otimes \mathbb{B}, (\mathbb{A} \otimes \mathbb{B})^3) := \{x \in (\mathbb{A} \otimes \mathbb{B})^3 \mid Q(x) = 0\}.$$

Then $T_x G_Q(\mathbb{A} \otimes \mathbb{B}, (\mathbb{A} \otimes \mathbb{B})^3) = \{y \mid Q(x, y) = 0\}$. If $x = (1, 0, 0)$, we need $y_2 \in \text{Im}(\mathbb{A} \otimes \mathbb{B}) \simeq \mathbb{A}_0 \oplus \mathbb{B}_0$ and there is no restriction on $y_3 \in \mathbb{A} \otimes \mathbb{B}$. This suggests that the varieties in proposition 4.3 (not the varieties of F-points) admit a common interpretation as $G_Q(\mathbb{A} \otimes \mathbb{B}, (\mathbb{A} \otimes \mathbb{B})^3)$.

Regarding F-varieties for $n > 3$, we have the following proposition:

Proposition 3.9 *Let $n > 3$. We have the following interpretations:*

F-points of the first row: $\mathbb{A}\mathbb{P}_0^{n-1}$

F-points of the second row: $\mathbb{A}\mathbb{P}^{n-1}$

F-hyperplanes of the third row: $G_w(\mathbb{A}^n, \mathbb{A}^{2n})$.

In particular, we have the identification $\mathbb{S}_{2n} = G_w(\mathbb{H}^n, \mathbb{H}^{2n})$.

Proof. Note that $SL_n(\mathbb{H}) = SL_{2n}$ to obtain that the points of the second row are indeed $\mathbb{A}\mathbb{P}^{n-1}$ and the first are $\mathbb{A}\mathbb{P}_0^{n-1}$ as in the four by four case. For the third row, one uses the same argument as above, only note that the corresponding Plucker type mapping is of degree n . \square

4 Folding and hyperplane sections of Severi varieties

4.1 Severi varieties

We have little new to say about the F-points (or F-lines) of the second row, otherwise known as the *Severi varieties* $\mathbb{A}\mathbb{P}^2 \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ which may be described as the projectivization of the rank one elements of $\mathcal{J}_3(\mathbb{A})$. Their secant varieties $\sigma(\mathbb{A}\mathbb{P}^2)$ are the rank at most two elements, i.e., the hypersurface $\det = 0$. Throughout this section we let $m = 1, 2, 4, 8$.

We record the following known proposition:

Proposition 4.1 *$SL_3(\mathbb{A})$ has three orbits on $\mathbb{P}\mathcal{J}_3(\mathbb{A})$, namely $\mathbb{A}\mathbb{P}^2$, $\sigma(\mathbb{A}\mathbb{P}^2) \setminus \mathbb{A}\mathbb{P}^2$, and the open orbit $\mathbb{P}\mathcal{J}_3(\mathbb{A}) \setminus \sigma(\mathbb{A}\mathbb{P}^2)$, which respectively correspond to the matrices of rank one, two and three.*

The unirulings of $\mathbb{A}\mathbb{P}^2$ are described in [21]. They are all $SL_3(\mathbb{A})$ -homogeneous, i.e., given by Tits transforms. Here we give several descriptions of the ruling of $\sigma(\mathbb{A}\mathbb{P}^2)$ by \mathbb{P}^{m+1} 's:

The rulings of $\sigma(\mathbb{A}\mathbb{P}^2)$ were implicitly described by Zak as follows: for $p \in \sigma(\mathbb{A}\mathbb{P}^2) \setminus \mathbb{A}\mathbb{P}^2$, let

$$\Sigma_p = \overline{\{x \in \mathbb{A}\mathbb{P}^2 \mid \exists y \in \mathbb{A}\mathbb{P}^2 \text{ such that } p \in \mathbb{P}_{xy}^1\}}$$

the *entry locus* of p . Then Zak [29] shows that Σ_p is a quadric hypersurface in a \mathbb{P}^{m+1} , i.e., an $\mathbb{A}\mathbb{P}^1$, and that $\sigma(\mathbb{A}\mathbb{P}^2)$ is therefore ruled by these \mathbb{P}^{m+1} 's.

Another way to view this ruling is as follows: Let $\mathbb{A}\mathbb{P}_*^2 \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})^*$ denote the closed orbit in the dual projective space. Then $\sigma(\mathbb{A}\mathbb{P}^2)^* = \mathbb{A}\mathbb{P}_*^2$, i.e. the dual of the secant variety of $\mathbb{A}\mathbb{P}^2$ is the the closed orbit in the dual projective space (and $\sigma(\mathbb{A}\mathbb{P}^2) = (\mathbb{A}\mathbb{P}_*^2)^*$ by the reflexivity theorem). Let N^* denote the conormal bundle to $\mathbb{A}\mathbb{P}_*^2$. Given $H \in \mathbb{A}\mathbb{P}_*^2$,

$$\mathbb{P}N_H^* = \{p \in \sigma(\mathbb{A}\mathbb{P}^2) \mid \tilde{T}_p \sigma(\mathbb{A}\mathbb{P}^2) \subset H\}$$

is the corresponding \mathbb{P}^{m+1} for any $p \in \mathbb{P}N_H^* \setminus \mathbb{A}\mathbb{P}^1 \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$.

The rulings may also be seen from Freudenthal's perspective: $\mathbb{A}\mathbb{P}^2$ is the space of F-points and $\mathbb{A}\mathbb{P}_*^2$ is the space of F-lines. The F-Schubert variety of a $p \in \mathbb{A}\mathbb{P}_*^2$ is an $\mathbb{A}\mathbb{P}^1 = Q^m \subset \mathbb{A}\mathbb{P}^2$ and this $\mathbb{A}\mathbb{P}^1$ is the variety describe above.

The Severi varieties were constructed by Zak using a degree two map defined by the quadrics vanishing on $Y_{\mathbb{A}} \subset \mathbb{P}(\mathbb{A} \oplus \mathbb{A}) = \mathbb{P}^{n-1} \subset \mathbb{P}^n$, see [29]. This construction can be generalized to construct the varieties of F-lines in the third and fourth rows as well, using $Y = \text{Seg}(\mathbb{P}^1 \times Y_{\mathbb{A}})$ for the third row and $Y = \text{Seg}(\mathbb{P}^2 \times Y_{\mathbb{A}})$ for the fourth row. See [23].

4.2 Geometric folding

To deduce the first line of the magic square from the second one we use the *folding* of a root system. Consider some Dynkin diagram with a two-fold symmetry θ , and let \mathfrak{g} be the corresponding simple Lie algebra. If we choose a system of Chevalley generators for \mathfrak{g} , there is a uniquely defined algebra involution of \mathfrak{g} inducing the automorphism θ of the simple roots [25]. Hence a decomposition $\mathfrak{g} = \mathfrak{f} \oplus W$ into eigenspaces, where \mathfrak{f} is a Lie subalgebra, and W has a natural \mathfrak{f} -module structure. A case-by-case examination then gives:

Proposition 4.2 *There is a commutative (in general non-associative) \mathfrak{f} -equivariant multiplication on W , and $V = \mathbb{C} \oplus W$ is a simple \mathfrak{g} -module.*

In most cases V inherits an algebra structure for which $F \subset GL(W) \subset GL(V)$ is the group preserving the structure. Here is a chart summarizing the representations arising from folding:

W	\mathbb{C}^{2n+1}	$\mathcal{J}_3(\mathbb{A})_0$	$\Lambda^{(2)}\mathbb{C}^{2n} = \mathcal{J}_n(\mathbb{H})_0$	\mathfrak{sl}_n
V	\mathbb{C}^{2n+2}	$\mathcal{J}_3(\mathbb{A})$	$\Lambda^2\mathbb{C}^{2n} = \mathcal{J}_n(\mathbb{H})$	$M_n(\mathbb{C})$

Remark. It follows from the results in [1] that if $X \subset \mathbb{P}V$ is a homogeneous variety (under a semi-simple group) such that a generic hyperplane section of X is still homogeneous, and if $X \subset \mathbb{P}V$ is not $\mathbb{P}^m \subseteq \mathbb{P}^m$ or $Q^m \subset \mathbb{P}^{m+1}$ (the two self-reproducing cases), then it must be the variety of F-points in the second row of a magic chart and $X \cap H \subset H$ is the corresponding variety of F-points in the first row.

Note that there is a slight anomaly in that the chart does not exactly correspond to geometric folding. However the exception, $v_2(Q)$ can be seen as a special case of a different phenomenon: *any* homogeneous variety $X = G/P \subset \mathbb{P}V$ can be realized as $v_2(\mathbb{P}V) \cap \mathbb{P}(S^{(2)}V)$. In the case of $v_2(Q)$, $\mathbb{P}(S^{(2)}V)$ happens to be a generic hyperplane. Note also that the $Q^{2m} \cap H$ section is accounted for by geometric folding but the $Q^{2n+1} \cap H$ section is not.

Let $H_F \subset H_G$ denote the corresponding maximal semi-simple subgroups of the isotropy groups fixing a point of $x \in X \cap H \subset X$. In the case of $X = \mathbb{O}\mathbb{P}^2$, $H_G = Spin_{10}$ acts irreducibly on $TX = \Delta_+$, a sixteen-dimensional half-spin representation. As an $H_F = Spin_7$ -module $T_x\mathbb{O}\mathbb{P}_0^2 = \Delta \oplus \mathbb{C}^7$, the sum of the spin and the vector representations.

This decomposition is interesting because spin representations of spinor groups usually decompose into sums of spin representations when restricted to smaller spinor groups. The appearance of the vector representation may be understood in terms of the triality automorphism of $Spin_8$. We consider $Spin_7 \subset Spin_8$. The relevant embedding $Spin_8 \subset Spin_{10}$ is such that, because of triality, $\Delta_+^{D_5}$ decomposes as $\Delta_+^{D_4} \oplus V^{D_4}$ as a D_4 -module. When one restricts further to $Spin_7 \subset Spin_8$, $V^{D_4} = V^{B_3} \oplus \mathbb{C}$ and $\Delta_+^{D_4} = \Delta^{B_3}$.

One can see the situation pictorially by considering the fold of the Dynkin diagram of E_6 into the diagram of F_4 :



5 F-planes in the third row

5.1 Constructions

A special case of the minuscule algorithm in [22] constructs $G_w(\mathbb{A}^3, \mathbb{A}^6) \subset \mathbb{P}(\mathcal{Z}_2(\mathbb{A}))$ from $\mathbb{A}\mathbb{P}^2$ via a degree three mapping, as well as constructing $\mathfrak{sp}_6(\mathbb{A})$ from $\mathfrak{sl}_3(\mathbb{A})$ as $\mathfrak{sp}_6(\mathbb{A}) = \mathbb{C}^* \oplus \mathcal{J}_3(\mathbb{A})^* \oplus (\mathfrak{sl}_3\mathbb{A} + \mathbb{C}) \oplus \mathcal{J}_3(\mathbb{A}) \oplus \mathbb{C}$. The construction also produces the increasing filtration of $\mathcal{Z}_2(\mathbb{A})$ as a $U(\mathfrak{sl}_3\mathbb{A})$ -module, namely

$$\mathcal{Z}_2(\mathbb{A}) = \mathbb{C} \oplus \mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A})^* \oplus \mathbb{C}.$$

The action of $\mathfrak{sp}_6(\mathbb{A})$ can also be described in terms of creation and annihilation, see [22].

Remark. The F-planes in the fourth row can be constructed by a mapping defined by polynomials vanishing on $Y_1 = \mathbb{P}^1 \times \mathbb{A}\mathbb{P}^2$, $Y_2 = \mathbb{A}\mathbb{P}^2_*$ and their auxilliary varieties. See [22], [23] for details.

Here is an alternate construction of $\mathfrak{sp}_6(\mathbb{A})$ and $\mathcal{Z}_2(\mathbb{A})$ that makes no reference to composition algebras, one only uses the existence of an invariant cubic polynomial:

Theorem 5.1 (*geometric version*) *Let $Z = H/Q \subset \mathbb{P}W$ be a homogeneous variety with H simple having the properties that closure of the largest H -orbit in $\mathbb{P}W$ is a cubic hypersurface and that $I_2(Z)$ is an irreducible H -module.*

Then $\mathfrak{g} = W^ \oplus (\mathfrak{h} + \mathbb{C}) \oplus W$ is a simple Lie algebra and $V = \mathbb{C} \oplus W \oplus W^* \oplus \mathbb{C}$ has a natural structure of simple \mathfrak{g} -module. Moreover, if $X \subset \mathbb{P}V$ denotes the closed G -orbit, then the space of \mathbb{P}^1 's in X through a point x is isomorphic to Z .*

The significance of this theorem is due to the set of varieties satisfying its hypotheses:

Proposition 5.2 *The varieties satisfying the hypotheses of theorem 5.1 are $\emptyset \subset \mathbb{P}^1$ and $\mathbb{A}\mathbb{P}^2 \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$. The varieties X so produced are the varieties occurring as the space of lines through a point of G^{ad} where G is an exceptional group, i.e., X is $v_3(\mathbb{P}^1)$ and $G_w(\mathbb{A}^3, \mathbb{A}^6)$.*

Note that the hypotheses force $I_2(Z) \simeq W$ as \mathfrak{h} -modules because Z is contained in the cubic hypersurface, whose equation gives an equivariant inclusion $W \rightarrow S^2W^*$ (by contraction).

Theorem 5.3 (*algebraic version*) *Let \mathfrak{l} be reductive with one dimensional center and let W be an irreducible \mathfrak{l} -module, with a non trivial action of the center. Suppose that $\wedge^2 W$ is irreducible, so that $\mathfrak{g} = \mathfrak{l} \oplus W \oplus W^*$ is a simple Lie algebra (see [22]). Suppose moreover that W is endowed with an \mathfrak{l} -invariant cubic form, and that, as an \mathfrak{l} -module, $S^2W = W^* \oplus S$, with S irreducible. Then $V = \mathbb{C} \oplus W \oplus W^* \oplus \mathbb{C}$ has a natural structure of simple \mathfrak{g} -module.*

We thus recover the constructions of Freudenthal without using division algebras. Moreover, our proofs will show that the constructions work because of the irreducibility of $I_2(Z)$. This perspective simplifies the computations. In the same spirit, we construct below, the invariant symplectic and quartic forms from a unified perspective and without use of composition algebras.

The equivalence of the two versions is as follows: If $W \neq \mathbb{C}$, then $S = S^{(2)}W$ (the Cartan product of W with itself). In general, if $Z \subset \mathbb{P}W$ is a closed orbit, then $I_2(Z)$ is the complement to $S^{(2)}W^*$ in S^2W^* .

The theorem is proved in the same way as the results on minuscule varieties in [22] only the argument is simpler. The idea is to define the natural action on each factor and to normalize the actions such that the Jacobi identities hold.

To define the action of \mathfrak{g} on V , let $C \in S^3W^*$ denote the cubic and $C^* \in S^3W$ denote the dual cubic. Then $\mathfrak{g} = \mathfrak{l} \oplus W \oplus W^*$ acts on V in the following way: $\mathfrak{h} \subset \mathfrak{l}$ acts naturally on each factor, in particular trivially on \mathbb{C} and \mathbb{C}^* ; $\mathbf{1} \in \mathbb{C} = \mathfrak{z}(\mathfrak{l})$ acts by multiplication by $-3/2, -1/2, 1/2, 3/2$ on the four respective components of W ; finally, the actions of W and W^* are given by the following formulae:

$$\begin{aligned} t.(\alpha \oplus r \oplus s^* \oplus \beta^*) &= \mathbf{0} \oplus 3\alpha t \oplus C(rt) \oplus \frac{1}{2}\langle t, s^* \rangle, \\ t^*.(\alpha \oplus r \oplus s^* \oplus \beta^*) &= \frac{1}{2}\langle r, t^* \rangle \oplus C^*(s^*t^*) \oplus 3\beta^*t^* \oplus \mathbf{0}. \end{aligned}$$

With this notation, the application ϕ in [22] in the special case of the minuscule theorem may be written as in §1.2.

The same construction in theorem 5.1 works to construct $Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = G_w(\underline{0}^3, \underline{0}^6)$ out of $\mathbb{P}^0 \sqcup \mathbb{P}^0 \sqcup \mathbb{P}^0 \subset \mathbb{P}^2$, in fact, $Seg(\mathbb{P}^1 \times Q^m)$ out of $\mathbb{P}^0 \sqcup Q^{m-2}$ where Q^m is a quadric hypersurface. The presence of $\underline{0}\mathbb{P}^2 = \mathbb{P}^0 \sqcup \mathbb{P}^0 \sqcup \mathbb{P}^0 \subset \mathbb{P}^2$ should come as no surprise, as the Severi varieties also classify the smooth connected base schemes of the quadro-quadro Cremona transformations (see [10]) and $\mathbb{P}^0 \sqcup \mathbb{P}^0 \sqcup \mathbb{P}^0 \subset \mathbb{P}^2$ is the base scheme of the classical Cremona transform. Note that $\mathbb{P}^0 \sqcup Q^{m-2}$ is also the base scheme of a quadro-quadro Cremona transformation.

Proposition 5.4 *Let Ω be the symplectic form on $V = \mathbb{C} \oplus W \oplus W^* \oplus \mathbb{C}^*$ defined by*

$$\Omega(\alpha \oplus r \oplus r^* \oplus \alpha^*, \beta \oplus s \oplus s^* \oplus \beta^*) = 6(\alpha\beta^* - \beta\alpha^*) - (\langle r, s^* \rangle - \langle s, r^* \rangle).$$

Then Ω is \mathfrak{g} -invariant.

Proof. The form Ω is clearly symplectic and \mathfrak{h} -invariant. Moreover, if $u = \alpha \oplus r \oplus r^* \oplus \alpha^*$, $v = \beta \oplus s \oplus s^* \oplus \beta^* \in V$, and $t \in W$, then

$$\begin{aligned} \Omega(\mathbf{1}.u, v) &= -9(\alpha\beta^* + \beta\alpha^*) + \frac{1}{2}(\langle r, s^* \rangle + \langle s, r^* \rangle) = \Omega(\mathbf{1}.v, u), \\ \Omega(t.u, v) &= -3(\langle t, s^* \rangle \alpha + \langle t, r^* \rangle \beta) - C(rst) = \Omega(t.v, u). \end{aligned}$$

This means that Ω is \mathbb{C} and W -invariant, hence by symmetry W^* -invariant as well. \square

5.2 The quartic invariant

The five \mathfrak{g} -modules V constructed above have a free invariant algebra, generated in degree four (see e.g. [4]). We write down this quartic invariant in a unified way, in terms of the \mathfrak{h} -invariant cubic C on W .

Proposition 5.5 *The quartic polynomial defined for $w = \alpha \oplus r \oplus s^* \oplus \beta^* \in V$ by*

$$Q(w) = (3\alpha\beta^* - \frac{1}{2}\langle r, s^* \rangle)^2 + \frac{1}{3}(\beta^*C(r^3) + \alpha C^*(s^{*3})) - \frac{1}{6}\langle C^*(s^{*2}), C(r^2) \rangle$$

is a \mathfrak{g} -invariant form on V .

Proof. Q is obviously an \mathfrak{h} -invariant polynomial. It is also \mathbb{C} -invariant: it is easy to check that each of its three terms Q_1, Q_2, Q_3 is \mathbb{C} -invariant. Taking into account the symmetry of the expression of Q , we just need to check that it is invariant under the action of W , since it will immediately be invariant also under the action of W^* . We compute the action of $t \in W$ on Q_1, Q_2, Q_3 separately:

$$\begin{aligned} t.Q_1(w) &= -\frac{1}{2}(3\alpha\beta^* - \frac{1}{2}\langle r, s^* \rangle)C(r^2t), \\ t.Q_2(w) &= \frac{1}{6}\langle t, s^* \rangle C(r^3) + 3\alpha\beta^* C(r^2t) + \alpha C^*(s^*C(rt)), \\ t.Q_3(w) &= -\frac{1}{3}\langle C^*(s^*C(rt)), C(r^2) \rangle - \alpha C^*(s^*C(rt)). \end{aligned}$$

It is then straightforward to check that the invariance of Q is equivalent to the identity

$$2\langle C^*(s^*C(rt)), C(r^2) \rangle = \langle t, s^* \rangle C(r^3) + 3\langle r, s^* \rangle C(r^2t).$$

Let $\theta : W \otimes W^* \rightarrow \mathfrak{g}$ be the map dual to the action of \mathfrak{g} on W . The fact that the Jacobi identities hold in \mathfrak{g} amounts to the following lemma, which partly follows from [22], Proposition 5.1, and can be proved along the same lines.

Lemma 5.6 *We can normalize θ, C and C^* in such a way that the following identities hold:*

$$\begin{aligned} \theta(r \otimes t^*)s - \theta(s \otimes t^*)r &= \langle s, t^* \rangle r - \langle r, t^* \rangle s, \\ \theta(r \otimes t^*)s + \theta(s \otimes t^*)r &= 2(\langle s, t^* \rangle r + \langle r, t^* \rangle s) - 2C^*(t^*C(rs)). \end{aligned}$$

Taking the difference of these two identities, and then contracting with $C(r^2)$ gives precisely the equality we needed. \square

The above expression of the quartic invariant was rediscovered by several authors in special cases [19, 11]. One may wish to compare it with Freudenthal's uniform expression for the quartic [11] p. 166 (which is only defined for $V = \mathcal{Z}_2(\mathbb{A})$).

The variety of tangent directions to the lines through a point of D_4^{ad} is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$. In this case the quartic is the simplest instance of Cayley's *hyperdeterminant*, see [13].

In particular, the above formula for the quartic applies to induce the hyperdeterminant from the cubic on $W = \mathcal{J}_3(\mathbb{Q}) = \mathbb{C}^3$ defined by $C(a \oplus b \oplus c) = abc$.

When we restrict the quartic form on $\mathcal{Z}_2(\mathbb{A})$ to $\mathcal{Z}_2(\mathbb{Q})$ we obtain the hyperdeterminant and in turn, the hyperdeterminant determines a unique G -invariant quartic form on each $\mathcal{Z}_2(\mathbb{A})$. Let $\Delta_{\mathcal{Z}} \subset \mathcal{Z}_2(\mathbb{Q})$ denote the subspace induced by $\Delta_{\mathcal{J}}$.

Specializing further, we have:

Proposition 5.7 *The quartic Q on $\mathcal{Z}_2(\mathbb{A})$ is the unique $Sp_6(\mathbb{A})$ invariant polynomial whose restriction to the subalgebra $\Delta_{\mathcal{Z}} \subset \mathcal{Z}_2(\mathbb{A})$ is the classical discriminant.*

Note that taking $\mathbb{A} = \mathbb{Q}$, this gives a new characterization even of the hyperdeterminant.

Proof. It is sufficient to show that the vector space $\mathfrak{sp}_6(\mathbb{A})\Delta_{\mathcal{Z}}$ is $\mathcal{Z}_2(\mathbb{A})$. Suppose to the contrary that $\mathfrak{sp}_6(\mathbb{A})\Delta_{\mathcal{Z}} = U$ is a proper subspace. Since each of the four "matrix" components of $\mathcal{Z}_2(\mathbb{A})$ is weighted differently for the cubic C , we see the subspace must be the sum of linear subspaces of each of the four components. In fact the two one-dimensional components must be present as they are in $\Delta_{\mathcal{Z}}$. Moreover, the other two components are dual to one another so must be cut equally. So it is sufficient to consider the action on $\Delta_{\mathcal{J}} \subset \mathcal{J}_3(\mathbb{A})$. But the identity matrix is in an open orbit and so we obtain everything. \square

Remark. The same construction works equally well with $Seg(\mathbb{P}^1 \times Q^m)$, to obtain a symplectic and quartic form on $\mathbb{C}^2 \otimes \mathbb{C}^{m+2}$ from the cubic form on $\mathbb{C} \oplus \mathbb{C}^m$, $C(a, b) = aq(b)$, where \mathbb{C}^m is equipped with a quadratic form q .

5.3 Orbits

Our description of the closed orbit $G_w(\mathbb{A}^3, \mathbb{A}^6)$ as the image of ϕ has the following known consequence, which also follows from [21] and [22], so we omit the proof.

Let W be a vector space with a symplectic form ω . A variety $Y \subset \mathbb{P}W$ is *Legendrian* if for all $y \in Y$, the affine tangent space $T_y Y$ is a maximal ω -isotropic subspace.

Proposition 5.8 *$G_w(\mathbb{A}^3, \mathbb{A}^6) \subset \mathbb{P}\mathcal{Z}_2(\mathbb{A})$ is a Legendrian variety. Moreover, its tangential variety $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$ is naturally isomorphic to its dual variety $G_w(\mathbb{A}^3, \mathbb{A}^6)^* \subset \mathbb{P}\mathcal{Z}_2(\mathbb{A})^*$, and is the quartic hypersurface $(Q = 0)$. In other words, $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6)) = (G_w(\mathbb{A}^3, \mathbb{A}^6)_*)^*$.*

The isomorphism $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6)) = (G_w(\mathbb{A}^3, \mathbb{A}^6)_*)^*$ gives another connection between the quartic invariant Q and the theory of hyperdeterminants [13].

Remark. It is unusual to have a smooth variety whose dual has degree four and it would be interesting to classify such. Zak [29] has classified the smooth varieties whose duals have degree less than four; in degree two there is only the quadric hypersurface and in degree three there are only ten examples, $Seg(\mathbb{P}^1 \times \mathbb{P}^2)$, its hyperplane section, the Severi varieties, and the smooth projections of the Severi varieties. We are unaware of any general method for constructing varieties with duals of a given degree, but we record the following observation:

Proposition 5.9 *The varieties of F -points in the second row of the $n = d$ magic chart, namely $v_2(\mathbb{P}^{d-1})$, $Seg(\mathbb{P}^{d-1} \times \mathbb{P}^{d-1})$ and $G(2, 2d)$ have dual varieties of degree d .*

The orbit structure of each of the varieties $G_w(\mathbb{A}^3, \mathbb{A}^6)$ has already been studied (e.g. in [4]), but their similarities seem to have been overlooked. The following proposition follows from results in [4]. We give two different short proofs along the lines of our study.

Proposition 5.10 *For each of the varieties of F -planes in the third row of the magic chart, there are exactly four orbits, the closures of which are ordered by inclusion:*

$$G_w(\mathbb{A}^3, \mathbb{A}^6) \subset \sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6)) \subset \tau(G_w(\mathbb{A}^3, \mathbb{A}^6)) \subset \mathbb{P}V.$$

The equations of $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$ (respectively $G_w(\mathbb{A}^3, \mathbb{A}^6)$) are given by the first (respectively second) derivatives of the discriminant Q . The dimensions are respectively $3m + 3$, $5m + 3$ and $6m + 6$.

We also describe the intermediate orbit closure $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$:

Proposition 5.11 *$\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$ can be described as*

1. *the singular locus of $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$,*
2. *the locus of points on a family of secant lines to $G_w(\mathbb{A}^3, \mathbb{A}^6)$ (a $(m + 4)$ -dimensional family for smooth points),*
3. *the locus of points on a secant line to $G_w(\mathbb{A}^3, \mathbb{A}^6)$ isotropic for the symplectic form (unique if a smooth point), in other words, points on a secant line to two interwoven points in the sense of Freudenthal, i.e., two points in a same F -Schubert variety $\Sigma_{G_w(\mathbb{A}^3, \mathbb{A}^6), a}^{G_w(\mathbb{A}^1, \mathbb{A}^6)}$,*
4. *the locus of points on a tangent line to the distribution $\tilde{\sigma}(\mathbb{A}\mathbb{P}^2) \subset TG_w(\mathbb{A}^3, \mathbb{A}^6)$.*

Remark. For the other Legendrian varieties that arise as the space of lines through a point of an adjoint variety we have the following orbit structures: for $v_3(\mathbb{P}^1)$ (tangent directions to lines through a point of G_2^{ad}), there are only three orbits as $\sigma_+(v_3(\mathbb{P}^1)) = v_3(\mathbb{P}^1)$. For $\mathbb{P}^1 \times Q^m$ (tangent directions to lines through a point of $SO(m)^{ad}$), the structure is the same as above except that $\sigma_+(\mathbb{P}^1 \times Q^m)$ decomposes into two irreducible components, $Seg(\mathbb{P}^1 \times \mathbb{P}^{m+1})$ and $\{[e \otimes b + f \otimes c] \mid e, f \in \mathbb{P}^1, b \wedge c \in G_o(2, \mathbb{C}^{m+2})\}$, with the exception of $Q^2 = \mathbb{P}^1 \times \mathbb{P}^1$ where there are three components.

Remark. We determine the orbit structure using an algorithm that is applicable in general. The idea is to infinitesimalize the study and reduce the problem to a lower dimensional question. Let $X = G/P \subset \mathbb{P}V$ be the closed orbit and fix $x \in X$. Then every $v \in V$ is in some $\hat{T}_x^{(k)}X \setminus T_x^{(k-1)}X$, where $\hat{T}_x^{(k)}X$ denotes the k -th osculating space (see [21]). Letting H be a maximal semi-simple subgroup of P , each $N_k = \hat{T}_x^{(k)}X / T_x^{(k-1)}X$ is an H -module and has corresponding orbits. Say there are p_k H -orbits in N_k and $T^{(d)} = V$. Then there are at most $p_1 + \dots + p_d$ G -orbits and in fact there are strictly less because different H -orbits will lead to the same G -orbit.

Proofs. Write $G_w(\mathbb{A}^3, \mathbb{A}^6) = G/P$ and let $Sl_3(\mathbb{A}) \subset P$ be a maximal semi-simple subgroup. Then there are four $Sl_3(\mathbb{A})$ -orbits in $T_x G_w(\mathbb{A}^3, \mathbb{A}^6)$, namely 0 , $\hat{\mathbb{A}}\mathbb{P}^2 \setminus 0$, $\hat{\sigma}(\mathbb{A}\mathbb{P}^2) \setminus \hat{\mathbb{A}}\mathbb{P}^2$, and $T_x G_w(\mathbb{A}^3, \mathbb{A}^6) \setminus \hat{\sigma}(\mathbb{A}\mathbb{P}^2)$. Since $\mathbb{A}\mathbb{P}^2$ is the base-locus of the second fundamental form, it gives the same G -orbit as 0 . Thus there are at most three G -orbits in $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$. To see that there are indeed three, note that the space $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$ of tangent directions to the distribution $\tilde{\sigma}(\mathbb{A}\mathbb{P}^2)$ is G -invariant, strictly contains $G_w(\mathbb{A}^3, \mathbb{A}^6)$ (since the intersection of $G_w(\mathbb{A}^3, \mathbb{A}^6)$ with any of its tangent spaces is an $\hat{\mathbb{A}}\mathbb{P}^2$), and is properly contained in $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$ (its dimension being smaller). Using the rational map ϕ above, it is easy to check that the derivatives of the quartic Q vanish on $\hat{\sigma}(\mathbb{A}\mathbb{P}^2) \subset T_x G_w(\mathbb{A}^3, \mathbb{A}^6)$ for $x = (1, 0, 0, 0)$, hence for any $x \in G_w(\mathbb{A}^3, \mathbb{A}^6)$. This implies that $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$ is the singular locus of $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$.

To prove that the equations of $G_w(\mathbb{A}^3, \mathbb{A}^6)$ are the second derivatives of Q , we just notice that this space of quadratic equations define a non empty G -stable subset of $\mathbb{P}V$, properly contained in the singular locus of $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$. Because of the orbit structure, this must be $G_w(\mathbb{A}^3, \mathbb{A}^6)$. Finally, since $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$ is a hypersurface, its complement must be an open orbit.

The second proposition follows by observing that each of these characterizations defines a union of orbits in $\mathbb{P}V$, which is properly contained in $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$, but different from $G_w(\mathbb{A}^3, \mathbb{A}^6)$. Hence each must coincide with $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$. \square

Our second proof gives more information about the entry loci and other geometric objects:

A generic point $p \in \tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$ lies on a unique tangent line so it will be sufficient to show there are points lying on a family of tangent lines but not on $G_w(\mathbb{A}^3, \mathbb{A}^6)$. Let p be on a tangent line to $x \in G_w(\mathbb{A}^3, \mathbb{A}^6)$ such that p corresponds to a vector $v \in T_x G_w(\mathbb{A}^3, \mathbb{A}^6)$ with the property that $[v] \in \tau(\mathbb{A}\mathbb{P}^2)_x \subset \mathbb{P}T_x G_w(\mathbb{A}^3, \mathbb{A}^6)$. In this case there exists $y \in \mathbb{A}\mathbb{P}^2_x$ such that v may also be considered an element of $T_y \mathbb{A}\mathbb{P}^2_x$. Moreover, we may consider $y \in G_w(\mathbb{A}^3, \mathbb{A}^6)$ (as tangent directions in $\mathbb{A}\mathbb{P}^2$ correspond to lines on $G_w(\mathbb{A}^3, \mathbb{A}^6)$). On the other hand since $\tau(\mathbb{A}\mathbb{P}^2)$ is degenerate, there is an $\mathbb{A}\mathbb{P}^1 = Q^{m'}$'s of choice of y (see §4 above), and any point $z \in \mathbb{A}\mathbb{P}^1 \subset G_w(\mathbb{A}^3, \mathbb{A}^6)$ has a tangent vector $w \in T_z G_w(\mathbb{A}^3, \mathbb{A}^6)$ corresponding to p . Thus p is on an $(m+1)$ -dimensional family of tangent lines and thus an $(m+2)$ -dimensional family of secant lines. By our explicit description, the equivalence of 1,2 and 4 follows and 3 follows from noticing that the $\mathbb{A}\mathbb{P}^2$ is an F-Schubert variety associated to an F-point $a \in G_w(\mathbb{A}^1, \mathbb{A}^6)$. \square

Note that we recover that there is no G -invariant polynomial on V (up to constants) other than Q and its powers. The geometric interpretation of $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$ has been investigated in [9] in the case of the Grassmannian $G(3, 6) \subset \mathbb{P}(\Lambda^3 \mathbb{C}^6)$.

Since there are only four orbits, we also have:

Proposition 5.12 *With the notations above, $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$ is self-dual.*

The following proposition can be proved in the same way that Proposition 3.2.

Proposition 5.13 *$\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$ is ruled by the \mathbb{P}^{m+3} 's that are the linear spans of the F-Schubert varieties*

$$\Sigma_{G_w(\mathbb{A}^3, \mathbb{A}^6)}^{Sp_6(\mathbb{A})^{ad}} \simeq Q^{m+2}.$$

In particular, a smooth point of $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$ lies on a unique $\mathbb{P}^{m+3} = \langle Q^{m+2} \rangle$.

Remark. Let V be an irreducible \mathfrak{g} -module. The decomposition of \mathfrak{g} -modules

$$S^2V = S^{(2)}V \oplus W \oplus \text{stuff}$$

with W irreducible implies that the closed G -orbit in $\mathbb{P}W$ induces a variety of quadrics of constant rank on V , and linear spaces on the closed orbit furnish linear systems of quadrics.

Linear systems of quadrics of constant rank arise as the second fundamental forms of degenerate dual varieties (see [20]), and few examples of such systems (or smooth varieties with degenerate duals) are known.

The interpretation of F-Schubert varieties associated to planes as a family of quadrics on $G_w(\mathbb{A}^3, \mathbb{A}^6)$ is related to the decomposition of the symmetric square of $\mathcal{Z}_2(\mathbb{A})$ as

$$S^2\mathcal{Z}_2(\mathbb{A}) = S^{(2)}\mathcal{Z}_2(\mathbb{A}) \oplus \mathfrak{sp}_6(\mathbb{A}).$$

An element $X \in \mathfrak{sp}_6(\mathbb{A})$ defines a quadratic form on $\mathcal{Z}_2(\mathbb{A})$, namely $q_X(u) = \omega(Xu, u)$. In particular, we obtain varieties of quadrics of constant rank.

Proposition 5.14 *Let $m = 1, 2, 4, 8$. The adjoint variety $Sp_6(\mathbb{A})^{ad}$, parametrizes a variety of dimension $4m + 1$ of quadrics of rank $m + 4$ on $\mathcal{Z}_2(\mathbb{A}) = \mathbb{C}^{6m+8}$.*

One can take linear spaces on these varieties on these varieties (except for $v_2(\mathbb{P}^5)$) to get linear systems of quadrics of constant rank.

6 Adjoint varieties of the exceptional groups

As shown in [22], $\mathfrak{e}(\mathbb{A})$ may be constructed from $\mathfrak{sp}_6(\mathbb{A})$, and the adjoint variety $X_{E(\mathbb{A})}^{ad}$ may be constructed from $G_w(\mathbb{A}^3, \mathbb{A}^6)$ via a rational map of degree four, given in terms of the quartic invariant Q on V . The construction also reproduces the five step \mathbb{Z} -grading of $\mathfrak{e}(\mathbb{A})$ and the filtration of $\mathfrak{e}(\mathbb{A})$ induced by $U(\mathfrak{sp}_6(\mathbb{A}))$. Continuing the notation of the previous section,

$$\mathfrak{e}(\mathbb{A}) = \mathbb{C}^* \oplus V^* \oplus (\mathbb{C} \oplus \mathfrak{sp}_6(\mathbb{A})) \oplus V \oplus \mathbb{C}$$

where using the composition algebra model, $V = \mathcal{Z}_2(\mathbb{A})$. The adjoint variety $E(\mathbb{A})^{ad}$ is the image of the rational mapping ψ described in §1.2.

While $\mathbb{P}(\mathfrak{e}(\mathbb{A}))$ does not have a finite number of $E(\mathbb{A})$ -orbits, there are only a finite number of $E(\mathbb{A})$ -orbits in $\sigma(E(\mathbb{A})^{ad}) \subset \mathbb{P}(\mathfrak{e}(\mathbb{A}))$ which we now describe. The following theorem improves upon recent results in [16] where it is shown that $\sigma(G^{ad})$ contains an open orbit for any simple group G . We show that it is actually the union of a finite number of orbits and exhibit them explicitly:

Theorem 6.1 Let $\mathfrak{e}(\mathbb{A})$ respectively denote $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, and $m = 1, 2, 4, 8$. Let $E(\mathbb{A})^{ad} \subset \mathbb{P}(\mathfrak{e}(\mathbb{A}))$ denote the adjoint variety, the closed $E(\mathbb{A})$ -orbit. Then $\sigma(E(\mathbb{A})^{ad}) = \tau(E(\mathbb{A})^{ad})$ and the $E(\mathbb{A})$ -orbit closures in $\sigma(E(\mathbb{A})^{ad})$ are as follows:

$$E(\mathbb{A})^{ad} \subset \sigma_{(2m+7)}(E(\mathbb{A})^{ad}) \subset \sigma_{(3)}(E(\mathbb{A})^{ad}) \subset \sigma_{(1)}(E(\mathbb{A})^{ad}) \subset \sigma(E(\mathbb{A})^{ad}).$$

These orbits are respectively of dimensions $6m+9, 10m+11, 12m+15, 12m+17$ and $12m+18$. Moreover, the open orbit in $\sigma(E(\mathbb{A})^{ad})$ is a semi-simple orbit, while the four others are projectivizations of nilpotent orbits.

With this notation, the orbit closure $\sigma_{(m)}(E(\mathbb{A})^{ad})$ has codimension m inside $\sigma(E(\mathbb{A})^{ad})$, so a general point of $\sigma_{(m)}(E(\mathbb{A})^{ad})$ has an $(m+1)$ -dimensional entry locus.

Proposition 6.2 The orbit closures above can be described as follows:

- $\sigma_{(1)}(E(\mathbb{A})^{ad}) = \sigma(E(\mathbb{A})^{ad}) \cap Q_{Killing}$, where $Q_{Killing}$ is the quadric hypersurface defined by the Killing form. Equivalently, it is the closure of the orbit of points belonging to a unique tangent line to the distribution $T_1E(\mathbb{A})^{ad}$ of contact hyperplanes in $TE(\mathbb{A})^{ad}$, and the points on a secant line of two hinged points in the sense of Freudenthal (see below).
- $\sigma_{(3)}(E(\mathbb{A})^{ad})$ is the closure of the orbit consisting of points on a tangent line to the distribution $\tilde{\tau}(G_w(\mathbb{A}^3, \mathbb{A}^6)) \subset TE(\mathbb{A})^{ad}$, a one dimensional family of such.
- $\sigma_{(2m+7)}(E(\mathbb{A})^{ad})$ is the closure of the orbit consisting of points belonging to an $(m+2)$ -dimensional family of tangent lines to the distribution $\tilde{\sigma}_+(G_w(\mathbb{A}^3, \mathbb{A}^6)) \subset TE(\mathbb{A})^{ad}$, equivalently of points on a secant line of two interwoven points in the sense of Freudenthal.

Note that $\sigma_{(3)}(E(\mathbb{A})^{ad})$ cannot be detected from Freudenthal's geometries. Thus, as in [21], the perspective of Freudenthal and Tits is extremely useful for understanding the projective geometry, but it does not reveal the full story.

Corollary 6.3 $\sigma_{(2m+7)}(E(\mathbb{A})^{ad})$ is ruled by the \mathbb{P}^{m+5} 's that are the linear spans of the F-Schubert varieties

$$\Sigma_{E(\mathbb{A})^{ad}}^{X_{F\text{-points}}^{E(\mathbb{A})}} \simeq Q^{m+4}.$$

In particular, a smooth point of $\sigma_{(2m+7)}(E(\mathbb{A})^{ad})$ lies on a unique $\mathbb{P}^{m+5} = \langle Q^{m+4} \rangle$.

Before entering into the proof of the theorem, we recall Freudenthal's incidence relations for points of F-symplecta, that is points of $E(\mathbb{A})^{ad}$. They can be:

joined, which means they are contained in a unique F-plane. In other words, two points $x, y \in E(\mathbb{A})^{ad}$ are joined if their secant line \mathbb{P}_{xy}^1 is contained in $E(\mathbb{A})^{ad}$;

interwoven, which means they intersect in an F-point. In other words, two points $x, y \in E(\mathbb{A})^{ad}$ are interwoven if they are contained in a $\Sigma_{E(\mathbb{A})^{ad}}^{X_{F\text{-points}}^{E(\mathbb{A})}} = Q^{m+4} \subset E(\mathbb{A})^{ad}$; if this F-Schubert variety is not unique, then x and y are joined;

hinged, which means they are joined to a third F-symplecton. In other words, $x, y \in E(\mathbb{A})^{ad}$ are (strictly) hinged if there exists (a unique) $z \in E(\mathbb{A})^{ad}$ such that the secant lines $\mathbb{P}_{zx}^1, \mathbb{P}_{zy}^1$ are contained in $E(\mathbb{A})^{ad}$. Equivalently, $x, y \in E(\mathbb{A})^{ad}$ are hinged if their secant line is contained in the quadric hypersurface defined by the Killing form;

generic, i.e., not hinged.

Proofs. By [16], $\sigma(E(\mathbb{A})^{ad}) = \tau(E(\mathbb{A})^{ad})$ and has secant defect one. Also, as noted in [16], fixing a Cartan subalgebra and a set of simple roots for $\mathfrak{e}(\mathbb{A})$, the orbit of $X_{\tilde{\alpha}} + X_{-\tilde{\alpha}}$ is open in $\sigma(E(\mathbb{A})^{ad})$, where $\tilde{\alpha}$ denotes the maximal root. This element is semi-simple, and conjugate to a multiple of $H_{\tilde{\alpha}}$. This proves that the open orbit is isomorphic to the (semi-simple) orbit of $H_{\tilde{\alpha}}$. Consider in $\mathfrak{e}(\mathbb{A})$ the cone C over this orbit, and an element x of its closure. The semi-simple part of x must be conjugate to $\lambda H_{\tilde{\alpha}}$ for some scalar λ . If $\lambda \neq 0$ and x is not semi-simple, the cone over the orbit it generates is of dimension strictly bigger than C , which is absurd. Hence x is semi-simple or nilpotent, and this proves that \overline{C}/C is a union of nilpotent orbits.

There potentially are four kinds of elements of $\tau(E(\mathbb{A})^{ad})$, corresponding to $v \in T_x E(\mathbb{A})^{ad} \setminus T_1$, $v \in T_{1x} \setminus \sigma G_w(\mathbb{A}^3, \mathbb{A}^6)_x$, $v \in \sigma(G_w(\mathbb{A}^3, \mathbb{A}^6))_x \setminus G_w(\mathbb{A}^3, \mathbb{A}^6)_x$ and $v \in G_w(\mathbb{A}^3, \mathbb{A}^6)_x$. The last type is the same as a point on $E(\mathbb{A})^{ad}$. Let $p \in \tau(E(\mathbb{A})^{ad})$.

Consider the case where there exists an $x \in E(\mathbb{A})^{ad}$ and $v \in T_{1x} E(\mathbb{A})^{ad}$ with p on the line corresponding to v . Since $\sigma(G_w(\mathbb{A}^3, \mathbb{A}^6))_x = \mathbb{P}T_{1x}$, there exist $y, z \in G_w(\mathbb{A}^3, \mathbb{A}^6)_x \subset E(\mathbb{A})^{ad}$ such that $p \in \mathbb{P}_{yz}^1$. Since $\mathbb{P}_{xz}^1, \mathbb{P}_{xy}^1 \subset E(\mathbb{A})^{ad}$, we see that p is indeed on a secant line of two hinged points, showing the equivalence of the first and third characterizations, modulo the unicity in the second: but this follows from Freudenthal's remark that if two F-symplecta are multiply hinged, that is joined to several others F-symplecta, they must be interwoven. Finally, the third and second characterizations are equivalent from Freudenthal's observations again.

Now consider the case where there exists an $x \in E(\mathbb{A})^{ad}$ and $v \in \hat{\tau}(G_w(\mathbb{A}^3, \mathbb{A}^6)_x) \subset T_{1x} E(\mathbb{A})^{ad}$ with p on the line corresponding to v . Let $y \in G_w(\mathbb{A}^3, \mathbb{A}^6)_x$ be the (in general unique) point such that v corresponds to a vector in $T_y G_w(\mathbb{A}^3, \mathbb{A}^6)_x$. By the same argument as in §5.3 above, p lies on a tangent line to all $z \in \mathbb{P}_{xy}^1$ and moreover these tangent lines are tangent to the distribution $\tilde{\tau}G_w(\mathbb{A}^3, \mathbb{A}^6)$. Moreover, for each x, y and v are uniquely determined, so the component of the locus tangent to $\tilde{\tau}(G_w(\mathbb{A}^3, \mathbb{A}^6))$ passing through x is a \mathbb{P}^1 . The dimension count follows.

Finally consider the case where there exists an $x \in E(\mathbb{A})^{ad}$ and $v \in \hat{\sigma}_+ G_w(\mathbb{A}^3, \mathbb{A}^6)_x \subset T_{1x} E(\mathbb{A})^{ad}$ with p on the line corresponding to v . Now there exist $y, z \in \hat{G}_w(\mathbb{A}^3, \mathbb{A}^6)_x \subset E(\mathbb{A})^{ad}$ such that $p \in \mathbb{P}_{yz}^1$, in fact a $\hat{Q}^{m+3} \subset \hat{G}_w(\mathbb{A}^3, \mathbb{A}^6)_x$ of such points so this orbit closure does not coincide with any of the others. The points y, z are interwoven as they are both contained in a $Q^{m+4} \subset E(\mathbb{A})^{ad}$, that is an F-Schubert variety

$$\Sigma_{E(\mathbb{A})^{ad}}^{X_{F-points}^{\mathfrak{e}(\mathbb{A})}}$$

showing the equivalence of the two characterizations. Each of these F-Schubert varieties generates a \mathbb{P}^{m+5} in $\mathbb{P}(\mathfrak{e}(\mathbb{A}))$.

We prove that two such \mathbb{P}^{m+5} 's, if they are not equal, can intersect only inside $E(\mathbb{A})^{ad}$. Suppose the contrary, and take a generic line l in their intersection. It cuts $E(\mathbb{A})^{ad}$ exactly in two points u and v , since the intersection of $E(\mathbb{A})^{ad}$ with each of our \mathbb{P}^{m+5} 's is a quadric. But then u and v are doubly interwoven, hence joined, thus the line l is contained in $E(\mathbb{A})^{ad}$, a contradiction.

This proves that a generic point of $\sigma_{(2m+7)}(E(\mathbb{A})^{ad})$ belongs to a unique \mathbb{P}^{m+5} generated by an F-Schubert variety. The dimension follows because $\dim X_{F-points}^{\mathfrak{e}(\mathbb{A})} + (m+5) = 10m+11$. \square

One can check from the tables in [6] that there exists nilpotent orbits with the dimensions claimed in the proposition (minus one, because of the projectivization). If we exclude F_4 , they are respectively labelled $A_1, 2A_1, 3A_1$ ($3A_1'$ in the case of E_7), and A_2 .

Remark. As noted in [7], the symmetric squares of the exceptional simple Lie algebras $\mathfrak{e}(\mathbb{A})$ have a uniform decomposition into irreducible components:

$$S^2 \mathfrak{e}(\mathbb{A}) = S^{(2)} \mathfrak{e}(\mathbb{A}) \oplus W \oplus \mathbb{C},$$

where $S^{(2)}\mathfrak{e}(\mathbb{A})$ denotes the Cartan product of $\mathfrak{e}(\mathbb{A})$ with itself, the \mathbb{C} component is given by the Killing form, and the other component W is the ambient space for F-points. In particular, we obtain:

Proposition 6.4 $X_{F\text{-points}}^{\mathfrak{e}(\mathbb{A})}$ parametrizes a variety of dimension $9m + 6$ of quadrics of rank $m + 6$ on $\mathfrak{e}(\mathbb{A})$.

We now describe the orbit structure of $\sigma(G^{ad})$ for the remaining simple groups. In each case the following properties hold: the open orbit is semi-simple and the others are nilpotent; there is the orbit $\sigma_{(1)}(G^{ad}) = \sigma(G^{ad}) \cap Q_{Killing}$, equivalently, the points on a tangent line of the distribution of contact hyperplanes $T_1 G^{ad}$; and $\dim \sigma(G^{ad}) = 2\dim G^{ad}$, so $\dim \sigma_{(m)}(G^{ad}) = 2\dim G^{ad} - m$.

Note that the orbit structure is not as uniform for the classical groups as for the exceptional groups.

G = G₂. The orbit structure inside the secant variety is

$$G_2^{ad} \subset \sigma_{(3)}(G_2^{ad}) \subset \sigma_{(1)}(G_2^{ad}) \subset \sigma(G_2^{ad}) = \tau(G_2^{ad}).$$

$\dim G_2^{ad} = 5$. The open orbit is semi-simple and the others are nilpotent. In particular, $\sigma_{(1)}(G_2^{ad})$ is the closure of the projectivization of the subregular nilpotent orbit. The orbit closure $\sigma_{(3)}(G_2^{ad})$ consists of points on a tangent line to the distribution $\tilde{\tau}(v_3(\mathbb{P}^1))$.

G = SL_n. Note that $SL_n^{ad} = \mathbb{F}_{1,n-1} \subset \mathbb{P}(\mathfrak{sl}_n)$ is a partial flag variety of dimension $2n - 3$. Here the orbit structure inside the secant variety is

$$SL_n^{ad} = \mathbb{F}_{1,n-1} \subset \sigma_{(3)}(\mathbb{F}_{1,n-1}) \subset \sigma_{(1)}(\mathbb{F}_{1,n-1}) \subset \sigma(\mathbb{F}_{1,n-1}) = \tau(\mathbb{F}_{1,n-1}).$$

The intermediate orbits correspond to endomorphisms that in Jordan normal form consist of one 3×3 nilpotent block, and two 2×2 nilpotent blocks respectively.

G = SO_n. Here $SO_n^{ad} = G_o(2, n) \subset \mathbb{P}(\Lambda^2 \mathbb{C}^n)$ is an isotropic Grassmanian for a quadratic form Q . The orbit structure is as follows:

$$SO_n^{ad} = G_o(2, n) \subset \begin{matrix} \sigma_{(2n-9)}(G_o(2, n)) \\ \sigma_{(7)}(G_o(2, n)) \end{matrix} \subset \sigma_{(1)}(G_o(2, n)) \subset \sigma(G_o(2, n)).$$

The orbit closures are *not* totally ordered by inclusion. This orbit structure is not surprising because $\text{Base II} = \mathbb{P}^1 \times \tilde{Q}$ and $\sigma_+(\mathbb{P}^1 \times \tilde{Q})$ has two irreducible components. The orbit closures $\sigma_{(2n-9)}(G_o(2, n))$ and $\sigma_{(7)}(G_o(2, n))$ correspond to points on a tangent line to one of the two corresponding distributions. (Here, if $E \in G_o(2, n)$, then $\tilde{Q} \subset \mathbb{P}E^\perp/E$ is a quadric hypersurface.) Note that, unlike in other cases, if a point lies on a tangent line to the distribution $\tilde{\tau}(\mathbb{P}^1 \times \tilde{Q})$, it is automatically also on a tangent line to the distribution $\tilde{\sigma}_+(\mathbb{P}^1 \times \tilde{Q})$.

We may see the orbits from the global geometry as follows: let $P, P' \in G_o(2, n)$ be distinct points. Let $M = P + P' \subset \mathbb{C}^n$. Then $\dim M = 3$ or 4 . If $\dim M = 3$, then $\text{rank } Q|_M = 0$ or 1 . When $\text{rank } Q|_M = 0$, P and P' are perpendicular and the corresponding secant is contained in $G_o(2, n)$. When $\text{rank } Q|_M = 1$, the corresponding orbit is $\sigma_{(2n-9)}(G_o(2, n))$. If $\dim M = 4$, $\text{rank } Q|_M = 0, 2$ or 4 . These cases determine orbit closures as follows: Rank zero occurs when P and P' are perpendicular; the corresponding orbit is an open subset of a \mathbb{P}^5 -bundle over $G_o(4, n)$, its closure is $\sigma_{(7)}(G_o(2, n))$. Rank two occurs when P' contains a line perpendicular to P , the corresponding orbit is an open subset of a $G(2, n - 4)$ -bundle over $G_o(2, n)$; its closure is $\sigma_{(1)}(G_o(2, n))$. Rank four is the generic case.

$\mathbf{G} = \mathbf{Sp}_{2n}$. Here $Sp_{2n}^{ad} = v_2(\mathbb{P}^{2n-1}) \subset \mathbb{P}S^2\mathbb{C}^{2n}$. Taking two distinct lines l and l' , the plane they generate is either isotropic or not. The orbit corresponding to the isotropic case is an open subset of a \mathbb{P}^2 -bundle over $G_w(2, 2n)$. The orbit structure of the secant variety of $Sp_{2n}^{ad} \subset \mathbb{P}(\mathfrak{sp}_{2n})$ is therefore

$$Sp_{2n}^{ad} = v_2(\mathbb{P}^{2n-1}) \subset \sigma_{(1)}(Sp_{2n}^{ad}) \subset \sigma(Sp_{2n}^{ad}).$$

This is the most degenerate case.

7 Desingularizations

Let $X \subset \mathbb{P}V$ be a smooth variety. If the tangential variety of X , $\tau(X) \subset \mathbb{P}V$ is nondegenerate, then it admits a desingularization $\tilde{T}X \rightarrow \tau(X)$ where $\tilde{T}X$ is the bundle of embedded tangent projective spaces. Similarly, if the dual variety $X^* \subset \mathbb{P}V^*$ is nondegenerate, it admits a desingularization $\mathbb{P}N^* \rightarrow X^*$ where N^* denotes the conormal bundle of X .

When X is homogeneous, both of these desingularizations are examples of what Kempf [18] calls the *collapsing of a vector bundle*. In particular, whenever $\tau(X)$ or X^* is nondegenerate, it has rational singularities and one can explicitly describe its desingularization via Tits transforms.

7.1 Orbits in $\mathbb{P}(\mathcal{J}_3(\mathbb{A}))$

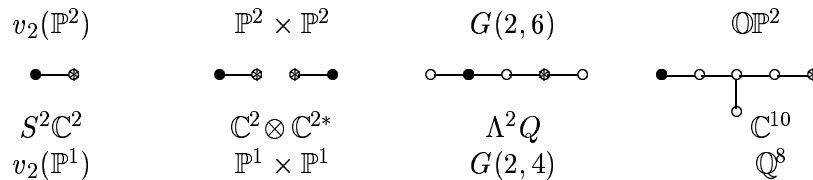
Here since $\sigma(\mathbb{A}\mathbb{P}^2) \simeq (\mathbb{A}\mathbb{P}_*^2)^*$ the above discussion applies to desingularize $\tau(\mathbb{A}\mathbb{P}^2) = \sigma(\mathbb{A}\mathbb{P}^2)$. Moreover, the bundle $M = N^*(\mathbb{A}\mathbb{P}_*^2)(-1)$ can be described as follows: its fiber over an F-line is the linear subspace of $\mathcal{J}_3(\mathbb{A})$ generated by the F-Schubert variety consisting of F-points incident to this F-line.

Proposition 7.1 *Let $G/P = \mathbb{A}\mathbb{P}^2 \subset \mathbb{P}V$ be a Severi variety. Let $\mathbb{A}\mathbb{P}_*^2 \subset \mathbb{P}V^*$ denote the Severi variety in the dual projective space, and M be as above. There is a natural diagram*

$$\begin{array}{ccc} E = \tilde{Q}^m & \longrightarrow & \mathbb{A}\mathbb{P}^2 \\ \cap & & \cap \\ \mathbb{P}M & \xrightarrow{f} & \sigma(\mathbb{A}\mathbb{P}^2) \\ \pi \downarrow & & \\ \mathbb{A}\mathbb{P}_*^2 & & \end{array}$$

where f is a desingularization of $\sigma(\mathbb{A}\mathbb{P}^2)$. The exceptional divisor E of f is naturally identified with the G -homogeneous space consisting of pairs of incident F-points and F-lines, with its two natural projections over $\mathbb{A}\mathbb{P}^2$ and $\mathbb{A}\mathbb{P}_*^2$. In other words, E is the set of points in $(\mathbb{A}\mathbb{P}_*^2)^*$ tangent to $\mathbb{A}\mathbb{P}_*^2$ along a quadric $\mathbb{A}\mathbb{P}^1 \simeq Q^m$.

In the four diagrams below, we indicate the nodes defining the space of F-points $\mathbb{A}\mathbb{P}^2$ with black dots and those defining the F-lines $\mathbb{A}\mathbb{P}_*^2 \simeq \mathbb{A}\mathbb{P}^2$ with stars. The bundle M on $\mathbb{A}\mathbb{P}_*^2$ (which is defined by the same node as $\mathbb{A}\mathbb{P}^2$), and the quadric inside the fibers of $\mathbb{P}M$ are below the diagrams.



7.2 Orbits in $\mathbb{P}(\mathcal{Z}_2(\mathbb{A}))$

The remarks at the beginning of this section apply to $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$, which is singular exactly along $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$. It can be desingularized by a collapsing which answers a question of Kempf in the case of E_7 , who failed to observe the orbit corresponds to a nondegenerate tangential variety.

Proposition 7.2 *Let $\tilde{T}G_w(\mathbb{A}^3, \mathbb{A}^6)$ be the bundle of embedded tangent spaces of $G_w(\mathbb{A}^3, \mathbb{A}^6)$, whose associated vector bundle has rank $\dim G_w(\mathbb{A}^3, \mathbb{A}^6) + 1$. There is a natural diagram*

$$\begin{array}{ccc} E = \tilde{\sigma}(\mathbb{A}\mathbb{P}^2) & \longrightarrow & \sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6)) \\ \cap & & \cap \\ \tilde{T}G_w(\mathbb{A}^3, \mathbb{A}^6) & \xrightarrow{g} & \tau(G_w(\mathbb{A}^3, \mathbb{A}^6)) \\ \pi \downarrow & & \\ G_w(\mathbb{A}^3, \mathbb{A}^6) & & \end{array}$$

where g is a desingularization of $\tau(G_w(\mathbb{A}^3, \mathbb{A}^6))$.

The exceptional divisor E is singular and it too can be desingularized by a homogeneous projective bundle over the G -homogeneous space of pairs of incident F-points and F-planes. The singular locus of E is $\mathbb{A}\tilde{\mathbb{P}}^2 \subset TG_w(\mathbb{A}^3, \mathbb{A}^6)$ and its fibers over $G_w(\mathbb{A}^3, \mathbb{A}^6)$ are cones over the Severi varieties $\mathbb{A}\mathbb{P}^2$. Outside this locus, E is a Q^{m+1} -bundle: a generic point p in $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$ is contained in the linear span \mathbb{P}^{m+3} of a unique F-Schubert variety $\Sigma_{G_w(\mathbb{A}^3, \mathbb{A}^6)}^{Sp_6(\mathbb{A})^{ad}, y}$, which is a quadric Q^{m+2} inside this \mathbb{P}^{m+3} . The fiber $g^{-1}(p)$ is then the section of Q^{m+2} by the hyperplane perpendicular to p .

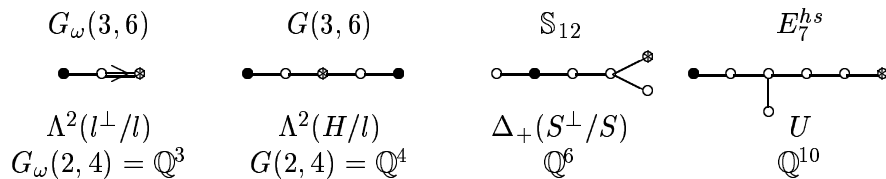
The orbit closure $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$ also admits a natural desingularization by a collapse given by Freudenthal geometry. Let S be the homogeneous vector bundle on $Sp_6(\mathbb{A})^{ad}$ (the space of F-points) defined by the node of the Dynkin diagram corresponding to F-planes, i.e., the bundle whose fiber at y is the linear span of the F-Schubert variety $\Sigma_{G_w(\mathbb{A}^3, \mathbb{A}^6)}^{Sp_6(\mathbb{A})^{ad}, y}$.

Theorem 7.3 *There is a natural diagram*

$$\begin{array}{ccc} E = \tilde{Q}^{m+2} & \longrightarrow & G_w(\mathbb{A}^3, \mathbb{A}^6) \\ \cap & & \cap \\ \mathbb{P}S & \xrightarrow{f} & \sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6)) \\ \pi \downarrow & & \\ Sp_6(\mathbb{A})^{ad} = G_w(\mathbb{A}^1, \mathbb{A}^6) & & \end{array}$$

where f is a desingularization of $\sigma_+(G_w(\mathbb{A}^3, \mathbb{A}^6))$. The exceptional divisor E of f is naturally identified with the G -homogeneous space consisting of pairs of incident F-points and F-planes, with its two natural projections over G^{ad} and $G_w(\mathbb{A}^3, \mathbb{A}^6)$. The intersection of this divisor with a fiber of π is a quadratic hypersurface.

Our four examples of the above situation are the following, where we indicate the nodes defining the space of F-points with black dots, and those defining the F-planes with stars. The vector bundle S and the quadric inside the fibers of $\mathbb{P}S$ are given below the diagrams.



7.3 Orbits in $\mathbb{P}(\mathfrak{e}(\mathbb{A}))$

First note that $\tilde{T}E(\mathbb{A})^{ad} \rightarrow \sigma(E(\mathbb{A})^{ad})$ provides a desingularization of $\sigma(E(\mathbb{A})^{ad})$ as $\sigma(E(\mathbb{A})^{ad})$ coincides with the tangential variety which is nondegenerate. The desingularization is as follows:

Proposition 7.4 *Let $E(\mathbb{A})^{ad}$ be a variety of F-symplecta and let $T_1E(\mathbb{A})^{ad} \subset TE(\mathbb{A})^{ad}$ be the bundle of contact hyperplanes. There is a natural diagram*

$$\begin{array}{ccc} \tilde{t}G_w(\mathbb{A}^3, \mathbb{A}^6) & \longrightarrow & \sigma_{(3)}(E(\mathbb{A})^{ad}) \\ \cap & & \cap \\ \tilde{T}_1E(\mathbb{A})^{ad} & \longrightarrow & \sigma_{(1)}(E(\mathbb{A})^{ad}) \\ \cap & & \cap \\ \tilde{T}E(\mathbb{A})^{ad} & \xrightarrow{g} & \sigma(E(\mathbb{A})^{ad}) \\ \pi \downarrow & & \\ E(\mathbb{A})^{ad} & & \end{array}$$

where g is a desingularization of $\sigma(E(\mathbb{A})^{ad})$.

Note that since $\sigma(E(\mathbb{A})^{ad})$ is normal (being the image of a collapsing), it is smooth in codimension one, hence the open orbit in the hypersurface $\sigma_{(1)}(E(\mathbb{A})^{ad})$ is contained in the smooth locus of $\sigma(E(\mathbb{A})^{ad})$. This implies that g above is also a desingularization of $\sigma_{(1)}(E(\mathbb{A})^{ad})$. In particular, we recover the fact that a general point of $\sigma_{(1)}(E(\mathbb{A})^{ad})$ belongs to a unique tangent line to the distribution $\tilde{T}_1E(\mathbb{A})^{ad}$.

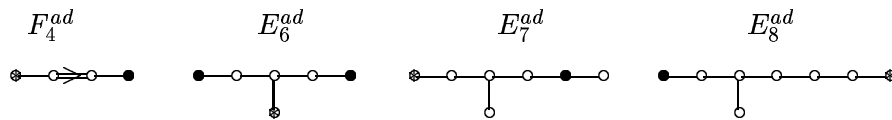
Finally, here is a desingularization of $\sigma_{(2m+7)}(E(\mathbb{A})^{ad})$. Let S be the bundle on $X_{F\text{-points}}^{E(\mathbb{A})}$ induced by the F-Schubert varieties $\Sigma_{E(\mathbb{A})^{ad}}^{X_{F\text{-points}}^{E(\mathbb{A})}} \simeq Q^{m+4}$.

Proposition 7.5 *Let $E(\mathbb{A})^{ad}$ be a variety of F-symplecta. There is a natural diagram*

$$\begin{array}{ccc} E = \tilde{Q}^{m+4} & \longrightarrow & E(\mathbb{A})^{ad} \\ \cap & & \cap \\ \mathbb{P}S & \xrightarrow{h} & \sigma_{(2m+7)}(E(\mathbb{A})^{ad}) \\ \pi \downarrow & & \\ X_{F\text{-points}}^{E(\mathbb{A})} & & \end{array}$$

where h is a desingularization.

Here the four examples of the above situation are the following, where we indicate the nodes defining the space of F-points with black dots, and those defining the vector bundle S on $X_{F\text{-points}}^{E(\mathbb{A})}$ with stars.



8 Hyperplane sections

Given an algebraic variety $X \subset \mathbb{P}V$, it is interesting to study the hyperplane sections $X \cap H$, for example the variation of topology or Hodge structure as one varies the hyperplane H . When $X = G/P$ is homogeneous one could hope to have explicit descriptions of all hyperplane sections, at least when there are a finite number of G -orbits on V . Donagi [9] gives such explicit descriptions for the Grassmanian $G(3, 6)$ and we generalize his description to the F-planes of the third row of the chart, as well as recording the sections of the Severi varieties.

Proposition 8.1 *There are three types of hyperplane sections of a Severi variety:*

1. *homogeneous sections, this is the generic case;*
2. *sections with a unique singular point, which is an ordinary quadratic singularity;*
3. *sections whose singular locus is an $\mathbb{A}\mathbb{P}^1 (= Q^m)$.*

Proposition 8.2 *The four types of hyperplane sections of $G_w(\mathbb{A}^3, \mathbb{A}^6)$ are:*

1. *smooth generic sections;*
2. *sections with a unique singularity, an ordinary quadratic singularity;*
3. *sections that are singular along a smooth quadric of dimension $m + 1$;*
4. *sections whose singular locus is a cone over an $\mathbb{A}\mathbb{P}^2$.*

These descriptions follow from our discussions above. More precisely, if G acts on V with a finite number of orbits and with closed orbit $X \subset \mathbb{P}V$, the orbit structure of G on V^* is the same as that on V . A generic hyperplane section is always smooth, and if the dual variety is nondegenerate, smooth points of the dual variety give rise to sections with a unique singularity that is an ordinary quadratic singularity. (In general, if X^* has defect δ_* , and $H \in X^*_{smooth}$, then $(X \cap H)_{sing}$ is a \mathbb{P}^{δ_*} .)

In the Severi case, the third type of section follows from Freudenthal's perspective, an F -line has contact with an $\mathbb{A}\mathbb{P}^1$. In the $G_w(\mathbb{A}^3, \mathbb{A}^6)$ case, the third and fourth types of sections again follow from Freudenthal where we identify $[y] \in \mathbb{P}V$ with $[y^*] = [\Omega(y, \cdot)] \in \mathbb{P}V^*$. We have $H \supset \tilde{T}_z G_w(\mathbb{A}^3, \mathbb{A}^6)$ if and only if $H^* \in \tilde{T}_z G_w(\mathbb{A}^3, \mathbb{A}^6)$ and H and H^* are in isomorphic orbit closures. The vertex of the cone in the last case is of course H^* .

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