ON SIMPLICIAL TORIC VARIETIES WHICH ARE SET-THEORETIC COMPLETE INTERSECTIONS

MARGHERITA BARILE, MARCEL MORALES AND APOSTOLOS THOMA

[1] ABSTRACT.In this paper we prove:

1. *In characteristic $p > 0$ every simplicial toric affine or projective variety with full parametrization is a set-theoretic complete intersection. This extends previous results by R. Hartshorne [2] and T.T. Moh[4].*
2. *In any characteristic, every simplicial toric affine or projective variety with full parametrization is an almost set-theoretic complete intersection. This extends previous known results by M. Barile-M. Morales [1] and A. Thoma[8].*
3. *In any characteristic, every simplicial toric affine or projective variety of codimension two is an almost set-theoretic complete intersection.*

Moreover the proofs are constructive and the equations we find are binomial ones.

Introduction

An important problem in Algebraic Geometry is to determine the minimum number of equations needed to define an algebraic variety V set-theoretically: if $I = I(V)$ is the defining ideal of V, this number is called the *arithmetical rank* of I and is denoted $ara(I)$. In this paper we only consider ideals generated by binomials. It is natural to define the *binomial arithmetical rank* of a binomial ideal I (written $bar(I)$) as the smallest integer s for which there exist binomials $f_1, \ldots, f_s$ in I such that $rad(I) = rad(f_1, \ldots, f_s)$. Hence the binomial arithmetical rank is an upper bound for the arithmetical rank of a binomial ideal. From the definitions we deduce the following inequality for a binomial ideal I:

$$h(I) \leq ara(I) \leq bar(I) \leq \mu(I).$$

Here $h(I)$ denotes the height and $\mu(I)$ denotes the minimal number of generators of I. When $h(I) = ara(I)$ the ideal I (and the variety V as well) is called a *set-theoretic complete intersection (s.t.c.i)*, when $h(I) = \mu(I)$ it is called a *complete intersection*. The ideal I is called an *almost set-theoretic complete intersection* if $ara(I) \leq h(I) + 1$. The binomial arithmetical rank was computed for the defining ideals of monomial curves in $\mathbf{P}^n_K$ in a

[1]A.M.S. Classification: 13A35,14M10,14M25

Mots clés : Toric variety, complete intersection, set-theoretic complete intersection

series of articles (see [1, 4, 8]). Here is a summary of the results.
Let C be a monomial curve in $\mathbf{P}^n_K$.
(i) If the characteristic of K is positive, then $bar(I(C)) = n-1$
(ii) If the characteristic of K is zero, then $bar(I(C)) = n-1$ if C is a complete intersection and $bar(I(C)) = n$, otherwise. In this article we extend these results and we prove that:

1. *in characteristic $p > 0$ any simplicial toric affine or projective variety with full parametrization is a set-theoretic complete intersection.*

2. *in any characteristic, any simplicial toric affine or projective variety with full parametrization is an almost set-theoretic complete intersection.*

3. *In any characteristic, every simplicial toric affine or projective variety of codimension two is an almost set-theoretic complete intersection.*

In the sequel we shall use the following notation: Let K be a field. A *toric variety* V of codimension $r \geq 2$ in K^{n+r} is a variety having the following parametrization:

$$\begin{aligned} x_1 &= u_1^{d_1} \\ &\vdots \\ x_n &= u_n^{d_n} \\ y_1 &= u_1^{a_{1,1}} \ldots u_n^{a_{1,n}} \\ &\vdots \\ y_r &= u_1^{a_{r,1}} \ldots u_n^{a_{r,n}} \end{aligned}$$

for some positive integers $d_1, \ldots, d_n$ and some integers $a_{i,j}$, where, for all $i = 1, \ldots, r$, at least one of $a_{i,1}, \ldots, a_{i,n}$ is non zero. Here we refer to the definition of toric variety given in [7], which also includes non normal varieties. The toric variety V is called *simplicial* if all the exponents are nonnegative. Let $I = I(V)$ be the ideal formed by the polynomials of $K[x_1, \ldots, x_n, y_1, \ldots, y_r]$ vanishing on V. We shall refer to I as to the defining ideal of V. The ideal I has a system of generators formed by binomials which are differences of two monomials with coefficient 1. A proof is given in [7].

1 Simplicial toric varieties with full support are s.t.c.i. in characteristic $p > 0$

1.1 General results

We refer to the variety V and its parametrization introduced above. Let

$$\Phi : \mathbb{Z}^r \to \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_n\mathbb{Z}$$

be the homomorphism of groups defined by

$$(s_1, \ldots, s_r) \mapsto ([s_1a_{1,1} + s_2a_{2,1} + \ldots - s_ra_{r,1}], \ldots, [s_1a_{1,n} + s_2a_{2,n} + \ldots - s_ra_{r,n}])$$

The elements of the lattice
$$\{(s_1, \ldots, s_r) \in Ker\Phi\}$$
are in a one-to-one correspondence with the binomials of I. Moreover $Ker\Phi$ admits a basis of the form

$$\{(s_{(-1,1)}, 0, \ldots, 0), \quad (s_{(0,1)}, s_{(0,2)}, 0, \ldots, 0), \quad \ldots, (s_{(r-2,1)}, s_{(r-2,2)}, \ldots, s_{(r-2,r)})\}.$$

These are simple generalizations of [5], Remark 2.1.2. For the sake of simplicity we shall put $t_0 = s_{(r-2,r)}$, $\vec{s} = (s_1, \ldots, s_{r-1})$, $\underline{y} = (y_1, \ldots, y_{r-1})$. In particular, if $(\vec{s}, t) \in Ker\Phi$, then $t \in t_0\mathbb{Z}$ and, conversely, for all multiples t of t_0 there is $\vec{s} \in \mathbb{Z}^{r-1}$ such that $(\vec{s}, t) \in Ker\Phi$.

Remark 1 *For all $\vec{s} \in \mathbb{Z}^{r-1}$ let $\vec{s}_+$ denote the positive part and $\vec{s}_-$ the negative part of $\vec{s}$. Fix an element $(\vec{s}, s_r) \in Ker\Phi$, and let*

$$s_1 a_{1,i} + s_2 a_{2,i} + \ldots - s_r a_{r,i} = v_i d_i,$$

for all $i = 1, \ldots, n$. Let $\vec{v}_+$ denote the positive part and $\vec{v}_-$ the negative part of $(v_1, \ldots, v_n)$. The binomial corresponding to $\vec{s}$ is then

$$\underline{y}^{\vec{s}_+} \underline{x}^{\vec{v}_-} - y_r^{s_r} \underline{y}^{\vec{s}_-} \underline{x}^{\vec{v}_+},$$

if $s_r \geq 0$, otherwise it is

$$\underline{y}^{\vec{s}_+} y_r^{-s_r} \underline{x}^{\vec{v}_-} - \underline{y}^{\vec{s}_-} \underline{x}^{\vec{v}_+}.$$

Let
$$J = I \cap K[x_1, \ldots, x_n, y_1, \ldots, y_{r-1}].$$
Then J is the defining ideal of the simplicial toric variety of codimension $r-1$ having the following parametrization:

$$\begin{aligned}
x_1 &= u_1^{d_1} \\
&\vdots \\
x_n &= u_n^{d_n} \\
y_1 &= u_1^{a_{1,1}} \ldots u_n^{a_{1,n}} \\
&\vdots \\
y_{r-1} &= u_1^{a_{r-1,1}} \ldots u_n^{a_{r-1,n}}.
\end{aligned}$$

We introduce one more piece of notation. Let M_1, M_2 be monomials, and let $h = M_1 - M_2$. For all positive integers q we set

$$h^{(q)} = M_1^q - M_2^q.$$

Lemma 1 *Let $\delta > 0$ be an integer for which there is a binomial*

$$f_r = y_r^{t_0\delta} - \underline{y}^{\vec{s}\delta} x_1^{l_1} \dots x_n^{l_n} \in I.$$

Then

$$h^{(\delta)} \in (J, f_r)$$

for all binomials h in I.

Proof .-Let $h \in I$ be a binomial. Since I is a prime ideal, we may assume that

$$h = y_r^{t_0\rho} g_1 - g_2$$

for some monomials $g_1, g_2 \in K[x_1, \dots, x_n, y_1, \dots, y_{r-1}]$. Then

$$\begin{aligned} h^{(\delta)} &= y_r^{t_0\rho\delta} g_1^\delta - g_2^\delta \\ &= (f_r^{(\rho)} + (\underline{y}^{\vec{s}\delta} x_1^{l_1} \dots x_n^{l_n})^\rho) g_1^\delta - g_2^\delta \\ &\in (J, f_r). \end{aligned}$$

1.2 Full parametrization

We say that the above parametrization of V is full if $a_{i,j} \neq 0$ for all (i,j). In this case the parametrization of the variety defined by J is full, too.

Lemma 2 *For all sufficiently large integers $\delta > 0$ there is a binomial*

$$f_r = y_r^{t_0\delta} - \underline{y}^{\vec{s}} x_1^{l_1} \dots x_n^{l_n} \in I.$$

Proof .-Let $\delta > 0$. There is $\vec{s'}$ such that $(\vec{s'}, t_0) \in Ker\Phi$. There are also some integers $r'_1, \dots, r'_n$ for which

$$\sum_{j=1}^{r-1} s'_j a_{j,i} - t_0 a_{r,i} = r'_i d_i$$

for all i. Multiplying this relation by $\delta > 0$ we obtain

$$\sum_{j=1}^{r-1} \delta s'_j a_{j,i} - t_0 \delta a_{r,i} = \delta r'_i d_i$$

for all i. Let $d = lcm\{d_1, ..., d_n\}$, then up to replacing $\delta s'_j$ with its residue modulo d, for all i we get a relation

$$\sum_{j}^{r-1} s_j a_{j,i} - t_0 \delta a_{r,i} = r_i d_i,$$

where $0 \leq s_j < d$ for all j. Thus, if δ is sufficiently large, we will have $r_i < 0$ for all i. But then

$$f_r = y_r^{t_0\delta} - \underline{y}^{\overrightarrow{s}} x_1^{-r_1} \dots x_n^{-r_n} \in I$$

is the binomial required.

As an immediate consequence we have:

Corollary 1 *Let p be a prime number. For all sufficiently large integers m there is a binomial*

$$f_r = y_r^{t_0 p^m} - \underline{y}^{\overrightarrow{s}_m} x_1^{l_1} \dots x_n^{l_n} \in I$$

Theorem 1 *Suppose that $\mathrm{char}\, K = p > 0$. Then every simplicial toric variety having a full parametrization is a set-theoretic complete intersection.*

Proof .-We proceed by induction on $r \geq 1$. Since the polynomial ring $K[x_1, \dots, x_n, y_1]$ is an UFD the claim is true for $r = 1$.
Suppose that $r \geq 2$ and the claim is true in codimension $r - 1$. Let $h \in I$ be a binomial, then by Corollary 1 and Lemma 1 we get

$$h^{p^m} = h^{(p^m)} \in (f_r, J)$$

for m sufficiently large. By the inductive hypothesis the ideal J is set-theoretically generated by $r - 1$ binomials $f_1, \dots, f_{r-1}$. Hence some power of h lies in $(f_1, \dots, f_r)$.

Remark 2 *Note that the proof of the preceding result yields a recursive construction of the defining equations of the simplicial toric variety for any field K of characteristic $p > 0$.*

2 Almost set-theoretic complete intersections

In this section we show that simplicial toric varieties having a full parametrization are almost set-theoretic complete intersections.

With respect to the notations introduced above, for all $i = 1, \dots, r$ let

$$A_i = \begin{pmatrix} d_1 & 0 & \dots & 0 & a_{1,1} & \dots & a_{i,1} \\ 0 & d_2 & \dots & 0 & a_{1,2} & \dots & a_{i,2} \\ & & & & & & \\ 0 & 0 & \dots & d_n & a_{1,n} & \dots & a_{i,n} \end{pmatrix}.$$

Moreover, let $D[j_1, \dots, j_n]$ be the determinant of the $n \times n$ submatrix consisting of the columns of A_r with the indices $j_1, \dots, j_n$, where $\{j_1, \dots, j_n\}$ is an n-subset of $\{1, 2, \dots, n+r\}$. For all $i = 1, \dots, r$ let $|A_i| := gcd\{D[j_1, \dots, j_n] : 1 \leq j_1 < j_2 < \dots < j_n \leq n + i\}$; for the sake of simplicity we set $g_i = |A_i|$. Moreover, let $e_i = g_{i-1}/g_i$, for all $i = 2, \dots, r$.

Lemma 3 *Let V be a simplicial toric variety. Then for every $i \in \{1, \dots, r\}$ there exist binomials*

$$M_i - N_i y_i^{e_i} \in I(V),$$

where M_i, N_i are monomials in $K[x_1, \ldots, x_n, y_1, \ldots, y_{i-1}]$. If the parametrization of V is full, then for every $i = 2, ..., r$ there exists a binomial

$$F_i = y_{i-1}^{\mu_i} - x_1^{\nu_{i,1}} \cdots x_n^{\nu_{i,n}} y_1^{\mu_{i,1}} \cdots y_{i-2}^{\mu_{i,i-2}} y_i^{e_i} \in I(V),$$

and there also exists a binomial

$$F_1 = y_1^{e_1} - x_1^{\nu_{1,1}} \cdots x_n^{\nu_{1,n}} \in I(V),$$

for some positive integers $\mu_{i,j}$ and $\nu_{i,j}$.

Proof .-In this proof $\mathbf{d_i}$ will denote the ith column vector of A_r for all $i = 1, \ldots, n$, and $\mathbf{a_i}$ will denote the $(n+i)$th column vector of A_r for all $i = 1, \ldots, r$. Set $\mu = lcm(d_1, \ldots, d_n)$ and $q_i = gcd(\mu, a_{i,1}, \ldots, a_{i,n})$ for all $i = 1, \ldots, r$. For all $i = 1, \ldots, r$ and all $j = 1, \ldots, n$ let $\rho_{i,j} = a_{i,j}\mu/d_j q_j$. Then, for all $i = 1, \ldots, r$, one has that

$$G_i = y_i^{\mu/q_i} - x_1^{\rho_{i,1}} \cdots x_n^{\rho_{i,n}} \in I(V).$$

It is easy to see that $e_1 = \mu/q_1$, then for $i = 1$ the preceding formula yields the required binomial F_1.
By a basic lemma in number theory (see [3]) the diophantine system $Ax = b$ has a solution iff $|A| \neq 0$ and $|A| = |Ab|$, whereAb is the augmented matrix.
Let $2 \leq i \leq r$. The integer g_{i-1} is a divisor of $D[1, \ldots, n] = d_0 d_1 \ldots d_n \neq 0$, hence $g_{i-1} \neq 0$. On the other hand it holds:

$$g_i = gcd\{g_{i-1}, D[j_1, \ldots, j_{n-1}, n+i] : 1 \leq j_1 < j_2 < \cdots < j_n \leq n+i-1\}, \qquad (1)$$

$$\begin{aligned} |A_{i-1}, e_i \mathbf{a_i}| &= gcd\{g_{i-1}, (g_{i-1}/g_i) D[j_1, \ldots, j_{n-1}, n+i] : 1 \leq j_k \leq n+i-1\} \\ &= (g_{i-1}/g_i) gcd\{g_i, D[j_1, \ldots, j_{n-1}, n+i] : 1 \leq j_k \leq n+i-1\} = g_{i-1}. \end{aligned}$$

Hence the diophantine system $A_{i-1}\mathbf{x} = e_i \mathbf{a_i}$ always has a solution. This means that the vector $e_i \mathbf{a_i}$ can be expressed as a linear combination of the vectors $\mathbf{d_1}, \ldots, \mathbf{d_n}, \mathbf{a_1}, \ldots, \mathbf{a_{i-1}}$ with integer coefficients, i.e., one has

$$e_i \mathbf{a_i} = t_1 \mathbf{d_1} + \ldots + t_n \mathbf{d_n} + t_{n+1} \mathbf{a_1} + \ldots + t_{n+i-1} \mathbf{a_{i-1}} \qquad (2),$$

for some integers $t_1, \ldots, t_{n+i-1}$. This expression give us monomials M_i, N_i in $K[x_1, \ldots, x_n, y_1, \ldots, y_{i-1}]$ such that $M_i - N_i y_i^{e_i} \in I(V)$.
Now suppose that the parametrization of V is full. From the binomial G_j we see that for each $\mathbf{a_j}$ there exist positive integers $\rho_j = \mu/q_i, \rho_{j,1}, \ldots, \rho_{j,n}$ such that $\rho_j \mathbf{a_j} = \rho_{j,1}\mathbf{d_1} + \ldots + \rho_{j,n}\mathbf{d_n}$. Furthermore, for all $1 \leq j \leq i-2$ there exists a positive integer ν_j such that, after adding all the zero vectors $\nu_j(\rho_{j,1}\mathbf{d_1} + \ldots + \rho_{j,n}\mathbf{d_n} - \rho_j \mathbf{a_j})$ to the right-hand side of (2), the new coefficient $\mu_{i,k}$ of $\mathbf{a_k}$ is negative for all $k = 1, \ldots, i-2$. There also exists a large positive integer ν_{i-1} such that after adding the zero vector $\nu_{i-1}(\rho_{i-1}\mathbf{a_{i-1}} - (\rho_{i-1,1}\mathbf{d_1} - \ldots - \rho_{i-1,n}\mathbf{d_n}))$ on the right-hand side of the new equation, for all $j = 1, \ldots, n$ the new coefficient $\nu_{i,j}$ of $\mathbf{d_j}$ is negative and the new coefficient μ_i of $\mathbf{a_i}$ is positive. It follows that for all $i = 2 \ldots, r$

$$F_i = y_{i-1}^{\mu_i} - x_1^{\nu_{i,1}} \cdots x_n^{\nu_{i,n}} y_1^{\mu_{i,1}} \cdots y_{i-2}^{\mu_{i,i-2}} y_i^{e_i} \in I(V).$$

Theorem 2 *Let V be a simplicial toric variety having a full parametrization. Then $r \leq bar(I(V)) \leq r+1$.*

Proof .-Consider the r binomials $F_1, F_2, \ldots, F_r$ which were defined in Lemma 3 and let F_{r+1} be any binomial monic in y_r, for example G_r. We claim that $I(V) = rad(F_1, \ldots, F_{r+1})$. By virtue of Hilbert Nullstellensatz the claim is proved once it has been shown that every point $\mathbf{x} = (x_1, \ldots, x_n, y_1, \ldots, y_r)$ which is a common zero of $F_1, \ldots, F_{r+1}$ in $\bar{K}^{n+r}$, where $\bar{K}$ denotes the algebraic closure of K, is also a point of V. First of all note that if $x_k = 0$ for some index k, then $y_j = 0$ for all indices j. It is then easy to find $u_1, \ldots, u_n \in \bar{K}$ which allow us to write $\mathbf{x}$ as a point of V. Now suppose that $x_k \neq 0$ for all indices k. By induction on i, $2 \leq i \leq r+1$, we show: if $\mathbf{x}$ is a zero of $F_1, \ldots, F_{i-1}$, then the coordinates of $\mathbf{x}$ fulfil the parametrization of V. The claim is easy for $i = 2$. Now fix an index i, $2 \leq i \leq r+1$. By the induction hypothesis there are nonzero $u_1, \ldots, u_n \in \bar{K}$ such that

$$x_1 = u_1^{d_1}, \ldots, x_n = u_n^{d_n}, \qquad y_1 = u_1^{a_{1,1}} \ldots u_n^{a_{1,n}}, \ldots, y_{i-1} = u_1^{a_{i-1,1}} \ldots u_n^{a_{i-1,n}}.$$

Since the point $\mathbf{x}$ is also a zero of F_i, we deduce that $y_i = \omega u_1^{a_{i,1}} \ldots u_n^{a_{i,n}}$, where ω is a suitable e_i-root of unity. Let $\zeta \in \bar{K}$ be such that $\zeta^{g_i} = \omega$, so that $\zeta^{g_{i-1}} = 1$. By (1) and Bézout's Identity there exist integers k_0 and $k_{j_1,\ldots,j_{n-1},n+i}$ such that

$$g_i = k_0 g_{i-1} + \sum k_{j_1,\ldots,j_{n-1},n+i} D[j_1, \ldots, j_{n-1}, n+i].$$

All the $D[j_1, \ldots, j_{n-1}, n+i]$ are linear combinations of $a_{i,1}, \ldots, a_{i,n}$. Therefore there exist $l_1, \ldots, l_n$ such that $g_i = k_0 g_{i-1} + l_1 a_{i,1} + \ldots + l_n a_{i,n}$. Setting $v_j = \zeta^{l_j} u_j$, we have that

$$x_1 = v_1^{d_1}, \ldots, x_n = v_n^{d_n}, \qquad y_1 = v_1^{a_{1,1}} \ldots v_n^{a_{1,n}}, \ldots, y_{i-1} = v_1^{a_{i-1,1}} \ldots v_n^{a_{i-1,n}}, \quad y_i = v_1^{a_{i,1}} \ldots v_n^{a_{i,n}},$$

since

$$\omega = \zeta^{g_i} = \zeta^{k_0 g_{i-1} + l_1 a_{i,1} + \ldots + l_n a_{i,n}} = \zeta^{l_1 a_{i,1} + \ldots + l_n a_{i,n}}$$

and also $1 = \zeta^{l_j d_j}$, since

$$l_j d_j = l_1 \times 0 + \ldots + l_j d_j + \ldots + l_n \times 0 = \sum k_{j_1,\ldots,j_{n-1},n+i} D[j_1, \ldots, j_{n-1}, j].$$

Moreover, $1 = \zeta^{l_1 a_{f,1} + \ldots + l_n a_{f,n}}$ for $f < i$. In fact one has that

$$l_1 a_{f,1} + \ldots + l_n a_{f,n} = \sum k_{j_1,\ldots,j_{n-1},n+i} D[j_1, \ldots, j_{n-1}, f+n]$$

and one of the following two cases occurs: either $f+n$ is one of the $j_1, \ldots, j_{n-1}$, and $D[j_1, \ldots, j_{n-1}, f+n] = 0$, or $f+n$ is different from $j_1, \ldots, j_{n-1}$, and $D[j_1, \ldots, j_{n-1}, f+n]$ is a multiple of g_{i-1}, since then $D[j_1, \ldots, j_{n-1}, f+n]$ is a subdeterminant of A_{i-1}. We have shown that the coordinates of the point $\mathbf{x}$ fulfil the parametrization of V.

3 Results in arbitrary characteristic and codimension 2

We show that Theorem 2 can be generalized: it can extended to the varieties which do not have a full parametrization, at least in codimension 2.
In this section we suppose that $r = 2$, i.e., V is a simplicial toric variety of codimension 2 in K^{n+2}. The parametrization of V now is:

$$\begin{aligned} x_1 &= u_1^{d_1} \\ &\vdots \\ x_n &= u_n^{d_n} \\ y_1 &= u_1^{a_{1,1}} \dots u_n^{a_{1,n}} \\ y_2 &= u_1^{a_{2,1}} \dots u_n^{a_{2,n}}, \end{aligned}$$

were the vectors $\mathbf{a}_1, \mathbf{a}_2$ may have zero components.

Theorem 3 *Let V be a simplicial toric variety of codimension 2. Then $2 \leq bar(I(V)) \leq 3$.*

Proof .-Let J_1 be the defining ideal of the simplicial toric variety having the following parametrization:

$$\begin{aligned} x_1 &= u_1^{d_1} \\ &\vdots \\ x_n &= u_n^{d_n} \\ y_1 &= u_1^{a_{1,1}} \dots u_n^{a_{1,n}}, \end{aligned}$$

and let J_2 be the defining ideal of the simplicial toric variety having the following parametrization:

$$\begin{aligned} x_1 &= u_1^{d_1} \\ &\vdots \\ x_n &= u_n^{d_n} \\ y_2 &= u_1^{a_{2,1}} \dots u_n^{a_{2,n}}. \end{aligned}$$

Both varieties have codimension one, therefore their defining ideals are principal. If we set

$$\mu = \text{lcm } (d_1, \dots, d_n),$$

$$q_1 = \text{gcd } (\mu, a_{1,1}, \dots, a_{1,n}), \quad e_1 = \mu/q_1 \qquad \text{and}$$
$$q_2 = \text{gcd } (\mu, a_{2,1}, \dots, a_{2,n}), \quad e_1' = \mu/q_2,$$

then J_1 is generated by

$$F_1 = y_1^{e_1} - x_1^{\frac{a_{1,1}\mu}{d_1 q_1}} \cdots x_n^{\frac{a_{1,n}\mu}{d_n q_1}}$$

and J_2 is generated by

$$F_3 = y_2^{e'_1} - x_1^{\frac{a_{2,1}\mu}{d_1 q_2}} \cdots x_n^{\frac{a_{2,n}\mu}{d_n q_2}}.$$

Note that F_1 is the difference of a power of y_1 and a monomial which only involves the variables x_k such that $k \in Supp(\mathbf{a}_1)$; similarly, F_3 is the difference of a power of y_2 and a monomial which only involves the variables x_k such that $k \in Supp(\mathbf{a}_2)$.
Ley ECt $F_2 = M_1 - M_2 y_2^{e_2} \in I(V)$ be the binomial given in in Lemma 3. We claim that $I(V) = rad(F_1, F_2, F_3)$.
Let $\mathbf{x} = (x_1, \ldots, x_n, y_1, y_2)$ be a common zero of F_1, F_2, F_3 in $\bar{K}^{n+2}$, where $\bar{K}$ denotes the algebraic closure of K. We show that $\mathbf{x}$ lies on V. If $x_k \neq 0$ for all indices k, then the claim can be easily proven.
Now suppose that $x_k = 0$ for at least one index $k \in Supp(\mathbf{a_1}) \cup Supp(\mathbf{a_2})$. One of the following cases occurs.

(i) If $k \in Supp(\mathbf{a}_1) \cap Supp(\mathbf{a}_2)$, then $F_1(\mathbf{x}) = 0$ implies that $y_1 = 0$, and $F_3(\mathbf{x}) = 0$ implies that $y_2 = 0$. For all $i = 1, \ldots, r$ let $u_i \in \bar{K}$ be such that $u_i^{d_i} = x_i$. These parameters allow us to write $\mathbf{x}$ as a point of V.

(ii) If $k \in Supp(\mathbf{a}_1)$, then, again, $F_1(\mathbf{x}) = 0$ implies that $y_1 = 0$; moreover, $F_3(\mathbf{x}) = 0$ implies that $\mathbf{x}$ is a point of $V(J_2)$, with respect to some $u_1, \ldots, u_n \in \bar{K}$: the same values of the parameters yield a representation of $\mathbf{x}$ as a point of V.

(iii) If $k \in Supp(\mathbf{a}_2)$, one can proceed as in (ii).

Remark 3 *1. It is easy to see that if α is an integer for which the equation $A_{i-1}\mathbf{x} = \alpha\mathbf{a_i}$ has an integer solution, then α is a multiple of e_i. We know from the results in Section 1 that there is a binomial $y_1^{s_0} M_1 - y_2^{t_0} M_2 \in I$, where M_1, M_2 are monomials of $K[x_1, \ldots, x_n]$. Hence t_0 is a multiple of e_2; on the other hand we know that in any binomial of I the exponent of y_2 is a multiple of t_0. Thus $t_0 = e_2$.*

2. It follows from [5], Th. 3.5. that the binomials F_1, F_2, F_3 belong to a Gröbner basis of I.

Corollary 2 *With respect to the notations introduced above, one has that*

1. if $e'_1 = p^m e_2$ for some prime $p > 0$, then V is a set-theoretic complete intersection in characteristic p.

2. if $e'_1 = e_2$ then V is a complete intersection in any characteristic.

Proof .-

1. It is clear that $F_2^{p^m} \in (F_1, F_3)$.

2. This is immediate from [5], Th. 3.5.

Example Let V be the simplicial toric variety of codimension 2 parametrized by

$$a = s^{20}, b = t^{20}, c = u^{20}, d = v^{20}, y = t^{12}u^5v^3, z = s^{10}t^3v^7.$$

Note that the parametrization is not full. The ideal I is minimally generated by

$$b^{12}c^5d^3 - y^{20}, a^8d^5y^4 - cz^{16}, a^2y^{16} - b^9c^4dz^4,$$

$$a^2b^3cd^2 - y^4z^4, a^4dy^{12} - b^6c^3z^8, a^6d^3y^8 - b^3c^2z^{12}, a^{10}b^3d^7 - z^{20}.$$

With respect to the notations of Theorem 3 one has that

$$g_0 = 20^4, g_1 = 20^3, g_2 = 5 \times 20^2, e_1 = 20, e_2 = 4$$

$$5 \times 20^2 = (7 \times 20 - 10 \times 12 - 3 \times 5) \times 20^2$$

$$l_1 = -12 \times 20^2, l_2 = -5 \times 20^2, l_3 = 0, l_4 = 20 \times 20^2$$

and

$$I = rad(b^{12}c^5d^3 - y^{20}, a^2b^3cd^2 - y^4z^4, a^{10}b^3d^7 - z^{20})$$

in any characteristic different from 5. In characteristic 5 we have that:

$$I = rad(b^{12}c^5d^3 - y^{20}, a^{10}b^3d^7 - z^{20})$$

Remark 4 *If V is arithmetically Cohen-Macaulay, then according to [5], Th. 3.5, and [6], the variety V is a s.t.c.i. on a binomial and a polynomial.*

References

[1] M. Barile, M. Morales, On the equations defining projective monomial curves. *Comm. Algebra*, **26** (1998), 1907–1912.

[2] R. Hartshorne, Complete intersections in characteristic $p > 0$. *Amer. J. Math.*,**101** (1979), 380–383.

[3] J. Heger, Denkschriften. Kais. Akad. Wissensch. Mathem. Naturw. Klasse, **14** (1858), II.

[4] T.T. Moh, Set-theoretic Complete Intersections, *Proc. AMS* **94** (1985), 217–220.

[5] M. Morales, Equations des Variétés Monomiales en codimension deux, *Journal of Algebra* **175** (1995), 1082–1095.

[6] L. Robbiano, G. Valla, Some curves in $\mathbf{P}^3$ are set-theoretic complete intersections, in Algebraic Geometry - Open Problems, Proc. Ravello 1982, Ciliberto, Ghione, Orecchia editors, Lecture Notes in Mathematics **997**, Springer 1983, 391–399.

[7] B. Sturmfels, Gröbner Bases and Convex Polytopes. University Lecture Series, No. 8, American Mathematical Society Providence, R.I. 1995.

[8] A. Thoma, On the Binomial Arithmetical Rank. Arch. Math. To appear.

Margherita Barile
Dipartimento di Matematica, Università degli Studi di Bari,
Via Orabona 4, 70125 Bari (ITALY)

Marcel Morales
Université de Grenoble I, Institut Fourier, UMR 5582, B.P.74,
38402 Saint-Martin D'Hères Cedex,
and IUFM de Lyon, 5 rue Anselme,
69317 Lyon Cedex (FRANCE)

Apostolos Thoma
Department of Mathematics,
University of Ioannina, Ioannina 45110 (GREECE)